Automated Theorem Proving

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Topics in Automated Reasoning

Thursday, Jan. 19, 2006

Introduction

- **Def. Automated Theorem Proving:**

  Proof of mathematical theorems by a computer program.

- **Depending on underlying logic, task varies from trivial to impossible:**
  - Simple description logic: Poly-time
  - Propositional logic: NP-Complete (3-SAT)
  - First-order logic w/ arithmetic: Impossible

Applications

- **Proofs of Mathematical Conjectures**
  - Graph theory: Four color theorem
  - Boolean algebra: Robbins conjecture
- **Hardware and Software Verification**
  - Verification: Arithmetic circuits
  - Program correctness: Invariants, safety
- **Query Answering**
  - Build domain-specific knowledge bases, use theorem proving to answer queries

Basic Task Structure

- **Given:**
  - Set of axioms (KB encoded as axioms)
  - Conjecture (assumptions + consequence)
- **Inference:**
  - Search through space of valid inferences
- **Output:**
  - Proof (if found, a sequence of steps deriving conjecture consequence from axioms and assumptions)

Many Logics / Many Theorem Proving Techniques

Focus on theorem proving for logics with a model-theoretic semantics (TBD)

- **Logics:**
  - Propositional, and first-order logic
  - Modal, temporal, and description logic
- **Theorem Proving Techniques:**
  - Resolution, tableaux, sequent, inverse
  - Best technique depends on logic and app.

Example of Propositional Logic Sequent Proof

- **Given:**
  - Axioms: None
  - Conjecture: $A \vee \neg A$?
- **Inference:**
  - Gentzen Sequent Calculus

- **Direct Proof:**

(I)	$\overline{A \vdash A}$
($\neg$R)	$\overline{\vdash \neg A, A}$
($\vee$R2)	$\overline{\vdash A \vee \neg A, A}$
(PR)	$\overline{\vdash A, A \vee \neg A}$
($\vee$R1)	$\overline{\vdash A \vee \neg A, A \vee \neg A}$
(CR)	$\overline{\vdash A \vee \neg A}$

Example of First-order Logic Resolution Proof

- **Given:**
  - **Axioms:**
    $\forall x\ Man(x) \Rightarrow Mortal(x)$
    $Man(Socrates)$
  - **Conjecture:**
    $\exists y\ Mortal(y)$?
- **Inference:**
  - **Refutation Resolution**

- **CNF:**
  $\neg Man(x) \vee Mortal(x)$
  $Man(Socrates)$
  $\neg Mortal(y)$ [Neg. conj.]

- **Proof:**
  1. $\neg Mortal(y)$ [Neg. conj.]
  2. $\neg Man(x) \vee Mortal(x)$ [Given]
  3. $Man(Socrates)$ [Given]
  4. $Mortal(Socrates)$ [Res. 2,3]
  5. $\perp$ [Res. 1,4]

  Contradiction $\Rightarrow$ Conj. is true

Example of Description Logic Tableaux Proof

- **Given:**
  - **Axioms:**
    None
  - **Conjecture:**
    $\neg\exists\ Child.\neg Male \Rightarrow \forall\ Child.Male$?
- **Inference:**
  - **Tableaux**

- **Proof:**
  Check unsatisfiability of
  $\exists Child.\neg Male \sqcap \forall\ Child.Male$

  x: $\exists Child.\neg Male \sqcap \forall\ Child.Male$
  x: $\forall\ Child.Male$ [$\sqcap$ -rule]
  x: $\exists Child.\neg Male$ [$\sqcap$ -rule]
  x: Child y [$\exists$-rule]
  y: $\neg Male$ [$\exists$-rule]
  y: Male [$\forall$-rule]
  <CLASH>

  Contradiction $\Rightarrow$ Conj. is true

Lecture Outline

- Common Definitions
  - Soundness, completeness, decidability
- Propositional and first-order logic
  - Syntax and semantics
  - Tableaux theorem proving
  - Resolution theorem proving
    - Strategies, orderings, redundancy, saturation optimizations, & extensions
- Modal, temporal, & description logics
  - Quick overview of logics / TP techniques

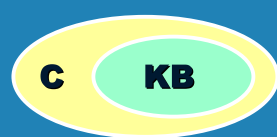
Entailment vs. Truth

- For each logic and theorem proving approach, we'll specify:
  - Syntax and semantics
  - Foundational axioms (if any)
  - Rules of inference
- Entailment vs. Truth
  - Let KB be the conjunction of axioms
  - Let F be a formula (possibly a conjecture)
  - We say KB |- F (read: KB entails F) if F can be derived from KB through rules of inference
  - We say KB |= F (read: KB models F) if semantics hold that F is true whenever KB is true

