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LOGICS WHICH CAPTURE COMPLEXITY CLASSES OVER THE REALS

FELIPE CUCKER AND KLAUS MEER

Abstract. In this paper we deal with the logical description of complexity classes arising in the real number model of computation introduced by Blum, Shub, and Smale [4]. We adapt the approach of descriptive complexity theory for this model developped in [14] and extend it to capture some further complexity classes over the reals by logical means. Among the latter we find $\mathrm{NC}_{\mathbb{R}}$, $\mathrm{PAR}_{\mathbb{R}}$, $\mathrm{EXP}_{\mathbb{R}}$ and some others more.

§1. Introduction. Decision problems considered in the classical theory of complexity are sets of finite structures satisfying a given property. For instance, the set of graphs which have a Hamiltonian circuit or the set of univariate polynomials with integer coefficients which have a complex root with norm one.

A common characteristic of these structures is that they can be coded as a finite string of letters from a finite alphabet, which in general is considered to be $\{0, 1\}$. The length of this string is said to be the size of the structure. Then, the complexity of the problem with respect to a model of computation and a designated resource is the function which measures, for every $n \in \mathbb{N}$, the necessary amount of this resource to decide whether a structure of size n satisfies the required property. A well known example of model of computation is the Turing machine. Common examples of resources for the Turing machine are the number of steps performed during the computation or the number of used tape cells.

Decision problems are then clustered in complexity classes. These are defined by selecting a computational model, a resource and a class of functions to which the complexity of the problem with respect to the first two choices is required to belong. Well known examples are P and PSPACE, the classes of problems decidable by a Turing machine which works in polynomial time and polynomial space respectively. Another well known example is NP, the class of problems decidable by a nondeterministic Turing machine in polynomial time. Yet another example is NC, the class of problems decidable by a uniform family of Boolean circuits having polylogarithmic depth and polynomial size.

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A large number of complexity classes have been studied in what is called computational complexity theory and the ultimate goal of the latter is to bring into light the relationships between the formers. A paradigmatic open problem in this line is whether $P \neq NP$. The books [1, 2, 22] present good pictures of this field.

The structures considered in computational complexity are simple. One would not consider problems for which it is more difficult to decide if a string of letters describes a structure than to decide if the described structure satisfies the property defining the problem. This simplicity translates in a natural way in terms of first-order logic. There exist a vocabulary L and a first-order sentence φ such that the set of structures of the problem can be seen as the set of finite models of φ. Consider for instance the set of finite graphs. Here L consists of a binary relation symbol E, and φ is the sentence

$$\forall u \, \forall v \; E(u,v) \Longrightarrow E(v,u)$$

which forces the interpretation of E to be symmetric. Any finite model $\mathfrak{A}$ of φ is a finite graph. Its universe A is the set of nodes of the graph and two such nodes u and v are joined by an edge if and only if $E(u,v)$ holds in the interpretation of E.

Descriptive complexity is a model-theoretic flavored branch of complexity which arises from the observation that some main complexity classes correspond to extensions of first-order logic. The foundational result in this branch was given by Fagin [11], who proved that a set of structures belongs to NP if and only if it can be defined in existential second-order logic. Later on Immerman [16] and Vardi [25] gave a similar characterization of P in terms of a fixed point extension of first-order logic. Today, most of the studied complexity classes have a characterization in terms of descriptive complexity. This provides a theory of complexity which is independent of machine models and restates most problems in computational complexity in terms of the expressive power of several extensions of first-order logic.

The preceding lines roughly describe a theory (or theories) which apply mainly to discrete structures as they appear in the design and analysis of algorithms in computer science. With the dawn of the computers and apart from the theory above, a different tradition arose around the subject of numerical computations as they are performed in numerical analysis. Here the problems are of an algebraic and analytic nature rather than combinatorial and they consider as inputs finite vectors over a field. A special emphasis is made on the field of the real numbers since this is the case numerical analysis deals with.

A cornerstone in this tradition is a paper by L. Blum, M. Shub and S. Smale [4] which introduced a machine model allowing the development of a complexity theory over the reals similar to the one built around the Turing machine, which we already called *classical*. Real versions of P and NP were defined in [4] and the existence of $NP_{\mathbb{R}}$-complete problems was proved. In the last few years several results were proved for this machine model by a variety of authors. For an overview of this see the book [3] as well as the survey paper [20].

The machine model introduced in [4] can be seen as a generalization of the Turing machine in which computations can be performed over an arbitrary base ring R. In case $R = \mathbb{F}_2$, the field of two elements, we obtain a model which is equivalent to the Turing machine.

A similar kind of generalization was done by Grädel and Gurevich in [13] for descriptive complexity. They introduced the notion of metafinite structure in which a finite structure is endowed with a set of functions into another structure, possibly infinite. Again, one important case of interest is when this second structure is $\mathbb{R}$ and Grädel and Meer specialize to this case in [14]. Here they characterize some complexity classes over the reals, such as $\mathrm{P}_{\mathbb{R}}$ or $\mathrm{NP}_{\mathbb{R}}$, in terms of logics for metafinite structures over $\mathbb{R}$, called $\mathbb{R}$-structures.

In this paper we continue the work initiated in [14]. Our starting point is the absence of a meaningful class of polynomial space over the reals. Michaux proved in [21] that such a class would contain all decidable problems. There are two natural candidates to ocuppy this vacancy. The class $\mathrm{PAR}_{\mathbb{R}}$ of sets decidable in parallel polynomial time and the class $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$ of sets decidable by machines which work simultaneously in exponential time and polynomial space. Both classes yield PSPACE over $\mathbb{F}_2$ but it turns out that the first one is weaker over $\mathbb{R}$. In Sections 4 and 5 we give logics for $\mathbb{R}$-structures which capture these two classes. Previously, in Section 3, we recall the basic concepts about $\mathbb{R}$-structures and their logics. Scattered along these sections some other complexity classes are characterized in terms of descriptive complexity. Among them, we find $\mathrm{NC}^k_{\mathbb{R}}$, $\mathrm{EXP}_{\mathbb{R}}$, $\mathrm{PH}_{\mathbb{R}}$ and $\mathrm{NP}^{[k]}_{\mathbb{R}}$.

The next section is intended to recall the reader the basic objects of the theory initiated by Blum, Shub and Smale.

§2. Machines and complexity classes over the reals. In the sequel we shall denote by $\mathbb{R}^\infty$ the disjoint union

$$\mathbb{R}^\infty = \bigsqcup_{n\in\mathbb{N}} \mathbb{R}^n.$$

This disjoint union is the set of finite sequences of real numbers. For any element $x \in \mathbb{R}^\infty$ we will denote by $|x|$ its *size*, i.e., the only n such that $x \in \mathbb{R}^n$.

The space $\mathbb{R}^\infty$ is the real analogue to Σ^* the set of all finite sequences of zeros and ones. It provides the inputs for machines over $\mathbb{R}$. For technical reasons we shall also consider the bi-infinite direct sum $\mathbb{R}_\infty$. Elements of this space have the form

$$(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$$

where $x_i \in \mathbb{R}$ for all $i \in \mathbb{Z}$ and $x_k = 0$ for k sufficiently large in absolute value. The space $\mathbb{R}_\infty$ has natural shift operations, shift left $\sigma_\ell : \mathbb{R}_\infty \to \mathbb{R}_\infty$ and shift right $\sigma_r : \mathbb{R}_\infty \to \mathbb{R}_\infty$ where

$$\sigma_\ell(x)_i = x_{i-1} \quad \text{and} \quad \sigma_r(x)_i = x_{i+1}.$$

We now recall the definition of machines over the reals. Instead of the original definition of [4] we take the more recent one appearing in [3].

Definition 1. A *machine over* $\mathbb{R}$ consists of an input space $\mathscr{I} = \mathbb{R}^\infty$, an output space $\mathscr{O} = \mathbb{R}^k$, $(k \leq \infty)$ and a state space $\mathscr{S} = \mathbb{R}_\infty$, together with a connected directed graph whose nodes labelled $1 \dots N$ correspond to the set of different instructions of the machine. These nodes are of one of the five following types: input, output, computation, branching and shift nodes. Let us describe them a bit more.

1. *Input nodes.* There is only one input node and is labelled with 1. Associated with this node there is a next node $\beta(1)$, and the input map $g_I : \mathcal{I} \to \mathcal{S}$.
2. *Output nodes.* There is only one output node and is labelled with N. It has no next nodes, once it is reached the computation halts, and the output map $g_O : \mathcal{S} \to \mathcal{O}$ places the result of the computation in the output space.
3. *Computation nodes.* Associated with a node m of this type there are a next node $\beta(m)$ and a map $g_m : \mathcal{S} \to \mathcal{S}$. The g_m is a polynomial or rational map on a finite number of coordinates, and the identity on the others.
4. *Branch nodes.* There are two nodes associated with a node m of this type: $\beta^+(m)$ and $\beta^-(m)$. The next node is $\beta^+(m)$ if $z_0 \geq 0$ and $\beta^-(m)$ otherwise. Here z_0 denotes the zeroth coordinate of $\mathcal{S}$.
5. *Shift nodes.* Associated with a node m of this type there is a next node $\beta(m)$ and a map $\sigma : \mathcal{S} \to \mathcal{S}$. The σ is either a left or a right shift.

The input map g_I places an input $(x_1, \ldots, x_n) \in \mathbb{R}^\infty$ in $(\ldots, 0, n, x_1, \ldots, x_n, 0, \ldots) \in \mathbb{R}_\infty$ with the size n in the zeroth coordinate. When $\mathcal{O} = \mathbb{R}^\infty$ we will take g_O as the identity map on the first m coordinates of $\mathcal{S}$. The value m is the number of consecutive ones stored in the negative coordinates of $\mathcal{S}$. Thus, for instance g_O maps

$$(\ldots, 3, 1, 1, 1, 1, x_1, x_2, x_3, x_4, x_5, \ldots)$$

to (x_1, x_2, x_3, x_4). In the case $\mathcal{O} = \mathbb{R}^k$ with finite k we shall take g_O as the identity restricted to the first k coordinates of $\mathcal{S}$.

An *instantaneous description* (or *configuration*) at any moment of the computation can be given by providing an element in $\mathcal{S}$ and the current node. After one computational step the first one changes according to the function associated with the current node and the node itself according to the function β.

For a given machine M, the function φ_M associating its output to a given input $x \in \mathbb{R}^\infty$ is called the *input-output function*. We shall say that a function $f : \mathbb{R}^\infty \to \mathbb{R}^k$, $k \leq \infty$, is *computable* when there is a machine M such that $f = \varphi_M$.

Also, a set $A \subseteq \mathbb{R}^\infty$ is *decided* by a machine M if its characteristic function $\chi_A : \mathbb{R}^\infty \to \{0, 1\}$ coincides with φ_M. So, for decision problems we consider machines whose output space is $\mathbb{R}$.

We can now introduce some central complexity classes.

DEFINITION 2. A machine M over $\mathbb{R}$ is said *to work in polynomial time* when there are constants c, $q \in \mathbb{N}$ such that for every input $y \in \mathbb{R}^\infty$, M reaches its output node after at most $c|y|^q$ steps. The class $\mathrm{P}_\mathbb{R}$ is then defined as the set of all subsets of $\mathbb{R}^\infty$ that can be accepted by a machine working in polynomial time. Also, M is said *to work in exponential time* when there are constants c, $q \in \mathbb{N}$ such that for every input $y \in \mathbb{R}^\infty$, M reaches its output node after at most $2^{c \cdot |y|^q}$ steps, and the class $\mathrm{EXP}_\mathbb{R}$ consists of all the subsets of $\mathbb{R}^\infty$ that can be accepted by a machine working in exponential time. Finally, a set A belongs to $\mathrm{NP}_\mathbb{R}$ if there is a machine M satisfying the following condition: for all y, $y \in A$ if and only if there is an $y' \in \mathbb{R}^\infty$ such that M accepts the input (y, y') within time polynomial in $|y|$.

In this model the element y' can be seen as the sequence of guesses used in the Turing machine model. However, we note that in this definition no nondeterministic machine is introduced as a computational model, and nondeterminism appears here as a new acceptance definition for the deterministic machine. Also, we note that the length of y' can be easily bounded by the time bound $p(|y|)$.

