Functions

We recall from our discussion of relations the following section.

Properties of Relations

D A relation $R: S \longrightarrow T$ is said to be functional and we write

$$Fncl(R) :\Leftrightarrow \ \forall s\epsilon S \ \ \exists!\, t\epsilon T \ such\ that\ sRt\ ,$$

that is, if and only if every source-element is assigned exactly one target-element as an output.

R Note that although every source-element serves as an input for a functional relation, not every target-element need serve as an output.

D Given sets S, T we define the set of all functions from S to T as follows:

$$Fnc(S,T) \coloneqq \ \{R\epsilon\mathcal{P}(S \times T)|Fncl(R)\}$$

R Since functions are relations of a certain type it is possible to specify functions as subsets, graphs, arrow-diagrams, digraphs, or matrices under those conditions under which relations may be so specified. It is, however, more convenient to specify functions using the following notational scheme.

S A function is *specified* by listing *five items*, separated by *specific symbols* as shown below:

$$\begin{pmatrix} Name\ of \\ of \\ function \end{pmatrix} \coloneqq \begin{pmatrix} Name\ of \\ generic \\ input \end{pmatrix} \longmapsto \begin{pmatrix} Output\ at \\ generic \\ input \end{pmatrix} : \begin{pmatrix} Source \\ of \\ Function \end{pmatrix} \longrightarrow \begin{pmatrix} Target \\ of \\ Function \end{pmatrix}$$

We use lowercase greek letters (such asα, as in the example below) or certain defined strings of latin letters (such as sin) to denote functions. We denote the generic input by lowercase latin letters; it is x in the example below. $\alpha(x)$, read alpha of x and not alpha times x, denotes the output at the generic input below. The set $Src(\alpha)$ denotes the source of the function; its elements are called source-elements. Every source-element serves as an input for a function. The set $Tgt(\alpha)$ denotes the target of the function; its elements are called target-elements. Not every target-element arises as an output, that is, as a value of the function at an input.

D The *image* of the set: $Src(\alpha)$ under the function

$$\alpha \coloneqq x \longmapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha),$$

denoted by $Im(Src(\alpha), \alpha)$, or in a more abbreviated form by $Im(\alpha)$, if there is no danger of confusion, is defined as follows:

$$Im(\alpha) \coloneqq \{y \in Tgt(\alpha) | \exists x \in Src(\alpha) such\ that\ y = \alpha(x)\}$$

R <u>Alternative terminology</u>

Source, Target, and Image are alternatively called Domain, Codomain, and Range respectively. Domain and Range for Source and Target respectively were also in use in the past.

D The graph of the function

$$\alpha \coloneqq x \longmapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha),$$

denoted by $Gr(\alpha)$ is defined as follows:

$$Gr(\alpha) \coloneqq \{(x, \alpha(x)) \in Src(\alpha) \times Tgt(\alpha) | x \in Src(\alpha)\}$$

R Note that:

$$Gr(\alpha) \subseteq Src(\alpha) \times Tgt(\alpha)$$

Special functions

R The following functions arise sufficiently often to deserve special names and symbols.

D Given a set S, we define the identity function of the set S as follows:

$$\iota_S \coloneqq x \longmapsto x : S \longrightarrow S$$

D Given a set S and a set T such that $S \subseteq T$, we define the inclusion function of the set S into the set T as follows:

$$\iota_{S \subseteq T} \coloneqq x \longmapsto x : S \longrightarrow T$$

D Given a set S and a set T and $t \in T$, we define the constant function from the set S into the set T with value t as follows:

$$\tilde{t} \coloneqq t_{S \longrightarrow T} \coloneqq x \longmapsto t : S \longrightarrow T$$

Given a family of sets $(S_k | k \in 1..n)$ we define $\forall k \in 1..n$

D the k^{th} projection

$$\pi_k \coloneqq (s_k | k \in 1..n) \longmapsto s_k : \prod_{k \in 1..n} S_k \longrightarrow S_k$$

D and the k^{th} insertion

$$\iota_k \coloneqq s \longmapsto (s, k) : S_k \longrightarrow \coprod_{k \in 1..n} S_k$$

