R Sets and membership constitute primitive notions in our discourse and as such cannot be defined. Our objective is to formulate all ideas precisely in terms of sets and membership to reduce the possibility of confusion to an absolute minimum.

Specification of sets

S Sets will be specified in two ways

either (1) by listing as in the examples that follow

$S \coloneqq \{\quad\},\ T \coloneqq \{a, @, ?\,, \#\}$

or (2) by using setbuilder notation as follows:

$S \coloneqq \ \{x \epsilon \mathcal{U} | P(x)\}$, the set of all x in the universe $\mathcal{U}$ that satisfy the conditon P

R The above definition, though simple, may, if adequate care is not taken, lead to curious conflicts that we shall have occasion to indicate later.

R In our discourse there will be two distinguished sets, the universe of all sets under consideration, denoted by $\mathcal{U}$ and the empty or the null set, denoted by { }.

R The empty set is also often denoted by ∅. We shall not use this symbol which some use for the number zero. Its use will lead one to lose points.

E The following are examples of the listing scheme:

$$\left\{\quad\right\},\ \mathcal{U},\quad \left\{a, b\right\},\quad \left\{\partial, \sqrt{\quad}, \infty\right\},\quad \left\{\{\quad\}, \{a\}, a\right\}$$

Q What are the elements of each of the sets listed in the above?

R Note that the elements can be anything at all. We shall agree not to use delimiting symbols used in sets as members of sets to prevent confusion. Sets can occur as members of other sets as in one of the examples in the above.

R For the listing specification scheme, an element is found by scanning the expression from left to right to find any symbol that is between the leftmost brace and a comma at the same level, between two commas at the same level as the leftmost and the rightmost brace, or between a comma at the same level at the rightmost brace and the rightmost brace. Although every symbol may occur as elements of sets, we shall not use {, }, [,], (,), , , | as elements of sets to avoid confusion and clashes.

R The empty set $\left\{ \quad \right\}$ and the universe $\mathcal{U}$ are distinguished sets. The empty set contains no elements while the universe contains every element.

E The following are examples of the setbuilder scheme:

$$S \coloneqq \left\{ x \in \mathbb{R} \middle| x > 2 \right\}.$$

We read this as the set of all x in the set of real numbers ℝ such that x is strictly bigger than 2. The left brace $\left\{ \right.$ is read as the set of all, ∈ is read as in, the vertical bar $\Big|$ is read as such that. We compare the above expression to the one immediately below, scanning from left to right and noting common symbols to conclude that:

$$S \coloneqq \left\{ x \in \mathcal{U} \middle| P(x) \right\}$$

the universe in this case is the set of all real numbers ℝ and the property P in question is the property of being bigger than 2.

R We shall have occasion to use the special notation for sets of counting or whole numbers starting at a specific number and ending at a specific number.

D We define:

$$\forall p, q \in \mathbb{N} \quad p..q \quad := \quad \left\{ n \in \mathbb{N} \middle| p \leq n \leq q \right\}$$

$$\forall x, y \in \mathbb{Z} \quad x..y \quad := \quad \left\{ z \in \mathbb{Z} \middle| x \leq z \leq y \right\}$$

E We note that:

$$2..5 \quad := \quad \left\{ 2, 3, 4, 5 \right\}$$

$$(-2)..3 \quad := \quad \left\{ -2, -1, 0, 1, 2, 3 \right\}$$

Q Compute the following:

$$0..1, \quad 1..(-1), \quad (-3)..1, \quad (-3)..(-1)$$

Q Consider the following sets and identify the universe and the property P :

$$A := \left\{ x \in \mathbb{Z} \middle| x = 2 \right\} \qquad C := \left\{ x \in \left\{ 1, 2, 3, 4 \right\} \middle| x \geq 2 \right\}$$

$$B := \left\{ x \in \mathbb{Q} \middle| x \neq 2 \right\} \qquad D := \left\{ x \in \left\{ a, b, 3, 4 \right\} \middle| x \text{ is a number} \right\}$$

E The following examples illustrates the construction of some sets using the setbuilder specification. One's J-number is used in the construction. Note that every entry in the J-number is a list of eight numbers each of which is in 0..9. Note that we do not include the letter J. The list $\left(J_k \in 0..9 \middle| k \in 1..8\right) = J_1J_2J_3J_4J_5J_6J_7J_8$

I shall use a fictitious J-number

$$JN \quad := \quad 00210519 \quad := \quad \left(J_k \in 0..9 \middle| k \in 1..8\right)$$

for this example. My goal is to find the set A defined below. using the above J-number. Here:

$$J_1 = 0, J_2 = 0, J_3 = 2, J_4 = 1, J_5 = 0, J_6 = 5, J_7 = 1, J_8 = 9$$

$$A \quad := \quad \left\{J_k \in 0..9 \middle| J_k < k\right\}$$

k		**1**		**2**		**3**		**4**		**5**		**6**		**7**		**8**	
J_k		$(\mathbf{0})$		$(\mathbf{0})$		$(\mathbf{2})$		$(\mathbf{1})$		$(\mathbf{0})$		$(\mathbf{5})$		$(\mathbf{1})$		**9**	
$J_k < k$		$(\boldsymbol{Y})$		$(\boldsymbol{Y})$		$(\boldsymbol{Y})$		$(\boldsymbol{Y})$		$(\boldsymbol{Y})$		$(\boldsymbol{Y})$		$(\boldsymbol{Y})$		$\boldsymbol{Y}$	
		$\boldsymbol{N}$		$\boldsymbol{N}$		$\boldsymbol{N}$		$\boldsymbol{N}$		$\boldsymbol{N}$		$\boldsymbol{N}$		$\boldsymbol{N}$		$(\boldsymbol{N})$	

We circle the $Y's$ and $N's$ depending upon whether the condition $J_k < k$ is satisfied or not. Then we circle the $J_k's$ above the circled $Y's$. Now we can fill in the elements of A in increasing order using the circled numbers.

$$A \quad := \quad \left\{J_k \in 0..9 \middle| J_k < k\right\} \quad = \quad \left\{0, 1, 2, 5\right\}$$

Next we have to compute the set B, defined as follows:

$$B \quad := \quad \left\{ k \in 1..8 \;\middle|\; k > J_k \right\}$$

We use the table as in the previous case.

k	$(\mathbf{1})$	$(\mathbf{2})$	$(\mathbf{3})$	$(\mathbf{4})$	$(\mathbf{5})$	$(\mathbf{6})$	$(\mathbf{7})$	$\mathbf{8}$
J_k	**0**	**0**	**2**	**1**	**0**	**5**	**1**	**9**
	$(\boldsymbol{Y})$	$(\boldsymbol{Y})$	$(\boldsymbol{Y})$	$(\boldsymbol{Y})$	$(\boldsymbol{Y})$	$(\boldsymbol{Y})$	$(\boldsymbol{Y})$	$\boldsymbol{Y}$
$k > J_k$								
	$\boldsymbol{N}$	$\boldsymbol{N}$	$\boldsymbol{N}$	$\boldsymbol{N}$	$\boldsymbol{N}$	$\boldsymbol{N}$	$\boldsymbol{N}$	$(\boldsymbol{N})$

We circle the $Y's$ and $N's$ depending upon whether the condition $k > J_k$ is satisfied or not. Then we circle the $J_k's$ above the circled $Y's$. Now we can fill in the elements of B in increasing order using the circled numbers.

$$B \quad := \quad \left\{ k \in 1..8 \;\middle|\; k > J_k \right\} = \quad \left\{ 1, 2, 3, 4, 5, 6, 7 \right\}$$

R The above examples will appear often on exams and is important to understand.

