Logic

In this section we study the algebra of reasoning. The unknowns in ordinary algebra are of type: number. Thus x represents an unknown number. In analogy with the word pronoun that serves as a space-holder for some specific noun or type of noun, we may call x a pro-number. In logic we deal with unknowns of type 'declarative sentence' such as: 'The moon is made of green cheese', 'It rains every night in this city' etc., that is, sentences that are capable of being assigned a truth-value such as true or false. Note that interrogative sentences, that is, questions, and imperative sentences, that is, orders, requests, and exclamatory sentences, are excluded from our discussion.

Declarative statements will be called by various names: proposition, assertion, sentence etc. Letters, p, q, r, s etc. will represent unknown propositions or assertions. We can call them pro-propositions if we want to make a distinction between a known proposition such as $2 = 3$ or an unknown propostion such as $\neg p$. Note that a natural language has a finite number of proper nouns and a very small number of pronouns. For example, in English we have several pronouns, of which he, she, and it are examples. If we say 'She is in the garden and she is eating an ice-cream.' it is not clear if the two instances of 'she' refer to the same person. If we say 'x is in the garden and y is eating an ice-cream', it is possible that x and y are distinct and it is also possible that x and y are the same. If we say 'x is in the garden and x is eating an ice-cream' then we are taling about the same person. We should be very careful if there is any danger of confusion.

The language used in basic algebra has infinitely many proper nouns of type number, that is, there are names (say in decimal notation) for every real number, and there are infinitely many pronouns: $x, x_1, x_2 \cdots$, $y, y_1, y_2 \cdots$ of type number. We shall define this language recursively. We shall also treat quantification but not in any detail. We shall set up a deductive structure using axioms (assertions that are assumed to be true) and rules of inference (rules that allow us to infer or deduce true assertions from other true assertions). A proof of an assertion deduces an assertion from axioms and other assertions either known to be true or assumed to be true using rules of inference given beforehand. In principle, one can check the steps and verify if a deduction is sound or correct.

This section is, to a large extent, a repetition of what you learnt in the section on sets in a different language. Since one is seeing this material a second time around, the material will appear simpler. One will see the material a third time, in the language of Boolean algebras. Each of these conceptual frameworks makes certain things easy and certain things not as easy. If one is conversant with all three points of view, one will find it easier to choose a framework that makes questions easier to answer. A table comparing all the laws in the three systems has been included so that you understand the correspondence very clearly.

All students agree that the course gets easier as the term progresses because one is seeing the same material three times in different languages.

Laws		Laws of the Algebra of Proposition (PC)		Laws of the Algebra of Sets (S)		Laws of the Boolean Algebra (BA)
PC S BA	1	$\perp$	1	$\{\}$	1	0
p A x	2	T	2	$\mathcal{U}$	2	1
q B y	3	$\vee$	3	$\cup$	3	$+$
r C z	4	$\wedge$	4	$\cap$	4	$*$
s D w	5	$\neg p$	5	$(\,)^c$	5	$(\,)'$
t E u	6	$p \wedge (\neg q)$	6	$A \setminus B$	6	$x * y'$
	7	$p \to q \equiv (\neg p) \vee q$	7	$(A)^c \cup (B)$	7	$(x)' + (y)$
	8	$\leftrightarrow$	8	$(A^c \cup B) \cap (B^c \cup A)$	8	$(x' + y) * (y' + x)$
Idempotent Laws		$p \vee p = p$ $p \wedge p = p$		$A \cup A = A$ $A \cap A = A$		$x + x = x$ $x * x = x$
Associative Laws		$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$		$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$		$(x + y) + z = x + (y + z)$ $(x * y) * z = x * (y * z)$
Commutative Laws		$p \vee q = q \vee p$ $p \wedge q = p \wedge q$		$A \cup B = B \cup A$ $A \cap B = B \cap A$		$x + y = y + x$ $x * y = y * x$
Distributive Laws		$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$		$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$		$x + (y * z) = (x + y) * (x + z)$ $x * (y + z) = (x * y) + (x * z)$
Identity Laws		$p \vee \perp = p$ $p \wedge T = p$ $p \vee T = T$ $p \wedge \perp = \perp$		$A \cup \{\} = A$ $A \cap \mathcal{U} = A$ $A \cup \mathcal{U} = \mathcal{U}$ $A \cap \{\} = \{\}$		$x + 0 = x$ $x * 1 = x$ $x + 1 = 1$ $x * 0 = 0$
Complement Laws		$p \vee (\neg p) = T$ $p \wedge (\neg p) = \perp$ $\neg T = \perp$ $\neg \perp = T$ Uniqueness of complement If $p \vee x = T$ and $p \wedge x = \perp$ then $x = \neg p$		$A \cup A^c = \mathcal{U}$ $A \cap A^c = \{\}$ $\mathcal{U}^c = \{\}$ $\{\}^c = \mathcal{U}$ Uniqueness of complement If $A \cup X = \mathcal{U}$ and $A \cap X = \{\}$ then $X = A^c$		$x + x' = 1$ $x * x' = 0$ $0' = 1$ $1' = 0$ Uniqueness of complement If $a + x = 1$ and $a * x = 0$ then $x = a'$
Involution Law		$\neg(\neg p) = p$		$(A^c)^c = A$		$(x')' = x$
De Morgan's Law		$\neg(p \vee q) = (\neg p) \wedge (\neg q)$ $\neg(p \wedge q) = (\neg p) \vee (\neg q)$		$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$		$(x + y)' = x' * y'$ $(x * y)' = x' + y'$
Absorption Law		$p \vee (p \wedge q) = p$ $p \wedge (p \vee q) = p$		$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$		$x + (x * y) = x$ $x * (x + y) = x$
Boundedness		$p \vee T = T$ $p \wedge \perp = \perp$		$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \{\} = \{\}$		$x + 1 = 1$ $x * 0 = 0$

