Relations

R The mathematical notion of a (binary) relation is an abstraction from the ordinary notion of human relationships which identity, parenthood, siblinghood, ancestry, descendancy, and non-identity are examples. If P denotes the relation of parenthood, then we denote the assertion 'x is a parent of y' variously as: $P(x, y), xPy, Pxy$ or $(x, y) \in P$.

We define 'x to be a sibling of y' if and only if x and y have a commom biological mother. The above definition implies the following: (0) x is a sibling of x (reflexivity) (1) if x is a sibling of y then y is a sibling of x (symmetry), and (2) if x is a sibling of y and y is a sibling of z then x is a sibling of z (transitivity). One should check the assertions using the definition and always keep this example in mind when studying equivalence relations. A relation with the properties of reflexivity, symmetry, and transitivity is called an equivalence relation. An equivalence relation allows every element to distinguish elements equivalent to itself from those that are inequivalent. Siblinghood partitions the set of human beings into non-overlapping pieces of siblings indexed by the common biological mother. All equivalence relations exhibit this feature. The set of siblings indexed by the common mother is called an equivalence class. Every partition of a set may likewise be construed as an equivalence relation.

We define 'x to be a ancestor of y' if and only if there is a chain of biological parents starting from y and reaching x. The above definition implies the following: The above definition implies the following: (0) x is an ancestor of x (reflexivity) (1) if x is an ancestor of y and y is an ancestor of x then x and y are identical (antisymmetry), and (2) if x is an ancestor of y and y is an ancestor of z then x is an ancestor of z (transitivity). One should check the assertions using the definition and always keep this example in mind when studying partial orders. To achieve symmetry, we define x be a zeroth order ancestor of x. To assure antisymmetry we need certain restrictions and the reader is invited to determine what these restrictions are. A relation with the properties of reflexivity, antisymmetry, and transitivity is called partial order. A partial order allows every element to distinguish an ancestor from a non-ancestor. The reader is invited to formalise the opposite relation of descendancy at this level.

One should try to understand all properties of relations within the context of human relationships such as motherhood, grandfatherhood, friendhip, and so on. Functions are particular types of relations in which every source element is related to exactly one target element. Some find it useful to tie abstract ideas to concrete fragments of experience. If one is familiar with a large variety of such examples, then one will find it easy to follow the material and one is less likely to make absurd mistakes.

D Given a set S (called the source of the relation and denoted by $Src(R)$) and a set T(called the target of the relation and denoted by $Tgt(R)$.), we define a relation $R: S \longrightarrow T$ to be a subset of $S \times T$; the subset is called the graph of the relation and is denoted by $Gr(R)$.

R Every relation $R: Src(R) \longrightarrow Tgt(R)$ is coded as a triple

$$R \coloneqq \left(Src(R),\ \ Gr(R) \subseteq Src(R) \times Tgt(R),\ \ Tgt(R)\right)$$

and every triple $(S,\ \ G \subseteq S \times T,\ \ T)$ may be construed as a relation $R: S \longrightarrow T$

with the identifications: $S: = Src(R),\ \ G \coloneqq Gr(R) \subseteq S \times T,\ \ T \coloneqq Tgt(R)$

D The elements of $Src(R)$ are called source-elements. Not every source-element serves as an input for a relation. The elements of $Tgt(R)$ are called target-elements. Not every target-element arises as an output for the relation.

R We often use uppercase latin letters such as: R, S, and T, or certain defined strings of latin letters (such as sin), or certain special symbols: $=, \neq, \equiv, \not\equiv, \cong, \approx, <, \leq, >, \geq\ etc.$) to denote relations.

R Every relation on a finite set may be *specified* either as:

a subset $R \subseteq S \times T$, or as

a graph $Gr(R)$, or as

an arrow-diagram: $AD(R)$ or as

a matrix $M(R)$, and

if $Src(R) = Tgt(R)$, as

a directed graph $DG(R)$.

R These various methods of specification convey exactly the same information, but each of them makes certain properties of relations particularly easy to deduce.

R If an element $s\epsilon S = Src(R)$ is related under the relation $R \subseteq S \times T$ to an element $t\epsilon T = Tgt(R)$, we say that s is R-related to t and denote this state of affairs by the following equivalent notational devices:

$$(s, t)\epsilon R \iff sRt \iff R(s, t)$$

D Given sets S, T we define the set of all relations from S to T as follows:

$$Rln(S, T) \coloneqq \{R | R \subseteq S \times T\} = \mathcal{P}(S \times T)$$

D If, in the above, $S = T$ we define the set of all relations from S to S as follows:

$$Rln(S) \coloneqq Rln(S,S) \coloneqq \{R | R \subseteq S \times S\} = \mathcal{P}(S \times S)$$

E We consider the methods of specification in the following example:

Consider the relation

$$R \coloneqq \left(\{1,2,3\}, \quad \left\{ (1,b),(2,b),(2,d) \right\} \subseteq \{1,2,3\} \times \{a,b,c,d\}, \qquad \{a,b,c,d\} \right)$$

We note that:

$Src(R) = \{1,2,3\}$

$Gr(R) = \left\{ (1,b),(2,b),(2,d) \right\} \subseteq \{1,2,3\} \times \{a,b,c,d\} = Src(R) \times Tgt(R)$

$Tgt(R) = \{a,b,c,d\}$

R We shall abbreviate the above in the form:

$$R \coloneqq \left\{ (1,b),(2,b),(2,d) \right\} \subseteq \{1,2,3\} \times \{a,b,c,d\}$$

from which specification the remaining information can be extracted.

S We may also specify the relation as a graph $Gr(R)$ visually as follows:

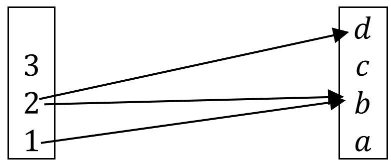
$Gr(R)$

$$\begin{array}{c|ccc}
d & \cdot & \odot & \cdot \\
c & \cdot & \cdot & \cdot \\
b & \odot & \odot & \cdot \\
a & \cdot & \cdot & \cdot \\
\hline
 & 1 & 2 & 3
\end{array}$$

Note that the above diagram is presenting information by having points at each of the twelve elements of the set $\{1,2,3\} \times \{a,b,c,d\} = Src(R) \times Tgt(R)$ and then circling the points at positions $(1,b),(2,b),(2,d)$ pretending that the members of $Src(R) = \{1,2,3\}$ lie along a horizontal line (the ghost of the $x-axis$) and that the members of $Tgt(R) = \{a,b,c,d\}$ lie along a vertical line (the ghost of the $y-axis$).

