Assignment - I

1. Show that --- $W \neq \mathbb{R}^n$

Case 1: $n = 1$:

$V = \mathbb{R} \Rightarrow$ only possible subspaces $\{0\}, \mathbb{R}$.

Case 2: $n \geq 2$

$V = \mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$

Take $W = \text{Span}(\{(1,0,\ldots 0)\}) = \langle (1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}) \rangle$

then $W \leq \mathbb{R}^n$ and $W \neq \mathbb{R}$

Hence, $\mathbb{R}^n$ contains a subspace $W \neq \{0\}$ such that $W \neq \mathbb{R}^n$.

2. Prove that --- independent.

$$\alpha_1(1,0,0,-1) + \alpha_2(0,1,0,-1) + \alpha_3(0,0,1,-1) + \alpha_4(0,0,0,1) = (0,0,0,0)$$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0,$

$-\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0$

$\Rightarrow \alpha_4 = 0$

Hence, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$\Rightarrow$ Set $\{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$ is Linearly Independent.

3. Find the rank --- matrices:

In Row Echelon form

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ $R_{12}(-4)$, $R_{13}(-7)$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_{23}(-2) \\ R_2\left(\frac{-1}{3}\right) \end{matrix}$$

Hence, $\rho(A) = 2$.

(b) $B = \begin{bmatrix} 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 10 & 4 & 7 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 4 & 1 & 3 \\ 5 & 10 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \\ \end{matrix} \quad R_{23}$$

Clearly, $R_1 \neq \alpha R_2$ for any $\alpha \in \mathbb{R}$

i.e. R_1 & R_2 are not dependent

Hence $\rho(B) = 2$.

(c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Already in row reduced echelon form

$\Rightarrow \rho(C) = 3$.

4. Let S --- $v \in \text{Span}(S)$

We have S as linearly independent set of $\mathbb{R}^n$

And $v \in \mathbb{R}^n$, $v \notin S$ (2)

i.e. $v \in \mathbb{R}^n \setminus S$

$\boxed{\Rightarrow}$ $S \cup \{v\}$ is LD, Let $S = \{x_1, -, x_m\}$

Suppose $v \notin \operatorname{span}(S)$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + - + \alpha_m x_m + \beta v = 0$

such that not all α_i or β are zero.

Now, if $\beta = 0$,

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + - + \alpha_m x_m = 0$

such that not all α_i are zero

But that implies

S is LD.

a that is contradiction

$\therefore \beta \neq 0$ (cannot be)

$$\Rightarrow v = \frac{-\alpha_1}{\beta} x_1 + \frac{(-\alpha_2)}{\beta} x_2 + - + \frac{(-\alpha_m)}{\beta} x_m$$

i.e. v is LC of vectors of S

$\Rightarrow v \in \operatorname{span}(S)$.

$\boxed{\Leftarrow}$ $v \in \operatorname{Span}(S)$

$\Rightarrow v = \alpha_1 x_1 + \alpha_2 x_2 + - + \alpha_m x_m$

for some $\alpha_i \in \mathbb{R}$

$\Rightarrow$ For $\beta = -1$,

$\alpha_1 x_1 + \alpha_2 x_2 + - + \alpha_m x_m + \beta v = 0$

Hence, not all α_i or β are zero

$\Rightarrow S \cup \{v\}$ is LD.

5. Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

(a) Find all e. values of A.

~~char/eq.~~

$\cancel{(4-\lambda)12 - 1(4-2\lambda+2) + 1(2-5+\lambda) = 0}$

char. eq.

$$\begin{vmatrix} (4-\lambda) & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left[(4-\lambda)\left[(5-\lambda)(2-\lambda)+2\right]\right] - 1\left[4-2\lambda+2\right] - 1\left[2-5+\lambda\right] = 0$$

$$\Rightarrow (4-\lambda)(12-7\lambda+\lambda^2) - 1(6-2\lambda) - 1(-3+\lambda) = 0$$

$$\Rightarrow (48-40\lambda+11\lambda^2-\lambda^3) - (6-2\lambda) - (-3+\lambda) = 0$$

$$\Rightarrow (-\lambda^3+11\lambda^2-39\lambda+45) = 0$$

$$\Rightarrow (\lambda-3)(\lambda-3)(\lambda-5) = 0$$

$\therefore$ Eigenvalues of A are 3 and 5.

(b) Find — of A.