D The language of propositional calculus is defined recursively as follows. The language of propositional calculus contains:

Propositional constants: $\top$ (Truth), $\bot$ (Falsehood)

A set of pro-propositions or propositional unknowns:

$$PU := \{p, p_1, p_2, p_3, \cdots, q, q_1, q_2, q_3, \cdots, r, r_1, r_2, r_3, \cdots, s, s_1, s_2, s_3, \cdots\}$$

Connectives:

$\neg$ (not, negation)

$\vee$ (or, disjunction),

$\wedge$ (and, conjunction),

$\longrightarrow$ (implies, implication)

$\longleftrightarrow$ (implies and is implied by, if and only if, bi-implication)

Logical constants:

$=$ (equality)

The set of well-formed formulas or propositions Prp is defined recursively as follows:

BC BC0 $\top, \bot \in Prp$

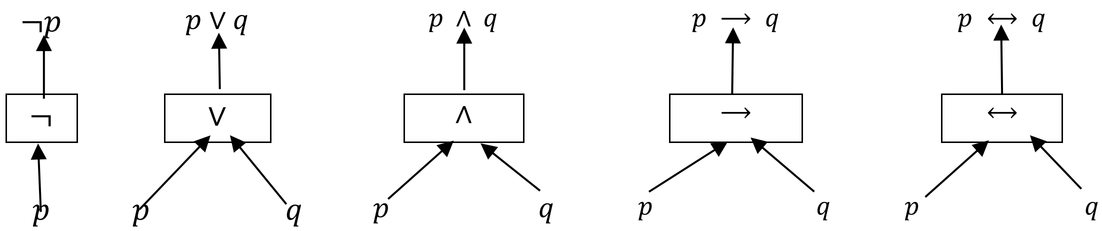
BC1 $\forall p \in PU\left(p \in Prp\right)$

RcS

$$\frac{\vartheta, \varphi \in Prp}{\neg\vartheta \in Prp \quad \vartheta \vee \varphi \in Prp \quad \vartheta \wedge \varphi \in Prp \quad \vartheta \longrightarrow \varphi \in Prp \quad \vartheta \longleftrightarrow \varphi \in Prp}$$

R The above definition can be understood to provide a formation-tree for each of member of Prp which can be exhibited as follows in the simplest cases. These basic trees are put together to build more complex expressions. The connectives should be enclosed in circles which my software does not allow me to do.

<u>Basic Formation Trees</u>



R Roughly speaking, we interpret propositions $\vartheta, \varphi \in Prp$ as maps from $\{0,1\}^n \longrightarrow \{0,1\}$ for appropriate values of n, where 0 represents the constant value for Falsehood $(\perp)$ and 1 the constant value for Truth $(\top)$. The following considerations explain the assignment of truth values.

R According to the above scheme, we have the following maps:

nullary	T	$:=$	$* \longmapsto T: \{0,1\}^0 \longrightarrow \{0,1\}$
nullary	$\perp$	$:=$	$* \longmapsto T: \{0,1\}^0 \longrightarrow \{0,1\}$
unary	$\neg$	$:=$	$p \longmapsto \neg p: \{0,1\}^1 \longrightarrow \{0,1\}$
binary	$\vee$	$:=$	$(p,q) \longmapsto p \vee q: \{0,1\}^2 \longrightarrow \{0,1\}$
binary	$\wedge$	$:=$	$(p,q) \longmapsto p \wedge q: \{0,1\}^2 \longrightarrow \{0,1\}$
binary	$\longrightarrow$	$:=$	$(p,q) \longmapsto p \longrightarrow q: \{0,1\}^2 \longrightarrow \{0,1\}$
binary	$\leftrightarrow$	$:=$	$(p,q) \longmapsto p \leftrightarrow q: \{0,1\}^2 \longrightarrow \{0,1\}$

R Note that:

$$\{0,1\}^0 \cong \{*\}, \{0,1\}^1 \cong \{0,1\}, \{0,1\}^2 \cong \{(0,0),(0,1),(1,0),(1,1)\}$$

Note the elements of $\{0,1\}^2$ are entered in the two columns of the truth-table with the headings p and q. The entries in each row correspond to the ordered pairs listed in the cartesian product in the above.