S The arrow diagram for the realtion $AD(R)$ is displayed as follows:

$AD(R)$



Note that the arrows should be assignment arrows which are drawn as $\longmapsto$; I do not seem to be able to do this with the software that I have. **Note that with each pair (x, y) in the relation, we draw an assignment arrow from x to y.**

Note also that the realtion R is not functional because not all source-element (for instance 3) has not been assigned an output, and also because the input 2 has been assigned two (rather than exactly one) output. Functionality fails on these two counts but on an exam, you are expected to provide just one reason so that I can make certain that you understand what information is enough to settle a claim; otherwise you may lose points.

S We carry out the following work in order to find the matrix $M(R)$.

$M(R)$

	a	b	c	d
1	0	1	0	0
2	0	1	0	1
3	0	0	0	0

$$M(R) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that we put the number 1 in the locations $(1, b), (2, b), (2, d)$corresponding to the elements of the set R and 0's at the other 9 (=12-3) locations. This algorithm yields a 3×4 matrix.

The above work needs to be shown on exams for computation of matrices of relations. The horizontal bar is a conclusion bar and indicates that the form of the matrix is deduced from the table that occurs above. Please note where and in what order the elements of the source and target are listed. **Follow exactly the same conventions on the exam.**

E We consider the methods of specification for the following example:

Consider the relation

$$S \coloneqq \left(\{1,2,3\},\ \left\{(1,2),(2,3),(3,1)\right\} \subseteq \{1,2,3\} \times \{1,2,3\}, \qquad \{1,2,3\} \right)$$

We note that:

$Src(S) = \{1,2,3\}$

$Gr(S) = \left\{(1,2),(2,3),(3,1)\right\} \subseteq \{1,2,3\} \times \{1,2,3\} = Src(S) \times Tgt(S)$

$Tgt(S) = \{1,2,3\}$

R We shall abbreviate the above in the form:

$$S \coloneqq \left\{(1,2),(2,3),(3,1)\right\} \subseteq \{1,2,3\} \times \{1,2,3\}$$

from which specification the remaining information can be extracted.

S We may also specify the relation as a graph $Gr(S)$ visually as follows:

$Gr(S)$

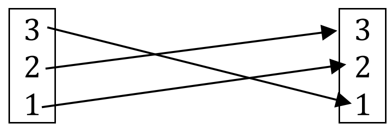
$$\begin{array}{c|ccc}
3 & \cdot & \odot & \cdot \\
2 & \odot & \cdot & \cdot \\
1 & \cdot & \cdot & \odot \\
\hline
 & 1 & 2 & 3
\end{array}$$

Note that, exactly as in the previous example, the above diagram is presenting information by having points at each of the members of the set $\{1,2,3\} \times \{1,2,3\} = Src(S) \times Tgt(S)$ and then circling the points at positions $(1,2),(2,3),(3,1)$ pretending that the members of $Src(S) = \{1,2,3\}$ lie along a horizontal line (the ghost of the $x - axis$) and that the members of $Tgt(R) = \{1,2,3\}$ lie along a vertical line (the ghost of the $y - axis$).

Note also that the relation S is functional, unlike the relation R is finctional because, every source-element is assigned exactly one output.

S The arrow diagram for the realtion $AD(S)$ is displayed as follows:

$AD(S)$



Note that the arrows should be assignment arrows which are drawn as $\mapsto$; I do not seem to be able to do this with the software that I have. Whether a relation is functional or not is (for small sources and targets) verified very quickly from arrow-diagrams, as this example illustrates.

S We carry out the following work in order to find the matrix $M(S)$.

$M(S)$

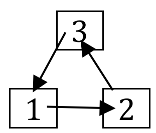
	1	2	3
1	0	1	0
2	0	0	1
3	1	0	0

$$M(S) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The above work needs to be shown on exams for computation of matrices of relations. The horizontal bar is a conclusion bar and indicates that the form of the matrix is deduced from the table that occurs above. Please note where and in Type equation here.what order the elements of the source and target are listed. **Follow exactly the same conventions on the exam**.

S Since for the relation S, $Src(S) = \{1, 2, 3\} = Tgt(S)$, one may specify S also using a directed graph, $DG(S)$, which is displayed as follows:

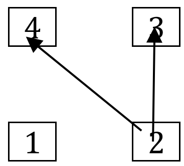
$DG(S)$



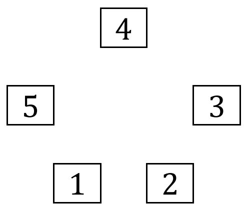
Note that we adopt the following conventions for constructing and dispalying directed graphs. If the set carrying the relation, that is, the common source and target have n elements, say with the names $1,2 \dots, n$, then we display the set as a regular $n - gon$, that is, a regular polygon with n sides with each element of the set enclosed in a $square$ (as I have drawn here because of convenience) or a *circle* (which you can draw using a small button or mechanical devices available for the purpose). The (horizontal) base always contains just the elements 1 and 2, as is the case in the example above. Note that a regular $3-gon$ is nothing other than an equilateral triangle as in the example. We label the squares or circles as the case may be, sequentially, and in the anticlockwise direction, with the numbers $1,2\ldots,n$ because we take the name of the $n-element\ set$ to be $1..n$ by convention.

If the carrying set is $\left\{1,2,3,4\right\}$ and the relation $P := \left\{(1,1),(2,3),(2,4))\right\}$ then the corresponging $4-gon$ is just a square on a horizontal base as shown, with a loop at 1 (**that I cannot draw but you must**) and with arrows from 2 to 3, and from 2 to 4. Loops are drawn without arrow as the direction is immaterial for our purposes.

$Gr(P)$



The carrying set $1..5$ will have its vertices form a regular pentagon as shown below. **This will be the case for Exam 1B.**



We have adopted the convention to name the elements of an n-element set using the first n natural numbers entirely because of convenience. In other classes and even for other situations in this course, we may have to abandon this convention.

Properties of Relations

D A relation $R: S \longrightarrow T$ is said to be functional and written

$$Fncl(R) :\Longleftrightarrow \forall s \epsilon S \; \exists! \, t \epsilon T \; such\ that\ sRt \,,$$

that is if and only if every source-element is assigned exactly one target-element as an output.

R Note that although every source-element serves as an input for a functional relation, not every target-element need serve as an output.

