Symbolic Variable Elimination in Discrete and Continuous Graphical Models

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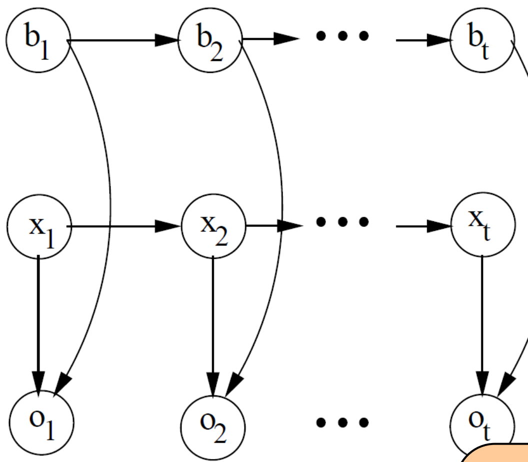
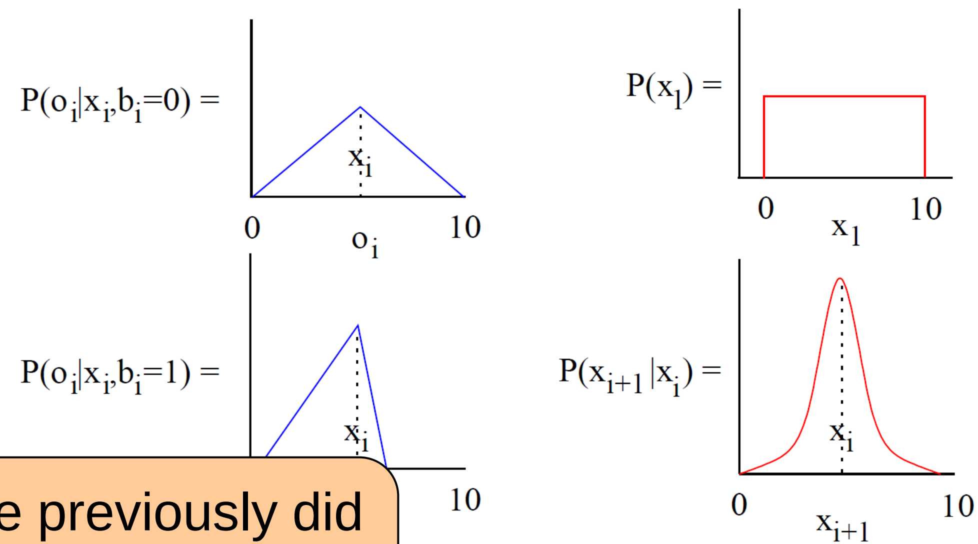
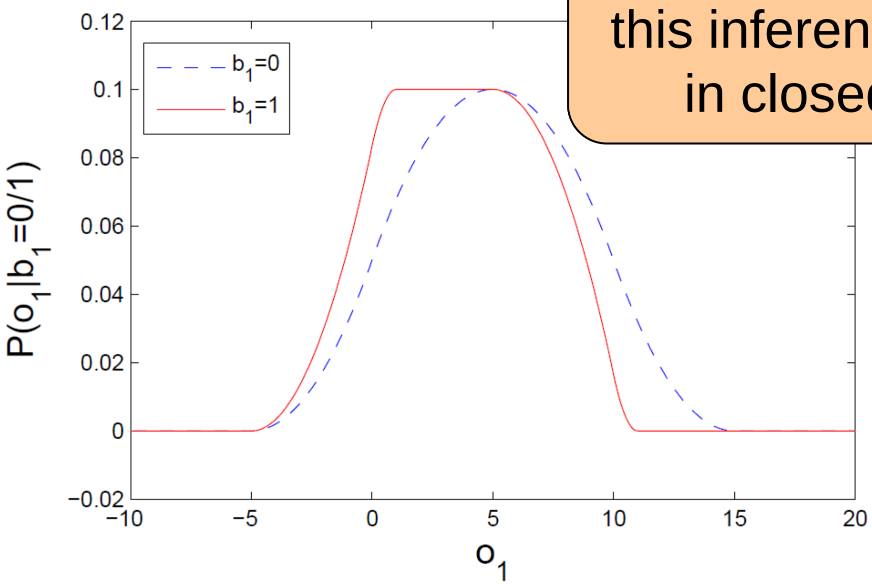
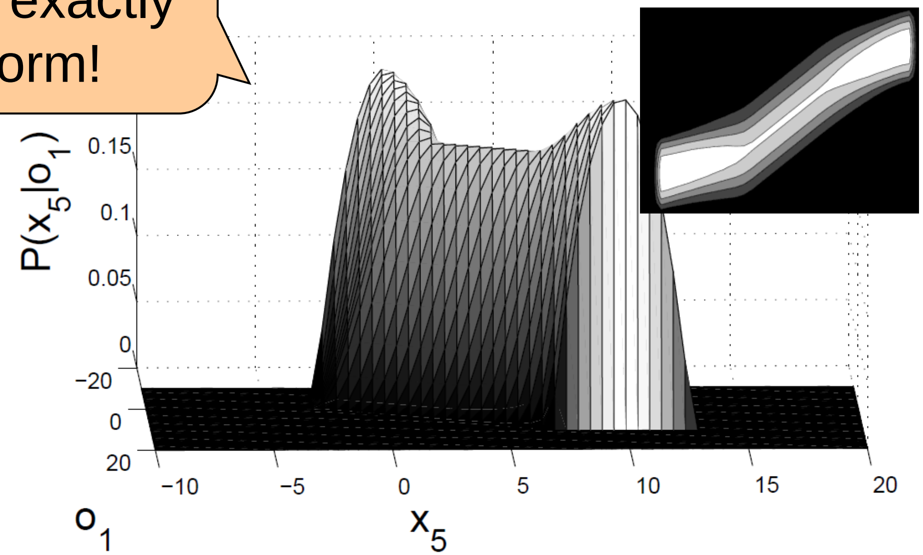
Inference for Dynamic Tracking





No one previously did this inference exactly in closed-form!





Exact Closed-form Continuous Inference

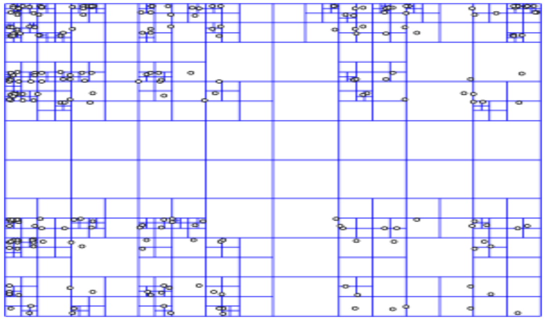
- Fully Gaussian
  - Most inference including conditional
- Fully Uniform
  - 1D, n-D hyperrectangular cases

Yes, but not a solution you can write on 1 sheet of paper

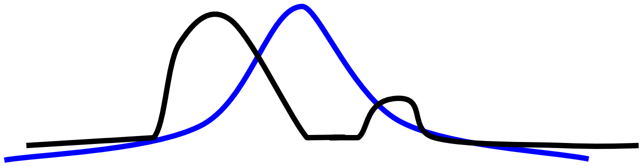
- General Uniform
- Piecewise, Asymmetrical
  - Exact (conditional) inference possible in closed-form?

Already Solved?

- How is inference done in piecewise models?
  - (Adaptively) discretize model:
    - Approximate, $O(N^D)$
    - Adaptivity is an artform



  - Projective approximation: variational, EP
    - Choose approximating class
    - Often Gaussian – good luck!