R A propositional expression $\varphi\left(p_1, p_2, \cdots, p_n\right) \in Prp$ with n propositional un knowns as arguments, gives rise to a map

$$\varphi := \left(p_1, p_2, \cdots, p_n\right) \longmapsto \varphi\left(p_1, p_2, \cdots, p_n\right) : \{0, 1\}^n \longrightarrow \{0,1\}$$

The truth table for φ will have 2^n rows. Thus if p and q are the only declared unknowns that occur in the expression, then there will be $2^2 = 4$ rows as we have already seen. An expression with exactly three declared unknowns, p, q, and r will have $2^3 = 8$ rows while an expression with exactly four unknowns, p, q, r, and s will have $2^4 = 16$ rows We have not been completely honest either in our notation or in our description. We shall not concern ourselves with these lapses at the moment. On the basis of the discussion above, it should be easy to understand the truth tables that follow.

<u>Truth Tables</u>

$\bot$	$\top$		p	q		$\neg p$		$p \vee q$	$p \wedge q$	$p \longrightarrow q$	$p \leftrightarrow q$
0	1		0	0		1		0	0	1	1
0	1		0	1		1		1	0	1	0
0	1		1	0		0		1	0	0	0
0	1		1	1		0		1	1	1	1

R Negation is simple; if p is true, then $\neg p$ is false and vice versa.

R The disjunction $p \vee q$ is false if each disjunct is false. In this form of disjunction we know that either one or the other or both p and q are true but we do not know which is the case.

R The conjunction $p \wedge q$ is true if each conjunct is true.

R Material implication $p \longrightarrow q$ is false if the antecedent p is true and the consequent q is false. This state of affairs provides evidence against the implication. If the antecedent is true and the consequent is also true, then that is not evidence against the implication. If the antecedent is false and the consequent is also false, then that is not evidence against the implication. If the antecedent is true and the consequent is false, that is not evidence against the implication either because the consequent may be implied by some other antecedent.

R Bi-implication $p \leftrightarrow q$ is true if and only if both p and q are true or if both p and q are false.

Q Find truth tables for the following propositional expressions.

$$(p \vee q) \longrightarrow (p)$$

$$(p \wedge q) \longrightarrow (\neg q)$$

$$(p \longrightarrow q) \longrightarrow (\neg p)$$

$$(p \longrightarrow q) \longrightarrow \big((\neg p) \longrightarrow q\big)$$

$$(p \vee q) \longrightarrow (r)$$

$$(p \wedge q) \longrightarrow (\neg r)$$

$$(p \longrightarrow q) \longrightarrow (\neg p)$$

$$(p \longrightarrow q) \longrightarrow \big((\neg p) \longrightarrow r\big)$$

R Examples and the method to be employed for the above problems appear in the below.

Q How can one find truth-tables of a propositional expression? To do this efficiently, we always start from the odd rows.

E We explain this procedure in the example below. Under $p \longrightarrow (\neg q)$ we first fill in the odd row out for implication, that is, when p is true $\neg q$ is false, then is $p \longrightarrow (\neg q)$ false and gets the value 0. The remaining rows will all get the value 1. In order to show the order in which the rows are filled, the table below has three columns that are filled in from left to right. On the exams, there will be exactly one column and one fills the entries in, in the correct order. Please use a ruler to ensure that the correct row is being used. The last grey column has the final answer.

p	q	$\neg p$	$\neg q$	$p \longrightarrow (\neg q)$	$p \longrightarrow (\neg q)$	$p \longrightarrow (\neg q)$
0	0	1	1		1	1
0	1	1	0		1	1
1	0	0	1		1	1
1	1	0	0	0		0

E Find the truth table for $p \longrightarrow (q \wedge r)$ Note that conjucntion gets 1 only if each conjunct has truth-value 1. Also implication gets 0 only of antecedent gets 0 and consequent gets 1. We have again used three columns for each formula to indicate the order in which the columns are filled from left to right. The last grey column has the final answer.

p	q	r	$q \wedge r$	$q \wedge r$	$q \wedge r$	$p \longrightarrow (q \wedge r)$	$p \longrightarrow (q \wedge r)$	$p \longrightarrow (q \wedge r)$
0	0	0		0	0		1	1
0	0	1		0	0		1	1
0	1	0		0	0		1	1
0	1	1	1		1		1	1
1	0	0		0	0	0		0
1	0	1		0	0		1	1
1	1	0		0	0	0		0
1	1	1	1		1		1	1

R We have addressed the problem of finding a truth-table for a propositional expression of formula. We shall now address the opposite question.

Q Given a truth-table, find a propositional expression of which it is the truth-table.