Q Compute the following sets:

$$C \quad := \quad \left\{ J_k \in 0..9 \;\middle|\; J_k is\ prime \right\}$$

$$D \quad := \quad \left\{ k \in 1..8 \;\middle|\; J_k is\ prime \right\}$$

$$E \quad := \quad \left\{ J_k \in 0..9 \,\middle|\, k \text{ is prime} \right\}$$

$$F \quad := \quad \left\{ k \in 1..8 \,\middle|\, k \text{ is prime} \right\}$$

You can use your own J-number or the one that I used in the examples below. You must use the method shown. We define the predicate:

$$\forall n \in \mathbb{N} \quad Prm\left(n \right) := n \text{ is a prime number}$$

Note that 0 and 1 are not primes (why?) and the list of prime numbers starts with: 2, 3, 5, 7, 11, 13, 17, 19 and is infinite. You can use the following table

k		**1**		**2**		**3**		**4**		**5**		**6**		**7**		**8**	
J_k																	
$Prm\left(J_k\right)$		***Y***		***Y***		***Y***		***Y***		***Y***		***Y***		***Y***		***Y***	
		N		***N***		***N***		***N***		***N***		***N***		***N***		***N***	

and three others that you need to construct yourself.

Containment (∈) membership, elementhood

S x is called a member (element) of a set S, written

$x \epsilon S$ where S ia specified as in (1) or (2) in the above

if and only if x is from the universe $\mathcal{U}$ of discourse and is:

either listed explicitly if specification (1) is employed

or can be verified to satisfy the condition P if specification (2) is employed.

E Circle the correct choices and answer:

$$a \epsilon \left\{a, b\right\} \quad (Y) \quad N \qquad (Pf)\ W$$

Why? By inspection

In this case we have to circle Y (Yes) denoted by Y because a is in the set $\left\{a, b\right\}$ because by scanning from left to right, we find it between a leftmost brace and a comma.

This long sentence is denoted by the code 'By inspection'.

E Circle the correct choices and answer:

$$c \epsilon \left\{a, b\right\} \quad Y \qquad (N) \quad (Pf)\ W$$

Why? By inspection

In this case we have to circle N (No) denoted by N because c is not in the set $\left\{a, b\right\}$ because by scanning from left to right, we do not find it between the leftmost brace and a comma at the same level, between two commas at the same level as the leftmost and the rightmost brace, or between a comma at the same level at the rightmost brace and the rightmost brace.

This long sentence is denoted by the code 'By inspection'.

Q Circle the correct choices and answer:

$$a \in \left\{\{a\}, b\right\} \qquad Y \qquad (N) \qquad (Pf) \;\; W$$

Why? You find the reason.

Q Circle the correct choices and answer:

$$\{a\} \in \left\{\{a\}, b\right\} \qquad (Y) \qquad N \qquad (Pf) \;\; W$$

Why? You find the reason.

T $$\forall x \in \mathcal{U}\left(x \notin \{\ \ \}\right)$$

The preceding assertion says that no element in the universe is contained in the empty set.

E Circle the correct choices and answer:

$$3 \in \left\{ x \in \mathbb{R} \middle| x > 2 \right\} \qquad (Y) \quad N \qquad (Pf) \;\; W$$

Why? $\qquad (3 \in R) \wedge (3 > 2)$

R In such a situation you are circling the answer Y and providing a proof for your assertion

E Circle the correct choices and answer:

$$2 \in \left\{ x \in \mathbb{R} \middle| x > 2 \right\} \qquad Y \quad (N) \quad (Pf) \;\; W$$

Why? $\qquad 2 \ngtr 2$

R In this case it is enough to say that 2 does not have the defining property for the set.

E Circle the correct choices and answer:

$$\pi \in \left\{ x \in \mathbb{N} \middle| x > 2 \right\} \qquad Y \quad (N) \quad (Pf) \;\; W$$

Why? $\qquad \pi \notin \mathbb{N}$

R In this case the number π does have the defining property but it is not in the universe.

Q Circle the correct choices and answer:

$$\pi \in \left\{ x \in \mathbb{R} \middle| x > 2 \right\} \qquad (Y) \quad N \qquad (Pf) \;\; W$$

Why? You find the reason.

R We have to use notation for quantification and inference.

N The **existential quantifier** is denoted by $\exists$ and is read variously as for some, there is some, there exists, depending on the context. For us, it will usually appear in the cluster $\exists x \epsilon S$ which is read as 'there is some x in S' or 'there exists (some) x in S'. The statement 'there is some x in S and there is also some y in S' is abbreviated to $\exists x, y\ \epsilon S$ for convenience. The negation of the existential quantifier is denoted by $\nexists$. The cluster $\nexists x \epsilon S$ is read 'there is no x in S' or 'there does not exist any x in S'.

N The **universal quantifier** is denoted by $\forall$ and is read variously as for every, for all, depending on the context. For us, it will usually appear in the cluster $\forall x \epsilon S$ which is read as 'for every x in S'. The statement 'for every x in S and also for every y in S' is abbreviated to $\forall x, y\ \epsilon S$ for convenience.

Or Consider the following assertions:

$$\exists x \in \mathcal{U}\left(P(x)\right)$$

For **some $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{U}}$** property P holds of x.

$$\forall x \in \mathcal{U}\left(P(x)\right)$$

For **every $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{U}}$** property P holds of x.

$$\exists x \in \mathcal{U}\ \forall y \in \mathcal{U}\left(Q(x,y)\right)$$

There is **some $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{U}}$** such that for every **$\boldsymbol{y}$ *in* $\boldsymbol{\mathcal{U}}$** property Q holds of x *and* y. **The same $\boldsymbol{x}$ works for every $\boldsymbol{y}$**.

$$\forall x \in \mathcal{U}\ \exists y \in \mathcal{U}\left(Q(x,y)\right)$$

For every $\boldsymbol{y}$ $\boldsymbol{in}$ $\boldsymbol{\mathcal{U}}$, there is **some** $\boldsymbol{x}$ $\boldsymbol{in}$ $\boldsymbol{\mathcal{U}}$ such that property Q holds of x and y. **Distinct** $\boldsymbol{x}$ **'s may be needed for distinct** $\boldsymbol{y's}$.

N **An inference** or **deduction** is displayed as $\left(\frac{\vartheta}{\varphi}\right)$ and is read as 'from the assertion (assumption, premise, antecedent) that ϑ is true, it follows that (we may infer that, we may deduce that, we may conclude that) the assertion (consequent) φ is true. ϑ and φ may have very complicated structure. The hoeizontal bar is called a conclusion bar.

