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Symbolic Dynamic Programming for Discrete and Continuous State MDPs

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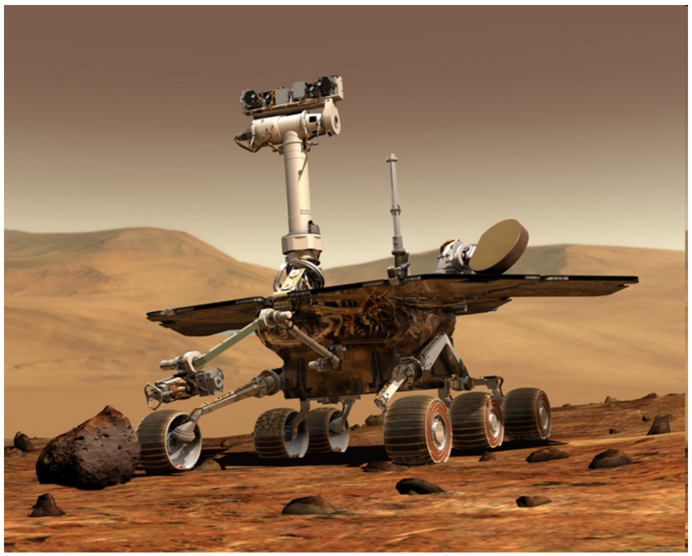
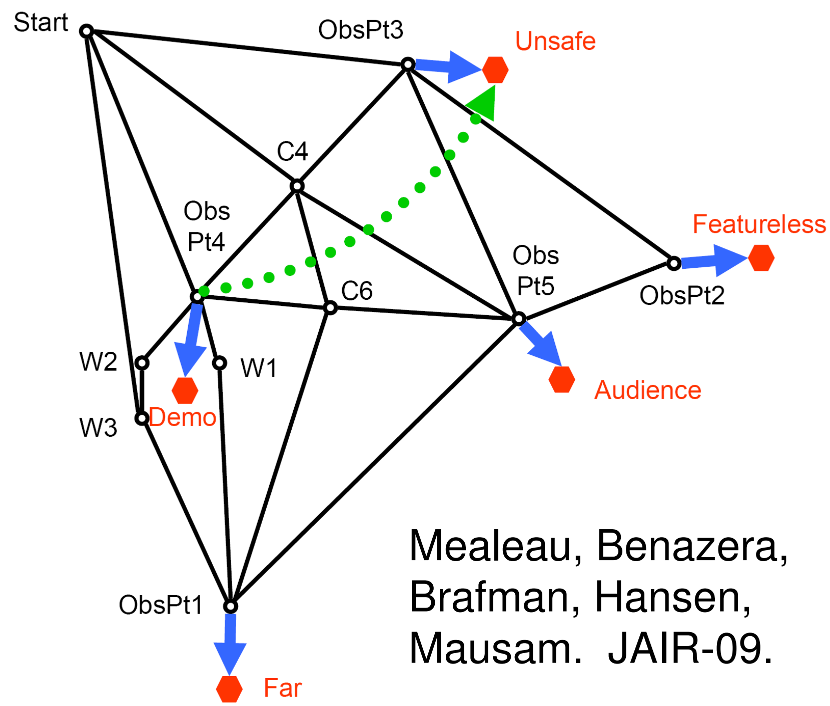
Continuous Domains: Mars Rovers





Mealeau, Benazera, Brafman, Hansen, Mausam. JAIR-09.

- Continuous state
  - Time (t), Energy (e), Robot position (x,y,θ)
- Closed-form exact solution?
  - Currently only if 1D or piecewise rectilinear solution exists ☹

Our work: exact for multidimensional nonlinear domains!

Previous Work

Discrete and Continuous (DC-)MDPs

- Mixed discrete / continuous state

$$(\vec{b}, \vec{x}) = (b_1, \ldots, b_n, x_1, \ldots, x_m) \in \{0,1\}^n \times \mathbb{R}^m$$

- Discrete action set $a \in \mathcal{A}$
- DBN factored transition model

$$P(\vec{b}', \vec{x}' | \vec{b}, \vec{x}, a) =$$

$$\underbrace{\left(\prod_{i=1}^{n} P(b_i' | \vec{b}, \vec{x}, a) \right)}_{\text{discrete}} \underbrace{\left(\prod_{j=1}^{m} P(x_j' | \vec{b}, \vec{b}', \vec{x}, a) \right)}_{continuous}$$

- Arbitrary action-dependent reward

$$R_a(\vec{b}, \vec{x}) = x_1 + x_2$$

Value Iteration for DC-MDPs

- Value of policy in state is expected sum of rewards
- Want optimal value $V^{h,*}$ over horizons $h \in 0..H$
  - Implicitly provides optimal horizon-dependent policy
- Compute inductively via **Value Iteration** for $h \in 0..H$
  - **Regression step:**

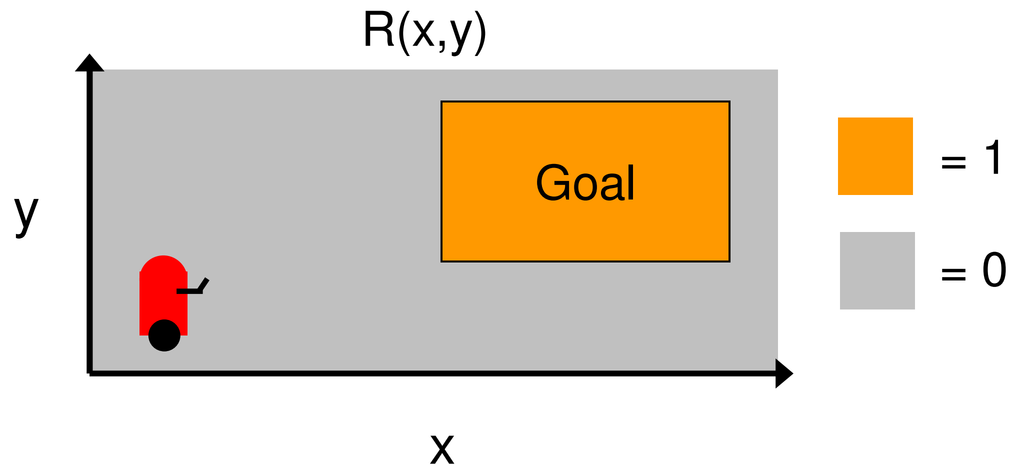
$$Q_a^{h+1}(\vec{b},\vec{x}) = R_a(\vec{b},\vec{x}) + \gamma \cdot$$

$$\sum_{\vec{b}'} \int_{\vec{x}'} \left(\prod_{i=1}^{n} P(b_i'|\vec{b},\vec{x},a) \prod_{j=1}^{m} P(x_j'|\vec{b},\vec{b}',\vec{x},a) \right) V^h(\vec{b}',\vec{x}')d\vec{x}'$$

