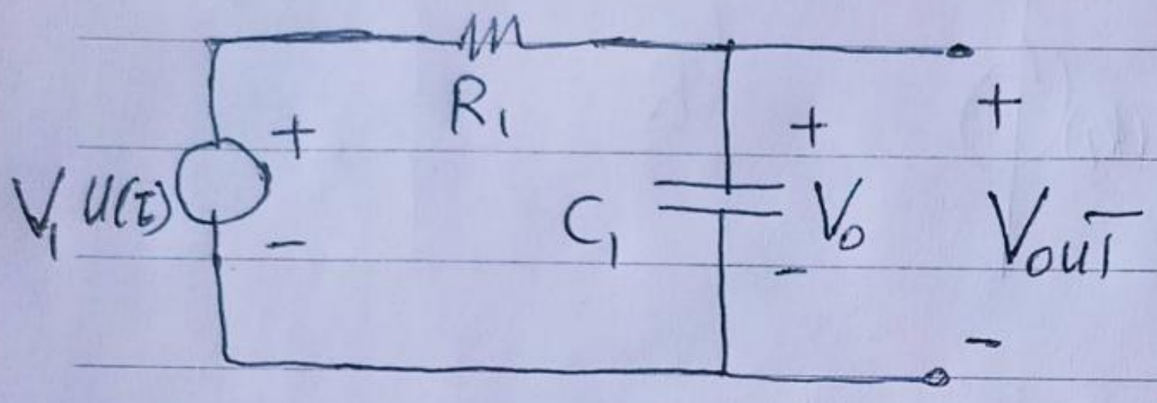
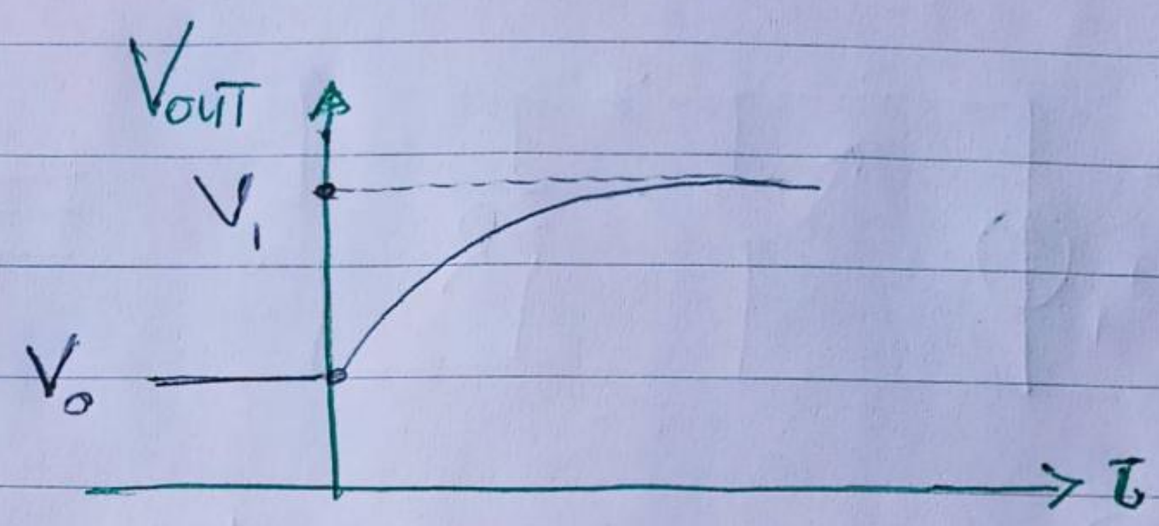
Method 2: Use superposition: ① Assume No input and find V_{out} for initial condition. ② Assume No initial condition and find V_{out} in response to the input. ③ add the two responses.

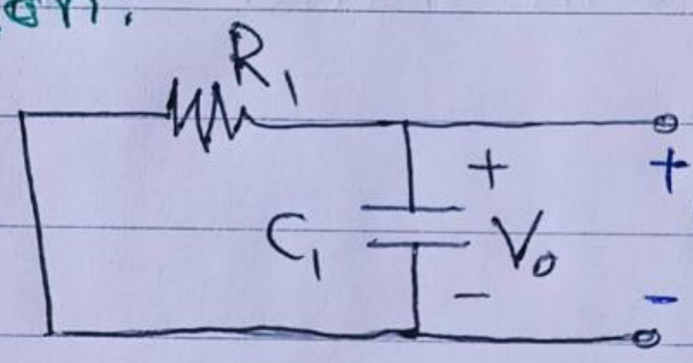
Example





Apply superposition:

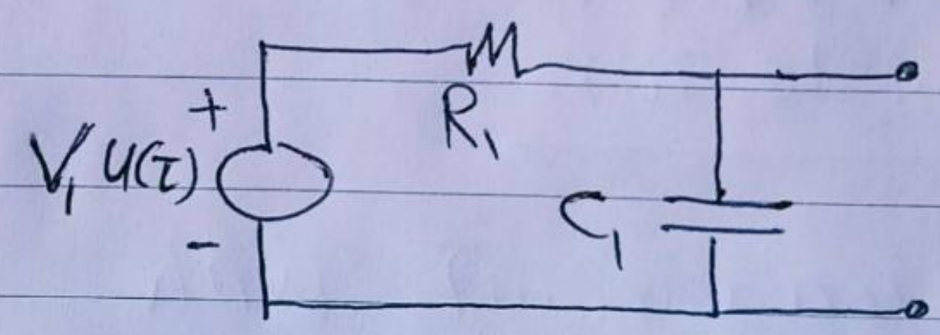
① No input:



$$V_{out_1} = V_0 \exp\frac{-t}{\tau}$$

$$\tau = R_1 C_1$$

② No initial condition:



$$V_{out2} = V_1\left(1 - \exp\frac{-t}{\tau}\right)$$

$$\tau = R_1 C_1$$

③ $V_{out} = V_{out_1} + V_{out2} = V_0 \exp\frac{-t}{\tau} + V_1\left(1 - \exp\frac{-t}{\tau}\right)$, $t > 0$

Alternative Approach: $y(t) = y_\infty + (y_0 - y_\infty)\exp\frac{-t}{\tau}$

$y_\infty = V_1$, $V_0 = y_0$ $\Rightarrow$ $V_{out}(t) = V_1 + (V_0 - V_1)\exp\frac{-t}{\tau}$

, $\tau = R_1 C_1$

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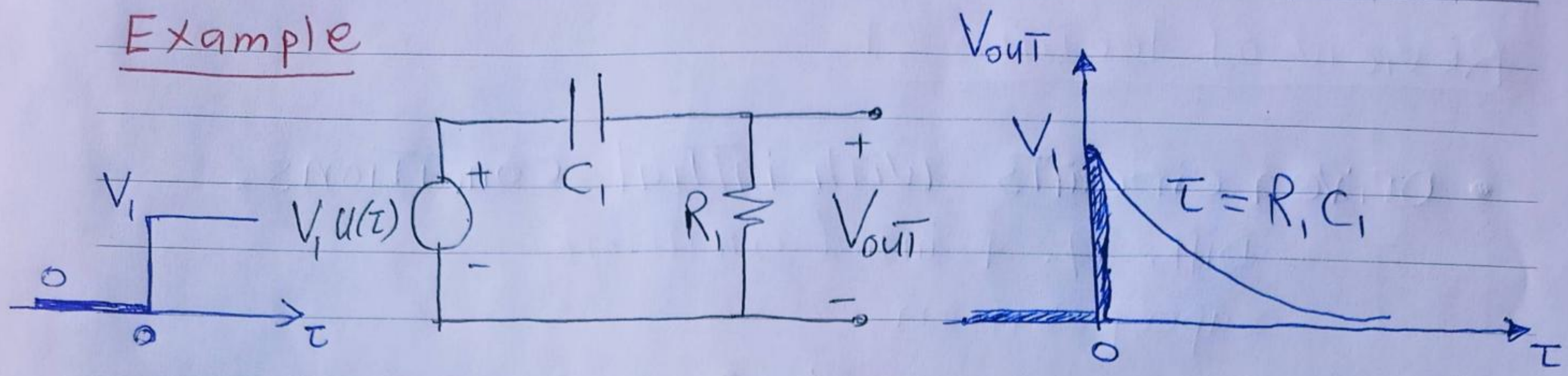
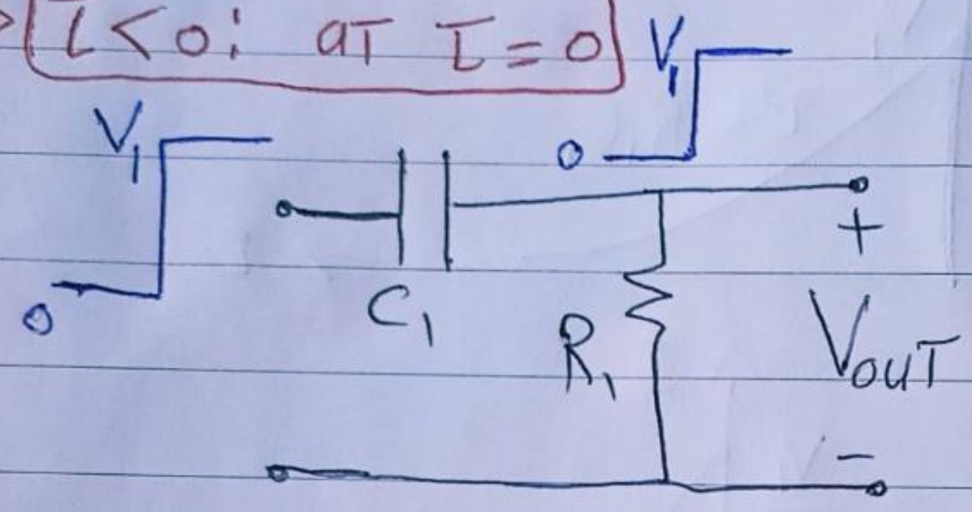
Example



