Review is based on:

A) Review of nonlocality: https://arxiv.org/pdf/1303.2849.pdf (ground work)
B) Example of using Bell inequalities to discriminate between causal structures: https://journals.aps.org/pra/pdf/10.1103/PhysRevA.95.022111(causal structure = how entanglement/LHV/postquantum states are distributed between devices)
C) Computational technique that we will use: https://arxiv.org/pdf/1609.00672.pdf
D) Classification of Bell inequalities: https://arxiv.org/pdf/1404.1306.pdf
E) New definition of causal structures for multipartite entanglement: see WIP drafts

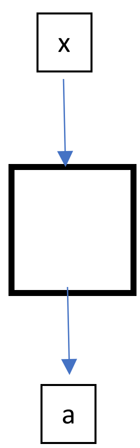
Bell Nonlocality Review (Apr 2014)

Nonlocal Correlations

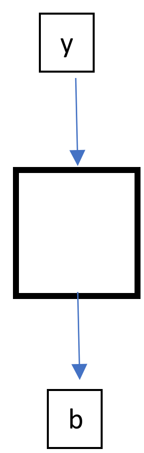
Definitions

Two observers:

Alice



Bob



$$a, b \in \{1, \ldots\ldots, \Delta\}$$
$$x, y \in \{1, \ldots\ldots, m\}$$

Joint probabilities are completely characterized by $\Delta^2 m^2$ each possible pair of inputs to each possible pair of outputs. The sum of the joint probabilities is a behavior and each individual probability is known as a correlation that is part of $\mathbb{R}^{\Delta^2 m^2}$. These are defined by the relativity constraints and the normalization constraints as follows $p(ab|xy) \geq 0$ *and* $\sum_{a,b=1}^{\Delta} p(ab|xy) = 1$. Due to normalization the probability space P *is a subspace of* $\mathbb{R}^{\Delta^2 m^2}$, $\dim P = (\Delta^2 - 1)m^2$.

There are three main types of correlations that can be obtained from using a physical model to describe the correlations obtained in a Bell Scenario.

1) No-signaling Correlations
    a. These correlations arise from the following constraints:
        i. $\sum_{b=1}^{\Delta} p(ab|xy) = \sum_{b=1}^{\Delta} p(ab|xy')\ for\ all\ a, x, y, y'$
        ii. $\sum_{a=1}^{\Delta} p(ab|xy) = \sum_{a=1}^{\Delta} p(ab|x'y)\ for\ all\ b, y, x, x'$
    b. The dimension of the affine subspace generate by the no-signaling correlations is $dimNS = 2(\Delta - 1)\text{m} + (\Delta - 1)^2 m^2 =: t$
    c. The physical interpretation is that the local marginal probabilities of Alice are independent of the measurement setting y of Bob's device (cannot signal)
    d. For the binary case ($\Delta = 2$), $dimNS = 2m + m^2$
        i. $\langle A_x \rangle = \sum_{a\ \in\{\pm 1\}} ap(a|x)$
        ii. $\langle B_y \rangle = \sum_{b\ \in\{\pm 1\}} bp(b|y)$
        iii. $\langle A_x B_y \rangle = \sum_{a,b\ \in\{\pm 1\}} abp(ab|xy)$
    e. Joint probabilities and correlators are related as follows:
        i. $p(ab|xy) = \left[1 + a\langle A_x \rangle + b\langle B_y \rangle + ab\langle A_x B_y \rangle\right]/4$
        ii. Therefore $1 + a\langle A_x \rangle + b\langle B_y \rangle + ab\langle A_x B_y \rangle \geq 0$
2) Local Correlations
    a. $p(ab|xy) = \int_{\Lambda} d\lambda q(\lambda) p(a|x,\lambda) p(b|y,\lambda)$
    b. λ is an arbitrary variable taking a value from Λ space and distributed according to the probability distribution q(λ) where $p(a|x,\lambda) and\ p(b|y,\lambda)$ are local response function for Alice and Bob, lambda is like shared randomness. Local behavior satisfies the no-signaling constraint but not all no-signaling constraints satisfy the locality constraints. L is a subset of NS.
3) Quantum Behaviors
    a. $p(ab|xy) = tr(\rho_{AB} M_{a|x} \otimes M_{b|y})$
        i. ρ_{AB} is a quantum state in a joint Hilbert space
        ii. $H_A \otimes H_B$ – the hilbert space
        iii. $M_{a|x}\ and\ M_{b|y}$ are the measurement operators
    b. Tensor Product Definition (Q): $p(ab|xy) = \langle \psi \mid M_{a|x} \otimes M_{b|y} \mid \psi \rangle$ given that the operators are orthogonal
        i. $M_{a|x} M_{a'|x} = \delta_{aa'} M_{a|x}$ and $\sum_a M_{a|x} = 1_A$
    c. Commutation Definition (Q'): $p(ab|xy) = \langle \psi \mid M_{a|x} M_{b|y} \mid \psi \rangle$
        i. $[M_{a|x}, M_{b|y}] = 0$
    d. $Q \subseteq Q'$
    e. $\dim L = \dim Q = \dim NS = t$

<u>Bell Inequalities</u>

Sets L, Q, and NS are closed, bounded, and convex.