N We use $\left(\frac{\vartheta}{\overline{\quad\varphi\quad}}\right)$ to abbreviate $\left(\left(\frac{\vartheta}{\varphi}\right) and \left(\frac{\varphi}{\vartheta}\right)\right)$ that is φ follows from ϑ and also that ϑ follows from φ. The double horizontal bar means that the inference is valid in both directions. The double horizintal bar is called an equivalence bar.

R The inference $\left(\frac{\vartheta \quad \varphi}{\psi}\right)$ is equivalent to $\left(\frac{\vartheta \wedge \varphi}{\psi}\right)$. If there are more than one premise, there are implicit and's between them.

R We shall display proofs as sequences of inferences.

Inclusion($\subseteq$), subsethood

D Given two sets A and B, A is said to be included in B, written

$$\left(A \subseteq B\right) :\Leftrightarrow \left(\forall x\epsilon\mathcal{U} \quad \left(\frac{x\epsilon A}{x\epsilon B}\right)\right)$$

R We read the above as follows:

A is included in B $\left(A \subseteq B\right)$ by definition, if and only if $\left(:\Leftrightarrow\right)$ for every x $\left(\forall x\right)$ in $\left(\in\right)$ the universe $\left(\mathcal{U}\right)$ $\left(\forall x\epsilon\mathcal{U}\right)$ from the fact that x is in A $\left(x\epsilon A\right)$ it follows that x is in B $\left(x\epsilon B\right)$ $\left(\left(\frac{x\epsilon A}{x\epsilon B}\right)\right)$

R The above means that every member of A is also member of B. Therefore, if the assertion of inclusion does not hold, then one should be able to produce an element of A that is not in B.

R If A and B are both 'small' sets, specified by listing elements as in (1), to check if $A \subseteq B$ one simply inspects both sides to verify that every element of A is also listed in B. If A and B are both specified using setbuilder notation as in (2) the following theorem is used.

T

$$\frac{\{a\epsilon S|P(a)\} \subseteq \{b\epsilon T|Q(b)\}}{\{a\epsilon S|P(a)\} \subseteq T \qquad \forall x\epsilon T \ \left(\frac{P(x)}{Q(x)}\right)}$$

R The reader is invited to investigate the other possibilities if any.

T $$\forall A \in \mathcal{U}\left(\{\ \ \} \subseteq A \subseteq \mathcal{U}\right)$$

T $$\forall A, B, C \in \mathcal{U} \quad \left(\frac{A\subseteq B \quad B\subseteq C}{A\subseteq C}\right)$$

T $$\forall A, B \in \mathcal{U} \quad \left(\left(A \subset B\right) \vee \left(A = B\right) \vee \left(B \subset A\right) \vee \left(\left(A \nsubseteq B\right) \wedge \left(B \nsubseteq A\right)\right)\right)$$

E Circle the correct choices and answer:

$$\left\{a, c\right\} \subseteq \left\{a, b\right\} \qquad Y \qquad (N) \quad (Pf) \quad (W)$$

Why? $(c \in LS) \wedge (c \notin RS)$

R In the above example, c is a witness to the falsehood of the assertion as it is in the left side (LS) of ⊆ and not on the right side (RS) of ⊆ and producing such an element provides a proof.

E Circle the correct choices and answer:

$$\left\{x \in \mathbb{R} \middle| x > 2\right\} \subseteq \left\{x \in \mathbb{N} \middle| x > 3\right\} \qquad Y \qquad (N) \quad (Pf) \quad (W)$$

Why? $$\frac{\dfrac{(\pi\in\mathbb{R})\wedge(x>2)}{(\pi\in LS)} \qquad \dfrac{(\pi\notin\mathbb{N})}{(\pi\notin RS)}}{(\pi\in LS)\wedge(\pi\notin RS)}$$

R It is important to understand this proof.

π is a real number therefore π and it is bigger than 2; therefore π is in LS. But π is not a natural number; therefore, it is not in RS. Therefore, π is in LS and not in the RS and constitutes a witness to the falsehood of the statement. Producing this witness to the false hood provides a proof of the assertion that the assertion is not true.

R It is also important to learn to read this proof.

From the fact that $\pi \in \mathbb{R}$ and the fact that $\pi > 2$, it follows that $\pi \in LS$. From the fact that $\pi \notin \mathbb{N}$, it follows that $\pi \notin RS$. From the facts that $\pi \in LS$ and $\pi \notin RS$, it follows that $(\pi \in LS) \wedge (\pi \notin RS)$; therefore, π is a witness to the falsehood of the assertion.

R You will be required to write proofs on exams symbolically as illustrated here.

E Circle the correct choices and answer:

$$\left\{x \in \mathbb{N} \middle| x > 3\right\} \subseteq \left\{x \in \mathbb{R} \middle| x > 2\right\} \qquad (Y) \quad N \qquad (Pf) \quad W$$

Why?

$$\begin{array}{c} x \in LS \\ \hline x \in \mathbb{N} \quad x > 3 \quad \mathbb{N} \subseteq \mathbb{R} \quad \forall x \in \mathbb{N}\left(\frac{x>3}{x>2}\right) \\ \hline x \in \mathbb{R} \quad x > 2 \\ \hline x \in RS \\ \hline \forall x \in \mathbb{R}\left(\frac{x \in LS}{x \in RS}\right) \\ \hline LS \subseteq RS \end{array}$$

R Again, it is very important to understand what the symbols are saying. The first line says: let $x \in LS$ be given. This is the move of the enemy, the sceptic. Our goal is to prove that then $x \in RS$. We collect relevant information about x. We conclude that $x \in \mathbb{N}$, $x > 3$. Now we also need some facts that we know from previous study of mathematics, namely that $\mathbb{N} \subseteq \mathbb{R}$ and that for every $x \in \mathbb{N}$, if $x > 3$ then $x > 2$; that is, every natural number is also a real number and that every natural number bigger than 3 is also bigger than 2. Therefore, we may conclude that $x \in \mathbb{R}$ and that $x > 2$. Therefore, we may conclude that $x \in RS$. Now comes a crucial observation. Our enemy the sceptic chose x; we had no control over it. Therefore, if the enemy had chosen some other x, my argument would still be valid for that new x. Therefore, we may conclude that for *every* $x \in LS$ it is also true that $x \in RS$. Therefore, every element of LS is in RS. Therefore, LS is included in RS.

R It is very important to get used to this level of clarity in one's reasoning.

Equality

D Given two sets A and B from the universe, A is said to be equal to B, written

$$\left(A = B\right) :\Leftrightarrow \left(\left(A \subseteq B\right) \wedge \left(B \subseteq A\right)\right)$$

C For finite sets specified by listing, the order in which the elements occur does not matter.

E For example:

$$\left\{a, b, c\right\} = \left\{a, c, b\right\} = \left\{b, a, c\right\} = \left\{b, c, a\right\} = \left\{c, a, b\right\} = \left\{c, b, a\right\}$$

Q In how many ways can one list the elements of the set $\left\{a, b\right\}$?