An example of a set in $\mathrm{NP}_{\mathbb{R}}$ is 4-FEAS, the set of polynomials of degree four which have a real root. A polynomial f is considered as an element in $\mathbb{R}^\infty$ by coding it by the sequence of its coefficients. Note that if f has n variables, then it has $\mathcal{O}(n^4)$ coefficients. Given such an f and a guess x (which we suppose of size n) a machine deciding 4-FEAS just computes $f(x)$ and accepts if this is zero rejecting otherwise.

The set 4-FEAS is also an example of a "difficult" $\mathrm{NP}_{\mathbb{R}}$ problem in a precise sense. The following is proven in [4] where the definition of $\mathrm{NP}_{\mathbb{R}}$-completeness can also be found.

THEOREM 1 ([4]). *The set* 4-FEAS *is* $\mathrm{NP}_{\mathbb{R}}$*-complete for reductions in* $\mathrm{P}_{\mathbb{R}}$.

Parallelism can also be considered for computations over the reals. We shall now briefly recall a parallel computational model. Let the sign function

$$\mathrm{sign}\colon \mathbb{R} \to \{0,1\}$$

be defined by $\mathrm{sign}(x) = 1$ if $x \geq 0$ and 0 otherwise.

DEFINITION 3. An *algebraic circuit* $\mathscr{C}$ over $\mathbb{R}$ is an acyclic directed graph where each node has indegree 0, 1 or 2. Nodes with indegree 0 are either labeled as *input nodes* or with elements of $\mathbb{R}$ (we shall call them *constant nodes*). Nodes with indegree 2 are labeled with the binary operators of $\mathbb{R}$, i.e., one of $\{+, \times, -, /\}$. They are called *arithmetic nodes*. Nodes with indegree 1 are either *sign nodes* or *output nodes*. All the output nodes have outdegree 0. Otherwise, there is no upper bound for the outdegree of the other kind of nodes. Occasionally, the nodes of an algebraic circuit will be called *gates*.

For an algebraic circuit $\mathscr{C}$, the *size* of $\mathscr{C}$, is the number of gates in $\mathscr{C}$. The *depth* of $\mathscr{C}$, is the length of the longest path from some input gate to some output gate.

Let $\mathscr{C}$ be an algebraic circuit with n input gates and m output gates. Then, to each gate g we inductively associate a function $f_g\colon \mathbb{R}^n \to \mathbb{R}$. We shall refer to the function $\varphi_{\mathscr{C}}\colon \mathbb{R}^n \to \mathbb{R}^m$ associated to the output gates as the function *computed by the circuit*.

REMARK 1. In order to $\varphi_{\mathscr{C}}$ be well defined it is necessary to allow division by zero. By convention we will understand that if a gate is labeled with / and its second argument is zero, the gate returns zero.

DEFINITION 4. Let $f\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$. We shall say that the family of algebraic circuits $\{\mathscr{C}_n\}_{n\geq 1}$ computes f, when for all $n \geq 1$ the function computed by $\mathscr{C}_n$ is the restriction of f to $\mathbb{R}^n \subset \mathbb{R}^\infty$.

We now require a condition on the whole family $\{\mathscr{C}_n\}_{n\geq 1}$ in order to ensure that its elements are not too unrelated as well as to ensure a finite description of the machine model. Gates of algebraic circuits can be described with five real numbers in the following way. If the gates of the circuit are $g_1, \dots, g_k$ then, gate g_j is described with the tuple $(j, t, i_\ell, i_r, c) \in \mathbb{R}^5$ where t represents the type of g_j according to the following association:

g_j	t
input	1
constant	2
$+$	3
$-$	4
$\times$	5
$/$	6
sign	7
output	8

For gates of indegree two, the numbers i_ℓ and i_r respectively denote the gates which provide left and right input to g_j. By convention, if g_j is an input or a constant gate then i_ℓ and i_r are 0 and, if g_j is a sign gate or an output gate then i_ℓ denotes the gate which provides the input to g_j and i_r is zero. Finally, if g_j is a constant gate c denotes the corresponding real constant. Otherwise, c is zero by convention. The whole circuit can then be described by a point in $\mathbb{R}^{5k}$. We shall suppose without loss of generality that, when describing a circuit, the first gates are the input gates.

DEFINITION 5. A family of circuits $\{\mathscr{C}_n\}_{n\in\mathbb{N}}$ is said to be *uniform* if there exists a machine M that returns the description of the ith gate of $\mathscr{C}_n$ with input (n, i). In case that $i > k$, the number of gates of $\mathscr{C}_n$, M returns $(i, 0, 0, 0, 0)$. If M works in time bounded by $\mathscr{O}(\log n)$ we shall say that the family is *L-uniform*, if M works in time $\mathscr{O}(n^k)$ for some positive integer k we shall say that the family is *P-uniform*.

We now define some parallel complexity classes by bounding the depth and size of uniform families of circuits. For further details about these classes see [3, 6, 7].

DEFINITION 6. Define $\mathrm{NC}_{\mathbb{R}}^k$ for $k \geq 1$ to be the class of sets $S \subseteq \mathbb{R}^\infty$ such that there is a L-uniform family of algebraic circuits $\{\mathscr{C}_n\}$ having size polynomial in n and depth $\mathscr{O}(\log^k n)$ that computes the characteristic function of S. The union of the $\mathrm{NC}_{\mathbb{R}}^k$ is denoted by $\mathrm{NC}_{\mathbb{R}}$. We define $\mathrm{PAR}_{\mathbb{R}}$ to be the class of all sets $S \subseteq \mathbb{R}^\infty$ such that there is a P-uniform family of algebraic circuits $\{\mathscr{C}_n\}$ having depth polynomial (and therefore size exponential) in n that computes the characteristic function of S.

§3. Logics on $\mathbb{R}$-structures. In this section we first recall basic notions of $\mathbb{R}$-structures and their logics. A main reference is [14] where these concepts were first introduced. We suppose the reader familiar with the main terminology of logic as well as with the concepts of vocabulary, first-order formula or sentence, interpretation and structure (see for example [10]).

DEFINITION 7. Let L_s, L_f be finite vocabularies where L_s may contain relation and function symbols, and L_f contains function symbols only. A $\mathbb{R}$-*structure of signature* $\sigma = (L_s, L_f)$ is a pair $\mathfrak{D} = (\mathfrak{A}, \mathscr{F})$ consisting of

(i) a finite structure $\mathfrak{A}$ of vocabulary L_s, called the *skeleton* of $\mathfrak{D}$, whose universe A will also be said to be the *universe* of $\mathfrak{D}$, and
(ii) a finite set $\mathscr{F}$ of functions $X\colon A^k \to \mathbb{R}$ interpreting the function symbols in L_f.

We shall denote the set of all $\mathbb{R}$-structures of signature σ by $\mathrm{Struct}_{\mathbb{R}}(\sigma)$.

DEFINITION 8. Let $\mathfrak{D}$ be a $\mathbb{R}$-structure of skeleton $\mathfrak{A}$. We denote by $|A|$ the cardinality of the universe A of $\mathfrak{A}$. A $\mathbb{R}$-structure $\mathfrak{D} = (\mathfrak{A}, \mathscr{F})$ is *ranked* if there is a unary function symbol $r \in L_f$ whose interpretation ρ in $\mathscr{F}$ bijects A with $\{0, 1, \ldots, |A|-1\}$. The function ρ is called *ranking*. A k-ranking on A is a bijection between A^k and $\{0, 1, \ldots, |A|^k - 1\}$.

3.1. First-order logic. Fix a countable set $V = \{v_0, v_1, \ldots,\}$ of variables. These variables range only over the skeleton; we do not use element variables taking values in $\mathbb{R}$.

DEFINITION 9. The language $\mathrm{FO}_{\mathbb{R}}$ contains, for each signature $\sigma = (L_s, L_f)$ a set of formulas and terms. Each term t takes, when interpreted in some $\mathbb{R}$-structure, values in either the skeleton, in which case we call it an *index term*, or in $\mathbb{R}$, in which case we call it a *number term*. Terms are defined inductively as follows

(i) The set of index terms is the closure of the set V of variables under applications of function symbols of L_s.
(ii) Any real number is a number term.
(iii) If $h_1, \ldots, h_k$ are index terms and X is a k-ary function symbol of L_f then $X(h_1, \ldots, h_k)$ is a number term.
(iv) If t, t' are number terms, then so are $t + t'$, $t - t'$, $t \times t'$, t/t' and $\mathrm{sign}(t)$.

Atomic formulas are equalities $h_1 = h_2$ of index terms, equalities $t_1 = t_2$ and inequalities $t_1 < t_2$ of number terms, and expressions $P(h_1, \ldots, h_k)$ where P is a k-ary predicate symbol in L_s and $h_1, \ldots, h_k$ are index terms.

The set of formulas of $\mathrm{FO}_{\mathbb{R}}$ is the smallest set containing all atomic formulas and which is closed under Boolean connectives and quantification $(\exists v)\psi$ and $(\forall v)\psi$. Note that we do *not* consider formulas $(\exists x)\psi$ where x ranges over $\mathbb{R}$.

REMARK 2. The interpretation of formulas in $\mathrm{FO}_{\mathbb{R}}$ on a $\mathbb{R}$-structure $\mathfrak{D}$ is clear. The only remark to be done is that, as with circuits, in order to have this interpretation well defined, we understand that $x/0 = 0$ for all $x \in \mathbb{R}$.

EXAMPLE 1. Let L_s be the empty set and L_f be $\{r, X\}$ where both function symbols have arity 1. Then, a simple class of ranked $\mathbb{R}$-structures with signature (L_s, L_f) is obtained by letting $\mathfrak{A}$ be a finite set A, $r^{\mathfrak{D}}$ any ranking on A and $X^{\mathfrak{D}}$ any unary function $X^{\mathfrak{D}}: A \to \mathbb{R}$. Since $r^{\mathfrak{D}}$ bijects A with $\{0, 1, \ldots, n-1\}$ where $n = |A|$, this $\mathbb{R}$-structure is a point $x_{\mathfrak{D}}$ in $\mathbb{R}^\infty$. Conversely, for each point $x \in \mathbb{R}^\infty$ there is an $\mathbb{R}$-structure $\mathfrak{D}$ such that $x = x_{\mathfrak{D}}$. Thus, this class of structures models $\mathbb{R}^\infty$.

On the other hand any $\mathbb{R}$-structure $\mathfrak{D} = (\mathfrak{A}, \mathscr{F})$ can be identified with a vector $e(\mathfrak{D}) \in \mathbb{R}^\infty$ using a natural encoding. To this aim choose a ranking on A. Without loss of generality the skeleton of $\mathfrak{D}$ can be assumed to consist of the plain set A only by replacing all functions and relations in L_s by their corresponding characteristic functions—the latter being considered as elements of the set $\mathscr{F}$. Now using the ranking each of the functions X in $\mathscr{F}$ can be represented by a vector $v_X \in \mathbb{R}^m$ for some appropriate m. The concatenation of all these v_X yields the encoding $e(\mathfrak{D}) \in \mathbb{R}^\infty$. Note that the length of $e(\mathfrak{D})$ is polynomially bounded in $|A|$; moreover for all $\mathbb{R}$-structures $\mathfrak{D}$, all rankings E on A and all functions $X: A^k \to \mathbb{R}$ the property that X represents the encoding $e(\mathfrak{D})$ of $\mathfrak{D}$ with respect to E is first-order expressible (see [14]).

Example 1 allows us to speak about complexity classes among $\mathbb{R}$-structures. If S is a set of $\mathbb{R}$-structures closed under isomorphisms, we say that S belongs to a complexity class $\mathscr{C}$ over the reals if the set $\{ e(\mathfrak{D}) \mid \mathfrak{D} \in S \}$ belongs to $\mathscr{C}$.