Model-theoretic semantics

- Model-theoretic semantics for logics
  - An interpretation is a truth assignment to atomic elements of a KB: $I\langle C,D\rangle = \{\langle F,F\rangle, \langle F,T\rangle, \langle T,F\rangle, \langle T,T\rangle\}$
  - A model of a formula is an interpretation where it is true: $I\langle C,D\rangle = \langle F,T\rangle$ models $C \lor D, C \Rightarrow D$, but not $C \land D$
  - Two properties of a formula F w.r.t. axioms of KB:
    - Validity: F is true in all models of KB
    - Satisfiability: F is true in ≥ 1 model of KB
- Think of truth in a set-theoretic manner

KB |= C



Models of KB
$\subseteq$ Models of C

Soundness, Completeness, and Decidability

- Two properties of ATP inference systems:
  - Soundness: If KB |- C then KB |= C
  - Completeness: If KB |= C then KB |- C
- For a given logic, an ATP *decision procedure* returns true or false for KB |- C
- For a logic, a *sound* and *complete decision procedure* has one of following properties:
  - Decidable: Decision procedure guaranteed to terminate in finite time
  - Semidecidable: Decision procedure guaranteed to terminate for either true or false, but not both
  - Undecidable: No termination guarantee

Prop. Logic Syntax

- Propositional variables: p, rain, sunny
- Connectives: $\Rightarrow$ $\Leftrightarrow$ $\neg$ $\wedge$ $\vee$
- Inductive definition of well-formed formula (wff):
  - Base: All propositional vars are wffs
  - Inductive 1: If A is a wff then $\neg$A is a wff
  - Inductive 2: If A and B are wffs then A $\wedge$ B, A $\vee$ B, A $\Rightarrow$ B, A $\Leftrightarrow$ B are wffs
- Examples:
  - rain, rain $\Rightarrow$ $\neg$ sunny
  - (rain $\Rightarrow$ $\neg$ sunny) $\Leftrightarrow$ (sunny $\Rightarrow$ $\neg$rain)

Prop. Logic Semantics

- For a formula F, the truth I(F) under interpretation I is recursively defined:
  - Base:
    - F is prop var A then I(F)=true iff I(A)=true
  - Recursive:
    - F is $\neg$C then I(F)=true iff I(C)=false
    - F is C $\wedge$ D then I(F)=true iff I(C)=true & I(D)=true
    - F is C $\vee$ D then I(F)=true iff I(C)=true or I(D)=true
    - F is C $\Rightarrow$ D then I(F)=true iff I($\neg$C $\vee$ D)=true
    - F is C $\Leftrightarrow$ D then I(F)=true iff I(C $\Rightarrow$ D)=true & I(D $\Rightarrow$ C)=true
- Truth defined recursively from ground up!

CNF Normalization

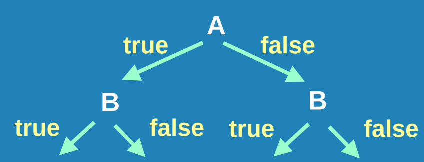
- Many prop. theorem proving techniques req. KB to be in clausal normal form (CNF):
  - Rewrite all C ⇔ D as C ⇒ D ∧ D ⇒ C
  - Rewrite all C ⇒ D as ¬C ∨ D
  - Push negation through connectives:
    - Rewrite ¬(C ∧ D) as ¬C ∨ ¬D
    - Rewrite ¬(C ∨ D) as ¬C ∧ ¬D
  - Rewrite double negation ¬ ¬ C as C
  - Now NNF, to get CNF, distribute ∨ over ∧:
    - Rewrite (C ∧ D) ∨ E as (C ∨ E) ∧ (D ∨ E)
- A clause is a disj. of literals (pos/neg vars)
- Can express KB as conj. of a set of clauses

CNF Normalization Example

- Given KB with single formula:
  - ¬ (rain ⇒ wet) ⇒ (inside ∧ warm)
- Rewrite all C ⇒ D as ¬C ∨ D
  - ¬ ¬ (¬ rain ∨ wet) ∨ (inside ∧ warm)
- Push negation through connectives:
  - (¬ ¬ ¬ rain ∨ ¬ ¬ wet) ∨ (inside ∧ warm)
- Rewrite double negation ¬ ¬ C as C
  - (¬ rain ∨ wet) ∨ (inside ∧ warm)
- Distribute ∨ over ∧:
  - (¬rain ∨ wet ∨ inside) ∧ (¬rain ∨ wet ∨ warm)
- CNF KB: {¬rain ∨ wet ∨ inside, ¬rain ∨ wet ∨ warm}

Prop. Theorem Proving

- $A \Rightarrow B$ iff $A \wedge \neg B$ is unsatisfiable
- Decision procedure for propositional logic is decidable, but NP-complete (reduction to 3-SAT)
- State-of-the-art prop. unsatisfiability methods are DPLL-based