Eigenvector corresponding to $\lambda = 3$

$Ax = \lambda x$

$$\Rightarrow \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e. $(A - \lambda I)x = \bar{0}$

i.e. solving ③

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_{13}(-1) \\ R_{12}(-2) \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 - x_3 = 0$$

$$\Rightarrow x_3 = x_1 + x_2$$

$\therefore$ Eigenspace corresponding to $\lambda = 3$

$$E_3 = \{ (x_1, x_2, x_1 + x_2) \mid x_1, x_2, x_3 \in \mathbb{R} \}$$

$$= \langle (1,0,1), (0,1,1) \rangle$$

Eigenvectors corr. to $\lambda = 3$

Eigenvector corresponding to $\lambda = 5$

$$(A - \lambda I)x = \bar{0}$$

i.e. solving

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & \\ 2 & 0 & -2 & 0 \\ 1 & 1 & -3 & \end{array}\right] \qquad \begin{array}{l} R_{13}(1) \\ R_{12}(+2) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & +2 & -4 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_{23}(-1) \\ R_2\left(\frac{1}{2}\right) \end{array}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_2 - 2x_3 = 0 \quad \Rightarrow \boxed{x_2 = 2x_3}$

$x_1 - x_2 + x_3 = 0$

$\Rightarrow \quad x_1 - 2x_3 + x_3 = 0$

$\Rightarrow \quad x_1 - x_3 = 0$

$\Rightarrow \boxed{x_1 = x_3}$

$\Rightarrow \quad E_5 = \{(x, 2x, x) \mid x \in \mathbb{R}\}$

$= \langle (1, 2, 1) \rangle$

$\hookrightarrow$ Eigenvector.

Hence, Maximum set of LI e.vectors of A:

$\{(1,0,1), (0,1,1), (1,2,1)\}$.

(c) Is A .... diagonal.

Yes A is diagonalizable

$\because$ cardinality of maximum set of LI e.vector of $A_{n \times n} = 3 = n$.

i.e. $A_{n\times n}$ has 3 L.I. eigenvectors

$n = 3$.

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

S.t. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$

6. Consider — of W. ④

$v_1 = (1,1,1,1)$, $v_2 = (1,1,2,4)$, $v_3 = (1,2,-4,-3)$

Using Gram-Schmidt Orthonormalization process,

$W = \cancel{\ldots}\ \langle v_1, v_2, v_3 \rangle$

let $w_1 = v_1 = (1,1,1,1)$

and $\boxed{u_1} = \dfrac{w_1}{\|w_1\|} = \dfrac{(1,1,1,1)}{\sqrt{4}} = \boxed{\dfrac{1}{2}(1,1,1,1)}$

Now,

let $w_2 = v_2 - \sum_{j=1}^{1} \langle v_2, u_j \rangle . u_j$

$= v_2 - \langle v_2, u_1 \rangle . u_1$

$= (1,1,2,4) - \dfrac{(4)}{2}(1,1,1,1)$

$= (-1,-1,0,2)$

And $\boxed{u_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{1}{\sqrt{6}}(-1,-1,0,2)}$

Now,

let $w_3 = v_3 - \sum_{j=1}^{2} \langle v_3, u_j \rangle . u_j$

$= (1,2,-4,-3) - \langle (1,2,-4,-3), \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \rangle \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$- \dfrac{1}{6} \langle (1,2,-4,-3), (-1,-1,0,2) \rangle . (-1,-1,0,2)$

$$= (1, 2, -4, -3) - \frac{1}{4} \cdot (-4)(1, 1, 1, 1) - \frac{1}{6}(-9)(-1, -1, 0, 2)$$

$$= (1, 2, -4, -3) + (1, 1, 1, 1) + \frac{3}{2}(-1, -1, 0, 2)$$

$$= \left(\frac{1}{2}, \frac{3}{2}, -\frac{9}{3}, 1\right) = \frac{1}{2}(1, 3, -6, 2)$$

$\therefore w_3 = \frac{1}{2}(1, 3, -9, 2)$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{\frac{1}{2}(1, 3, -6, 2)}{\frac{1}{2}\sqrt{50}}$$

$$= \frac{1}{\sqrt{50}}(1, 3, -6, 2)$$

$\{u_1, u_2, u_3\}$ are orthogonal basis.

$$\therefore \boxed{u_3 = \frac{1}{\sqrt{50}}(1, 3, -6, 2)}$$

Hence, $\{u_1, u_2, u_3\}$ is an orthonormal or orthogonal basis of W.

7. T/F :

a : M' : Symmetric matrices in M. $(A^T = A)$ form a subspace

True

Take $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & b_4 & b_5 \\ b_3 & b_5 & b_6 \end{bmatrix}$

$A, B \in M'$

$\alpha A + \beta B$

$$= \begin{bmatrix} \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 & \alpha a_3 + \beta b_3 \\ \alpha a_2 + \beta b_2 & \alpha a_4 + \beta b_4 & \alpha a_5 + \beta b_5 \\ \alpha a_3 + \beta b_3 & \alpha a_5 + \beta b_5 & \alpha a_6 + \beta b_6 \end{bmatrix} \in M'$$

b: skew-symmetric matrices in M ($A^T = -A$) M'' form a subspace.

$\boxed{\text{True}}$ Take $A = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{bmatrix}$

$A, B \in M''$.

$$\alpha A + \beta B = \begin{bmatrix} 0 & \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 \\ -(\alpha a_1 + \beta b_1) & 0 & \alpha a_3 + \beta b_3 \\ -(\alpha a_2 + \beta b_2) & -(\alpha a_3 + \beta b_3) & 0 \end{bmatrix}$$

$\in M''$

c: Unsymmetric matrices in M ($A^T \neq A$) forms a subspace M'''.