Q In how many ways can one list the elements of the set $\left\{a, b, c, d\right\}$?

Q Prove that $\left((-3)..1\right) = \{\quad\}$.

R If $A \neq B$ then at least one of the sets has an element that is not in the other.

Q Prove that:

$$\forall A, B \in \mathcal{U}\left(\left(A \neq B\right) :\Leftrightarrow \left(\left(A \nsubseteq B\right) \vee \left(B \nsubseteq A\right)\right)\right)$$

Operations on Sets

R For small finite sets, the operations described here can be visualised using Venn diagrams, named after the mathematician Venn. These are very useful and allow us to use a 'geometric' faculty of the brain to process this sort of information. To gain this facility, one possibility is to read the Wikipedia article on sets.

Power-set

D
$$\mathcal{P}(A) := \left\{S \epsilon \mathcal{U} \middle| S \subseteq A\right\}$$

Note that the power-set of A is the set of subsets of A.

E We compute the following:

$$\mathcal{P}\left(\{\quad\}\right) = \left\{\{\quad\}\right\}$$

$$\mathcal{P}\left(\{a\}\right) = \left\{\begin{matrix}\{\quad\}, \\ \{a\}\end{matrix}\right\}$$

$$\mathcal{P}\left(\{a, b\}\right) = \left\{\begin{matrix}\{\quad\}, \\ \{a\}, \{b\}, \\ \{a, b\}\end{matrix}\right\}$$

$$\mathcal{P}\left(\{a, b, c\}\right) = \left\{\begin{matrix}\{\quad\}, \\ \{a\}, \{b\}, \{c\}, \\ \{a, b\}, \{a, c\}, \{b, c\}, \\ \{a, b, c\}\end{matrix}\right\}$$

Q Compute the following: $\mathcal{P}\left(\{a,b,c,d\}\right), \mathcal{P}\left(\{a,b,c,d,e\}\right)$

R Note that a set with n elements can have subsets of sizes: $0, 1, 2 \dots n$. In each of the above computations, we list on the first row the only subset of size 0, namely the empty set, then on the second row, subsets of size one, then on the third row, subsets of size 2, and so on. Note that every row has a comma in the end to indicate that there is another row below; only the last row does not have a comma at the end to indicate that it is the last row. You will be required to maintain exactly the same syntax and conventions on your exam. Otherwise you will lose points.

R We shall see later that the power-set of an n-element set contains exactly 2^n elements. Please count the number of elements in the preceding examples to verify this is the case. For instance, $\{a,b,c\}$ has 3 elements while $\mathcal{P}\left(\{a,b,c\}\right)$ has $2^3 = 8$ elements.

R The notes on counting has more information on such counting.

<u>Union</u>

D $$A \cup B \coloneqq \{x \epsilon \mathcal{U} | (x \epsilon A) \vee (x \epsilon B)\}$$

R Note that the connective or ($\vee$) is used in its inclusive sense in mathematics. Therefore, an element x is in $A \cup B$ if it is either in A, or in B, or in both. Equivalently, an element is in the union of two sets if and only if it is in at least one of them; this version applies to the union of arbitrarily many sets. An element is in the union of a collection of sets if and only if, it is in at least one of them. To compute a union of a number of sets, we simply list every element that appears in at least one of them exactly once. If there is some 'obvious' ordering on the elements of the sets, then we maintain that ordering in the answer, just to reduce the possibility of error.

E Compute:
$$\begin{aligned} &\{1,2\} \cup \{2,3\} \\ = &\{1,2,3\} \end{aligned}$$

E Compute:
$$\begin{aligned} &\{1,2\} \cup \{2,3\} \cup \{3,4\} \\ = &\{1,2,3\} \end{aligned}$$

E Compute:
$$\begin{aligned} &\{1,2,3\} \cup \{2,3,4\} \cup \{3,4\} \\ = &\{1,2,3,4\} \end{aligned}$$

R You are required to line up your signs of equality on the left as shown here.

Intersection

D
$$A \cap B := \{x \epsilon \mathcal{U} | (x \epsilon A) \wedge (x \epsilon B)\}$$

R Note that an element x is in $A \cap B$ if and only if it is in both A and B. An element is in the intersection of a collection of arbitrarily many sets if and only if it is in all of them.

E Compute:
$$\begin{aligned} &\{1,2\} \cap \{2,3\} \\ = &\{2\} \end{aligned}$$

E Compute:
$$\begin{aligned} &\{1,2\} \cap \{2,3\} \cap \{3,4\} \\ = &\{\ \ \} \end{aligned}$$

E Compute:
$$\begin{aligned} &\{1,2,3\} \cap \{2,3,4\} \cap \{3,4\} \\ = &\{3\} \end{aligned}$$

R You are required to line up your signs of equality on the left as shown here.

Relative Complementation

D
$$A \backslash B := \{x \epsilon \mathcal{U} | (x \epsilon A) \wedge (x \notin B)\}$$

R Note that $A\backslash B$contains those elements of A that are not in B and as such is not symmetric with respect to A and B. To emphasize this asymmetry, we note below the definition of $B\backslash A$ even though it is redundant to do so.

D $$B\backslash A \coloneqq \{x \epsilon \mathcal{U} | (x \epsilon B) \wedge (x \notin A)\}$$

R Note that in general: $A\backslash B \neq B\backslash A$

E Compute:
$$\begin{aligned} & \{1,2\}\backslash\{2,3\} \\ = & \{1,\cancel{2}\}\backslash\{2,3\} \\ = & \{1\} \end{aligned}$$

R In the preceding computation, we scan the elements of the set $\{1,2\}$ from left to right and ask of each of them if it is in $\{2,3\}$as follows.

Is 1 in $\{2,3\}$ and the answer is no. The next question is:

Does 1 get removed and the answer is no. Then we leave 1 alone.

Since we are scanning from left to right, we then ask of the next element 2, is 2 in $\{2,3\}$ and the answer is yes. The next question is:

Does 2 get removed and the answer is yes. Then we note removal by a horizontal line through the element removed. On the exam removal of an element must be shown with a horizontal bar through the removed element as indicated here.

Finally, one records the element 1 from the set $\{1,2\}$ that has not been removed as shown earlier as the only element of the set $\{1,2\}\backslash\{2,3\}$.

R Those who do not follow the above algorithm systematically as indicated here very often make mistakes. One needs to pay very close attention to the minutest details in this class so that one does not make mistake.

E Compute:

$$\left(\{1,2\}\backslash\{2,3\}\right)\backslash\{3,4\}$$
$$= \left(\{1,\cancel{2}\}\backslash\{2,3\}\right)\backslash\{3,4\}$$
$$= \{1\}\backslash\{3,4\}$$
$$= \{1\}$$

E Compute:

$$\{1,2\}\backslash\left(\{2,3\}\backslash\{3,4\}\right)$$
$$= \{1,2\}\backslash\left(\{2,\cancel{3}\}\backslash\{3,4\}\right)$$
$$= \{1,\cancel{2}\}\backslash\{2\}$$
$$= \{1\}$$

R Note that in the previous two examples the answer turns out to be the same. The question is whether this is always the case, no matter what the three sets are. One example says nothing more or less than that this can happen. But as there are infinitely many sets, one does not know if it is always the case. It is therefore important to know what is generally true. The chart that appears later lists all the basic rules for operations on sets.