R We shall solve the above problem algorithmically. The idea is to realise the formula as a disjunction of conjunctions. For every set $\left\{p_k \in P \middle| k \in 1..n\right\} = \left\{p_1, p_2, \cdots, p_n\right\}$ of propositional unknowns, it is possible to construct 2^n conjunctions $\left\{\varphi_l \middle| l \in 1..2^n\right\}$using exactly n propositions from the set $\left\{p_k,\ \neg(p_k) \middle| k \in 1..n\right\}$ such that for every $l \in 1..2^n,\ \varphi_l$ has exactly one 1 and $(2^n - 1)$ 0's in its truth table. Since there are exactly 2^nrows in the truth table, we shall have a complete list. Now if a truth table has 1's only at the positions $\left\{k_i \in \mathbb{N} \middle| i \in 1..m \subseteq 1..2^n\right\}$ then the formula $\varphi \coloneqq \bigvee_{i\in 1..m}\left(\varphi_{k_i}\right)$ will have precisely the required truth table. Disjuctions get a 1 preceisely where at least one disjunct has a 1 and we have arranged matters so that there is exactly one φ_{k_i} with a 1 in the correct location. A complete list $\left\{\varphi_l \middle| l \in 1..2^n\right\}$ for $n = 2$ and $n = 3$ appear in the below. The reader should construct the case $n = 4$.

R This algorithm yields a solution to the problem but the solution is not easy to use in applications such as the design of circuits. In the section on Boolean Algebras we shall learn how one may get 'simpler' solutions. We shall need to define an appropriate notion of simplicity.

R The algorithm yields the disjunctive normal form $DNF(\varphi)$ for a $\varphi\left(p_1, p_2, \cdots, p_n\right) \in Prp$

Tables for Finding Formulas with Prescribed Truth Tables

					$\varphi_1(p,q)$	$\varphi_2(p,q)$	$\varphi_3(p,q)$	$\varphi_4(p,q)$
p	q	$\neg p$	$\neg q$		$p \wedge q$	$p \wedge (\neg q)$	$(\neg p) \wedge q$	$(\neg p) \wedge (\neg q)$
0	0	1	1		0	0	0	1
0	1	1	0		0	0	1	0
1	0	0	1		0	1	0	0
1	1	0	0		1	0	0	0

						φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \wedge q \wedge r$	$p \wedge q \wedge (\neg r)$	$p \wedge (\neg q) \wedge r$	$p \wedge (\neg q) \wedge (\neg r)$	$(\neg p) \wedge q \wedge r$	$(\neg p) \wedge q \wedge (\neg r)$	$(\neg p) \wedge (\neg q) \wedge r$	$(\neg p) \wedge (\neg q) \wedge (\neg r)$
0	0	0	1	1	1	0	0	0	0	0	0	0	1
0	0	1	1	1	0	0	0	0	0	0	0	1	0
0	1	0	1	0	1	0	0	0	0	0	1	0	0
0	1	1	1	0	0	0	0	0	0	1	0	0	0
1	0	0	0	1	1	0	0	0	1	0	0	0	0
1	0	1	0	1	0	0	0	1	0	0	0	0	0
1	1	0	0	0	1	0	1	0	0	0	0	0	0
1	1	1	0	0	0	1	0	0	0	0	0	0	0

In this case, the we have $\varphi_1(p,q,r)$, $\varphi_2(p,q,r)$ and so on but we have suppressed the arguments.

E To illustrate the algorithm we discussed we consider the following example. Find $\varphi(p,q)$ with the following truth table.

p	q	$\neg p$	$\neg q$		$\varphi(p,q)$	$\varphi_1(p,q)$	$\varphi_3(p,q)$
0	0	1	1		0	0	0
0	1	1	0		1	0	1
1	0	0	1		0	0	0
1	1	0	0		1	1	0

Note that we decompose $\varphi(p,q)$ into constituent columns each of which have exactly one 1 in in it. We start from the bottommost 1 and make our way to the top starting from left to right. We with the complete table that we have already constructed to identify the columns as $\varphi_1(p,q)$ and $\varphi_3(p,q)$. We conclude that:

$$\begin{array}{rcc} & & \varphi(p,q) \\ = & & \left(\varphi_1(p,q)\right) \\ & \vee & \left(\varphi_3(p,q)\right) \end{array}$$

$$\begin{array}{rcc} & & \varphi(p,q) \\ = & & (p \wedge q) \\ = & \vee & \left((\neg p) \wedge \mathrm{q}\right) \end{array}$$

E To illustrate the algorithm we discussed we consider the following example. Find $\varphi(p,q,r)$ with the following truth table.

						φ	φ_1	φ_3	φ_5	φ_8
p	q	r	$\neg p$	$\neg q$	$\neg r$	$\varphi(p,q,r)$				
0	0	0	1	1	1	1	0	0	0	1
0	0	1	1	1	0	0	0	0	0	0
0	1	0	1	0	1	0	0	0	0	0
0	1	1	1	0	0	1	0	0	1	0
1	0	0	0	1	1	0	0	0	0	0
1	0	1	0	1	0	1	0	1	0	0
1	1	0	0	0	1	0	0	0	0	0
1	1	1	0	0	0	1	1	0	0	0

$$
\begin{array}{ccl}
 & & \varphi(p,q) \\
= & & \big(\varphi_1(p,q)\big) \\
 & \vee & \big(\varphi_3(p,q)\big) \\
 & \vee & \big(\varphi_5(p,q)\big) \\
 & \vee & \big(\varphi_8(p,q)\big)
\end{array}
$$

$$
\begin{array}{ccl}
 & & \varphi(p,q) \\
= & & (p \wedge q \wedge r) \\
 & \vee & (p \wedge (\neg q) \wedge r) \\
 & \vee & \big((\neg p) \wedge q \wedge r\big) \\
 & \vee & \big((\neg p) \wedge (\neg q) \wedge (\neg r)\big)
\end{array}
$$

E What we find here is called the disjunctive normal form for the proposition $\varphi \in Prp$.

