Efficient Solutions to Factored MDP with Imprecise Transition Probabilities

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Outline

1. Markov Decision Processes (MDPs)

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2. MDPs with **Imprecise Probabilities** (MDP**IPs**)

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2. MDPs with **Imprecise Probabilities** (MDP**IPs**)
3. Representing **Factored** MDPIPs
4. Efficient **Solutions** for Factored MDPIPs
5. Summary

MDP - Formal model

MDP is defined by $\mathcal{M} = (S, A, R, P, \gamma)$:

- S is a set of states.
- A is a set of actions.
- $R(s, a)$ is a reward function.
- $P(s'|s, a)$ are transition probabilities $\forall s, s' \in S$ and $\forall a \in A$
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How to act in an MDP?

- Policy $\pi : S \to A$
- But what criteria to optimize?

MDP - Value Function

- Define value of a policy π:

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- MDP optimal policy π^*:

$$V_{\pi^*}(s) \geq V_{\pi'}(s) \;\; \forall \pi', s$$

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  - At $t = \infty$, convergence:

$$\lim_{t \to \infty} \max_{s} |V^t(s) - V^{t-1}(s)| = 0$$

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Why imprecision in transition probabilities?

- Imprecise or conflicting **expert elicitations**
- **Insufficient data** to estimate precise transition models
- **Non-stationary** (but bounded) transition probabilities

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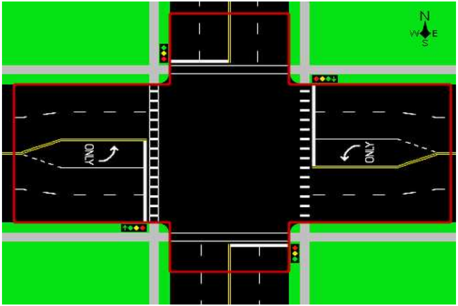

Example: **non-stationarity** in **traffic arrival & turn probabilities**

- fluctuate each hour of the day
- drift over time (probabilities measured every 2-3 years)

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- What's new?
  - **Credal set** $K = \{P\}$ of possible transition probabilities

MDPIP Value Iteration

- Be as robust as possible, given uncertainty:

$$V^t(s) = \max_{a \in A} \min_{P \in K} \left\{ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{t-1}(s') \right\}$$

Representing large MDPIPs

- **Compact representation:**
  - Factored state and action variables
  - Decision diagrams (DDs) for reward and transition

Algebraic Decision Diagrams (ADDs)

a	b	c	F(a,b,c)
0	0	0	0.00
0	0	1	0.00
0	1	0	0.00
0	1	1	1.00
1	0	0	0.00
1	0	1	1.00
1	1	0	0.00
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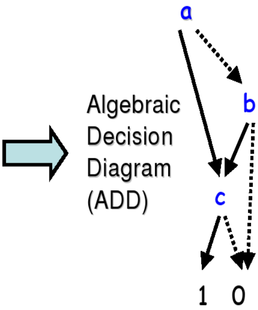
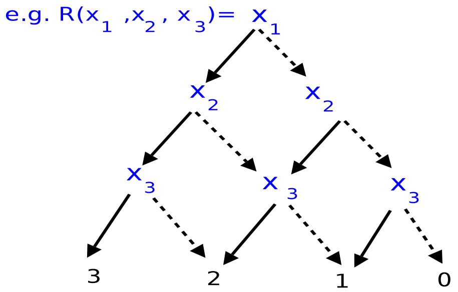


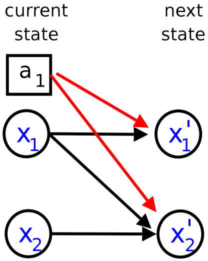
Compact representation for R: ADD

Reward R as an ADD:



Compact representation for K: DCN

Dynamic Credal Networks (DCN) [Cozman00] (DBN extension)



a)

CPT action a_1

x_1	$x_1' = 0$
F	p_{11}
T	p_{12}

$$0 \leqslant p_{11} \leqslant 0.6$$

$$0 \leqslant p_{12} \leqslant 1$$

$$0.3 \leqslant p_{11} + p_{12} \leqslant 0.8$$

b)

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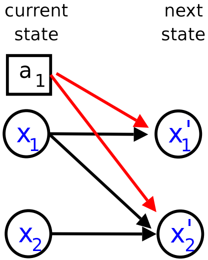