R **You are required to line up your signs of equality on the left as shown here**.

<u>Absolute Complementation</u>

D $$(A)^c \coloneqq A^c_{\mathcal{U}} \coloneqq \mathcal{U}\backslash A$$

R Note that the customary notation $(A)^c$ is deficient in that it omits the universe of discourse. A more complete notation which includes the universe of discourse as a subscript should be used if there is any danger of confusion.

R We shall not do any special examples for absolute complementation because as soon as $\mathcal{U}$ is specified the computation is carried out in the same way as with relative omplementation.

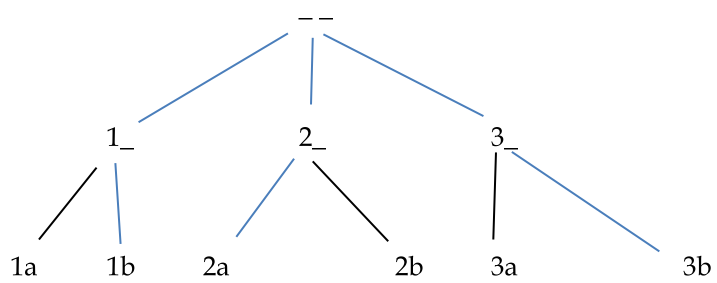
<u>Product</u>

D $$A \sqcap B \coloneqq A \times B \coloneqq \{(a, b)\epsilon\mathcal{U}|(a\epsilon A) \wedge (b\epsilon B)\}$$

R The product is also called the cartesian product. We use a tree to generate the elements of the product systematically.

E Compute: $\left\{1, 2, 3\right\} \times \left\{a, b\right\}$

<u>Tree</u>



The diagram above, labelled tree, is the calculation that needs to accompany the answer below. The tree records that you have carried out an algorithm to arrive at the answer. Since there are two sets of which we are taking the product, we start out with two empty slots or blanks (represented by _). The left slot must be filled with the elements 1, 2, or 3 of the left factor $\left\{1, 2, 3\right\}$ of the cartesian product. Since there are three choices, we get three branches ending in 1_, 2_, and 3_. The right slot must be filled with the elements a, b of the right factor $\left\{a, b\right\}$. Therefore, at the second level of the tree, we get two branches from each node. The two branches 1_ end in 1a and 1b, the two branches from 2_ end in 2a and 2b, and the three branches from 3_ end in 3a and 3b. This gives a total of six elements of the product set. The words 2a etc. must be decorated with parentheses and commas as $(2, a)$ to qualify as elements of the produce.

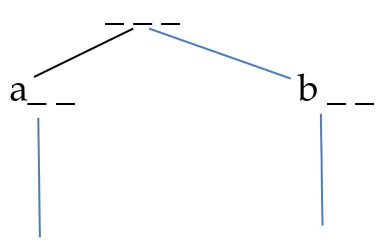
$$\left\{1, 2, 3\right\} \times \left\{a, b\right\}$$

$$= \quad \left\{\begin{matrix} (1, a), & (2, a), (3, a), \\ (1, b), & (2, b), (3, b) \end{matrix}\right\}$$

E Compute: $\left\{a, 1\right\} \times \left\{b\right\} \times \left\{1, 2, 3\right\}$

As in the previous example, we have to draw a tree. Since there are three factors, we start with three empty slots. The first slot may be filled with two elements a and 1 of, $\left\{a, 1\right\}$. Therefore, there ar two branches at the top. Since the second factor has only one element b in $\left\{b\right\}$ the second slot can be filled only with b and, therefore, there will be one branch from each node at the second level. Each node at the third level will have three branches because the third factor $\left\{1, 2, 3\right\}$ has three elements 1, 2, and 3. Lastly, the words have to be decorated with parentheses and commas, so that they are of the right type. The tree appears below.

<u>Tree</u>



Please check very carefilly for errors

$$\left\{a, 1\right\} \times \left\{b\right\} \times \left\{1, 2, 3\right\}$$

$$= \left\{\begin{matrix} (a, b, 1), & (a, b, 2), & (a, b, 3), \\ (b, b, 1), & (b, b, 2), & (b, b, 3) \end{matrix}\right\}$$

<u>Coproduct</u>

D $$A \sqcup B \coloneqq A \dot{\cup} B \coloneqq \left(A \times \{0\}\right) \cup \left(B \times \{1\}\right)$$

R The coproduct is also called the disjoint union (denoted usually by: $\dot{\cup}$) and uses a device for rendering two sets with a non-empty overlap disjoint, before taking a union so that the elements of the overlap effectively occur twice in the coproduct.

E Compute:

$$\begin{aligned} & \left\{a, b\right\} \sqcup \left\{b, 1\right\} \\ = \quad & \left(\left\{a, b\right\} \times \{0\}\right) \cup \left(\left\{b, 1\right\} \times \{1\}\right) \\ = \quad & \left\{(a, 0), (b, 0)\right\} \cup \left\{(a, 1), (b, 1)\right\} \\ = \quad & \left\{\begin{matrix} (a, 0), & (b, 0), \\ (b, 1), & (1, 1) \end{matrix}\right\} \end{aligned}$$

For this computation you will need to show two trees for computing the two cartesian understanding to get full credit.

R One can now make various expressions using these basic building blocks and compute them.

You try the following:

Q Compute: $\left\{a, b\right\} \sqcup \left\{b, c\right\} \sqcup \left\{a, c\right\}$

E Compute:

$$\begin{aligned} & \left\{a, b\right\} \cup \left(\left\{b, c\right\} \setminus \left\{a, c\right\}\right) \\ = \quad & \left\{a, b\right\} \cup \left(\left\{b, \cancel{c}\right\} \setminus \left\{a, c\right\}\right) \\ = \quad & \left\{a, b\right\} \cup \left(\left\{b\right\}\right) \\ = \quad & \left\{a, b\right\} \end{aligned}$$

You try the following:

Q Compute: $\left\{a,b\right\} \cup \left(\left\{b,c\right\} \cap \left\{a,c\right\}\right)$

Q Compute: $\left(\left\{b,c\right\} \cap \left\{a,c\right\}\right) \times \left\{a,b\right\}$

Q Compute: $\left(\left\{b,c\right\} \cap \left\{a,c\right\}\right) \times \left\{\left(\left\{b,c\right\} \cap \left\{a,c\right\}\right)\right\}$

Q Compute: $\left\{a,b\right\} \times \left(\left\{b,c\right\} \cap \left\{a,c\right\}\right)$

Q Compute: $\left\{a,b\right\} \setminus \left(\left\{b,c\right\} \cap \left\{a,c\right\}\right)$

Q Compute: $\left(\left\{b,c\right\} \cap \left\{a,c\right\}\right) \setminus \left\{a,b\right\}$

Q Compute: $\mathcal{P}\left(\left\{b,c\right\} \cap \left\{a,c\right\}\right) \cap \mathcal{P}\left(\left\{a,b\right\}\right)$

E Prove that $\forall A, B \in \mathcal{U}\left((A \cap B) \subseteq A\right)$

SPf

$$A \in \mathcal{U} \quad B \in \mathcal{U} \quad x \in \mathcal{U} \quad x \in A \cap B$$

$$(x \in A) \wedge (x \in B)$$

$$x \in A$$

$$\forall x \in \mathcal{U}\left(\frac{x \in A \cap B}{x \in A}\right)$$

$$(A \cap B) \subseteq A$$

$$\forall A, B \in \mathcal{U}\left((A \cap B) \subseteq A\right)$$

EPf

R In order to learn the syntax, it is crucial that you read this in words. It is essentially like a programming language.