T Every $\varphi \in Prp$ can be expressed uniquely as a disjunction of conjunctions of propositional unknowns and negated propositional unknowns. This is called the disjunctive normal form and is written as $DNF(\varphi)$.

R We use the disjunctive normal form to prove that two propositions are equal, using the following theorem.

T $$\forall \varphi_1, \varphi_2 \in Prp \quad \left(\frac{DNF(\varphi_1) = DNF(\varphi_2)}{\varphi_1 = \varphi_2} \right)$$

Q Make up truth-tables for propositional expressions in two, three, and four unknowns and find the corresponding formulas.

Special propositional expressions

D Given $\varphi \in Prp$, we denote the truth table of φ by $TT(\varphi\,)$; when we talk about values in $TT(\varphi\,)$, we mean the numbers occurring under the column with the heading φ.

D $\varphi \in Prp$ called a tautology and written

$$Tlg(\varphi\,) \quad :\Leftrightarrow \quad 0 \notin TT(\varphi\,), \text{ that is, } TT(\varphi\,) \text{ contains only 1's.}$$

D $\varphi \in Prp$ is called a a contradiction and written

$$Cdn(\varphi\,) \quad :\Leftrightarrow \quad 1 \notin TT(\varphi\,), \text{ that is, } TT(\varphi\,) \text{ contains only 0's.}$$

D $\varphi \in Prp$ is called a contingency and written

$$Cng(\varphi\,) \quad :\Leftrightarrow \quad 0{,}1 \in TT(\varphi\,), \text{ that is, } TT(\varphi\,) \text{ contains both 0's and 1's}$$

E

p	$\neg p$	$p \vee (\neg p)$	$p \wedge (\neg p)$	$p \rightarrow (\neg p)$
0	1	1	0	1
1	0	1	0	0
		$Tlg(p \vee (\neg p))$	$Cdn(p \wedge (\neg p))$	$Cng(p \rightarrow (\neg p))$

E

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \vee (q \rightarrow p)$	$(p \rightarrow q) \leftrightarrow (q \rightarrow p)$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	0	1	1	0
1	1	1	1	1	1
				$Tlg((p \rightarrow q) \vee (q \rightarrow p))$	$Cng((p \rightarrow q) \leftrightarrow (q \rightarrow p))$

Relations between Propositional Expressions

Given $\varphi, \psi \in Prp$

D φ is said to logically imply ψ , and written

$$\varphi \Rrightarrow \psi \quad :\Leftrightarrow \quad Tlg(\varphi \longrightarrow \psi)$$

D φ is said to be logically equivalent to ψ , and written

$$\varphi \Lleftarrow\!\Rrightarrow \psi \quad :\Leftrightarrow \quad Tlg(\varphi \leftrightarrow \psi)$$

E

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \longrightarrow (q \vee p)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	1
1	1	1	1	1
				$Tlg\big((p \wedge q) \longrightarrow (q \vee p)\big)$

$$\frac{Tlg\big((p \wedge q) \longrightarrow (q \vee p)\big)}{(p \wedge q) \Rrightarrow (q \vee p)}$$

E

p	q	$\neg p$	$\neg q$	$p \longrightarrow q$	$(\neg q) \longrightarrow (\neg p)$	$(p \longrightarrow q) \leftrightarrow \big((\neg q) \longrightarrow (\neg p)\big)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1
						$Tlg\Big((p \longrightarrow q) \leftrightarrow \big((\neg q) \longrightarrow (\neg p)\big)\Big)$

$$\frac{Tlg\Big((p \longrightarrow q) \leftrightarrow \big((\neg q) \longrightarrow (\neg p)\big)\Big)}{(p \longrightarrow q) \Lleftarrow\!\Rrightarrow \big((\neg q) \longrightarrow (\neg p)\big)}$$

R The preceding equivalence constitutes the basis for the proof principle known as 'proof by contraposition'; because the expression $(\neg q) \longrightarrow (\neg p)$ is called the contrapositive of the expression $(p \longrightarrow q)$. In this method of proof, in order to prove

$(p \longrightarrow q)$, we assume $(\neg q)$ and establish $(\neg p)$. We have already seen prroofs by contraposition.

