Problem Sheet-1

1. Let $X_1, \ldots, X_n$ be a random sample drawn from $N(\mu, \sigma^2)$. Find

   (a) MLE for (μ, σ^2);

   (b) MLE for σ^2 when μ is known;

   (c) MLE for μ when σ^2 is known.

2. Let $X_1, \ldots, X_n$ be a random sample drawn from $U(\theta - 1/2, \theta + 1/2)$. Find the MLE for θ.

3. Let $X_1, \ldots, X_n$ be a random sample drawn from a population having PMF

$$P(N = k) = \frac{1}{N}, \quad k = 1, 2, \ldots, N.$$

   Find MLE for N.

4. Let $X_1, \ldots, X_n$ be a random sample drawn from $N(\mu, \mu^2)$. Find the MLE for μ.

5. Let $X_1, \ldots, X_n$ be a random sample drawn from $f(x; \theta)$. Find the MLE for θ in each of the following cases:

   (a) $f(x; \theta) = e^{-x+\theta}, \quad \theta \leq x < \infty$;

   (b) $f(x; \theta) = (\theta\alpha) x^{\alpha-1} e^{-\theta x^{\alpha}}, \quad x > 0$ and α is known;

   (c) $f(x; \theta) = \theta(1 - x)^{\theta-1}, \quad 0 \leq x \leq 1, \ \theta > 1$;

   (d) $f(x; \theta) = 1 \quad \theta \leq x \leq \theta + 1$;

6. Let $X_1, \ldots, X_n$ be a random sample drawn from $N(\mu, \sigma^2)$. Show that the sample mean $\bar{X}_n$ and the sample variance S^2 are unbiased estimators of μ and σ^2, respectively.

7. Let $X_1, X_2, ..., X_n$be a random sample from a $U(-\theta, \theta)$ population. Find the MLE of θ.

8. A random sample of 12 shearing pins is taken in a study of the Rockwell hardness of the head on the pin. Measurements on the Rockwell hardness were made for each of the 12, yielding an average value of 48.50 with a sample standard deviation of 1.5. Assuming the measurements to be normally distributed, construct a 90%confidence interval for the mean Rockwell hardness.

9. An electric scale gives a reading equal to the true weight plus a random error that is normally distributed with mean 0 and standard deviation $\sigma = .1$ mg. Suppose that the results of five successive weighings of the same object are as follows: $3.142, 3.163, 3.155, 3.150, 3.141$.

   (a) Determine a 90 percent confidence interval estimate of the true weight;

(b) Determine a 99 percent confidence interval estimate of the true weight.

10. The PCB concentration of a fish caught in Lake Chilika was measured by a technique that is known to result in an error of measurement that is normally distributed with a standard deviation of .08 ppm (parts per million). Suppose the results of 10 independent measurements of this fish are: $11.2, 12.4, 10.8, 11.6, 12.5, 10.1, 11.0, 12.2, 12.4, 10.6$

    (a) Estimate the population mean μ;

    (b) Give a 95 percent confidence interval for the PCB level of this fish;

    (c) Give a 95 percent lower confidence interval;

    (d) Give a 95 percent upper confidence interval.

11. A sample of 20 cigarettes is tested to determine nicotine content and the average value observed was 1.2 mg.

    (a) Compute a 99 percent two-sided confidence interval for the mean nicotine content of a cigarette nbif it is known that the standard deviation of a cigarette's nicotine content is $\sigma = .2$ mg;

    (b) Compute a value c for which we can assert "with 99 percent confidence" that c is larger than the mean nicotine content of a cigarette.

12. Consider the number of 2nd year B.Tech. students of IIT Jodhpur who have registered the course MA221 in this ongoing semester. Let $X_1, X_2, ..., X_{30}$ denote the number students attended in randomly chosen 30 lecture classes. Assume that the population have normal distribution with mean μ and variance σ^2. Suppose the observed data is as follows:

$$200, 190, 185, 195, 188, 182, 192, 184, 188, 187, 182, 188, 182, 191, 192,$$
$$191, 182, 190, 182, 187, 192, 170, 184, 179, 188, 181, 182, 178, 181, 182$$

    (a) Using method of moments, find the estimators for μ and σ;

    (b) Find a 99% two sided confidence interval for μ and σ.