  - Sampling: Monte Carlo… Gibbs
    - May not converge for (near) deterministic distributions
    - Not for unobserved evidence – E[x | o=?]

No expressive, exact, closed-form solutions! Yet.

What has everyone been missing?

Symbolic representations and operations on piecewise functions

Continuous variant of work started in (Boutilier, Reiter, Price, IJCAI-01)

Piecewise Functions (Cases)

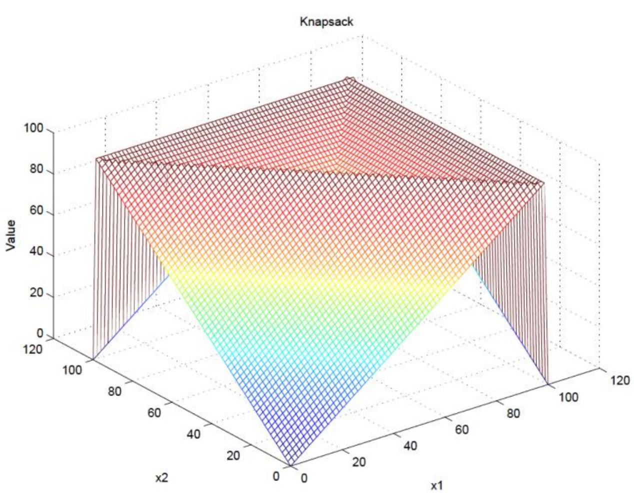
$$z = f(x,y) = \begin{cases} \boxed{(x > 3) \wedge (y \cdot \ x) : \quad x + y} \\ (x \cdot \ 3) \vee (y > x) : \quad x^2 + xy^3 \end{cases}$$

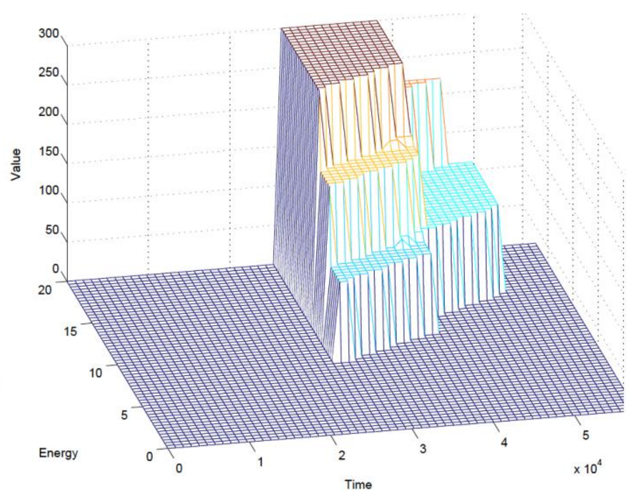
Partition

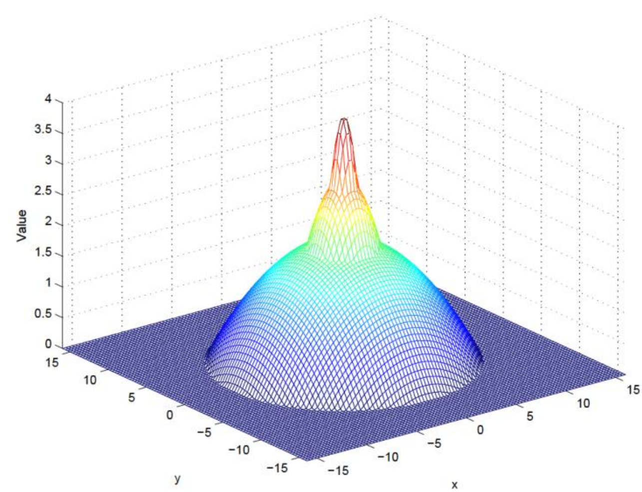
Constraint

Value

Linear constraints and value	Linear constraints, constant value	Quadratic constraints and value







Formal Problem Statement

- General continuous graphical models represented by piecewise functions (cases)

$$f = \begin{cases} \phi_1 : & f_1 \\ \vdots & \vdots \\ \phi_k : & f_k \end{cases}$$

- For $\sum \prod$ variable elimination, need exact closed-form solution inferred via the following piecewise calculus:
  - $f_1 \oplus f_2$, $f_1 \otimes f_2$
  - $\max(f_1, f_2)$, $\min(f_1, f_2)$
  - $\int_x f(x)$

Question: how do we perform these operations in closed-form?

Polynomial Case Operations: $\oplus$, $\otimes$

$$\begin{cases} \phi_1 : & f_1 \\ \phi_2 : & f_2 \end{cases} \oplus \begin{cases} \psi_1 : & g_1 \\ \psi_2 : & g_2 \end{cases} = \qquad ?$$

Polynomial Case Operations: ⊕, ⊗

$$\begin{cases} \phi_1 : & f_1 \\ \phi_2 : & f_2 \end{cases} \oplus \begin{cases} \psi_1 : & g_1 \\ \psi_2 : & g_2 \end{cases} = \begin{cases} \phi_1 \wedge \psi_1 : & f_1 + g_1 \\ \phi_1 \wedge \psi_2 : & f_1 + g_2 \\ \phi_2 \wedge \psi_1 : & f_2 + g_1 \\ \phi_2 \wedge \psi_2 : & f_2 + g_2 \end{cases}$$

- Similarly for ⊗
  - Polynomials closed under +, *
- What about max?
  - Max of polynomials is not a polynomial ☹

Polynomial Case Operations: max

$$\max\left(\begin{cases} \phi_1 : & f_1 \\ \phi_2 : & f_2 \end{cases}, \begin{cases} \psi_1 : & g_1 \\ \psi_2 : & g_2 \end{cases} \right) = \qquad ?$$

Polynomial Case Operations: max

$$
\max\left(\begin{cases} \phi_1 : & f_1 \\ \phi_2 : & f_2 \end{cases}, \begin{cases} \psi_1 : & g_1 \\ \psi_2 : & g_2 \end{cases} \right) = \begin{cases} \phi_1 \wedge \psi_1 \wedge f_1 > g_1 : & f_1 \\ \phi_1 \wedge \psi_1 \wedge f_1 \cdot g_1 : & g_1 \\ \phi_1 \wedge \psi_2 \wedge f_1 > g_2 : & f_1 \\ \phi_1 \wedge \psi_2 \wedge f_1 \cdot g_2 : & g_2 \\ \phi_2 \wedge \psi_1 \wedge f_2 > g_1 : & f_2 \\ \phi_2 \wedge \psi_1 \wedge f_2 \cdot g_1 : & g_1 \\ \phi_2 \wedge \psi_2 \wedge f_2 > g_2 : & f_2 \\ \phi_2 \wedge \psi_2 \wedge f_2 \cdot g_2 : & g_2 \end{cases}
$$

- Still a piecewise polynomial!

Size blowup?
We'll get to that…

Integration: $\int_x$

- $\int_x$ closed for polynomials
  - But how to compute for case?

$$\int_x \begin{cases} \phi_1 : & f_1 \\ \vdots & \vdots \\ \phi_k : & f_k \end{cases} dx = \int_x \sum_{i=1}^{k} [\phi_i] \cdot f_i \, dx$$

$$= \sum_i \int_x [\phi_i] \cdot f_i \, dx$$

  - Just integrate case partitions, $\oplus$ results!