D Given sets S, T we define the set of all functions from S to T as follows:

$$Fnc(S,T) \coloneqq \{R \epsilon \mathcal{P}(S \times T) | Fncl(R)\}$$

Special Relations

Empty relation

D Given sets S, T we define the empty relation:

$$Emp_{S\times T} \coloneqq \{ \quad \}_{S\times T} \coloneqq \{ \quad \} \subseteq S \times T$$

Total Relation

D Given sets S, T we define the total relation:

$$Ttl_{S\times T} \coloneqq S \times T \subseteq S \times T$$

Diagonal Relation

D Given a set S, we define the diagonal (or identity) relation $\Delta_S: S \longrightarrow S$ on S as follows:

$$\Delta_S \coloneqq \{(s,t) \epsilon S \times S | s = t\} \subseteq S \times S$$

R Note that the diagonal relation is functional.

Operations on Relations

R Since a relation is defined as a subset of a set, the following set-theoretic operations $\left(\mathcal{P}(\quad), (\quad)^c \cup,\cap, (\quad)\backslash(\quad)\right)$ may be performed on relations as well.

Opposition

D Given a relation $R: S \longrightarrow T$, its opposite or converse: $R^{op}: T \longrightarrow S$ is defined as follows:

$$R^{op} := \{(t,s)\epsilon T \times S | R(s,t)\} \subseteq T \times S$$

R Note that: $\forall s\epsilon S, \forall t\epsilon T \left(\left(R(s,t)\right) \Leftrightarrow \left(R^{op}(t,s)\right)\right)$

E We consider the idea of opposition in the following example by constructing R^{op}:

$$R := \left(\{1,2,3\}, \quad \left\{(1,b),(2,b),(2,d)\right\} \subseteq \{1,2,3\} \times \{a,b,c,d\}, \qquad \{a,b,c,d\}\right)$$

We note that:

$Src(R) = \{1,2,3\} = Tgt(R^{op})$

$Gr(R) = \left\{(1,b),(2,b),(2,d)\right\} \subseteq \{1,2,3\} \times \{a,b,c,d\} = Src(R) \times Tgt(R)$

$Tgt(R) = \{a,b,c,d\} = Src(R^{op})$

$Gr(R^{op}) = \left\{(b,1),(b,2),(d,2)\right\} \subseteq \{a,b,c,d\} \times \{1,2,3\} = Src(R^{op}) \times Tgt(R^{op})$

R We shall abbreviate the above in the form: $R^{op} := \left\{(b,1),(b,2),(d,2)\right\}$

from which specification the remaining information can be extracted.

R To produce a visual display for $Gr(R^{op})$, we may either start from scratch or we may use $Gr(R)$ that we already have; this is usually more efficient. Note that we have interchanged the source and target and we must interchange the elements of every pairs that is in the relation. Thus every $(x,y) \in R$ leads to a $(y,x) \in R^{op}$.

S We may also specify the relation as a graph $Gr(R)$ visually as follows:

$Gr(R)$

$$
\begin{array}{c|ccc}
d & \cdot & \odot & \cdot \\
c & \cdot & \cdot & \cdot \\
b & \odot & \odot & \cdot \\
a & \cdot & \cdot & \cdot \\
\hline
 & 1 & 2 & 3
\end{array}
$$

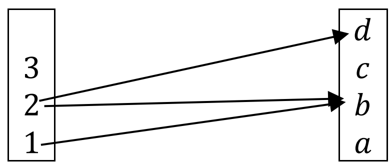
$Gr(R^{op})$

$$
\begin{array}{c|cccc}
3 & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \odot & \cdot & \odot \\
1 & \cdot & \odot & \cdot & \cdot \\
\hline
 & a & b & c & d
\end{array}
$$

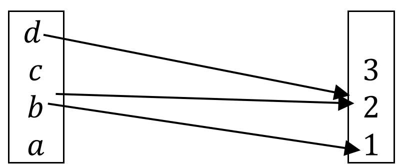
Note that in $Gr(R^{op})$ the 'axes' are interchanged and the graph of $Gr(R)$ is reflected in the 'diagonal' line. This comment should be understood at the level of 'precision' at which it is formulated. The purpose is to point out that there is a very specific relationship between $Gr(R)$ and $Gr(R^{op})$ and this relation is clearly visible if one looks the displays.

S The arrow diagram $AD(R)$ is displayed as follows:

$AD(R)$



$AD(R^{op})$



R We interchange source and target and reverse every arrow. The symmetry is plain if one looks at the two diagrams together. One may, however, produce the

S We may use matrix $M(R)$ to produce $M(R^{op})$ mechanically. We shall point out the mechanical procedure after finding $M(R^{op})$ from scratch.

$M(R)$

	a	b	c	d
1	0	1	0	0
2	0	1	0	1
3	0	0	0	0

$$M(R) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We get the following by following the same procedure as before.

$M(R^{op})$

	1	2	3
a	0	0	0
b	1	1	0
c	0	0	0
d	0	1	0

$$M(R^{op}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

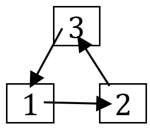
R Note that $M(R^{op})$is obtained from $M(R)$ simply by interchanging the rows and columns on $M(R)$, an operation that is called transposition. We record the result as in the below.

T

$$M(R^{op}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)^T = \left(M(R)\right)^T$$

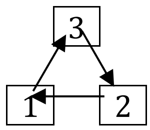
S For a relation with the same source and target, one may mechanically construct $DG(R^{op})$ from $DG(R)$ simply by reveresing all the arrows. We apply this idea to the example from before. We recall $DG(S)$ from before.

$DG(S)$



We produce $DG(S^{op})$ mechanically below.

$DG(S^{op})$



Note that since loops are not drawn with arrows, we do not have to do anything to them.

<u>Composition</u>

D Given a relation $Q: S \longrightarrow U$, and a relation $P: U \longrightarrow T$, that is: $S \overset{Q}{\longrightarrow} U \overset{P}{\longrightarrow} T$ the composition $P \circ Q: S \longrightarrow T$ is defined as follows:

$$P \circ Q \ \coloneqq \left\{ (s,t) \epsilon S \times T \middle| \exists u \epsilon U \ such\ that\ \left(\big(Q(u,t)\big) \wedge \big(P(s,u)\big) \right) \right\} \subseteq S \times T$$

T Given relations: $S \overset{R}{\longrightarrow} U \overset{Q}{\longrightarrow} V \overset{P}{\longrightarrow} T$ we have $P \circ (Q \circ R) = (P \circ Q) \circ R$ that is composition is associative.

T Given $S \overset{Q}{\longrightarrow} U \overset{P}{\longrightarrow} T$ we have $M(P \circ Q) = \big(M(Q)\big)\big(M(P)\big)\left(mod\ 2 \right)$.