Operations on Functions

D Composition of Functions

Given two functions:

$$\alpha := x \longmapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha), \text{ and}$$

$$\beta := x \longmapsto \beta(x): Src(\beta) \longrightarrow Tgt(\beta)$$

where $Tgt(\alpha) = Src(\beta)$

that is: $$Src(\alpha) \xrightarrow{\alpha} Tgt(\alpha) = Src(\beta) \xrightarrow{\beta} Tgt(\beta)$$

we define the composite $\beta \circ \alpha$ (read β circle α) as follows:

$$\beta \circ \alpha := x \longmapsto \beta(\alpha(x)): Src(\alpha) \longrightarrow Tgt(\beta)$$

Note that: $\forall x \in Src(\alpha), (\beta \circ \alpha)(x) = \beta(\alpha(x))$

D Given a function: $\alpha \in Fnc(S)$, we may compose α with itself; the n-fold composite of α with itself, denoted by $R^{\circ n}$ is defined recursively as follows:

BC $\alpha^{\circ 0} := \iota$

RcS $\forall n \in \mathbb{N}$ $\alpha^{\circ(n+1)} := \left(\alpha^{\circ(n)}\right) \circ \alpha$

R The notion of composition may be modified to include situations in which $Tgt(\alpha) \neq Src(\beta)$.

Products and Multiples

Given two functions:

$$\alpha \coloneqq x \;\longmapsto\; \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha), \text{ and}$$

$$\beta \coloneqq x \;\longmapsto\; \beta(x): Src(\beta) \longrightarrow Tgt(\beta)$$

D we define the product $\alpha \times \beta$ as follows:

$$\alpha \times \beta \coloneqq (x,y) \;\longmapsto\; \big(\alpha(x), \beta(y)\big): \big(Src(\alpha)\big) \times \big(Src(\beta)\big) \longrightarrow \big(Tgt(\alpha)\big) \times \big(Tgt(\beta)\big)$$

and if: $Src(\alpha) = Src(\beta) = S$

D we define the pair $(\alpha, \beta\,)$ as follows:

$$(\alpha, \beta) \coloneqq s \;\longmapsto\; \big(\alpha(s), \beta(s)\big): S \longrightarrow \big(Tgt(\alpha)\big) \times \big(Tgt(\beta)\big)$$

Restriction of Source, Extension of Target, and Adjustment of Target to Image

R In some situations, one encounters functions that may be made bijective by first restricting the source, and then by adjusting the target to the image.

Given a function $\alpha \coloneqq x \;\longmapsto\; \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha)$,

a set S, such that $S \subseteq Src(\alpha)$, and

a set T, such that $Tgt(\alpha) \subseteq T$, we define:

D the restriction of α,

$$\alpha\Big|_{S} \coloneqq x \;\longmapsto\; \alpha(x): S \longrightarrow Tgt(\alpha)$$

D the extension of α,

$$\alpha\Big|^{T} \coloneqq x \;\longmapsto\; \alpha(x): Src(\alpha) \longrightarrow T$$

D the adjustment of α,

$$\alpha\Big|^{Im}, \coloneqq x \;\longmapsto\; \alpha(x): Src(\alpha) \longrightarrow Im(\alpha)$$

<u>Functions induced by $\alpha \in Fnc(Src(\alpha), Tgt(\alpha))$</u>

D Given $\alpha \in Fnc(Src(\alpha), Tgt(\alpha))$, and $S \in \mathcal{P}(Src(\alpha))$, $T \in \mathcal{P}(Tgt(\alpha))$,we define:

$\alpha_{>}(A)$ the image of A under α and

$\alpha^{<}(B)$, the pre-image of B under α as follows:

D $$\alpha_{>}(S) \coloneqq \{t \in Tgt(\alpha) | \exists s \in S(t = \alpha(s))\} = \{\alpha(s) | s \in S\}$$

and

D $$\alpha^{<}(T) \coloneqq \{s \in Src(\alpha) | \exists t \in T(\alpha(s) = t)\} = \{s \in Src(\alpha) | \alpha(s) \in T\}$$

D Given $\alpha \in Fnc(Src(\alpha), Tgt(\alpha))$, we define:

the image function induced by α:

$\alpha_{>}: \mathcal{P}(Src(\alpha)) \longrightarrow \mathcal{P}(Tgt(\alpha))$ and

the pre-image function induced by α:

$\alpha^{<}: \mathcal{P}(Tgt(\alpha)) \longrightarrow \mathcal{P}(Src(\alpha))$

as follows:

D $$\alpha_{>} \coloneqq S \mapsto \alpha_{>}(S): \mathcal{P}(Src(\alpha)) \longrightarrow \mathcal{P}(Tgt(\alpha))$$

and

D $$\alpha_{>} \coloneqq S \mapsto \alpha_{>}(S): \mathcal{P}(Src(\alpha)) \longrightarrow \mathcal{P}(Tgt(\alpha))$$

Properties of Functions

D Equality

Two functions: $\alpha := x \mapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha)$ and

$\beta := x \mapsto \beta(x): Src(\beta) \longrightarrow Tgt(\beta)$

are said to be *equal*, if the following holds:

$$(\alpha = \beta) :\Leftrightarrow \begin{pmatrix} \big((1)\ Src(\alpha) = Src(\beta)\big) \\ \wedge \big((2)\ Tgt(\alpha) = Tgt(\beta)\big) \\ \wedge \big((3)\ \forall x \in Src(\alpha)(\alpha(x) = \beta(x))\big) \end{pmatrix}$$

D Injectivity

The function $\alpha := x \mapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha)$ is said to be *injective*, denoted by the predicate *Inj,* if the following holds:

$$Inj\ (\alpha): \Leftrightarrow \forall x, y \in Src(\alpha), \begin{pmatrix} (1)(x \neq y) \Longrightarrow (\alpha(x) \neq \alpha(y)) \\ or\ equivalently, \\ (2)(\alpha(x) = \alpha(y)) \Longrightarrow (x = y) \end{pmatrix}$$

R An injective function preserves the distinctness of inputs, that is, an injective function does not assign the same output to more than one input.

A To prove that a function is injective, if $Src(\alpha)$ and $Tgt(\alpha)$ are both finite sets, use a display.

If $Src(\alpha)$is not a finite set, see if: $\frac{\alpha(x)=\alpha(y)}{x=y}$ is valid.

Otherwise, to establish non-injectivity, find a counterexample, that is $x, y \in Src(\alpha)$, such that $x \neq y$ but $\alpha(x) = \alpha(y)$.

D Surjectivity

The function $\alpha := x \mapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha)$ is said to be *surjective*, denoted by the predicate *Srj*, if the following holds:

$Srj\ (\alpha): \Leftrightarrow \forall y \in Tgt(\alpha), \exists x \in Src(\alpha)$, such that: $y = \alpha(x)$.

R A surjective function hits every target-element, that is, every target element is realized as an output; that is if $(\alpha) = Tgt(\alpha)$.

A To prove that a function is surjective, if $Src(\alpha)$ and $Tgt(\alpha)$ are both finite sets, use a display.

If $Src(\alpha)$is not a finite set, see if: $\frac{\alpha(x)=y}{x=\beta(y)}$ is valid, that is, see if you can solve the equation $\alpha(x) = y$ for x in terms of y.

Otherwise to establish non-surjectivity, find a counterexample, that is, some element b $\in Tgt(\alpha)$ that is not hit.

D Bijectivity

The function $\alpha := x \longmapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha)$ is said to be *bijective*, denoted by the predicate *Bij*, if the following holds:

$$Bij\ (\alpha) :\Leftrightarrow\ Inj\ (\alpha)\ \wedge Srj(\alpha).$$

D Invertibility

The function $\alpha := x \longmapsto \alpha(x): Src(\alpha) \longrightarrow Tgt(\alpha)$ is said to be *invertible*, denoted by the predicate *Inv*, if the following holds:

$$Inv\ (\alpha) :\Leftrightarrow\ \exists\beta: Tgt(\alpha)\ (=\ Src(\beta)) \longrightarrow Src(\alpha), \begin{pmatrix} (1)(\beta \circ \alpha = \iota_{Src(\alpha)}) \\ and \\ (2)(\alpha \circ \beta =\ \iota_{Src(\beta)}) \end{pmatrix}$$

Under these conditions, β is called *the inverse of* α, and α is called *the inverse of* β.

