<u>Relations</u>

D Given a set S (called the source of the relation and denoted by $Src(R)$) and a set T(called the target of the relation and denoted by $Tgt(R)$.), we define a relation $R: S \longrightarrow T$ to be a subset of $S \times T$; the subset is called the graph of the relation and is denoted by $Gr(R)$.

R Every relation $R: Src(R) \longrightarrow Tgt(R)$ is coded as a triple

$$R \coloneqq \big(Src(R),\ \ Gr(R) \subseteq Src(R) \times Tgt(R),\ \ Tgt(R)\big)$$

and every triple $(S,\ \ G \subseteq S \times T,\ \ T)$ may be construed as a relation $R: S \longrightarrow T$

with the identifications: $S: = Src(R),\ \ G \coloneqq Gr(R) \subseteq S \times T,\ \ T \coloneqq Tgt(R)$

D The elements of $Src(R)$ are called source-elements. Not every source-element serves as an input for a relation. The elements of $Tgt(R)$ are called target-elements. Not every target-element arises as an output for the relation.

R We often use uppercase latin letters such as: R, S, and T, or certain defined strings of latin letters (such as sin), or certain special symbols: $=, \neq, \equiv, \not\equiv, \cong, \approx, <, \leq, >, \geq\ etc.)$ to denote relations.

R Every relation on a finite set may be *specified* either as:

a subset $R \subseteq S \times T$, or as

a graph $Gr(R)$, or as

an arrow-diagram: $AD(R)$ or as

a matrix $M(R)$, and

if $Src(R) = Tgt(R)$, as

a directed graph $DG(R)$.

R These various methods of specification convey exactly the same information, but each of them makes certain properties of relations particularly easy to deduce.

R If an element $s\epsilon S = Src(R)$ is related under the relation $R \subseteq S \times T$ to an element $t\epsilon T = Tgt(R)$, we say that s is R-related to t and denote this state of affairs by the following equivalent notational devices:

$$(s,t)\epsilon R \ \Leftrightarrow sRt \Leftrightarrow R(s,t)$$

D Given sets S, T we define the set of all relations from S to T as follows:

$$Rln(S,T) \coloneqq \ \{R | R \subseteq S \times T\} = \mathcal{P}(S \times T)$$

D If, in the above, $S = T$ we define the set of all relations from S to S as follows:

$$Rln(S) \coloneqq Rln(S,S) \coloneqq \{R | R \subseteq S \times S\} = \mathcal{P}(S \times S)$$

Properties of Relations

D A relation $R: S \longrightarrow T$ is said to be functional and written

$$Fncl(R) :\Leftrightarrow \forall s \epsilon S \ \exists!\, t \epsilon T \ such \ that \ sRt \,,$$

that is if and only if every source-element is assigned exactly one target-element as an output.

R Note that although every source-element serves as an input for a functional relation, not every target-element need serve as an output.

D Given sets S, T we define the set of all functions from S to T as follows:

$$Fnc(S,T) \coloneqq \{R \epsilon \mathcal{P}(S \times T) | Fncl(R)\}$$

Special Relations

Empty relation

D Given sets S, T we define the empty relation:

$$Emp_{S \times T} \coloneqq \{\quad\}_{S \times T} \coloneqq \{\quad\} \subseteq S \times T$$

Total Relation

D Given sets S, T we define the total relation:

$$Ttl_{S \times T} \coloneqq S \times T \subseteq S \times T$$

Diagonal Relation

D Given a set S, we define the diagonal (or identity) relation $\Delta_S: S \longrightarrow S$ on S as follows:

$$\Delta_S \coloneqq \{(s,t) \epsilon S \times S | s = t\} \subseteq S \times S$$

R Note that the diagonal relation is functional.

Operations on Relations

R Since a relation is defined as a subset of a set, the following set-theoretic operations $\big(\mathcal{P}(\quad), (\quad)^c \cup, \cap, (\quad)\backslash(\quad)\big)$ may be performed on relations as well.