R Note that the order in which P and Q occur are different on the two sides of equality. Mod 2 means that the computation of matrix entries used arithmetic modulo 2. The entries are computed as in ordinary marix multiplication but the final entry is the remainder that is obtained upon dividing the normal entry by 2.

R You have to learn matrix multiplication by yourself for Exam 1B.

R $M(P \circ Q)$ may be computed in two different ways, first by computing the composite relation and then finding its matrix, or by multiplying computing the matrices of the factors and then multiplying the matrices in opposite order as shown.

R Composition is, however, not commutative, in general, that is:

Given $S \xrightarrow{Q} S \xrightarrow{P} S$ $\quad$ $P \circ Q \neq Q \circ P$ in general.

R Note that the relation of motherhood composed with itself yields one form of the relation of grandmotherhood.

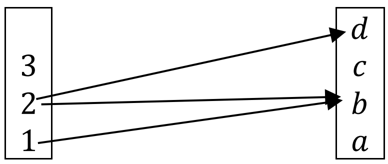
R The idea of composition is conveniently illustrated using arrow diagrams for specifying relations. Our example shows that the two orders of composition need not be defined.

E To illustrate the idea of composition we use the following examples:

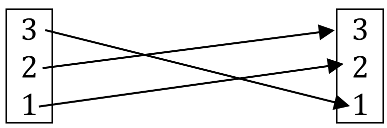
$$P := \left\{(1, b), (2, b), (2, d)\right\} \subseteq \{1, 2, 3\} \times \{a, b, c, d\}$$

$$Q := \left\{(1,2), (2, 3), (3, 1)\right\} \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

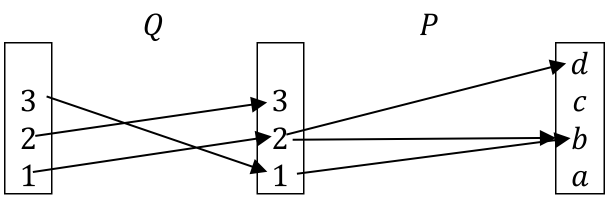
$AD(P)$



$AD(Q)$

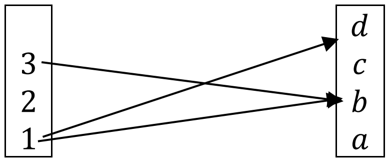


Note that since $Tgt\ (Q) = \{1, 2, 3\} = Src(P)$, the composition $P \circ Q$ is defined and leads to the following situation:



By following the arrows from each element get the arrow diagram for $P \circ Q$ as shown below.

$AD(P \circ Q\ )$



Therefore $P \circ Q = \left\{(1, b), (1, d), (3, d)\right\} \subseteq \{1, 2, 3\} \times \{a, b, c, d\}$

Since $Tgt\ (P) = \{a, b, c, d\} \neq \{1, 2, 3\} = Src(P)$, the $Q \circ P$ is not defined.

E We compute $M(P \circ Q)$ below:

$M(P \circ Q)$

	a	b	c	d
1	0	1	0	1
2	0	0	0	0
3	0	0	0	1

$$M(P \circ Q) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

E We note the following:

$$M(P) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M(Q) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore,

$M(Q)M(P)$

$$= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (0)(0) + (1)(0) + (0)(0) & (0)(1) + (1)(1) + (0)(0) & (0)(0) + (1)(0) + (0)(0) & (0)(0) + (1)(1) + (0)(0) \\ (0)(0) + (0)(0) + (1)(0) & (0)(1) + (0)(1) + (1)(0) & (0)(0) + (0)(0) + (1)(0) & (0)(0) + (0)(1) + (1)(0) \\ (1)(0) + (0)(0) + (0)(0) & (1)(1) + (0)(1) + (0)(0) & (1)(0) + (0)(0) + (0)(0) & (1)(0) + (0)(1) + (0)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+1+0 & 0+0+0 & 0+1+0 \\ 0+0+0 & 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 1+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \left(mod\ 2 \right)$$

$$= M(P \circ Q)$$

R **In the above the reduction modulo 2 is trivial as all the numbers are 0 or 1. If there is any other number, all odd numbers should be replaced by 1 and all even numbers replaced by 0; this is precisely what reduction modulo 2 does.**

R If the equality $M(P \circ Q) = \big(M(Q)\big)\big(M(P)\big)\left(mod\ 2 \right)$ is violated, then, something is amiss and one should redo the computation.

R **All computations with matrices must be displayed in the above mnner with all steps exactly as shown. The indicidual entries may be computed in a separate part of the document as it may be difficult to fit in everything in a single matrix.** In this case one should show separate calculations for each element and label them with the name of the entry such as $\big(M(R)\big)_{kl}$where k and l have particular numerical values such as: $\big(M(R)\big)_{11}$, $\big(M(R)\big)_{23}$, $\big(M(R)\big)_{34}$etc.

Iteration

D Given a relation: $R \in Rln(S)$, we may compose R with itself; the n-fold composite of R with itself, denoted by $R^{\circ n}$ is defined recursively as follows:

BC $R^{\circ 0} := \Delta_S$

RcS $\forall n \in \mathbb{N}$ $R^{\circ(n+1)} := \left(R^{\circ(n)}\right) \circ R$

D Given a functional relation: $\alpha \in Fnc(S)$, we may compose α with itself; the n-fold composite of α with itself, denoted by $R^{\circ n}$ is defined recursively as follows:

BC $\alpha^{\circ 0} := \iota_S$

RcS $\forall n \in \mathbb{N}$ $\alpha^{\circ(n+1)} := \left(\alpha^{\circ(n)}\right) \circ \alpha$

Properties of a relation from a set to itself ($R: S \longrightarrow S$)

Reflexivity

D A relation $R: S \longrightarrow S$ from a set to itself is said to be reflexive, and written:

$Rflx(R): \Leftrightarrow \ \forall s \epsilon S(sRs)$

T Given a set S and a relation $R \epsilon\ Rln(S)$, there exists a smallest relexive relation that includes R, called the reflexive closure of R denoted by $RflxClsr(R)$.

$RflxClsr(R) = R \cup \Delta_S$

R A relation $R \in Rln\ (S, S)$ is not reflexive if and only if in $DG\ (R)$ every vertex has a loop. Therefore, to prove that a relation is not reflexive, one must exhibit a vertex that does not have a loop. To make a non-reflexive relation reflexive one must add loops to every vertex that does not have a loop; the relation so obtained is preceisely the reflexive closure of R.