  - **Maximization step:**

$$V_{h+1} = \max_{a \in A} Q_a^{h+1}(\vec{b},\vec{x})$$

Exact Solutions to DC-MDPs: Domain

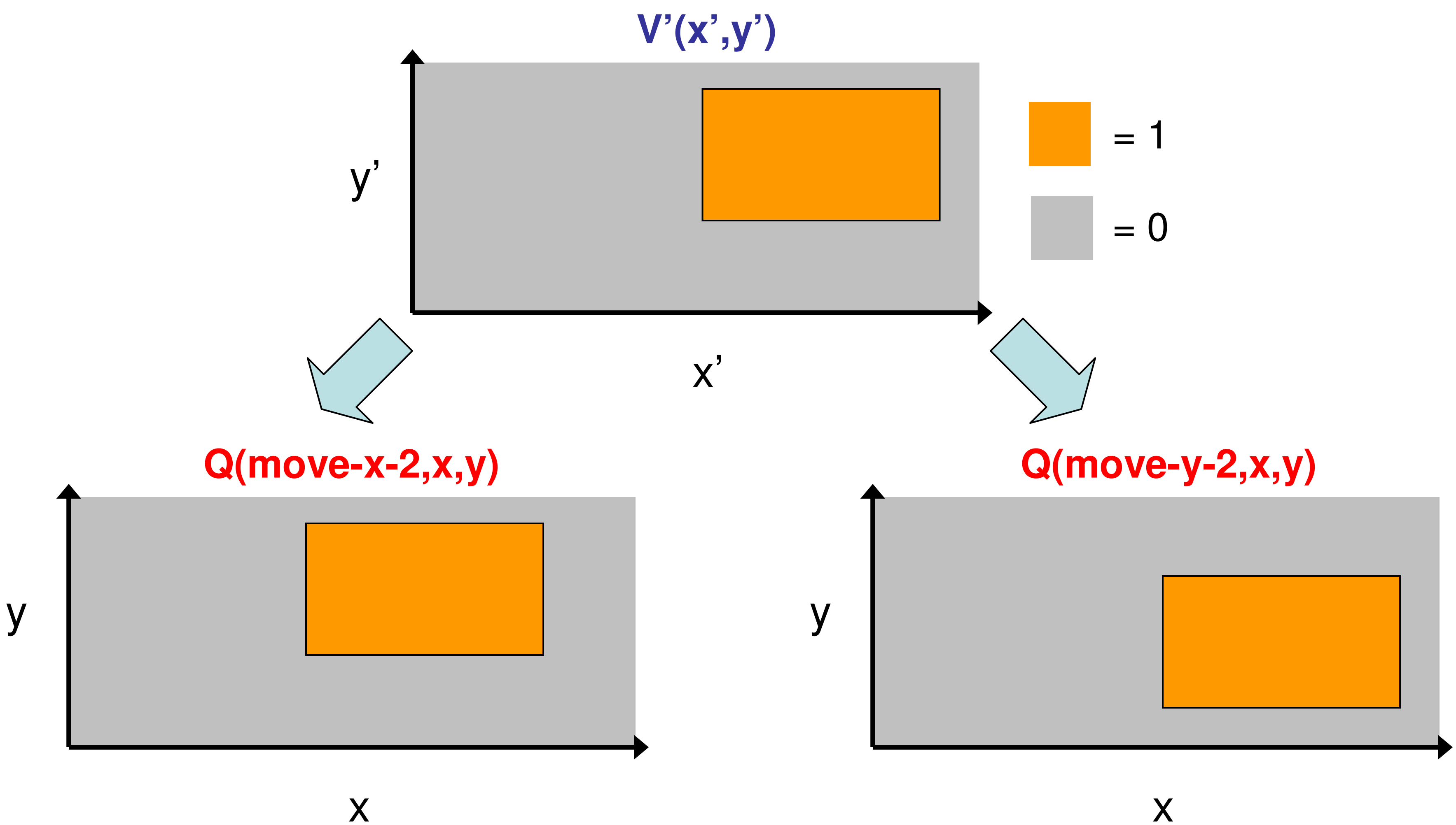


- 2-D Navigation
- State: (x,y) $\in \mathbb{R}^2$
- Actions:
  - move-x-2
    - x' = x + 2
    - y' = y
  - move-y-2
    - x' = x
    - y' = y + 2
- Reward:
  - R(x,y) = I[(x > 5) ^ (x < 10) ^ (y > 2) ^ (y < 5)]

Assumptions:
1. Continuous transitions are deterministic and linear
2. Discrete transitions can be stochastic
3. Reward is piecewise rectilinear

Exact Solutions to DC-MDPs: Regression

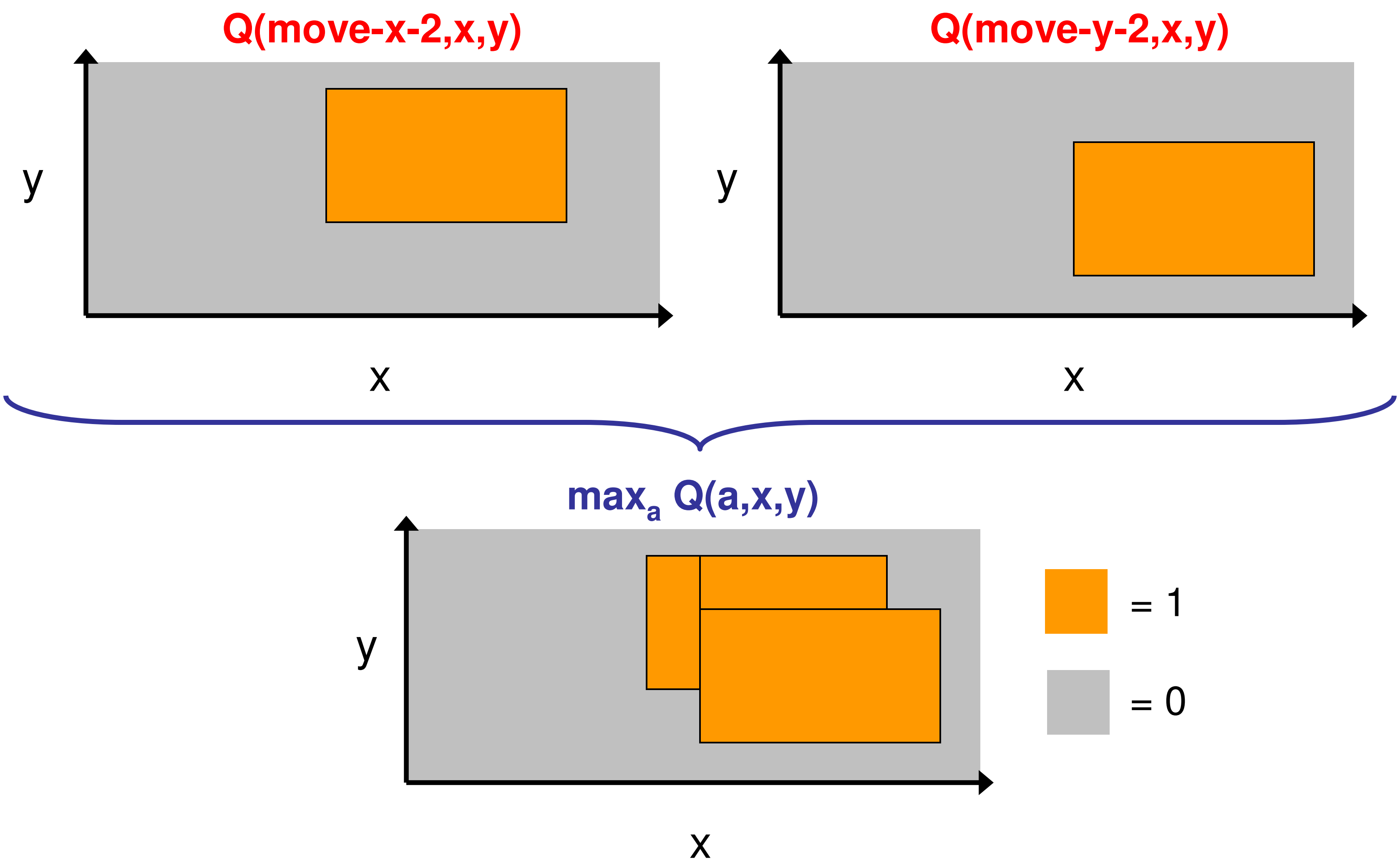
- Continuous regression is just translation of "pieces"