13. The following are scores on IQ tests of a random sample of 18 students at IIT Jodhpur.

$$130, 122, 119, 142, 136, 127, 120, 152, 141, 132, 127, 118, 150, 141, 133, 137, 129, 142$$

    (a) Construct a 95 percent confidence interval estimate of the average IQ score of all students at IIT Jodhpur;

    (b) Construct a 95 percent lower confidence interval estimate;

    (c) Construct a 95 percent upper confidence interval estimate.

14. A random sample of 300 HDFC Bank VISA cardholders accounts indicated a sample mean debt of \$1, 220 with a sample standard deviation of \$840.

    (a) Construct a 95 percent confidence interval estimate of the average debt of all cardholders;

    (b) Find the smallest value v that "with 90 percent confidence," exceeds the average debt per cardholder.

15. The capacities (in ampere-hours) of 10 batteries were recorded as follows:

$$140, 136, 150, 144, 148, 152, 138, 141, 143, 151$$

    (a) Estimate the population variance σ^2;

    (b) Compute a 99 percent two-sided confidence interval for σ^2;

    (c) Compute a value v that enables us to state, with 90 percent confidence, σ^2 that is less than v.

16. The amount of beryllium in a substance is often determined by the use of a photometric filtration method. If the weight of the beryllium is μ, then the value given by the photometric filtration method is normally distributed with mean μ and standard deviation σ. A total of eight independent measurements of 3.180 mg of beryllium gave the following results.

$$3.166, 3.192, 3.175, 3.180, 3.182, 3.171, 3.184, 3.177$$

    Use the preceding data to

    (a) estimate σ;

    (b) find a 99 percent confidence interval estimate of σ.

17. A civil engineer wishes to measure the compressive strength of two different types of concrete. A random sample of 10 specimens of the first type yielded the following data (in psi)

$$\text{Type 1: } 3250, 3268, 4302, 3184, 32663297, 3332, 3502, 3064, 3116,$$

    whereas a sample of 10 specimens of the second yielded the data

$$\text{Type 2: } 3094, 3106, 3004, 3066, 2984, 3124, 3316, 3212, 3380, 3018.$$

    If we assume that the samples are normal with a common variance, determine

    (a) a 95 percent two-sided confidence interval for $\mu_1 - \mu_2$, the difference in means;

    (b) a 95 percent one-sided upper confidence interval for $\mu_1 - \mu_2$;

    (c) a 95 percent one-sided lower confidence interval for $\mu_1 - \mu_2$.

18. Independent random samples are taken from the output of two machines on a production line. The weight of each item is of interest. From the first machine, a sample of size 36 is taken, with sample mean weight of 120 grams and a sample variance of 4. From the second machine, a sample of size 64 is taken, with a sample mean weight of 130 grams and a sample variance of 5. It is assumed that the weights of items from the first machine are normally distributed with mean μ_1 and variance σ_1^2 and that the weights of items from the second machine are normally distributed with mean μ_2 and variance σ_2^2.

    (a) Find a 99 percent confidence interval for $\mu_1 - \mu_2$, when $\sigma_1 = \sigma_2$;

    (b) Find a 99 percent confidence interval for $\mu_1 - \mu_2$, when it is known in advance that the population variances are 4 and 5, respectively:

    (c) Find a 99 percent confidence interval for $\mu_1 - \mu_2$, when there is no information is provided for population variances.

19. In a study conducted by the Department of Chemistry of IIT Jodhpur, fifteen "samples" of water were collected from a certain station in a river in order to gain some insight regarding the amount of orthophosphorous in the river. The concentration of the chemical is measured in milligrams per liter. Le us suppose that the mean at the station is not as important as the upper extremes of the distribution of the chemical at the station. Concern centers around whether the concentrations at these extremes are too large. Readings for the fifteen water samples gave a sample mean of 3.84 milligrams per liter and sample standard deviation of 3.07 milligrams per liter. Assume that the readings are a random sample from a normal distribution. Calculate a prediction interval (upper 95% prediction limit) and a tolerance limit (95% upper tolerance limit that exceeds 95% of the population of value). Interpret both; that is, tell what each communicates to us about the upper extremes of the distribution of orthophosphorous at the sampling station.

20. In a hatch chemical process, two catalysts arc being compared for their effect on the output of the process reaction. A sample of 12 batches was prepared using catalyst 1 and a sample of 10 batches was obtained using catalyst 2. The 12 batches for which catalyst 1 was used gave an average yield of 85 with a sample standard deviation of 4, and the second sample gave an average of 81 and a sample standard deviation of 5. Find a 99% confidence interval for the difference between the population means, assuming that, the populations art approximately normally distributed with equal variances.