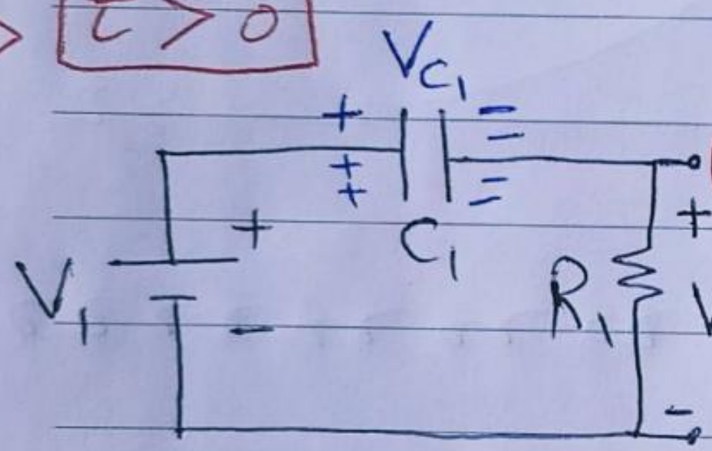
$\Rightarrow$ $\boxed{t<0:\ \text{at } t=0}$



$I < \infty \Rightarrow$ cap voltage can't jump $\Rightarrow V_{C_1}(0^+) = V_{C_1}(0^-) = 0$

$\therefore V_{out}(0) = V_1$

$\Rightarrow$ $\boxed{t>0}$



$\Rightarrow +V_1 - V_{C_1} - V_{out} = 0 \Rightarrow V_{out} = V_1 - V_{C_1}$

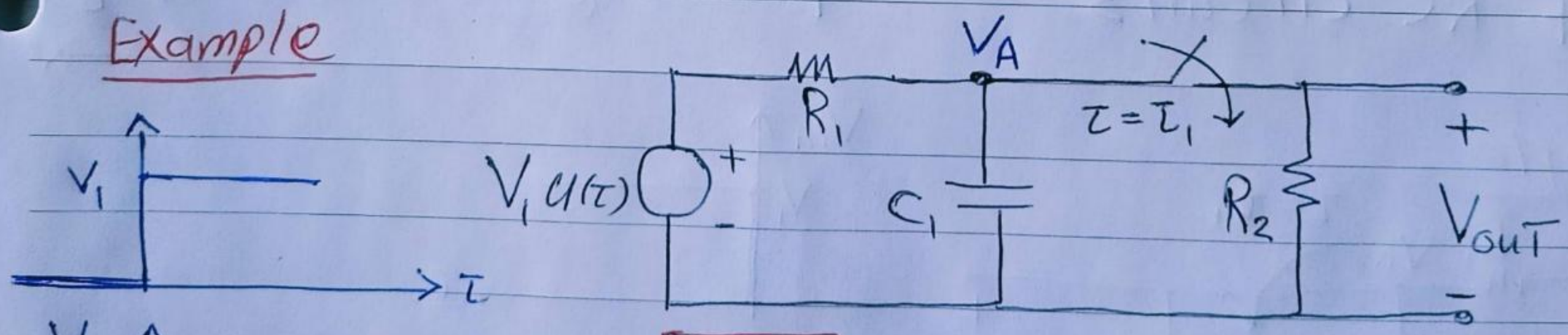
or $\Rightarrow$ general response:

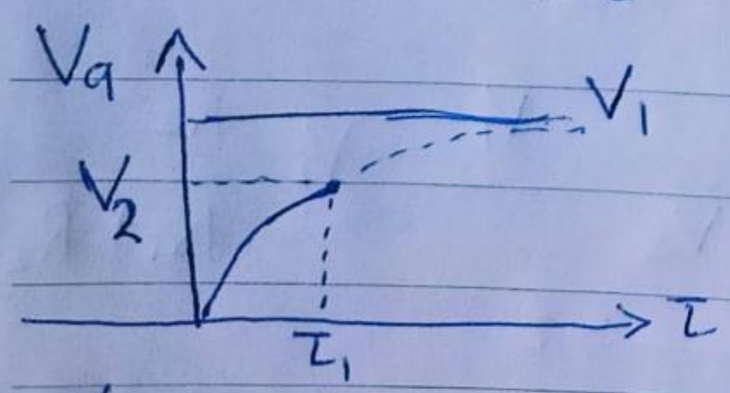
$$y(t) = y_\infty + (y_0 - y_\infty)\exp\frac{-t}{\tau}$$

$V_{out}(0) = y_0 = V_1$ & $V_{out}(\infty) = y_\infty = 0$

$$\therefore V_{out}(t) = \left(0 + (V_1 - 0)\exp\frac{-t}{\tau}\right)u(t), \quad \tau = R_1 C_1 \quad *$$

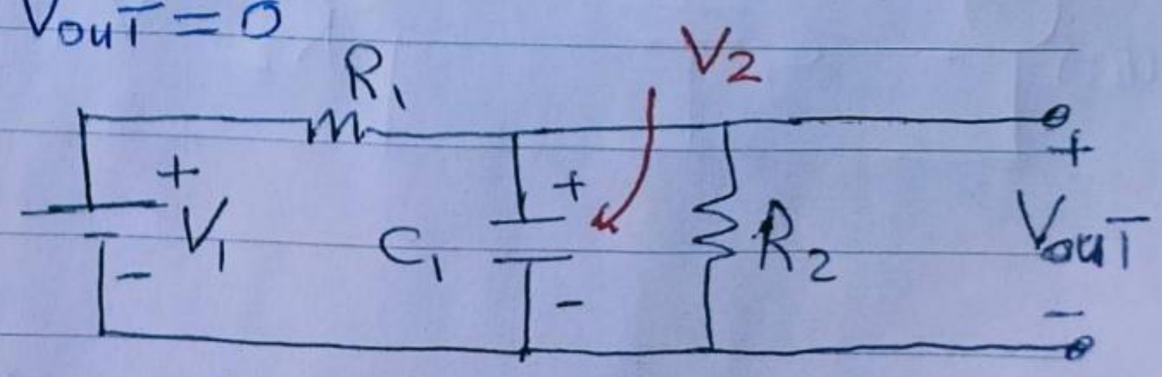
Example

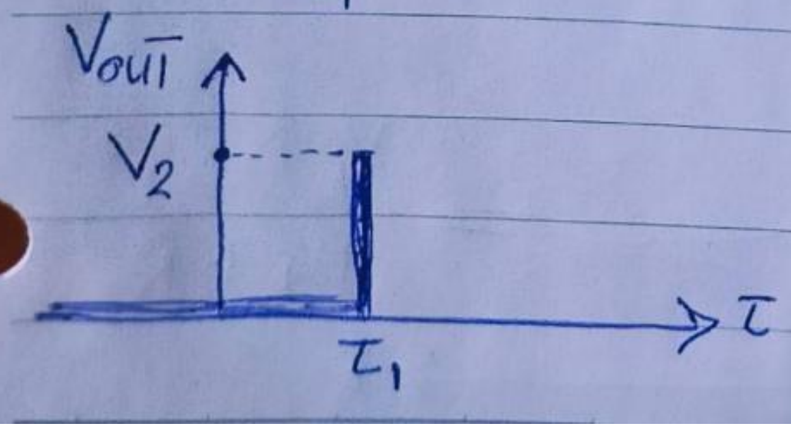




$\Rightarrow$ $\boxed{t < \tau_1}$: $V_{out} = 0$

$t > \tau_1$:





$$V_{out}(t) = V_1\frac{R_2}{R_1+R_2} + \left(V_2 - V_1\frac{R_2}{R_1+R_2}\right)\exp\frac{-t}{\tau}$$

$\tau = ?$

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Lec 22: Additional RC Circuit Examples

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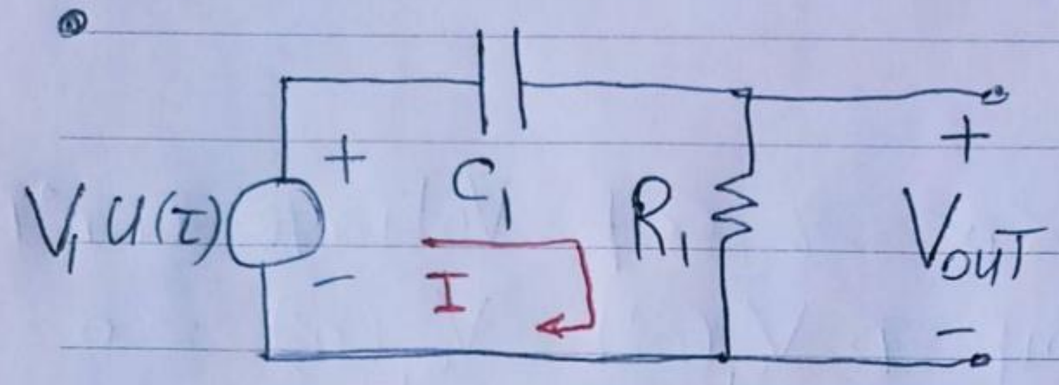
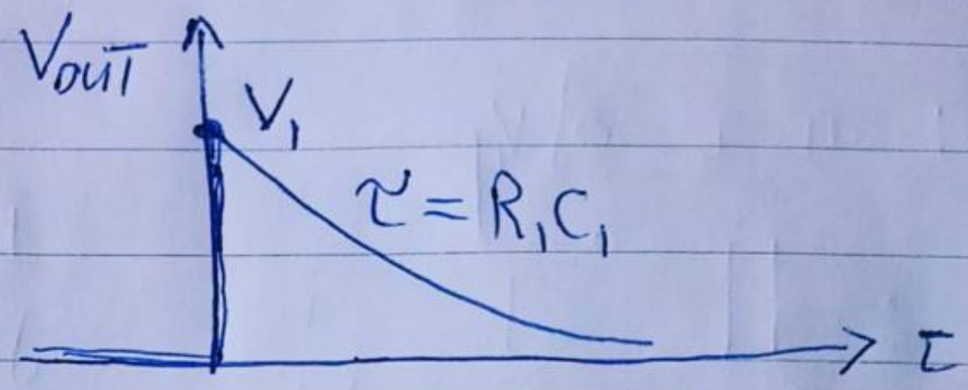
Review of lecture 21

