MATH 8820

Homework Assignment 2

Problem 1: In this problem we will explore an alternate class of priors.

(a) Find a resource and write a basic description of what a Jeffrey's prior is. Be thorough, your exposition here needs to be such that someone who knows nothing of what a Jeffrey's prior is could learn the salient features. Identify why people like Jeffery's priors and discuss drawbacks.

(b) Derive the Jeffery's prior for the Bernoulli model (i.e., assuming $Y_i \stackrel{iid}{\sim} \text{Bernoulli}(p)$), and find the posterior distribution of p under this prior. Is it a known distribution? Discuss how you would proceed to conduct posterior inference.

(c) Derive the Jeffery's prior for the Poisson model (i.e., assuming $Y_i \stackrel{iid}{\sim} \text{Poisson}(\lambda)$), and find the posterior distribution of λ under this prior. Is it a known distribution? Discuss how you would proceed to conduct posterior inference.

(d) Derive the Jeffery's prior for the exponential model (i.e., assuming $Y_i \stackrel{iid}{\sim} \text{Exponential}(\beta)$), and find the posterior distribution of β under this prior. Is it a known distribution? Discuss how you would proceed to conduct posterior inference.

Problem 2: Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, such that X_1 and X_2 are independent. Define $Y = X_1 + X_2$. The goal of this problem is to determine the distribution of Y; i.e., the sum of two independent normal random variables.

(a) Obviously, this is trivial, so simply state the distribution of Y.

(b) Approach 1: Consider using Monte Carlo sampling to obtain a histogram and kernel density estimate (see the code that I have provided) of the pdf of Y by directly sampling both X_1 and X_2. Over plot the true density of Y and comment. Note, you should make use of large enough Monte Carlo sample that your results are reasonable.

(c) Approach 2: Note, the distribution of Y can also be obtained through the convolution of the probability distributions of X_1 and X_2. Sketch out theoretically how this would be done. Based on this idea, create a Monte Carlo sampling technique which can be used to approximate the pdf of Y evaluated at any point in the support. Use this function and add the approximation based on this technique to the Figure described in the part (b) above.

(d) Repeat the two approaches described in (b) and (c) above for $X_1 \sim \text{Gamma}(\alpha_1, \beta_1)$ and $X_2 \sim \text{Gamma}(\alpha_2, \beta_2)$. Note, unless $\beta_1 = \beta_2$, the resulting distribution of Y is not friendly. Also be cautious of the support.

Problem 3: Game of chance, the game of craps and a betting strategy are going to be explored in this Monte Carlo simulation experiment.

(a) Do a search and outline the rules of craps.

(b) Your initial bet will be 10 dollars. The return on any bet will be the amount that you bet; e.g., if you bet 10 dollars and win, then you win 10 dollars. You are going to play exactly 10 games of craps. If you lose on the previous game, your next bet requires you to "double down," that is, if on one game you bet X dollars and lose, on the next game you have to bet $2X$ dollars. If you win on the previous game, your next bet will be 10 dollars. Write a Monte Carol simulation to find your expected winnings after the 10 games. Also examine the distribution of your expected winnings. Does it appear that this is a good betting strategy?

Problem 4: There are three six-sided dice in a bag, all physically balanced but with different labellings on the sides. In particular, each die has 6 sides, with Die A's sides being labeled $\{1, 1, 2, 2, 3, 4\}$, Die B's sides being labeled $\{1, 2, 2, 3, 3, 4\}$, Die C's sides being labeled $\{1, 2, 3, 4, 4, 4\}$. You are blindfolded and you randomly select one die from the bag. You roll this die 29 times, and your friend tells you that your rolls produced the following frequency table of results:

Outcome	Frequency
1	5
2	11
3	6
4	7

You are not told which die you rolled.

(a) The parameter of interest is $\mathbf{p} = (p_1, p_2, p_3, p_4)$, the probability vector for the die that you rolled. What is the parameter space of the problem?

(b) Find the maximum likelihood estimate of $\mathbf{p} = (p_1, p_2, p_3, p_4)$.

(c) Set up a size α hypothesis test of $H_0 : \mathbf{p} = \mathbf{p}_C$ versus $H_1 : \mathbf{p} \neq \mathbf{p}_C$, where $\mathbf{p}_C$ denotes the probability vector corresponding to Die C. Hint: you should consider a likelihood ratio test. Use Monte Carlo techniques to compute the p-value of your test.