E $\exists A, B \in \mathcal{U}\left((A \backslash B) = A\right) \quad (Y) \quad N \quad (Pf) \;\; (W)$

Spf

$$A \coloneqq \{\quad\} \quad B \coloneqq \{\quad\}$$

$$A \backslash B = \{\quad\} \backslash \{\quad\} = \{\quad\} = A$$

$$\exists A, B \in \mathcal{U}\left((A \backslash B) = A\right)$$

Epf

E Prove that $\forall A, B \in \mathcal{U}\left((A \backslash B) = A \cap (B^c)\right)$

Spf

$$
\begin{array}{c}
A \in \mathcal{U} \quad B \in \mathcal{U} \quad x \in \mathcal{U} \quad x \in A\backslash B \\
\hline\hline
x \in A \quad x \notin B \\
\hline\hline
x \in A \quad x \in (B^c) \\
\hline\hline
(x \in A) \wedge \left(x \in (B^c)\right) \\
\hline\hline
x \in A \cap (B^c) \\
\hline\hline
\forall x \in \mathcal{U}\left(\begin{array}{c} x \in A\backslash B \\ \hline\hline x \in A \cap (B^c) \end{array}\right) \\
\hline
(A\backslash B) = A \cap (B^c) \\
\hline
\forall A, B \in \mathcal{U}\left((A\backslash B) = A \cap (B^c)\right)
\end{array}
$$

EPf

R The proof is read and understood as follows:

$$A \in \mathcal{U} \quad B \in \mathcal{U} \quad x \in \mathcal{U} \quad x \in A\backslash B$$

Let $x, A, B \in \mathcal{U}$ be given such that $x \in A\backslash B$; this is the sceptic's move.

$$x \in A \quad x \notin B$$

$x \in A\backslash B$ is equivalent to $x \in A$ and $x \notin B$.

$$x \in A \quad x \in (B^c)$$

$x \in \mathcal{U}$ and $x \notin B$ is equivalent to $x \in \mathcal{U}\backslash B = B^c$.

$$(x \in A) \wedge \left(x \in (B^c)\right)$$

$(x \in A)$ and $\left(x \in (B^c)\right)$ is equivalent to $(x \in A) \wedge \left(x \in (B^c)\right)$.

$$x \in A \cap (B^c)$$

$(x \in A) \land \left(x \in (B^c)\right)$ is by definition equivalen to $x \in A \cap (B^c)$.

$$\forall x \in \mathcal{U}\left(\frac{\frac{x \in A \backslash B}{}}{x \in A \cap (B^c)}\right)$$

Since $x \in \mathcal{U}$ was picked by the sceptic, and since the argument doe not use any special property of $x \in \mathcal{U}$, the argument will work for any other $x \in \mathcal{U}$. It follows that the argument is valid for every $x \in \mathcal{U}$. Hence $\forall x \in \mathcal{U}$, $x \in A \backslash B$ is equivalent to $x \in A \cap (B^c)$.

$$(A \backslash B) = A \cap (B^c)$$

It follows from the above that the set $(A \backslash B)$ is equal to the set $A \cap (B^c)$.

$$\forall A, B \in \mathcal{U}\left((A \backslash B) = A \cap (B^c)\right)$$

Since $A, B \in \mathcal{U}$ were picked by the sceptic and the argument does not use any special properties of $A, B \in \mathcal{U}$, the argument is valid no matter which $A, B \in \mathcal{U}$ are picked. Therefore, the argument is valid for every $A, B \in \mathcal{U}$. Hence $(A \backslash B) = A \cap (B^c)$ holds for every $A, B \in \mathcal{U}$.

R In order to understand proofs properly, one must first write proofs in words, using the minimilastic vocabulary used here. It is crucial that one do this because otherwise one will not learn the syntax. It is essentially like a programming language. Please use the minimalistic vocabulary that I used to maintain the structure of the argument.

<u>Counting</u>

D Given a universe of sets $\mathcal{U}$ and a finite set S we define a cardinality function as follows:

$$\nu \coloneqq S \;\longmapsto\; \nu(S) \colon \mathcal{U} \longrightarrow \mathbb{N},$$

R $\nu(S)$ is the number of elements in S and provides one possible formalization, among many, of the notion of 'size of S'.

E $\nu\left(\{\ \ \}\right) = 0 \qquad \nu\left(\{a\}\right) = 1 \qquad \nu\left(\{a, b, c\}\right) = 3$

<u>Comparing Cardinalities</u>

Given two sets S and T

T $$\frac{S \subseteq T}{\nu(S) \leq \nu(T)}$$

T $$\nu\big(\mathcal{P}(S)\big) = 2^{(\nu(S))}$$

T $$\nu(S \times T) = \nu(S) \times \nu(T)$$

T $$\nu(S \sqcup T) = \nu(S) + \nu(T)$$

T $$\nu(S \cup T) = \begin{array}{cc} & \nu(S) + \nu(T) \\ - & \nu(S \cap T) \end{array}$$

T $$\nu(S \cup T \cup U) = \begin{array}{cccccc} & \nu(S) & + & \nu(T) & + & \nu(U) \\ - & \nu(S \cap T) & - & \nu(S \cap U) & - & \nu(T \cap U) \\ + & S \cap T \cap U & & & & \end{array}$$

R The formula generalizing the above formulas to the case of an indefinitely finite family of finite sets is given below; it is more intricate and is said to embody the inclusion-exclusion principle. It is in the handout on counting.