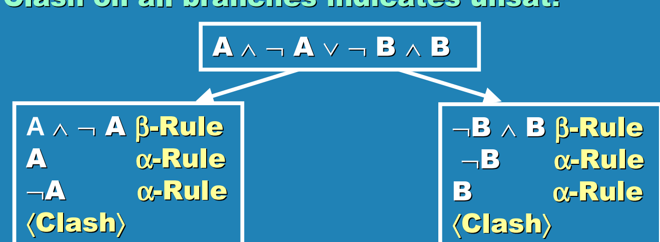
Instantiate prop vars until all clauses falsified, backtrack and do for all instantiations $\Rightarrow$ unsat!

- Many optimizations, more next week

Prop. Tableaux Methods

Given negated query F (in NNF), use rules to recursively break down:

- α-Rule: Given $A \wedge B$ add A and B
- β-Rule: Given $A \vee B$ branch on A and B
- $\langle$Clash$\rangle$: If A and $\neg A$ occur on same branch
- Clash on all branches indicates unsat!



Note: Inverse method is inverse of tableaux - bottom up

Propositional Resolution

- One rule:

Rule:

$$\frac{A \lor B \quad \neg B \lor C}{A \lor C}$$

Example application:

$$\frac{\neg precip \lor \neg freezing \lor snow \qquad \neg snow \lor slippery}{\neg precip \lor \neg freezing \lor slippery}$$

- Simple strategy is to make all possible resolution inferences
- Refutation resolution is sound and complete

Resolution Strategies

Need strategies to restrict search:

- Unit resolution:
  - Only resolve with unit clauses
  - Complete for Horn KB
  - Intuition: Decrease clause size
- Set of support:
  - SOS starts with query clauses
  - Only resolve SOS clauses with non-SOS clauses and put resolvents in SOS
  - Intuition: KB should be satisfiable so refutation should derive from query
- Input resolution:
  - At each step resolve only with input (KB or query)
  - I.e., don't resolve non-input clauses
  - Linear input: also allow ancestor ⇒ complete

Ordering Strategies

- Refutation of a clause requires refutation of all literals
- Enforce an ordering on proposition elimination to restrict search
  - Example order: p then r then q
  - General idea behind Davis-Putnam (DP) & directional resolution (Dechter & Rish)
- Effective, but does not work with all resolution strategies, e.g. SOS + ordered resolution is incomplete

Prop. Inference Software

- Mainly DPLL SAT algorithms
  - zChaff – highly optimized & documented DPLL solver, source available
  - siege – best performing DPLL solver, source not available
  - 2clseq – DPLL solver with constraint propagation (balance search / reasoning)
- For some applications: BDDs
  - BDDs maintain all possible models in a canonical data structure
  - CUDD ADD/BDD Package – very efficient

First-order logic

- Refer to objects and relations b/w them
- Propositional logic requires all relations to be propositionalized
  - Scott-at-home, Scott-at-work, Jim-at-subway, etc…
- Really want a compact relational form:
  - at(Scott, home), at(Scott, work), at(Jim, subway), etc…
- Then can use variables and quantify over all objects:
  - $\forall x$ person(x) $\Rightarrow \exists y$ at(x,y) ^ place(y)

First-order Logic Syntax

- Terms (technical definition is inductive b/c of fns)
  - Variables: w, x, y, z
  - Constants: a, b, c, d
  - Functions over terms: f(a), f(x,y), f(x,c,f(f(z)))
- Predicates: P(x), Q(f(x,y)), R(x, f(x,f(c,z),c))
- Connectives: $\Rightarrow \Leftrightarrow \neg \wedge \vee$
- Quantifiers: $\forall$ $\exists$
- Inductive wff definition:
  - Same as prop. but with following modifications…
  - Base: All predicates over terms are wffs
  - Inductive: If A is a wff and x is a variable term then $\forall$x A & $\exists$x A are wffs

First-order Logic Semantics

- Interpretation $I = (\Delta^I, \bullet^I)$
  - Δ^I is a non-empty domain
  - $\bullet^I$ maps from predicate symbols P of arity n into a subset of $\times_{1\ldots n} \Delta^I$ (where P is true)
- Example
  - Δ^I is {Scott, Jim}
  - $\bullet^I$ maps at(•,•) into { ⟨Scott, loc(Scott)⟩, ⟨Jim, loc(Jim)⟩ }
  - All other ground predicates are false in I, e.g. at(Scott, loc(Jim)), at(Scott, Scott)
- NB: FOL has ∞ interpretations/models!