Partition Integral

1. Determine integration bounds

$$\int_x [\phi_1] \cdot f_1 \, dx$$

$$\phi_1 := [x > -1] \wedge [x > y - 1] \wedge [x \cdot \; z] \wedge [x \cdot \; y + 1] \wedge [y > 0]$$

$$f_1 := x^2 - xy$$

$$LB := \begin{cases} y - 1 > -1 : & y - 1 \\ y - 1 \cdot \; -1 : & -1 \end{cases} \qquad UB := \begin{cases} z < y + 1 : & z \\ z \geq y + 1 : & y + 1 \end{cases}$$

UB and LB are symbolic!

$$[\phi_{cons}] \int_{x = LB}^{UB} f_1 \, dx$$

What constraints here?
- independent of x
- pairwise UB > LB

How to evaluate?

Definite Integral Evaluation

- How to evaluate integral bounds?

$$\int_{x=LB}^{UB} x^2 - xy \;=\; \frac{1}{3}x^3 - \frac{1}{2}x^2 y \Big|_{LB}^{UB}$$

$$LB := \begin{cases} y-1 > -1 : & y-1 \\ y-1 \cdot\; -1 : & -1 \end{cases} \qquad UB := \begin{cases} z < y+1 : & z \\ z \geq y+1 : & y+1 \end{cases}$$

- Can do polynomial operations on cases!

$$f_1\Big|_{LB}^{UB} = \left[\frac{1}{3}UB \quad UB \quad UB \ominus \frac{1}{2}UB \quad UB \quad (y)\right]$$

$$\ominus \left[\frac{1}{3}LB \quad LB \quad LB \ominus \frac{1}{2}LB \quad LB \quad (y)\right]$$

Symbolically, exactly evaluated!

Exact Graphical Model Inference!

(directed and undirected)

- Can do general probabilistic inference

$$p(x_2|x_1) = \frac{\int_{x_3} \cdots \int_{x_n} \bigotimes_{i=1}^{k} case_i \; dx_n \cdots dx_3}{\int_{x_2} \cdots \int_{x_n} \bigotimes_{i=1}^{k} case_i \; dx_n \cdots dx_2}$$

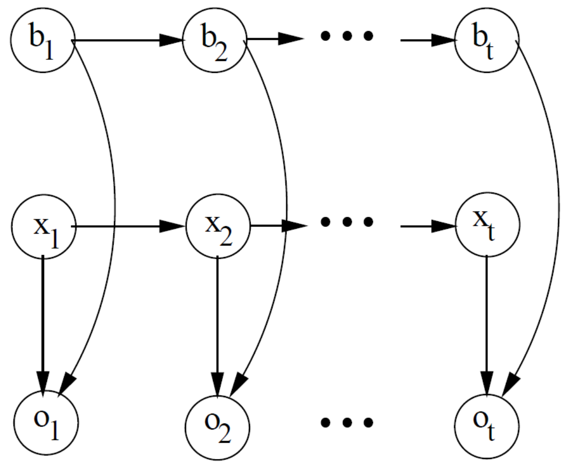
- Or an exact expectation of *any* polynomial

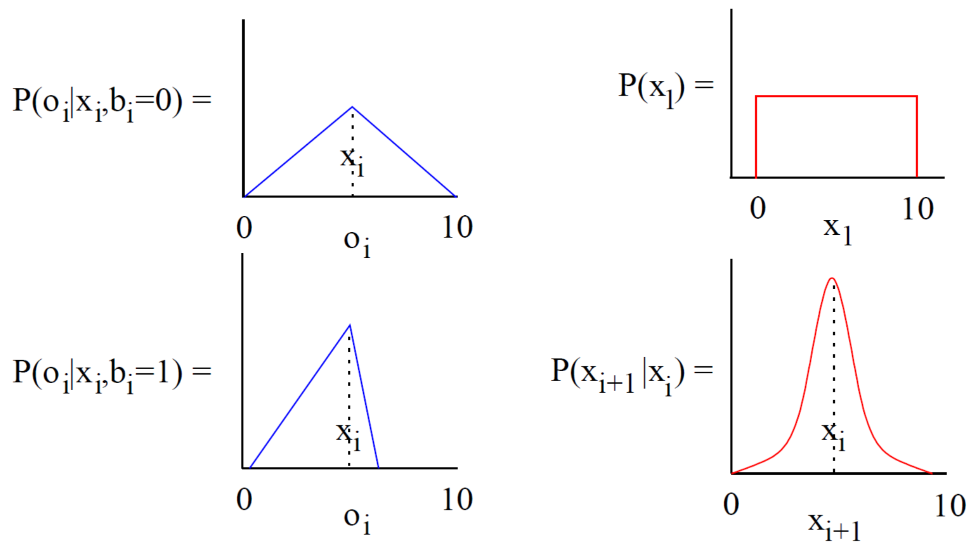
$$\mathbb{E}_{\mathbf{x} \sim p(\mathbf{x}|\mathbf{o})}[poly(\mathbf{x})|\mathbf{o}] = \int_{\mathbf{x}} p(\mathbf{x}|\mathbf{o}) poly(\mathbf{x}) d\mathbf{x}$$

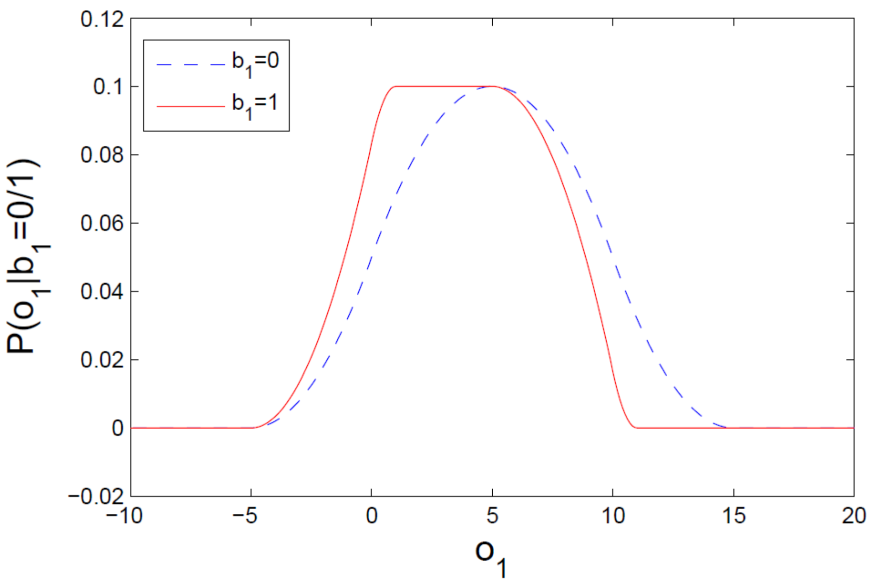
  – *poly* = Mean, variance, skew, curtosis, …, x^2+y^2+xy

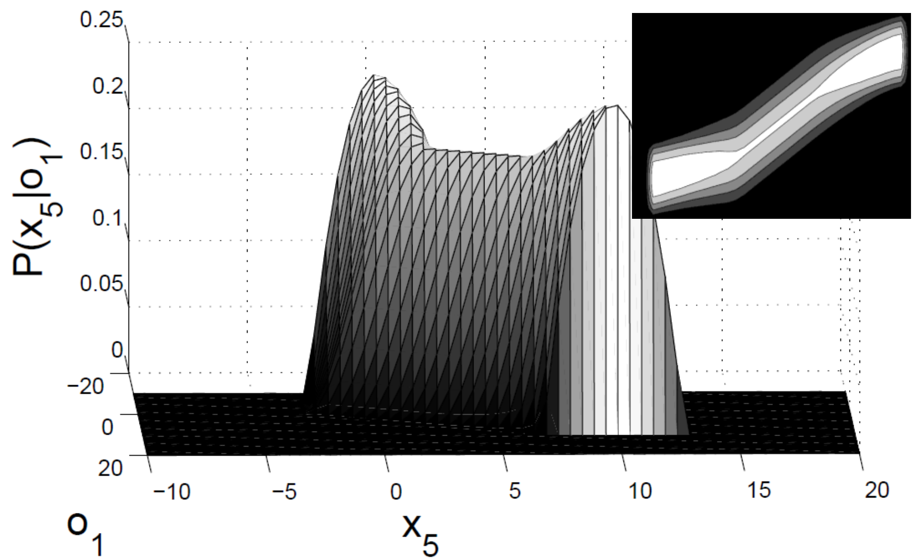
All computed by **Symbolic Variable Elimination (SVE)**

Voila: Closed-form Exact Inference via SVE!