$\boxed{\text{False}}$ Take $\bar{0} \in M$.

But $\bar{0} \notin M'''$

since $\bar{0}$ is symmetric

Hence M''' cannot be subspa

8. Find the SVD for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Alternate way for (SVD) :-

SVD of A :

$$\underset{2\times 3}{A} = \underset{2\times 2}{U}\ \underset{2\times 3}{\Sigma}\ \underset{3\times 3}{V^T}$$

where $\Sigma \to$ diagonal

$U, V \to$ orthonormal

For U :

$$A A^T = U \Sigma^2 U^T$$

$$A A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\lambda_1 = 1$ (Row sum of both rows)

$\Rightarrow \lambda_2 = 4 - \lambda_1 = 4 - 1 = 3$

Corresponding eigenvectors:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

For V :

$$A^T A = V \Sigma^2 V^T$$

$$A^T A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Eigenvalues : 1, 3, 0 . .

Eigenvectors :

For 0:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x - y + z = 0$$

$$-x + y = 0 \qquad \Rightarrow x - y = 0$$

$$x + z = 0 \qquad \Rightarrow x = y$$

$$\Rightarrow z = -x.$$

$$\Rightarrow (1, 1, -1) \qquad \Rightarrow v_1 = \frac{1}{\sqrt{3}} (1, 1, -1)$$

For 1:

$$\cancel{\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x = 0, \quad y = z.$$

$$\Rightarrow (0, 1, 1) \qquad \Rightarrow v_2 = \frac{1}{\sqrt{2}} (0, 1, 1)$$

For 3:

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0 & +2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \Rightarrow 2z = x$$

$$y = -z$$

$$\Rightarrow V = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 2 & \sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \\ \sqrt{3} & 1 & -\sqrt{2} \end{bmatrix}$$ (13)

Hence,

$$A = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & \sqrt{2} \\ \sqrt{3} & -1 & \sqrt{2} \\ \sqrt{3} & 1 & \sqrt{2} \end{bmatrix}^T$$

$$A = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ 2 & -1 & 1 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}$$

is the SVD of A.

9. Show that every square real matrix has a polar decomposition.

$A \to$ square matrix (real)

Polar Decomposition, $A = QP$ — $Q \to$ orthogonal
$P \to$ +ve semidef

(1) $P \to$ Positive semi Def. (To show)

$P^2 = A^2 A \Rightarrow P = \sqrt{A^T A}$

$\because A^T A$ is symmetric & +ve semi-def.

$\Rightarrow P$ is also symmetric & positive semidef.

(2) $Q \to$ Orthogonal (To show) $Q^T Q = I$

$Q = AP^{-1}$

$\therefore Q^T Q = (AP^{-1})^T (AP^{-1}) = (P^{-1})^T A^T A P^{-1}$

$\because P$ is symmetric $\Rightarrow (P^{-1})^T = P^{-1}$

$\therefore Q^T Q = P^{-1} A^T A P^{-1}$ $[\because A^T A = P^2]$

$\Rightarrow Q^T Q = P^{-1} P^2 P^{-1} = (P^{-1} P)(P P^{-1}) = I$

$\Rightarrow Q$ is orthogonal

$\therefore QP = (AP^{-1})P = A \to$ Proved.

10. Find the pseudo inverse

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In general, first do SVD and then check for pseudo inverse.

SVD of A:

$$\underset{3\times 2}{A} = \underset{3\times 3}{U} \; \underset{3\times 2}{\Sigma} \; \underset{2\times 2}{V^T}$$

Here, it can be clearly seen that

$U = I_3$, $\Sigma = A$, $V = I_2$

$\Rightarrow A = I . \Sigma \; I$

Psuedo Inverse of Σ:

$$\Sigma^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Psuedo Inverse of A:

$$A^+ = V \Sigma^+ U^T = \Sigma^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Pseudo Inverse of A :

$$A^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

11. Find — orthogonal.

u & v should be orthogonal

$\Rightarrow \langle u, v \rangle = 0$ (inner product)

$\Rightarrow \langle (1,2,k,3), (3,k,7,-5) \rangle = 0$

$\Rightarrow 3 + 2k + 7k - 15 = 0$

$\Rightarrow 9k = 12$

$\Rightarrow \boxed{k = \frac{4}{3}}$

12. Suppose $\longrightarrow$ $ad-bc$?

$$\begin{vmatrix} a-Lc & b-Ld \\ c-la & d-lb \end{vmatrix}$$

$$= ad - alb - dLc + lLcb - (bc - bla - cLd + lLad)$$

$$= ad - \cancel{alb} - \cancel{dLc} + lLcb - bc + \cancel{bla} + \cancel{cLd} - lLad$$

$$= ad - bc - lL(ad-bc)$$

$$= (ad-bc)(1-lL) \neq ad-bc.$$

and only $= ad-bc$

if $\boxed{lL = 1}$

$\therefore$ No, not in general.