Feng *et al*, UAI-04

Exact Solutions to DC-MDPs: Maximization

- Q-value maximization yields piecewise rectilinear solution



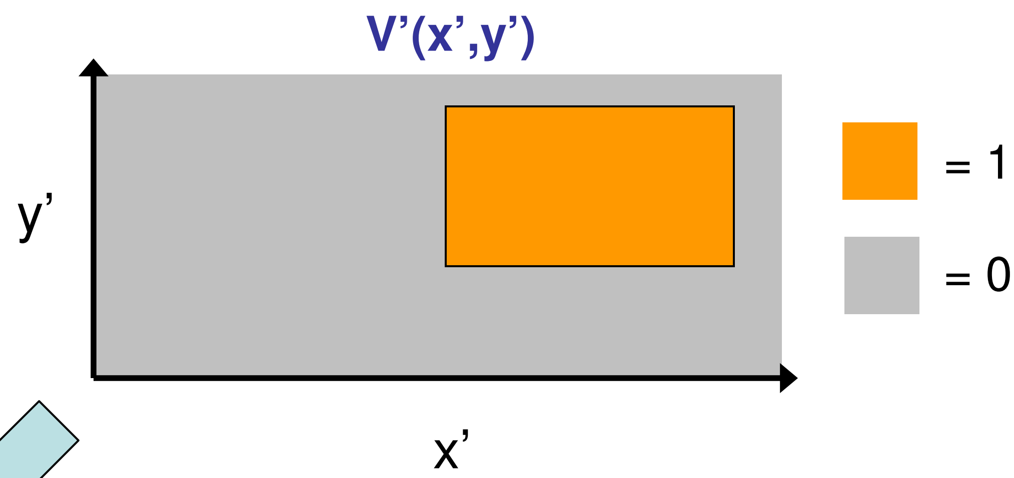
Previous Work Limitations I

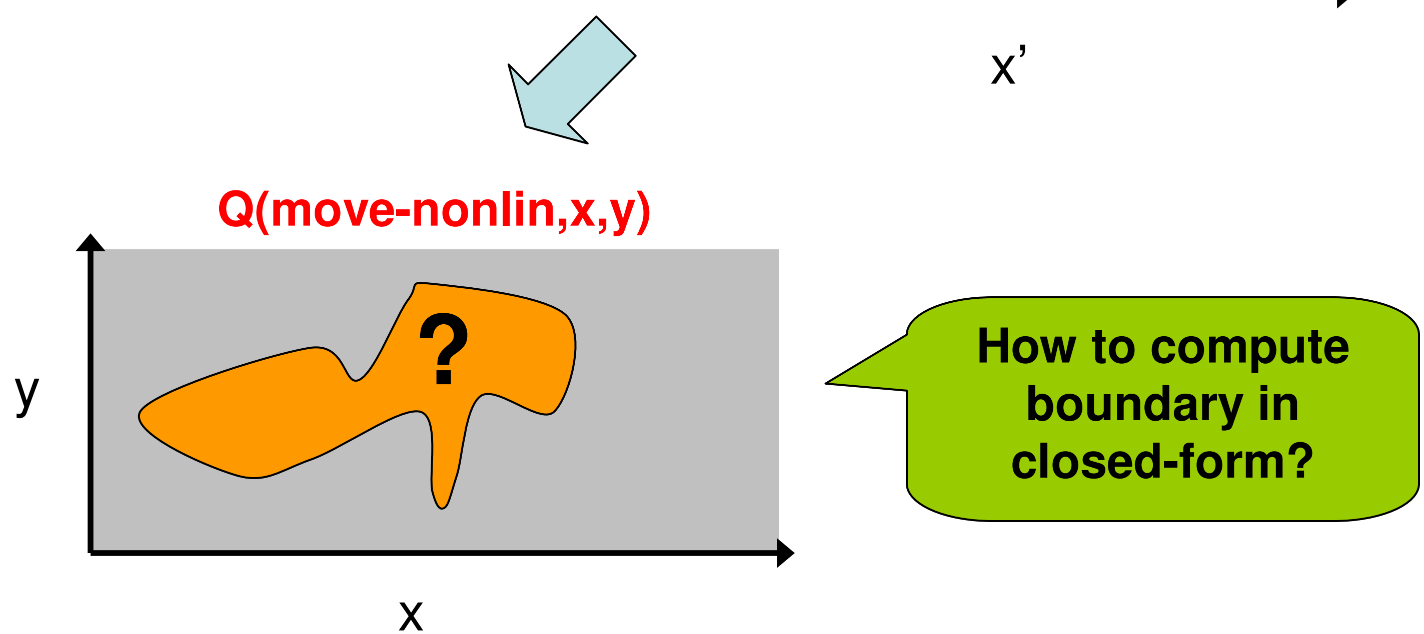
- Exact regression when transitions nonlinear?

Action move-nonlin:

– $x' = x^3y + y^2$

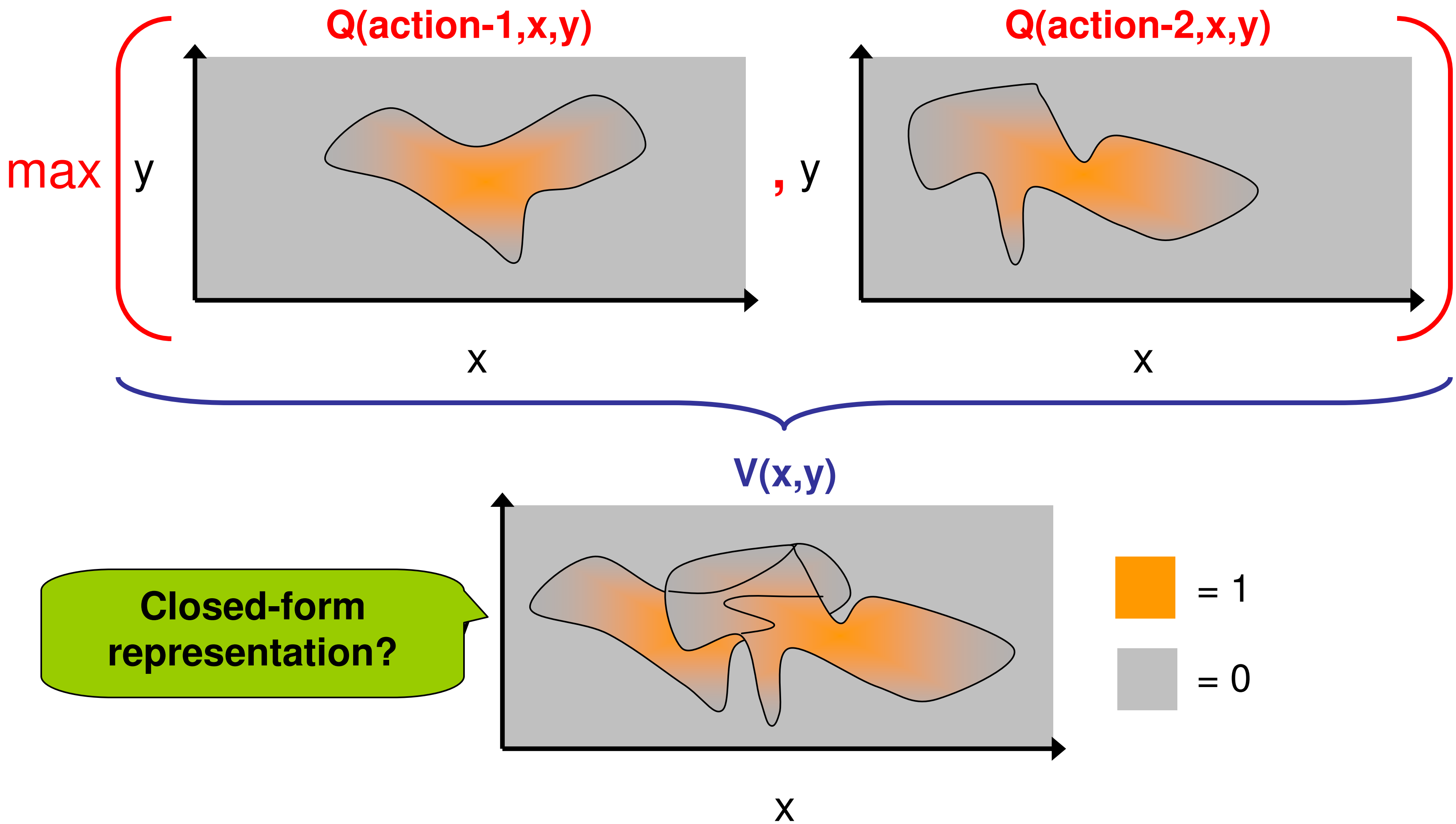
– $y' = y * \log(x^2y)$





Previous Work Limitations II

- max(.,.) when reward/value arbitrary piecewise?