Q Find formulas for $\nu(A \cup B \cup C \cup D)$ and $\nu(A \cup B \cup C \cup D \cup E)$

Laws	Laws of the Algebra of Proposition (PC)		Laws of the Algebra of Sets (S)	
PC S BA p A x q B y r C z s D w t E u	1	$\bot$	1	$\{ \}$
	2	T	2	$\mathcal{U}$
	3	$\vee$	3	$\cup$
	4	$\wedge$	4	$\cap$
	5	$\neg p$	5	$(\ )^c$
	6	$p \wedge (\neg q)$	6	$A \backslash B$
	7	$p \rightarrow q \equiv (\neg p) \vee q$	7	$(A)^c \cup (B)$
	8	$\leftrightarrow$	8	$(A^c \cup B) \cap (B^c \cup A)$
Idempotent Laws	$p \vee p = p$ $p \wedge p = p$		$A \cup A = A$ $A \cap A = A$	
Associative Laws	$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$		$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	
Commutative Laws	$p \vee q = q \vee p$ $p \wedge q = p \wedge q$		$A \cup B = B \cup A$ $A \cap B = B \cap A$	
Distributive Laws	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$		$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
Identity Laws	$p \vee \bot = p$ $p \wedge T = p$ $p \vee T = T$ $p \wedge \bot = \bot$		$A \cup \{ \} = A$ $A \cap \mathcal{U} = A$ $A \cup \mathcal{U} = \mathcal{U}$ $A \cap \{ \} = \{ \}$	
Complement Laws	$p \vee (\neg p) = T$ $p \wedge (\neg p) = \bot$ $\neg T = \bot$ $\neg \bot = T$ Uniqueness of complement If $p \vee x = T$ and $p \wedge x = \bot$ then $x = \neg p$		$A \cup A^c = \mathcal{U}$ $A \cap A^c = \{ \}$ $\mathcal{U}^c = \{ \}$ $\{ \}^c = \mathcal{U}$ Uniqueness of complement If $A \cup X = \mathcal{U}$ and $A \cap X = \{\}$ then $x = A^c$	
Involution Law	$\neg(\neg p) = p$		$(A^c)^c = A$	
De Morgan's Law	$\neg(p \vee q) = (\neg p) \wedge (\neg q)$ $\neg(p \wedge q) = (\neg p) \vee (\neg q)$		$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	
Absorption Law	$p \vee (p \wedge q) = p$ $p \wedge (p \vee q) = p$		$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	

Please check very carefilly for errors

Boundedness	p ∨ T = T p ∧ ⊥ = ⊥	A ∪ $\mathcal{U}$ = $\mathcal{U}$ A ∩ { } = { }

Home Work

It is given that the sets:

$$X, Y, Z\ \epsilon \big\{\{\quad\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\,\big\}$$

and that the operations: $\odot, \circledast, \circledcirc\ \epsilon\{\cup,\ \cap,\ \backslash,\ \times,\ \sqcup\,\}$

0 The following computation shows that using substitution, one may easily generate a very large number of problems so that one has a vast collection.

$$\left(X \circledast Y\right)\left\langle\begin{matrix} X \longleftarrow \{b, c\} \\ \circledast \longleftarrow \quad \cap \\ Y \longleftarrow \{b, c\}\end{matrix}\right\rangle = \left\{b, c\right\} \cap \left\{b, c\right\}$$

The expression $\left\{b, c\right\} \cap \left\{b, c\right\}$ that one may compute is being generated from the expression $\left(X \circledast Y\right)$ by making the substitutions $X \longleftarrow \{b, c\}$ that is: X replaced with $\{b, c\}$, $\circledast$ replaced with $\circledast$ replaced with $\cap$.

Compute the following expressions using particular choices for the sets $X, Y$and Z from the collections in the above, and count the number of elements in the resulting set before carrying out the computation, using counting formulas, and after carrying out the computation, by counting elements.

R You must use the same value for every occurrence of the same letter (such as X) or every occurrence of the same operation (such as $\odot$) in every expression. You may use same or different values in the place of distinct letters (such as X and Y) or in the place of distinct connetives (such as $\odot$and$\circledast$).

R Count how many variants there are of each problem. Do as many of these variants as you need to in order to understand what is going on.

R No answers will be provided; you should try instead to work out each problem in more than one way to see if all methods lead to the same answer. Devising several ways of computing the same expression will help you understand the material battier.

R If the answers for two expressions turn out to be the same for different choices of the 'unknown' sets and operations, determine if this equality is fortuitous, or if it always holds, no matter what the values of the unknowns are.

10 $X \odot Y$

11 $X \odot (Y \circledast Z)$

12 $(X \odot Y) \circledast Z$

13 $(X \odot Y) \circledast (Y \circledcirc Z)$

14 $(X \odot Y) \circledast (X \circledcirc Z)$

15 $(X \odot Z) \circledast (Y \circledcirc Z)$

16 $\big(\mathcal{P}(X)\big)$

17 $\big(\mathcal{P}(X)\big) \odot Y$

18 $X \odot \big(\mathcal{P}(Y)\big)$

19 $\big(\mathcal{P}(X)\big) \odot \big(\mathcal{P}(Y)\big)$

Appendix

Recursive specifications

R Families of sets may be specified recursively. Even though the following material is to be found in the notes on induction and recursion, it is reproduced here to allow the students to familiarise themselves with the concepts informally.

D The factorial function φ, is defined recursively as follows:

BC		$\varphi(0) \coloneqq 1$
RcS	$\forall n \in \mathbb{N}$	$\varphi(n+1) \coloneqq (n+1)\varphi(n)$

R Note that: $\forall n \in \mathbb{N}$, $\varphi(n) = n! \coloneqq \prod_{k \in 1..n} k$, but we are not using the the notation $n!$ to conform to the conventions in this section.

To calculate $\varphi(1)$, we may use the recursive definition and proceed as follows. We shall repeatedly use Euclid's rule, namely: **equals substituted into equals yield equals**.

The first task is to match $\varphi(1)$ to the left or the right side of one of the defining equations. We shall show the match explicitly.

$$\varphi(1) = \varphi(0+1) = \Big(\varphi(n+1)\Big)\Big\langle n \quad \leftarrow \quad 0\Big\rangle$$

The above means that $\varphi(1)$ may be obtained by substituting 0 in the place of n in the expression $\Big(\varphi(n+1)\Big)$. Since $\varphi(n+1)$ is equal by definition to $(n+1)\varphi(n)$, we may, using Euclid's rule, substitute 0 in the expression $(n+1)\varphi(n)$ to obtain a new value $\varphi(n+1)$. This will reduce the argument of φ to 0 and this is to be used repeatedly to compute $\varphi(n)$. We record the process as follows.

$$\begin{aligned} & \varphi(1) \\ = \quad & \varphi(0+1) \end{aligned}$$

$$= \quad \Big(\varphi(n+1)\Big)\Big\langle n \quad \longleftarrow \; 0\Big\rangle$$

$$= \quad \Big((n+1)\varphi(n)\Big)\Big\langle n \quad \longleftarrow \; 0\Big\rangle$$

$$= \quad (0+1)\varphi(0)$$

$$= \quad (1)1 \qquad \Big(\because \varphi(0) \coloneqq 1\Big)$$

$$= \quad 1$$

Similarly, $\varphi(2)$

$$= \quad \varphi(1+1)$$

$$= \quad \Big(\varphi(n+1)\Big)\Big\langle n \quad \longleftarrow \; 1\Big\rangle$$

$$= \quad \Big((n+1)\varphi(n)\Big)\Big\langle n \quad \longleftarrow \; 1\Big\rangle$$

$$= \quad (1+1)\varphi(1)$$

$$= \quad (2)1 \qquad \Big(\because \varphi(1) \coloneqq 1\Big)$$

$$= \quad 2!$$

$$= \quad 2$$

Similarly, $\varphi(3)$

$$= \quad \varphi(2+1)$$

$$= \quad \Big(\varphi(n+1)\Big)\Big\langle n \quad \longleftarrow \; 2\Big\rangle$$

$$= \quad \Big((n+1)\varphi(n)\Big)\Big\langle n \quad \longleftarrow \quad 2\Big\rangle$$

$$= \quad (2+1)\varphi(2)$$

$$= \quad (3)(2)1 \qquad\qquad \Big(\because \varphi(2) \coloneqq (2)1\Big)$$

$$= \quad 3!$$

$$= \quad 6$$

R This process of repeatedly 'running backwards' justifies the name recursion.