E

$\perp$	p	q	$\neg q$	$p \wedge (\neg q)$	$(p \wedge (\neg q)) \longrightarrow \perp$	$p \longrightarrow q$	$(p \longrightarrow q) \leftrightarrow \big((p \wedge (\neg q)) \longrightarrow \perp\big)$
0	0	0	1	0	1	1	1
0	0	1	0	0	1	1	1
0	1	0	1	1	0	0	1
0	1	1	0	0	1	1	1
							$Tlg\Big((p \longrightarrow q) \leftrightarrow \big((p \wedge (\neg q)) \longrightarrow \perp\big)\Big)$

$$\frac{Tlg\Big((p \longrightarrow q) \leftrightarrow \big((p \wedge (\neg q)) \longrightarrow \perp\big)\Big)}{(p \longrightarrow q) \Longleftarrow\!\!\Longrightarrow \big((p \wedge (\neg q)) \longrightarrow \perp\big)}$$

R The preceding equivalence constitutes the basis for the proof principle known as 'proof by contradiction'; because the expression $\perp$ is called falsehood or contradiction. In this method of proof, in order to prove $(p \longrightarrow q)$, we assume (p) and $(\neg q)$ and deduce $\perp$, that is, a contradiction. We have already seen proofs by contradiction; for example the irrationality of $\sqrt{2}$.

R We record the two preceding results and a third in the form of a theorem because they are used very frequently.

T
$$\boxed{\Big(p \longrightarrow q\Big) = \Big((\neg p) \vee q\Big) = \Big((\neg q) \longrightarrow (\neg p)\Big) = \Big((p \wedge (\neg q)) \longrightarrow \perp\Big)}$$

R We are being imprecise in the above by writing = rather than $\Longleftarrow\!\!\Longrightarrow$.

In the algebra of propositions, however, we may interpret logical equivalence as equality and, therefore, on the chart we state the rules in terms of equality.

R If, in order to prove $(p \longrightarrow q)$, we assume (p) and deduce (q), then we are said to employ the method of direct proof. A proof by contraposition is also called an indirect proof.

Q Prove that $(p \longrightarrow \perp) \Longleftrightarrow (\neg p)$

Q Prove that $(p \longrightarrow q) \Longleftrightarrow \big((\neg p) \vee q\big)$

Q Classify as a tautology, contradiction, or contingency:

0 $(p \vee p) \longrightarrow p$

1 $p \longrightarrow (p \vee q)$

2 $(p \vee q) \longrightarrow (q \vee p)$

3 $(p \longrightarrow q) \longrightarrow \big((r \vee p) \longrightarrow (r \vee q)\big)$

Q Express, if possible, all the connectives in terms of the following sets of c:onnectives: $\left\{\vee, \neg\right\}, \left\{\wedge, \neg\right\}, \left\{\vee, \longrightarrow\right\}, \left\{\wedge, \longrightarrow\right\}$

S An deduction or inference is specified by a list of premises, assumptions, or hypotheses $\varphi_1, \varphi_2, \cdots, \varphi_n \in Prp$ and a conclusion $\psi \in Prp$ that is inferred or deduced from the premises; it is displayed as follows: $\Delta \coloneqq \left(\dfrac{\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n}{\psi}\right)$

D A deduction or inference $\Delta \coloneqq \left(\dfrac{\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n}{\psi}\right)$ is said to be valid and written

$$Vld\left(\Delta\right) \Leftrightarrow Vld\left(\frac{\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n}{\psi}\right) :\Leftrightarrow Tlg\left(\left(\bigwedge\nolimits_{k \in 1..n} (\varphi_k)\right) \longrightarrow \psi\right)$$

D If a deduction or inference Δ is not valid, we write $\cancel{Vld}\left(\Delta\right)$.

E Prove that $Vld\left(\frac{p \quad p\to q}{q}\right)$

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to (q)$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1
				$Tlg\big((p \wedge (p \to q)) \to (q)\big)$

$$\frac{Tlg\big((p \wedge (p \to q)) \to (q)\big)}{Vld\left(\frac{p \quad p\to q}{q}\right)}$$

Q Prove that $Vld\left(\frac{p\wedge q}{q}\right)$, $Vld\left(\frac{p}{p\vee q}\right)$, $Vld\left(\frac{p\to q \quad q\to r}{p\to r}\right)$

D One may define the validity of a deduction using an inferential structure recursively, where the bases cases essentially corresponds to axioms, namely propositions that are declared to be true at the outset, and sound rules of inference that essentially list atomic deductions.

D Validity of a deduction can be defined within the preceding scheme. We shall not do so but we shall introduce the process through examples and readers will be expected to mimic the process. One should regard this as an introduction to a game in which one has been given the pieces (the language), the allowed combinations of pieces (members of Prp), and the allowed moves (axioms and rules of inference). One must start from the given premises and propositional expressions that are either known to be true (theorems) or declared to be true (axioms) and use rules of inference to make sound deductions from the premises to arrive at the conclusion.

