Tutorial 3 (Abubakari Sumaila Salpawuni)

September 26, 2021

Q_1 – solution

1.(a) Given $X \sim U(0,1)$. Define $g(x)$ as the area the square such that;

$$\begin{aligned}
\text{Area} &= g(x) = X^2 \\
E(g(X)) &= \int_0^1 g(x) f_X(x) dx \\
E(X^2) &= \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 \\
&= \frac{1}{3}\text{sq. units}
\end{aligned}$$

1.(b) If $X_i \sim Poi(\lambda_i)$, then it moment generating function, m.g.f, is $M_X(t) = e^{[\lambda_i(e^t-1)]}$. Since the variables are independent, we have;

$$\begin{aligned}
M_Y &= M_{X_1}(t) \cdot M_{X_2}(t) \ldots \cdot M_{X_n}(t) \\
&= e^{[\lambda_1(e^t-1)]} \cdot e^{[\lambda_2(e^t-1)]} \cdot \ldots \cdot e^{[\lambda_n(e^t-1)]} \\
&= e^{[(\lambda_1+\lambda_2+\ldots+\lambda_n)(e^t-1)]} \\
&= e^{[\sum_{i=1}^n \lambda_i(e^t-1)]}
\end{aligned}$$

$\therefore \sum_{i=1}^n X_i \sim \text{Poi}(\sum_i^n \lambda_i)$

1.(c) Given a density function, one way to solve this is to use the method of transformation.

Let $y = g(x) = \frac{1}{x}$. The density of $g(y)$ can be computed as;

$$g(y) = \left|\frac{dx}{dy}\right| \cdot f(W(y))$$

where $W(y)$ is the inverse function of g(x).

$$\begin{aligned}
y &= \frac{1}{x} \implies x = \frac{1}{y} \\
\left|\frac{dx}{dy}\right| &= -\frac{1}{y^2} = \frac{1}{y^2}
\end{aligned}$$

$$
\begin{aligned}
g(x) &= \frac{1}{y^2} \times \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-\frac{1}{y\beta}} \\
&= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{(n+2)-1} e^{-\frac{1}{y\beta}} \\
\therefore g\left(\frac{1}{x}\right) &= \text{Gamma}(n+2, \beta)
\end{aligned}
$$

Q_2 – solution

This is clearly a Bayesian problem. Define D as an event that a subject has the disease; T^+ be event that a subject's test result returns positive, and the event that a subject's test result T^- returns negative. Thus far, we proceed with the following pieces of information;

$$
\begin{aligned}
P(D) &= 0.015 \quad \text{(prevalence)} \\
P(T^+|D) &= 0.97 \quad \text{(sensitivity)} \\
P(T^-|D') &= 0.95 \quad \text{(specificity)}
\end{aligned}
$$

2.(a) i. We partition T into T^+ and T^-. Thus,

$$
\begin{aligned}
P(T^+|D) &= P(D)P(T^+|D) + P(D')P(T^+|D') \\
&= 0.015 \times 0.97 + (1 - 0.015) \times (1 - 0.95) \\
&= 0.905975
\end{aligned}
$$

ii. We're interested in $P(D|T^+)$, the probability you have the disease given you tested positive to the disease. Using the Bayesian formulation, we proceed as follows:

$$
\begin{aligned}
P(D|T^+) &= \frac{P(D)P(T^+|D)}{P(D)P(T^+|D) + P(D')P(T^+|D')} \\
&= \frac{0.015 \times 0.97}{0.905975} \\
&= 0.0161(\mathbf{1.61}\%)
\end{aligned}
$$

2.(b) $P(A) = 0.40, P(B) = 0.10, P(C) = 0.50$
Let D be the event that a defective product is produced
$P(D|A) = 0.02$
$P(D|B) = 0.03$
$P(D|C) = 0.04$

i.

$$
\begin{aligned}
P(D) &= P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C) \\
&= 0.40(0.02) + 0.10(0.03) + 0.50(0.04) \\
&= 0.031
\end{aligned}
$$

ii. α. $P(A|D) = \frac{P(A)P(D|A)}{P(D)} = \frac{0.40 \times 0.02}{0.031} = \frac{8}{31}$

β. $P(B|D) = \frac{P(B)P(D|A)}{P(D)} = \frac{0.10 \times 0.03}{0.031} = \frac{3}{31}$

γ. $P(C|D) = \frac{P(C)P(D|A)}{P(D)} = \frac{0.50 \times 0.04}{0.031} = \frac{20}{31}$

