The Binomial Theorem

$\forall a, b \in \mathbb{R}, \forall n \in \mathbb{N}$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k}$$

D The coefficient of $a^{n-k}b^k$ in the expansion of $(a+b)^n$, denoted by $Cfcnt\left(a^{n-k}b^k, (a+b)^n\right)$ is $\binom{n}{k}$.

$$Cfcnt\left(a^{n-k}b^k, (a+b)^n\right) = \binom{n}{k}$$

R We are multiplying n instances of $(a+b)$. Therefore, we have n slots, and in each slot, we must put either an a or a b. If we want to put b in k of the slots, we have to choose k slots out of n and this may be done in $\binom{n}{k}$ ways. The remaining slots are each filled with an a. Therefore, the term is $a^{n-k}b^k$ and since such a term arises in exactly $\binom{n}{k}$ ways, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$.

E
$$\begin{aligned}
& Cffnt\left(a^7 b^3, (a+b)^{10}\right) \\
= \quad & Cfcnt\left(a^{10-3} b^3, (a+b)^{10}\right) \\
= \quad & \left(Cfcnt\left(a^{n-k}b^k, (a+b)^n\right)\right)\left\langle \begin{matrix} n \longleftarrow 10 \\ k \longleftarrow 3 \end{matrix} \right\rangle \\
= \quad & \left(\binom{n}{k}\right)\left\langle \begin{matrix} n \longleftarrow 10 \\ k \longleftarrow 3 \end{matrix} \right\rangle \\
= \quad & \binom{10}{3}
\end{aligned}$$

R The above is a good answer. There is no need to calculate the decimal value for this number unless one needs to.

E Find the expansion of $(2-3x)^3$ using the binomial theorem.

0 $$(2-3x)^3$$

$$= \left(2+(-3)x\right)^3$$

$$= \left((a+b)^n\right)\left\langle \begin{matrix} n \leftarrow 5 \\ a \leftarrow 2 \\ b \leftarrow \left((-3)x\right) \end{matrix} \right\rangle$$

1 $$\frac{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}{\left(2+(-3)x\right)^3 = \sum_{k=0}^{3} \binom{3}{k} 2^{3-k} \left((-3)x\right)^k} \left\langle \begin{matrix} n \leftarrow 3 \\ a \leftarrow 2 \\ b \leftarrow \left((-3)x\right) \end{matrix} \right\rangle$$

2

$\sum_{k=0}^{3} (3) a^{3-k} b^k$

$=$	$\binom{3}{0} 2^{3-0} \left((-3)x\right)^0$	$=$	8	$=$	8
$+$	$\binom{3}{1} 2^{3-1} \left((-3)x\right)^1$	$+$	$(-1)12x$	$+$	$(-1)12x$
$+$	$\binom{3}{2} 2^{3-2} \left((-3)x\right)^2$	$+$	$54xx$	$+$	$54x^2$
$+$	$\binom{3}{3} 2^{3-3} \left((-3)x\right)^3$	$+$	$(-1)27xxx$	$+$	$(-1)27x^3$

R The calculations for the above appear later. **All calculations must be exhibited exactly as shown here**. **Please place the columns side by side and please draw a vertical line with a ruler under the signs of equality to separate the columns**. The different pieces should be indexed with numbers so that the order in which the calculations are made is absolutely clear.

R **We adopt the following ordering on the set of terms**. Note that within each term or word, the ordering adopted is numero-alphabetical, that is, all numbers precede letters that are ordered alphabetically. The only negative number allowed is -1. -1 precedes all other numbers. The words so formed are then ordered lexicographically (as in the dictionary, again, with numbers by themselves coming first) after suppressing the numerical coefficients at the beginning of the terms. Please pay attention to this bookkeeping scheme.