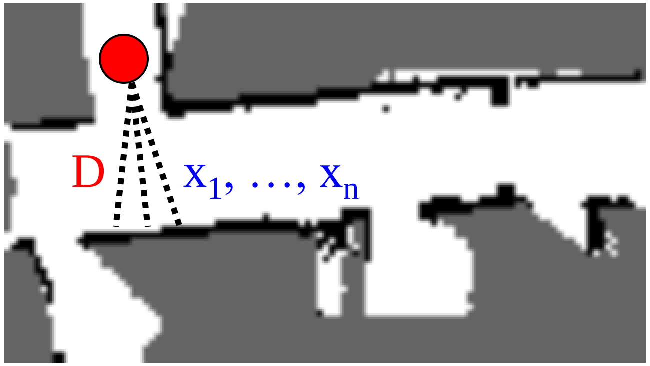




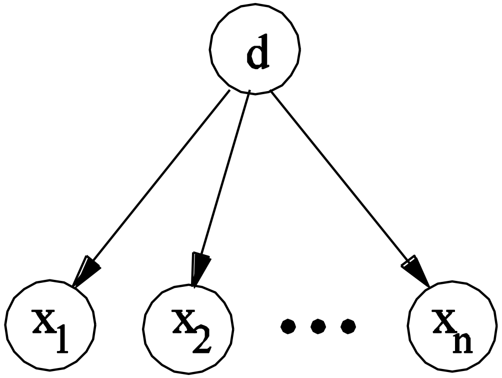


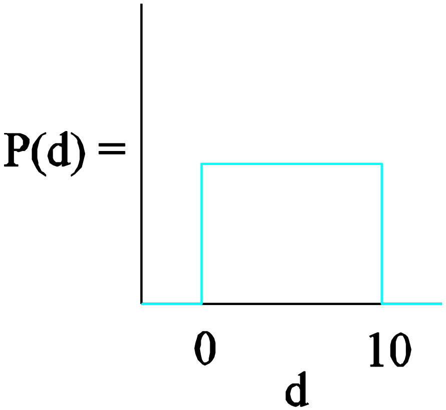


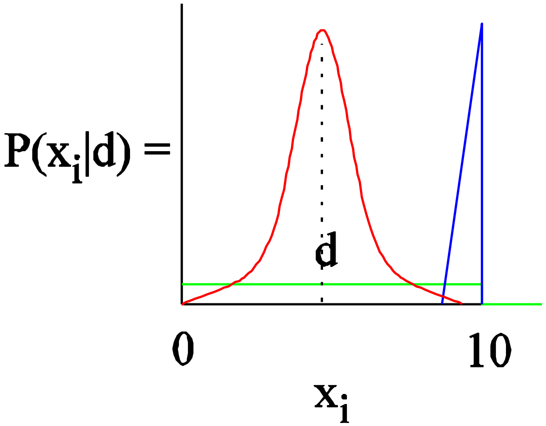
Bayesian Robotics



- D: true distance to wall
- $x_1, \ldots, x_n$: measurements
- want: $E[D \mid x_1,\ldots,x_{n-1}, x_n=?]$

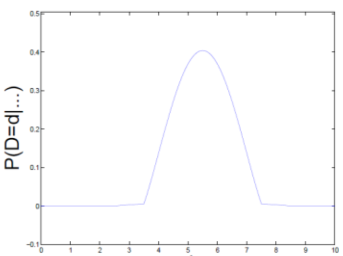




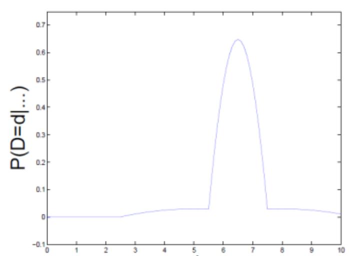


Bayesian Robotics: Inference Examples

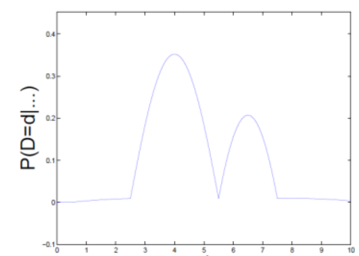
- Posterior distance shown in graph
- Expected distance calculation shown below



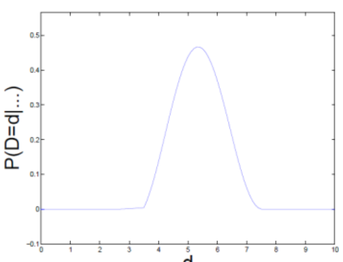
$$\mathbb{E}[D|x_1 = 6, x_2 = 5] = 4.98$$



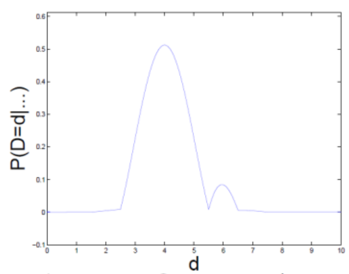
$$\mathbb{E}[D|x_1 = 8, x_2 = 5] = 6.0$$



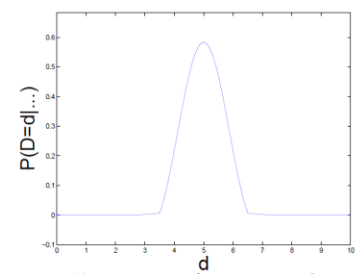
$$\mathbb{E}[D|x_1 = 5, x_2 = 3, x_3 = 8] = 4.39$$