EXAMPLE 2. If $\mathfrak{D}$ is a $\mathbb{R}$-structure of signature (L_s, L_f) and $r \in L_f$ is a unary function symbol we can express in first-order logic the requirement that r is interpreted as a ranking in $\mathfrak{D}$. This is done by the sentence

$$r \text{ is injective} \;\&\; \exists \mathtt{o}\; r(\mathtt{o}) = 0 \;\&\; \forall u\; [u \neq \mathtt{o} \Longrightarrow (r(\mathtt{o}) < r(u) \;\&\; \exists v\; r(u) = r(v) + 1)].$$

REMARK 3. If ρ is a ranking on A and $|A| = n$ then, there are elements $\mathtt{o}, \mathtt{1} \in A$ such that $\rho(\mathtt{o}) = 0$ and $\rho(\mathtt{1}) = n - 1$. Note that these two elements are first-order definable in the sense that they are the only elements in A satisfying

$$\forall v\; (v \neq \mathtt{o} \Longrightarrow \rho(\mathtt{o}) < \rho(v)) \quad \text{and} \quad \forall v\; (v \neq \mathtt{1} \Longrightarrow \rho(v) < \rho(\mathtt{1}))$$

respectively. We shall take advantadge of this property by freely using the symbols $\mathtt{o}$ and $\mathtt{1}$ as symbol constants that are to be interpreted as the first and last elements in A with respect to the ranking ρ. Note that, in particular, this allows us to use the symbol n to denote the cardinality of A since $n = \rho(\mathtt{1}) + 1$.

REMARK 4. Any ranking ρ induces, for all $k \geq 1$ a k-ranking ρ^k on A by lexicographical ordering. Note that ρ^k is definable in the sense that for all $(v_1, \ldots, v_k) \in A^k$

$$\rho^k(v_1, \ldots, v_k) = \rho(v_1)n^{k-1} + \rho(v_2)n^{k-2} + \cdots + \rho(v_k).$$

Again, we will take advantage of this to freely use the symbol ρ^k to denote the k-ranking induced by ρ on A.

EXAMPLE 3. Let us see that we can describe algebraic circuits with first-order logic. To do so, we consider the vocabularies $L_s = \{f_l, f_r\}$ where f_l and f_r are function symbols of arity one and $L_f = \{r, C, F_t\}$ whose three symbols are also of arity one. If $\mathfrak{D}$ is a $\mathbb{R}$-structure with signature (L_s, L_f) we intend to interpret its universe A as the set of nodes of an algebraic circuit. Left and right predecessors are to be given by the functions f_l and f_r respectively. The type of each node will be given by F_t and the real numbers associated to the constant nodes by the function C. Finally, we will require r to be interpreted as a ranking.

We now want to express all these requirements with first-order sentences. To interprete r as a ranking we use the sentence in Example 2. The sentence

$$\forall v \left[\bigvee_{k=1}^{8} F_t(v) = k \right]$$

ensures that the elements of A are of one of the eight kinds described in the table above.

The sentence

$$\forall v\, \forall w\; [(F_t(v) = 1 \;\&\; F_t(w) \neq 1) \Longrightarrow r(v) < r(w)]$$

requires that input gates are the first gates of the circuit. Finally, the sentence

$$\forall v\; [(F_t(v) \neq 1 \;\&\; F_t(v) \neq 2) \Longrightarrow (r(f_l(v)) < r(v) \;\&\; r(f_r(v)) < r(v))]$$

requires that when a gate v is not of input or constant type its predecessors are really so, that is, they are ranked before v. It also ensures that A, considered as a directed graph, is acyclic.

The expressive power of first-order logic is not too big.

PROPOSITION 1. *Let σ be a signature, φ a first-order sentence and* $S = \{\mathfrak{D} \in \mathrm{Struct}_{\mathbb{R}}(\sigma) \mid \mathfrak{D} \models \varphi\}$. *Then* $S \in \mathrm{NC}^1_{\mathbb{R}}$.

PROOF. We can assume that φ is in prenex form that is, that it has the form

$$\varphi = Q_1 v_1 \dots Q_s v_s \ \psi(v_1, \dots, v_s)$$

where $Q_1, \dots, Q_s$ are first-order quantifiers and $\psi(v_1, \dots, v_s)$ is a quantifier-free formula.

Consider now a $\mathbb{R}$-structure $\mathfrak{D}$ and let A be its universe. Given a point

$$(v_1, \dots, v_s) \in A^s$$

it takes constant time to check whether $\psi(v_1, \dots, v_s)$ holds over $\mathbb{R}$. On the other hand, if $|A| = n$, there are n^s possible tuples $(v_1, \dots, v_s)$ in A^s. Thus, we can decide whether $\mathfrak{D}$ satisfies φ in parallel logarithmic time by independently checking the validity of ψ for all the n^s possibilities and accept when the resulting n^s truth values satisfy the prefix of quantifiers. ⊣

3.2. Fixed point first-order logic. A first-order number term $F(\bar{t})$ with free variables $\bar{t} = (t_1, \dots, t_r)$ is interpreted in a $\mathbb{R}$-structure $\mathfrak{D}$ with universe A as a function $F^{\mathfrak{D}} : A^r \to \mathbb{R}$. Fixed point first-order logic enhances first-order logic with the introduction of two grammatical rules to build number terms: the *maximization rule* and the *fixed point rule*. The first one, allows some form of quantification for describing $F^{\mathfrak{D}}$ and the second one, the definition of $F^{\mathfrak{D}}$ in an inductive way.

For simplicity, in the rest of this paper we restrict attention to *functional* $\mathbb{R}$-structures, i.e., $\mathbb{R}$-structures whose signatures do not contain relation symbols. This represents no loss of expressive power since we can replace any relation $P \subseteq A^k$ by its characteristic function $\chi_P : A^k \to \mathbb{R}$.

We first define the maximization rule $\mathrm{MAX}^1_{\mathbb{R}}$.

DEFINITION 10. Let $F(s, \bar{t})$ be a number term with free variables s and $\bar{t} = (t_1, \dots, t_r)$. Then

$$\max_s F(s, \bar{t})$$

is also a number term with free variables $\bar{t}$. Its interpretation in any $\mathbb{R}$-structure $\mathfrak{D}$ and for any point $u \in A^r$ interpreting $\bar{t}$ is the maximum of $F^{\mathfrak{D}}(a, u)$ where a ranges over A.

EXAMPLE 4. If the signature contains a symbol r which is interpreted as a ranking, then we can define the size n of the universe with the number term $\max_s r(s) + 1$.

DEFINITION 11. We denote by $\mathrm{FO}_{\mathbb{R}} + \mathrm{MAX}^1_{\mathbb{R}}$ the logic obtained by adding to $\mathrm{FO}_{\mathbb{R}}$ the maximization rule.

The expressive power gained by allowing the maximization rule lies in the possibility of writing characteristic functions as number terms. If $\varphi(v_1, \ldots, v_r)$ is a first-order formula we define its characteristic function $\chi[\varphi]$ on a structure $\mathfrak{D}$ by

$$\chi[\varphi](a_1, \ldots, a_r) = \begin{cases} 1 & \text{if } \mathfrak{D} \models \varphi(a_1, \ldots, a_r) \\ 0 & \text{otherwise} \end{cases}$$

where $a_1, \ldots, a_r \in A$, the universe of $\mathfrak{D}$.

Proposition 2. *For every first-order formula $\varphi(v_1, \ldots, v_r)$ there is a number term in $\mathrm{FO}_{\mathbb{R}} + \mathrm{MAX}^1_{\mathbb{R}}$ describing $\chi[\varphi]$.*

Proof. The proof is done by induction on the construction of φ. If φ is atomic it must have the form $F = G$ or $F < G$ with F and G number terms. This follows from the assumption that signatures do not contain relational symbols. Then

$$\chi[F = G] = \mathrm{sign}[-(F - G)^2]$$
$$\chi[F < G] = 1 - [\mathrm{sign}(F - G)].$$

If φ has the form $\exists x\ \psi(x)$ then

$$\chi[\varphi] = \max_x \chi[\psi(x)].$$

Finally, if $\varphi = \neg\psi$ then $\chi[\varphi] = 1 - \chi[\psi]$. ⊣

We now define the fixed point rule.

Definition 12. Fix a signature $\sigma = (L_s, L_f)$, an integer $r \geq 1$, and a pair (Z, D) of function symbols both of arity r and not contained in this signature. Let $F(Z, \bar{t})$ and $H(D, \bar{t})$ be number terms of signature $(L_s, L_f \cup \{Z, D\})$ and free variables $\bar{t} = (t_1, \ldots, t_r)$. Note that Z can appear several times in F and we do not require that its arguments are $t_1, \ldots, t_r$. The only restriction is that the number of free variables in F coincides with the arity of Z. A similar remark holds for H and D.

For any $\mathbb{R}$-structure $\mathfrak{D}$ of signature σ and any interpretation $\zeta\colon A^r \to \mathbb{R}$ of Z and $\Delta\colon A^r \to \mathbb{R}$ of D respectively the number terms $F(Z, \bar{t})$ and $H(D, \bar{t})$ define functions

$$F^{\mathfrak{D}}_{\zeta}, H^{\mathfrak{D}}_{\Delta} : A^r \to \mathbb{R}.$$

Let us consider the sequence of pairs $\{\Delta^i, \zeta^i\}_{i \geq 0}$ with $\zeta^i\colon A^r \to \mathbb{R}$ and $\Delta^i\colon A^r \to \mathbb{R}$ inductively defined by

$$\Delta^0(x) = 0 \quad \text{for all } x \in A^r,$$
$$\zeta^0(x) = 0 \quad \text{for all } x \in A^r,$$

$$\Delta^{i+1}(x) = \begin{cases} H^{\mathfrak{D}}_{\Delta^i}(x) & \text{if } \Delta^i(x) = 0 \\ \Delta^i(x) & \text{otherwise,} \end{cases}$$

$$\zeta^{i+1}(x) = \begin{cases} F^{\mathfrak{D}}_{\zeta^i}(x) & \text{if } \Delta^i(x) = 0 \\ \zeta^i(x) & \text{otherwise.} \end{cases}$$

Since $\Delta^{i+1}(x)$ only differs from $\Delta^i(x)$ in case the latter is zero, one has that $\Delta^j = \Delta^{j+1}$ for some $j < |A|^r$. In this case, moreover, we also have that $\zeta^j = \zeta^{j+1}$.

We denote these fixed points by Z^∞ and D^∞ and call them the *fixed points of* $F(Z,\bar{t})$ *and* $H(D,\bar{t})$ *on* $\mathfrak{D}$. We say that $F^{\mathfrak{D}}$ *updates* ζ.

Note that D plays the role of the characteristic function for the domain of Z and the different Δ^i determine the successive updatings of this domain. We say that Z^∞ *is defined on* x if $D^\infty(x) \neq 0$.

The fixed point rule is now stated in the following way. If $F(Z,\bar{t})$ and $H(D,\bar{t})$ are number terms as above then

$$\mathbf{fp}[Z(\bar{t}) \leftarrow F(Z,\bar{t}), H(D,\bar{t})](\bar{u}) \quad \text{and} \quad \mathbf{fp}[D(\bar{t}) \leftarrow H(D,\bar{t})](\bar{u})$$

are number terms of signature (L_s, L_f). Their interpretations on a given $\mathbb{R}$-structure $\mathfrak{D}$ are $Z^\infty(\bar{u})$ and $D^\infty(\bar{u})$ respectively.

A simple example of a function definable with the fixed point rule, which we will use in the next section, is the exponential.

EXAMPLE 5. Consider a signature of ranked $\mathbb{R}$-structures and let r be the symbol for the ranking. We will define a function 2^r by means of the above fixed point rule such that for all ranked $\mathbb{R}$-structure $\mathfrak{D}$ the interpretation of 2^r is

$$\begin{aligned} 2^\rho : A &\rightarrow \{0,\ldots,2^{n-1}\} \\ x &\rightarrow 2^{\rho(x)} \end{aligned}$$

where A is the universe of $\mathfrak{D}$, $n = |A|$ and ρ is the ranking which interprets r.

To do so, consider the number terms $F(Z,x)$

$$\chi[r(x)=0] + \max_s \chi[r(x) = r(s)+1]2Z(s)$$

and $H(D,x)$

$$\chi[r(x)=0] + \max_s \chi[r(x) = r(s)+1]D(s).$$

One can check that Z^∞ is the function 2^ρ described above. In a similar way, for $k \geq 2$ one defines a function 2^{r^k} which is interpreted as

$$\begin{aligned} 2^{\rho^k} : A &\rightarrow \{0,\ldots,2^{n^k-1}\} \\ x &\rightarrow 2^{\rho^k(x)}. \end{aligned}$$

Another example of fixed point is the evaluation of algebraic circuits.