30		$\binom{3}{3}$	31		$\binom{3}{2}$
	$=$	$\binom{3}{3-3}$		$=$	$\binom{3}{3-2}$
	$=$	$\binom{3}{0}$		$=$	$\binom{3}{1}$
	$=$	1		$=$	3
40			41		
		$\binom{3}{0}2^{3-0}\left((-3)x\right)^0$			$\binom{3}{1}2^{3-1}\left((-3)x\right)^1$
	$=$	$(1)8(1)$		$=$	$(1)\ 2^2(-3)x$
	$=$	8		$=$	$(-1)12x$
42			43		

	$\binom{3}{2}2^{3-2}\left((-3)x\right)^2$		$\binom{3}{3}2^{3-3}\left((-3)x\right)^3$
$=$	$(3)(2^1)9xx$	$=$	$(1)\ 2^0(-27)xxx$
$=$	$54xx$	$=$	$(-1)27xxx$

Q Find the expansion of $(3x - 2y)^5$ using the binomial theorem.

The Multinomial Theorem

$\forall x_1, x_2, \dots, x_p \in \mathbb{R}, \forall p, n \in \mathbb{N}$

$$\boxed{\left(x_1 + x_2 + \cdots + x_p\right)^n = \sum_{n_1+n_2+\cdots+n_p=n} \binom{n}{n_1 \quad n_2 \quad \dots \quad n_p} \left(x_1{}^{n_1}\right)\left(x_2{}^{n_2}\right)\dots\left(x_p{}^{n_p}\right)}$$

D The coefficient of $(x_1{}^{n_1})(x_2{}^{n_2})\dots(x_p{}^{n_p})$, in the expansion of

$\left(x_1 + x_2 + \cdots + x_p\right)^n$, denoted by

$Cfcnt\left((x_1{}^{n_1})(x_2{}^{n_2})\dots(x_p{}^{n_p}), \left(x_1 + x_2 + \cdots + x_p\right)^n\right)$ is $\binom{n}{n_1 \quad n_2 \quad \dots \quad n_p}$.

$$Cfcnt\left((x_1{}^{n_1})(x_2{}^{n_2})\dots(x_p{}^{n_p})\ ,\ \left(x_1 + x_2 + \cdots + x_p\right)^n\right) = \binom{n}{n_1 \quad n_2 \quad \dots \quad n_p}$$

E $Cfcnt\left((x_1{}^3)(x_2{}^2)(x_3{}^1)\ ,\ (x_1 + x_2 + x_3)^6\right) \quad = \quad \binom{3}{3 \quad 2 \quad 1}$

Please check very carefully for errors

R To use the same line of reasoning (used for the binomial theorem) for the expansion of $(x_1 + x_2 + \cdots + x_p)^n$, we notice that to get $(x_1{}^{n_1})(x_2{}^{n_2}) \ldots (x_p{}^{n_p})$, we choose n_1 slots out of n, fill them in with $x_1's$, then we choose n_2 slots out of the remaining $n - n_1$slots, fill them in with $x_2's$, then n_3 slots out of the remaining $n - n_1 - n_2$ slots, and so on. The expression for the coefficient that we get this way is:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-\cdots-n_p}{n_3}.$$

Q Decide if:

$$\binom{n}{n_1 \quad n_2 \quad \ldots \quad n_p}$$

$$= \quad \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\ldots\binom{n-n_1-\cdots-n_p}{n_3}$$

E Find the expansion of $(1 - x + y)^2$ using the multinomial theorem.

0

$$
\begin{aligned}
& (1 - x + y)^2 \\
= \quad & \Big(1 + (-1)x + y\Big)^2 \\
= \quad & \Big((x_1 + x_2 + x_3)^n\Big)\left\langle \begin{matrix} n \longleftarrow 2 \\ x_1 \longleftarrow 1 \\ x_2 \longleftarrow (-1)x \\ x_2 \longleftarrow y \end{matrix} \right\rangle
\end{aligned}
$$