Hyperplane separation theorem describes that for each behavior $\hat{p} \in R^t$ that is not part of the sets L, Q, or NS, there exists a hyperplane that separates this vector from the corresponding set.

$$s \cdot p = \sum_{abxy} s_{xy}^{ab} p(ab|xy) \leq S_k$$

This inequality is satisfied by all $p \in \mathcal{K}$ and is not satisfied for other values of p. For the local set L, such inequalities are known as *Bell Inequalities*. The inequalities for the quantum set are often called *quantum Bell* inequalities or *Tsirelson* inequalities.

1) Local Polytope
   a. To identify the hyperplanes that characterize the set L we express the local correlations as probability functions of a hidden variable. This hidden variable can also have a randomized effect. Thus we can defined a hidden variable as follows where $\lambda' = (\lambda, \mu_1, \mu_2)$

$$p'(a|x, \lambda') = \begin{cases} 1 & if\ F(a-1|x,\lambda) \leq \mu_1 < F(a|x,\lambda) \\ 0 & otherwise \end{cases}$$

   Where

$$F(a|x,\lambda) = \sum_{\tilde{a} \leq a} p(\tilde{a}|x,\lambda)$$

   We choose $q'(\lambda') = q'(\lambda, \mu_1, \mu_2) = q(\lambda)$

   This new model is stochastic but deterministic.

   b. Rephrase the local model as $\lambda = (a_1, \dots, a_m;\ b_0, \dots, b_m)$

$$d_\lambda(ab|xy) = \begin{cases} 1 & if\ a = a_x\ and\ b = b_y \\ 0 & otherwise \end{cases}$$

   There are Δ^{2m} possible output assignments and therefore Δ^{2m} local deterministic behaviors.

   A behavior p is local if it can be written as a convex combination of these deterministic points.

$$p = \sum_\lambda q_\lambda d_\lambda\ with\ q\lambda \geq 0, \sum_\lambda q_\lambda = 1$$

   This is typically a ***linear programming*** problem.

   c. **Linear programming** form of the convex combination:

$$\begin{matrix} \max(s, s_l) \\ s.t. \end{matrix} \begin{cases} s \cdot p - S_l \\ s \cdot d_\lambda - S_l \leq 0, \quad \lambda = 1, \dots, \Delta^{2m} \\ s \cdot p - S_l \leq 1 \end{cases}$$

If p is local the maximum, the maximum S of the program is S <= 0. If p is non-local, the maximum is S = 1.

Since the set L is the convex hull of a finite number of points, it is a *polytope*.

d. Minkowski's theorem states that a polytope can, equivalently to the representation as the convex hull of it vertices be represented as the intersection of finitely many half-spaces.
   i. $p \in L \quad iff \quad s^i \cdot p \leq S_l^i \quad \forall i \in I$
   ii. I indexes a finite set of linear inequalities. Thus, the set L can be defined by a finite number of Bell Inequalities.

2) Facet Bell Inequalities
   a. $s \cdot p \leq S_l$ is a valid inequality for the polytope L, then $F = \{p \in L \mid s \cdot p = S_l\}$ is called a face of L. Face of dimension dim F = dim L-1 = t-1 are called facets of L and the corresponding inequalities are called facet Bell inequalities. The term "tight bell inequalities" is also used. Provide a minimal representation of the set L. Any bell inequality can be written as a non-negative combination of facet inequalities.
   b. **Computer Science aspect:** The problem of determining the facet inequalities from a set of vertices is called the convex hull problem (can use computer codes cdd and porta for relatively simple cases). This problem becomes hard for larger m's and same goes for identifying the points of nonlocality. It was proven that deciding whether a behavior is local for a class of Bell scenarios with binary outputs and m inputs is NP-complete.

3) Examples of bell inequalities:
   a. For $\Delta = 2 \quad m = 2$ the CHSH completely defines the set of bell inequalities because there is only one facet inequality needed to describe the scenario.
   b. For $\Delta = 2 \quad m = 3$ was computationally solved in 1981 and contains the CHSH inequality along with $p_1^A + p_1^B - p_{11} - p_{12} - p_{13} - p_{21} - p_{31} - p_{22} - p_{23} + p_{32} \geq -1$
      i. This additional inequality is known as the Froissard inequality, and it can be generalized for the case of an arbitrary number of measurements m with binary outcomes, a family known as I_{mm22}.
   c. For $\Delta = \mathrm{x} \quad m = 2$, the following inequality exists $[a_1 - b_1] + [b_1 - a_2] + [a_2 - b_2] + [b_2 - a_1 - 1] \geq d - 1$
      i. Where $[a_x - b_y] = \sum_{j=0}^{\Delta-1} jp(a - b = j mod \Delta | \mathrm{xy})$
      ii. What is d?
      iii. This inequality is known as the CGLMP inequality. It is shown to be facet defining for all Δ
   d. The above inequality can be extended to an arbitrary number of inputs:
      i. $[a_1 - b_1] + [b_1 - a_2] + [a_2 - b_2] + \cdots + [a_m - b_m] + [b_m - a_1 - 1] \geq d - 1$
      ii. Not a facet inequality
   e. For m=2 and m=3, number of inequalities is 1 and 2, for m = 4, unknown, and for m=10 at least 44,368,793 (gross underestimate!!!)
   f. Many bell inequalities have been solved using the convex hull problem for small values of Δ and m. A relationship between $\Delta = 2$ and higher dimensional polytopes like the cut polytope can be used to identify large families of bell inequalities. Methods exploiting symmetries to generate Bell inequalities are also used.
      i. What are quadratic inequalities?