A solution to previous limitations:

Symbolic Dynamic Programming (SDP)

n.b., motivated by SDP from Boutilier *et al* (IJCAI-01) but here continuous instead of relational

SDP uses Symbolic *Case* Representation

$$f = \begin{cases} \phi_1 : & f_1 \\ \vdots & \vdots \\ \phi_k : & f_k \end{cases}$$

$$P(x'|x,y) = \delta\left(x' - \begin{cases} (x < xy^2) \wedge (x > y) : & x + y \\ (x \geq xy^2) \vee (x \leq y) : & (x - y)^2 + 1 \end{cases}\right)$$

Deterministic transitions represented by δ over (conditional) equation

Logical combinations of inequalities of arbitrary expressions

Arbitrary expressions

Case Operations: ⊕, ⊗

$$\begin{cases} \phi_1 : & f_1 \\ \phi_2 : & f_2 \end{cases} \oplus \begin{cases} \psi_1 : & g_1 \\ \psi_2 : & g_2 \end{cases} = \qquad ?$$

Case Operations: ⊕, ⊗

$$\begin{cases} \phi_1 : & f_1 \\ \phi_2 : & f_2 \end{cases} \oplus \begin{cases} \psi_1 : & g_1 \\ \psi_2 : & g_2 \end{cases} = \begin{cases} \phi_1 \wedge \psi_1 : & f_1 + g_1 \\ \phi_1 \wedge \psi_2 : & f_1 + g_2 \\ \phi_2 \wedge \psi_1 : & f_2 + g_1 \\ \phi_2 \wedge \psi_2 : & f_2 + g_2 \end{cases}$$

- Similarly for ⊗
  - Expressions trivially closed under +, *
- What about max?
  - $\max(f_1, g_1)$ not pure arithmetic expression ☹

Case Operations: max

$$\max\left(\begin{cases}\phi_1: & f_1\\ \phi_2: & f_2\end{cases},\begin{cases}\psi_1: & g_1\\ \psi_2: & g_2\end{cases}\right) = \qquad ?$$

Case Operations: max

$$\max\left(\begin{cases}\phi_1: & f_1\\ \phi_2: & f_2\end{cases}, \begin{cases}\psi_1: & g_1\\ \psi_2: & g_2\end{cases}\right) = \begin{cases}\phi_1 \wedge \psi_1 \wedge f_1 > g_1: & f_1\\ \phi_1 \wedge \psi_1 \wedge f_1 \leq g_1: & g_1\\ \phi_1 \wedge \psi_2 \wedge f_1 > g_2: & f_1\\ \phi_1 \wedge \psi_2 \wedge f_1 \leq g_2: & g_2\\ \phi_2 \wedge \psi_1 \wedge f_2 > g_1: & f_2\\ \phi_2 \wedge \psi_1 \wedge f_2 \leq g_1: & g_1\\ \phi_2 \wedge \psi_2 \wedge f_2 > g_2: & f_2\\ \phi_2 \wedge \psi_2 \wedge f_2 \leq g_2: & g_2\end{cases}$$

Key point: still in case form!

Size blowup? We'll get to that…

Symbolic Dynamic Programming

- In a nutshell
  - $R(.)$, $P(.|.)$ defined as case statements
  - Value iteration uses case operations
    - $\oplus$, $\otimes$, max
  - Then provably:
    - $V^h(.)$ is also in case form for all horizons h!
- Only "tricky part" is continuous regression

SDP Regression Step

- **Binary variables** b_i
  - Factored $\sum_{bi \in \{0,1\}}$ (e.g, SPUDD: Hoey *et al*, UAI-99)

- **Continuous variables** x_j
  - $\int_x \delta[x - y] f(x) dx = f(y)$ triggers symbolic *substitution*, so

$$\int_{x_j'} \delta[x_j' - g(\vec{x})] V' dx_j' \; = \; V'\{x_j'/g(\vec{x})\}$$

  - e.g.,

$$\int_{x_1'} \delta[x_1' - (x_1^2 + 1)] \left(\begin{cases} \underline{x_1'} < 2 : & \underline{x_1'} \\ \underline{x_1'} \geq 2 : & \underline{x_1'^2} \end{cases} \right) dx_1' \; = \; \begin{cases} \underline{x_1^2 + 1} < 2 : & \underline{x_1^2 + 1} \\ \underline{x_1^2 + 1} \geq 2 : & \underline{(x_1^2 + 1)^2} \end{cases}$$

  - If g is *case*: need *conditional substitution*, see paper

Case $\rightarrow$ XADD

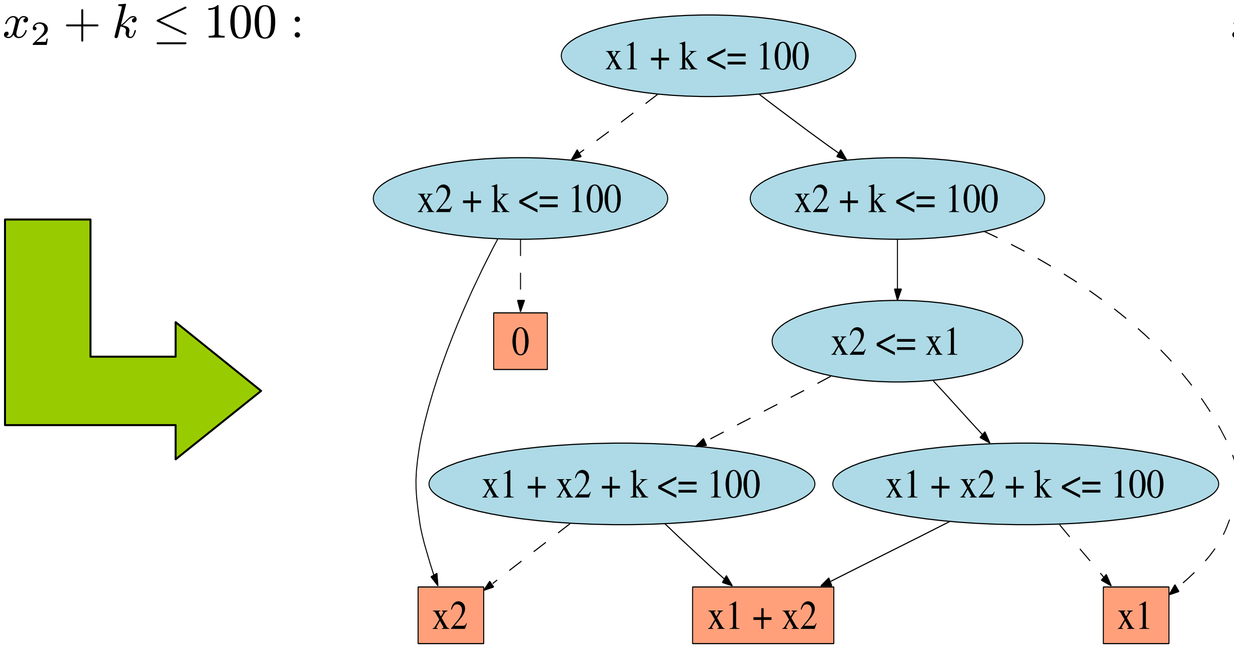
SDP needs an efficient data structure for