EXAMPLE 6. Let us consider the description of algebraic circuits done in Example 3 and let us extend this description by supposing that the constant $C(v)$ associated to an input gate v is the actual input to the circuit. We are then interested in defining the evaluation function $E: A \rightarrow \mathbb{R}$ that assigns to each gate of the circuit the value it computes for this input. To do so, let us denote by T the set $\{1,2,\ldots,8\}$ and by g_i the polynomial

$$g_i(x) = \prod_{\substack{j \in T \\ j \neq i}} \frac{x-j}{i-j}$$

that maps i to 1 and j to 0 for $i, j \in T$, $i \neq j$.

Now we consider the number term $F(E, v)$ given by

$$\begin{aligned}((g_1+g_2)(F_t(v)))C(v) &+ g_3(F_t(v))(E(f_l(v))+E(f_r(v))) \\ &+ g_4(F_t(v))(E(f_l(v))-E(f_r(v))) + g_5(F_t(v))(E(f_l(v))\times E(f_r(v))) \\ &\quad + g_6(F_t(v))(E(f_l(v))/E(f_r(v))) + g_7(F_t(v))\mathrm{sign}(E(f_l(v))) \\ &\qquad + g_8(F_t(v))E(f_l(v))\end{aligned}$$

as well as the number term $H(D, v)$ given by

$$\begin{aligned}((g_1+g_2)(F_t(v))) &+ (g_3+g_4+g_5+g_6)(F_t(v))(D(f_l(v))\times D(f_r(v))) \\ &+ (g_7+g_8)(F_t(v))(D(f_l(v))).\end{aligned}$$

The updating of D by H is clear. The function Δ^1 will return 1 if v is an input or constant node and 0 otherwise. Then, Δ^{i+1} updates v from 0 to 1 if the $\Delta^i(x) = 1$ for the predecessors x of v. Eventually, all nodes are updated to 1 and D^∞ is the constant function 1.

The function $E^\infty : A \to \mathbb{R}$ defined by $F(E, v)$ is the evaluation function we are looking for. Notice that since D^∞ is the constant function 1, we can ensure that the function E is total.

DEFINITION 13. *Fixed point logic* for $\mathbb{R}$-structures, denoted $\mathrm{FP}^1_{\mathbb{R}}$, is obtained by augmenting first-order logic $\mathrm{FO}_{\mathbb{R}}$ with the maximization rule and the fixed point rule.

To see an example of a set describable within fixed point logic, we recall the Circuit Evaluation Problem. This is the set of pairs $(\mathscr{C}, b)$ such that $\mathscr{C}$ is an algebraic circuit and $b \in \mathbb{R}^n$ (n is the number of input gates in $\mathscr{C}$) satisfying $\varphi_{\mathscr{C}}(b) = 1$. It is known from [9] that this is a $\mathrm{P}_{\mathbb{R}}$-complete problem.

EXAMPLE 7. Example 6 allows us to describe the Circuit Evaluation Problem within fixed point first-order logic for $\mathbb{R}$-structures. Since we are only considering circuits with a unique output gate we add to the axioms shown in Example 3 the sentence

$$\forall u\ (F_t(u) = 8 \Rightarrow u = \mathtt{1})$$

requiring that this is so and that such gate is the last one. Now we require the unique output gate to return 1 with

$$\mathbf{fp}[E(v) \leftarrow F(E, v), H(D, v)](\mathtt{1}) = 1.$$

The fact that we can describe a $\mathrm{P}_{\mathbb{R}}$-complete problem with fixed point first-order logic is not fortuitous.

THEOREM 2 ([14]). *Let σ be a signature and S be a decision problem of ranked $\mathbb{R}$-structures over the signature σ. Then the following two statements are equivalent.*

(i) $S \in \mathrm{P}_{\mathbb{R}}$
(ii) *there exists a sentence ψ in $\mathrm{FP}^1_{\mathbb{R}}$ such that* $S = \{\mathfrak{D} \in \mathrm{Struct}_{\mathbb{R}}(\sigma) \mid \mathfrak{D} \models \psi\}$.

REMARK 5. We will abreviate in the sequel a statement like the one of Theorem 2 in the following way. For ranked $\mathbb{R}$-structures, $\mathrm{FP}^1_{\mathbb{R}} = \mathrm{P}_{\mathbb{R}}$.

Some simple modifications in the proof of Theorem 2 allow to prove the following result.

PROPOSITION 3. *Let $\varphi\colon \mathbb{N}^2 \to \mathbb{R}$ be a function such that $\varphi(n,i)$ is computable in time $t(n) \leq n^q$ for $q \in \mathbb{N}$. Then there exists a number term $\widehat{\varphi}(v_1,\dots,v_q)$ in $\mathrm{FP}^1_{\mathbb{R}}$ such that for every ranked $\mathbb{R}$-structure $\mathfrak{D}$ of size n*

$$\widehat{\varphi}^{\mathfrak{D}} : A^q \to \mathbb{R}$$
$$(a_1,\dots,a_q) \to \varphi(n, \rho^q(a_1,\dots,a_q)).$$

Here, we recall ρ^q stands for the interpretation of the ranking on A^q. Moreover the fixed point operator is used only one time in $\widehat{\varphi}(v_1,\dots,v_q)$ and the number of updatings needed to reach the fixed point is bounded by $t(n)$.

In the rest of this section we will capture the classes $\mathrm{NC}^k_{\mathbb{R}}$ with some extensions of first-order logic. Since we already know that fixed point first-order logic captures the class $\mathrm{P}_{\mathbb{R}}$, we shall look at some logic between $\mathrm{FO}_{\mathbb{R}}$ and $\mathrm{FP}^1_{\mathbb{R}}$. We shall do so by limiting the number of updatings in the definition of fixed points.

DEFINITION 14. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. The formulas of f-bounded fixed point logic $\mathrm{FP}^1_{\mathbb{R}}[f]$ are those in $\mathrm{FP}^1_{\mathbb{R}}$. The interpretation of a formula φ in a $\mathbb{R}$-structure $\mathfrak{D}$ of size n is as in $\mathrm{FP}^1_{\mathbb{R}}$ with only one difference; while defining a function Z as a fixed point, the updating procedure performs at most $f(n)$ steps. If after this number of steps the fixed point has not been reached, the $f(n)$th update of the function is taken as the interpretation of Z.

If $\mathscr{F}$ is a set of functions $f\colon \mathbb{N} \to \mathbb{N}$ we denote by $\mathrm{FP}^1_{\mathbb{R}}[\mathscr{F}]$ the union

$$\bigcup_{f \in \mathscr{F}} \mathrm{FP}^1_{\mathbb{R}}[f].$$

For instance

$$\mathrm{FP}^1_{\mathbb{R}}[\mathscr{O}(\log^k n)] = \bigcup_{c,d \in \mathbb{N}} \mathrm{FP}^1_{\mathbb{R}}[c(\log^k n) + d].$$

THEOREM 3. *For all $k \geq 1$, $\mathrm{FP}^1_{\mathbb{R}}[\mathscr{O}(\log^k n)] = \mathrm{NC}^k_{\mathbb{R}}$.*

PROOF. The inclusion $\mathrm{FP}^1_{\mathbb{R}}[\mathscr{O}(\log^k n)] \subseteq \mathrm{NC}^k_{\mathbb{R}}$ easily follows.

To see the reversed inclusion we consider ranked $\mathbb{R}$-structures as points in $\mathbb{R}^\infty$ as in Example 1.

Let S be a set of $\mathbb{R}$-structures in $\mathrm{NC}^k_{\mathbb{R}}$. Then, there is a family of circuits $\{C_n\}_{n\in\mathbb{N}}$ with size at most n^q and depth $c\log^k n$ which decides S. Moreover there is a machine M which, on input $(n,i) \in \mathbb{N}^2$ returns a point in $\mathbb{R}^5$ encoding the ith gate of C_n in time $\mathscr{O}(\log n)$.

We first describe, for an input structure $\mathfrak{D}$ of size n, the circuit C_n. By Proposition 3 there are fixed point first-order number terms $\hat{t}(v)$, $\hat{l}(v)$, $\hat{r}(v)$ and $\hat{c}(v)$ with $v = (v_1,\dots,v_q)$ such that $(r(v), \hat{t}(v), \hat{l}(v), \hat{r}(v), \hat{c}(v))$ describes the $r(v)$th gate of circuit C_n. Moreover, these terms are actually in $\mathrm{FP}^1_{\mathbb{R}}[\mathscr{O}(\log n)]$ since M works in logarithmic time.

We can now describe the evaluation of circuit C_n over an input structure $\mathfrak{D}$ of size n. This is done by describing the evaluation of the circuit as in Example 6 with some care since we can not assume to have the function symbols f_l and f_r in the signature.

Let v be a point in A^r where A is the universe of $\mathfrak{D}$ and assume for instance that $\hat{t}(v) = 3$, that is, that v represents an addition gate. Then, the number term $T_3(v)$ defined by

$$\begin{aligned}\max_{u_\ell}\max_{u_r}\ &\chi[r(u_\ell) = \hat{l}(v)]\,\chi[r(u_r) = \hat{r}(v)]\\ &\qquad\chi[(E(u_\ell) + E(u_r) \geq 0]\,(E(u_\ell) + E(u_r))\\ -\max_{u_\ell}\max_{u_r}\ &\chi[r(u_\ell) = \hat{l}(v)]\,\chi[r(u_r) = \hat{r}(v)]\\ &\qquad\chi[E(u_\ell) + E(u_r) < 0]\,(-E(u_\ell) - E(u_r))\end{aligned}$$

defines $E(v)$. One can define in a similar way terms $T_i(v)$ for $i = 1, \dots, 8,\ i \neq 3$. Note that now $T_1(v)$ is simply $X(v)$. Then, the evaluation function E is the fixed point associated to the number term $F(E, v)$ given by

$$\sum_{i=1}^{8} g_i(v)(\hat{t}(v))T_i(v).$$

The $\mathbb{R}$-structure $\mathfrak{D}$ is in S if and only if the following sentence

$$\forall v\ (\hat{t}(v) = 8 \Longrightarrow E(v) = 1)$$

requiring the evaluation of the output gate to be one, holds. ⊣

§4. **Complexity classes beyond $\mathrm{P}_{\mathbb{R}}$.** The catalog of complexity classes over the reals is not so vast as in the classical case. Nevertheless, it includes several natural complexity classes. For instance $\mathrm{P}_{\mathbb{R}}$ and $\mathrm{EXP}_{\mathbb{R}}$ are the real versions for polynomial and exponential time respectively. Also, $\mathrm{NP}_{\mathbb{R}}$ is the class of sets decided in nondeterministic polynomial time, and the possibility of alternating a bounded number of existential and universal guesses leads to the polynomial hierarchy $\mathrm{PH}_{\mathbb{R}}$ over the reals.

A difference which stands out is the power of space in the real and classical settings. While in the latter one has that PH $\subseteq$ PSPACE $\subseteq$ EXP, in the former it has been proved in [21] that every decidable set can be decided using a polynomial number of registers. Yet we can consider two classes between $\mathrm{PH}_{\mathbb{R}}$ and $\mathrm{EXP}_{\mathbb{R}}$ whose classical versions coincide with PSPACE.

On the one hand there is the class $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$ of sets decided by machines which use exponential time and polynomial space. On the other hand there is the class $\mathrm{PAR}_{\mathbb{R}}$ defined in Section 2. The classical version of these two classes coincide with PSPACE. Over the reals, it is shown in [23, 24] that $\mathrm{PH}_{\mathbb{R}} \subset \mathrm{PAR}_{\mathbb{R}}$ and obvious modifications in Borodin's argument in [5] yield that $\mathrm{PAR}_{\mathbb{R}} \subseteq (\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$. Moreover, this latter inclusion is strict according to [6]. Therefore, we can summarize this situation with the diagram

$$\begin{array}{ccccccccc} & \mathrm{NP}_{\mathbb{R}} \ldots & & & & & & & \\ \nearrow & & \searrow & & & & & & \\ \mathrm{P}_{\mathbb{R}} & & & \mathrm{PH}_{\mathbb{R}} & \longrightarrow & \mathrm{PAR}_{\mathbb{R}} & \xrightarrow{\neq} & (\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}}) \longrightarrow & \mathrm{EXP}_{\mathbb{R}} \\ \searrow & & \nearrow & & & & & & \\ & \mathrm{coNP}_{\mathbb{R}} \ldots & & & & & & & \end{array}$$

where an arrow $\rightarrow$ means inclusion and an arrow $\xrightarrow{\neq}$ means strict inclusion.