- Driven circuits with initial conditions.
  - Diff. Eq. + initial condition
  - superposition





$$I = C_1 \frac{dv}{dt}$$

$t = 0$: $C_1 \equiv$ short

$t = \infty$: $C_1 =$ open

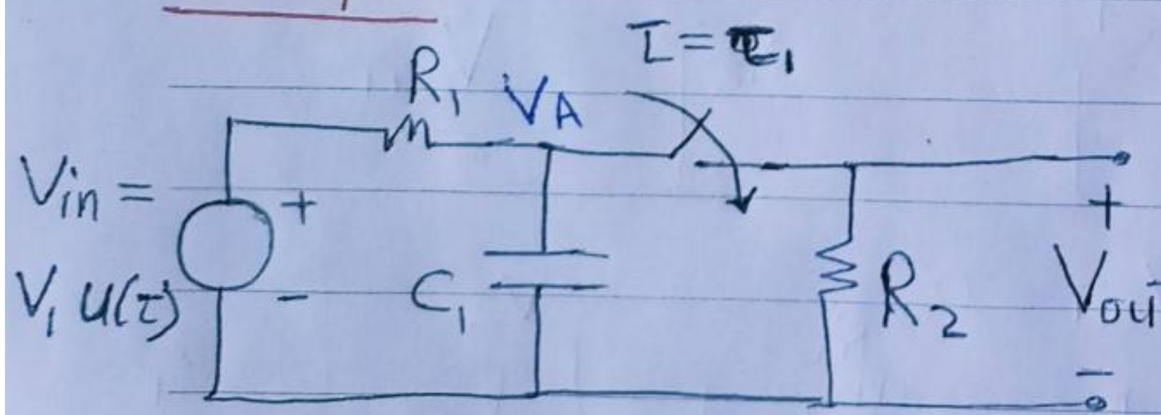
Today's lecture

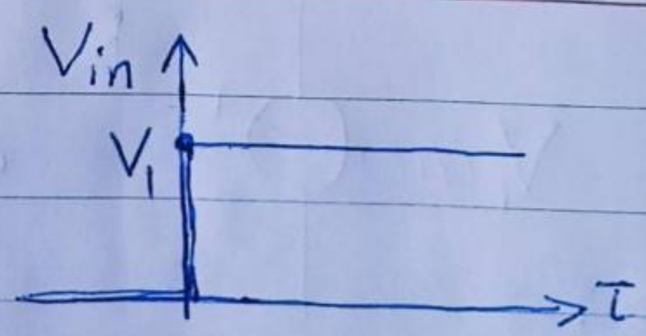
- additional Examples of RC circuits

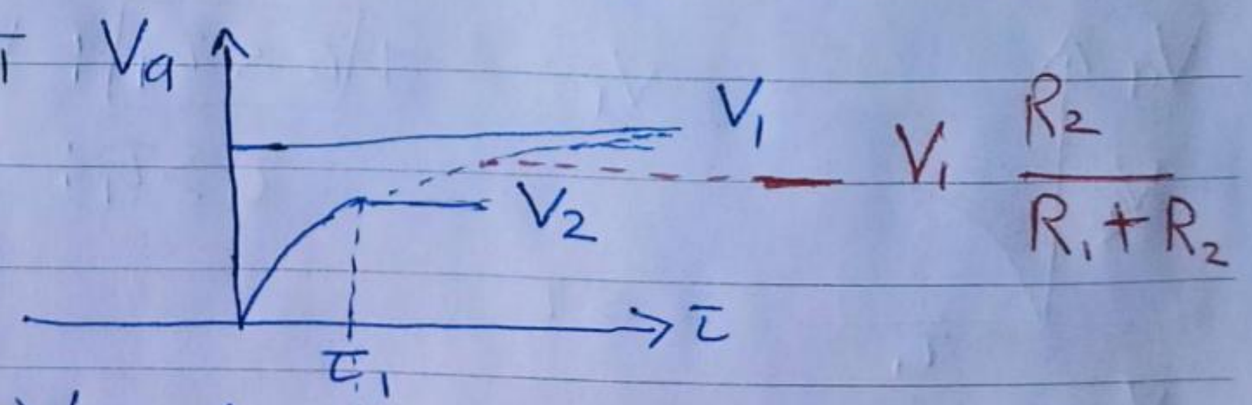
For a 1st-order system:

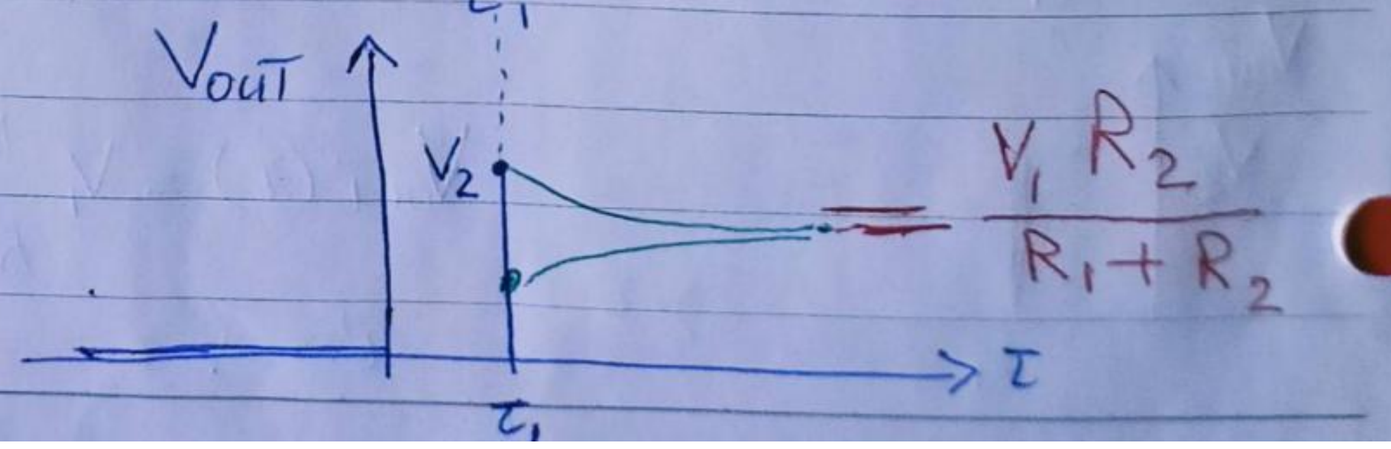
$$y(t) = y_\infty + (y_0 - y_\infty) \exp^{-\frac{t}{\tau}}$$

Example









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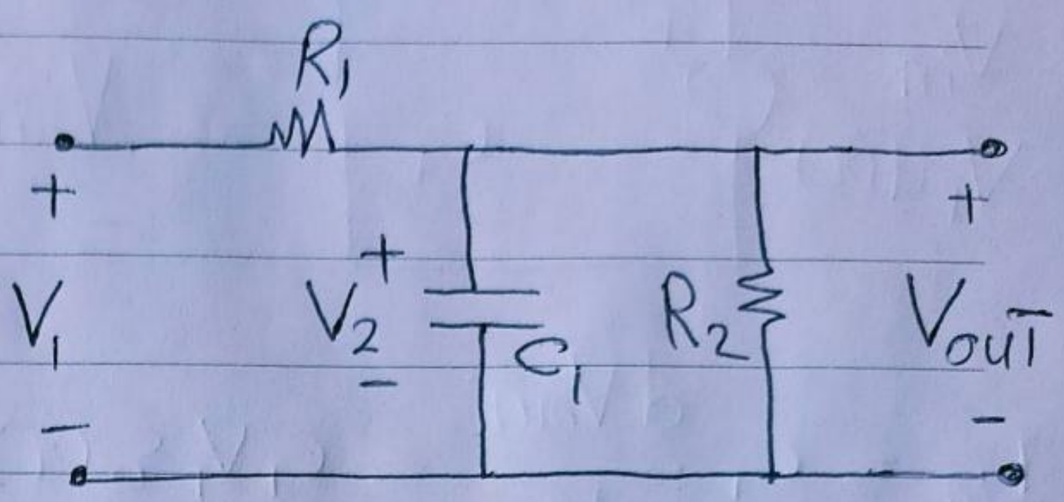
For $t < t_1$: $V_A = V_1 \left(1 - \exp \frac{-t}{\tau_1}\right) u(t)$

$\tau_1 = R_1 C_1$

after $t = t_1$:



$V_A(t_1) = V_1 \left(1 - \exp \frac{-t_1}{\tau_1}\right)$

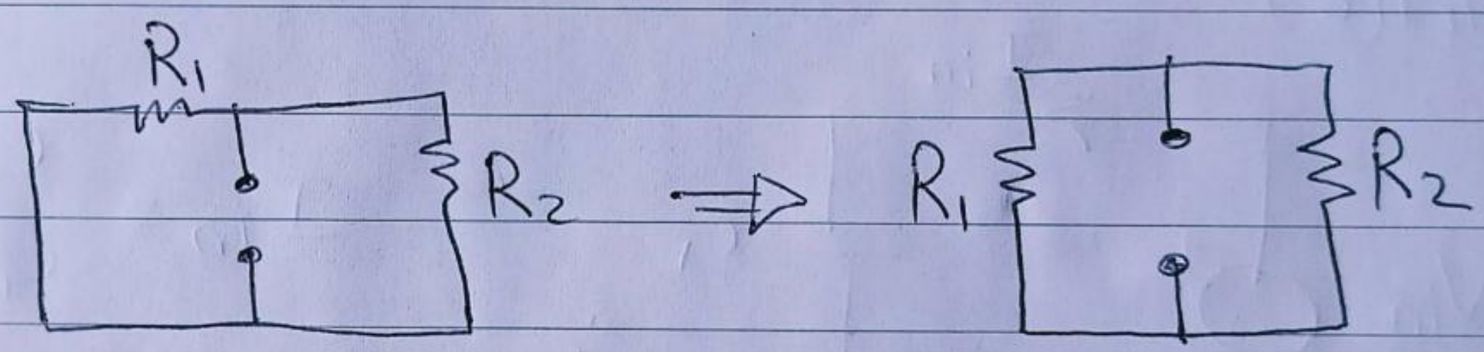
$= V_2$

$y_\infty = V_{out}(t = \infty) = V_1 \frac{R_2}{R_1 + R_2}$

$y(t_1^+) = V_2 = V_1 \left(1 - \exp \frac{-t_1}{\tau_1}\right)$

$\tau_2 = ?$

$\tau_2 = (R_1 \| R_2) C_1$



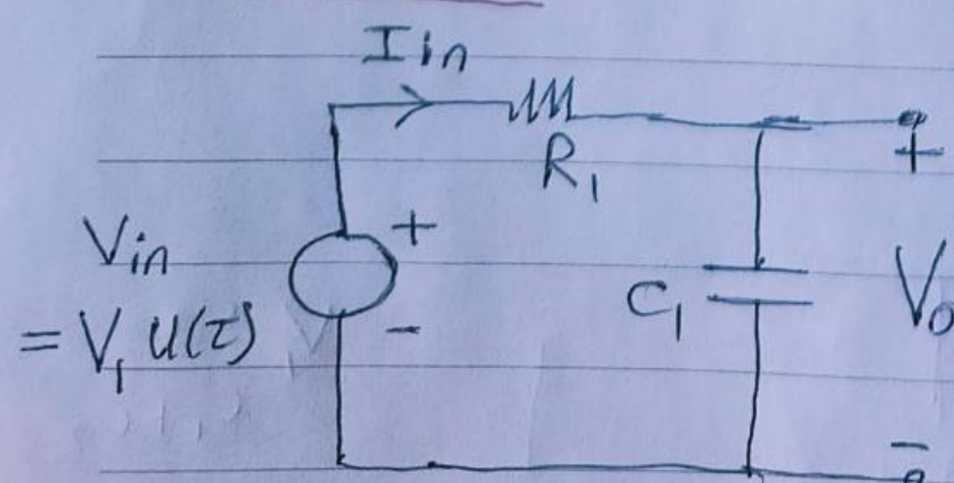
$t > t_1$:

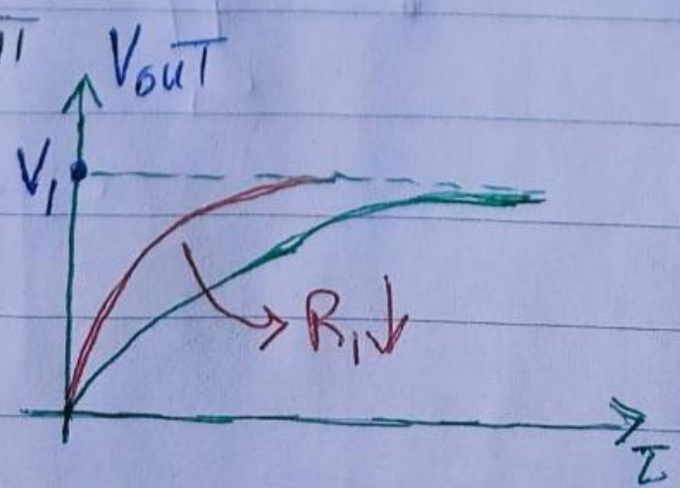
$$V_{out}(t) = V_1 \frac{R_2}{R_1 + R_2} + \left(V_1 \left(1 - \exp \frac{-t_1}{\tau_1}\right) - \frac{V_1 R_2}{R_1 + R_2} \right) \exp \frac{-t}{\tau_2}$$

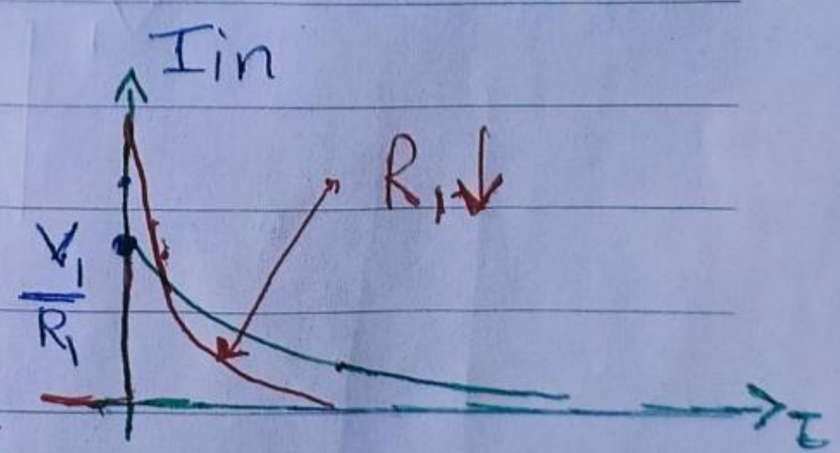
$$V_2 = V_1 \left(1 - \exp \frac{-t_1}{\tau_1}\right) \lesseqgtr V_1 \frac{R_2}{R_1 + R_2} \quad *$$

Example

Part (a)







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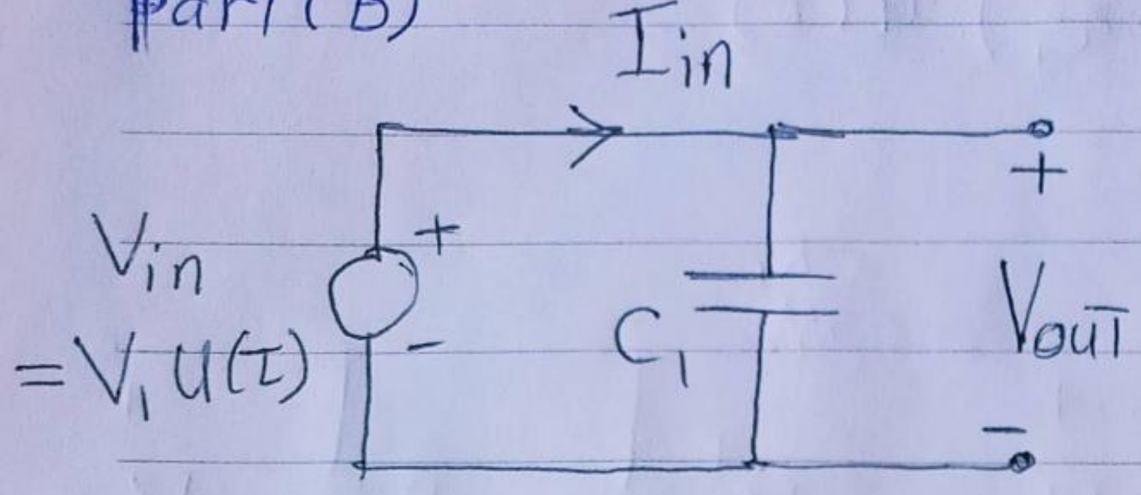
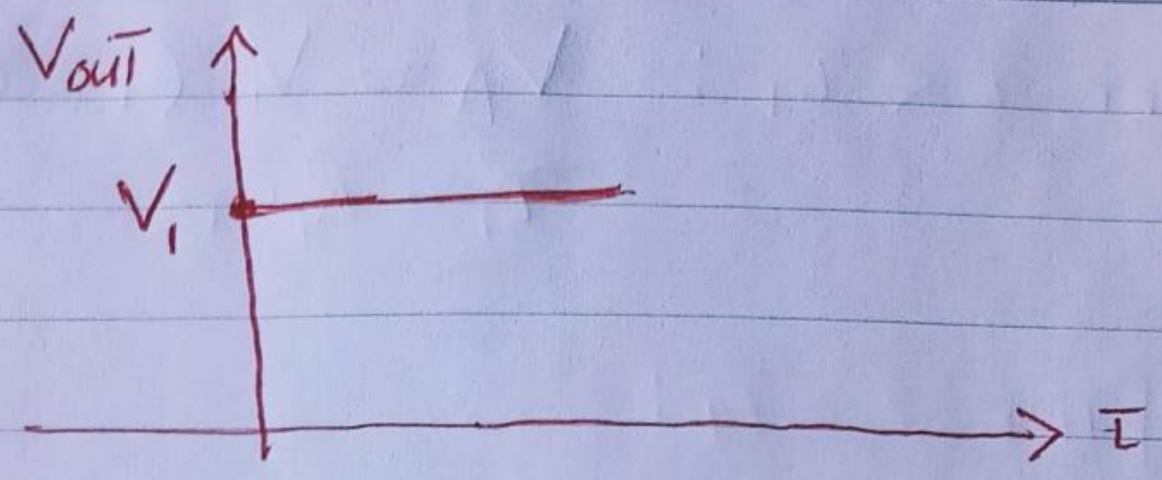
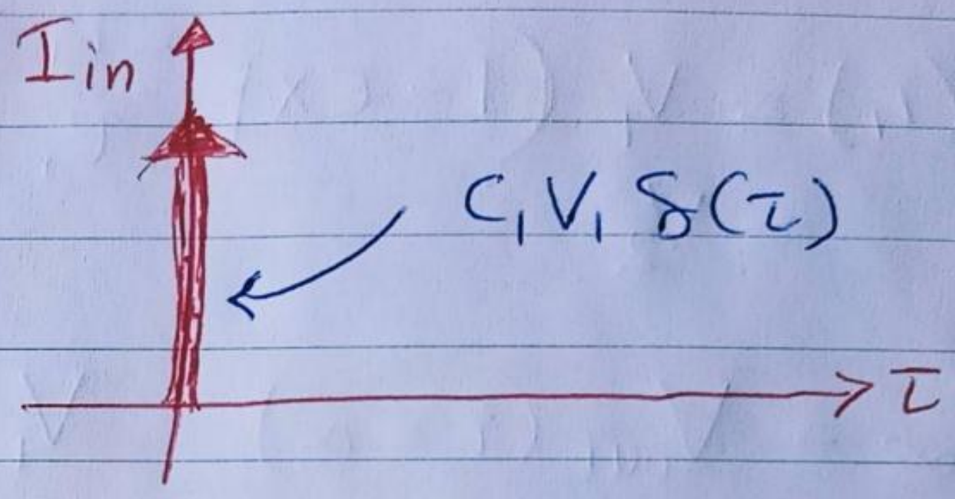
Part (b)