R The recursive scheme should be understood by reading the definition in words. There is a function $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ whose output at the input 0 is 1, and to calculate its output at the input $n+1$, we need to multiply the current input, namely $n+1$, by the output at the immediately preceding number, namely, $\varphi(n)$.

E As a second example we consider the Fibonacci numbers which are defined recursively as follows.

$$\begin{aligned} &\rho(0) \coloneqq 0 \\ &\rho(1) \coloneqq 1 \\ \forall n \in \mathbb{N} \qquad &\rho(n+2) \coloneqq \rho(n+1) + \rho(n) \end{aligned}$$

R The recursive scheme should be understood by reading the definition in words. There is a function $\rho\colon \mathbb{N} \longrightarrow \mathbb{N}$ whose output at the input 0 is 1 and at input 1 is 1. To calculate its output at the input $n+2$, we need to add the outputs at the two immediately preceding inputs, namely, $\rho(n+1)$ and $\rho(n)$.

We can record our calculation for $\rho(2)$ following the same bookkeeping scheme as before.

$$\rho(2)$$

$$
\begin{aligned}
&= \quad \rho(0+2) \\
&= \quad \Big(\rho(n+2)\Big)\Big\langle n \quad \longleftarrow 0 \Big\rangle \\
&= \quad \Big(\rho(n+1)+\rho(n)\Big)\Big\langle n \quad \longleftarrow \ 0 \Big\rangle \\
&= \quad \rho(0+1)+\rho(0) \\
&= \quad \rho(0+1)+\rho(0) \qquad\qquad \Big(\because \rho(0) \coloneqq 0, \rho(1) \coloneqq 1\Big) \\
&= \quad \rho(1)+\rho(0) \\
&= \quad 1+0 \\
&= \quad 1
\end{aligned}
$$

Even without doing the calculation, one can immediately see that $\rho(3) = \rho(2) + \rho(1) = 1 + 1$=2

Q Compute $\rho(10)$ recursively as shown earlier, by looking at the first 9 outputs, and also using a formula that you need to find.

Q $\exists n \in \mathbb{N}\Big(\rho(n) = 123456789\Big) \qquad Y \qquad N \qquad Pf \qquad W$

Q Try to understand the following recursive scheme for the function $\varphi: \mathbb{N} \longrightarrow \mathbb{N}$ by interpreting its meaning in a manner analogous to the one used for Fibonacci numers. Then calculate the value of $\varphi(2), \varphi(3), \varphi(4)$, and $\varphi(5)$. Find a general formula for $\varphi(n)$.

$$
\begin{gathered}
\varphi(0) \coloneqq 1 \\
\varphi(1) \coloneqq 2 \\
\forall n \in \mathbb{N} \qquad \varphi(n+2) \coloneqq \Big(\varphi(n+1)\Big)\Big(\varphi(n)\Big)
\end{gathered}
$$

Operations on indexed families

R We shall need to consider operations on indexed families where the number of members of the family is very large or indefinite. The operations are usually commutative and associative. We shall have occasion to use binary operations such as sums of numbers, products of numbers, unions, intersections, products, and coproducts of sets, disjunctions and conjunctions of propositions, and so on. In all such cases, the idea is that we perform the operation as the index runs over the index set. The index may be denoted by any letter, as long as there is no clash. We shall not develop the matter any more formally here but provide some examples to illustrate how one uses the notation which is compact and convenient.

E $\sum_{k\in 1..4}(a_k) :=\ \sum_{k=1}^{4}(a_k) :=\ a_1 + a_2 + a_3 + a_4$

This is a sum of $(a_k)'s$ as k runs from 1 to 4.

E $\sum_{l\in\{3,4,6\}}(a_l) :=\ a_3 + a_4 + a_6$

This is a sum of $(a_k)'s$ as l runs over the elements of the set $\{3, 4, 6\}$.

E $\sum_{i\ \in 1..4}(a_n) := \sum_{i=1}^{n}(a_i) :=\ \ a_1 + a_2 + \cdots + a_n$

This is a of $(a_k)'s$ as k runs from 1 to n.

E $\sum_{k\in 1..4}(k) :=\ \sum_{k=1}^{4}(k) :=\ 1 + 2 + 3 + 4$

E $\sum_{k\in 1..n}(k^2) :=\ \sum_{k=1}^{n}(k^2) :=\ 1^2 + 2^2 + \cdots + n^2$

E $\sum_{k\in 1..n}(k^{k+1}) \coloneqq \sum_{k=1}^{n}(k^{k+1}) \coloneqq 1^{1+1} + 2^{2+1} + \cdots + n^{n+1}$

E $\prod_{k\in 1..n}(a_k) \coloneqq \prod_{k=1}^{n}(a_k) = (a_1)(a_2)\cdots(a_n)$

This is a product of the numbers a_k as k runs from 1 to n.

E $\prod_{k\in 1..n}(A_k) \coloneqq \prod_{k=1}^{n}(A_k) = A_1 \times A_2 \times \cdots \times A_n$

This is a product of the sets A_k as k runs from 1 to n.

E $\coprod_{k\in 1..n}(A_k) \coloneqq \prod_{k=1}^{n}(A_k) = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$

This is a coproduct of the sets A_k as k runs from 1 to n.

E $\bigcup_{k\in 1..n}(A_k) \coloneqq \bigcup_{k=1}^{n}(A_k) = A_1 \cup A_2 \cup \cdots \cup A_n$

This is a union of the sets A_k as k runs from 1 to n.

E $\bigcap_{k\in 1..n}(A_k) \coloneqq \bigcap_{k=1}^{n}(A_k) \coloneqq A_1 \cap A_2 \cap \cdots \cap A_n$

This is a union of the sets A_k as k runs from 1 to n.

E $\bigvee_{k\in 1..n}(p_k) \coloneqq \bigvee_{k=1}^{n}(p_k) \coloneqq p_1 \vee p_2 \vee \cdots \vee p_n$

This is a union of the sets A_k as k runs from 1 to n.

E $\bigwedge_{k\in 1..n}(p_k) \coloneqq \bigwedge_{k=1}^{n}(p_k) \coloneqq p_1 \wedge p_2 \wedge \cdots \wedge p_n$

This is a union of the sets A_k as k runs from 1 to n.