- compact, minimal case representation
- efficient case operations

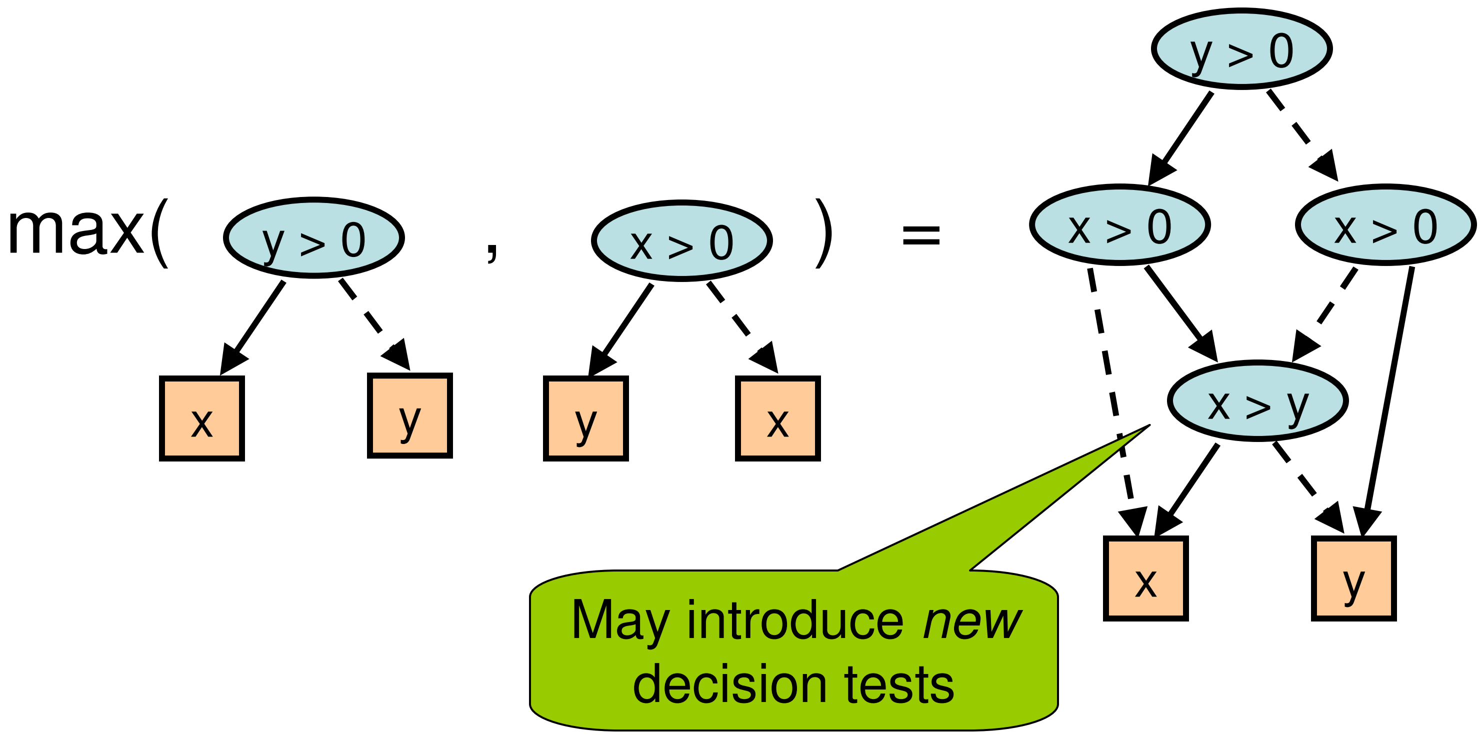
XADDs

- Extended ADD representation of case statements

$$V = \begin{cases} x_1 + k > 100 \wedge x_2 + k > 100 : & 0 \\ x_1 + k > 100 \wedge x_2 + k \leq 100 : & x_2 \\ x_1 + k \leq 100 \wedge x_2 + k > 100 : & x_1 \\ x_1 + x_2 + k > 100 \wedge x_1 + k \leq 100 \wedge x_2 + k \leq 100 \wedge x_2 > x_1 : & x_2 \\ x_1 + x_2 + k > 100 \wedge x_1 + k \leq 100 \wedge x_2 + k \leq 100 \wedge x_2 \leq x_1 : & x_1 \\ x_1 + x_2 + k \leq 100 : & x_1 + x_2 \end{cases}$$



XADD Maximization

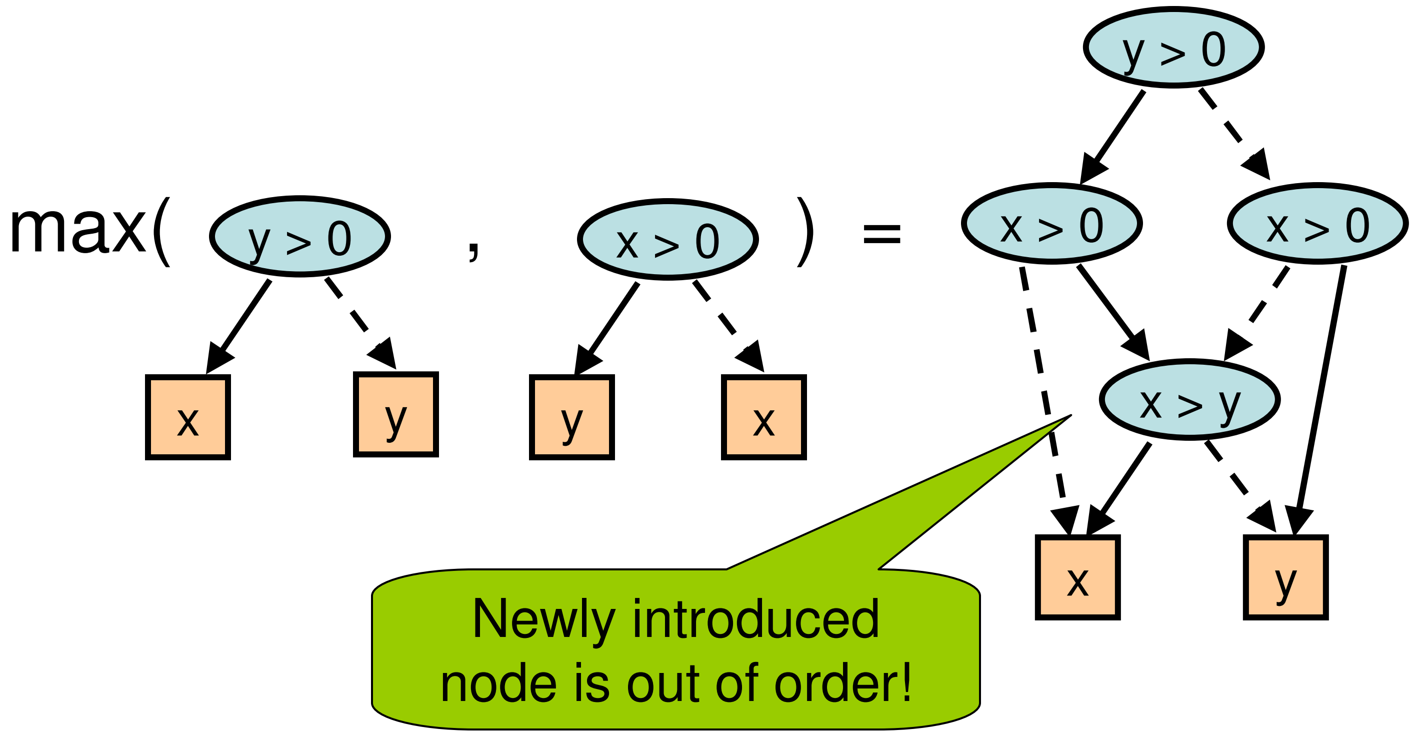


Maintaining XADD Orderings I

- Max may get variables out of order

Decision ordering (root→leaf)