Substitution and Unification

- Substitution
  - A substitution list θ is a list of variable-term pairs
    - e.g., θ={x/3,y/f(z)}
  - When θ is applied to an FOL formula, every free occurrence of a variable in the list is replaced with the given term
    - e.g. (P(x,y) ^ ∃x P(x,y))θ = P(3,f(z)) ^ ∃x P(x,f(z))
- Unification / Most General Unifier
  - The unifier UNIF(x,y) of two predicates/terms is a substitution that makes both arguments identical
    - e.g. Unif(P(x,f(x)), P(y, f(f(z)))) = {x/f(1), y/f(1), z/1}
  - The most general unifier MGU(x,y) is just that… all other unifiers can be obtained from the MGU by additional subst. (MGU exists for unifiable args)
    - e.g. MGU(P(x,f(x)), P(y, f(f(z)))) = {x/f(z), y/f(z)}

Skolemization

- Skolemization is the process of getting rid of all $\exists$ quantifiers from a formula while preserving (un)satisfiability:
  - If $\exists x$ quantifier is the outermost quantifier, remove the $\exists$ quantifier and substitute a new constant for x
  - If $\exists x$ quantifier occurs inside of $\forall$ quantifiers, remove the $\exists$ quantifier and substitute a new function of all $\forall$ quantified variables for x
- Examples:
  - Skolemize($\exists w\ \exists x\ \forall y\ \forall z\ P(w,x,y,z)$) = $\forall y\ \forall z\ P(c,d,y,z)$
  - Skolemize($\forall w\ \exists x\ \forall y\ \exists z\ P(w,x,y,z)$) = $\forall w\ \forall y\ P(w,f(w),y,f(x,y))$

CNF Conversion

CNF conversion is the same as the propositional case up to NNF, then do:

- Standardize apart variables (all quantified variables should have different names)
  - e.g. $\forall x\ A(x) \wedge \exists x\ \neg A(x)$ becomes $\forall x\ A(x) \wedge \exists y\ \neg A(y)$
- Skolemize formula
  - e.g. $\forall x\ A(x) \wedge \exists y\ \neg A(y)$ becomes $\forall x\ A(x) \wedge \neg A(c)$
- Drop universals
  - e.g. $\forall x\ A(x) \wedge \neg A(c)$ becomes $A(x) \wedge \neg A(c)$
- Distribute $\vee$ over $\wedge$

First-order Theorem Proving

- Tableaux methods
  - Preferred for some types of reasoning and for subsets of FOL (guarded fragment, set theory)
  - Highly successful for description and modal logics which conform to guarded fragment of FOL
- Resolution Methods
  - Most successful technique for a variety of KBs
  - But... search space grows very quickly
  - Need a variety of optimizations in practice
    - strategies, ordering, redundancy elimination
- FOL TP complete ☺, but semidecidable ☹
  - Will return in finite time if formula entailed
  - May run forever if not entailed

First-order Tableaux

Given negated query F (in NNF), use rules to recursively break down:

- α-Rule, β-Rule: Same as for prop tableaux
- γ-Rule: Given $\forall x\ A(x)$ add $A(?v)$ for variable $?v$
- δ-Rule: Given $\exists x\ A(x)$ add $A(f)$ for Skolem function f
- ⟨Clash⟩: If unifiable A and $\neg A$ occur on same branch

$\forall x\ A(x) \wedge \exists x\ \neg A(x) \vee \exists x,y\ \neg B(x,y) \wedge \forall x,y\ B(x,y)$

$\forall x\ A(x) \wedge \exists x\ \neg A(x)$	β-Rule
$A(?y)$	α/γ -Rule
$\neg A(c)$	α/δ -Rule
⟨Clash⟩	

$\exists x,y\ \neg B(x,y) \wedge \forall x,y\ B(x,y)$	β-Rule
$\neg B(c,d)$	$\alpha/\delta/\delta$ -Rule
$B(?y,?z)$	$\alpha/\gamma/\gamma$ -Rule
⟨Clash⟩	

First-order Resolution

- **Binary Resolution Rule**

Rule:

$$\frac{C \vee D \quad \neg E \vee F}{(C \vee F)\theta} \quad \theta = MGU(D,E)$$

Example application:

$$\frac{P(3) \vee Q(f(x)) \vee R(y) \quad \neg Q(y)}{P(3) \vee R(f(x))}$$

- **Factoring Rule**

Rule:

$$\frac{C \vee D \vee E}{C\theta \vee E} \quad \theta = MGU(C,D)$$

Example application:

$$\frac{P(z) \vee Q(3) \vee Q(z)}{P(3) \vee Q(3)}$$

Example of First-order Logic Resolution Proof

- **Given:**
  - **Axioms:**
    $\forall x\ Man(x) \Rightarrow Mortal(x)$
    $Man(Socrates)$
  - **Conjecture:**
    $\exists y\ Mortal(y)$?
- **Inference:**
  - **Refutation Resolution**

- **CNF:**
  $\neg Man(x) \vee Mortal(x)$
  $Man(Socrates)$
  $\neg Mortal(y)$ [Neg. conj.]

