Indian Institute of Technology Jodhpur

Matrix Theory(MAL7051)

Academic Year 2024-25, Semester I

Assignment 1

Due Date: Sep 05, 2024 **Maximum Marks: 100**

Instructions:

- Read each question carefully before answering.
- Plagiarism of any kind will not be tolerated and will result in zero marks.
- Late submissions will incur a penalty of 10% deduction per day.

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1. Show that the space R^n contains a subspace $W(\neq \{0\})$ such that $W \neq R^n$
2. Prove that the set $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$ is linearly independent.
3. Find the rank of each of the following matrices:
   (a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
   (b) $B = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 10 & 4 & 7 \end{pmatrix}$
   (c) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
4. Let S be a linearly independent subset of R^n, and let v be a vector in R^n that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$.
5. Let
$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}.$$
   (a) Find all the eigenvalues of A.
   (b) Find a maximum set of linearly independent eigenvectors of A.
   (c) Is A diagonalizable? If yes, find P such that $D = P^{-1}AP$ is diagonal.
6. Consider the subspace W of R^4 spanned by the vectors
$$v_1 = (1,1,1,1),\ v_2 = (1,1,2,4),\ v_3 = (1,2,-4,-3).$$
   Find (a) an orthogonal basis of W; (b) an orthonormal basis of W.
7. True or false (check addition in each case by an example):
   (a) The symmetric matrices in $\mathbf{M}$ (with $A^T = A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^T = -A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^T \neq A$) form a subspace.

8. Find the singular value decomposition for

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

9. Show that every square real matrix has a polar decomposition.

10. Find the pseudo inverse for the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

11. Find k so that $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ in R^4 are orthogonal.

12. Suppose you do two row operations at once, going from:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} a - Lc & b - Ld \\ c - la & d - lb \end{bmatrix}$$

Find the second determinant. Does it equal $ad - bc$?

13. Consider the rotation of vectors around the x-axis in vector space R^2. Find the corresponding matrix, its eigenvalues, and the corresponding eigenvectors.

14. Let $A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

(a) Find eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal.

(c) Find A^8 and $f(A)$ where $f(t) = t^4 - 5t^3 + 7t^2 - 2t + 5$.

(d) Find a matrix B such that $B^2 = A$.

15. Find the eigenvalues for matrix B:

$$B = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 \end{bmatrix}$$

16. Find the characteristic roots of the 2-rowed orthogonal matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and verify that they are of unit modulus.

17. Find a real matrix that has no real eigenvalues or eigenvectors.

18. Answer the following :

(a) Given the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, compute its polar decomposition $A = UP$, where U is a unitary matrix and P is a positive semi-definite matrix.

(b) For the matrix $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, find the matrices U and P in the polar decomposition $B = UP$.

(c) Calculate the polar decomposition of the matrix $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and verify that the decomposition is unique.

19. Which of these transformations are not linear? The input is $\mathbf{v} = (v_1, v_2)$:

    (a) $T(\mathbf{v}) = (v_2, v_1)$
    (b) $T(\mathbf{v}) = v_1 v_2$

20. Suppose a linear transformation T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(\mathbf{v})$:

    (a) $\mathbf{v} = (-1, 1)$
    (b) $\mathbf{v} = (a, b)$

21. Let $T : R^3 \to R^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3).$$

    Find the dimension of the range space of T^2.

22. Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 6 \end{pmatrix}$, a real symmetric matrix. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

23. Find the characteristic polynomial $\Delta(t)$ of each of the following linear operators:

    (a) $\mathbf{F} : R^2 \to R^2$ defined by $\mathbf{F}(x, y) = (3x - 7y, \quad 2x + 5y)$.
    (b) $\mathbf{D} : V \to V$ defined by $\mathbf{D}(f) = \frac{df}{dt}$, where V is the space of functions with basis $S = \{\sin t, \cos t\}$.

24. Find a linear map $\mathbf{F} : R^3 \to R^4$ whose image is spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -5)$.

25. Let $G : R^2 \to R^3$ be defined by $G(x, y) = (x + y, x - 2y, 3x + y)$.

    (a) Show that G is nonsingular.
    (b) Find a formula for G^{-1}.

26. Find the trace and determinant of each of the following linear maps on R^3:

    (a) $F(x, y, z) = (x + 3y, \ 3x - 2z, \ x - 4y - 3z)$.
    (b) $G(x, y, z) = (y + 3z, \ 2x - 4z, \ 5x + 7y)$.

27. Find a basis (and the dimension) for each of these subspaces of 3×3 matrices:

    (a) All diagonal matrices.
    (b) All symmetric matrices ($A^T = A$).
    (c) All skew-symmetric matrices ($A^T = -A$).

28. Given the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

    Find all the variables of the matrix

$$A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

29. Let T be a linear operator. Show that the following statements are equivalent:

   (a) A scalar λ is an eigenvalue of T.

   (b) The linear operator $\lambda I - T$ is singular.

   (c) The scalar λ is a root of the characteristic polynomial $\Delta(t)$ of T.

30. Consider the following matrix:

$$A = \begin{bmatrix} -3 & 8 & 8 \\ -1 & 5 & -2 \\ -1 & -2 & 9 \end{bmatrix},$$

   Find A^{50}(**Think Wisely!!!**).

Best of Luck!!!