$$\mathbb{E}[D|x_1 = 5, x_2 = 5, x_3 = 6] = 4.98$$



$$\mathbb{E}[D|x_2 = 1, x_2 = 3, x_3 = 4, x_4 = 8] = 5.45$$



$$\mathbb{E}[D|x_1 = 5, x_2 = 4, x_3 = 6, x_4 = 5] = 4.89$$

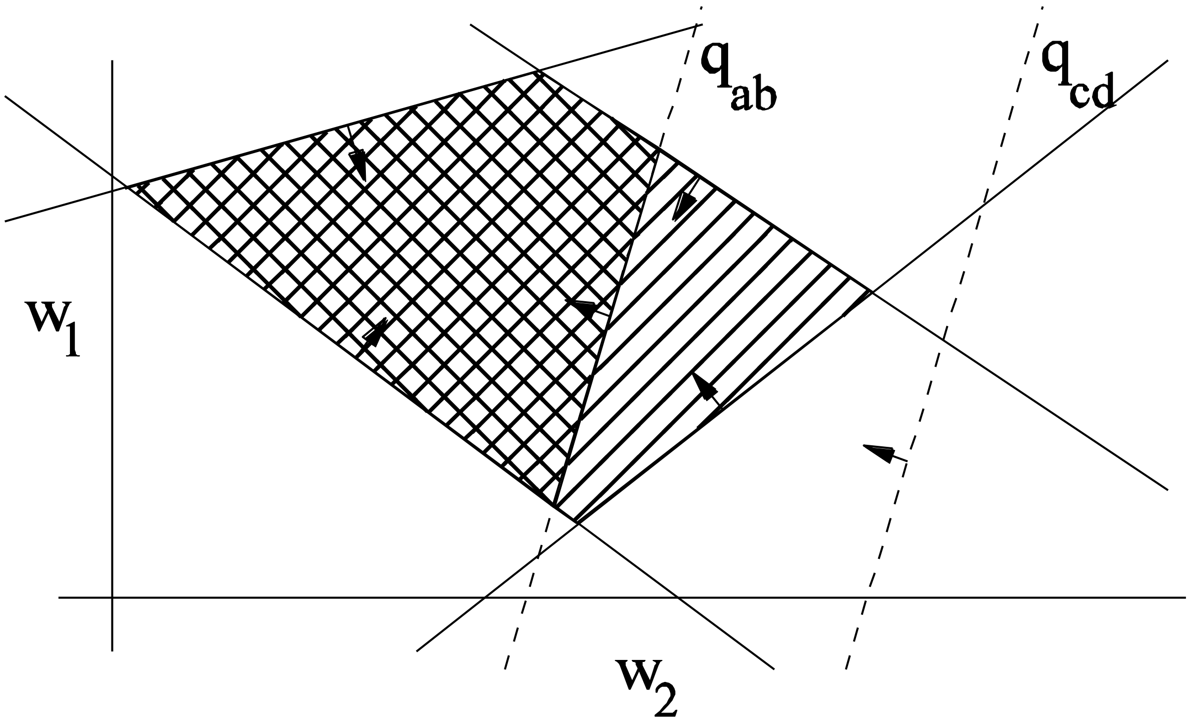
Extra: A New Conjugate Prior for Bayesian Inference

- General Bayesian Inference

$$p(\vec{\theta}|D_{n+1}) \propto p(d_{n+1}|\vec{\theta})p(\vec{\theta}|D_n)$$

- Prior and likelihood are case statements
  - ➔ then posterior is a case statement

Example: Bayesian Preference Learning for Linear Utilities



- If prior is uniform over weights
  and each preference likelihood linearly constrains weights
  ➔ posterior is uniform over convex polytope

- All case statements!

Inference is equivalent to finding volume of a convex polytope!

Computational Complexity?

- In theory for SVE on graphical models
  - Best-case complexity Ω(#operations)
  - Worst-case complexity is O(exp(#operations))
    - Not explicitly tree-width dependent!
    - **But worse:** integral may invoke 100's of operations!

Fortunately data structures mitigate worst-case complexity

BDD / ADDs

Quick Introduction

Function Representation (Tables)

- How to represent functions: $B^n \rightarrow R$?
- How about a fully enumerated table…
- …OK, but can we be more compact?

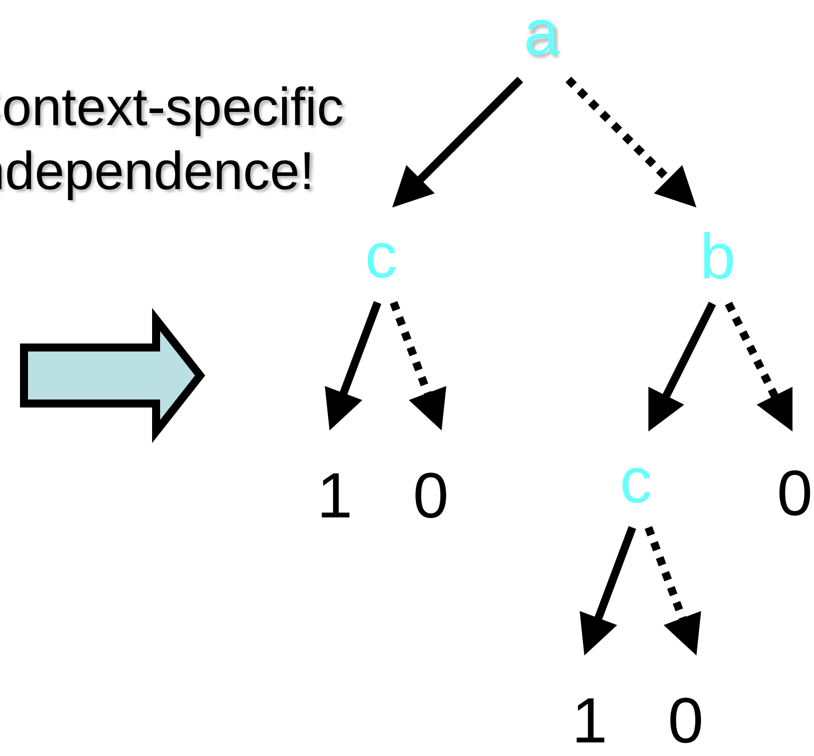
a	b	c	F(a,b,c)
0	0	0	0.00
0	0	1	0.00
0	1	0	0.00
0	1	1	1.00
1	0	0	0.00
1	0	1	1.00
1	1	0	0.00
1	1	1	1.00

Function Representation (Trees)

- How about a tree? Sure, can simplify.

a	b	c	F(a,b,c)
0	0	0	0.00
0	0	1	0.00
0	1	0	0.00
0	1	1	1.00
1	0	0	0.00
1	0	1	1.00
1	1	0	0.00
1	1	1	1.00

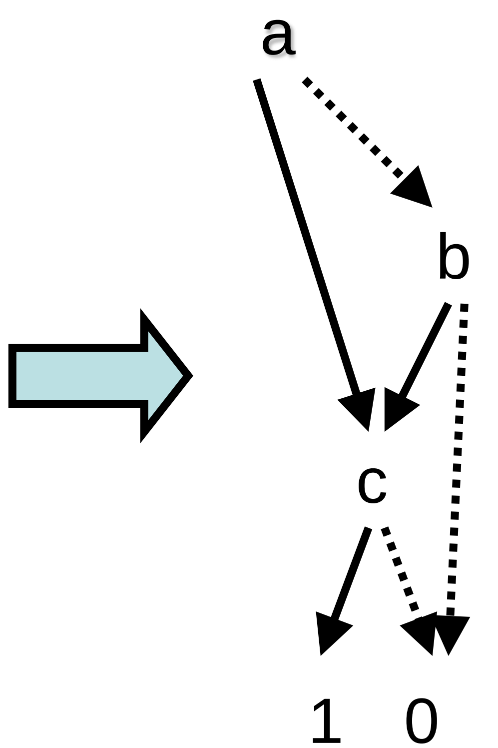
Context-specific independence!



