Artificial Intelligence

Lec 19: First Order Logic (contd.)

Pratik Mazumder

Content taken from various sources

First-Order Logic: Universal Quantifiers

- Mammals drink milk.
  - $\forall x\ (mammal(x) \Rightarrow drink(x, Milk))$
- Man is mortal.
  - $\forall x\ (man(x) \Rightarrow mortal(x))$
- Man is a mammal.
  - $\forall x\ (man(x) \Rightarrow mammal(x))$
- Tom is a man.
  - $man(Tom)$

First-Order Logic: Existential Quantifiers

- At least one planet has life on it.

  planet(x): x is a planet
  haslife(x): x has life on it

  $\exists x\ (\text{planet}(x) \land \text{haslife}(x)))$

- “There exists a bird that cannot fly”

  fly(x): x can fly
  bird(x): x is a bird

  - $\exists x\ (\text{bird}(x) \land \neg\text{fly}(x)))$

- Generally, $\Rightarrow$ appears to be the natural connective to use with $\forall$, $\land$ is the natural connective to use with $\exists$.

First-Order Logic: Duality of Quantifiers

- "All man are mortal" is logically equivalent to "No man is immortal".

- Formally,

  $\forall x\ (man(x) \Rightarrow mortal(x))$

  Equivalent to

  $\neg(\exists x\ (man(x) \land \neg mortal(x)))$

- "There exist birds that can fly" is logically equivalent to "It is not the case that all birds cannot fly".

  $\exists x\ (bird(x) \land fly(x)))$
  $\neg(\forall x\ (bird(x) \Rightarrow \neg fly(x)))$

First-Order Logic: Sentences

- A predicate is a sentence.
- If s1, s2 are sentences and x is a variable, then

  (s1), ¬s1, ∃x s1, ∀x s1, s1∧s2, s1∨s2, s1 ⇒ s2 are sentences
- Almost similar to propositional logic.

First-Order Logic: Sentences

- Some dogs bark.
- All dogs have four legs.
- All barking dogs are irritating.
- No dogs purr.
- Students are people who are enrolled in courses.

Solutions:

- $\exists x\ (dog(x) \wedge bark(x))$
- $\forall x\ (dog(x) \Rightarrow fourlegs(x))$, $\forall x\ (dog(x) \Rightarrow legs(x,4))$
- $\forall x\ (dog(x) \wedge bark(x)) \Rightarrow irritating(x)$
- $\forall x\ (dog(x) \Rightarrow \neg purr(x))$, $\neg(\exists x\ (dog(x) \wedge purr(x)))$
- $\forall x\ (student(x) \Rightarrow (people(x) \wedge enrolled(x)))$

First-Order Logic: Semantics

- In Propositional Logic, the truth value of a formula depends on the truth assignments to the propositions.
- In FOL, the truth value of a formula depends on the interpretation of predicate symbols and variables over some domain D (universe of discourse).
- Consider the FOL formula $\neg P(x)$.
- Given domain $D=\{x_1,x_2\}$, a possible interpretation: $P(x_1)=True$, $P(x_2)=False$
  - say $x=x_1$ then $\neg P(x)=?$

Nested Quantifiers

- Sometimes it may be necessary to use multiple quantifiers

- E.g., it is difficult to express “Everybody loves some” using a single quantifier.

- Suppose predicate loves(x,y) means “Person x loves person y”.

- What does $\forall x \ \exists y$ loves(x,y) mean?
    - For all person x, there exists some person y who x loves.

- What does $\exists y \ \forall x$ loves(x,y) mean?
    - There exists some person y, who is loved by all persons.

- Therefore, order of quantifiers is very important.