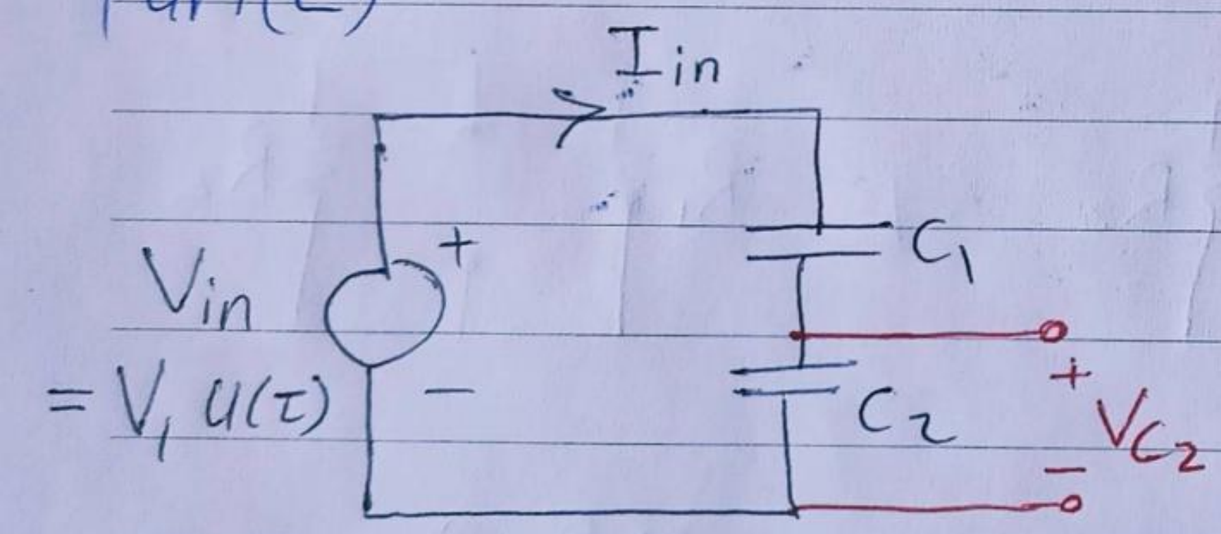


$$I_{in} = C_1 \frac{dV_{out}}{dt} = C_1 V_1 \delta(t)$$



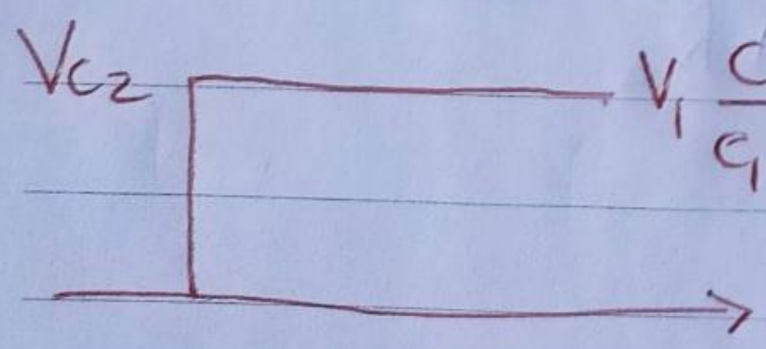
" We Charge C_1 in zero Time"

Part (C)

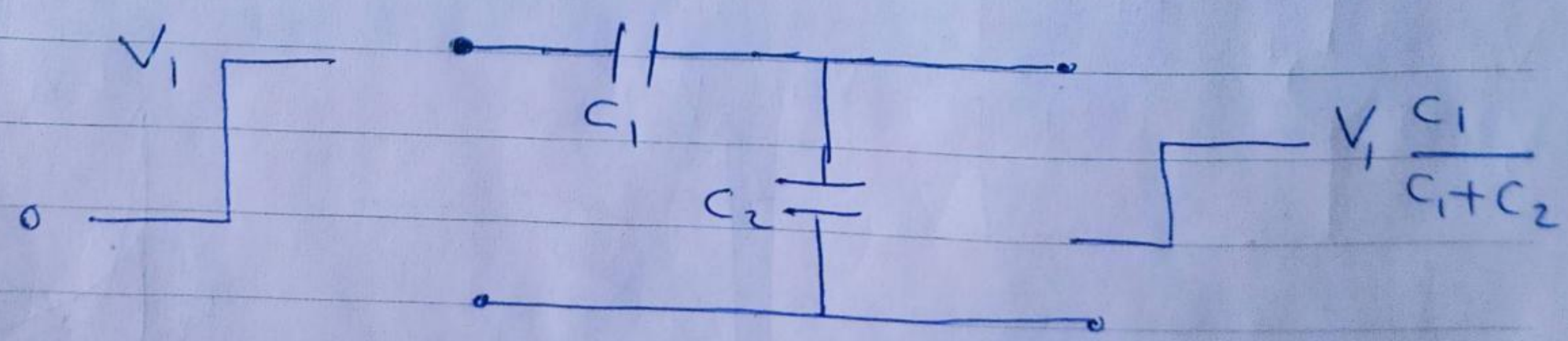


$$I_{in} = \frac{C_1 C_2}{C_1 + C_2} V_1 \delta(t)$$

$$V_{C_2} = \frac{1}{C_2} \int_0^t I_{in}\, dt = \frac{1}{C_2} \int_{0^-}^{0^+} \frac{C_1 C_2}{C_1 + C_2} V_1 \delta(t)\, dt$$



$$= \frac{C_1 V_1}{C_1 + C_2} \#$$



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Lec 23: RC Circuit Example; Intro. to Inductors

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Review of lecture 22

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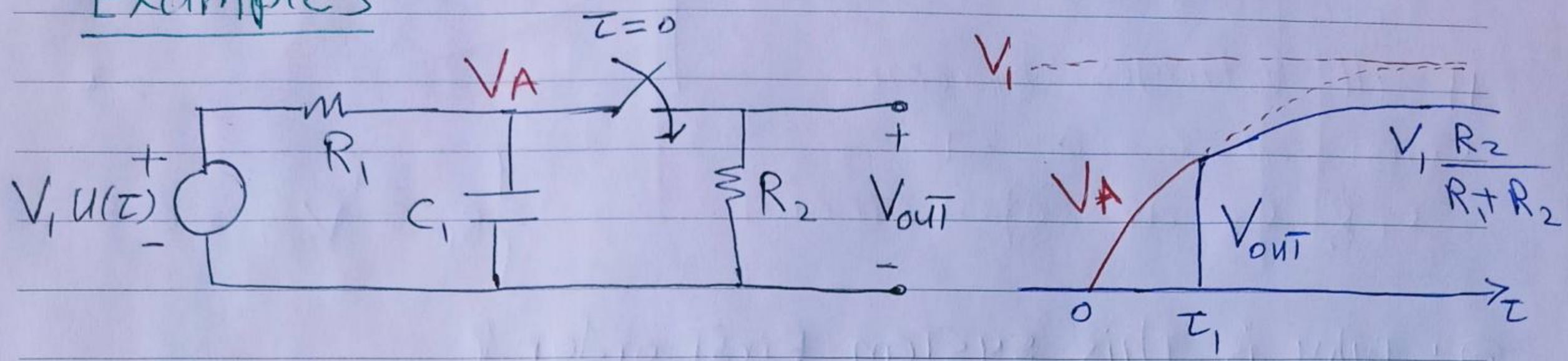
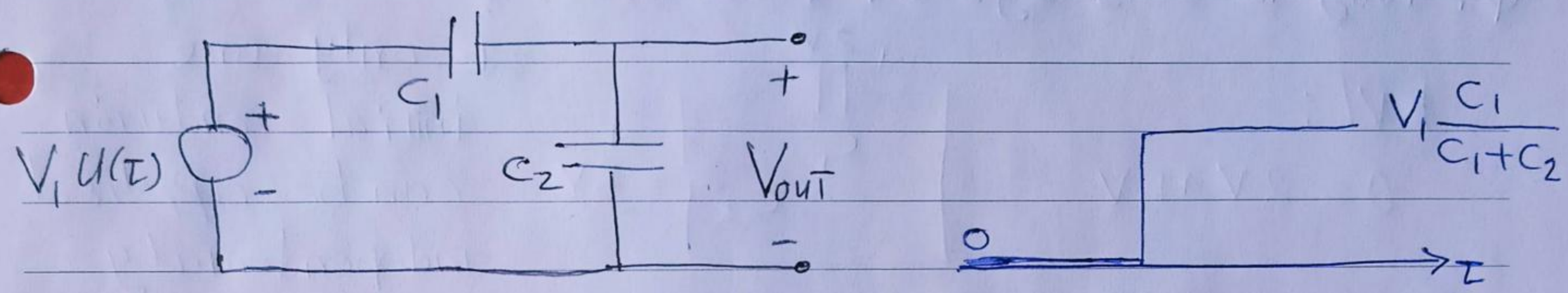
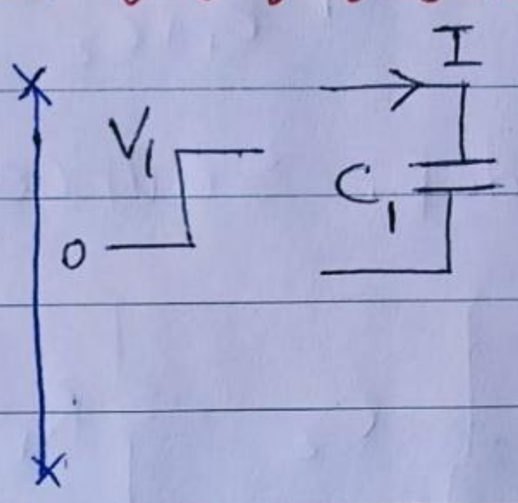
Examples