- **Proof:**
  1. $\neg Mortal(y)$ [Neg. conj.]
  2. $\neg Man(x) \vee Mortal(x)$ [Given]
  3. $Man(Socrates)$ [Given]
  4. $Mortal(Socrates)$ [Res. 2,3]
  5. $\perp$ [Res. 1,4]

  Contradiction $\Rightarrow$ Conj. is true

Importance of Factoring

- Without the factoring rule, binary resolution is incomplete
- For example, take the following refutable clause set:
  – { A(w) v A(z), ~A(y) v ~A(z) }
- All binary resolutions yield clauses of the same form
- Clause set is only refutable if one of the clauses is first factored

Search Control

Additional refinements of prop strategies yield goal-directed / bottom-up search:

– SLD Resolution
  - KB of definite clauses (i.e. Horn rules), e.g. Uncle(?x,?y) := Father(?x,?z) $\wedge$ Brother(?x,?y)
  - Resolution backward chains from goal of rules
  - With negation-as-failure semantics, SLD-resolution is logic programming, i.e. Prolog

– Negative and Positive Hyperresolution
  - All negative (positive) literals in nucleus clause are *simultaneously* resolved with completely positive (negative) satellite clauses
  - Positive hyperres yields backward chaining
  - Negative hyperres yields forward chaining

Tabled Inference

- Naïve approaches to resolution perform one inference per step
- For **SLD or neg. hyperres** and **KBs w/ large numbers of constants / functions**, can store clause terms and perform **tabled res**, e.g.
  - CNF KB = { R(a,b), R(b,a), R(b,c), R(c,b), ¬R(x,y) ∨ ¬R(y,z) ∨ R(x,z) }
  - Perform DB-like join during SLD or neg. hyperres:

R(x,y)		R(y,z)		R(x,z)
{ ⟨a,b⟩, ⟨b,a⟩, ⟨b,c⟩, ⟨c,b⟩ }	×	{ ⟨a,b⟩, ⟨b,a⟩, ⟨b,c⟩, ⟨c,b⟩ }	⟹	{ ⟨a,a⟩, ⟨a,c⟩, ⟨b,b⟩, ⟨c,c⟩, ⟨c,a⟩, ⟨c,c⟩ }

- Can cache tabled inferences for reuse
- Huge improvement for instance-heavy KBs

Term Indexing

- **Term indexing** is another general technique for **fast retrieval** of **sets of terms / clauses** matching criteria
- Common uses in modern theorem provers:
  - Term q is unifiable with term t, i.e., $\exists\theta$ s.t. $q\theta = t\theta$
  - Term t is an instance of q, i.e., $\exists\theta$ s.t. $q\theta = t$
  - Term t is a generalization of q, i.e., $\exists\theta$ s.t. $q = t\theta$
  - Clause q subsumes clause t, i.e., $\exists\theta$ s.t. $q\theta \subseteq t$
  - Clause q is subsumed by clause t, i.e., $\exists\theta$ s.t. $t\theta \subseteq q$
- Techniques: (Google for "term indexing")
  - Path indexing
  - Code, context, & discrimination trees

Age-weight Ratio

- During a resolution strategy, have two sets:
  - Active: Set of active clauses for resolving with
  - Frontier: Candidate clauses to resolve with Active
- Idea: Store the frontier in two queues
  - Age queue: Standard FIFO queue
  - Weight queue: Priority queue where clause priority determined by heuristic measure:
    - Number of literals, number of terms, etc...
- A:W ratio: Choose A clauses from age queue for every W chosen from weight queue
  - Retains completeness of strategy if A is non-zero
    - I.e., fair b/c all clauses eventually selected
  - Can speed up inference by orders of magnitude!

Redundancy Control

- Redundancy of clauses is a huge problem in FOL resolution
  - For clauses C & D, C is redundant if $\exists\theta$ s.t. $C\theta \subseteq D$ as a multiset, a.k.a. θ-subsumption
  - If true, D is redundant and can be removed
    - Intuition: If D used in a refutation, $C\theta$ could be substituted leading to even shorter refutation
- Two types of subsumption where N is a new resolvent and $A \in$ Active:
  - Forward subsumption: A θ-subsumes N, delete N
  - Backward subsumption: N θ-subsumes A, delete A
- Forward / backward subsumption expensive but saves many redundant inferences

Saturation Theorem Proving

- Given a set of clauses S:
  - S is saturated if all possible inferences from clauses in S generate forward subsumed clauses
  - Thus, all new inferences can be deleted without sacrificing completeness
  - If S does not contain the empty clause then S is satisfiable
- Saturation implies no proof possible!
- Usually need ordering restrictions to reach saturation (if possible)...