Nested Quantifiers

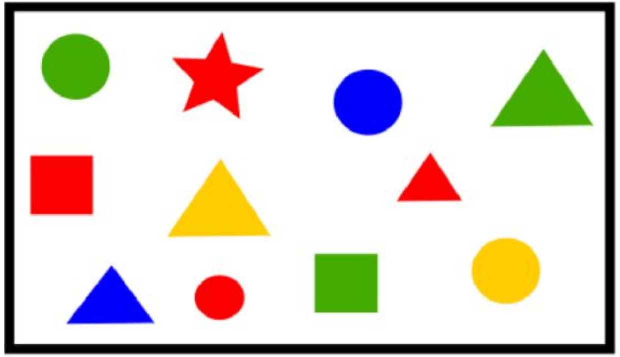
Using Loves(x,y) express the following in FOL

- Someone loves everyone
- There is someone who doesnot love anyone.
- There is someone who is not loved by anyone.
- Everyone loves everyone.
- There is someone who doesn’t love himself/herself.

Solution

- $\exists x\ \forall y$ Loves(x,y)
- $\exists x\ \forall y\ \neg$Loves(x,y)
- $\exists x\ \forall y\ \neg$Loves(y,x)
- $\forall x\ \forall y$ Loves(x,y)
- $\exists x\ \neg$Loves(x,x)

Nested Quantifiers



Which formulas are true/false? If false, give a counterexample

- $\forall x.\exists y.\ (\text{sameShape}(x, y) \wedge \text{differentColor}(x, y))$
- $\forall x.\exists y.\ (\text{sameColor}(x, y) \wedge \text{differentShape}(x, y))$
- $\forall x.\ (\text{triangle}(x) \rightarrow (\exists y.\ (\text{circle}(y) \wedge \text{sameColor}(x, y))))$

FOL Sentences

- Birthday(x,y) - x celebrates birthday on date y

- $\forall y \ \exists x$ Birthday(x,y)
  - For all dates y, there exists some person x who celebrate his/her birthday on that date
  - “Every day someone celebrates his/her birthday”

- Brother(x,y) - y is x’s brother
  Loves(x,y) - x loves y

  $\forall x \ \forall y$ Brother(x,y) $\Rightarrow$ Loves(x,y)
  Everyone loves his/her brother

- Let M(x) represent the mother of X then “Everyone loves his/her mother” can be represented in FOL as

  $\forall x$ Loves(x,M(x))

FOL Sentences

- Any number is the successor of its predecessor, e.g., 9 is the successor of its predecessor 8
  successor(x), predecessor(x), equal(x,y)

  $\forall x$ equal(x, successor(predecessor(x)))

  Or,

- $\forall x$ (successor(predecessor(x)) **=** x)

  Not generally allowed in FOL/predicate logic

FOL with Equality

- In FOL with equality, we are allowed to use the equality sign (=) between two functions.
- This is just for representational ease.
- The definition of the sentence is modified to include equality as

  term=term is also a sentence.

FOL with Equality

$$
\begin{array}{rcl}
\mathit{Sentence} & \rightarrow & \mathit{AtomicSentence} \mid \mathit{ComplexSentence} \\
\mathit{AtomicSentence} & \rightarrow & \mathit{Predicate} \mid \mathit{Predicate}(\mathit{Term}, \ldots) \mid \mathit{Term} = \mathit{Term} \\
\mathit{ComplexSentence} & \rightarrow & (\ \mathit{Sentence}\ ) \mid [\ \mathit{Sentence}\ ] \\
 & \mid & \neg\ \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \land \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \lor \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \ \Rightarrow\ \mathit{Sentence} \\
 & \mid & \mathit{Sentence} \ \Leftrightarrow\ \mathit{Sentence} \\
 & \mid & \mathit{Quantifier}\ \mathit{Variable}, \ldots\ \mathit{Sentence} \\
 & & \\
\mathit{Term} & \rightarrow & \mathit{Function}(\mathit{Term}, \ldots) \\
 & \mid & \mathit{Constant} \\
 & \mid & \mathit{Variable} \\
 & & \\
\mathit{Quantifier} & \rightarrow & \forall \mid \exists \\
\mathit{Constant} & \rightarrow & A \mid X_1 \mid \mathit{John} \mid \cdots \\
\mathit{Variable} & \rightarrow & a \mid x \mid s \mid \cdots \\
\mathit{Predicate} & \rightarrow & \mathit{True} \mid \mathit{False} \mid \mathit{After} \mid \mathit{Loves} \mid \mathit{Raining} \mid \cdots \\
\mathit{Function} & \rightarrow & \mathit{Mother} \mid \mathit{LeftLeg} \mid \cdots \\
\textsc{Operator Precedence} & : & \neg, =, \land, \lor, \Rightarrow, \Leftrightarrow
\end{array}
$$