1

$$
\frac{\left(x_1 + x_2 + \cdots + x_p\right)^n = \sum_{n_1+n_2+\cdots+n_p=n} \binom{n}{n_1 \quad n_2 \quad \dots \quad n_p} \left(x_1{}^{n_1}\right)\left(x_2{}^{n_2}\right) \dots \left(x_p{}^{n_p}\right)}{\Big(1 + (-1)x + y\Big)^2 = \sum_{n_1+n_2+\cdots+n_p=2} \binom{2}{n_1 \quad n_2 \quad n_3} \left(1^{n_1}\right)\left(\left((-1)x\right)^{n_2}\right)\left(y^{n_3}\right)} \left\langle \begin{matrix} p \longleftarrow 3 \\ n \longleftarrow 2 \\ x_1 \longleftarrow 1 \\ x_2 \longleftarrow (-1)x \\ x_2 \longleftarrow y \end{matrix} \right\rangle
$$

2

$$
\begin{aligned}
2 \quad &= \quad 0 + 0 + 2 \quad = \quad 0 + 1 + 1 \\
&= \quad 0 + 2 + 0 \quad = \quad 1 + 0 + 1 \\
&= \quad 2 + 0 + 0 \quad = \quad 1 + 1 + 0
\end{aligned}
$$

R Note that we consider these decompositions in 'lexicographic' ordering using the alphabet $\{0, 1, 2\}$. The ordereing is: $002, 011, 020, 101, 110, 200$. We shall write the terms in this order first. But, before we do so, we shall compute all the coefficients. This is an intermediate step.

3

$$\left(1 + (-1)x + y\right)^2$$

$$\begin{array}{llll}
= & \binom{2}{0\;\,0\;\,2} 1^0\left((-1)x\right)^0 y^2 & = & yy \\
+ & \binom{2}{0\;\,1\;\,1} 1^0\left((-1)x\right)^1 y^1 & + & (-1)12xy \\
+ & \binom{2}{0\;\,2\;\,0} 1^0\left((-1)x\right)^2 y^0 & + & xx \\
+ & \binom{2}{1\;\,0\;\,1} 1^1\left((-1)x\right)^0 y^1 & + & 2y \\
+ & \binom{2}{1\;\,1\;\,0} 1^1\left((-1)x\right)^1 y^0 & + & (-1)2x \\
+ & \binom{2}{2\;\,0\;\,0} 1^1\left((-1)x\right)^1 y^0 & + & 1
\end{array}$$

$$\begin{array}{llll}
= & 1 & = & 1 \\
+ & (-1)2x & + & (-1)2x \\
+ & xx & + & x^2 \\
+ & (-1)2xy & + & (-1)2xy \\
+ & 2y & + & 2y \\
+ & yy & + & y^2
\end{array}$$

R The above should be listed as columns side by side with vertical line drawn below the signs of equality to separate the columns.

R The above procedure implicitly uses various orderings and produces a normal form for polynomial expressions. Note that we first compare terms by exhibiting them explicitly in expanded form. But we write the final expression in condensed form by using exponential notation, for example, for instance, x^2 rather than x

R **The answer should be displayed in the form**:

$$
\begin{aligned}
\left(1+(-1)x+y\right)^2 &= \quad 1 \\
&+ \quad (-1)2x \\
&+ \quad x^2 \\
&+ \quad (-1)2xy \\
&+ \quad 2y \\
&+ \quad y^2
\end{aligned}
$$

R **The calculations for the above appear later. All calculations must be exhibited exactly as carried out here. The different pieces should be indexed with numbers so that the order in which the calculations are made is absolutely clear.**

R **The following ordering is adopted on the set of terms**. Note that within each term or word, the ordering adopted is numero-alphabetical, that is, numbers precede letters that are ordered alphabetically. The only negative number allowed is -1. -1 precedes all other numbers. The words so formed are then ordered lexicographically (as in the dictionary, again, with numbers by themselves coming first) after suppressing the numerical coefficients at the beginning of the terms. Please pay attention to this bookkeeping scheme.