In this and the next section we will capture the classes in the diagram above with specific logics on $\mathbb{R}$-structures.

DEFINITION 15. We say that ψ is an *existential second-order sentence* (of signature $\sigma = (L_s, L_f)$) if $\psi = \exists Y_1 \ldots \exists Y_r\ \phi$ where ϕ is a first-order sentence in $\mathrm{FO}_{\mathbb{R}}$ of signature $(L_s, L_f \cup \{Y_1, \ldots, Y_r\})$. The symbols $Y_1, \ldots, Y_r$ will be called *function variables*. The sentence ψ is true in a $\mathbb{R}$-structure $\mathfrak{D}$ of signature σ when there exist interpretations of $Y_1, \ldots, Y_r$ such that ϕ holds true on $\mathfrak{D}$. The set of existential second-order sentences will be denoted by $\exists\mathrm{SO}_{\mathbb{R}}$. Together with the interpretation above it constitutes *existential second-order logic*.

EXAMPLE 8 ([14]). Let us see how to describe 4-FEAS with an existential second-order sentence.

Consider the signature $(\emptyset, \{r, c\})$ where the arities of r and c are 1 and 4 respectively, and require that r is interpreted as a ranking.

Let $\mathfrak{D} = (\mathfrak{A}, \mathscr{F})$ be any $\mathbb{R}$-structure where $\mathscr{F}$ consists of interpretations $C: A^4 \rightarrow \mathbb{R}$ and $\rho: A \rightarrow \mathbb{R}$ of c and r. Let $n = |A| - 1$ so that ρ bijects A with $\{0, 1, \ldots, n\}$. Then $\mathfrak{D}$ defines a homogeneous polynomial $\widehat{g} \in \mathbb{R}[X_0, \ldots, X_n]$ of degree four, namely

$$\widehat{g} = \sum_{(i,j,k,\ell) \in A^4} C(i, j, k, \ell) X_i X_j X_k X_\ell.$$

We obtain an arbitrary, that is, not necessarily homogeneous, polynomial $g \in \mathbb{R}[X_1, \ldots, X_n]$ of degree four by setting $X_0 = 1$ in $\widehat{g}$. We also say that $\mathfrak{D}$ defines g. Notice that for every polynomial g of degree four in n variables there is a $\mathbb{R}$-structure $\mathfrak{D}$ of size $n + 1$ such that $\mathfrak{D}$ defines g.

Denote by $\mathtt{o}$, $\mathtt{1}$, $\bar{\mathtt{o}}$ and $\bar{\mathtt{1}}$ the first and last elements of A and A^4 with respect to ρ and ρ^4 respectively. The following sentence quantifies two functions $X: A \rightarrow \mathbb{R}$ and $Y: A^4 \rightarrow \mathbb{R}$

$$\begin{aligned} \psi \equiv (\exists X)(\exists Y)\ \big(Y(\bar{\mathtt{o}}) = C(\bar{\mathtt{o}})\ \&\ Y(\bar{\mathtt{1}}) = 0\ \&\ X(\mathtt{o}) = 1 \\ \forall u_1 \ldots \forall u_4\ [u \neq \bar{\mathtt{o}} \Longrightarrow \exists v_1 \ldots \exists v_4 (r^4(u) = r^4(v) + 1) \\ \&\ Y(u) = Y(v) + C(u) X(u_1) X(u_2) X(u_3) X(u_4)]\big). \end{aligned}$$

Here, if $a_i = \rho^{-1}(i)$ for $i = 1, \ldots, n$ then, $(X(a_1), \ldots, X(a_n)) \in \mathbb{R}^n$ describes the zero of g. The latter polynomial is now evaluated monomial by monomial at the point $(X(a_1), \ldots, X(a_n))$. This is done by letting u cycle through A^4 and using Y in order to hold the sum of all monomials of g up to the current u (according to the considered ordering on A^4). The update of Y is described by the last row of the formula. The sentence ψ describes 4-FEAS in the sense that for any $\mathbb{R}$-structure $\mathfrak{D}$ one has that $\mathfrak{D} \models \psi$ if and only if the polynomial g of degree four defined by $\mathfrak{D}$ has a real zero.

Again, the fact that existential second order logic describes a $\mathrm{NP}_{\mathbb{R}}$-complete problem is not fortuitous.

THEOREM 4 ([14]). $\exists\mathrm{SO}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$.

Denote by $\mathrm{SO}_{\mathbb{R}}$ the logic obtained by allowing quantifiers in Definition 15 to be both universal and existential. We then have the following result.

COROLLARY 1. $\mathrm{SO}_{\mathbb{R}} = \mathrm{PH}_{\mathbb{R}}$.

Nondeterminism for real machines as introduced in [4] appears as a sequence of "guesses" $y_1, \dots, y_m$ where the y_i are real numbers. It formalizes the idea of finding a witness in a continuous space and problems like 4-FEAS capture the difficulty of such a search. However, this kind of search is not the only possibility. In many situations a witness is found in a discrete space and we only need to guess one among a finite set of possibilities. Let us define this more formally (for further details see [8]).

DEFINITION 16. A set $S \subseteq \mathbb{R}^\infty$ belongs to $\mathrm{DNP}_{\mathbb{R}}$ when it can be decided in polynomial time by a nondeterministic machine whose guesses are restricted to be in the set $\{0, 1\}$. We say that the machine above is a *digital nondeterministic* machine.

Natural examples of problems in $\mathrm{DNP}_{\mathbb{R}}$ exist. The real versions of the Knapsack and the Travelling Salesman problems belong to $\mathrm{DNP}_{\mathbb{R}}$.

Theorems 2 and 4 suggest a logic for capturing $\mathrm{DNP}_{\mathbb{R}}$.

DEFINITION 17. Let (L_s, L_f) be a signature. A symbol $Y \in L_f$ is said to be *digital* when for every $\mathbb{R}$-structure its interpretation has image included in $\{0, 1\}$. We say that ψ is an *existential digital second-order sentence* (of signature (L_s, L_f) if $\psi = \exists Y_1 \dots \exists Y_r \phi$ where ϕ is a fixed point first-order sentence of signature $(L_s, L_f \cup \{Y_1, \dots, Y_r\})$. The symbols $Y_1, \dots, Y_r$ here are digital and will be called *subset variables*. As above, the set of existential second-order formulas will be denoted by $\exists\mathrm{DSO}_{\mathbb{R}}$. Digital second-order logic $\mathrm{DSO}_{\mathbb{R}}$ is obtained by allowing the quantifiers above to be universal as well.

A simple variation of Theorem 2 yields the following.

THEOREM 5. $\exists\mathrm{DSO}_{\mathbb{R}} = \mathrm{DNP}_{\mathbb{R}}$.

4.1. A fixed point rule for $\mathrm{DSO}_{\mathbb{R}}$. The last goal of this section is to capture the classes $\mathrm{EXP}_{\mathbb{R}}$ and $\mathrm{PAR}_{\mathbb{R}}$. To do so we will enhance $\mathrm{DSO}_{\mathbb{R}}$ with some fixed point rule to inductively define functions which take as arguments subsets of the universe as well as its elements. As with first-order logic, we also define a maximization rule to be able to write characteristic functions of formulas in $\mathrm{DSO}_{\mathbb{R}}$ as number terms.

In order to apply these functions we need to define the right class of terms to pass them as arguments. Index terms correspond to functions valued in the universe and number terms to real valued functions. None of them represent subsets of the universe. Subset terms as defined below represent subsets of A^k in a natural way.

DEFINITION 18. Let $\tau(x_1, \ldots, x_r)$ be a number term with free element variables $x_1, \ldots, x_r$. For any $\mathbb{R}$-structure $\mathfrak{D}$ consider the digital function $\bar{\tau}^{\mathfrak{D}}$

$$A^r \to \{0, 1\},$$
$$(v_1, \ldots, v_r) \to \begin{cases} 1 & \text{if } \tau^{\mathfrak{D}}(v_1, \ldots, v_r) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A *subset term* is either a digital function symbol X or an expression of the form $[\![\tau(x_1, \ldots, x_r)]\!]$. The interpretation of the subset term $[\![\tau(x_1, \ldots, x_r)]\!]$ is the function $\bar{\tau}^{\mathfrak{D}}$ above.

We also extend the definition of number term to include expressions of the form $F(x_1, \ldots, x_r)$ where F is a subset term of arity r and $x_1, \ldots, x_r$ are index terms.

EXAMPLE 9. Some simple examples of subset terms are given by the formulas

$$\tau_\emptyset(x_1, \ldots, x_r) = \bigwedge_{j=1}^{r} \rho(x_j) < 0$$

which defines the empty set,

$$\tau_T(x_1, \ldots, x_r) = \bigwedge_{j=1}^{r} \rho(x_j) \geq 0$$

which defines the total set $T = A^r$,

$$\tau_I(x_1, \ldots, x_r) = \bigwedge_{j=1}^{r} \rho(x_j) = 0$$

which defines the set $I = \{(\mathtt{o}, \ldots, \mathtt{o})\}$,

$$\tau_M(x_1, \ldots, x_r) = \bigwedge_{j=1}^{r} \rho(x_j) = n - 1$$

which defines the set $M = \{(\mathtt{1}, \ldots, \mathtt{1})\}$, and

$$\tau_{M+1}(x_1, \ldots, x_r) = \tau_M(x_1, \ldots, x_r) \vee \tau_I(x_1, \ldots, x_r)$$

which defines the set $M + 1 = \{(\mathtt{1}, \ldots, \mathtt{1}), (\mathtt{o}, \ldots, \mathtt{o})\}$. We shall denote by `O`, `T`, `I`, `M` and `M+1` the subset terms defined by them.

We now define the maximization and fixed point rules for digital second-order logic. The definition closely follows the one for first-order logic done in Section 3.

DEFINITION 19. Let S and $\bar{T} = (T_1, \ldots, T_s)$ be subset variables with arities k and $a_1, \ldots, a_s$ respectively. Let also $\bar{t} = (t_1, \ldots, t_r)$ be element variables and $F(S, \bar{t}, \bar{T})$ be a number term having S and $\bar{T}$ as free subset variables and $\bar{t}$ as free element variables. Then

$$\max_S F(S, \bar{t}, \bar{T})$$

is also a number term with free variables $\bar{t}, \bar{T}$. Its interpretation in any $\mathbb{R}$-structure $\mathfrak{D}$ for any sets $U_i \in \mathscr{P}(A^{a_i})$, $i = 1, \ldots, s$, interpreting $\bar{T}$ and any elements $v_1, \ldots, v_r$ in A interpreting $\bar{t}$ is the maximum of $F^{\mathfrak{D}}(V, \bar{U}, \bar{v})$ where V ranges over $\mathscr{P}(A^k)$.

DEFINITION 20. We denote by $\mathrm{DSO}_{\mathbb{R}} + \mathrm{MAX}^2_{\mathbb{R}}$ the logic obtained by adding to $\mathrm{DSO}_{\mathbb{R}}$ the maximization rule.

Again, the maximization rule allows us to write characteristic functions of digital second-order formulas as number terms.

PROPOSITION 4. *For all digital second-order formula φ there is a number term in $\mathrm{DSO}_{\mathbb{R}} + \mathrm{MAX}^2_{\mathbb{R}}$ describing $\chi[\varphi]$.*

We now define the fixed point rule. In order to do so we need to use a new kind of signature. Before introducing it, notice that digital function symbols can be interpreted in a natural way as elements in $\mathscr{P}(A^r)$, the set of parts of A^r where r is the arity of the symbol.

DEFINITION 21. A *second-order signature* is a triple (L_s, L_f, L_o) where (L_s, L_f) is a signature as defined in Definition 7 and L_o is a finite set of function symbols called *second-order functionals*. Each of them has associated a tuple $(r, s, a_1, \dots, a_s)$ of natural numbers. The number r is the *element arity*, s is the *subset arity*, and the numbers $a_1, \dots, a_s$ are the *argument arities*. A second-order functional takes as argument tuples of the form $(x_1, \dots, x_r, X_1, \dots, X_s)$ where x_i is an index term for $i = 1, \dots, r$ and X_i is a subset term of arity a_i for $i = 1, \dots, s$. Thus a symbol $\mathscr{Z}$ in L_o is interpreted on a $\mathbb{R}$-structure $\mathfrak{D}$ of universe A as a function

$$\mathscr{Z}^{\mathfrak{D}} : A^r \times \mathscr{P}(A^{a_1}) \times \dots \times \mathscr{P}(A^{a_s}) \to \mathbb{R}.$$

REMARK 6.