Opposition

D Given a relation $R: S \longrightarrow T$, its opposite or converse: $R^{op}: T \longrightarrow S$ is defined as follows:

$$R^{op} := \{(t,s) \epsilon T \times S | R(s,t)\} \subseteq T \times S$$

R Note that: $\forall s \epsilon S, \forall t \epsilon T \Big(\big(R(s,t)\big) \Leftrightarrow \big(R^{op}(t,s)\big)\Big)$

Composition

D Given a relation $Q: S \longrightarrow U$, and a relation $P: U \longrightarrow T$, that is: $S \xrightarrow{Q} U \xrightarrow{P} T$ the composition $P \circ Q: S \longrightarrow T$ is defined as follows:

$$P \circ Q := \Big\{(s,t) \epsilon S \times T \Big| \exists u \epsilon U \text{ such that } \Big(\big(Q(u,t)\big) \wedge \big(P(s,u)\big)\Big)\Big\} \subseteq S \times T$$

T Given $S \xrightarrow{R} U \xrightarrow{Q} V \xrightarrow{P} T$ we have $P \circ (Q \circ R) = (P \circ Q) \circ R$ that is composition is associative

R Composition is, however, not commutative, in general, that is:

Given $S \xrightarrow{Q} S \xrightarrow{P} S$ $P \circ Q \neq Q \circ P$ in general.

Iteration

D Given a relation: $R \in Rln(S)$, we may compose R with itself; the n-fold composite of R with itself, denoted by $R^{\circ n}$ is defined recursively as follows:

BC $R^{\circ 0} := \Delta_S$

RcS $\forall n \in \mathbb{N}$ $R^{\circ(n+1)} := \big(R^{\circ(n)}\big) \circ R$

D Given a functional relation: $\alpha \in Fnc(S)$, we may compose α with itself; the n-fold composite of α with itself, denoted by $R^{\circ n}$ is defined recursively as follows:

BC $\alpha^{\circ 0} := \iota_S$

RcS $\forall n \in \mathbb{N}$ $\alpha^{\circ(n+1)} := \big(\alpha^{\circ(n)}\big) \circ \alpha$

Properties of a relation from a set to itself ($R: S \longrightarrow S$)

Reflexivity

D A relation $R: S \longrightarrow S$ from a set to itself is said to be reflexive, and written:

$$Rflx(R): \Leftrightarrow \ \forall s \epsilon S(sRs)$$

T Given a set S and a relation $R\epsilon\ Rln(S)$, there exists a smallest relexive relation that includes R, called the reflexive closure of R denoted by $RflxClsr(R)$.

$$RflxClsr(R) = R \cup \Delta_S$$

Symmetry

D A relation $R: S \longrightarrow S$ from a set to itself is said to be symmetric, and written:

$$Sym(R): \Leftrightarrow \ \forall s, t\ \epsilon S \left(\frac{sRt}{tRs}\right)$$

T Given a set S and a relation $R\epsilon\ Rln(S)$ there exists a smallest symmetric relation that includes R, called the symmetric closure of R, denoted by $SymClsr(R)$.

$$SymClsr(R) = R \cup R^{op}$$

Antisymmetry

D A relation $R: S \longrightarrow S$ from a set to itself is said to be antisymmetric, and written:

$$AntSym(R): \Leftrightarrow \ \forall s, t\ \epsilon S \left(\frac{(sRt) \wedge (tRs)}{s = t}\right)$$

Transitivity

D A relation $R: S \longrightarrow S$ from a set to itself is said to be transitive, and written:

$$Trns(R): \Leftrightarrow \ \forall r, s, t\ \epsilon S \left(\frac{(rRs) \wedge (sRt)}{rRt}\right)$$

T Given a set S and a relation $R\epsilon\ Rln(S)$ there exists a smallest transitive relation that includes R, called the transitive closure of R, denoted by $TransClsr(R)$.

$$TrnsClsr(R) = \bigcup_{n=0}^{\nu(S)} \left(R^{(\circ n)}\right)$$

Equivalence

D A relation $R: S \longrightarrow S$ from a set to itself is said to be an equivalence, and written:

$$Eqv(R) :\Leftrightarrow \big(Rflx(R)\big) \wedge \big(Sym(R)\big) \wedge \big(Trns(R)\big)$$

T Given a set S and a relation $R \epsilon\ Rln(S)$ there exists a smallest transitive relation that includes R, called the equivalence closure of R, denoted by $EqvClsr(R)$.

$$EqvClsr(R) = \bigcup_{n=0}^{\nu(S)} \big((R \cup R^{op})^{(\circ n)}\big)$$

D Given an equivalence relation $E \in Rln(S)$, and $s \in S$ we define the equivalence class of s under the equivalence E as follows:

$$EqvCl(s, E) \coloneqq \{t \in S \mid tEs\}$$

T Given an equivalence relation $E \in Rln(S)$,

$$\forall s, t \in S \left(\frac{\Big(\big(EqvCl(s,E)\big) \cap \big(EqvCl(t,E)\big)\Big) \neq \{\ \ \}}{sEt} \right)$$