30

$$\begin{aligned} & \binom{2}{0\quad 0\quad 2} \\ = \quad & \binom{2}{0\quad 2\quad 0} \\ = \quad & \binom{2}{0\quad 2\quad 0} \\ = \quad & \frac{\cancel{2!}}{(0!)\cancel{(2!)}(0!)} \\ = \quad & \frac{1}{(1)(1)(1)} \\ = \quad & 1 \end{aligned}$$

31

$$\begin{aligned} & \binom{2}{0\quad 1\quad 1} \\ = \quad & \binom{2}{1\quad 0\quad 1} \\ = \quad & \binom{2}{1\quad 1\quad 0} \\ = \quad & \frac{2!}{(1!)(1!)(0!)} \\ = \quad & \frac{2}{(1)(1)(1)} \\ = \quad & 2 \end{aligned}$$

40

$$\begin{aligned} & \binom{2}{0\quad 0\quad 2} 1^0 \big((-1)x\big)^0 y^2 \\ = \quad & (1)1(1)yy \\ = \quad & yy \end{aligned}$$

41

$$\begin{aligned} & \binom{2}{0\quad 1\quad 1} 1^0 \big((-1)x\big)^1 y^1 \\ = \quad & (2)\ 1(-1)xy \\ = \quad & (-1)2xy \end{aligned}$$

42

$$\begin{aligned} & \binom{2}{0\quad 2\quad 0} 1^0 \big((-1)x\big)^2 y^0 \\ = \quad & (1)1xx1 \\ = \quad & xx \end{aligned}$$

43

$$\begin{aligned} & \binom{2}{1\quad 0\quad 1} 1^1 \big((-1)x\big)^0 y^1 \\ = \quad & (2)\ 1(1)y \\ = \quad & 2y \end{aligned}$$

44

$$\begin{aligned} & \binom{2}{1\quad 1\quad 0} 1^1 \big((-1)x\big)^1 y^0 \\ = \quad & (2)1(-1)x1 \\ = \quad & (-1)2x \end{aligned}$$

45

$$\begin{aligned} & \binom{2}{2\quad 0\quad 0} 1^2 \big((-1)x\big)^0 y^0 \\ = \quad & (1)\ 1(1)1 \\ = \quad & 1 \end{aligned}$$

<u>The Binomial Theorem</u>

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k}$$

We list below various points related to the binomial theorem

<u>The Binomial Theorem and various consequences</u>

Note that Pascal's triangle, a part of which is exhibited below, is symmetric about the central vertical line.

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
⋮																

The identity:

$$\forall n \in 0..n, \forall k \in 0..k \quad \binom{n}{k} = \binom{n}{n-k}$$

guarantees the symmetry of Pascal's triangle about the central vertical line.

The identity:

$$\forall n \in 0..n, \forall k \in 0..k \quad \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

guarantees that every entry is obtained by adding the entries immediately to the left and right on the preceding row, provided there is one, as seen from the n^{th} row and the from the $(n+1)^{st}$ row as exhibited below.

Various other combinatorial identities are listed later.

nth row and (n +1)st row

	$\binom{n}{0}$		$\binom{n}{1}$	…		$\binom{n}{k}$		$\binom{n}{k+1}$	…		$\binom{n}{n-1}$		$\binom{n}{n}$	
$\binom{n+1}{0}$		$\binom{n+1}{1}$		…	$\binom{n+1}{k}$		$\binom{n+1}{k+1}$		…	$\binom{n+1}{n-1}$		$\binom{n+1}{n}$		$\binom{n+1}{n+1}$