(i) A digital function symbol, say X, can occur in a formula in two different ways. This is the case for instance in the term

$$\mathscr{Z}(X) + X(x)$$

in which $\mathscr{Z}$ denotes a second-order functional and x an element variable.

(ii) We won't consider $\mathbb{R}$-structures over second-order signatures in what follows but only as a technical trick to define the second-order fixed point rule.

DEFINITION 22. Let s, $a_1, \dots, a_s$ be natural numbers and X_i be subset variables of arity a_i for $i = 1, \dots, s$. Let also $x_1, \dots, x_r$ be r element variables, $r \in \mathbb{N}$. Denote $\bar{x} = (x_1, \dots, x_r)$ and $\bar{X} = (X_1, \dots, X_s)$.

Fix a signature $\sigma = (L_s, L_f)$ and a pair $(\mathscr{Z}, \mathscr{D})$ of second-order functionals both of element arity r, subset arity s and argument arities $a_1, \dots, a_s$. Thus $\widetilde{\sigma} = (L_s, L_f, \{\mathscr{Z}, \mathscr{D}\})$ is a second-order signature. Let $F(\mathscr{Z}, \bar{x}, \bar{X})$ and $H(\mathscr{D}, \bar{x}, \bar{X})$ be number terms of signature $(L_s, L_f, \{\mathscr{Z}, \mathscr{D}\})$ which have $\bar{X}$ as free subset variable and $\bar{x}$ as free element variables.

For any $\mathbb{R}$-structure $\mathfrak{D}$ of signature σ and interpretations

$$\zeta, \Delta\colon A^r \times \mathscr{P}(A^{a_1}) \times \dots \times \mathscr{P}(A^{a_s}) \to \mathbb{R}$$

of $\mathscr{Z}$ and $\mathscr{D}$ respectively the number terms $F(\mathscr{Z}, \bar{x}, \bar{X})$ and $H(\mathscr{D}, \bar{x}, \bar{X})$ define functions

$$F^{\mathfrak{D}}_{\zeta}, H^{\mathfrak{D}}_{\Delta} : A^r \times \mathscr{P}(A^{a_1}) \times \dots \times \mathscr{P}(A^{a_s}) \to \mathbb{R}.$$

Consider the sequence of pairs $\{\Delta^i, \zeta^i\}_{i\geq 0}$ with $\zeta^i : A^r \times \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s}) \to \mathbb{R}$ and $\Delta^i : A^r \times \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s}) \to \mathbb{R}$ inductively defined by

$$\Delta^0(\bar{u}, \bar{U}) = 0 \quad \text{for all } (\bar{u}, \bar{U}) \in A^r \times \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s}),$$
$$\zeta^0(\bar{u}, \bar{U}) = 0 \quad \text{for all } (\bar{u}, \bar{U}) \in A^r \times \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s}),$$

$$\Delta^{i+1}(\bar{u}, \bar{U}) = \begin{cases} H^{\mathfrak{D}}_{\Delta^i}(\bar{u}, \bar{U}) & \text{if } \Delta^i(\bar{u}, \bar{U}) = 0 \\ \Delta^i(\bar{u}, \bar{U}) & \text{otherwise,} \end{cases}$$

$$\zeta^{i+1}(\bar{u}, \bar{U}) = \begin{cases} F^{\mathfrak{D}}_{\zeta^i}(\bar{u}, \bar{U}) & \text{if } \Delta^i(\bar{u}, \bar{U}) = 0 \\ \zeta^i(\bar{u}, \bar{U}) & \text{otherwise.} \end{cases}$$

Fixed points $\mathscr{Z}^\infty$ and $\mathscr{D}^\infty$ are defined as in the preceding section. Now however. the number of steps in the inductive definition is bounded by 2^{n^ℓ} where $n = |A|$ for a suitable ℓ.

EXAMPLE 10. Consider a signature of ranked $\mathbb{R}$-structures and let r be the symbol for the ranking. We will define a function $\widetilde{r^k}$ such that for all ranked $\mathbb{R}$-structures $\mathfrak{D}$ the interpretation of $\widetilde{r^k}$ is

$$\widetilde{\rho^k} : \mathscr{P}(A^k) \to \{0, \ldots, 2^{n^k} - 1\}$$
$$U \to \sum_{x \in U} 2^{\rho^k(x)}$$

where A is the universe of $\mathfrak{D}$, $n = |A|$ and ρ is the interpretation of the ranking.

To do so, consider the number terms $F(Z, X)$

$$\max_U \chi[\exists \bar{x}\ X = U \cup \{\bar{x}\}](Z(U) + 2^{r^k}(\bar{x}))$$

and $H(D, X)$

$$\chi[X = \emptyset] + \max_U \chi[\exists \bar{x}\ X = U \cup \{\bar{x}\}]D(U)$$

where $\bar{x} = (x_1, \ldots, x_k)$. One can check that Z^∞ is the function $\widetilde{\rho^k}$ above. Here we are using the exponential function 2^{r^k} described in Example 5.

Notice that $\widetilde{\rho^k}(\emptyset) = 0$, $\widetilde{\rho^k}(T) = 2^{n^k} - 1$, $\widetilde{\rho^k}(I) = 1$, $\widetilde{\rho^k}(M) = 2^{n^k-1}$, and $\widetilde{\rho^k}(M+1) = 2^{n^k-1} + 1$ for the sets T, I, M and $M+1$ defined in Example 9.

DEFINITION 23. *Fixed point digital second-order logic* for $\mathbb{R}$-structures, denoted $\mathrm{FP}^2_{\mathbb{R}}$, is obtained by augmenting $\mathrm{DSO}_{\mathbb{R}}$ with the maximization rule and the fixed point rule described above.

The maximization rule can also be used in conjunction with $\widetilde{\rho}$ to express some arithmetic in the universe. The following example of this capacity will be useful in Theorem 6.

EXAMPLE 11. Let S, $\bar{t}$ and $\bar{T}$ as in the definition above and let $F(S, \bar{t}, \bar{T})$ be a number term. Suppose that r is a function symbol interpreted as a ranking ρ and consider the number term

$$\max_U \chi[\widetilde{r^k}(U) = \widetilde{r^k}(S) + 1]F(U, \bar{t}, \bar{T}) - \max_U \chi[\widetilde{r^k}(U) = \widetilde{r^k}(S) + 1](-F(U, \bar{t}, \bar{T}))$$

also with free variables S, $\bar{t}$ and $\bar{T}$.

For every $\mathbb{R}$-structure $\mathfrak{D}$ with universe A and every elements $U \in \mathscr{P}(A^k)$, $v \in A^r$, and $W \in \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s})$ interpreting U, $\bar{t}$ and $\bar{T}$ respectively and such that $\widetilde{\rho^k}(U) \neq 0$ this new term is interpreted as $F^{\mathfrak{D}}(V, v, W)$ where V is the only element in $\mathscr{P}(A^k)$ such that $\widetilde{\rho^k}(V) = \widetilde{\rho^k}(U) - 1$. If $\widetilde{\rho^k}(U) = 0$ then it is interpreted as zero. We will denote this term by $F(S-1, \bar{t}, \bar{T})$.

THEOREM 6. *For ranked* $\mathbb{R}$*-structures* $\mathrm{FP}^2_{\mathbb{R}} = \mathrm{EXP}_{\mathbb{R}}$.

PROOF. Let S be a set of ranked $\mathbb{R}$-structures and M be a machine which decides S in exponential time. That is, for any $\mathbb{R}$-structure $\mathfrak{D}$ of size n the computation of M with input $\mathfrak{D}$ runs in time bounded by 2^{n^m} for a fixed m. We can assume that the state space of M is $\mathbb{R}^{2^{n^m}}$. A point in this space has coordinates $(x_{-2^{n^m-1}}, \ldots, x_0, x_1, \ldots, x_{2^{n^m-1}-1})$. Notice that the zeroth coordinate x_0 is in the 2^{n^m-1}th position in this vector.

We can also assume without loss of generality that the functions associated to the computation nodes of M are single arithmetical operations between two coordinates of the state space and that the result of such an operation is stored in the zeroth coordinate of its state space.

Let $\mathfrak{D}$ be a ranked $\mathbb{R}$-structure of size n and A be its universe. Denote by x the point $e(\mathfrak{D})$ refered to in Example 1.

Given a subset $J \in \mathscr{P}(A^k)$ denote by $\widetilde{J}$ the integer $\widetilde{\rho^k}(J)$ where ρ denotes the ranking on A.

Consider the function $\zeta\colon A \times \mathscr{P}(A^m) \times \mathscr{P}(A^m) \to \mathbb{R}$ such that

- $\zeta(\mathtt{0}, V, T)$ is the node reached by M after $\widetilde{T}+1$ steps
- $\zeta(\mathtt{1}, V, T)$ is the content of the $\widetilde{V}$th register of M after $\widetilde{T}+1$ steps

where we denote by $\mathtt{0}$ and $\mathtt{1}$ the two elements in A which are mapped to 0 and 1 by ρ.

If ζ can be defined within $\mathrm{FP}^2_{\mathbb{R}}$ by a number term Z then the inclusion $\mathrm{EXP}_{\mathbb{R}} \subset \mathrm{FP}^2_{\mathbb{R}}$ follows. The $\mathbb{R}$-structure $\mathfrak{D}$ is accepted by M if and only if $Z(\mathtt{1}, \mathtt{M+1}, \mathtt{T}) = 1$ where $\mathtt{M+1}$ and $\mathtt{T}$ are the subset terms defined in Example 9.

Our goal is to define ζ inductively. As a first attempt to do so we can write

$$Z(u, V, T) = \chi[\rho(u) = 0] Z(\mathtt{0}, V, T) + \chi[\rho(u) = 1] Z(\mathtt{1}, V, T). \tag{1}$$

Moreover, we have that

$$Z(\mathtt{0}, V, \mathtt{0}) = \beta(1)$$

where $\beta(1)$ is the succesor of the input node, and that

$$Z(\mathtt{1}, V, \mathtt{0}) = x_V$$

where x_V is the $\widetilde{V}$th coordinate of the image of x under the input map of M. This value can be described with a first-order term according to Example 1.

Therefore, we only need to find number terms defining $Z(\mathtt{0}, V, T)$ and $Z(\mathtt{1}, V, T)$ for $\widetilde{T} \geq 1$.

The first function above is simply dealt with since for $\widetilde{T} \geq 1$

$$Z(0, V, T) = \sum_{r=2}^{N} \chi[Z(0, V, T-1) = r]\beta(r)$$

where if the rth node is not a branching node then $\beta(r)$ is a well defined value and otherwise it is $\beta^+(r)$ or $\beta^-(r)$ according to whether $x_0 \geq 0$ or $x_0 < 0$. The value $\beta(r)$ in this latter case can be described with the number term

$$\chi[Z(1, \mathtt{M}, T-1) \geq 0]\beta^+(r) + \chi[Z(1, \mathtt{M}, T-1) < 0]\beta^-(r).$$

Here N is the number of nodes of the machine M.

Thus, the term

$$\chi[\widetilde{T} = 0]\beta(1) + \chi[\widetilde{T} \neq 0] \sum_{r=2}^{N} \chi[Z(0, V, T-1) = r]\beta(r) \tag{2}$$

describes $Z(0, V, T)$.

We pass now to the value $Z(1, V, T)$. Denote by $\mathscr{C}$ the set of computation nodes. Then at a node $r \in \mathscr{C}$ the value $Z(1, V, T)$ is given by the term $Z_r(1, V, T)$

$$\chi[\widetilde{V} = 0](Z(1, \mathtt{K}, T-1) \circ_r Z(1, \mathtt{G}, T-1)) + \chi[\widetilde{V} \neq 0]Z(1, V, T-1)$$

where the $\circ_r$ is the operation performed at node r and it operates the $\widetilde{\mathtt{K}}$th and $\widetilde{\mathtt{G}}$th registers of M.