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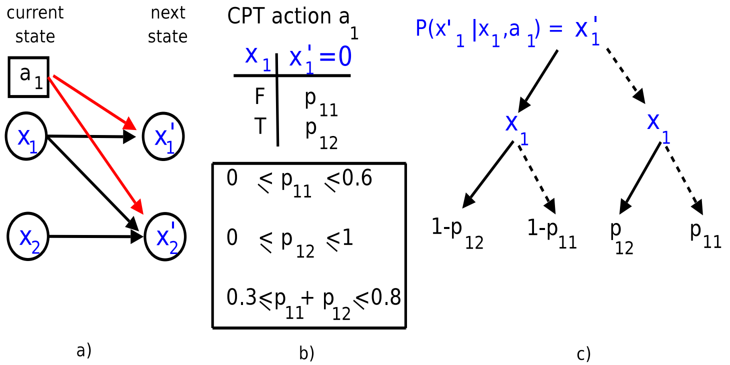
Note: **product of CPTs** yields **polynomial** expressions ($p_1^2 p_2$)
$\Longrightarrow$ restricted to **multilinear** ($p_1 p_2 p_3$) if CPTs do not share p_i

Compact representation for K: DCN + PADD

CPTs represented as **Parameterized ADDs (PADDs)**



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Extending ADDs to PADDs:

- **Equality testing** on leaf expressions
- **Binary operations** on leaves:
  - Easy to do +, *, - on algebraic expressions
  - Division does not yield a PADD (fractional leaves)
  - max and min do not yield PADDs (inequality decision nodes)
    - but no need to perform these for factored MDPIPs!

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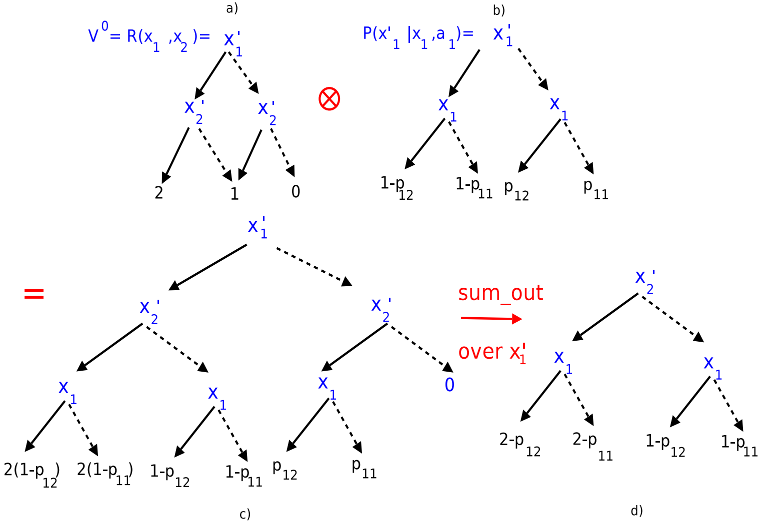
- **Compact and efficient solutions:**
  - *SPUDD-IP*: Factored value iteration with DDs
- **Bounded error approximations:**
  - *APRICODD-IP*: Naive pruning approach
  - *OBJECTIVE-IP*: Pruning where it counts

Factored MDPIP Value Iteration: SPUDD-IP

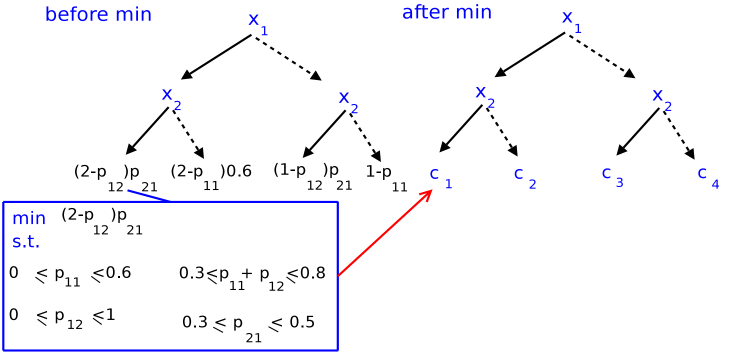
***SPUDD-IP*: Extend SPUDD [HoeyStHuBout99] to MDPIPs**

$$V^t(\vec{x}) = \max_{a \in A} \left\{ R(\vec{x}, a) \oplus \gamma \min_{\vec{p}} \boxed{\sum_{\vec{x}'} \bigotimes_{i=1}^{n} P(x_i' | pa_a(x_i'), a, \vec{p}) V^{t-1}(\vec{x}')} \right\}$$

First iteration:



First iteration (continued):