Axioms and Rules of Inference for Propositional Calculus

Primitive Operations:

0. $\{\vee, \longrightarrow\}$

Axioms

0 $(p \vee p) \longrightarrow p$

1 $p \longrightarrow (p \vee q)$

2 $(p \vee q) \longrightarrow (q \vee p)$

3 $(p \longrightarrow q) \longrightarrow \big((r \vee p) \longrightarrow (r \vee q)\big)$

Rules of Inference

0 Rule of Substitution: From x to infer any substitution-instance of x

1 $\dfrac{p \quad p \longrightarrow q}{q}$ (Modus Ponens)

Derived Rules of Inference:

0 $\dfrac{}{\top}$ 1 $\dfrac{\bot}{\top}$ 2 $\dfrac{\bot}{p}$ 3 $\dfrac{p}{p}$

4 $\dfrac{p \wedge q}{p}$ 5 $\dfrac{p \wedge q}{q}$ 6 $\dfrac{p \longrightarrow q \quad q \longrightarrow r}{p \longrightarrow r}$

Proof-Schemes for $p \longrightarrow q$

Direct Proof $\dfrac{p}{q}$

Indirect Proof or Proof by Contraposition $\dfrac{\neg q}{\neg p}$

Proof by Contradiction $\dfrac{p \wedge (\neg q)}{\bot}$

E Prove that $Vld\left(\frac{p \wedge q}{q}\right)$ using deduction and rules of inference.

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\top}{p \vee (\neg p)}}{\big(p \vee (\neg p)\big) \vee (\neg q)}}{p \vee \big((\neg p) \vee (\neg q)\big)}}{\big((\neg p) \vee (\neg q)\big) \vee p}}{\big(\neg(p \wedge q)\big) \vee p}}{(p \wedge q) \longrightarrow p}$$

$$Vld\left(\frac{p \wedge q}{p}\right)$$

R First, we note that whatever can be deduced from $\top$ is true. We note that the above deduction yields that $(p \wedge q) \longrightarrow p$ is true. Hence it is valid to deduce p from $p \wedge q$.It is important to understand each step in the deduction. Since $p \vee (\neg p)$ is the law of the excluded middle which we found to be a tautology, it follows that $Vld\left(\frac{\top}{p \vee (\neg p)}\right)$. The next deduction is the substitution indicated below.

$p \longrightarrow (p \vee q)$ is an axiom that indicates the validity of deduction $Vld\left(\frac{p}{p \vee q}\right)$.

Note that we have the following substitution to justify the following step: $\frac{\big(p \vee (\neg p)\big)}{\big(p \vee (\neg p)\big) \vee (\neg q)}$

$$\frac{Vld\left(\dfrac{p}{p\vee q}\right)}{Vld\left(\dfrac{\left(p\vee(\neg p)\right)}{\left(p\vee(\neg p)\right)\vee(\neg q)}\right)}\left\langle\begin{array}{l}p\longleftarrow\left(p\vee(\neg p)\right)\\ \quad q\longleftarrow(\neg q)\end{array}\right\rangle$$

The next steps: $\left(\dfrac{p\vee\left((\neg p)\vee(\neg q)\right)}{\left((\neg p)\vee(\neg q)\right)\vee p}\right)$ and $\left(\dfrac{p\vee\left((\neg p)\vee(\neg q)\right)}{\left((\neg p)\vee(\neg q)\right)\vee p}\right)$ amount to regroupings of terms which use of the associative and commutative laws for disjunction.

The next step: $\left(\dfrac{\left((\neg p)\vee(\neg q)\right)\vee p}{\left(\neg(p\wedge q)\right)\vee p}\right)$ uses De Morgan's law: $\left((\neg p)\vee(\neg q)\right)=\left(\neg(p\wedge q)\right)$.

The next step: $\left(\dfrac{\left(\neg(p\wedge q)\right)\vee p}{\left(\neg(p\wedge q)\right)\vee p}\right)$ is again a substitution instance of the equivalence: $(p\longrightarrow q)=\ \left((\neg p)\vee q\right)$

$$\frac{(p\longrightarrow q)=\left((\neg p)\vee q\right)}{\left(\left(\neg(p\wedge q)\right)\vee p\right)=\left((p\wedge q)\longrightarrow p\right)}\left\langle\begin{array}{l}p\longleftarrow(p\wedge q)\\ \quad q\longleftarrow(p)\end{array}\right\rangle$$

Hence $Vld\left(\frac{p\wedge q}{p}\right)$

R Roughly speaking, we now have three methods with which to handle questions regarding propositional calculus; these are:

(0) the method of truth tables which students seem to prefer as it is completely mechanical

(1) the rules of propositional calculus

(2) axioms and rules of inference of propositional calculus

R Students are responsible for each of these methods and should gain proficiency in each.

Q Prove by each of the three methods mentioned in the above that each of the derived rules of inference is valid.

R In the next example we shall produce the formation tree of an expression $\varphi \in Prp$. We convert the given expression into a 'boxed' expression that has the following features.

0 Inside the innermost boxes, in between any box and boxes nested at the level immediately below, and outside all boxes, there is exactly one connective.