E Condsider the relation $R := \left\{(1,1), (2,3), (3,4)\right\} \subseteq \{1,2,3,4\} \times \{1,2,3,4\}$

$Rflx(R)$ Y (N)

Why? $2 \not R 2$

R The above means that 2 is not related to 2.

R In $DG(R)$ only 1 has a loop on it; **to construct $\boldsymbol{RflxClsr(R)}$ we have to add a loop to every vertex of $\boldsymbol{R}$**. This is best shown on $DG(R)$ except that I do not know how to add a loop with my software. The formula, hwoever, is not difficult to write:

$$RflxClsr(R) = \left\{(1,1),\ (\mathbf{2},\mathbf{2}), (2,3),\ (\mathbf{3},\mathbf{3}), (3,4), (\mathbf{4},\mathbf{4})\right\}$$

R The new elements added are in boldface. Note that $RflxClsr(R)$ may either be computed using the formula or by adding loops to every vertex that does not have a loop. The two procedures must yield the same result.

<u>Symmetry</u>

D A relation $R: S \longrightarrow S$ from a set to itself is said to be symmetric, and written:

$$Sym(R) :\Leftrightarrow\ \forall s, t \,\epsilon S \left(\frac{sRt}{tRs}\right)$$

T Given a set S and a relation $R \epsilon\, Rln(S)$ there exists a smallest symmetric relation that includes R, called the symmetric closure of R, denoted by $SymClsr(R)$.

$$SymClsr(R) = R \cup R^{op}$$

R A relation $R \in Rln\,(S, S)$ is not symmetric if and only if in $DG\,(R)$, every pair of distinct vertices in $DG\,(R)$ has either no arrows or exactly two arrows in opposite directions between them. To prove that a relation is not symmetric, one has to find a pair of vertices in $DG\,(R)$ with exactly one arrow between them. To make a non-symmetric relation symmetric, in $DG\,(R)$ one must add to every pair of distinct vertices with just one arrow between them, a second arrow in the opposite direction; what one obtains thereby is the symmetric closure.

E Condsider the relation $R \coloneqq \left\{(1,1), (2,3), (3,4)\right\} \subseteq \{1,2,3,4\} \times \{1,2,3,4\}$

$Sym(R)$ $\quad Y \quad (N)$

Why? $\quad (2R3) \wedge (3\not R2)$

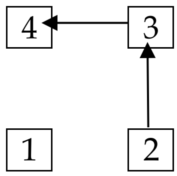
R The above means that 2 is related to 3 but 3 is not related to 2. To construct the symmetric closure for every arrow between two distinct vertices, one must add an arrow in the opposite direction.

$$SymClsr(R) = \left\{(1,1), (2,3), (\mathbf{3},\mathbf{2}), (3,4), (\mathbf{4},\mathbf{3})\right\}$$

The new elements added are in boldface. Note that $SymClsr(R)$ may either be computed using the formula or by adding arrows to every vertex-pair that have exactly one arrow between them. The two procedures must yield the same result.

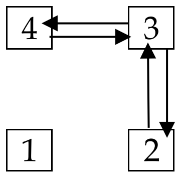
We draw $DG(R)$ in the below.

$DG(R)$



Note that there is a loop at 1 that I have not succeeded in drawing. The digraph of the symmetric closure $DG(SymClsr(R))$ is displayed below. We have to add two arrows opposite to the ones that are already there

$DG(SymClsr(R))$



R Note that the loop at 1 needs to be indicated as it is part of the specification of R.

<u>Antisymmetry</u>

D A relation $R: S \longrightarrow S$ from a set to itself is said to be antisymmetric, and written:

$$AntSym(R) :\Leftrightarrow \ \forall s, t \in S \left(\frac{(sRt) \wedge (tRs)}{s = t} \right)$$

E The relation 'less than or eqaul to' $\leq \in Rln(\mathbb{R}, \mathbb{R})$ is an example of an antisymmetric relation because:

$$\forall s, t \in \mathbb{R} \left(\frac{(s \leq t) \wedge (t \leq s)}{s = t} \right)$$

E The relation 'is included in' $\subseteq \in Rln(\mathcal{U}, \mathcal{U})$ for every universe of sets is an example of an antisymmetric relation because:

$$\forall S, T \in \mathcal{U} \left(\frac{(S \subseteq T) \wedge (T \subseteq S)}{S = T} \right)$$

Transitivity

D A relation $R: S \longrightarrow S$ from a set to itself is said to be transitive, and written:

$$Trns(R) :\Leftrightarrow \ \forall r, s, t \in S \left(\frac{(rRs) \wedge (sRt)}{rRt} \right)$$

T Given a set S and a relation $R \in Rln(S)$ there exists a smallest transitive relation that includes R, called the transitive closure of R, denoted by $TransClsr(R)$.

$$TrnsClsr\left(R^{(\circ n)}\right) = \cup_{n=1}^{\nu(S)} \left(R^{(\circ n)}\right)$$

R A relation R is not transitive if and only if, roughly speaking, in $DG(R)$ there is at least one 'incomplete directed triangle' that is one which has two sides but not the third. The example below illustrates the situation. To prove that a relation is not transitive one has to exhibit an 'incomplete directed triangle'. To make a non-transitive relation transitive, one must complete every 'incomplete directed triangles' by adding the missing arrows and completing every new 'incomplete directed triangle' that may arise in the process until there are no more 'incomplete directed triangles' left.

E Condsider the relation $R \coloneqq \left\{(1,1), (2,3), (3,4)\right\} \subseteq \{1,2,3,4\} \times \{1,2,3,4\}$

$Trns(R)$ $\quad Y \quad$ (N)

Why? $\quad \left((2R3) \wedge (3R4)\right) \wedge (2\not R4)$

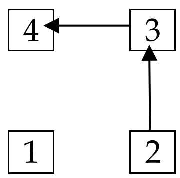
The transsitive closure of R is formed by adding in the missing arrow between 2 and 4 for the 'incomplete directed triangle' between the vertices: 2, 3, and 4.

$$TrnsClsr(R) = \left\{(1,1), (2,3), (3,4), (\mathbf{2}, \mathbf{4})\right\}$$

The new element added is in boldface. Note that $TrnsClsr(R)$ may either be computed using the formula or by adding arrows to complete every 'incomplete directed triangle' and repeating the procedure until there are no 'incomplete directed triangles' left. The two procedures must yield the same result.

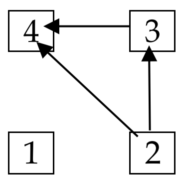
We draw $DG(R)$ in the below.