- x > y
- y > 0
- x > 0

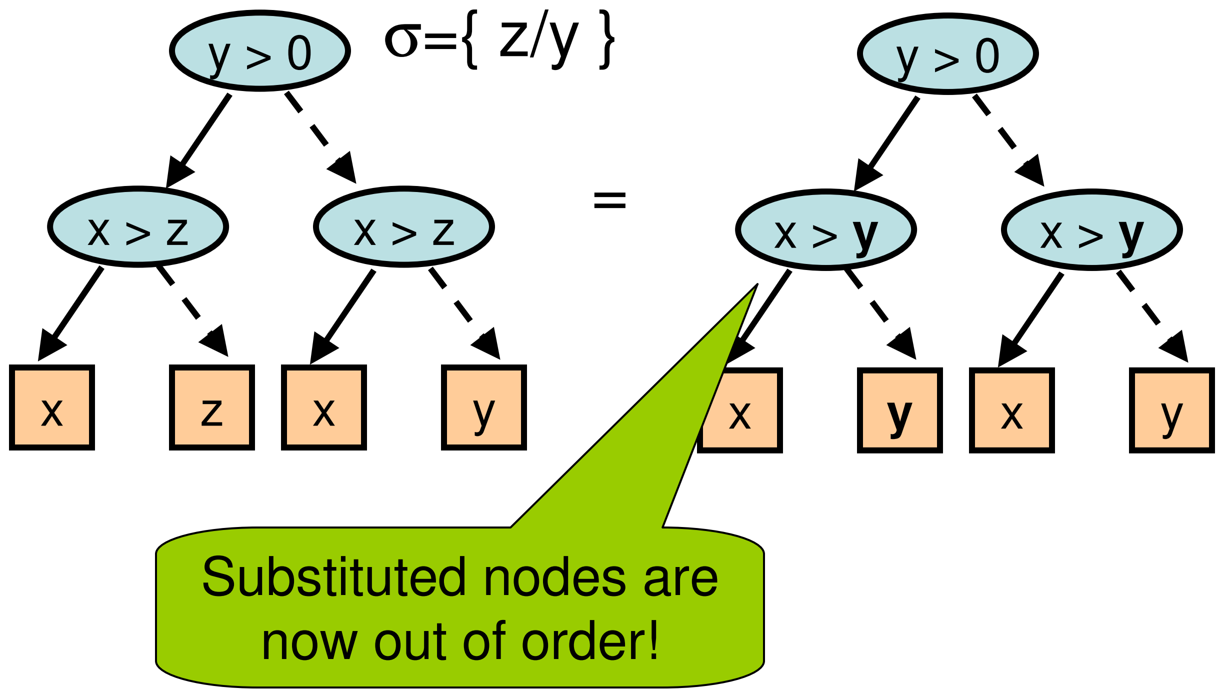


Maintaining XADD Orderings II

- Substitution may get vars out of order

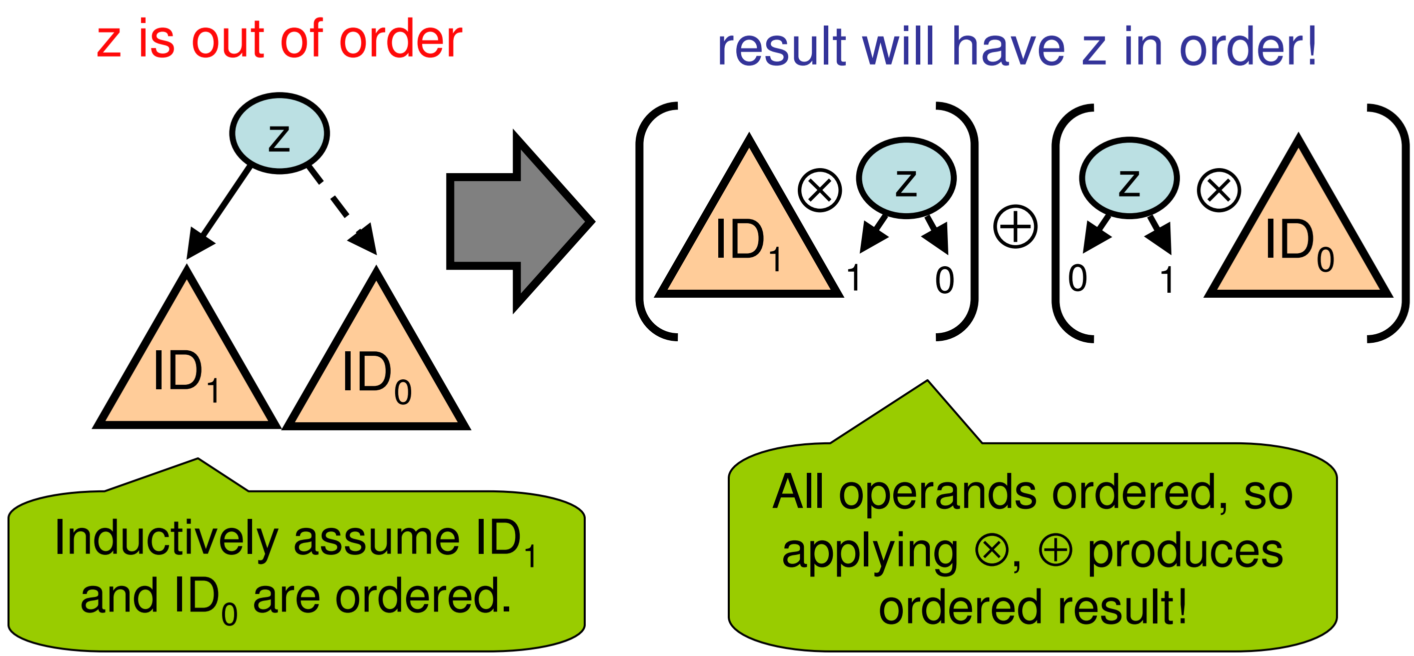
Decision ordering ($root \rightarrow leaf$):