Function Representation (ADDs)

- Why not a directed acyclic graph (DAG)?

a	b	c	F(a,b,c)
0	0	0	0.00
0	0	1	0.00
0	1	0	0.00
0	1	1	1.00
1	0	0	0.00
1	0	1	1.00
1	1	0	0.00
1	1	1	1.00

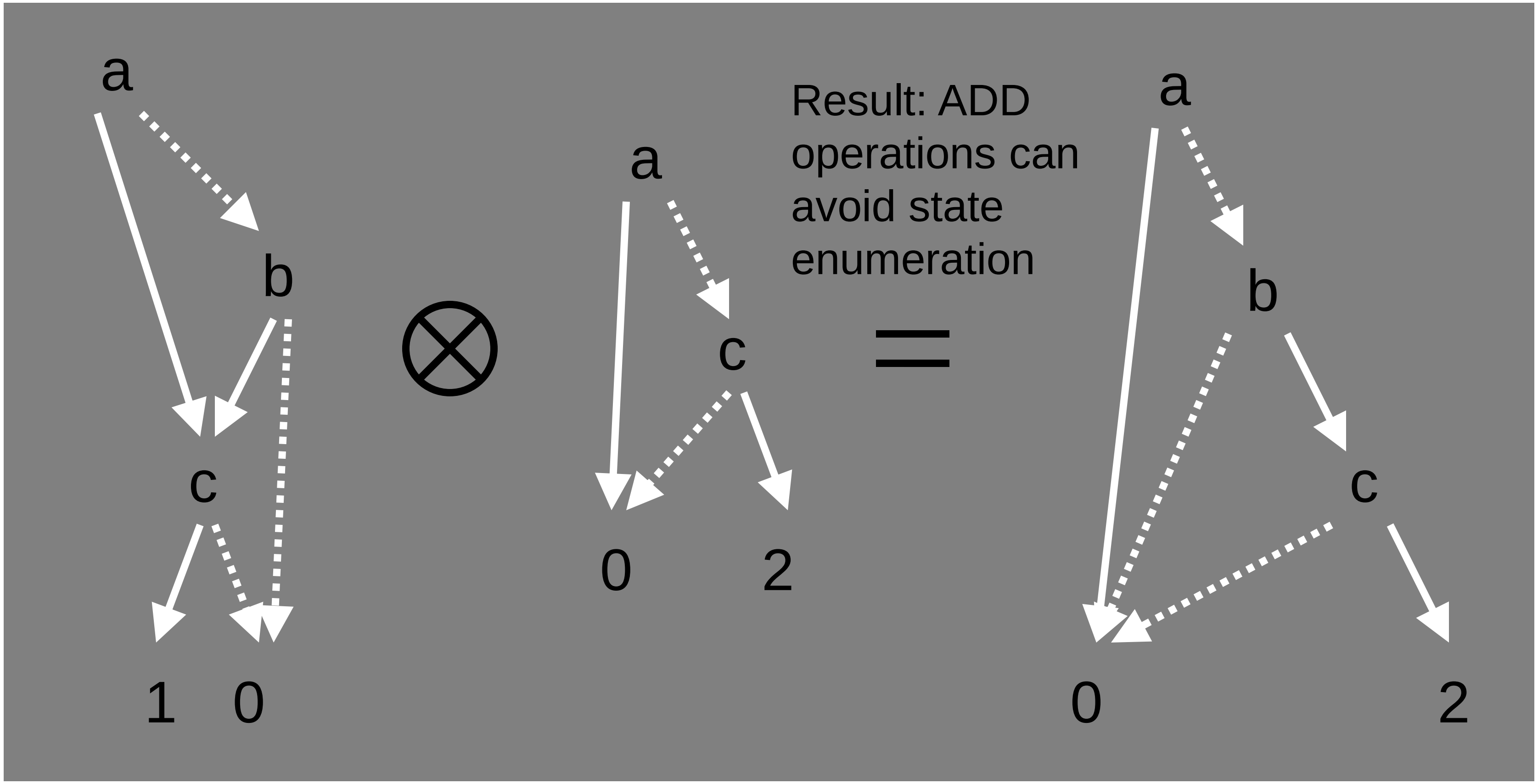


Algebraic Decision Diagram (ADD)

Think of BDDs as {0,1} subset of ADD range

Binary Operations (ADDs)

- Why do we order variable tests?
- Enables us to do efficient binary operations…



Case $\rightarrow$ XADD

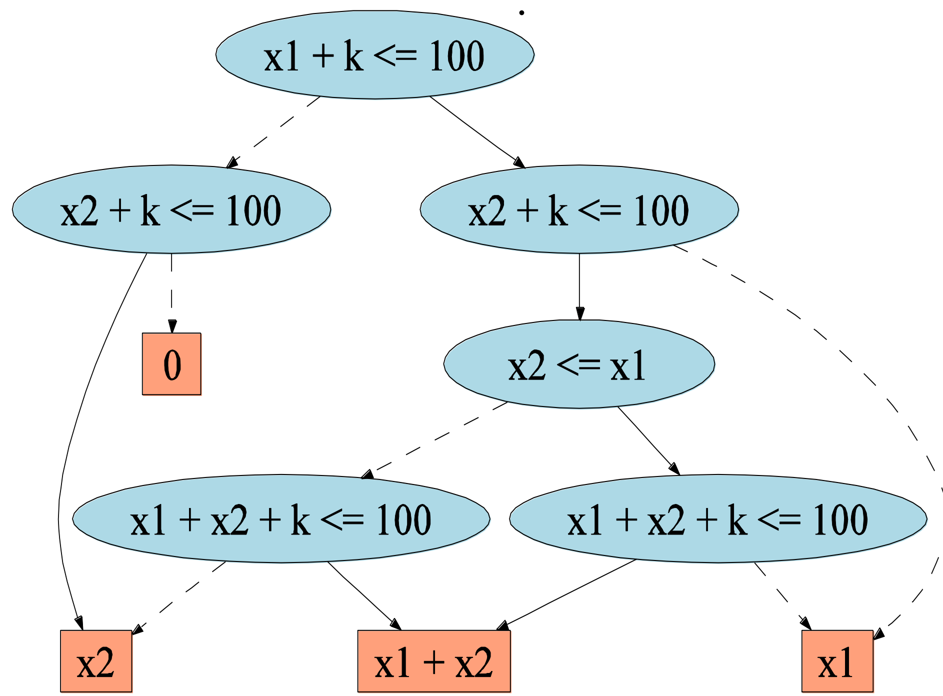
XADD = continuous variable e**x**tension
of **a**lgebraic **d**ecision **d**iagram

Efficient XADD data structure for cases
• strict ordering of atomic inequality tests

➔ compact, minimal case representation
➔ efficient case operations

Case → XADD

$$V = \begin{cases} x_1 + k > 100 \wedge x_2 + k > 100 : & 0 \\ x_1 + k > 100 \wedge x_2 + k \leq 100 : & x_2 \\ x_1 + k \leq 100 \wedge x_2 + k > 100 : & x_1 \\ x_1 + x_2 + k > 100 \wedge x_1 + k \leq 100 \wedge x_2 + k \leq 100 \wedge x_2 > x_1 : & x_2 \\ x_1 + x_2 + k > 100 \wedge x_1 + k \leq 100 \wedge x_2 + k \leq 100 \wedge x_2 \leq x_1 : & x_1 \\ x_1 + x_2 + k \leq 100 : & x_1 + x_2 \\ \vdots & \vdots \end{cases}$$

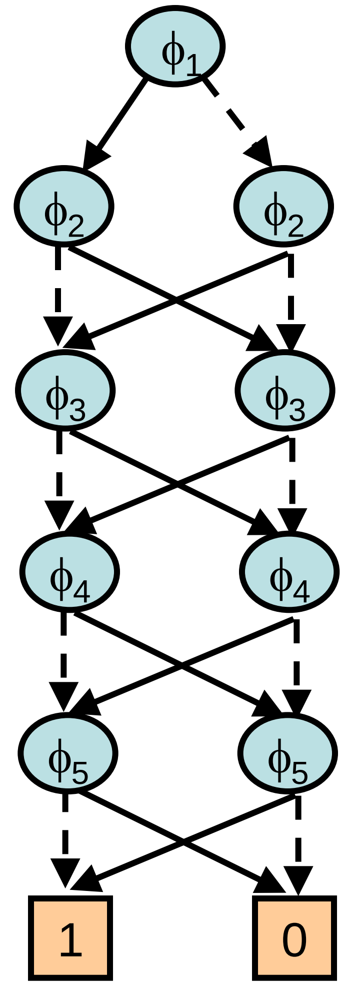


*With non-trivial extensions over ADD, can reduce to a minimal canonical form!