$DG(R)$



Note that there is a loop at 1 that I have not succeeded in drawing. The digraph of the transitive closure $DG\big(TrnsClsr(R)\big)$ is displayed below. We have to add an arrow from 2 to 4 to complete the 'incomplete direced tringle' between 2, 3, and 4. **Please note the direction of the added arrow because it should not make a 'cycle'. One must be careful to direct the added arrow correctly.**

$DG\big(TrnsClsr(R)\big)$



R Note that the loop at 1 needs to be indicated as it is part of the specification of R.

Equivalence

D A relation $R{:}\, S \longrightarrow S$ from a set to itself is said to be an equivalence, and written:

$$Eqv(R){:}\Leftrightarrow \big(Rflx(R)\big) \wedge \big(Sym(R)\big) \wedge \big(Trns(R)\big)$$

T Given a set S and a relation $R\epsilon\ Rln(S)$ there exists a smallest transitive relation that includes R, called the equivalence closure of R, denoted by $EqvClsr(R)$.

$$EqvClsr\left(R\right) = \cup_{n=0}^{\nu(S)}\left(\left(R \cup R^{op}\right)^{(\circ n)}\right)$$

D Given an equivalence relation $E \in Rln(S)$, and $s \in S$ we define the equivalence class of s under the equivalence E as follows:

$$EqvCl(s, E) \coloneqq \{t \in S \mid tEs\}$$

T Given an equivalence relation $E \in Rln(S)$,

$$\forall s, t \in S \left(\frac{\big((EqvCl(s,E)) \cap (EqvCl(t,E))\big) \neq \{\ \}}{sEt} \right)$$

T Given an equivalence relation $E \in Rln(S)$

$$S = \left(\cup_{s \in S} \left(EqvCl(s, E) \right) \right)$$

D The set of equivalence classes, $EqvClsss(S, E)$ of S under E is called the quatient of S by E and defined as follows:

$$\frac{S}{E} \coloneqq EqvClsss(S, E) \coloneqq \{EqvCl(s, E) | s \in S\}$$

R Sometimes the structure of the equivalence classes $EqvCl(s, E)$ of S under E permit the (uniform) choice of exactly one distinguished element from every equivalence class; if this is the case, then the distinguished element so chosen for every class, is called the canonical representative or normal form for the equivalence class.

E Condsider the relation $R \coloneqq \left\{ (1,1), (2,3), (3,4) \right\} \subseteq \{1,2,3,4\} \times \{1,2,3,4\}$

$Eqv(R)$ Y (N)

Why? $\cancel{Rflx}(R)$

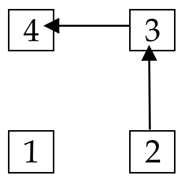
The equivlence closure of R is formed by adding in the missing arrow between 2 and 4 for the 'incomplete directed triangle' between the vertices: 2, 3, and 4.

$$EqvClsr(R) = \left\{ (1,1), (\mathbf{2},\mathbf{2}), (2,3), (\mathbf{3},\mathbf{2}), (\mathbf{2},\mathbf{4}), (\mathbf{3},\mathbf{3}), (3,4), (\mathbf{4},\mathbf{2}), (\mathbf{4},\mathbf{3}) \right\}$$

The new elements added are in boldface. Note that $EqvClsr(R)$ may either be computed using the formula or by adding arrows to complete every 'incomplete directed triangle' and repeating the procedure until there are no vertex-pair with exactly one arrow between them and no 'incomplete directed triangles' left. Note that these two procedures must yield the same result.

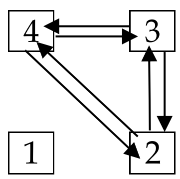
We draw $DG(R)$ in the below.

$DG(R)$



Note that there is a loop at 1 that I have not succeeded in drawing. The digraph of the transitive closure $DG\big(TrnsClsr(R)\big)$ is displayed below. We have to add an arrow from 2 to 4 to complete the 'incomplete direced tringle' between 2, 3, and 4. **Please note the direction of the added arrow because it should not make a 'cycle'. One must be careful to direct the added arrow correctly.**

$DG\big(EqvClsr(R)\big)$



R Note that the loop at 1, 2, 3, and 4 need to be indicated as they are in $EqvClsr(R)$.

R Note that:

$$\frac{S}{EqvClsr(R)} = EqvClsss(S, E) = \left\{\{1\}, \{2, 3, 4\}\right\}.$$

Each equivalence class is a set of connected elements in the digraph. In this example, there are exactly two equivalence classes, namely: $\{1\}$ and $\{2, 3, 4\}$.

R Note also that the equivalence classes partition the set that carries the relation, that is, the set $\{1, 2, 3, 4\}$ that carries the relation is the disjoint union of the equivalence clases, namely, $\{1\}$ and $\{2, 3, 4\}$ as shown below:

$$\{1, 2, 3, 4\} = \{1\} \uplus \{2, 3, 4\}$$

Partial Order

D A relation $R: S \longrightarrow S$ from a set to itself is said to be a partial order, and written:

$$PrtOrd(R) :\Leftrightarrow \big(Rflx(R)\big) \wedge \big(AntSym(R)\big) \wedge \big(Trns(R)\big)$$

R It is convenient to display a partial order P by its Hasse Diagram which is denoted by $HsDgrm(R)$.

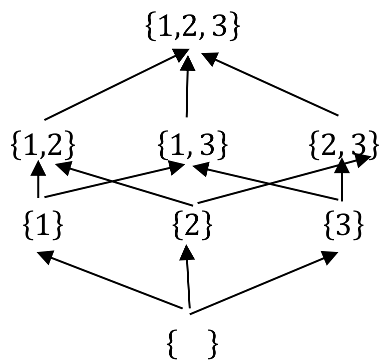
E Consider the inclusion relation $\subseteq \in \left(\mathcal{P}\left(\{1,2,3,4\}\right)\right) \times \left(\mathcal{P}\left(\{1,2,3,4\}\right)\right)$

We prove that $\subseteq$ is a partial order on $\mathcal{S} \coloneqq \mathcal{P}\left(\{1,2,3,4\}\right)$.

$$\dfrac{A, B, C \in \mathcal{S} \coloneqq \mathcal{P}\left(\{1,2,3,4\}\right)}{\dfrac{\dfrac{\dfrac{A \subseteq A}{\forall A \in \mathcal{S}\left(A \subseteq A\right)}}{Rflx(\subseteq)} \qquad \dfrac{\dfrac{\left(\dfrac{A \subseteq B \;\; B \subseteq A}{A = B}\right)}{\forall A \in \mathcal{S}\left(\dfrac{A \subseteq B \;\; B \subseteq A}{A = B}\right)}}{AntSym(\subseteq)} \qquad \dfrac{\dfrac{\left(\dfrac{A \subseteq B \;\; B \subseteq C}{A \subseteq C}\right)}{\forall A \in \mathcal{S}\left(\dfrac{A \subseteq B \;\; B \subseteq C}{A \subseteq C}\right)}}{Trns(\subseteq)}}{PO(\subseteq)}}$$

We draw the $HsDgrm\left(\mathcal{P}\left(\{1,2,3,4\}\right), \subseteq\right)$ in the below.

$$HsDgrm\left(\mathcal{P}\left(\{1,2,3,4\}\right),\subseteq\right)$$



R Before we consider the next example, we define the divisibility relation, denoted by the symbol $|$ on the natual numbers $\mathbb{N}$. We read $a|b$ as a divides b.