Denote now by $\mathscr{S}_\ell$ the set of shift nodes which shift to the left and by $\mathscr{S}_r$ the set of shift nodes which shift to the right. For $\widetilde{T} \geq 1$ the value $Z(1, V, T)$ is then given by

$$\begin{aligned} &\sum_{r \notin (\mathscr{C} \cup \mathscr{S}_\ell \cup \mathscr{S}_r)} \chi[Z(0, V, T-1) = r]Z(1, V, T-1) \\ &\quad + \sum_{r \in \mathscr{C}} \chi[Z(0, V, T-1) = r]Z_r(1, V, T) \\ &\quad + \sum_{r \in \mathscr{S}_\ell} \chi[Z(0, V, T-1) = r]Z(1, V-1, T-1) \\ &\quad + \sum_{r \in \mathscr{S}_r} \chi[Z(0, V, T-1) = r]Z(1, V+1, T-1). \end{aligned} \tag{3}$$

Replacing terms (2) and (3) in (1) we get the number term which defines Z as a fixed point. The condition $Z(1, \mathtt{M}, \mathtt{T}) = 1$ is now be expressed in fixed point digital second-order logic. This proves that sets of $\mathbb{R}$-structures decidable in exponential time are definable within $\mathrm{FP}^2_{\mathbb{R}}$.

To prove the converse is simpler. We only need to prove that number terms defined with the second-order maximization and fixed point operator can be evaluated in exponential time. For the first operator, one only evaluates the term to maximize over all the elements of $\mathscr{P}(A^k)$ and selects the maximum. Since this number of elements is exponential in $|A|$ the assertion follows. For the second operator, one applies the inductive definition. Again, since the number of updates is exponentially bounded so is the total time required to do this. ⊣

COROLLARY 2. *Let $\varphi\colon \mathbb{N}^2 \to \mathbb{R}$ be a function such that $\varphi(n,i)$ is computable in time $t(n) \le 2^{n^q}$ for $q \in \mathbb{N}$. Then there exists a number term $\widehat{\varphi}(U)$ in $\mathrm{FP}^2_{\mathbb{R}}$ such that for all ranked structure $\mathfrak{D}$ of size n*

$$\widehat{\varphi}^{\mathfrak{D}} : \mathscr{P}(A^q) \to \mathbb{R}$$
$$U \to \varphi(n, \widetilde{\rho^q}(U)).$$

Moreover the fixed point operator is used only one time in $\widehat{\varphi}(U)$ and the number of updatings needed to reach the fixed point is bounded by $t(n)$.

4.2. A bounded depth fixed point rule for $\mathrm{DSO}_{\mathbb{R}}$. We define the *bounded depth fixed point rule* for digital second order logic as we defined the fixed point rule for this logic but with the following updating scheme.

$$\Delta^0(\bar{u}, \bar{U}) = 0 \quad \text{for all } (\bar{u}, \bar{U}) \in A^r \times \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s}),$$
$$\zeta^0(\bar{u}, \bar{U}) = 0 \quad \text{for all } (\bar{u}, \bar{U}) \in A^r \times \mathscr{P}(A^{a_1}) \times \ldots \times \mathscr{P}(A^{a_s}),$$

$$\Delta^{i+1}(\bar{u}, \bar{U}) = \begin{cases} H^{\mathfrak{D}}_{\Delta^i}(\bar{u}, \bar{U}) & \text{if } \Delta^i(\bar{u}, \bar{U}) = 0 \text{ and } i \le n^\ell \\ \Delta^i(\bar{u}, \bar{U}) & \text{otherwise,} \end{cases}$$

$$\zeta^{i+1}(\bar{u}, \bar{U}) = \begin{cases} F^{\mathfrak{D}}_{\zeta^i}(\bar{u}, \bar{U}) & \text{if } \Delta^i(\bar{u}, \bar{U}) = 0 \\ \zeta^i(\bar{u}, \bar{U}) & \text{otherwise.} \end{cases}$$

Notice that the only difference in this new sheme is that the number of steps in the inductive definition is forced to be at most n^ℓ where $n = |A|$ and ℓ is a natural number.

EXAMPLE 12. The function $\widetilde{r}$ defined in Example 10 actually uses n updates and therefore it is defined with only the bounded depth fixed point rule.

DEFINITION 24. *Bounded depth fixed point digital second-order logic* for $\mathbb{R}$-structures, denoted $\mathrm{BFP}^2_{\mathbb{R}}$, is obtained by augmenting $\mathrm{DSO}_{\mathbb{R}}$ with the maximization rule and the bounded depth fixed point rule.

REMARK 7. The number of updates is closely related to the running time of algorithms. That is why for $\mathrm{EXP}_{\mathbb{R}}$ we needed second-order structures to describe the entire computation. For $\mathrm{PAR}_{\mathbb{R}}$ we only need polynomially many updates. On the other hand, for $\mathrm{PAR}_{\mathbb{R}}$ exponentially many objects must be described—the gates of a circuit. Here second-order structures are used to name these objects.

THEOREM 7. $\mathrm{BFP}^2_{\mathbb{R}} = \mathrm{PAR}_{\mathbb{R}}$.

PROOF. We only sketch the proof since it follows the same lines of that of Theorem 3. Let S be a set of $\mathbb{R}$-structures in $\mathrm{PAR}_{\mathbb{R}}$. Then, there is a family of circuits $\{C_n\}_{n\in\mathbb{N}}$ with size at most 2^{n^q} and depth at most cn^k which decides S. Moreover there is a machine M which, on input $(n,i) \in \mathbb{N}^2$ returns a point in $\mathbb{R}^5$ encoding the ith gate of C_n in time polynomial in n.

By Corollary 2 there are $\mathrm{FP}^2_{\mathbb{R}}$ number terms $\hat{t}(v)$, $\hat{l}(v)$, $\hat{r}(v)$ and $\hat{c}(v)$ with $v = (v_1, \ldots, v_q)$ such that $(r(v), \hat{t}(v), \hat{l}(v), \hat{r}(v), \hat{c}(v))$ describes the $r(v)$th gate of circuit C_n. One now extends the idea of Theorem 3 and Example 6 to evaluate C_n over input $\mathfrak{D}$. ⊣

REMARK 8. As a consequence of the above theorem the following problem is complete in $\mathrm{PAR}_{\mathbb{R}}$ for polynomial time reductions: given a $\mathbb{R}$-structure and a $\mathrm{BFP}^2_{\mathbb{R}}$ sentence, decide whether the structure satisfies the sentence. This is, to the best of our knowledge, the first mention of a $\mathrm{PAR}_{\mathbb{R}}$-complete problem.

COROLLARY 3. *Let σ be a signature of ranked $\mathbb{R}$-structures. For every sentence φ in $\mathrm{SO}_{\mathbb{R}}$ over σ there exists a sentence $\widehat{\varphi}$ in $\mathrm{BFP}^2_{\mathbb{R}}$ over σ equivalent to φ. That is, for every $\mathbb{R}$-structure $\mathfrak{D}$*

$$\mathfrak{D} \models \varphi \iff \mathfrak{D} \models \widehat{\varphi}.$$

PROOF. It follows from Corollary 1, Theorem 7 and the fact that $\mathrm{PH}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$ shown in [24]. ⊣

REMARK 9. Theorems 5, 6 and 7 (and Corollary 3) were shown for ranked $\mathbb{R}$-structures. It is possible to avoid this hypothesis and define the ranking within $\exists\mathrm{DSO}_{\mathbb{R}}$. Roughly, one first considers a digital function $R\colon A^2 \to \{0, 1\}$ describing a total order in A. The existence of R is stated within $\exists\mathrm{DSO}_{\mathbb{R}}$. Then, a number term $r(u)$ for the ranking is defined as a fixed-point for first-order logic.

§5. A logic for $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$.

As was already mentioned over $\mathbb{R}$ it does not make sense to consider space resources alone. In order to find a reasonable analogue of class PSPACE over the reals one possibility is provided by using other characterizations of PSPACE in the discrete setting. In the preceding section we did so for the class $\mathrm{PAR}_{\mathbb{R}}$. In this section we want to consider the class of sets being computable by machines over $\mathbb{R}$ which use a polynomial number of registers and work in exponential time. This class will be denoted by $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$. Whereas in the discrete theory the demand of working in exponential time is inherently satisfied for any PSPACE algorithm, in the real setting the situation is different. It is undecidable whether a computation terminates even for a constant space computation.

Theorem 6 already captured exponential time computations without any further restriction to place resources. The $\mathrm{FP}^2_{\mathbb{R}}$ logic defined above captures computations which can use an exponential amount of registers. In order to deal with computations for sets in class $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$ we will therefore have to restrict the $\mathrm{FP}^2_{\mathbb{R}}$ logic (unless $\mathrm{EXP}_{\mathbb{R}} = (\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$, which is widely conjectured to be false). The way to do this is quite natural. We simply force the updating terms F and H used in the fixed point construction to have a very special structure; for updating some $\zeta^{i+1}(\bar{u}, \bar{U})$ and $\Delta^{i+1}(\bar{u}, \bar{U})$ respectively, the terms F and H are allowed to depend on special former values of ζ^i and Δ^i only. This restriction will concern the dependence on the subset variables.

In order to make the following definition more clear let us first revisit Example 9. We want to define the ranking $\widetilde{\rho}$ of that example once again, this time in a slightly different way. The reason for doing so will become apparent in the subsequent definition and the proof of Theorem 8.

EXAMPLE 13. Consider a signature of ranked $\mathbb{R}$-structures and let r be the symbol for the ranking. We will define a function $\widetilde{r}$ such that for all ranked $\mathbb{R}$-structures $\mathfrak{D}$

the interpretation of $\widetilde{r}$ is

$$\widetilde{\rho}\colon \mathscr{P}(A) \to \{0, \dots, 2^n - 1\}$$
$$U \to \sum_{x \in U} 2^{\rho(x)}$$

where A is the universe of $\mathfrak{D}$, $n = |A|$ and ρ is the interpretation of the ranking. For $U = \emptyset$ the sum equals 0 by convention.

(a) We want to show for two elements $X, Y \in \mathscr{P}(A)$ that the property "$\widetilde{\rho}(X) - 1 = \widetilde{\rho}(Y)$" is expressible in $\mathrm{DSO}_{\mathbb{R}} + \mathrm{MAX}^2_{\mathbb{R}}$. To do so, note that $\widetilde{\rho}(X) > \widetilde{\rho}(Y)$ if and only if the following formula holds:

$$\exists x \; X(x) \;\&\; \forall y \; Y(y) \Longrightarrow r(y) \neq r(x)$$
$$\&\; \forall y \; (Y(y) \;\&\; r(y) > r(x)) \Longrightarrow (\exists z \; X(z) \;\&\; r(y) = r(z)).$$

Thus the property $\widetilde{\rho}(Y) = \widetilde{\rho}(X) - 1$ is expressible via

$$\widetilde{\rho}(X) > \widetilde{\rho}(Y) \;\&\; \forall Z \; \widetilde{\rho}(Z) > \widetilde{\rho}(Y) \Longrightarrow \widetilde{\rho}(Z) \geq \widetilde{\rho}(X).$$

In the sequel we will write $Y = X - 1$ having in mind that this property is expressible in the above way.

(b) Now let us define $\widetilde{\rho}$ as a fixed point. Consider the terms

$$F(\mathscr{Z}, X) = \mathscr{Z}(X - 1) + \chi[\psi(X)] \cdot 2^{r(x)}$$

and

$$H(\mathscr{D}, X) = \chi[X = \emptyset] + \chi[\psi(X)] \cdot (\mathscr{D}(X - 1) + 1)$$

where

$$\psi(X) := \exists x \; X(x) \;\&\; (Y = X - 1 \Longrightarrow \neg Y(x)).$$

To check that $\mathscr{Z}^\infty$ is the function $\widetilde{\rho}$ above note that

$$\zeta^i(U) = \sum_{\substack{x \in U \\ r(x) \leq i-1}} 2^{r(x)} \quad \text{and} \quad \Delta^i(X) = \begin{cases} i & \text{if } U \subset \{0, \dots, i-1\} \\ 0 & \text{otherwise.} \end{cases}$$

Here we are using the exponential function 2^r described in Example 5. In a similar manner one can extend r to a ranking in $\mathscr{P}(A^s)$ for any $s \in \mathbb{N}$.