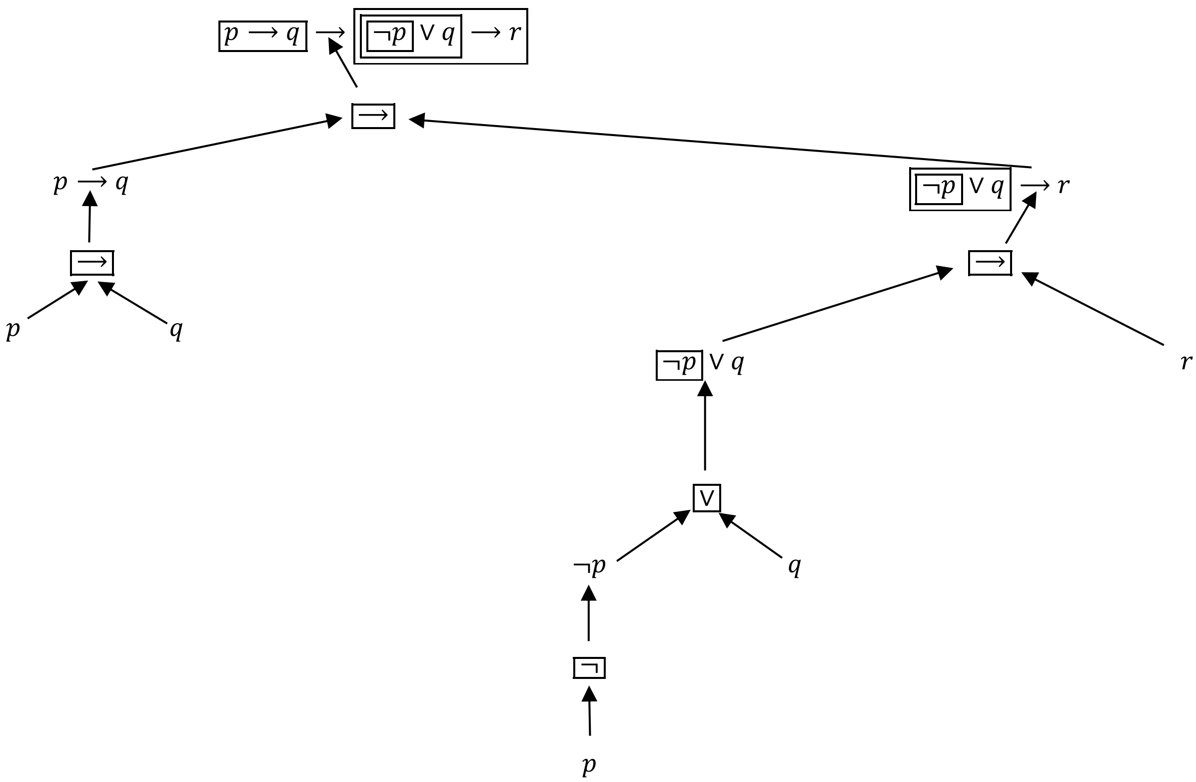
1 The root of the tree is drawn at the connective that is outside all the boxes. At the next stage one writes the connected expressions without the outermost boxes and applies this procedure to each connected subexpression. One stops when each branch ends in a propositional unknown. What is described here is essentially an algorithm for producing formation tree starting from a propositional expression.

E Find the formation tree for the following expression:

$$(p \longrightarrow q) \longrightarrow \big((\neg p) \longrightarrow r\big)$$

$$(p \longrightarrow q) \longrightarrow \Big(\big((\neg p) \vee q\big) \longrightarrow r\Big)$$

$$= \qquad \boxed{p \longrightarrow q} \longrightarrow \boxed{\boxed{\boxed{\neg p} \vee q} \longrightarrow r}$$



R The above is a very badly drawn formation tree and I do not know if the problem lies with me or with the software. Since you will be drawing these by hand, please draw arrows that have the bottom edge on a boxed connective, vertically.

R On the very last exam, you will need to draw a formation tree.

Axioms and Rules of Inference for First-order Logic

Axioms

0. All tautologies
1. All sentences of the form: $\forall x \varphi(x) \longrightarrow \varphi(a)$
2. All sentences of the form: $\varphi(a) \longrightarrow \exists x \varphi(x)$

Rules of Inference

0. $\dfrac{x \quad x \longrightarrow y}{y}$
1. $\dfrac{\varphi(a) \longrightarrow Y}{\exists x \varphi(x) \longrightarrow Y}$
2. $\dfrac{\forall x \varphi(x) \longrightarrow Y}{\varphi(a) \longrightarrow Y}$

In 1 and 2, Y is closed and a is a parameter that does not occur in either Y or $\varphi(x)$.

Derived Rules of Inference:

0. $\dfrac{}{\top}$ 1. $\dfrac{\bot}{\top}$ 2. $\dfrac{\bot}{p}$ 3. $\dfrac{p}{p}$

4. $\dfrac{p \wedge q}{p}$ 5. $\dfrac{p \wedge q}{q}$ 6. $\dfrac{p \longrightarrow q \quad q \longrightarrow r}{p \longrightarrow r}$

Proof-Schemes for $p \longrightarrow q$

Direct Proof Indirect Proof or Proof by Contraposition Proof by Contradiction

$\dfrac{p}{q}$ $\dfrac{\neg q}{\neg p}$ $\dfrac{p \wedge (\neg q)}{\bot}$