4) Nonlocal games (Another name for bell inequalities) → Interactive proof systems (w/wo entanglement)
   a. When talking about a game:
      i. There is a referee (outside party), parties or systems are referred to as *players*. Referee is also known as the *verifier* and the players as *provers*. Referee chooses a question $x \epsilon X$ for Alice and $y \epsilon Y$ for Bob according to some probability distribution $\pi: X \times Y \rightarrow \{0,1\}$ upon receiving x and y, Alice and Bob return an answer a and b that belong to the set of possible answers R_A and R_B.
      ii. The referee then decides if the answers are winning according to the rules.
      iii. Rules are expressed as $V: R_A \times R_B \times X \times Y \rightarrow \{0,1\}$ where V(a,b,x,y) = 1 iff Alice and Bob win against the referee by giving answers a,b for questions x and y.
      iv. Cannot share information once the game starts, in the classical setting the strategy is known as shared randomness, which is known as local hidden variables.
   b. Relation between games and inequalities: questions are measurements (we assume the questions between the two systems are the same), answers are measurement outcomes.
   c. Probability that Alice and Bob win:

$$p_{win} = \sum_{x,y} \pi(x,y) \sum_{a,b} V(a,b|x,y) p(a,b|x,y)$$

   d. Maximum winning probability: $\max p_{win} = S_l \quad p_{win} = s \cdot p \leq S_l$
      Where the coefficients are given by
      $s_{x,y}^{a,b} = \pi(x,y) V(a,b|x,y)$
   e. Hence games form a subset of general Bell Inequalities
5) Types of games
   a. XOR games:
      i. Equivalent to *correlation Bell inequalities* with binary outcomes.
      ii. What are correlation inequalities?
      iii. REVIEW!!!
   b. Projection and unique games
      i. A projection game is such that for every pair of questions x and y to Alice and Bob, and for every answer a of Alice, there exists a unique winning answer for Bob. In quantum information literature, there are often called *unique games.*
      ii. k-to-k' games (a projection game is k = k' = 1)
   c. Other classes of games:
      i. A linear game is a game for the set of possible answers is associated with an Abelian group G of size Δ, find a function $W: \{1, \dots, m\}^{x2} \rightarrow G$ such that V(a,b|x,y) = 1 iff a - b = W(x,y)
         1. Any linear game is a unique game and has been show to be a uniform game.

2. CHSH is both free and symmetric
ii. Kochen-Specker or GHZ games
1. Optimal quantum strategy yields

Bell Inequality Violations (Quantum and No-signaling)

1) Quantum Bounds
a. Properties of Quantum Correlations
i. In general $S_q > S_l$
ii. Two requirements for a quantum behavior to be nonlocal (non-commutativity and entanglement)
1. Alice's different measurements must be non-commuting and vice versa
2. State ρ must be entangled
iii. Not a polytope (Can't be described by external points)
iv. All extremal points of L are extremal points of Q
v. Certain faces of L are also faces of Q
1. Example: $\Delta - 1$-dimensional face associated to the hyperplanes p(ab|xy) = 0
vi. The local and quantum states may have common faces that are only violated by NS (still an open question)
vii. In m=2 and m=3 all non-local extremal behaviors are unique
viii. $S_q = \max_{\boldsymbol{p} \in \boldsymbol{Q}} \boldsymbol{s} \cdot \boldsymbol{p} = \max_{S} ||S||$

$$S = \sum_{abxy} s_{xy}^{ab} M_{a|x} M_{b|y}$$

$\left||S|\right|$ denotes the spectral norm (largest eigenvalue) of S. This is done over all possible Bell operators S associated with s. In the case of the CHSH expression, the Bell operator takes the form $S = \widehat{A_1}\left(\widehat{B_1} + \widehat{B_2}\right) + \widehat{A_2}(\widehat{B_1} - \widehat{B_2})$ where $\widehat{A_x}, \widehat{B_y}$ are arbitrary +- 1-eigenvalued observables

Why is the last term -$\widehat{B_2}$?