Simplification Orderings

For complete ordered resolution in FOL, must use term simplification orderings:

- Well-founded (Noetherian): If there is no infinitely decreasing chain of terms s.t. $t_0 \succ t_1 \succ t_2 \succ \ldots \succ t_\infty$
- Monotonic: If $s \succ t$ then $f[s] \succ f[t]$ ($f[s]$ and $f[t]$ are identical except for [term])
- Stable under Subst.: If $s \succ t$ then $s\theta \succ t\theta$

Examples: (Google for following keywords)

- Knuth-Bendix ordering
- Lexicographic path ordering

Literal Ordering & Selection

- Can extend term ordering to literals $\succ_{lit}$:
  - If literals equal but opposite sign, then negative literal $\succ_{lit}$ positive literal
  - Otherwise, treat literals as terms (modulo sign) and literal ordering $\succ_{lit}$ is just term ordering $\succ$
- A selection function selects literals, and must adhere to following rules:
  - At least one literal must be selected
  - Either a negative literal is among the selection, or all maximal positive literals w.r.t. $\succ_{lit}$ are selected
- Show selected literals by underscore
  - e.g., $\{ A \vee \underline{\neg B} \vee \neg C\ ,\ \underline{D} \vee E \vee \neg F,\ \neg G \vee \underline{H} \vee \underline{I} \}$

Ordered Resolution w/ Selection

- Binary Ordered Res w/ Selection

Rule:

$$\frac{C \vee \underline{D} \qquad \underline{\neg E} \vee F}{(C \vee F)\theta}\ \theta = MGU(D,E)$$

Example application:

$$\frac{P(3) \vee \underline{Q(f(x))} \vee R(y) \quad \underline{\neg Q(y)}}{P(3) \vee R(f(x))}$$

- Ordered Factoring w/ Selection

Rule:

$$\frac{\underline{C} \vee \underline{D} \vee E}{C\theta \vee E}\ \theta = MGU(C,D)$$

Example application:

$$\frac{P(z) \vee \underline{Q(3)} \vee \underline{Q(z)}}{P(3) \vee Q(3)}$$

Clause Orderings & Redundancy

- Must define **specialized redundancy criterion** for forward and backward subsumption / deletion when using **ordered resolution:**
  - Define bag (clause) extension of literal ordering:
    - $\{x, y_1, \ldots, y_m\} \succ_{bag} \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ if $\forall i\ x \succ_{lit} x_i$
  - Can define redundancy w.r.t. $\succ$ bag ordering:
    - Clause C is redundant w.r.t. set of clauses S, if $\exists\ C_1, \ldots, C_n \in S$, $n \geq 0$, s.t. $\forall i\ C_i \prec_{bag} C$ and $C_1, \ldots, C_n \models C$
  - Under ordered res, even if C θ-subsumes D, D is not redundant (and can't be deleted) unless $C \prec_{bag} D$
- NB: Search restrictions of ordered res far outweigh weakened notion of redundancy
- **Ordered res is effective saturation strategy!**

Equality

- A predicate w/ special interpretation
- Could axiomatize:
  - $x=x$ (reflexive)
  - $x=y \Rightarrow y=x$ (symmetric)
  - $x=y \wedge y=z \Rightarrow x=z$ (transitive)
  - For each function f:
    - $x_1=y_1 \wedge \ldots \wedge x_n=y_n \Rightarrow f(x_1,\ldots,x_n)=f(y_1,\ldots,y_n)$
  - For each predicate P:
    - $x_1=y_1 \wedge \ldots \wedge x_n=y_n \wedge P(x_1,\ldots,x_n) \Rightarrow P(y_1,\ldots,y_n)$
- Too many axioms... better to reason about equality in inference rules

Inference Rules for Equality

- **Demodulation (incomplete)**

Rule: Literal containing z

$$\frac{x=y \quad L[z] \vee D}{L[y\theta] \vee D}\ \theta=MGU(x,z)$$

Example application:

$$\frac{x=f(x) \quad P(3) \vee Q}{P(f(3)) \vee Q}\ \theta=\{x/3\}$$

- **Paramodulation (complete)**

Rule: Literal containing z

$$\frac{x=y \vee C \quad L[z] \vee D}{(L[y] \vee C \vee D)\theta}\ \theta=MGU(x,z)$$

Example application:

$$\frac{x=f(x) \vee C \quad P(3) \vee Q}{P(f(3)) \vee C \vee Q}\ \theta=\{x/3\}$$