13. Consider the rotation — eigenvectors.

$T_\phi : \mathbb{R}^2 \to \mathbb{R}^2$. → rotates every vector by an angle of ϕ (anti-clockwise)

$T_\phi(x, y) = (x\cos\phi - y\sin\phi,\ y\cos\phi + x\sin\phi)$

Around x-axis ⇒ $\phi = 90$

⇒ $T(x, y) = (-y, x)$

∴ Matrix : $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

⇒ $\lambda^2 - 0\lambda + 1 = 0$ char. eq.

⇒ $\lambda^2 + 1 = 0$

No real e. values

complex eigenvalues : $\lambda = \pm i$

No eigenvectors <u>in $\mathbb{R}^2$</u> since no proper. nontrivial subspace is invariant

∵ every vector is rotated by angle of $90°$.

14. $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$

(a). e. values & e. vectors

$A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \left. \begin{matrix} 2-1 = 1 \\ -2+3 = 1 \end{matrix} \right\}$ Row sum $= 1$

$\Rightarrow$ One e.value, $\boxed{\lambda_1 = 1}$

$\text{Trace} = \lambda_1 + \lambda_2$

$\Rightarrow 5 = \lambda_1 + \lambda_2 = 1 + \lambda_2$

$\Rightarrow \boxed{\lambda_2 = 4}$

e.vector corresponding to $\lambda = 1$

$Ax = \lambda x$

$$\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$

$\Rightarrow x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2$

$\therefore E_1 = \{(x_1, x_1) \mid x_1 \in \mathbb{R}\}$

e.vector : $(1, 1)$

e.vector corresponding to $\lambda = 4$.

$(A - \lambda I)x = \bar{0}$

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$

$\Rightarrow 2x_1 + x_2 = 0 \quad \Rightarrow x_2 = -2x_1$

$\therefore E_4 = \{(x_1, -2x_1) \mid x_1 \in \mathbb{R}\}$

e.vector : $(1, -2)$

(b.) Find non-singular P s.t. $D = P^{-1}AP$.

Set $P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

then $A = P \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} P^{-1}$.

where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ ← diagonal.

(c.) Find A^8 & $f(A)$ s.t $f(t) = t^4 - 5t^3 + 7t^2 - 2t + 5$

$A = PDP^{-1}$

$\Rightarrow A^8 = (PDP^{-1})^8 = \underbrace{(PDP^{-1})(PDP^{-1}) \ldots (PDP^{-1})}_{8 \text{ times}}$

$= (PD)(P^{-1}P)(DP^{-1}) \ldots (PDP^{-1})$

$= PD^8P^{-1}$

$\Rightarrow A^8 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4^8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$

$= \frac{-1}{3} \begin{bmatrix} 1 & 4^8 \\ 1 & -2 \times 4^8 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \frac{-1}{3} \begin{bmatrix} -2 - 4^8 & -1 + 4^8 \\ -2 + (2 \times 4^8) & -1 - (2 \times 4^8) \end{bmatrix}$

$= \frac{-1}{3} \begin{bmatrix} -65538 & 65535 \\ 131070 & -131073 \end{bmatrix}$

$A^8 = \begin{bmatrix} 21846 & -21845 \\ -43690 & 43691 \end{bmatrix}$

Also, characteristic equation, (7)

$ch(\lambda) = (\lambda-1)(\lambda-4)$

$= \lambda^2 - \lambda - 4\lambda + 4$

$ch(\lambda) = \lambda^2 - 5\lambda + 4.$

$f(t) = t^4 - 5t^3 + 7t^2 - 2t + 5.$

$= t^2(t^2 - 5t + 4) + 3t^2 - 2t + 5$

$= t^2(t^2-5t+4) + 3(t^2 - 5t + 4)$
$+ 13t - 7$

Also, we know from Cayley Hamilton theorem,

$ch(A) = 0$ where $ch(A)$ is Characteristic equation of A

$\Rightarrow f(A)$

$= A^2(A^2-5A+4) + 3(A^2-5A+4)$
$+ 13A - 7I$

$= A^2(0) + 3(0) + 13A - 7I$

$= 13A - 7I$

$= 13\begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 19 & -13 \\ -26 & 32 \end{bmatrix}$

Hence, $\boxed{f(A) = \begin{bmatrix} 19 & -13 \\ -26 & 32 \end{bmatrix}}$

(d)

$$\Rightarrow (b_4 - b_1)(b_4 + b_1) = 1$$

$$A = PDP^{-1}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \left(\frac{-1}{3}\right) \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\cancel{\Rightarrow D = P^{-1} A P}$$

$$\cancel{\Rightarrow D^2 = P^{-1} A^2 P}$$

$$A = P\sqrt{D} \cdot \sqrt{D} P^{-1}$$

$$= P\sqrt{D} P^{-1} \cdot P\sqrt{D} P^{-1}$$

let $B = P\sqrt{D}P^{-1} \Rightarrow A = B^2$

$$\Rightarrow B = -\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{-1}{3}\begin{bmatrix} -4 & 1 \\ +2 & -5 \end{bmatrix} = \begin{bmatrix} 1.3333 & -0.3333 \\ -0.6667 & 1.6667 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3333 & -0.3333 \\ -0.6667 & 1.6667 \end{bmatrix}.$$