3.(c) Using Matrix approach, Let $\hat{y} = X\beta + \epsilon$ such that;

$$\beta = (X'X)^{-1}X'y$$

$$X'X = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 5 & 6 & \dots & 10 \\ 1 & 1 & \dots & 6 \end{pmatrix} \begin{pmatrix} 1 & 5 & 1 \\ 1 & 6 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 10 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 70 & 40 \\ 70 & 514 & 298 \\ 40 & 298 & 194 \end{pmatrix}$$

$$(X'X)^{-1} = \begin{pmatrix} 2.2179 & -0.3374 & -0.0610 \\ -0.3374 & 0.0691 & 0.0366 \\ -0.0610 & 0.0366 & 0.0488 \end{pmatrix}$$

$$X'y = \begin{pmatrix} 60 \\ 449 \\ 264 \end{pmatrix}$$

$$\hat{\beta} = \begin{pmatrix} 2.2179 & -0.3374 & -0.0610 \\ -0.3374 & 0.0691 & 0.0366 \\ -0.0610 & 0.0366 & 0.0488 \end{pmatrix} \begin{pmatrix} 60 \\ 449 \\ 264 \end{pmatrix}$$

$$= \begin{pmatrix} -2.3146 \\ 1.1195 \\ 0.1098 \end{pmatrix}$$

$$\therefore \hat{\beta} = -2.3146 + 1.1195X_1 + 0.1098X_2$$

Q_3 – solution

3.(a) i.

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{0} \frac{2x|x|}{5}dx + \int_{0}^{2} \frac{2x|x|}{5}dx$$

$$E(X) = -\frac{2}{5}\int_{-1}^{0} x^2dx + \frac{2}{5}\int_{0}^{2} x^2dx$$

$$= -\frac{2}{15}x^3\Big|_{-1}^{0} + \frac{2}{15}x^3\Big|_{0}^{2}$$

$$= -\frac{2}{15} + \frac{16}{15}$$

$$= \frac{14}{15}.$$

ii.

$$\text{Var(X)} = E(X^2) - [E(X)]^2$$

$$= -\int_{-1}^{0} \frac{2x^2|x|}{5}dx + \int_{0}^{2} \frac{2x^2|x|}{5}dx - \left(\frac{14}{15}\right)^2$$

$$= -\frac{1}{10}x^4\Big|_{-1}^{0} + \frac{1}{10}x^4\Big|_{0}^{2} - \left(\frac{14}{15}\right)^2$$

$$= -\frac{1}{10} + \frac{16}{10} - \left(\frac{14}{15}\right)^2 = \frac{17}{10} - \frac{196}{225}$$

$$= \frac{95}{102}.$$

3.(b)

$$Y \sim U(-1,8) = \begin{cases} \frac{1}{9}, & -1 < x < 8 \\ 0, & \text{otherwise} \end{cases}$$

For the equation $2x^2 + 4Yx + 3Y + 2 = 0$ to have real roots, the *discriminant* must be non-negative, thus;

$$b^2 - 4ac \geq 0 \implies (4Y)^2 - 4(2)(3Y+2) \geq 0$$
$$\implies 16Y^2 - 24Y + 16 \geq 0$$
$$\implies 2Y^2 - 8Y + 2 \geq 0$$
$$\implies Y \geq 2 \quad \text{or} \quad Y \leq -\frac{1}{2}$$

Using the probability density function above, and the idea of mutually exclusive events, we have;

$$
\begin{aligned}
P(Y \leq -\frac{1}{2} \cup Y \geq 2) &= P(Y \leq -\frac{1}{2}) + P(Y \geq 2) \\
&= \int_{-1}^{\frac{1}{2}} f_Y(y)dy + \int_{2}^{8} f_Y(y)dy \\
&= \frac{1}{9}(\frac{1}{2} + 6) \\
&= \frac{13}{18}.
\end{aligned}
$$

3.(c) Note that the moment generating function for the Exponential distribution is given as $M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$. Comparing this with the above function, we may infer that;

$$
f(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}\lambda t}, & t < \lambda \\ \\ 0, & \text{otherwise} \end{cases}
$$

i. The cumulative density function for the Exponential distribution is thus;

$$1 - F(x) = e^{-\lambda t}, \quad t < \lambda$$

$\therefore P(X > \text{In}4) = e^{-\frac{1}{2}\text{In}4} = \frac{1}{2}.$

ii.

$$
\begin{aligned}
E(X) &= \frac{d}{dx} M_X(t)\bigg|_{t=0} \\
&= \frac{0.5}{(0.5 - t)^2}\bigg|_{t=0} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \frac{d^2}{dx^2} M_X(t)\bigg|_{t=0} \\
&= \frac{2(0.5)}{(0.5 - t)^3}\bigg|_{t=0} \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
\text{Var(X)} &= E(X^2) - [E(X)]^2 \\
&= 8 - 2^2 = 4 \\
\therefore \text{Var(X)} &= 4.
\end{aligned}
$$