R Sets and membership constitute primitive notions in our discourse and as such cannot be defined.

Specification of sets

Sets will be specified in two ways:

S either (1) by listing elements as in the examples that follow

$S \coloneqq \{\quad\}$, $T \coloneqq \{a, @, ?, \#\}$

or (2) by using setbuilder notation as follows

$S \coloneqq \{x\epsilon\mathcal{U}|P(x)\}$, the set of all x in the universe U that satisfy the conditon P

R The above definition, though simple, may, if adequate care is not taken, lead to curious conflicts that we shall have occasion to indicate later.

Membership, elementhood

S x is called a member (element) of a set S, written

$x \epsilon S$ where S ia specified as in (1) or (2) in the above

if and only if x is from the universe $\mathcal{U}$ of discourse and is:

either listed explicitly if specification (1) is employed

or can be verified to satisfy the condition P if specification (2) is employed.

Inclusion, subsethood

D Given two sets A and B, A is said to be included in B, written

$$\left(A \subseteq B\right) :\Leftrightarrow \left(\forall x \epsilon \mathcal{U} \quad \left(\frac{x\epsilon A}{x\epsilon B}\right)\right)$$

R The above means that every member of A is also member of B.

R If A and B are both 'small' sets, specified by listing elements as in (1), to check if $A \subseteq B$ one simply inspects both sides to verify that every element of A is also listed in B. If A and B are both specified using setbuilder notation as in (2) the following theorem is used.

T

$$\frac{\{a\epsilon S|P(a)\} \subseteq \{b\epsilon T|Q(b)\}}{\{a\epsilon S|P(a)\} \subseteq T \qquad x\epsilon T \ \left(\frac{P(x)}{Q(x)}\right)}$$

R The reader is invited to investigate the remaining two possibilities.

Operations on Sets

Power-set

D $$\mathcal{P}(A) := \left\{ S \epsilon \mathcal{U} \middle| S \subseteq A \right\}$$ Note: BW uses Pwr(S)

Note that the power-set of A is the set of subsets of A.

Union

D $$A \cup B := \{x \epsilon \mathcal{U} | (x \epsilon A) \vee (x \epsilon B)\}$$

R Note that the connective or ($\vee$) is used in its inclusive sense in mathematics. Therefore an element x is in $A \cup B$ if it is either in A, or in B, or in both. Equivalently, an element is in the union of two sets if and only if it is in at least one of them; this version applies to the union of arbitrarily many sets. An element is in the union of a collection of sets if and only if, it is in at least one of them.

Intersection

D $$A \cap B := \{x \epsilon \mathcal{U} | (x \epsilon A) \wedge (x \epsilon B)\}$$

R Note that an element x is in $A \cap B$ if and only if it is in both A and B. An element is in the intersection of a collection of arbitrarily many sets if and only if it is in all of them.

Relative Complementation

D $$A \backslash B := \{x \epsilon \mathcal{U} | (x \epsilon A) \wedge (x \notin B)\}$$

R Note that $A \backslash B$contains those elements of A that are not in B and as such is not symmetric with respect to A and B. To emphasize this asymmetry, we note below the definition of $B \backslash A$ even though it is redundant to do so.

D $$B \backslash A := \{x \epsilon \mathcal{U} | (x \epsilon B) \wedge (x \notin A)\}$$

R Note that in general: $A \backslash B \neq B \backslash A$

Absolute Complementation

D $$(A)^c := A^c_{\mathcal{U}} := \mathcal{U} \backslash A$$

R Note that the customary notation $(A)^c$ is deficient in that it omits the universe of discourse. A more complete notation which includes the universe of discourse as a subscript should be used if there is any danger of confusion.

Product

D $$A \sqcap B \coloneqq A \times B \coloneqq \{(a,b)\epsilon\mathcal{U} | (a\epsilon A) \wedge (b\epsilon B)\}$$

R The product is also called the cartesian product. We use a tree to generate the elements of the product systematically.

Coproduct

D $$A \sqcup B \coloneqq A \uplus B \coloneqq \left(A \times \{0\}\right) \cup \left(B \times \{1\}\right)$$

R The coproduct is also called the disjoint union (denoted usually by: $\uplus$) and uses a device for rendering two sets with a non-empty overlap disjoint, before taking a union so that the elements of the overlap effectively occur twice in the coproduct.