D $\forall a, b \in \mathbb{N} \qquad a|b :\Longleftrightarrow \exists c \in \mathbb{N}\left(b = ca\right)$

T $\forall n \in \mathbb{N} \qquad n|0$

Spf

$$\frac{\left(\dfrac{\dfrac{n \in \mathbb{N}}{0 = (0)n}}{n|0}\right)}{\forall n \in \mathbb{N}\left(n|0\right)}$$

Epf

R A curious feature of the above definition is that $0|0$; it does not, however, follow that this allows us to ascribe a meaning to $\frac{0}{0}$, because $0 = (1)0 = (2)0 = 3(0)$ etc.

E We illustrate the idea by producing the Hasse Diagram for the factors of the number 100, partially ordered by divisibility. First we compute the factors of 100, using the following bookkeeping scheme :

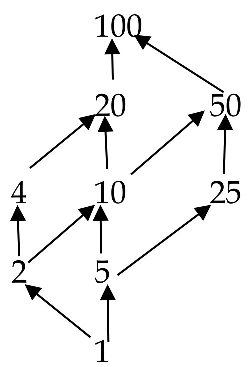
2	100
2	50
5	25
5	5
	1

The computation, which involves divinding by each prime sequentially starting with 2, shows that $100 = 2^2 5^2$. Therefore all factors can be fond by considering all 2- letter words (that appear as exponents of 2 and 5) in the alphabet $\{0, 1, 2\}$ (because the exponents of 2 and 5 may be 0, 1 or 2) such that order matters and repitition is allowed. These words are memebers of the set : $\begin{Bmatrix} 00, & 01, & 02, \\ 10, & 11, & 12, \\ 20, & 21, & 22 \end{Bmatrix}$ which we count using a tree (that you will learn in the counting section). The factors are therefore elements of the set :

$$\begin{Bmatrix} 2^0 5^0, & 2^0 5^1, & 2^0 5^2, \\ 2^1 5^0, & 2^1 5^1, & 2^1 5^2, \\ 2^2 5^0, & 2^2 5^1, & 2^2 5^2 \end{Bmatrix} = \begin{Bmatrix} 1, & 5, & 25, \\ 2, & 10, & 50, \\ 4, & 20, & 100 \end{Bmatrix} = \{1, 2, 4, 5, 10, 20, 25, 50, 100\};$$

in the last set, the factors are written in increasing order. We call the set of factors of 100, $Fctrs(100)$. $Fctrs(100)$ is partially ordered by divisibility and we may therefore draw: $HssDgrm\left(Fctrs(100), |\right)$

$$HssDgrm\left(Fctrs(100), |\right)$$



R Note that a number a is a factor of a number b if one can go up from a to b following a sequence of arrows. For instance, $25 \nmid 20$. Therefore, there is no arrow from 25 to 20. Similarly, $4 \nmid 50$.

Q Find $HssDgrm\left(Fctrs(1000), |\right)$ exactly as shown in the above.

T $\mathbb{N}\backslash\{0\}$ equipped with the divisibility relation $|$ is a partial order, in symbols $PO\left(\mathbb{N}\backslash\{0\}, |\right)$.

$$\dfrac{a, b, c \in \mathbb{N}\backslash\{0\}}{\dfrac{\dfrac{\dfrac{a = 1a}{a|a}}{\forall a \in \mathbb{N}\left(a|a\right)}}{Rflx(|)} \qquad \dfrac{\dfrac{\left(\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a|b \quad b|a}{b = ca \quad a = db}}{a = db = d(ca) = (dc)a}}{dc = 1}}{d = 1 \quad c = 1}}{b = ca = 1a = a}}{b = a}\right)}{\forall a, b \in \mathbb{N}\left(\dfrac{a|b \quad b|a}{a = b}\right)}}{AntSym(|)} \qquad \dfrac{\dfrac{\left(\dfrac{\dfrac{\dfrac{a|b \quad b|c}{b = ca \quad c = db}}{c = db = d(ca) = (dc)a}}{a|c}\right)}{\forall a, b, c \in \mathbb{N}\left(\dfrac{a|b \quad b|c}{a|c}\right)}}{Trns(|)}}$$

$$PO(\mathbb{N}\backslash\{0\}, |)$$

Q Prove that $\mathbb{R}$ equipped with the relation $\leq$ is a partial order, in symbols $PO\left(\mathbb{R}, \leq\right)$.

Q Prove that $\mathbb{R}$ equipped with the relation $\geq$ is a partial order, in symbols $PO\left(\mathbb{R}, \geq\right)$.

Partitions

D Given a set S, a family $\mathcal{S} := \left(S_k \in \mathcal{P}(S) \middle| k \epsilon K\right)$ of subsets of S is said to consrtitute a partition (of S), and written

$$Prtn(\mathcal{S}, S)$$

$$:\Leftrightarrow \left(\begin{array}{ll} & \left(\left(\forall k \epsilon K\right)\left(S_k \neq \{\ \ \}\right)\right) \\ \wedge & \left(\left(\forall l, k \epsilon K\right)\left(\frac{l \neq k}{S_l \cap S_k = \{\ \ \}}\right)\right) \\ \wedge & \left(\bigcup_{k \epsilon K}(S_k) = S\right) \end{array}\right)$$

R Given a partition $\mathcal{S} := \left(S_k \in \mathcal{P}(S) \middle| k \epsilon K\right)$ each S_k is called a block of the partition $\mathcal{S}$

D $$Prtns\left(S\right) := \left\{\mathcal{S} \epsilon \mathcal{P}\left(\mathcal{P}(S)\right) \middle| Prtn(\mathcal{S}, S)\right\}$$

is the set of partitions of the set S.

D $$EqvRlns\left(S\right) := \left\{R \epsilon Rln(S) \middle| EqvRln(S)\right\}$$

is the set of equivalence relations on the set S.