Todays lecture

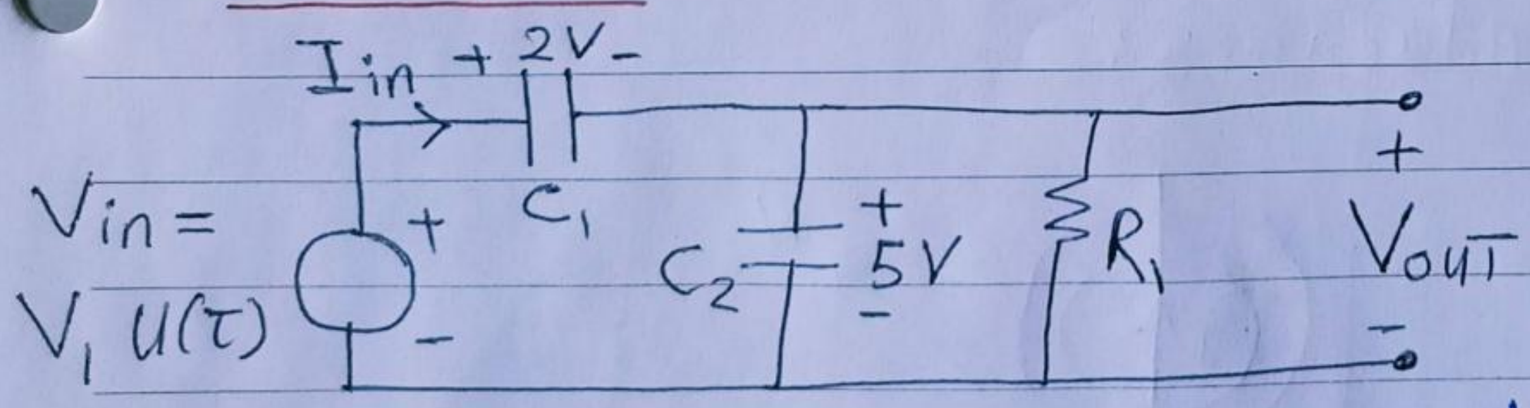
- Rc circuit Examples.
- intro. to inductor.



$I = C_1 V_1 \delta(\tau)$

IF $V = \text{const} \Rightarrow C = \text{open}$

Example



- if V_{OUT} doesn't jump: C_1 has a voltage change of V_1 in zero time; $I_{in} \propto \delta(\tau) \Rightarrow$ The impulse can not flow through R_1; otherwise, it would create an infinite voltage. $\Rightarrow$ impulse of current must flow through C_2.

- charge delivered to R_1 from 0^- to 0^+: $\int_{0^-}^{0^+} I_{R_1}\, d\tau \to 0$

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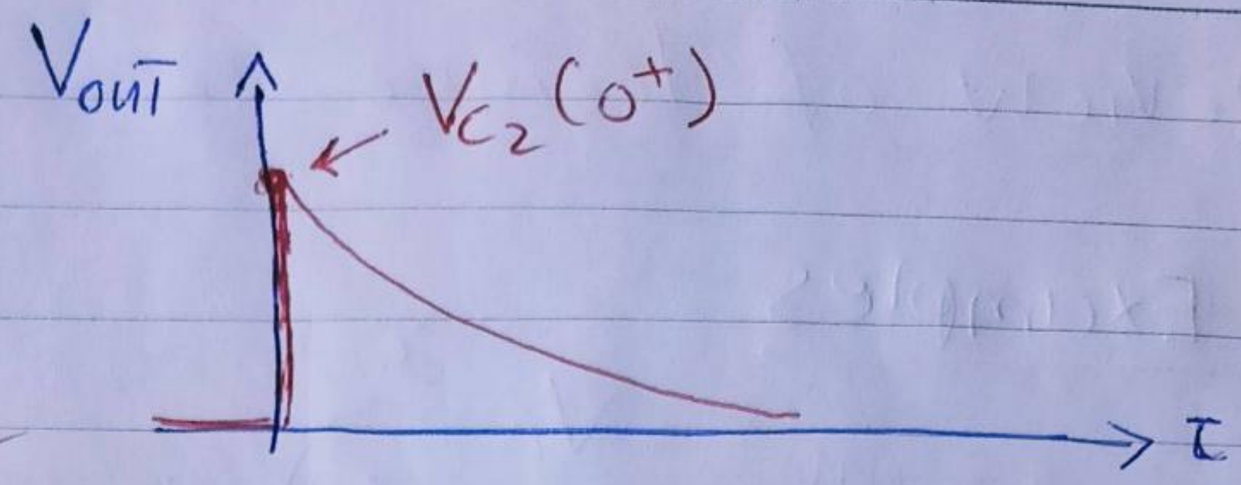
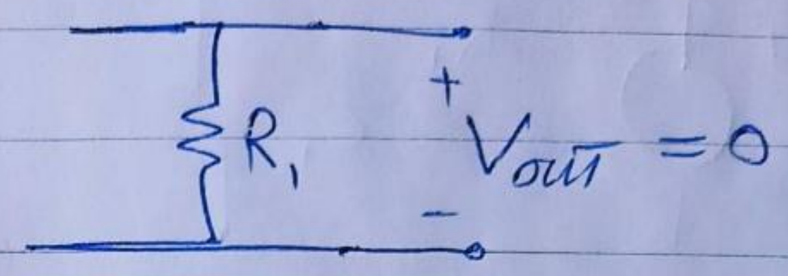
$$V_{C_2}(0^+) = \frac{V_1 C_1}{C_1 + C_2}$$

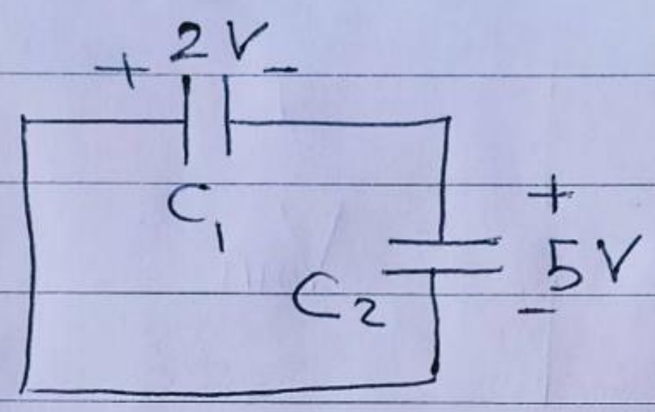


if all Voltages become const, at $\tau = \infty$ ⇒
C_1 & C_2 are open circuit



Q: why is This system First order?

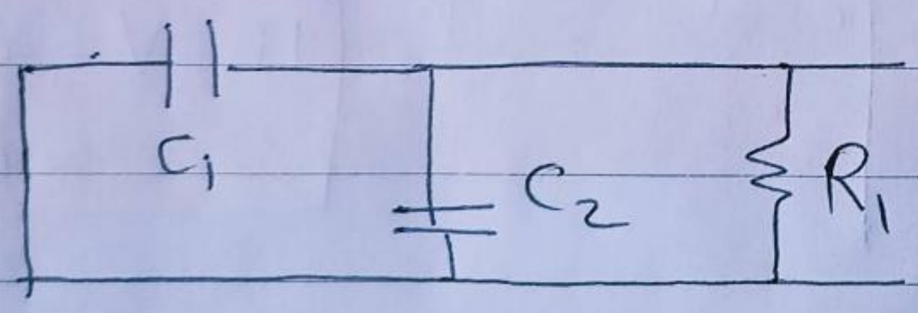
KVL:
$0 = 2V + 5V$



⇒ only one initial condition can be applied independentaly

$$y_\infty = 0, \quad y_{0^+} = V_1 \frac{C_1}{C_1 + C_2}$$

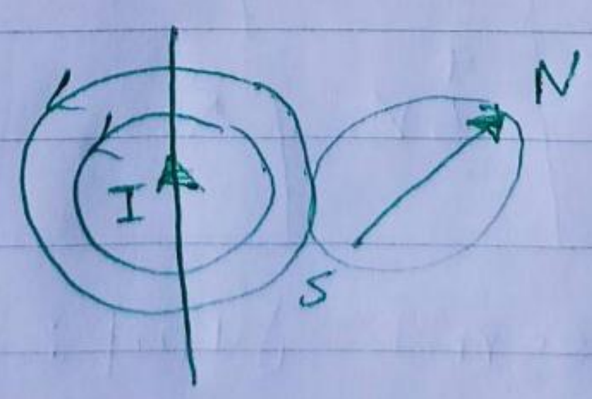
⇒ τ: set indep. sources To zero: $\tau = R_1(C_1 + C_2)$



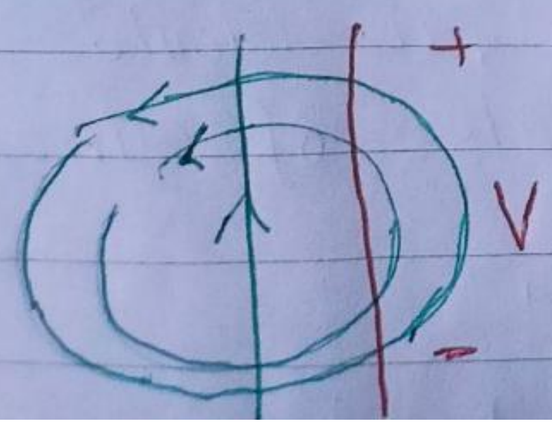
$$\boxed{V_{out}(t) = V_1 \frac{C_1}{C_1 + C_2} \exp\frac{-t}{\tau} u(t)}$$

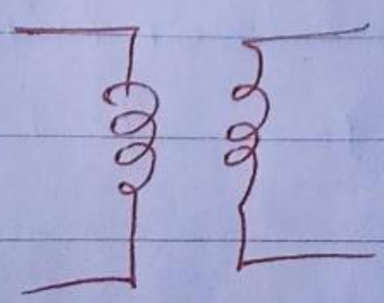
<u>introduction on inductors:</u>

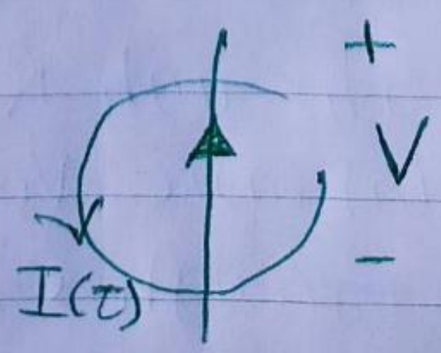
1820: oyested.