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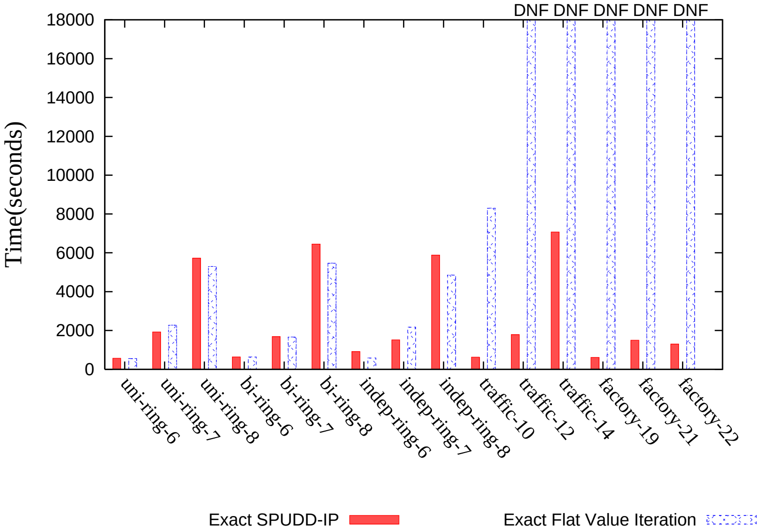
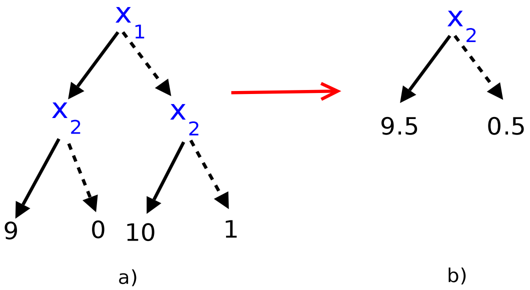
SPUDD-IP vs. Flat MDPIP Value Iteration



Approximate solution for MDPIPs: APRICODD-IP

***APRICODD-IP*: APRICODD [StHoeyBout00] for MDPIPs**

- After each iteration, prune the values that are similar
- Achieves a **bounded** approximate solution



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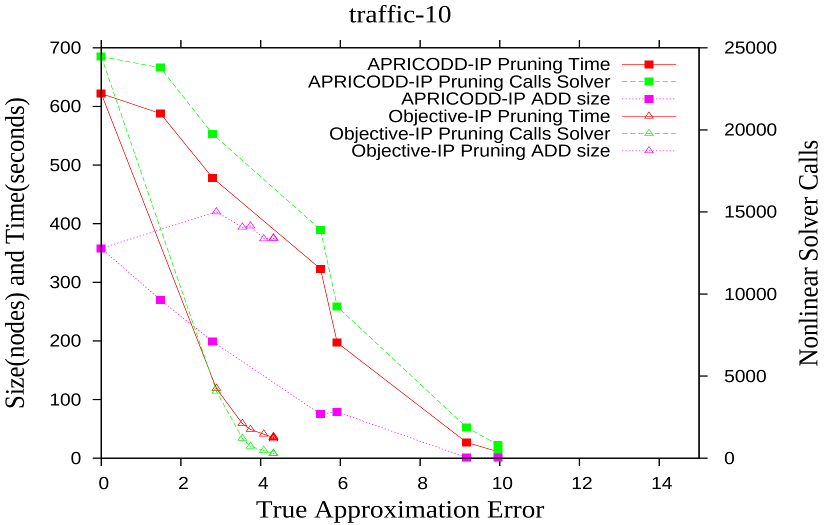
- Produces **bounded** approximately optimal solution

Objective-IP vs. APRICODD-IP



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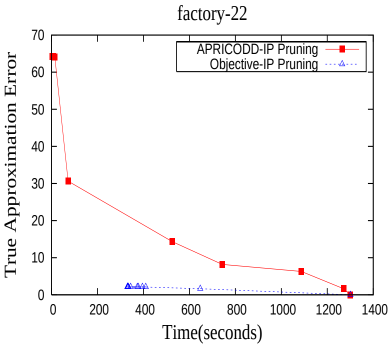

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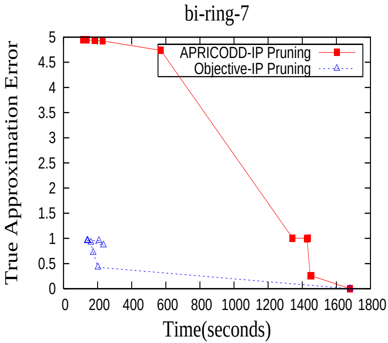

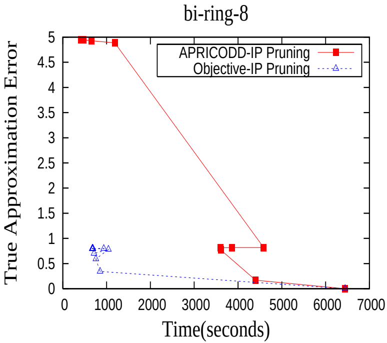

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$\Longrightarrow$ **Above are strict subsets of Factored MDPIPs**

- Transition probabilities are polynomial expressions of linearly constrained variables (i.e., from multiplying CPTs in DCN)

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