$$S^2 = 4 + [\widehat{A_1}, \widehat{A_2}][\widehat{B_1}, \widehat{B_2}]$$

$$\left||S^2|\right| \leq 8 \left||S|\right| \leq 2\sqrt{2}$$

NP-hard in tripartite case
b. Correlation Inequalities
i. Tsirelson showed that m^2correlators are quantum iff there exist 2m unit vectors $\langle A_x B_y \rangle = \hat{v}_x \cdot \hat{w}_y$
ii. This is useful because this problem can be cast as a semidefinite program (SDP)
iii. In the $\Delta = 2, m = 2$, this SDP approach can be used to yield a complete description of $Q \cap C$
iv. For CHSH $|asin\langle A_1 B_1 \rangle + asin\langle A_1 B_2 \rangle + \mathrm{asin}\langle A_2 B_1 \rangle - asin\langle A_2 B_2 \rangle| \leq \pi$
v. More information on solving inequalities in correlation space
1. Avis et al., 2009; Tsirelson, 1987, 1993
vi. It is much harder to gather the local bound for a correlation Bell inequality than the quantum bound
c. State and measurement dependent bounds

i. Simple approach: Gather the family of Bell operators S in Hilbert space H with dimH and maximize ||S|| over all operators.
    1. No guaranteed convergence
    2. Typically only provides lower bounds
    3. Good for finding quantum violations of Bell inequalities
ii. Easier to compute the quantum bound for a fixed quantum state
    1. Use explicit parameterization of the bell operators
    2. Can also use an iterative approach find eigenvector, repeat
iii. Exploit the fact that for a given quantum state a Bell expression is bilinear in measurement operators (linear in it's operator for A compared to fixed B operator and vice versa). Iterative approach: fix Alice's optimize Bob's, fix Bob's optimize over Alice's, back and forth till convergence. True for any Bell expression with binary outcomes.

d. General Bounds
    i. Finding the quantum bound is an instance of polynomial optimization.
    ii. Any polynomial optimization problem in commutative variables can be solved using a hierarchy of SDPs – Two methods have been shown by Lassere, 2001, and Parrilo 2003

    iii. These techniques yield a powerful approach to obtaining upper-bounds on S_q
    iv. Upper Bounds Method using k operators:
        1. Let O be a set of k operators consisting of the a and b operators together with finite products of them. Denote by O_i (i = 1, …, k) the elements of O and introduce the k x k matrix Γ with entries $\Gamma_{ij} = \langle\psi|O_i^\dagger O_j|\psi\rangle$, called the moment matrix associated to O.
            a. $\Gamma \geq 0$ is semidefinite positive
            b. The entries of Γ satisfy a series of linear inequalities
            c. The probabilities p(ab|xy) defining the behavior p correspond to a subset of the entries of Γ. Necessary condition for a behavior p to be quantum is therefore that there exist a moment matrix Γ with the above properties, can be determined using SDP.
            d. For any O, the set of behaviors p that belongs to probability space for which there exist such a moment matrix thus define a set Q_O that contains the quantum set Q'. Optimizing a Bell expression (which is linear in p) over this set Q_O is also a SDP and yields an upper-bound on S_q. Can provide decreasing approximations of the upper-bounds
    v. SDP Hierarchy:
        1. Relies on the fact that for any Bell operator S we have $\xi = \hat{b}_q 1_{matrix} - S \geq 0 \quad i.e. \left||S|\right| \leq \hat{b}_q$ iff polynomial ξ can be written as a weighted sum of squares of other polynomials
        2. If we limit the degree of these polynomials the problem can be cast as an SDP, better and better bounds for increasing values of l.

3. The hierarchy of SDP relaxations converges in the asymptotic limit to the set Q'. It is also possible to certify that a behavior p belongs to the quantum set Q or to obtain the optimal bound S_q of a Bell expression at a finite step in the hierarchy.
4. Can compare the SDP upper bounds to a lower bound.
5. Numerical examples show that low-order steps of the hierarchy approximate the bound well.

2) No-signaling bounds
   a. Easier task than the other two. The no-signaling behaviors are captured by the set of $\Delta^2 m^2$ positivity inequalities p(ab|xy) >= 0. Easily be verified by checking that all positivity inequalities are satisfied. Complexity of this scales polynomially with Δ, m.
   b. Linear programming can be used to identify S_{ns} to obtain a crude bound for S_q.
   c. NS is defined by a finite number of linear inequalities, it is a polytope, and can also be described as a convex hull. Can also use facets to describe the surface.
   d. The vertices of L are also vertices of NS since they cannot be written as a convex combination of any other behavior. All other vertices of NS are non-local.
   e. The geometry of the no-signaling set and its relation to L is particularly simple for the $\Delta = 2, \mathrm{m} = 2$ scenario.
      i. No-signaling forms an 8-dimensional subspace of the full probability space. The local polytope consists of 16 vertices, and 24 facets. 16 of these facets are positivity inequalities and 8 are different versions. No signaling polytope consists of 16 facets, the positivity inequalities, and 24 vertices. 16 of these vertices are local and the other 8 are non-local vertices.