T For every set S, $Prtns(S)$ and $EqvRlns(S)$ are in bijective correspondence.

E We compute below all the partitions of the set $S := \{1, 2, 3\}$. The partitions are: $\left\{\left\{1\right\}, \left\{2, 3\right\}\right\}, \left\{\left\{2\right\}, \left\{1, 3\right\}\right\}, \left\{\left\{3\right\}, \left\{1, 2\right\}\right\}, \left\{\left\{1\right\}, \left\{2\right\}, \left\{3\right\}\right\}$

Therefore

$$Prtns\left(\{1,2,3\}\right)$$

$$=\left\{\left\{\left\{1\right\},\left\{2,3\right\}\right\},\left\{\left\{2\right\},\left\{1,3\right\}\right\},\left\{\left\{3\right\},\left\{1,2\right\}\right\},\left\{\left\{1\right\},\left\{2\right\},\left\{3\right\}\right\}\right\}$$

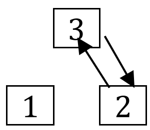
and

$$\nu\left(Prtns(S \coloneqq \{1,2,3\})\right) = 4.$$

Therefore, there are exactly 4 equivalence relations on $\{1,2,3\}$; the blocks of the partition are precisely the equivalence classes for the corresponding equivalence relation.

For instance, the partition $\left\{\left\{1\right\},\left\{2,3\right\}\right\}$ corresponds to the equivalence relation E on $\{1,2,3\}$ and $DG(E)$ is displayed below:

$DG(E)$



Note that there are loops on on 1 that I cannot draw at the moment.

Q Draw the digraphs of the remaining equivalence relations on $\{1,2,3\}$.

Q Find all the partitions and equivalence relations on $\{1,2,3,4\}$ and $\{1,2,3,4,5\}$.

E Define the relation $R \in Rln\left(\mathbb{Z}\right)$ as follows:

$$\forall x,y \in \mathbb{Z} \qquad xRy :\Leftrightarrow \left(|x| = |y|\right)$$

Q Prove that R is an equivalence relation on $\mathbb{Z}$, that is $R \in EqvRln\left(\mathbb{Z}\right)$ and find equivalence classes.

$$\cfrac{x,y,z \in \mathbb{Z}}{\cfrac{\left(\cfrac{\cfrac{\cfrac{|x| = |x|}{xRx}}{\forall x \in \mathbb{Z}(xRx)}}{Rflx(R)}\right) \left(\cfrac{\cfrac{\cfrac{\cfrac{xRy}{|x| = |y|}}{|y| = |x|}}{yRx}}{\cfrac{\forall x \in \mathbb{Z}\left(\frac{xRy}{yRx}\right)}{Sym(R)}}\right) \left(\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{xRy\ yRz}{xRy \quad yRz}}{|x| = |y| \quad |y| = |z|}}{|x| = |z|}}{xRz}}{\cfrac{\forall x \in \mathbb{Z}\left(\frac{xRy \quad yRz}{xRz}\right)}{Trns(R)}}\right)}{\cfrac{\left(Rflx(R)\right) \wedge Sym(R) \wedge \left(Trns(R)\right)}{Eqv(\mathbb{Z}, R\ )}}}$$

R We note that:

$$\cfrac{\cfrac{\cfrac{x,y \in \mathbb{Z}}{|x| = |y|}}{(x = y) \vee (x = -y)}}{\forall x,y \in \mathbb{Z}\left(\cfrac{|x| = |y|}{(x = y) \vee (x = -y)}\right)}$$

Since $\forall x \in \mathbb{Z}\left(|x| \in \mathbb{N}\right)$ each natural number serves as the 'mother' (using the analogy of siblinghood) and may be used as an index for an equivalence class. The number 0 has exactly one 'child' namely 0. Every nonzero natural number n has two children: $-n$ and n; such pairs form equivalence classes. Therefore, the equivalence classes are $\{0\}, \{-1, 1\}, \{-2, 2\}, \{-3,3\}, \ldots$ and so on. More formally we write

$$\frac{\mathbb{Z}}{R} = \left\{\{0\}, \{-1, 1\}, \{-2, 2\}, \{-3,3\}, \ldots\right\} = \bigcup_{n \in \mathbb{N}} \left\{\{-n, n\}\right\}$$

R There are several problems in the Problem Set that are similar to the above example but the function defining the equivalence relation is different from the absolute value function.

Home Work

It is given that the relations:

$$R, S, T$$

$$\epsilon \quad \{\{\ \}, \{(a,a)\}, \{(c,c)\}, \{(a,b),(b,a)\}, \{(a,b),(a,c)\}, \{(a,b),(b,c),(c,a)\}\}$$

$$\subseteq \quad \mathcal{P}\left(\{a,b,c\} \quad \times \quad \{a,b,c\}\right)$$

and that the operations: $\odot, \circledast, \circledcirc \ \epsilon \{\circ,\ \cup\ ,\ \cap\ ,\ \backslash\ ,\ \times,\ \sqcup\ \}$

R Count how many variants there are of each problem. Do as many of these variants as you need to in order to understand what is going on.

0 Find for every R: $Gr(R)$, $AD(R)$, $DG(R\ )$

1 Determine for every R, if:

$$Rflx(R),\ Sym(R),\ AntSym(R\ ), Trns(R\ )\ Eqv(R\ ),\ PO(R\ ),\ Fnctl(R\ )$$

and find

$$RflxClsr(R),\ SymClsr(R), ,TrnsClsr(R\ ),\ EqvClsr(R\ ), \text{ and}$$

$$EqvCls\big(EqvClsr(R\ )\big)$$

2 Compute the following expressions using particular choices for the relations $R, S,$ and T from the collection above.

R No answers will be provided; you should try instead to work out each problem in more than one way to see if all methods lead to the same answer. Devising several ways of computing the same expression will help you understand the material battier.

R If the answers for two expressions turn out to be the same for different choices of the 'unknown' sets and operations, determine if this equality is fortuitous, or if it always holds, no matter what the values of the unknowns are.

20 $R \odot S$

21 $R \odot (S \circledast T)$

22 $(R \odot S) \circledast T$

3 Prove that the following relations are equivalence relations and find the equivalence classes:

31 $\forall(s,t)\epsilon\mathbb{Z}\times\mathbb{Z}\left(\left(R(s,t)\right):\Leftrightarrow s^2=t^2\right)$

4 Prove that the following relations are partial orders:

41 $\forall(s,t)\epsilon(\mathbb{N}\backslash\{0\})\times(\mathbb{N}\backslash\{0\})\left(\left(R(s,t)\right):\Leftrightarrow s|t\right)$