T Given an equivalence relation $E \in Rln(S)$

$$S = \Big(\bigcup_{s \in S} \big(EqvCl(s, E)\big)\Big)$$

D The set of equivalence classes of S under E is called the quatient of S by E and defined as follows:

$$\frac{S}{E} \coloneqq \{EqvCl(s, E) \mid s \in S\}$$

R Sometimes the structure of the equivalence classes $EqvCl(s, E)$ of S under E permit the (uniform) choice of exactly one distinguished element from every equivalence class; if this is the case, then the distinguished element so chosen for every class, is called the canonical representative or normal form for the equivalence class.

Partial Order

D A relation $R: S \longrightarrow S$ from a set to itself is said to be a partial order, and written:

$$PrtOrd(R) :\Leftrightarrow \big(Rflx(R)\big) \wedge \big(AntSym(R)\big) \wedge \big(Trns(R)\big)$$

R It is convenient to display a partial order P by its Hasse Diagram which is denoted by $HsDgrm(R)$.

Partitions

D Given a set S, a collection $\mathcal{S} := (S_k | k \epsilon K)$ of subsets of S is said to consrtitute a partition (of S), and written

$$Prtn(\mathcal{S}, S) :\Leftrightarrow \left((\forall k \epsilon K)(S_k \neq \{\ \ \})\right) \wedge \left((\forall l, k \epsilon K)\left(\frac{l \neq k}{S_l \cap S_k = \{\ \ \}}\right)\right) \wedge \left(\bigcup_{k \epsilon K}(S_k) = S\right)$$

R Given a partition $\mathcal{S} := (S_k | k \epsilon K)$ each $S_k$is called a block of the partition $\mathcal{S}$

D $Prtn(S) := \{\mathcal{S} \epsilon \mathcal{P}(\mathcal{P}(S)) | Prtn(\mathcal{S}, S)\}$ is the set of partitions of the set S.

D $EqvRln(S) := \{R \epsilon Rln(S) | R \text{ is an equivalence relation}\}$

T For every set S, $Prtn(S)$ and $EqvRln(S)$ are in bijective correspondence.

Home Work

It is given that the relations:

$$R, S, T$$

$$\in \quad \{\{\ \}, \{(a,a)\}, \{(c,c)\}, \{(a,b),(b,a)\}, \{(a,b),(a,c)\}, \{(a,b),(b,c),(c,a)\}\}$$

$$\subseteq \quad \mathcal{P}\left(\{a,b,c\} \quad \times \quad \{a,b,c\}\right)$$

and that the operations: $\odot, \circledast, \circledcirc \ \in \{\circ,\ \cup,\ \cap,\ \backslash,\ \times,\ \sqcup\}$

R Count how many variants there are of each problem. Do as many of these variants as you need to in order to understand what is going on.

0 Find for every R: $Gr(R)$, $AD(R)$, $DG(R)$

1 Determine for every R, if:

$$Rflx(R), Sym(R), AntSym(R), Trns(R)\ Eqv(R), PO(R), Fnctl(R)$$

and find

$RflxClsr(R)$, $SymClsr(R)$, , $TrnsClsr(R)$, $EqvClsr(R)$, and

$$EqvCls\big(EqvClsr(R)\big)$$

2 Compute the following expressions using particular choices for the relations R, S, and T from the collection above.

R No answers will be provided; you should try instead to work out each problem in more than one way to see if all methods lead to the same answer. Devising several ways of computing the same expression will help you understand the material battier.

R If the answers for two expressions turn out to be the same for different choices of the 'unknown' sets and operations, determine if this equality is fortuitous, or if it always holds, no matter what the values of the unknowns are.

20 $R \odot S$

21 $R \odot (S \circledast T)$

22 $(R \odot S) \circledast T$

3 Prove that the following relations are equivalence relations and find the equivalence classes:

31 $$\forall(s,t)\epsilon\mathbb{Z}\times\mathbb{Z}\left(\left(R(s,t)\right):\Leftrightarrow s^2=t^2\right)$$

4 Prove that the following relations are partial orders and find the equivalence classes:

41 $$\forall(s,t)\epsilon(\mathbb{N}\backslash\{0\})\times(\mathbb{N}\backslash\{0\})\left(\left(R(s,t)\right):\Leftrightarrow s|t\right)$$