- $x > y$
- $y > 0$
- $x > z$



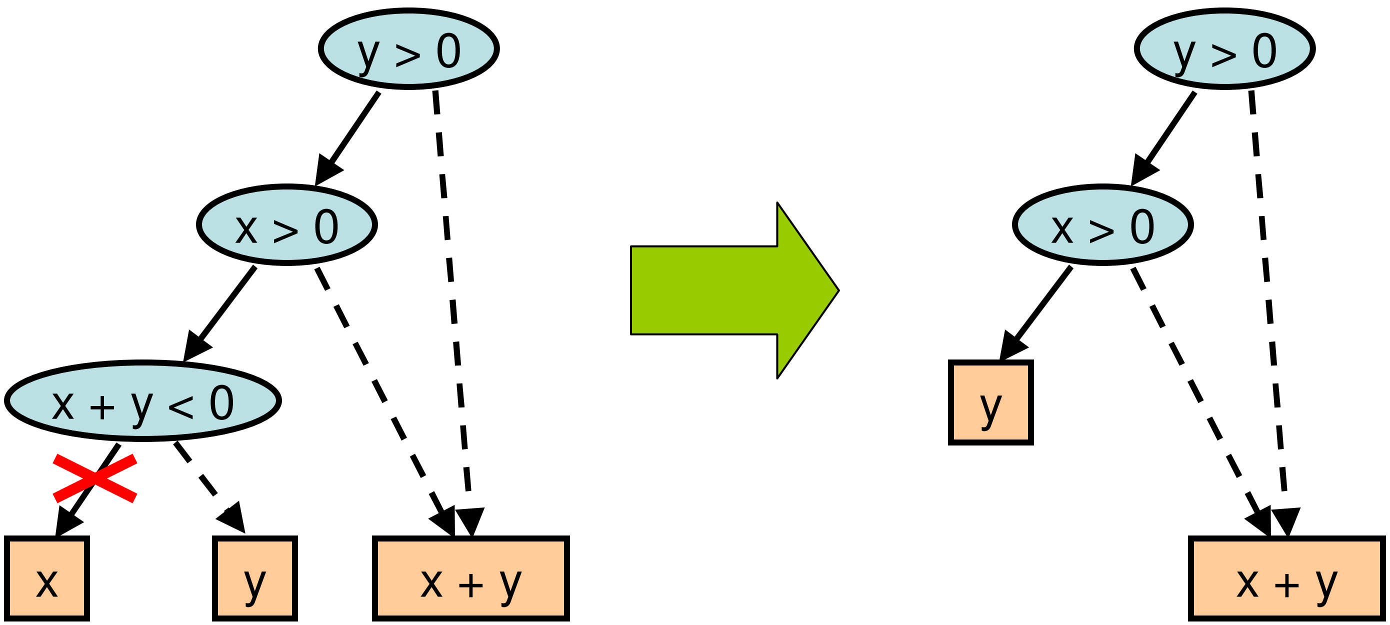
Correcting XADD Ordering

- Build *ordered* XADD from *unordered* XADD



Similar to Penberthy & Weld, AAAI-94

XADD Pruning



Node unreachable – $x + y < 0$ always false if $x > 0$ & $y > 0$

If **linear**, can detect with feasibility checker of LP solver & prune

Empirical Results

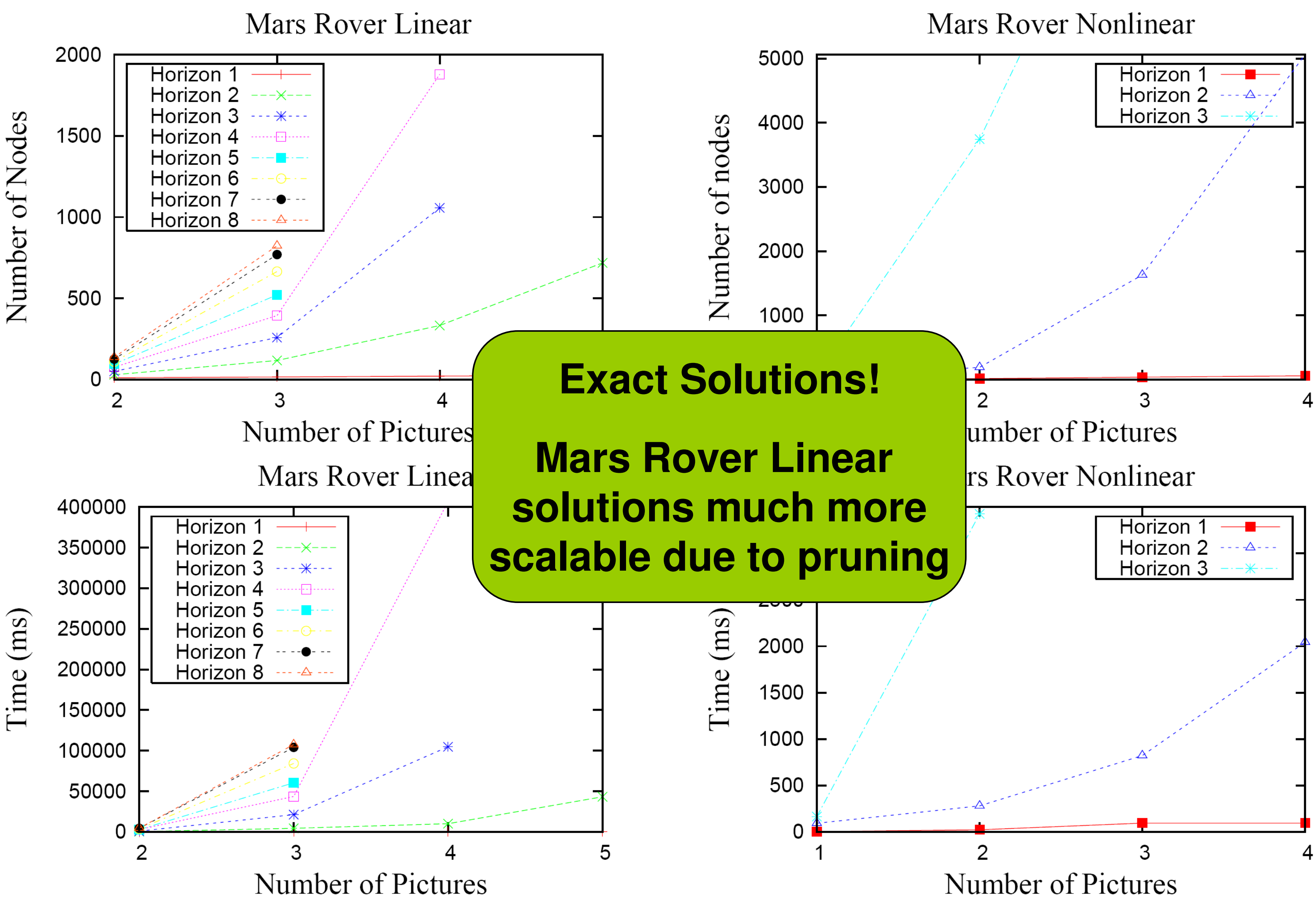
Problem Domains

- Knapsack problem (high-dimensional toy problem)
  - Transfer continuous resources to knapsack
    - Subject to capacity constraints
    - Reward for amount transferred
  - Can solve for optimal ∞-horizon solution

- Mars Rover (variants of Bresina *et al*, UAI-02)
  - Linear
    - take pictures with linear time/energy constraints
  - Nonlinear
    - move to target (x,y) position, taking pictures along way
    - reward is truncated quadratic

- All problem domains / code online:
  - http://code.google.com/p/xadd-inference/

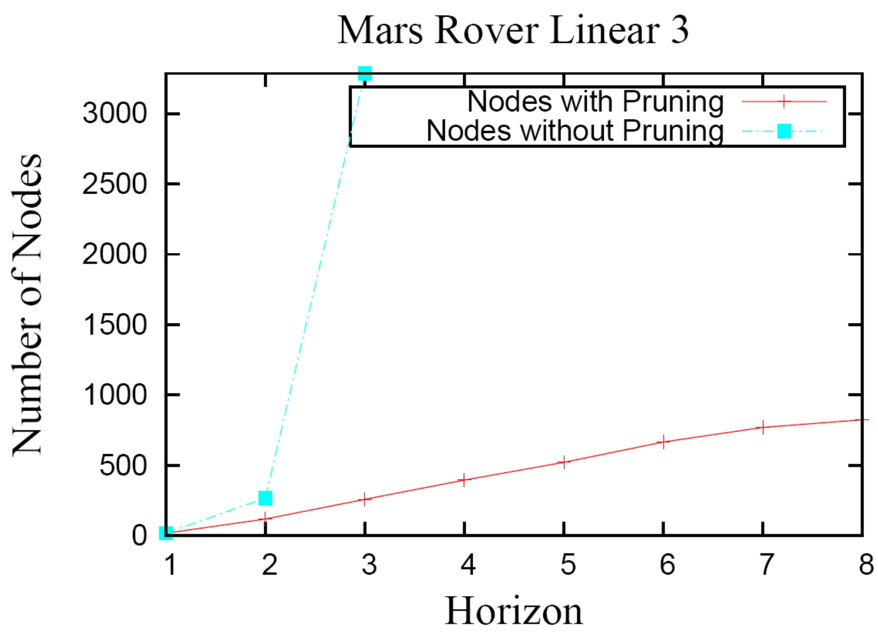
Results: Time and Space for Mars Rover

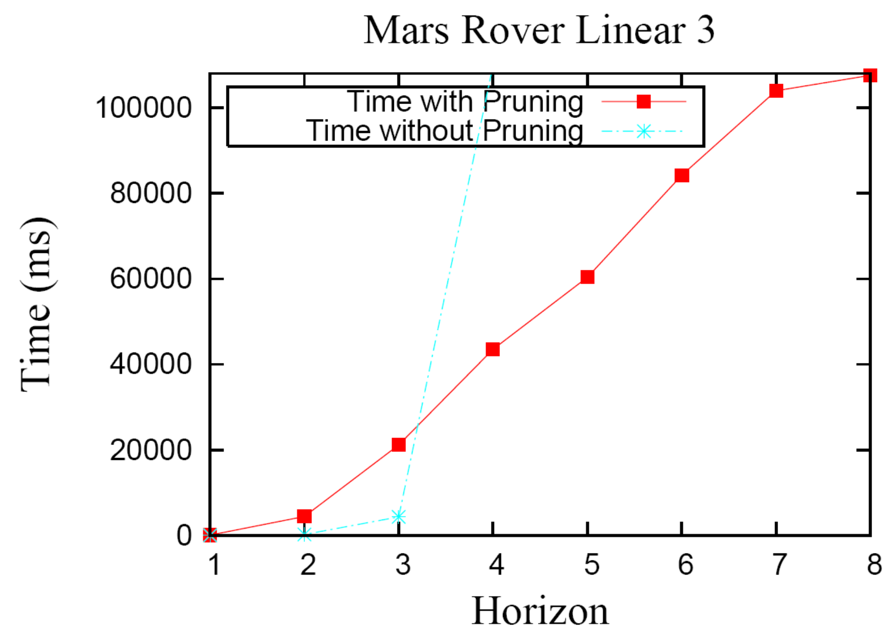


Exact Solutions!