Nonlocality and Quantum Theory

<u>Nonlocality vs. Entanglement</u>

Nonlocal correlations require an entangled state

State cannot be in

$$\rho AB = \sum_{\lambda} p_{\lambda}\, \rho_A^{\lambda} \otimes \rho_B^{\lambda}$$

Do all entangled states lead to nonlocality?

For pure states, yes, shown for two qubits and bipartite states of Hilbert space. Only pure states that do not violate the Bell inequalities are product states

For mixed states, it's more complicated,

1) Werner discovered a class of mixed entangled states which admit to a local model
2) Werner's observations were extended to general measurements

WHY?

The situation is complicated because directly measuring a mixed state ρ is not always the best way to reveal it's non-locality. It may be required that we must make multiple joint measurements and pre-processing. There exist different possible scenarios for revealing the nonlocality of mixed entangled states. Any state from which pure bipartite entanglement can be distilled will lead to nonlocality. For undistillable (or bound) bipartite entangled states it is not known whether Bell inequality violations can be obtained. Recent studies suggest nonlocality might in fact be generic for all entangled states.

Types of nonlocality:

1) Single copy nonlocality
   a. Mixed states:
      i. Associate ρ with a correlation matrix T_ρ with entries $t_{ij} = tr[\rho(\sigma_i \otimes \sigma_j)]$ for i,j = 1,2,3 where σ_i are the Pauli matrices
      ii. Maximum CHSH value S for ρ is $S_p = 2\sqrt{m_{11}^2 + m_{22}^2}$ where m_{11}^2 and m_{22}^2 are the two largest eigenvalues of the matrix $T_\rho T_\rho^T$
      iii. Werner state for two qubits for 16 dimensions:
      $$\rho_W = p|\phi_+\rangle\langle\phi_+| + (1-p)\frac{identity}{4}$$
      iv. For higher dimensional Hilbert space d
      $$\rho_W = p\frac{2P_{anti}}{d(d-1)} + (1-p)\frac{identity}{d}$$
         1. P_{anti} denotes the projector on the antisymmetric subspace
            a. These states have a particular symmetry, being invariant under unitary operations of the form $U \otimes U$
      v. Almeida expands Werner States to isotropic states
      $$\rho_{iso} = p|\Phi_+\rangle\langle\Phi_+| + (1-p)\frac{identity}{d^2}$$
         1. Where $|\Phi_+\rangle = (\frac{1}{\sqrt{d}})\sum_{i=0}^{d-1}|ii\rangle$ is a maximally entangled state of local dimension d
         2. Note: ρ_{iso} violates CGLMP, when p is above a critical value, p_{NL}. Decrease the local dimension d. In particular, p_{NL} → 0.67 as d → infinity
         3. Allows one to construct a local model for general states of isotropic form. It is found for $p \leq \Theta(\frac{\log(d)}{d^2})$ admits a local model for projective measurements. Upper bound $p \geq \Theta(\frac{4}{(\sqrt{2}-1)d})$ tends to zero as d → infinity.
         4. If a symmetric extension exists
      vi.

2) Hidden Nonlocality
   a. Each observer performs a sequence of measurements. First, some states that show a local model can display nonlocality if judicious filtering is applied before measurement. Popescu showed this occurs for Werner States and it shows hidden non-locality. Shared mixed entangled state with a mixture of local and nonlocal. To extract nonlocality, first a

local measurement is made for which the outcome can only occur for a nonlocal state in the mixture. By performing the appropriate local measurements they can violate a Bell inequality. Choose the measurement basis after a successful filter operation.