Table with row entries written from left to right and right to left

								$\binom{0}{0}$								
							$\binom{1}{0}$		$\binom{1}{1}$							
							$\binom{1}{1}$		$\binom{1}{0}$							
						$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$						
						$\binom{2}{2}$		$\binom{2}{1}$		$\binom{2}{0}$						
					$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$					
					$\binom{3}{3}$		$\binom{3}{2}$		$\binom{3}{1}$		$\binom{3}{0}$					
				$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$				
				$\binom{4}{4}$		$\binom{4}{3}$		$\binom{4}{2}$		$\binom{4}{1}$		$\binom{4}{0}$				
			$\binom{5}{0}$		$\binom{5}{1}$		$\binom{5}{2}$		$\binom{5}{3}$		$\binom{5}{4}$		$\binom{5}{5}$			
			$\binom{5}{5}$		$\binom{5}{4}$		$\binom{5}{3}$		$\binom{5}{2}$		$\binom{5}{1}$		$\binom{5}{0}$			
		$\binom{6}{0}$		$\binom{6}{1}$		$\binom{6}{2}$		$\binom{6}{3}$		$\binom{6}{4}$		$\binom{6}{5}$		$\binom{6}{6}$		
		$\binom{6}{5}$		$\binom{6}{5}$		$\binom{6}{4}$		$\binom{6}{3}$		$\binom{6}{2}$		$\binom{6}{1}$		$\binom{6}{0}$		
⋮																

Note that the same identity:

$$\forall n \epsilon\ 0..n, \forall k \in 0..k \quad \binom{n}{k} = \binom{n}{n-k}$$

again guarantees that every entry in the same column is equal to the entry in the cell directly below it in the two tables below.

(2n)th row

$\binom{2n}{0}$		$\binom{2n}{1}$		$\binom{2n}{2}$	$\cdots$	$\binom{2n}{n}$	$\cdots$	$\binom{2n}{2n-2}$		$\binom{2n}{2n-1}$		$\binom{2n}{2n}$
$\binom{2n}{2n}$		$\binom{2n}{2n-1}$		$\binom{2n}{2n-2}$	$\cdots$	$\binom{2n}{n}$	$\cdots$	$\binom{2n}{2}$		$\binom{2n}{1}$		$\binom{2n}{0}$

(2n +1)st row

$\binom{2n+1}{0}$	$\binom{2n+1}{1}$	$\binom{2n+1}{2}$	$\cdots$	$\binom{2n+1}{n}$	$\binom{2n+1}{n+1}$	$\cdots$	$\binom{2n+1}{2n-1}$	$\binom{2n+1}{2n}$	$\binom{2n+1}{2n+1}$
$\binom{2n+1}{2n+1}$	$\binom{2n+1}{2n}$	$\binom{2n+1}{2n-1}$	$\cdots$	$\binom{2n+1}{n+1}$	$\binom{2n+1}{n}$	$\cdots$	$\binom{2n+1}{2}$	$\binom{2n+1}{1}$	$\binom{2n+1}{0}$

The above considerations allow us to write the binomial theorem in certain forms that are convenient for certain applications.

We first note that:

$$\frac{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k}{(a+b)^n = (b+a)^n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k}\, a^k = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$$

The above inference allows us to carry out the calculations of the following pages:

0	1	2	3	4
$(a+b)^{2n}$	=		=	
		$\binom{2n}{0}$ a^{2n}		$\binom{2n}{0}$ b^{2n}
	+	$\binom{2n}{1} a^{2n-1}b$	+	$\binom{2n}{1} b^{2n-1}a$
	+	$\binom{2n}{2} a^{2n-2}b^2$	+	$\binom{2n}{2} b^{2n-2}a^2$
	+	$\vdots$	+	$\vdots$
	+	$\binom{2n}{n} a^n b^n$	+	$\binom{2n}{n} b^n a^n$
	+	$\vdots$	+	$\vdots$
	+	$\binom{2n}{2n-2} a^2 b^{2n-2}$	+	$\binom{2n}{2n-2} b^2 a^{2n-2}$
	+	$\binom{2n}{2n-1} ab^{2n-1}$	+	$\binom{2n}{2n-2} ba^{2n-1}$
	+	$\binom{2n}{2n} b^{2n}$	+	$\binom{2n}{2n} a^{2n}$

Please check very carefully for errors

Adding columns 2 and 4 in the previous table and dividing by 2, we have the desired formula:

1	2		3
	$(a+b)^{2n}$		
=	$\binom{2n}{0}(\mathrm{a}^{2n-2(0)}+\mathrm{b}^{2n-2(0)})$	=	$\binom{2n}{0}(\mathrm{a}^{2n}+\mathrm{b}^{2n})$
+	$\binom{2n}{1}(ab)^1(a^{2n-2(1)}+b^{2n-2(1)})$	+	$\binom{2n}{1}(ab)^1(a^{2n-2}+b^{2n-2})$
+	$\binom{2n}{2}(ab)^2(a^{2n-2(2)}+b^{2n-2(2)})$	+	$\binom{2n}{2}(ab)^2(a^{2n-4}+b^{2n-4})$
+		+	
+	$\vdots$	+	$\vdots$
+		+	
+	$\binom{2n}{n-2}(ab)^{n-1}(a^{2n-2(n-2)}+b^{2n-2(n-2)})$	+	$\binom{2n}{n-2}(ab)^{n-1}(a^4+b^4)$
+	$\binom{2n}{n-1}(ab)^{n-2}(a^{2n-2(n-1)}+b^{2n-2(n-1)})$	+	$\binom{2n}{n-1}(ab)^{n-2}(a^2+b^2)$
+	$\binom{2n}{n}(ab)^n(a^{2n-2(n)}+b^{2n-2(n)})$	+	$\binom{2n}{n}(ab)^n(a^0+b^0)$

The general formula is:

$$\boxed{(a+b)^{2n} = \left(\sum_{k=0}^{n} \binom{2n}{k}\left((ab)^k\left(a^{2n-2k} + b^{2n-2k}\right)\right)\right) - \binom{2n}{n}}$$

Making the substitution: $b \leftarrow (-b)$, we get:

$$\boxed{(a-b)^{2n} = \left(\sum_{k=0}^{n} \binom{2n}{k}(-1)^k\left((ab)^k\left(a^{2n-2k} + b^{2n-2k}\right)\right)\right) - \binom{2n}{n}}$$

0	1	2	3	4
$(a+b)^{2n+1}$	=	a^{2n+1}	=	b^{2n+1}
	+	$\binom{2n+1}{1}a^{2n}b$	+	$\binom{2n+1}{1}b^{2n}a$
	+	$\binom{2n+1}{2}a^{2n-1}b^2$	+	$\binom{2n+1}{2}b^{2n-1}a^2$
	+	$\vdots$	+	$\vdots$
	+	$\binom{2n+1}{n}a^{n+1}b^n$	+	$\binom{2n+1}{n}b^{n+1}a^n$
	+	$\binom{2n+1}{n+1}a^n b^{n+1}$	+	$\binom{2n+1}{n+1}b^n a^{n+1}$
	+	$\vdots$	+	$\vdots$
	+	$\binom{2n+1}{2n-2}a^2 b^{2n-1}$	+	$\binom{2n+1}{2n-2}b^2 a^{2n-1}$
	+	$\binom{2n+1}{2n-1}ab^{2n}$	+	$\binom{2n}{+12n-1}ba^{2n}$
	+	$\binom{2n+1}{2n}b^{2n+1}$	+	$\binom{2n+1}{2n}a^{2n+1}$

Adding columns 2 and 4 and dividing by 2, we have the desired formula:

	2		3
	$(a+b)^{2n+1}$		
=	$\binom{2n+1}{0}(\mathrm{a}^{2n-2(0)+1}+\mathrm{b}^{2n-2(0)+1})$	=	$\binom{2n+1}{0}(\mathrm{a}^{2n+1}+\mathrm{b}^{2n+1})$
+	$\binom{2n+1}{1}(ab)^1(a^{2n-2(1)+1}+b^{2n-2(1)+1})$	+	$\binom{2n+1}{1}(ab)^1(a^{2n-1}+b^{2n-1})$
+	$\binom{2n+1}{2}(ab)^2(a^{2n-2(2)+1}+b^{2n-2(2)+1})$	+	$\binom{2n+1}{2}(ab)^2(a^{2n-3}+b^{2n-3})$
+	$\vdots$	+	$\vdots$
+	$\binom{2n+1}{n-1}(ab)^{n-1}(a^{2n-2(n-1)+1}+b^{2n-2(n-1)+1})$	+	$\binom{2n+1}{n-1}(ab)^{n-1}(a^3+b^3)$
+	$\binom{2n+1}{n}(ab)^n(a^{2n-2(n)+1}+b^{2n-2(n)+1})$	+	$\binom{2n+1}{n}(ab)^n(a^1+b^{1)})$