15. e.values for B (8)

$$B = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & & & & & | \\ 10 & & & & & | \\ 10 & & & & & | \\ 10 & & & & & | \\ 10 & & \text{———} & & & 10 \end{bmatrix}_{6\times 6}$$

Let $A = [\sqrt{10}\ \sqrt{10}\ \sqrt{10}\ \sqrt{10}\ \sqrt{10}\ \sqrt{10}]$

then $A^t A = B$

And $A A^t = [6 \times 10] = [60]$

We know, non-zero e.values of AA^t & A^tA are same

and e.value of AA^t is 60

$\Rightarrow$ e.value of $A^tA = B$ is

60, 0, 0, 0, 0, 0.

16. Find — modulus.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\text{trace}(A) = 2\cos\theta$ $\quad$ $\det(A) = \cos^2\theta + \sin^2\theta$
$= 1$

characteristic eq^n:

$$ch(\lambda) = \lambda^2 - tr(A).\lambda + \det(A)$$
$$= \lambda^2 - 2\cos\theta . \lambda + 1 = 0$$

$$\Rightarrow \lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$
$$= \cos\theta \pm i\sqrt{\sin^2\theta}$$
$$= \cos\theta \pm i\sin\theta.$$

$\Rightarrow \lambda_1 = \cos\theta + i\sin\theta$

$\lambda_2 = \cos\theta - i\sin\theta.$

$|\lambda_1| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

$|\lambda_2| = \sqrt{\cos^2\theta + (-\sin\theta)^2} = \sqrt{1} = 1$

$\therefore$ λ_1, λ_2 are of unit modulus.

17. Find — e. vectors.

Rotation matrix in $\mathbb{R}^2$ with $\theta = 90$.

(Above ex.)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has no. real e. values on e. vectors

18. Ans. the foll.

(a) Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, compute its polar decomposition $A = UP$ where U is a unitary matrix & P is positive semi-definite matrix.

$$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

Eigenvalues of $A^T A$:

$$\lambda^2 - 30\lambda + 4 = 0$$

$$\Rightarrow \lambda = \frac{30 \pm \sqrt{884}}{2} = 29.866,\ 0.134.$$

Eigenvector corresponding to $\lambda_1 = 29.865$

$$\begin{bmatrix} -19.865 & 14 \\ 14 & 0.135 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{0}$$

$$\Rightarrow -19.865 x_1 + 14 x_2 = 0$$

$$\Rightarrow v_1 = \begin{bmatrix} 0.705 \\ 1 \end{bmatrix}$$

$\|v_1\| = \sqrt{0.705^2 + 1} = 1.223$

Eigenvector for $\lambda_2 = 0.135$

$$\begin{bmatrix} 9.865 & 14 \\ 14 & 19.865 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 = \frac{-14}{9.865} x_2$$

$$\Rightarrow x_1 = -1.419 x_2$$

$$\Rightarrow v_2 = \begin{bmatrix} -1.419 \\ 1 \end{bmatrix} \rightarrow \|v_2\| = 1.736.$$

Matrix $V \rightarrow$ Matrix of Normalized e. vectors.

$$\Rightarrow V = \begin{bmatrix} 0.577 & -0.818 \\ 0.818 & 0.577 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0.577 & 0.818 \\ -0.818 & 0.577 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 29.865 & 0 \\ 0 & 0.135 \end{bmatrix}$$

$$D^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 5.465 & 0 \\ 0 & 0.367 \end{bmatrix}$$

By Eigen Decomposition

$$P = V D^{1/2} V^T$$

$$P = \begin{bmatrix} 0.577 & -0.818 \\ 0.818 & 0.577 \end{bmatrix} \begin{bmatrix} 5.465 & 0 \\ 0 & 0.367 \end{bmatrix} \begin{bmatrix} 0.577 & 0.818 \\ -0.818 & 0.577 \end{bmatrix}$$

$$= \begin{bmatrix} 2.065 & 2.407 \\ 2.407 & 3.782 \end{bmatrix} \rightarrow \text{Positive semi def.}$$

Now, Unitary Matrix $= AP^{-1}$

$$\therefore P^{-1} = \frac{1}{\det P} \begin{bmatrix} 3.782 & -2.407 \\ -2.407 & 2.065 \end{bmatrix}$$

$$= \frac{1}{2.012} \begin{bmatrix} 3.782 & -2.407 \\ -2.407 & 2.065 \end{bmatrix} = \begin{bmatrix} 1.880 & -1.192 \\ -1.197 & 1.026 \end{bmatrix}$$

$$U = AP^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1.880 & -1.197 \\ -1.197 & 1.026 \end{bmatrix}$$

$$= \begin{bmatrix} -0.514 & 0.855 \\ 0.852 & 0.513 \end{bmatrix}$$

Hence, Polar decomposition of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A = UP$$

$$= \begin{bmatrix} -0.514 & 0.855 \\ 0.852 & 0.513 \end{bmatrix} \begin{bmatrix} 2.065 & 2.407 \\ 2.407 & 3.782 \end{bmatrix}$$

(b) For matrix $B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, find polar decom. $B = UP$.