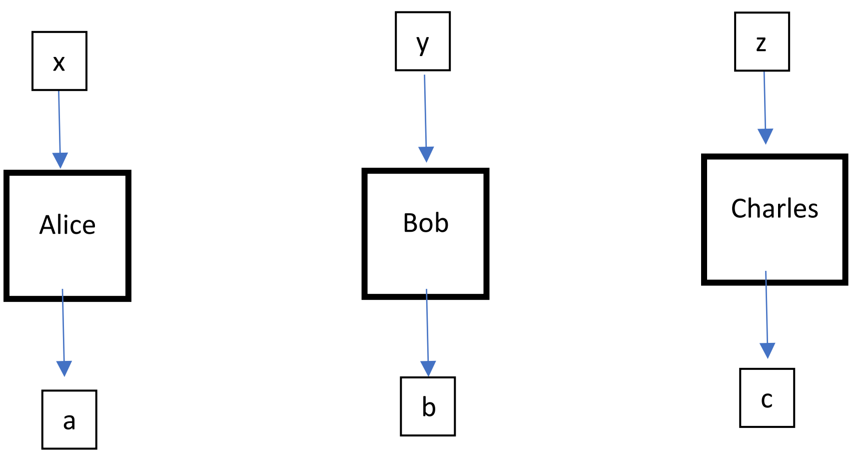
3) Multi-copy nonlocality
    a. Perform measurements on multiple copies of the state ρ. No initial filtering is allowed. The maximal violation of the CHSH inequality can be increased if several copies of the state are jointly measured, for specific states. Possibility of performing measurements on several copies of a state leads to a phenomenon known as activation of nonlocality.
    b. Quantum nonlocality can be superactivated, nonlocality is not an additive quantity. For every state that belongs to a d-dimensional complex Hilbert space, with singlet fidelity > 1/d, there exists a k such that the state measured over k copies introduces nonlocality. Every entangled isotropic state is a nonlocal resource and establishes a direct connection between the usefulness of a state in quantum teleportation and its nonlocality.
4) More general scenarios
    a. Distribution of a state across n observers, activation of nonlocality can also occur here. One or more observers can perform a joint measurement on several subsystems.
    b. Examples of activation of nonlocality in networks
        i. Concatenating multiple copies of a nonviolating CHSH state can cause violation of CHSH
        ii. Many copies of Werner states in a star network violate a Bell inequality for p >~ 0.64
5) Entanglement distillation and nonlocality
    a. Entanglement distillation followed by a standard (single-copy) Bell test
    b. Peres conjecture: every state with a positive partial transposition (PPT → undistillable) admits a local model
    c. State can be distilled if it violates an inequality for CHSH and Mermin inequalities
    d. Method for upper-bounding the possible violation of a given Bell inequality for PPT states shows that many bipartite states cannot be violated by PPT
    e. In the case of more parties, nonlocality does not imply distillability of entanglement, disproving the Peres conjecture in the multipartite case
6) Nonlocality and teleportation
    a. Quantum teleportation is a nonlocal phenomenon based on entanglement. Not every entangled state can be used for teleportation. Is a state's teleportability related to it's nonlocality.
    b. Popescu showed for a specific system two qubits Werner states that are part of a local model can be used for teleportation → Deemed teleportability and nonlocality unrelated
    c. The local teleportability vanishes when considering more general scenarios for revealing nonlocality. Recently shown that in the multi-copy scenario, any teleportable state is a nonlocal resource → Direct link between teleportation and nonlocality
    d. More qualitative relation between the amount of CHSH violation and teleportability was recently made. Maximal violation of the CHSH inequality is shown to lower bound the average fidelity for teleportation

$$F_{telep} \geq \frac{1}{2}(1 + \frac{S_\rho^2}{12})$$

Optimal classical strategy Ftelep = 2/3

7) More nonlocality with less entanglement
   a. Maximal entanglement for certain inequalities can only be achieved with partially entangled states
   b. There exist simple Bell inequalities the maximal violation of which cannot be obtained from maximally entangled states, requires partially entangled states.
   c. Some partially entangled give violations that are much larger than maximally entangled
   d. This phenomenon is known as the anomaly of nonlocality

Multipartite Nonlocality



<u>Defining multipartite nonlocality</u>

Refinements to nonlocality

a) Charles is uncorrelated to Alice and Bob. Correlations can clearly violate the locality condition can cause bipartite nonlocality.
b) Genuine multipartite nonlocality – All three are nonlocally correlated.

1) Genuine multipartite nonlocality (by Svetlinchy)

$$p(abc|xyz) = \int d\lambda q(\lambda) p_\lambda(ab|xy) p_\lambda(c|z) + \int d\mu q(\mu) p_\mu(bc|yz) p_\mu(a|x)$$
$$+ \int d\nu q(\nu) p_\nu(ac|xz) p_\nu(b|y)$$
$$where \int d\lambda q(\lambda) + \int d\mu q(\mu) + \int d\nu q(\nu) = 1$$

a. Convex combination of three terms, each term at most two of the parties are nonlocally correlated
b. If the three parties cannot be written in the above form, we say they are 3-way nonlocal