Mars Rover Linear solutions much more scalable due to pruning

Results: XADD Pruning vs. No Pruning





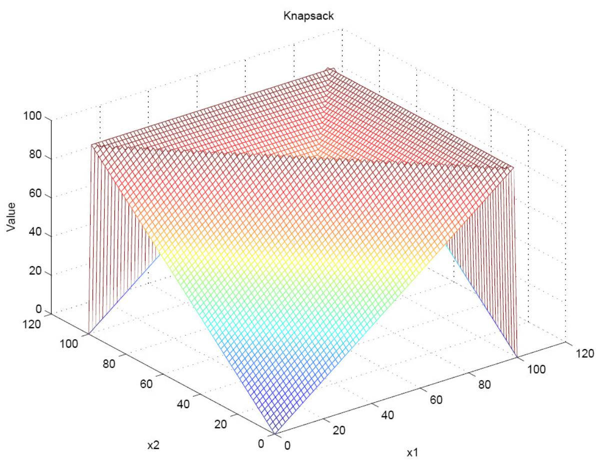
Summary:

- **without pruning: superlinear vs. horizon**
- **with pruning: linear vs. horizon**

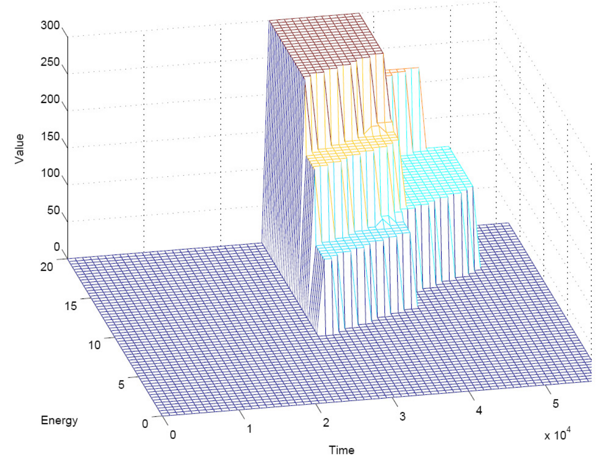
Worth the effort to prune!

Obligatory 3D Value Function Gallery

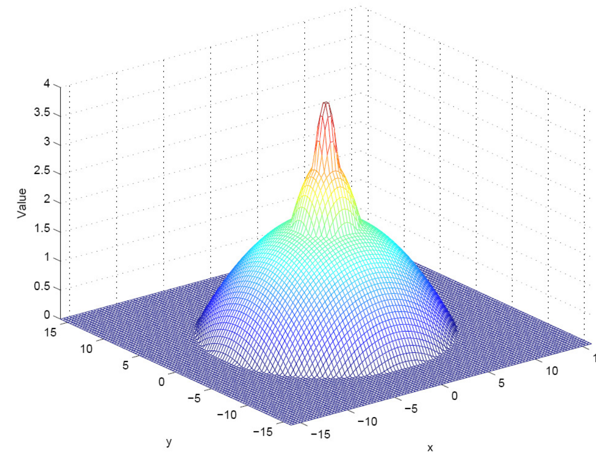
Knapsack



Mars Rover Linear



Mars Rover Nonlinear



Exact value functions in case form:

- **linear & nonlinear piecewise boundaries!**
- **nonlinear function surfaces!**

Conclusions

- First exact, closed-form solutions to subset of **multidimensional, nonlinear DC-MDPs**
- Key insights
  - Symbolic *case* representation
  - DP in terms of case $\otimes$, $\oplus$, max
  - $\int \delta$ triggers (conditional) substitution
- Need compact case, efficient operations
  - Case $\rightarrow$ Extended ADD (XADD)
  - $\otimes$, $\oplus$ technique for efficient decision reordering
  - Advantages of pruning

Future Work

- Efficiency
  - XADD pruning in nonlinear case
  - XADD Approximation?
    - Extend APRICODD (St-Aubin *et al,* NIPS-00)
- Expressivity
  - Full continuous stochastic extension
    - Currently, continuous transitions are mixture of δ's
    - Ideally want Gaussian noise, etc.
  - Continuous actions?
  - Partial observability?

Thank you!

Questions?

Extra Slides

XADD: Details

- The XADD is an ADD allowing
  - Arbitrary expressions at leaves
  - Arbitrary expression inequalities at decision nodes
    - If expressions polynomial, decisions & leaves have canonical form
    - Enforce ordering on all decision tests: $(x < y)$ before $(x < 3)$
- Operations same as XADD
  - But leaf operations may produce XADDs themselves!
    - May also require introduction of new decisions
    - E.g., maximization
- Can introduce support for substitution
  - Needed for SDP regression

1D: Boyan and Littman (1999)

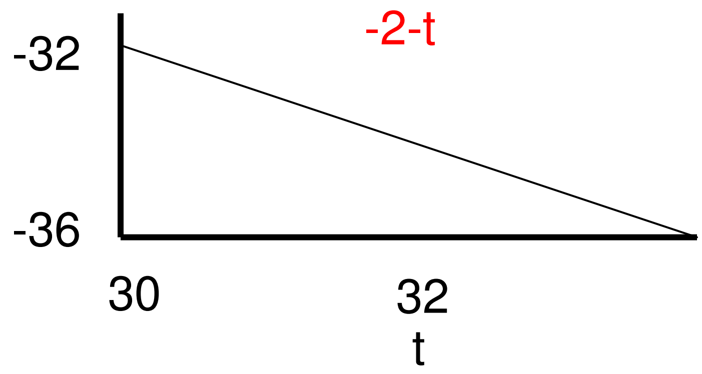
- Exact Solutions to Time-dependent MDPs
  - Assume actions / transitions as follows

Action = bus:

t' = t + 30

s' = office

R(s,t) = -2 – t + 20*I[s'=office]

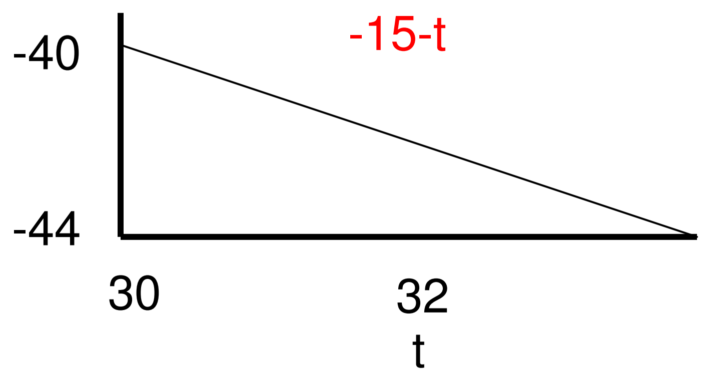


Action = taxi:

t' = t + 10

s' = office

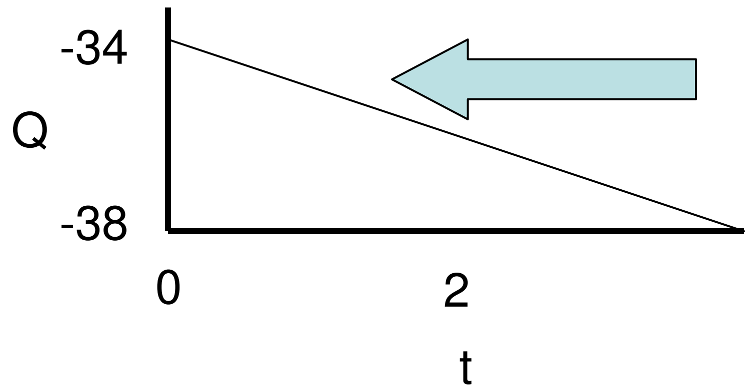
R(s,t) = -15 - t + 20*I[s'=office]



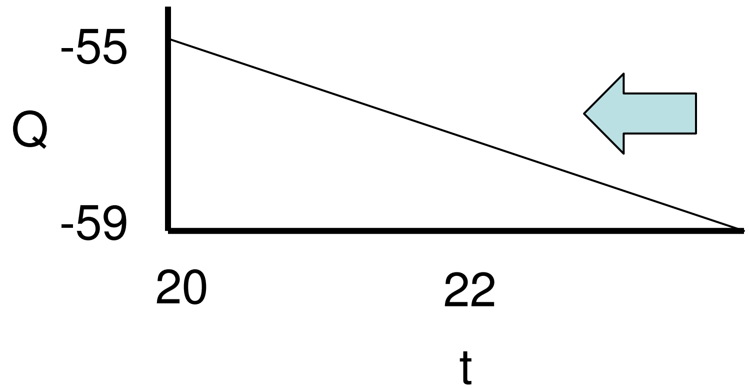
1D: Boyan and Littman (1999)

- Continuous transitions are δ-functions
  - Regressions just sums & translations of value

Q(a=bus,s=office,t)



Q(a=taxi,s=office,t)



1D: Boyan and Littman (1999)

- Value max is just piecewise partitioning

max(Q(a=bus,s=office,t), Q(a=taxi,s=office,t))