$B = UP$ $\quad U \to$ unitary $\quad P \to$ positive semi definite.

$$B^T B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}$$

To find P:

$$\lambda^2 - 15\lambda + 49 = 0$$

$$\Rightarrow \lambda = \frac{15 \pm \sqrt{29}}{2} = 10.192,\ 4.807$$

Eigenvectors corresponding to $\lambda_1 = 10.192$:

$v_1 = \begin{bmatrix} 1 \\ 5.192 \end{bmatrix}$ $\quad \|v_1\| = 5.282$ $\quad$ Normalized $v_1 = \begin{bmatrix} 0.189 \\ 0.981 \end{bmatrix}$

and $\lambda_2 = 4.807$

$v_2 = \begin{bmatrix} 1 \\ 0.193 \end{bmatrix}$ $\quad \|v_2\| = 1.081$ $\quad$ Normalized $v_2 = \begin{bmatrix} 0.981 \\ -0.1895 \end{bmatrix}$

$$\Rightarrow V = \begin{bmatrix} 0.189 & 0.981 \\ 0.981 & -0.189 \end{bmatrix} \Rightarrow V^T = V$$

$$D = \begin{bmatrix} 10.192 & 0 \\ 0 & 4.807 \end{bmatrix} \qquad D^{1/2} = \begin{bmatrix} 3.192 & 0 \\ 0 & 2.192 \end{bmatrix}$$

Eigen Decomposition:

$$P = VD^{1/2}V^T = \begin{bmatrix} 2.2234 & 0.18513 \\ 0.45308 & 3.15008 \end{bmatrix} \to \text{Positive semi def. matrix.}$$

Unitary Matrix $U = BP^{-1}$

$$P^{-1} = \frac{1}{6.92004}\begin{bmatrix} 3.15005 & -0.18513 \\ -0.45308 & 2.2234 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4552 & -0.02677 \\ -0.06547 & 0.32198 \end{bmatrix}$$

$$U = BP^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0.45 & -0.026 \\ -0.06 & 0.321 \end{bmatrix}$$

$$= \begin{bmatrix} 0.96 & -0.373 \\ 0.27 & 0.937 \end{bmatrix}$$

Polar Decomposition of $A = UP$

$$A = \begin{bmatrix} 0.96 & -0.373 \\ 0.27 & 0.937 \end{bmatrix}\begin{bmatrix} 2.223 & 0.183 \\ 0.453 & 3.15 \end{bmatrix}$$

(C) Calculate polar decomposition of the matrix $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ & verify that the decomposition is unique.

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$C^{T}C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$P = I \rightarrow$ semi def. matrix.

Unitary matrix $U = CP^{-1}$

$P^{-1} = I^{-1} = I \qquad U = C.$

To verify U is ~~unique~~ : unitary :

$U^{T}U = C^{T}C = I$ ✓

Verification of uniqueness

$C = UP$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$P = I$ is fixed then $U = C$ is fixed & (unique). satisfied $C = UP$

$\Rightarrow$ Decomposition is unique.

19. Which — $v = (v_1, v_2)$

(a) $T(v) = (v_2, 2v_1)$

Linear

(b) $T(v) = v_1 v_2$

Non-linear.

20. T is LT. —

$T(1,1) = (2,2)$

$T(2,0) = (0,0)$

(a) $v = (-1,1)$

$(-1,1) = +1(1,1) - 1(2,0)$

Since T is LT

$\Rightarrow T(-1,1) = 1\,T(1,1) - 1\,T(2,0)$

$= 1 \times (2,2) - 1 \times (0,0)$

$= (2,2)$.

(b) $v = (a, b)$

$$(a,b) = b(1,1) - \frac{(b-a)}{2}(2,0)$$

$$\Rightarrow T(a,b) = b\,T(1,1) - \frac{b-a}{2}T(2,0)$$

$$= b(2,2)$$

$$= (2b, 2b)$$

21. Let ——— of T^2

$$T(x_1, x_2\, x_3) = (x_1 + 3x_2 + 2x_3,\ 3x_1 + 4x_2 + x_3,\ 2x_1 + x_2 - x_3)$$

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(1,0,0) = (1, 3, 2)$

$T(0,1,0) = (3, 4, 1)$

$T(0,0,1) = (2, 1, -1)$

$$\Rightarrow [T] = A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

And

Null space :

$$N(T) = \{ x \in \mathbb{R}^3 \mid Tx = \bar{0} \}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + 4x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \end{aligned}\right\}$$

$\Rightarrow x_1 - 3x_1 + 2x_3 = 0$

$5x_1 + 5x_2 = 0$

$\Rightarrow \boxed{x_1 = -x_2}$

$\Rightarrow -2x_1 + 2x_3 = 0$

$\Rightarrow \boxed{x_1 = x_3}$

$$\Rightarrow N(T) = \{ (x_1, -x_1, x_1) \mid x_1 \in \mathbb{R} \}$$

And $\dim(N(T)) = \eta(T) = 1$

Then by Rank nullity theorem,

$$\rho(T) = 3 - \eta(T)$$
$$= 3 - 1 = 2$$

$\therefore \boxed{\rho(T) = 2}$ = Dim. of Range

22. $A = \begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix}$ ← Real symmetric matrix.