2) Beyond Svetlinchy's Model
   a. In the above tripartite example, the bipartite probability distributions are unconstrained aside from normalization. We have not imposed the no-signaling constraints. These conditions guarantee that even with lambda, Alice cannot send a message to Bob by choosing her measurement setting and vice versa. If at least one of the following is not satisfied it allows for signaling from Alice to Bob or vice versa:
   No signaling constraints:
   $$p_\lambda(a|xy) = p_\lambda(a|xy') \ \forall a, x, y, y'$$
   $$p_\lambda(b|xy) = p_\lambda(b|x'y) \ \forall a, y, x, x'$$
   b. Such signaling terms are inconsistent from a physical perspective and can lead to grandfather-type paradoxes.
   c. Svetlinchy's definition from the perspective of classical simulations of quantum correlations in terms of shared random data and communication. The decomposition corresponds to simulation models where all parties receive their measurement setting at the same time, there are several rounds of communication between only two of the parties, and finally all parties produce a measurement outcome.
   d. To address such shortcomings, there are two alternatives:
      i. Require all bipartite correlations in the decomposition to satisfy the no-signaling conditions. Set of such correlations is $s_{2|1}^{ns}$, correlations that cannot be written in this form are genuine tripartite nonlocal
      ii. Require time-ordering of bipartite correlations in decomposition. Specifically, $s_{2|1}^{to}$ of 2-way time-ordered correlations contains all distributions that can be written in the form
      $$p(abc|xyz) = \int d\lambda q(\lambda) p_\lambda^{T_{AB}}(ab|xy) p_\lambda(c|z)$$
      $$+ \int d\mu q(\mu) p_\lambda^{T_{AC}}(bc|yz) p_\mu(a|x)$$
      $$+ \int d\nu q(\nu) p_\lambda^{T_{BC}}(ac|xz) p_\nu(b|y)$$
      $$where\ p_\lambda^{T_{AB}}(ab|xy) = p_\lambda^{A>B}(ab|xy)\ or\ p_\lambda^{B>A}(ab|xy)$$
      Only allows for one way signaling → solves problem
      If cannot be described by above is genuine tripartite nonlocal
   e. The three relations are related like so:
   $$\mathcal{L} \subset S_{2|1}^{ns} \subset S_{2|1}^{to} \subset S_{2|1}^{svet}$$

While violation of Svetlinchy's always guarantees genuinely tripartite nonlocal, there are some correlations whose tripartite nature only manifests when considering the weaker cases.

Nonlocality of multipartite quantum states

1) Multipartite nonlocality vs multipartite entanglement
   a. All pure entangled n-partite states are nonlocal
      i. Follows from the fact that n-2 parties can always project the remaining two parties in a pure entangled state
   b. Genuine multipartite entanglement occurs when a quantum state cannot be decomposed as a convex combination of biseparable states
   c. The presence of genuine multipartite nonlocality witnesses the presence of genuine multipartite entanglement (device independent)
   d. It is not known if all pure genuine multipartite entangled states are genuine multipartite nonlocal
   e. All connected graph states are fully genuine nonlocal
   f. tangle (specific measure of multipartite entanglement) is closely related to the violation of Svetlinchy's inequality
   g. From this connection it can be shown that there exist pure entangled states in the GHZ class which do not violate Svetlinchy's inequality
   h. Connection between genuine multipartite entanglement and nonlocality may depend on which definition of genuine multipartite nonlocality is used
   i. For time-ordering and no-signaling, numerical evidence shows that all pure genuine tripartite entangled qubit states are genuine tripartite nonlocal
2) Greenberger-Horne-Zeilinger (GHZ) states
   a. GHZ display one of the most striking forms of nonlocality according to the Mermin-GHZ paradox
$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
   b. One can obtain correlations which are maximally nonlocal, interestingly these particular GHZ correlations do not feature genuine multipartite nonlocality, because they can be reproduced by a bi-separable model of the form. However, you can show the violation of Svetlinchy's inequality of
$$S_3 = 4\sqrt{2} > 4$$
   c. This violation can be intuitively understood by considering the form of Svetlinchy's inequality
   d. The correlations of generalized GHZ states of the form:
$$|GHZ_n^d\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle^{\otimes n}$$
   Featuring n parties and systems of local dimension d have also been investigated. Show full genuine multipartite nonlocal correlations, monogamous, and locally random. Analogues of Mermin-GHZ paradox were reported for combinations of n and d, and was

also presented for the case of continuous variable systems. All qubit GHZ states violate the generalization of Svetlinchy's inequality for an arbitrary number of parties and hence display genuine multipartite nonlocal correlations.

3) Graph States
   a. Form an important family of multipartite quantum states (including GHZ and cluster states). All codeword states used in the standard quantum error correcting codes correspond to graph states and one-way quantum computation uses graph states
   b. Definition:
      i. Let G be a graph featuring n vertices and a certain number of edges connecting them.
      ii. For each vertex i, we define neigh(i) as the neighborhood of i, set of vertices which are connected to i by an edge
      iii. Associate to each vertex i a stabilizing operator

$$g_i = X_i \bigotimes_{j \,\in\, neigh(i)} Z_j$$

      iv. Where Xi, Yi, and Zi denote the Pauli matrices applied to qubit i. The graph state |G> associated with the graph G is then the unique common eigenvector to all stabilizing operators g_i (i.e. g_i|G> = |G> for all i).
      v. $\langle G|g_i|G\rangle = 1\, for\ all\ i$ by
      vi. Create the stabilizer group for the stabilizer operators by creating all possible products of the operators, producing the commutative group with 2^n

$$S(G) = \{s_j\}_{j=1,\dots,2^n}\ where\ s_j = \prod_{i\,\in\, I_j(G)} s_i$$

$$where\ I_j(G)\ denotes\ any\ of\ the\ 2^n\ subsets\ of\ the\ vertices\ of\ the\ graph\ G$$

   c. This fundamental structure of graph states underpins a strong form of nonlocality. All graph states feature nonlocal correlations. Tree graphs the violation of Bell inequalities increases exponentially with the number of vertices.
   d. While the generality of the above approach is remarkable, it is possible for certain important classes of graph states to derive stronger proofs of nonlocality. Cluster states form a subclass of graph states based on square lattice graphs.
   e. In this discussion we have considered Bell nonlocality, much less is known about genuine multipartite nonlocality. However, it is known that if a graph is connected it will display fully genuine multipartite nonlocality.