Laws		Laws of the Algebra of Proposition (PC)		Laws of the Algebra of Sets (S)
PC S BA	1	$\perp$	1	$\{ \}$
p A x	2	T	2	$\mathcal{U}$
q B y	3	$\vee$	3	$\cup$
r C z	4	$\wedge$	4	$\cap$
s D w	5	$\neg p$	5	$(\ )^c$
t E u	6	$p \wedge (\neg q)$	6	$A \backslash B$
	7	$p \rightarrow q \equiv (\neg p) \vee q$	7	$(A\ )^c \cup (B\ )$
	8	$\leftrightarrow$	8	$(A^c \cup B) \cap (B^c \cup A)$
Idempotent Laws		$p \vee p = p$ $p \wedge p = p$		$A \cup A = A$ $A \cap A = A$
Associative Laws		$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$		$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws		$p \vee q = q \vee p$ $p \wedge q = p \wedge q$		$A \cup B = B \cup A$ $A \cap B = B \cap A$
Distributive Laws		$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$		$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws		$p \vee \perp = p$ $p \wedge T = p$ $p \vee T = T$ $p \wedge \perp = \perp$		$A \cup \{ \} = A$ $A \cap \mathcal{U} = A$ $A \cup \mathcal{U} = \mathcal{U}$ $A \cap \{ \} = \{ \}$
Complement Laws		$p \vee (\neg p) = T$ $p \wedge (\neg p) = \perp$ $\neg T = \perp$ $\neg \perp = T$ Uniqueness of complement If $p \vee x = T$ and $p \wedge x = \perp$ then $x = \neg p$		$A \cup A^c = \mathcal{U}$ $A \cap A^c = \{ \}$ $\mathcal{U}^c = \{ \}$ $\{ \}^c = \mathcal{U}$ Uniqueness of complement If $A \cup X = \mathcal{U}$ and $A \cap X = \{\}$ then $x = A^c$
Involution Law		$\neg(\neg p) = p$		$(A^c)^c = A$
De Morgan's Law		$\neg(p \vee q) = (\neg p) \wedge (\neg q)$ $\neg(p \wedge q) = (\neg p) \vee (\neg q)$		$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$
Absorption Law		$p \vee (p \wedge q) = p$ $p \wedge (p \vee q) = p$		$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Boundedness		$p \vee T = T$ $p \wedge \perp = \perp$		$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \{ \} = \{ \}$

Home Work

It is given that the sets: $X, Y, Z \in \{\{\ \ \}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

and that the operations: $\odot, \circledast, \circledcirc \in \{\cup,\ \cap,\ \backslash,\ \times,\ \sqcup\}$

1 Compute the following expressions using particular choices for the sets X, Y and Z from the collection above, and count the number of elements in the resulting set before carrying out the computation, using counting formulas, and after carrying out the computation, by counting elements.

R You must use the same value for every occurrence of the same letter (such as X) or ebery occurrence of the same opeartion (such as $\odot$) in every expression. You may use same or different values in the place of distinct letters (such as X and Y) or in the place of distinct connetives (such as $\odot$and$\circledast$).

R Count how many variants there are of each problem. Do as many of these variants as you need to in order to understand what is going on.

R No answers will be provided; you should try instead to work out each problem in more than one way to see if all methods lead to the same answer. Devising several ways of computing the same expression will help you understand the material battier.

R If the answers for two expressions turn out to be the same for different choices of the 'unknown' sets and operations, determine if this equality is fortuitous, or if it always holds, no matter what the values of the unknowns are.

10 $X \odot Y$

11 $X \odot (Y \circledast Z)$

12 $(X \odot Y) \circledast Z$

13 $(X \odot Y) \circledast (Y \circledcirc Z)$

14 $(X \odot Y) \circledast (X \circledcirc Z)$

15 $(X \odot Z) \circledast (Y \circledcirc Z)$

16 $\left(\mathcal{P}(X)\right)$

17 $\left(\mathcal{P}(X)\right) \odot Y$

18 $X \odot \left(\mathcal{P}(Y)\right)$

19 $\left(\mathcal{P}(X)\right) \odot \left(\mathcal{P}(Y)\right)$