Equational Programming

- **Used extensively for algebraic group theory proofs**
- **All axioms and conjectures are unit equality predicates with arithmetic functions on the LHS and RHS, e.g.**
  - a*(x+y) = a*x+a*y ?
- **In this case, associative-commutative (AC) unification (Stickel) important for efficiency, e.g.**
  - MGU(x+3*y*y, z*3*z+1) = {x/1, y/z}

First-order theorem proving software

Many highly optimized first-order theorem proving implementations:

- **Vampire** (1st place for many years in CADE TP competition)
- **Otter** (Foundation for modern TP, still very good, usually 2nd place in CADE)
- **SPASS** (Specialized for sort reasoning)
- **SETHEO** (Connection tableaux calculus)
- **EQP** (Equational theorem proving system, proved Robbins conjecture)

First-order TP Progress

- Ever since the 1970s I at various times investigated using automated theorem-proving systems. But it always seemed that extensive human input--typically from the creators of the system--was needed to make such systems actually find non-trivial proofs.
- In the late 1990s, however, I decided to try the latest systems and was surprised to find that some of them could routinely produce proofs hundreds of steps long with little or no guidance. … the overall ability to do proofs--at least in pure operator systems--seemed vastly to exceed that of any human.

--Steven Wolfram, "A New Kind of Science"

On the other hand...

- Success of modern theorem provers relies largely on heuristic tuning
- Input KBs are analyzed for properties which determine strategies and various parameters of inference
- Still an art as much as a science, much room for more principled tuning of parameters, e.g.
  - Automatic partitioning of KBs to induce good literal orderings (McIlraith and Amir)

Gödel's Incompleteness Theorem

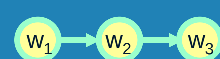
- FOL inference is complete (Gödel)
- So what is Gödel's incompleteness theorem (GIT) about?
- GIT: Inference in FOL with arithmetic (+,*,exp) is incomplete b/c set of axioms for arithmetic is not recursively enumerable.
- Read: Inference rules are sound and complete, but no way to generate all axioms required for arithmetic!

Modal Logic

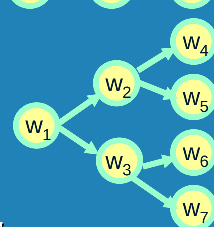
- Logic of **knowledge** and/or **belief**, e.g.
  - English: Scott knows that you know that Scott knows this lecture is boring
  - Modal Logic K_n (n agents): $K_{Scott}K_{you}K_{Scott}$ LIB
- Possible worlds (Kripke) semantics
  - Each modal operator K_i corresponds to a set of possible interpretations (i.e., possible worlds)
  - Different axioms (T,D,4,5,...) correspond to relations b/w worlds, Axiom 4: $K_i\varphi$ => $K_iK_i\varphi$
  - Semantics: $K_i\varphi$ iff φ is true in all worlds agent i considers possible according to axioms & KB
- Postpone reasoning until DL...

Temporal Logic

- A modal logic where the possible worlds are linked by time:
  - LTL: Linear temporal logic



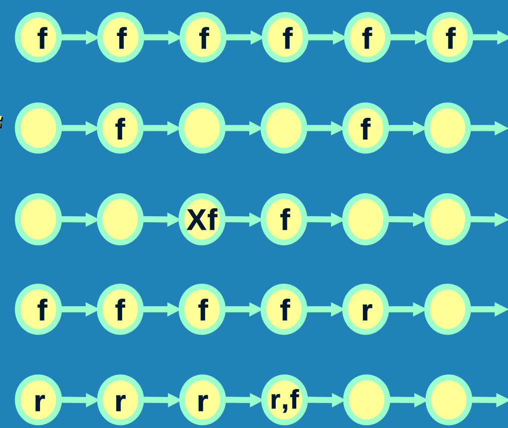
    - World states evolve deterministically
    - State can involve action
  - CTL: Computation tree logic



    - World states can evolve non-deterministically
- Temporal operators specify conditions on world evolution
- Used for verification, safety checks

LTL Temporal Operators

- G f: always f
- F f: eventually f
- X f: next state
- f U r: until
- f R r: releases



Temporal Logic Inference

- Because time evolves infinitely, propositional SAT methods won't work for LTL/CTL verification (will branch infinitely)
- However, LTL/CTL inference is monotonic!
  - To check condition, start with set of all worlds
  - Evolve world one step, remove states not satisfying condition
  - Continue evolution until set does not change… this is set of all states for which condition holds
- For propositional temporal logic, number of worlds is finite ⇒ termination ⇒ decidable!
- BDD data structure used to compactly encode sets of worlds and evolve worlds.

