Ex 1

- $\log_2 n = \mathcal{O}(n)$

$\log_2 n \leq c \cdot n \quad | 2^x$ for all $n \geq n_0$

$n \leq 2^{c \cdot n}$ for $c = 1, n_0 = 1$

$\Rightarrow \log_2 n = \mathcal{O}(n)$ ✓

$$\lim_{n \to \infty} \frac{\log_2 n}{n} = \lim_{n \to \infty} \frac{1/n \cdot \ln 2}{n}$$

$$= \lim_{n \to \infty} \frac{1}{n^2 \ln 2} = 0$$

- $\log_2 n = \Omega(n)$?

$\log_2 n \geq c \cdot n \quad | 2^x$ for all $n \geq n_0$

$\cancel{n \geq 2^{c \cdot n}}$

$$\lim_{n \to \infty} \left(\frac{d}{dn} \log_2 n\right) = \lim_{n \to \infty} \frac{1}{n \cdot \log(2)} = 0$$

$$\lim_{n \to \infty} \frac{d}{dn} c \cdot n = \lim_{n \to \infty} c = c \text{ for } c > 0$$

$\Rightarrow$ at some point, gradient of $c \cdot n$ exceeds gradient of $\log_2 n$, even if $c \to 0$

$\Rightarrow$ at a further point, the graphs will cross due to the higher gradient of $c \cdot n$

Ex 2

- $\log_b n \leq C \cdot n$

$$\lim_{n \to \infty} \frac{\log_b n}{C \cdot n} \qquad \log_b n = \frac{\log_{10} n}{\log_{10} b}$$

$$\Leftrightarrow \lim_{n \to \infty} \frac{\log_{10} n / \log_{10} b}{C \cdot n} \rightarrow \text{const.!}$$

$$\Leftrightarrow \lim_{n \to \infty} \frac{\log_{10} n \cdot C_2}{C_1 \cdot n} = \lim_{n \to \infty} \frac{\frac{1}{n \cdot \log_{10}} \cdot C_2}{C_1} = 0$$

C_1 can be chosen freely: $C_1 = C_2$

$$\lim_{n \to \infty} \frac{1}{n \cdot \log_{10}} = 0$$

- for $b = 1$:

$$\log_1 n \begin{cases} \text{undefined} & \text{for } n = 1 \\ \infty & \text{for } n > 1 \end{cases}$$

for $b < 1$:

$$\log_b n \begin{cases} > 0 & \text{for } n < 1 \\ 0 & \text{for } n = 1 \\ < 0 & \text{for } n > 1 \end{cases}$$

- Proposition:

$\log_b n \leq C \cdot n$ for $C = 1$ and $n_0 = 1$

if $b = \sqrt[4]{n}$:

$$\log_{\sqrt[4]{n}} n \leq n \qquad \log_{\sqrt[4]{n}} n = \frac{\log n}{\log n^{1/4}} = \frac{1}{1/4} \cdot \frac{\log n}{\log n}$$

$$= 4$$

$4 \leq n$ for $n_0 > 1$ ✓

Ex 3

order by runtime complexity :

f_1: $\sqrt{n}$

f_2: $n \log_{10} n$

f_3: $n \log_2 n^2$

f_4: n^2

f_5: $n^2 \log_2 n^2$

$f_1 = O(f_2)$

$$i=1 \quad \lim_{n\to\infty} \frac{\sqrt{n}}{n \log_{10} n} = \lim_{n\to\infty} \frac{1}{\sqrt{n} \log_{10} n} = 0 \quad ✓$$

$$i=2 \quad \lim_{n\to\infty} \frac{\cancel{n} \log_{10} n}{\cancel{n} \log_2 n^2} = \lim_{n\to\infty} \frac{\log 2 (\log n + 1)}{\log 10 (\log n^2 + 2)}$$

$$= \lim_{n\to\infty} \frac{\log 2}{2 \log 10} < \infty \quad ✓$$

$$i=3 \quad \lim_{n\to\infty} \frac{\cancel{n} \log_2 n^2}{n^{\cancel{2}}} \lim_{n\to\infty} \frac{2/n \log(2)}{1} = 0 \quad ✓$$

$$i=4 \quad \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2} \log_2 n^2} = \lim_{n\to\infty} \frac{1}{\log_2 n^2} = 0 \quad ✓$$

for $f_i = O(f_{i+1})$ to hold: $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

for $f_i = \Theta(f_{i+1})$ to hold : $a \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$\Rightarrow$ $f_i = \Theta(f_i + 1)$ does not hold for $i = 1, 3, 4$

$f_i = \Theta(f_i + 1)$ holds for $i = 2$:

$f_2 \in \Theta(f_3)$