DEFINITION 25. With the notations of Definition 22 consider number terms $F(\mathscr{Z}, \bar{x}, \bar{X})$ and $H(\mathscr{D}, \bar{x}, \bar{X})$ of the following form: whenever in $F(\mathscr{Z}, \bar{x}, \bar{X})$ the symbol $\mathscr{Z}$ is applied to an argument

$$(T_1, \dots, T_s) \in \mathscr{P}(A^{a_1}) \times \cdots \times \mathscr{P}(A^{a_s})$$

then $T_i = X_i - 1, \; i = 1, \dots, s$, i.e.,

$$F(\mathscr{Z}, \bar{x}, \bar{X}) = \tilde{F}(\bar{x}, \mathscr{Z}(X_1 - 1, \dots, X_s - 1))$$

for some number term $\tilde{F}$ (cf. Example 13, part (a)). The same should hold true for $H(\mathscr{D}, \bar{x}, \bar{X})$ with respect to applications of $\mathscr{D}$.

Fixed points $\mathscr{Z}^\infty$ and $\mathscr{D}^\infty$ now are defined by means of F and H just as it is done in Definition 22.

Primitive fixed-point second-order logic for ranked $\mathbb{R}$-structures, denoted by $\mathrm{pFP}^2_{\mathbb{R}}$, now is obtained by restricting $\mathrm{FP}^2_{\mathbb{R}}$ logic as explained above.

REMARK 10. The above restriction on the values F and H are allowed to fall back upon is quite natural, for example in defining primitive recursion. Related updating rules are already used in [14] to characterize constant space polynomial time computations as primitive recursive global functions, and also in discrete descriptive complexity theory (f.e. in [12]).

Before stating the next theorem note that the ranking on $\mathscr{P}(A^r)$ induced by that on A according to Example 13 is defined in $\mathrm{pFP}^2_{\mathbb{R}}$ logic. This will be needed below.

THEOREM 8. $\mathrm{pFP}^2_{\mathbb{R}} = (\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$.

PROOF. The inclusion "$\subseteq$" can be seen as follows. If $F(\mathscr{Z}, \bar{x}, \bar{X})$ and $H(\mathscr{D}, \bar{x}, \bar{X})$ are number terms computable in $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$ then so are the fixed points $\mathscr{Z}^\infty$ and $\mathscr{D}^\infty$. In order to compute some $\zeta^{i+1}(\bar{x}, \bar{X})$, $\Delta^{i+1}(\bar{x}, \bar{X})$ one only has to know the polynomially many values

$$\zeta^i(\bar{y}, X_1 - 1, \dots, X_s - 1), \quad \Delta^i(\bar{y}, X_1 - 1, \dots, X_s - 1),$$

$\bar{y} \in A^k$ and evaluate F and H in exponential time and polynomial space. The assertion now follows by closedness of $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$ under the $\mathrm{MAX}^2_{\mathbb{R}}$ rule.

For the reversed inclusion we closely follow the proof of Theorem 6. We express the computation of a machine M computing a function in $(\mathrm{EXP}_{\mathbb{R}}, \mathrm{PSPACE}_{\mathbb{R}})$ by means of a fixed point $(\mathscr{Z}^\infty, \mathscr{D}^\infty)$. Assume M to work in time 2^{n^m} and space n^k. This time the computation is described by means of a fixed point construction for a function $\zeta : A \times A^k \times \mathscr{P}(A^m) \to \mathbb{R}$. Now the intermediate updates ζ^i and Δ^i should have the following "interpretations"; here $\widetilde{\rho}$ once again denotes the interpretation of the ranking on $\mathscr{P}(A^m)$. If $\widetilde{T} := \widetilde{\rho}(T) \leq i$ then $\zeta^i(0, \bar{x}, T)$ describes the node reached after $\widetilde{T} + 1$ many steps, whereas $\zeta^i(1, \bar{x}, T)$ denotes the content of register $\rho(\bar{x})$ after $\widetilde{T} + 1$ many steps of M's computation. Applying the proof of Theorem 6 it is an easy exercise to express the function ζ in $\mathrm{pFP}^2_{\mathbb{R}}$ logic. The only thing worth mentioning here is that the updates only rely on the values at the previous time-step; thus the updating formulae just have to fall back upon $T - 1$. This yields the claim. ⊣

§6. Miscellaneous.

In this closing section we want to outline briefly some more results on how other important complexity classes in the real setting can be captured by logics on $\mathbb{R}$-structures.

6.1. Capturing functional versions of $\mathrm{DNP}_{\mathbb{R}}$ and $\mathrm{NP}_{\mathbb{R}}$. Along the preceding sections we have seen two different kind of descriptions. Describing decision problems leads to the examination of *sentences* in certain logics, whereas the description of functions gives rise to study *number terms*. In order to switch from the former to the latter (i.e., capturing characteristic functions of decision problems) maximization rules turn out to be useful in replacing quantifiers. Typical examples appeared in Propositions 2 and 4 above. Note that according to Definition 20 a number term in $\mathrm{DSO}_{\mathbb{R}} + \mathrm{MAX}^2_{\mathbb{R}}$ logic can contain the $\mathrm{MAX}^2_{\mathbb{R}}$ operator at different places within the term; it does not necessarily appear only at the left most end. This is the reason why the characteristic function of any digital second-order sentence φ can be expressed in this logic (see Proposition 4). If we want to define number terms capturing characteristic functions of *existential* second-order sentences only we have to restrict the use of maximization operators.

In the following we identify $NP_{\mathbb{R}}$ as well as $DNP_{\mathbb{R}}$ machines with the characteristic functions of the languages they accept.

REMARK 11. An extension of the above definition to arbitrary nondeterministically computable functions is straightforward if digital nondeterminism is considered. This is due to the fact that the maximum always exists in that case. For arbitrary machines over $\mathbb{R}$ this may cause trouble. That is the reason why we restrict ourselves to characteristic functions.

In order to define a class of number terms representing exactly these classes of characteristic functions let us revisit Definition 12. Exept with the function symbols $\mathscr{Z}$ and $\mathscr{D}$ we enlarge the set L_f further by a function symbol Y. Thus F and H may be number terms additionally depending on Y, too and the same holds true for the fixed point. Consider all number terms obtained by closing first-order logic $FO_{\mathbb{R}}$ with the maximization rule and the extended fixed point rule (cf. Definition 13). These terms now may contain free subset variables $Y_1, \ldots, Y_r$, being introduced via the fixed point rule. In order to bound them another maximization operator is applied: if $F(Y_1, \ldots, Y_r, t)$ is such a number term $\max_{Y_1, \ldots, Y_r} F(Y_1, \ldots, Y_r, t)$ yields a term with free variable t.

DEFINITION 26. Let $\exists MAX^2_{\mathbb{R}}$ be the set of terms obtained from $FP^1_{\mathbb{R}}$ by applying the above maximization rule; let $\exists DMAX^2_{\mathbb{R}}$ denote that subset of $\exists MAX^2_{\mathbb{R}}$ obtained if all Y_i are digital function symbols.

Note that we allow the appearance of the $\exists MAX^2_{\mathbb{R}}$ rule only once at the left most end of a number term in $FP^1_{\mathbb{R}}$. That is the reason why $\exists MAX^2_{\mathbb{R}}$ and $\exists DMAX^2_{\mathbb{R}}$ will only catch existential quantifiers.

THEOREM 9. *The terms in* $\exists MAX^2_{\mathbb{R}}$ *representing characteristic functions equal* $NP_{\mathbb{R}}$; *those in* $\exists DMAX^2_{\mathbb{R}}$ *representing characteristic functions equal* $DNP_{\mathbb{R}}$.

PROOF. The proof basically follows the same lines as that of Theorem 2. The inclusions in $NP_{\mathbb{R}}$ and $DNP_{\mathbb{R}}$ resp. follow by guessing those interpretations for the Y_i yielding the maximum. For the converse consider a function Z describing the content of the machine registers. Z is updated via a number term $F(x, Y, Z)$ representing the evolution of a computation if Y is guessed. $\dashv$

REMARK 12. A similar characterization of nondeterministic computations in the discrete setting can be found in [12].

6.2. Capturing $NP^{[k]}_{\mathbb{R}}$. Whereas for arbitrary nondeterministic computations the number of real guesses can increase polynomially with the input dimension, the class $DNP_{\mathbb{R}}$ represents a (probably) very restricted notion of nondeterminism: each guessed vector $v \in \{0,1\}^\infty$ can be coded in a single real. We thus have $DNP_{\mathbb{R}} \subset NP^{[1]}_{\mathbb{R}}$, where the "exponent" 1 indicates that for any problem instance guessing a single real number suffices to verify membership in the given language. In the same way we can define classes $NP^{[k]}_{\mathbb{R}}$ for any natural number k (see [18]). Note that

$$P_{\mathbb{R}} \subset DNP_{\mathbb{R}} \subset NP^{[1]}_{\mathbb{R}} \subset NP^{[2]}_{\mathbb{R}} \subset \cdots \subset NP^{[k]}_{\mathbb{R}} \subset \cdots \subset \bigcup_{n \in \mathbb{N}} NP^{[n]}_{\mathbb{R}} \subset NP_{\mathbb{R}}$$

and none of the above inclusions is known to be strict.

We'll close this section by presenting logics which capture the classes $\mathrm{NP}_{\mathbb{R}}^{[k]}$.

DEFINITION 27 (cf. Definition 17). Let (L_s, L_f) be a signature. We say that ψ is a *nullary existential second-order* sentence of signature (L_s, L_f) if $\psi = \exists Y_1 \ldots \exists Y_r \phi$ where ϕ is a fixed point first-order sentence of signature $(L_s, L_f \cup \{Y_1, \ldots, Y_r\})$ and all Y_i are nullary functions, i.e., of arity 0. The set of nullary existential second-order sentences is denoted by *null*-$\exists\mathrm{SO}_{\mathbb{R}}$.

For a given $k \in \mathbb{N}$ the set *null*-$\exists_k\mathrm{SO}_{\mathbb{R}}$ is given by all nullary existential second-order sentences ψ, where the number r of existential quantifiers does not exceed k.

THEOREM 10. *Let S be a decision problem of ranked $\mathbb{R}$-structures.*

(a) *The following two statements are equivalent for all $k \in \mathbb{N}$.*
  (i) $S \in \mathrm{NP}_{\mathbb{R}}^{[k]}$,
  (ii) *there exists a sentence ψ in null-$\exists_k\mathrm{SO}_{\mathbb{R}}$ such that $S = \{\mathfrak{D} \mid \mathfrak{D} \models \psi\}$.*

(b) *The following two statements are equivalent.*
  (i) $S \in \bigcup_{n\in\mathbb{N}} \mathrm{NP}_{\mathbb{R}}^{[n]}$,
  (ii) *there exists a sentence ψ in null-$\exists\mathrm{SO}_{\mathbb{R}}$ such that $S = \{\mathfrak{D} \mid \mathfrak{D} \models \psi\}$.*

PROOF. Straightforward from the proof of Theorem 4 in [14]. The nullary existential quantifiers are used to represent a guess, whereas the subsequent polynomial time computation can be expressed via a fixed point first-order formula. ⊣

REMARK 13.

(i) First-order logic for finite structures already allows guessing a special state of a Turing machine working for example in logarithmic space. This is used in many characterizations for classes like LOGSPACE or NLOGSPACE (together with so-called transitive closure logics, see [17]). First-order logic for $\mathbb{R}$ does not share this property since the quantifiers do not range over symbols from the second part of the signature. The extension to nullary second-order logic at least allows to guess a particular state of a machine working in constant space. With respect to the meaning of constant space computations over $\mathbb{R}$ see again [21] and [14].

(ii) Descriptive complexity over the reals as presented here can also be used in order to provide a logical study of counting problems over $\mathbb{R}$. The interested reader is refered to [19] and [15].

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DEPARTMENT OF MATHEMATICS
CITY UNIVERSITY OF HONG KONG
83 TAT CHEE AVENUE, KOWLOON TONG, HONG KONG
E-mail: macucker@sobolev.cityu.edu.hk

LEHRSTUHL C FÜR MATHEMATIK
RWTH AACHEN
D-52056 AACHEN, GERMANY
E-mail: meer@rwth-aachen.de