DeMorgan's Laws for Quantifiers

- DeMorgan's law: earlier seen in Propositional Logic:

  $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

  $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

- DeMorgan's law extends to First-Order Logic

  $\neg(\mathrm{even}(x) \vee \mathrm{div4}(x)) \equiv \neg\mathrm{even}(x) \wedge \neg\mathrm{div4}(x)$

- DeMorgan's law for Quantifiers

  $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$
  $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$

DeMorgan’s Laws for Quantifiers

- No one in B3 hostel is enrolled in AI

  $\forall x\ (B3(x) \Rightarrow \neg Enrolled(x, AI))$

  $\equiv \forall x\ (\neg B3(x) \vee \neg Enrolled(x, AI))$

  $\equiv \neg \exists x\ (B3(x) \wedge Enrolled(x, AI))$

Equivalence in FOL

- In FOL, two formulas F_1 and F_2 are equivalent if $F_1 \Leftrightarrow F_2$ is valid
- In Propositional Logic, we could have proved equivalence using truth tables, but not possible in FOL.
  - because FOL deals with quantifiers that range over an infinite domain of objects, whereas
  - propositional logic deals with a finite set of propositional variables.
- However, we can still use known equivalences to re-write one formula as another.
- Equivalent: $\neg \forall x\ \exists y\ P(x,y) \equiv \exists x\ \forall y\ \neg P(x,y)$

Equivalence in FOL

The **name** of the equivalence used **MUST** be mentioned.

Otherwise No Considerations

$$
\begin{array}{rcll}
(\alpha \land \beta) & \equiv & (\beta \land \alpha) & \text{commutativity of } \land \\
(\alpha \lor \beta) & \equiv & (\beta \lor \alpha) & \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) & \equiv & (\alpha \land (\beta \land \gamma)) & \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) & \equiv & (\alpha \lor (\beta \lor \gamma)) & \text{associativity of } \lor \\
\neg(\neg\alpha) & \equiv & \alpha & \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg\beta \Rightarrow \neg\alpha) & \text{contraposition} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg\alpha \lor \beta) & \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) & \equiv & ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) & \text{biconditional elimination} \\
\neg(\alpha \land \beta) & \equiv & (\neg\alpha \lor \neg\beta) & \text{De Morgan} \\
\neg(\alpha \lor \beta) & \equiv & (\neg\alpha \land \neg\beta) & \text{De Morgan} \\
(\alpha \land (\beta \lor \gamma)) & \equiv & ((\alpha \land \beta) \lor (\alpha \land \gamma)) & \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) & \equiv & ((\alpha \lor \beta) \land (\alpha \lor \gamma)) & \text{distributivity of } \lor \text{ over } \land
\end{array}
$$

Equivalence in FOL

The name of the equivalence used MUST be mentioned.

Otherwise No Considerations

- Law of double negation: $\neg\neg\phi \equiv \phi$
- Identity Laws: $\phi \wedge T \equiv \phi \qquad \phi \vee F \equiv \phi$
- Domination Laws: $\phi \vee T \equiv T \qquad \phi \wedge F \equiv F$
- Idempotent Laws: $\phi \vee \phi \equiv \phi \qquad \phi \wedge \phi \equiv \phi$
- Negation Laws: $\phi \wedge \neg\phi \equiv F \qquad \phi \vee \neg\phi \equiv T$
- Absorption Laws: $\phi_1 \wedge (\phi_1 \vee \phi_2) \equiv \phi_1 \qquad \phi_1 \vee (\phi_1 \wedge \phi_2) = \phi_1$