The general formula is:

$$\boxed{(a+b)^{2n+1}=\sum_{k=0}^{n}\binom{2n+1}{k}\left((ab)^k(a^{2n-2k+1}+\ b^{2n-2k+1})\right)}$$

Please check very carefully for errors

Making the substitution $b \leftarrow (-b)$, we get:

$$\boxed{(a-b)^{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{k} \left((-1)^k (ab)^k (a^{2n-2k+1} - b^{2n-2k+1})\right)}$$

We also get the following formulas:

$$\boxed{(a+b)^{2n} + (a-b)^{2n} = \left(\sum_{k=0}^{n} \binom{2n}{k} \left((ab)^k (1 + (-1)^k)(a^{2n-2k} + b^{2n-2k})\right)\right) - 2\binom{2n}{n}}$$

$$\boxed{(a+b)^{2n} - (a-b)^{2n} = \left(\sum_{k=0}^{n} \binom{2n}{k} \left((ab)^k (1 - (-1)^k)(a^{2n-2k} + b^{2n-2k})\right)\right)}$$

$$\boxed{(a+b)^{2n+1} + (a-b)^{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{k} (1 + (-1)^k)\left((ab)^k (a^{2n-2k+1} + b^{2n-2k+1})\right)}$$

$$\boxed{\begin{gathered}(a+b)^{2n+1} - (a-b)^{2n+1} \\ = \sum_{k=0}^{n} \binom{2n+1}{k} \left((ab)^k \left((1 + (-1)^k) a^{2n-2k+1} + (1 - (-1)^k) b^{2n-2k+1}\right)\right)\end{gathered}}$$

Making the substitution $a \leftarrow x$ and $b \leftarrow x^{-1}$, since $ab \leftarrow 1$ we get:

$$\boxed{(x+x^{-1})^{2n} = \left(\sum_{k=0}^{n} \binom{2n}{k}\left(\left((x)^{2n-2k} + (x)^{-(2n-2k)}\right)\right)\right) - \binom{2n}{n}}$$

$$\boxed{(x-x^{-1})^{2n} = \left(\sum_{k=0}^{n} \binom{2n}{k}(-1)^k\left(\left((x)^{2n-2k} + (x)^{-(2n-2k)}\right)\right)\right) - \binom{2n}{n}}$$

$$\boxed{(x+x^{-1})^{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{k}\left(\left(x^{2n-2k+1} + x^{-(2n-2k+1)}\right)\right)}$$

$$\boxed{(x-x^{-1})^{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{k}\left((-1)^k\left(x^{2n-2k+1} - x^{-(2n-2k+1)}\right)\right)}$$

Combinatorial Identities

$$\forall m, n, p \in \mathbb{N}, \forall k \in 0..n, \forall l \in 0..k$$

0. Symmetry $\binom{n}{k} = \binom{n}{n-k}$

1. Absorption $\binom{n}{k} = \left(\frac{n}{k}\right)\binom{n-1}{k-1} \quad (k \neq 0)$

2. Trinomial Revision $\binom{n}{l}\binom{n-l}{k-l} = \binom{k}{l}\binom{n}{k}$

3. Parallel Summation $\sum_{k=0}^{n}\binom{m+k}{k} = \binom{m+n+1}{n}$

4. Upper Summation $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1} \qquad 0 \leq m \leq n$

5. Vandermonde Convolution $\sum_{k=0}^{n}\binom{m}{k}\binom{p}{n-k} = \binom{m+p}{n} \quad n \leq m+p$

6. Addition $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Please check very carefully for errors