Find an orthogonal matrix P s.t. $P^{-1}AP$ is diagonal.

eigen values of A:

$$\lambda^2 - 9\lambda + 17 = 0$$

$$\lambda = \frac{9 \pm \sqrt{13}}{2} \approx 6.30278 \text{ and } 2.69722$$

Eigenvector: $\lambda = 6.30278$

$$\begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} -0.30278 \\ 1 \end{bmatrix} = 6.30278 \begin{bmatrix} -0.30278 \\ 1 \end{bmatrix}$$

$\lambda = 2.69722$

$$\begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 3.30278 \\ 1 \end{bmatrix} = 2.69722 \begin{bmatrix} 3.30278 \\ 1 \end{bmatrix}$$

$$\Rightarrow \cancel{P = \begin{bmatrix} -0.30278 & 1 \\ 1 & 3.30278 \end{bmatrix}}$$

$$P = \begin{bmatrix} -0.30278 & 3.30278 \\ 1 & 1 \end{bmatrix}$$

orthogonal Matrix

$\Rightarrow P^{-1} = P^T$

$$\therefore A = \begin{bmatrix} -0.30278 & 3.30278 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6.30278 & 0 \\ 0 & 2.69722 \end{bmatrix} \cdot \begin{bmatrix} -0.30278 & 1 \\ 3.30278 & 1 \end{bmatrix}$$

Hence we got Orthogonal matrix P s.t.:

$A = PDP^{-1} \Rightarrow P^{-1}AP = D$

i.e. diagonal.

23. Find char. poly —.

(a) $F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = (3x - 7y, 2x + 5y)$

$F(1,0) = (3,2)$

$F(0,1) = (-7,5)$

$\Rightarrow [F] = A = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$ $\quad tr(A) = 8$, $\det(A) = 29$

$\Rightarrow$ characteristic polynomial of F

= char. poly. of A

$ch(\lambda) = \lambda^2 - 8\lambda + 29$.

(b) $D: V \to V$

$D(f) = \frac{df}{dt}$ $\quad V$ = space of func. spanned by $S = \{\sin t, \cos t\}$

$D(\sin t) = -\cos t = 0.\sin t - 1\cos t$

$D(\cos t) = \sin t$.

$[D] = A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $tr(A) = 0$, $det(A) = 1$ ⑩. ~~23~~

$\Rightarrow ch(\lambda) = \lambda^2 + 1$

24. Find a linear --- $(2,0,-1,-5)$

Basis of $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be a map s.t.

$F(1,0,0) = (1,2,0,-4)$

$F(0,1,0) = (2,0,-1,-5)$

$F(0,0,1) = (0,0,0,0)$.

Then $R(F) = \langle (1,2,0,-4), (2,0,-1,-5) \rangle$

25. Let G ——.

$G : \mathbb{R}^2 \to \mathbb{R}^3$

$G(x,y) = (x+y, x-2y, 3x+y)$

(a) $N(G) = \{(x,y) : T(x,y) = (0,0,0)\}$

$\Rightarrow x+y=0 \quad \Rightarrow 3y = 0 \Rightarrow y=0$

$x-2y=0 \quad \Rightarrow x = 2y$

$3x+y=0 \quad \Rightarrow x=0$

$\Rightarrow N(G) = \{(0,0)\}$

$\Rightarrow$ G is non-singular.

(b) Formula for G^{-1}:

$G(x,y) = (u,v,w) = (x+y, x-2y, 3x+y)$

$\Rightarrow x+y = u \Rightarrow x = u-y$

~~$x-2y = v \Rightarrow u-y-2y = v$~~

$$\Rightarrow u - v = 3y \rightarrow y = \frac{(u-v)}{3}$$

$$\Rightarrow x = u - \frac{(u-v)}{3} = \frac{2u+v}{3}$$

$$\rightarrow G^{-1}(u, v, w) = \left(\frac{2u+v}{3}, \frac{u-v}{3}\right).$$

is the formula of G^{-1}

Although G is non singular, still G is non-invertible

$\because$ $\dim(\mathbb{R}^2) \neq \dim(\mathbb{R}^3)$

G^{-1} Does not exist here.

