

Ex 1

• $\log_2 n = \mathcal{O}(n)$

$$\log_2 n \leq C \cdot n \quad | 2^x \text{ for all } n \geq n_0$$

$$n \leq 2^{C \cdot n} \quad \text{for } C=1, n_0=1$$

$$\Rightarrow \log_2 n = \mathcal{O}(n) \quad \checkmark$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log_2 n}{n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n \cdot \ln 2}}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2 \ln 2} = 0 \end{aligned} \quad \square$$

• $\log_2 n = \Omega(n)$?

$$\log_2 n \geq C \cdot n \quad | 2^x \text{ for all } n \geq n_0$$

~~$n \geq 2^{C \cdot n}$~~

$$\lim_{n \rightarrow \infty} \left(\frac{d}{dn} \log_2 n \right) = \lim_{n \rightarrow \infty} \frac{1}{n \cdot \log 2} = 0$$

$$\lim_{n \rightarrow \infty} \frac{d}{dn} (C \cdot n) = \lim_{n \rightarrow \infty} C = C \quad \text{for } C > 0$$

\Rightarrow at some point, gradient of $C \cdot n$ exceeds gradient of $\log_2 n$, even if $C \rightarrow 0$

\Rightarrow at a further point, the graphs will cross due to the higher gradient of $C \cdot n$



Ex 2

$$\log_b n \leq C \cdot n$$

$$\lim_{n \rightarrow \infty} \frac{\log_b n}{C \cdot n}$$

$$\log_b n = \frac{\log_{10} n}{\log_{10} b}$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \frac{\log_{10} n / \log_{10} b}{C \cdot n} \rightarrow \text{const.}!$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \frac{\log_{10} n \cdot C_2}{C_1 \cdot n} = \lim_{n \rightarrow \infty} \frac{1}{n \cdot \log_{10}} \cdot C_2 = 0$$

C_1 can be chosen freely: $C_1 = C_2$

$$\lim_{n \rightarrow \infty} \frac{1}{n \cdot \log_{10}} = 0$$

for $b = 1$:

$$\log_1 n \begin{cases} \text{undefined} & \text{for } n=1 \\ \infty & \text{for } n > 1 \end{cases}$$

for $b < 1$:

$$\log_b n \begin{cases} > 0 & \text{for } n < 1 \\ = 0 & \text{for } n = 1 \\ < 0 & \text{for } n > 1 \end{cases}$$

Proposition:

$$\log_b n \leq C \cdot n \quad \text{for } C=1 \text{ and } n_0=1$$

if $b = \sqrt[4]{n}$:

$$\log_{\sqrt[4]{n}} n \leq n \quad \log_{\sqrt[4]{n}} n = \frac{\log n}{\log n^{1/4}} = \frac{1}{1/4} \cdot \frac{\log n}{\log n} = 4$$

$$4 \leq n \quad \text{for } n_0 > 1$$

Ex 3

order by runtime complexity :

$$f_1: \sqrt{n}$$

$$f_2: n \log_{10} n$$

$$f_3: n \log_2 n^2$$

$$f_4: n^2 \log_2 n^2$$

$$f_5: n^2 \log_2 n^2$$

$$f_1 = O(f_2)$$

$$i=1 \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n \log_{10} n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n} \log_{10} n} = 0 \quad \checkmark$$

$$i=2 \lim_{n \rightarrow \infty} \frac{n \log_{10} n}{n \log_2 n^2} = \lim_{n \rightarrow \infty} \frac{1}{\log_2 10 \log_2 (n^2 + 1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2 \log_2 10} < \infty \quad \checkmark$$

$$i=3 \lim_{n \rightarrow \infty} \frac{\sqrt{n} \log_2 n^2}{n^2 \log_2 n^2} = \lim_{n \rightarrow \infty} \frac{1}{n \log_2 (2)} = 0 \quad \checkmark$$

$$i=4 \lim_{n \rightarrow \infty} \frac{n^2}{n^2 \log_2 n^2} = \lim_{n \rightarrow \infty} \frac{1}{\log_2 n^2} = 0 \quad \checkmark$$

for $f_i = O(f_{i+1})$ to hold: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty$

for $f_i = \Theta(f_{i+1})$ to hold: $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty$

$\Rightarrow f_i = \Theta(f_{i+1})$ does not hold for $i = 1, 3, 4$

$f_i = \Theta(f_{i+1})$ holds for $i = 2$:

$f_2 \in \Theta(f_3)$