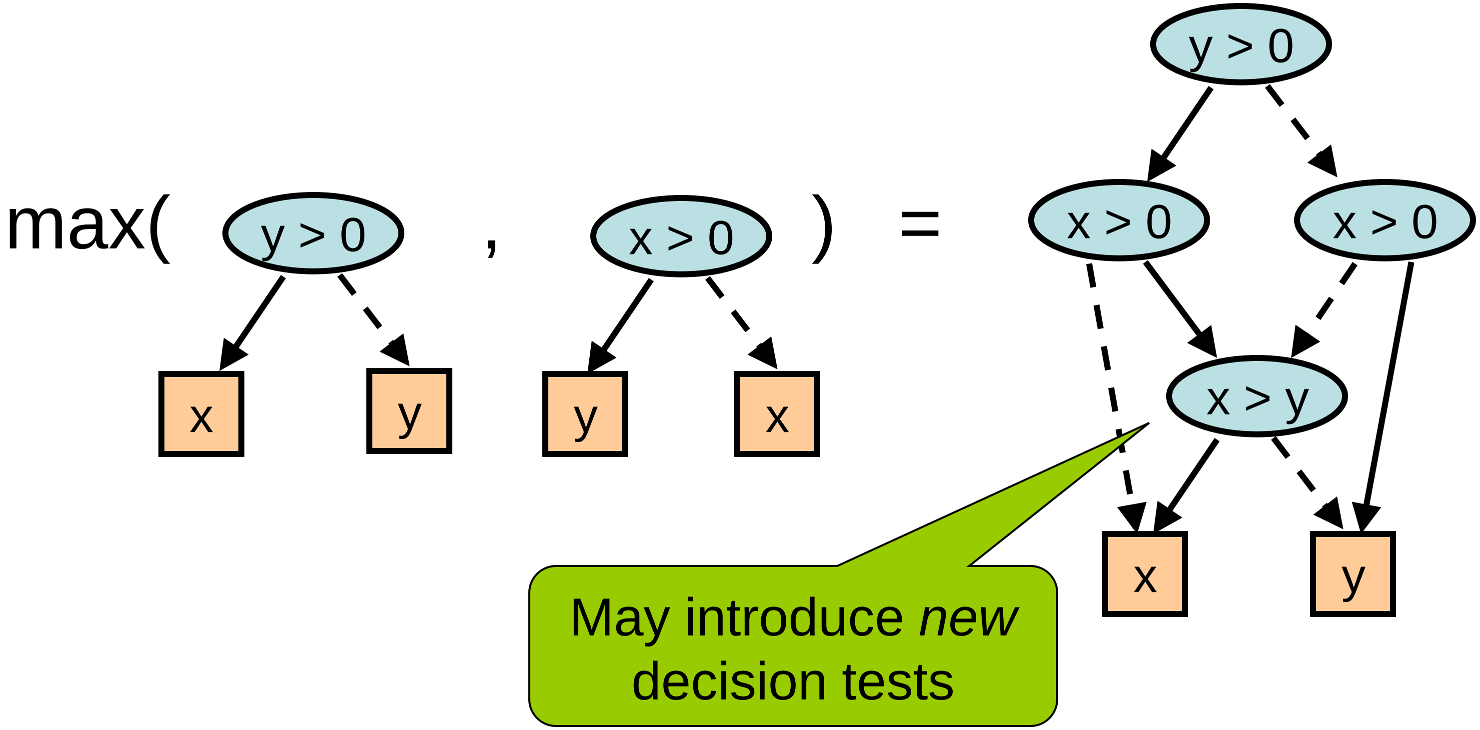
Compactness of (X)ADDs

- Linear in number of decisions ϕ_i

- Case version has exponential number of partitions!



XADD Maximization



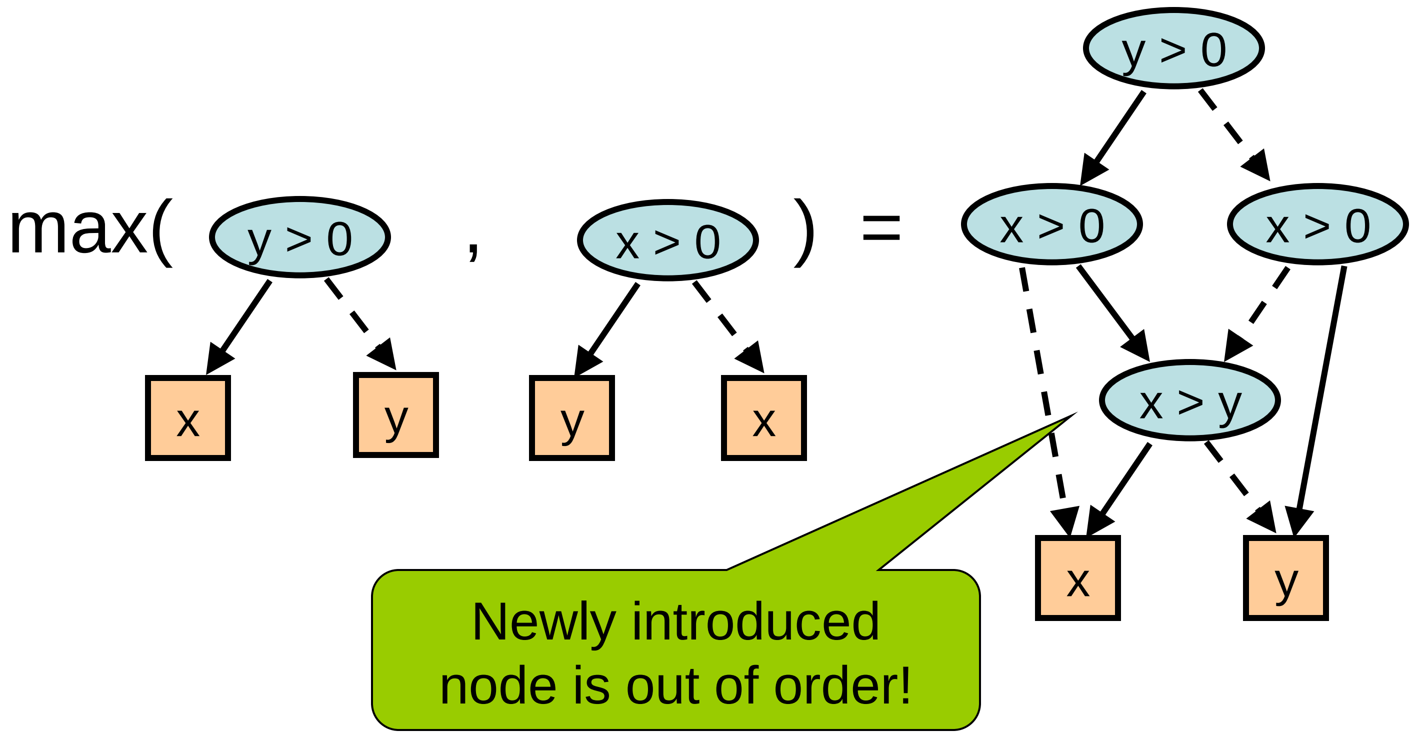
Operations exploit structure: O(|f||g|)