26. Find the trace — on $\mathbb{R}^3$.

(a) $F(x, y, z) = (x + 3y, 3x - 2z, x - 4y - 3z)$

$$[F] = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & -2 \\ 1 & -4 & -3 \end{bmatrix}$$

$Tr(F) = -2$

$Det(F) = 1(-8) - 3(-9+2)$

$= -8 + 21 = 13.$

(b) $G(x, y, z) = (y + 3z, 2x - 4z, 5x + 7y)$

$$[G] = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & -4 \\ 0 & 7 & 0 \end{bmatrix}$$

$Tr(G) = 0$

$Det(G) = -2(-21) = 42.$

27. Find a basis (& dim) — $(M_3(\mathbb{R}))$

(a) All diagonal matrix.

$$M_1 = \left\{ \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \middle| \; x_i \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$\dim(M_1) = 3.$

(b) All symmetric matrix.

$$M_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(c) All skew-symmetric matrix:

$$M_3 = \left\{ \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix} \middle| \; a_i \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

28. Given the matrix :

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{Find } A^{-1}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 3 & 5 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & 5 & 2 & 0 & 1 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \end{array}\right]$$

R_{13}

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & -3 \\ 0 & -2 & -3 & 1 & 0 & -4 \end{array}\right] \quad \begin{array}{l} R_{12}(-3) \\ R_{13}(-4) \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & -3 \\ 0 & 0 & -4 & 1 & 1 & -7 \end{array}\right] \quad R_{23}(1)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -0.5 & 0 & +0.5 & -1.5 \\ 0 & 0 & 1 & -0.25 & -0.25 & 1.75 \end{array}\right] \quad R_2\left(\frac{1}{2}\right), R_3\left(\frac{-1}{4}\right)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & +0.25 & +0.25 & -0.75 \\ 0 & 1 & 0 & -0.125 & +0.375 & -0.625 \\ 0 & 0 & 1 & -0.25 & -0.25 & 1.75 \end{array}\right] \quad \begin{array}{l} R_{32}(0.5) \\ R_{31}(-1) \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.375 & -0.125 & -0.125 \\ 0 & 1 & 0 & -0.125 & +0.375 & -0.625 \\ 0 & 0 & 1 & -0.25 & -0.25 & 1.75 \end{array}\right] \quad R_{21}(-1)$$

$$= [\,I \mid A^{-1}\,]$$

Hence $A^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -5 \\ -2 & -2 & 14 \end{bmatrix}$

Hence, $a = 3/8$ $\quad b = -1/8$ $\quad c = -1/8$

$d = -1/8$ $\quad e = 3/8$ $\quad f = -5/8$

$g = -2/8$ $\quad h = -2/8$ $\quad i = 14/8.$

29. Let T be a L.O. Show that TFAE:

(a) A scalar λ is an e.value of T.

Then by definition

$$Ax = \lambda x \qquad x \neq 0 \qquad \text{for some } x.$$

$$\Leftrightarrow (A - \lambda I)x = 0$$

$$\Leftrightarrow (\lambda I - A)x = 0$$

i.e. $(\lambda I - A)$ is singular.

$\therefore$ $\boxed{(a) \Leftrightarrow (b)}$

(b) The LO $\lambda I - T$ is singular.

$\Leftrightarrow$ For some x, $x \neq 0$,

$$(\lambda I - A)x = 0$$

let $\lambda I - A = B$

$\Leftrightarrow Bx = 0$

$\Leftrightarrow |B| = 0$

$\Leftrightarrow |A - \lambda I| = 0.$ characteristic poly.

i.e. λ is a root of characteristic polynomial. of T

$\boxed{(b) \Leftrightarrow (c)}$

Hence proved.

30. $A = \begin{bmatrix} -3 & 8 & 8 \\ -1 & 5 & -2 \\ -1 & 2 & 9 \end{bmatrix}$ Find A^{50}

characteristics eq^n of A:

$\lambda^3 - 11\lambda^2 + 23\lambda + 35 = 0$

$\Rightarrow (\lambda+1)(\lambda-5)(\lambda-7) = 0$

$\Rightarrow \lambda = -1, 5, 7.$

e. vector for $\lambda = -1$: $v_1 = \begin{bmatrix} 16 \\ 3 \\ 1 \end{bmatrix}$. | $\lambda = 5$ $v_2 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ | $\lambda = 7$ $v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 16 & -2 & 0 \\ 3 & -3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0.08333 & -0.08333 & -0.08333 \\ 0.16667 & -0.66667 & -0.6666 \\ -0.025 & 0.75 & 1.75 \end{bmatrix}$

$$A = \overset{P}{\begin{bmatrix} 16 & -2 & 0 \\ 3 & -3 & -1 \\ 1 & 1 & 1 \end{bmatrix}} \overset{D}{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}} \overset{P^{-1}}{\begin{bmatrix} 0.08333 & -0.08333 & 0.08333 \\ 0.16667 & -0.6667 & -0.6667 \\ -0.25 & 0.75 & 1.75 \end{bmatrix}}$$

$A^{50} = P D^{50} P^{-1}$

$$= \begin{bmatrix} 16 & -2 & 0 \\ 3 & -3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5^{50} & 0 \\ 0 & 0 & 7^{50} \end{bmatrix} \begin{bmatrix} 0.08333 & -0.0833 & -0.083 \\ 0.1667 & -0.667 & -0.667 \\ -0.25 & 0.75 & 1.75 \end{bmatrix}$$