N Note that α and β appear symmetrically in the above definition and, therefore, it makes no sense to talk of 'inverse' functions, as it is impossible to distinguish between the 'direct' and the 'inverse' function. It may be the case that one of the functions came to be considered earlier, leading to a higher degree of familiarity with it, thereby earning the one that came to be considered later the qualifier 'inverse'. This event, however, is a historical accident. Two functions that satisfy the conditions of the above definition are called inverses of each other.

N We denote the *inverse of* α, by the symbol $\alpha^{\leftarrow}$, that is:

$$\beta = \alpha^{\leftarrow},$$

$\alpha =\ \beta^{\leftarrow}$, and

$$(\alpha^{\leftarrow})^{\leftarrow} =\ \alpha$$

R Note that in the definition of invertibility, the conditions (1) and (2) are respectively equivalent to the conditions (1)' and (2)' that appear below:

(1)' $\forall x \in Src(\alpha), (\beta \circ \alpha)(x) = \beta\big(\alpha(x)\big) = \iota_{Src(\alpha)}(x) = x$ and

(2)' $\forall\, y\ \in Tgt(\alpha), (\alpha \circ \beta)(x) = \alpha\big(\beta(x)\big) = \iota_{Tgt(\alpha)}(y) = y.$

R The function α is invertible, if and only if, there is a function β going backwards that undoes the effect of α, and in such a way, that α likewise undoes the effect of β.

T $Bij\ (\alpha) \iff Inv\ (\alpha)$

Pf Omitted.

R If the function: $\alpha \coloneqq x\ \longmapsto\ \alpha(x)\colon Src(\alpha) \longrightarrow Tgt(\alpha)$ is invertible, to find the inverse $\alpha^{\leftharpoondown}$of α, we first solve the equation $\alpha(x) = y$ for x in terms of y, as shown below:

$$\frac{\alpha(x)=y}{x=\beta(y)},$$

and then set: $\alpha^{\leftharpoondown} \coloneqq y\ \longmapsto\ \beta(y)\colon Tgt(\alpha) \longrightarrow Src(\alpha).$

<u>Home Work</u>

It is given that the relations:

$$Gr(\alpha), Gr(\beta), Gr(\gamma)$$

$$\in \quad \left\{\begin{matrix} \{(a,a),(b,a),(c,a)\}, & \{(a,a),(b,b),(c,a)\}, & \{(a,a),(b,b),(c,c)\}, \\ \{(a,b),(b,a),(c,c)\}, & \{(a,b),(b,c),(c,a)\}, & \{(a,c),(b,a),(c,b)\}, \\ \{(a,b),(b,c),(c,b)\}, & \{(a,a),(b,c),(c,b)\}, & \{(a,c),(b,c),(c,a)\} \end{matrix}\right\}$$

$$\subseteq \quad \mathcal{P}\left(\{a,b,c\} \quad \times \quad \{a,b,c\}\right)$$

are **graphs of functions**, that is: $\alpha, \beta, \gamma \in Fnc(\{a,b,c\})$, and that

$$S \in \mathcal{P}(\{a,b,c\}), T \in \big\{\{a,b,c,d\},\{a,b,c,d,e\}\big\} \ \odot, \circledast, \circledcirc \ \in \left\{\circ, \Big|_S, \Big|^T, \Big|^{Im}\right\}$$

R Count how many variants there are of each problem. Do as many of these variants as you need to in order to understand what is going on.

0 Find for every α: its specification as a function.

1 Determine for every α, if:

$$Inj(\alpha), Srj(\alpha), Bij(\alpha)$$

and if $Bij(\alpha)$, find

$$\alpha^{\leftarrow}.$$

2 Compute the following expressions using particular choices for functions: α, β, and γ , for sets: S, and T, and operations: $\odot, \circledast$, and $\circledcirc$ from appropriate sets in the above.

20 $\odot\,(R)$

21 $R \odot S$

22 $(R \odot S) \circledast T$

3 For each of your answers αin 2 determine if:

$Inj(\alpha), Srj(\alpha), Bij(\alpha\ )$

and if $Bij(\alpha\ )$, find

$\alpha^{\leftarrow}$.

4 Determine for every α, how many bijecive restrictions there are and find their specifications.

5 Determine for every α, $\alpha_{>}$ and $\alpha^{<}$.