Maintaining XADD Orderings

- Max may get decisions out of order

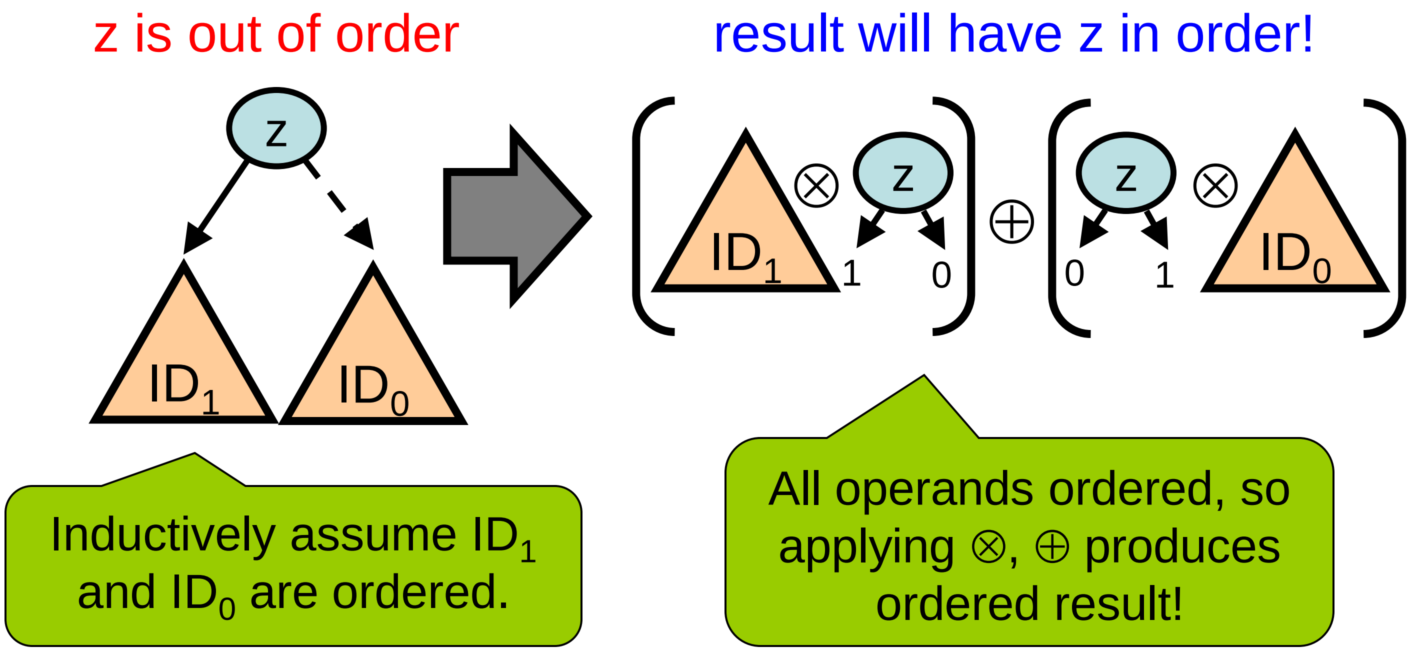
Decision ordering (root→leaf)

- x > y
- y > 0
- x > 0

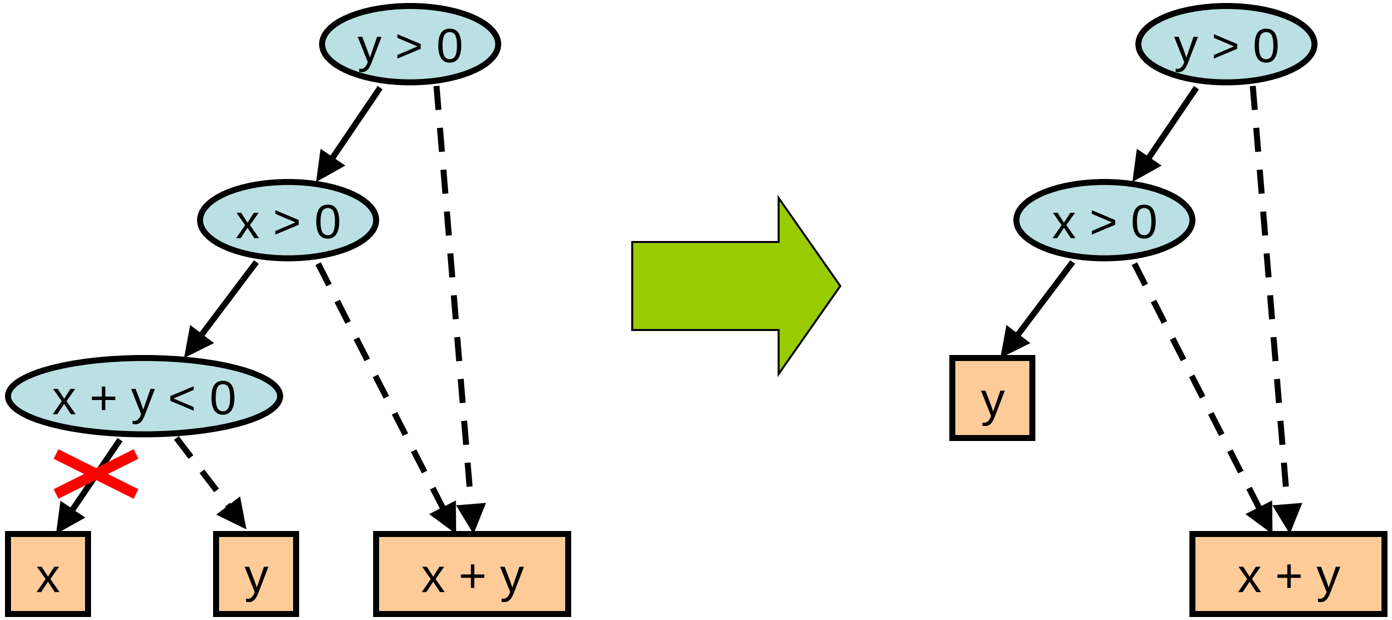


Correcting XADD Ordering

- Obtain *ordered* XADD from *unordered* XADD
  - key idea: binary operations maintain orderings



Maintaining Minimality



Node unreachable –
x + y < 0 always
false if x > 0 & y > 0

If **linear**, can detect with
feasibility checker of LP
solver & prune

More subtle
prunings as
well.

XADD Makes Possible all Previous Inference

Could not even do a single integral or maximization without it!

Recap

- Defined a calculus for piecewise functions
    - $f_1 \oplus f_2$, $f_1 \otimes f_2$
    - $\max(f_1, f_2)$, $\min(f_1, f_2)$
    - $\int_x f(x)$
    - See Zamani & Sanner (AAAI-12) for $\max_x f(x)$, $\min_x f(x)$
- **Defined XADD to efficiently compute with cases**
- **Makes possible**
    - **Closed-form inference in continuous graphical models**
    - **Exact (conditional) expectations of any polynomial**
    - **New paradigms for Bayesian inference**

Piecewise Calculus + XADD

= Expressive, Exact, Closed-form Inference in Discrete & Continuous Variable Graphical Models

Thank you!

Questions?