4) Nonlocality of other multipartite quantum states
   a. Very little is known beyond the case of graph states
   b. An important class of multipartite entangled states are Dicke states, states with a fixed number of excitations and symmetric under permutation of the parties, important for light and matter. Symmetric state of n particles with a single excitation known as the W state is:

$$|W_n\rangle = \frac{1}{\sqrt{n}}(|0 \dots 01\rangle + \cdots + |10 \dots 0\rangle)$$

   c. Such states are relevant to quantum memories
   d. To detect nonlocality

    i. have n-2 parties performing a measurement in the logical basis {|0>, |1>}, when all project onto |0> they wait for the remaining two parties a Bell state on which the CHSH inequality can be tested and violated.
    ii. Another case of nonlocality based on robustness to losses. When k << n particles are lost, the state remains basically unchanged.
        1. For ex. $Tr_k(|W_n><W_n|)$ ~~ $|W_n><W_n|$ where tr_k denotes the partial trace on the k particles that are lost
        2. Has a high persistence of nonlocality

Correlations in a tripartite Bell scenario (Jul 2018)

Study the Mermin and GYNI inequalities. Can we tell apart the sets of quantum and almost quantum correlations by looking at their violation of facet Bell inequalities. Rich yet simple tripartite Bell scenario allows us to do so. Study the relations between the set of almost quantum correlations and those sets of postquantum correlations. In the n-partite scenario, the (n-1)th level of the NPA hierarchy is not contained in the almost quantum set.

Simplest Tripartite Bell Scenario

Computation of Local Bounds

1) Computing the maximum violation of a Bell inequality for correlations restricted to the set of almost quantum correlations turns out to be a semidefinite program (SDP), which is a kind of convex optimization problem that can be efficiently solved on a computer.
2) Additionally, computing an upper bound on the maximal quantum violation can also be achieved by solving the hierarchy of SDPs introduced by NPA.
3) To ensure that the upper bound is a tight fit perform iterative optimization by considering local projective measurements on three-qubit states

Computation of Quantum, and Almost Quantum Bounds

1) Can characterize Q algorithmically using the NPA hierarchy of $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_k$, each set Q_k is characterized by a set of feasible solutions to an SDP, requires a Hermitian matrix Γ_k whose entries being all the moments up to order 2k can only have non-negative eigenvalues.

Inflation for Causal Inference of Latent Vars (Aug 2018)

Summary

1) Causal inference: determine which hypotheses can explain a given joint probability distribution
2) Model → Directed acyclic graphs
    a. Easy case: If all nodes correspond to observed variables
    b. Difficult case: confounders, there are latent unobserved variables (LHV)
        i. Latent variables result in insufficiency of the conditional independence

3) Instrumental inequality: necessary condition for compatibility of distribution with a causal structure known as the instrumental scenario
4) New technique → Inflation technique: for a given causal structure, one can construct many new causal structures (inflations), an inflation duplicates one or more of the nodes, and mirrors the form of the subgraph. Certain marginal probabilities on subsets of observed variables in the original causal structure are compatible with the original structure, then the same marginal probabilities may be compatible with the inflated causal structure.
5) Steps:
    a. Identify the inequalities for the marginal problem
    b. Look at sets of variables for the inflated causal structure which produce new relation
    c. Write down a factorization of their joint distribution
    d. Generate the inequalities for the marginal probabilities of the inflated causal structure
    e. If the inequalities do not hold for the original and inflated structure, the two are not compatible

Example 1:

Incompatibility of perfect three-way correlation

Example 2:

Incompatibility of the W-type distribution with the Triangle scenario (Cannot be shown any other way)

Classification of Bell Inequalities (Apr 2014)

Summary

1) Bell scenarios, Bell-like inequalities and oriented bell expressions
    a. Local, Quantum, No-signaling
2) Constraints: normalization and no-signaling
3) Superfluous parties: Cannot forego defining a n-partite by a <n-partite scenario
4) Composite inequalities: use lifting to generate a new inequality
5) Removing degeneracies
6) Relabelings
7) Generate canonical bell-like inequalities
8) Example of YAML files (python and MATLAB)

Ncpol2sdpa

Solves global polynomial optimization problems of either commutative variables or noncommutative operators using semidefinite programming (SDP) relaxation. The problem can be unconstrained or constrained. By equalities and inequalities and also be constraints on moments. Helps solve large scale optimization problems.

1) Calculating maximum quantum violation of Bell inequalities

2) Hierarchy for quantum steering