COS 516: Class Project Outline
Proving the Correctness of the Normal Equations with Dafny

Samuel Sanft

October 10, 2024

1 Introduction

The normal equations are the solution to the optimization problem: $\min_x ||Ax - b||^2$, where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n, b \in \mathbb{R}^m$. The normal equations state that the optimal solution to the optimization problem is the solution to the following linear equation: $A^T Ax^* = A^T b$, or $x^* = (A^T A)^{-1} A^T b$, when $A^T A$ is invertible. The goal of my project is to implement linear algebraic operations (vector addition, matrix-vector/matrix-matrix multiplication, dot-products/vector norms, etc) and then prove the correctness of the normal equations in Dafny.

2 Proposed Work

Since there are no native types for dealing with matrices and vectors in Dafny, I'll start by defining these types and operations dealing with them. I'll then verify the correctness of various linear algebraic identities that will be useful in ultimately proving the correctness of the normal equations.

- which tool(s) you plan to use: Dafny
- which examples you plan to target: The normal equations and any linear algebraic principles necessary to implement/prove their correctness.
- what modules you plan to create or implement:
  - Vector and Matrix types
  - Vector-vector operations (vector addition, dot-product, vector norm) with associated proofs ($v_1 + v_2 = v_2 + v_1, v_1^T v_1 = ||v_1||^2$, etc)
  - Matrix-vector multiplication with associated proofs
  - Matrix-matrix multiplication with associated proofs
  - Proofs related to matrix inverses
  - Proof of normal equations
- what deliverables you will provide (in addition to a final project report): All of the implemented modules and their related proofs. These could be used to implement and verify the correctness of other algorithms dealing with matrices and vectors in future work.

3 Proposed Schedule

- Week 1 (Oct 21 - 27): Begin defining types and basic operations
- Week 2 (Oct 28 - Nov 3): Finish defining all necessary operations and work on related proofs
- Week 3 (Nov 4 - Nov 10): Verify case where $A^T A = I$ as intermediate step
- Week 4 (Nov 11 - Nov 17): Verify normal equations
- Week 5 (Nov 18 - Nov 24): Finish verification + work on presentation
- project presentations: November 25, December 2 and 4 (in class, schedule is TBD).
- final project report: due on December 13.