Description Logic

- A concept oriented logic:

English	FOL	DL
Dog with a Spot (DWS)	DWS(x) $\Leftrightarrow$ Dog(x) ^ ($\exists$y.has(x,y) ^ Spot(y))	DWS $\Leftrightarrow$ Dog $\sqcap$ $\exists$has.Spot
Large Dog with a Dark Spot (LDWDS)	LDWDS(x) $\Leftrightarrow$ (Dog(x) ^ Large(x)) ^ ($\exists$y.has(x,y) ^ (Spot(y) ^ Dark(y))	LDWDS $\Leftrightarrow$ Dog $\sqcap$ Large $\sqcap$ $\exists$has.(Spot $\sqcap$ Dark)

- Guarded fragment subset of FOL

Description Logic (DL) Inference

- Natural correspondence between ALC DL and modal logic (Schild):
  - Modal propositions are concepts that hold in possible worlds w, e.g. lecture is boring: LIB(w)
  - Modal operators K_i are DL roles that link possible worlds: $K_{scott}(w_1, w_2)$
  - If Scott knows that the lecture-is-boring then $\forall w_2\ K_{scott}(w_1, w_2) \Rightarrow LIB(w_2)$ (w_1 is a free variable)
  - Or in DL notation $\forall K_{scott}.LIB$
- Since decidable tableaux methods known for modal logics, these were imported into DL and later extended to expressive DLs
- Benefit of DL: Decidable subset of FOL that is ideal for conceptual ontology reasoning!

Example of Description Logic Tableaux Proof

- **Given:**
  - **Axioms:**
    None
  - **Conjecture:**
    $\neg\exists$ Child.$\neg$Male $\Rightarrow$
    $\forall$ Child.Male ?
- **Inference:**
  - **Tableaux**

- **Proof:**

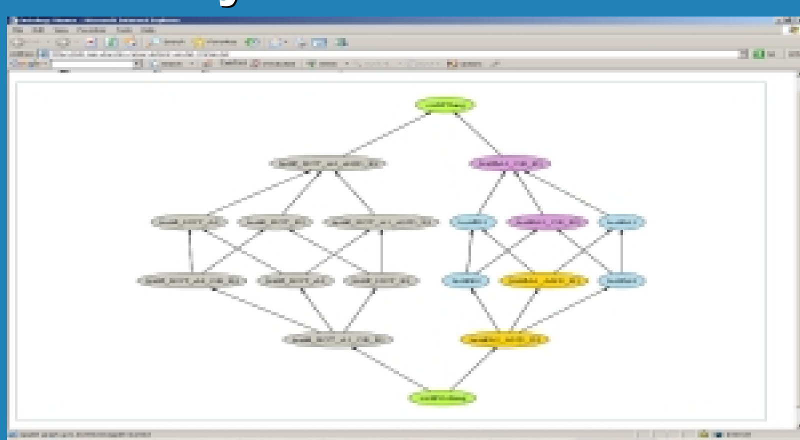
Check unsatisfiability of
$\exists$Child.$\neg$Male $\sqcap$ $\forall$ Child.Male

x: $\exists$Child.$\neg$Male $\sqcap$ $\forall$ Child.Male	
x: $\forall$ Child.Male	[$\sqcap$ -rule]
x: $\exists$Child.$\neg$Male	[$\sqcap$ -rule]
x: Child y	[$\exists$-rule]
y: $\neg$Male	[$\exists$-rule]
y: Male	[$\forall$-rule]
<CLASH>	

Contradiction $\Rightarrow$ Conj. is true

DL Reasoner Output (FaCT++)

Taxonomy encodes all $\Rightarrow$ relations



Modal, Verification, and DL Inference Software

- Modal logic
  - MSPASS (converts modal formula to FOL)
  - By correspondence, also DL reasoners
- Verification (temporal and non-temporal)
  - PVS (interactive TP for HW/SW verification)
  - ALLOY (first-order HW/SW model checker)
  - NuSMV (BDD-based LTL/CTL HW/SW verif.)
- DL Reasoning
  - Classic (limited DL, poly-time inference)
  - Racer (expressive DL, highly optimized)
  - FaCT++ (very expr. DL, highly optimized)

Repositories of TP Problems

Many repositories of theorem proving knowledge bases:

- TPTP: Thousands of Problems for TPs
  - Algebraic group theory, geometry, set theory, topology, software verification, NLP KBs
- SATLIB: Library of Prop. SAT problems
  - Hardware verification, industrial planning problems, hard randomized problems
- Open/ResearchCyc: Public version of Cyc
  - Large common-sense repository expressed in higher-order logic
- Semantic Web: DL ontologies in OWL
  - The web is the limit!

Concluding Thoughts

- Many logics, inference techniques, and computational guarantees
- Have to balance expressivity and computational tradeoffs with task-specific needs (Brachman & Levesque, 1985)
- Woods (1987): Don’t blame the tool!
  - A poor craftsman blames the tool when their efforts fail
  - An experienced craftsman uses the right tool for the job