- Faraday







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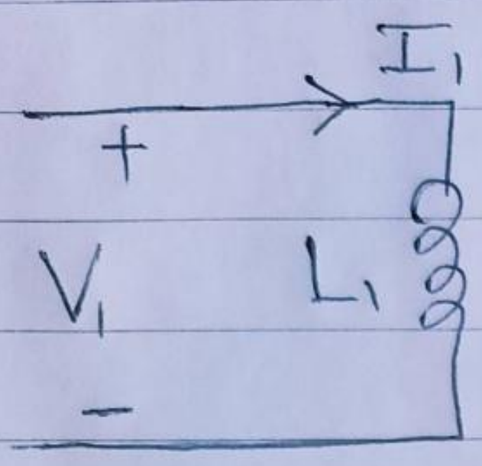
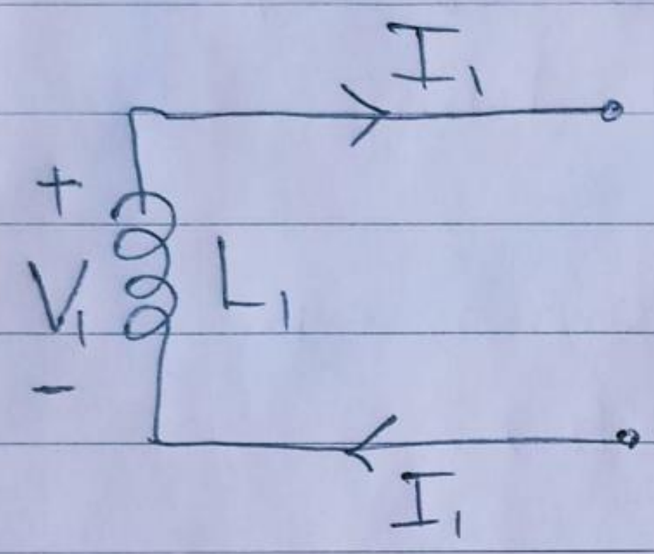
• inductor Basics



$$V_1 = L_1 \frac{dI_1}{dt}$$

If $V_1 < \infty$, I_1 can not jump in zero time.

If I_1 const $\Rightarrow$ $V_1 = 0 \Rightarrow L_1 = \text{short}$



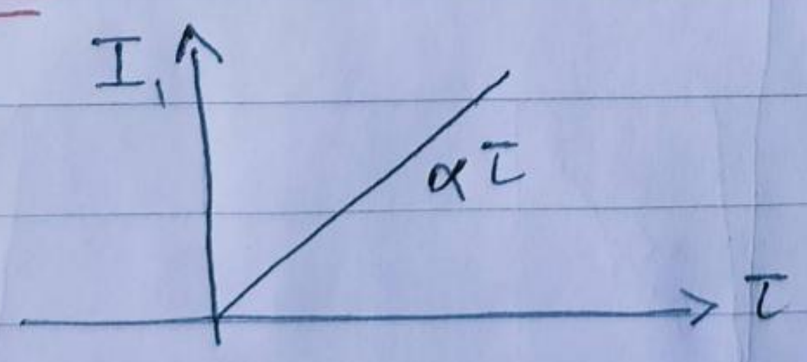
$$V_1 = -L_1 \frac{dI_1}{dt}$$

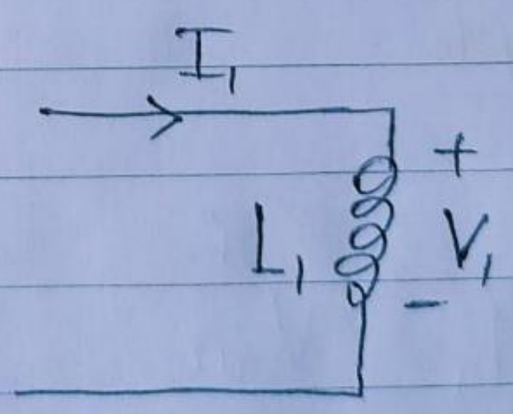
- To find the current of an inductor:

$$I_1 = \frac{1}{L_1} \int V_1 \, dt$$

one more point: initial condition: L_1 I_0

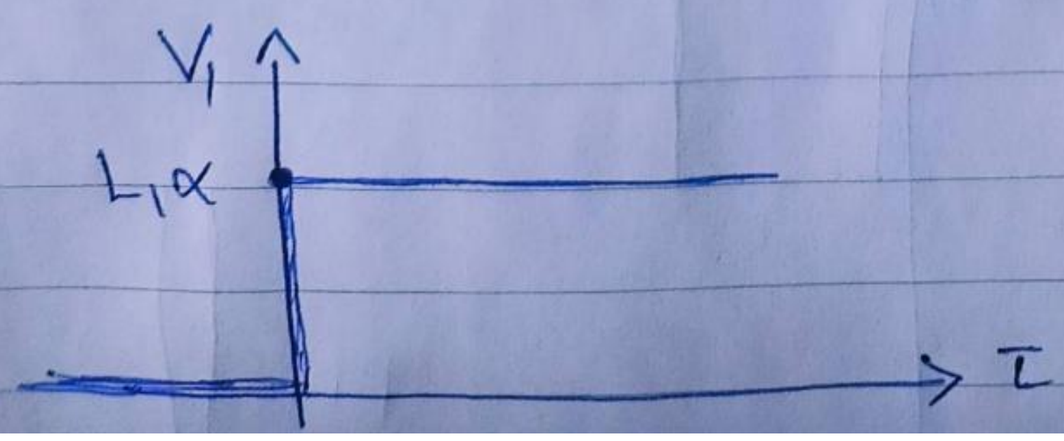
Example





$$V_1 = L_1 \frac{dI_1}{dt} = L_1 \propto$$

"Note": The faster we change the current the larger the voltage drop across inductor.



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Lec 24: Inductor Circuits

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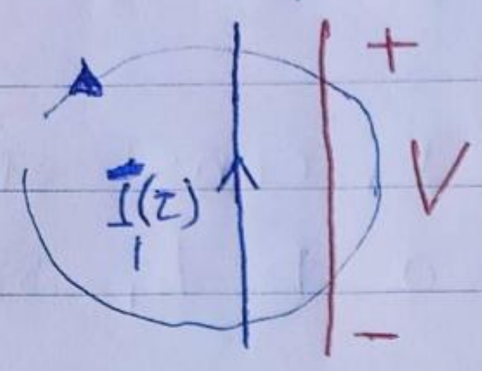
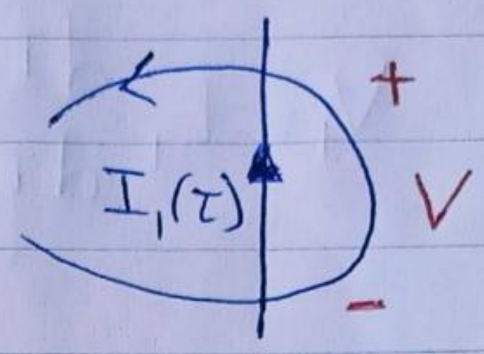
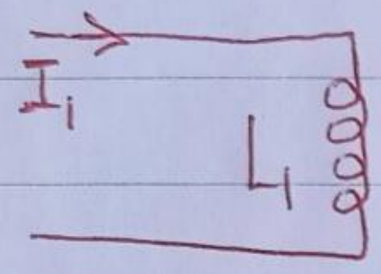
Review of lec. 23

Inductor:





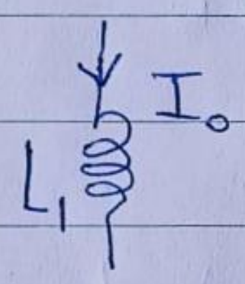


$$V_1 = L_1 \frac{dI_1}{dt}$$

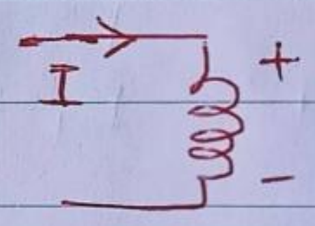
→ if $V_1 < \infty \Rightarrow I_1$ can not jump

→ If I_1 const. $\Rightarrow$ $L_1 \equiv$ short

Initial inductor:



Todays lecture:

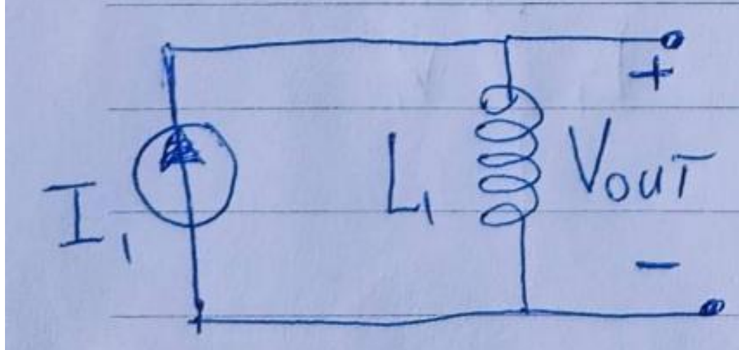


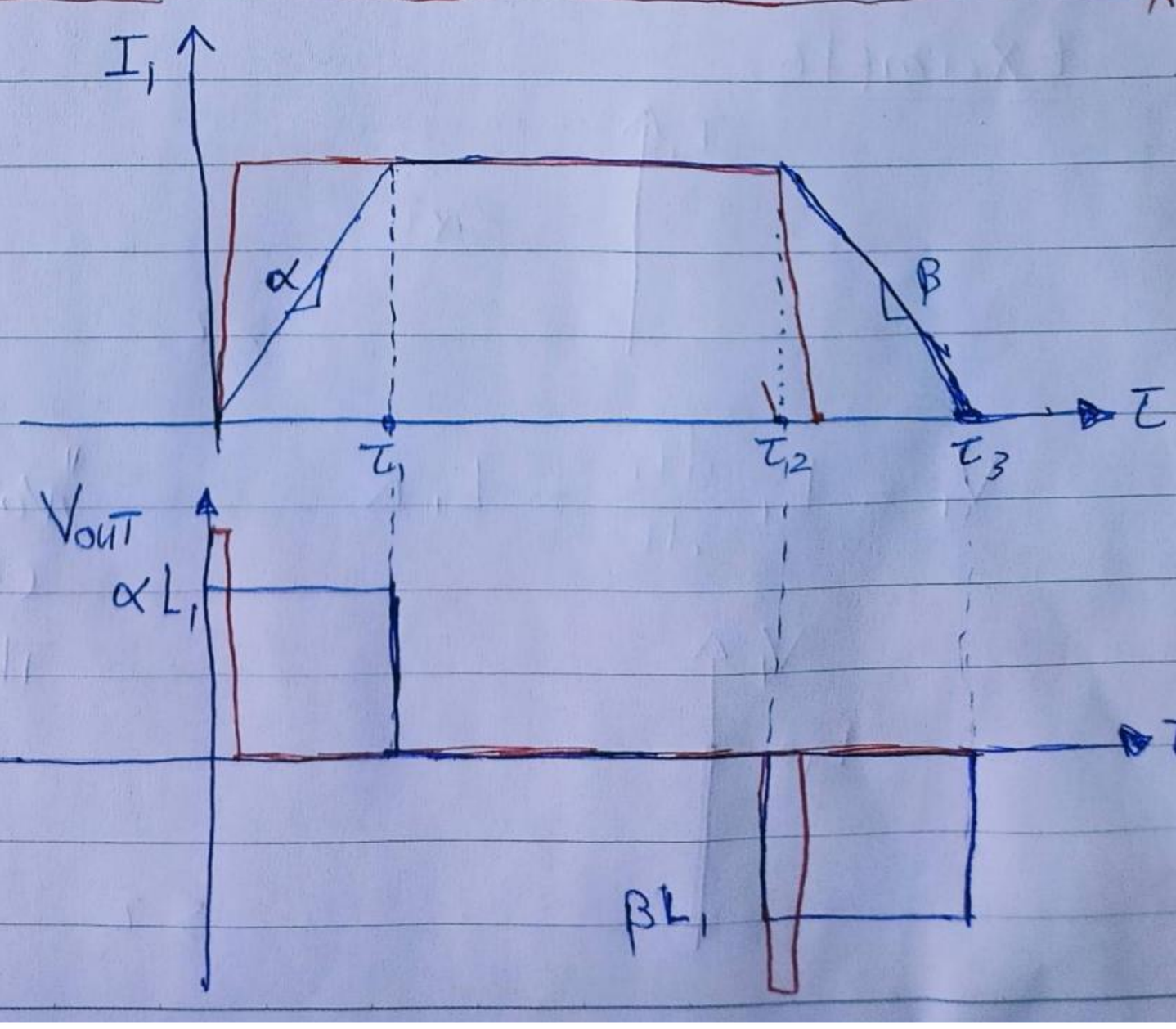
$$V = L \frac{dI}{dt}$$

- inductor circuit Examples.
- other properties of inductors.
- A simple RL circuit.

$$I = \frac{1}{L} \int V dt$$

Example





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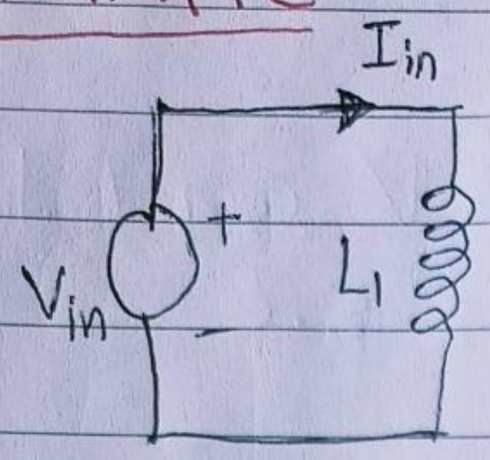
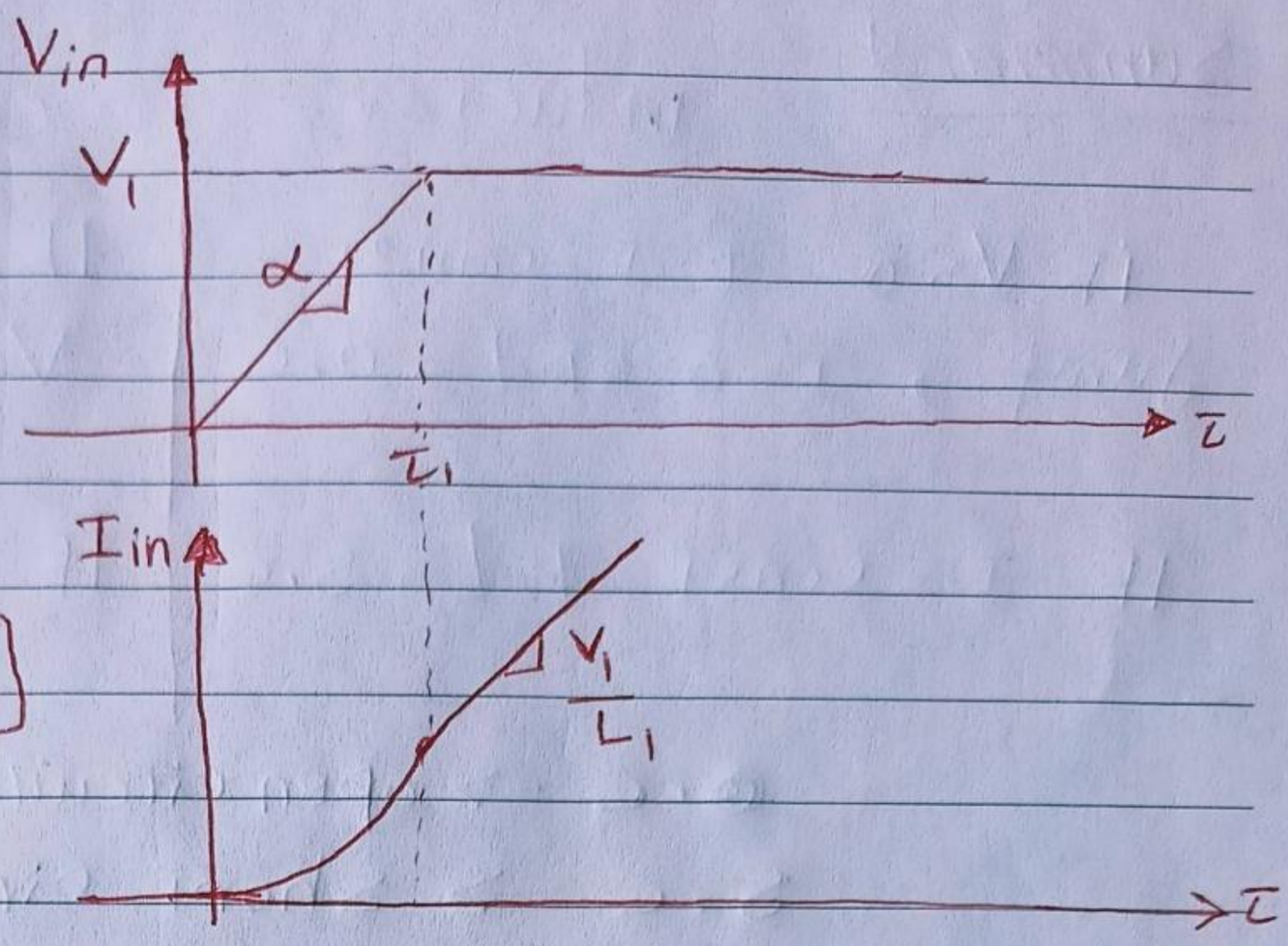
Example



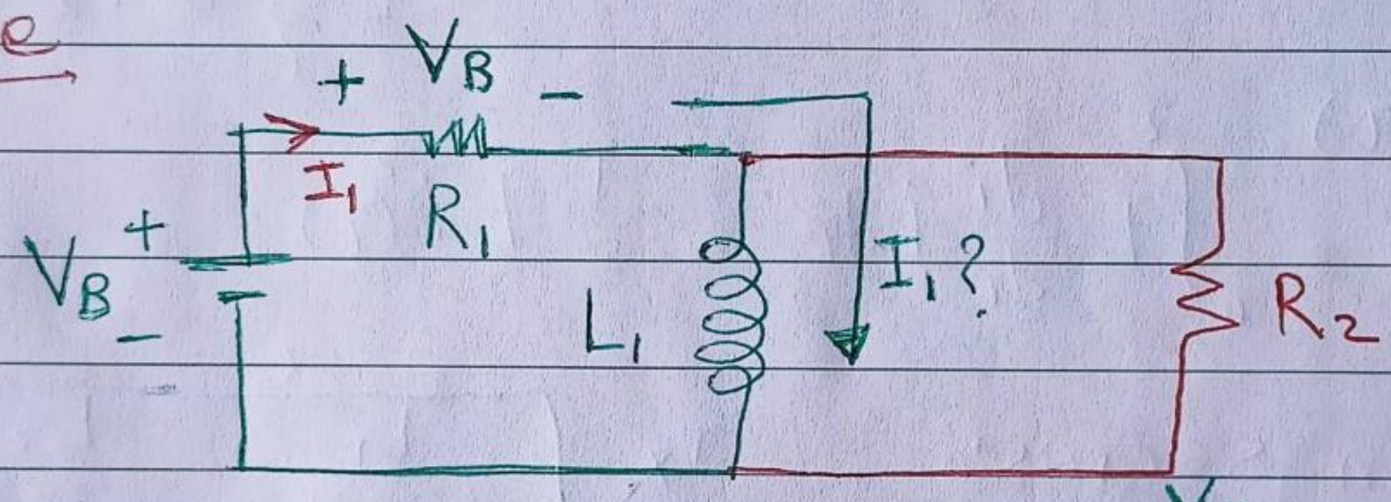


$I_{in} = \frac{1}{L_1}\int \alpha \tau \, d\tau$; $\boxed{0 < \tau < \tau_1}$

$= \frac{\alpha}{L}\tau^2$

$\boxed{\tau > \tau_1}$; $I_{in} = \frac{1}{L_1} V_1 \tau$

Example



I_1 is const. $\Rightarrow V_{L_1} = 0 \Rightarrow I_1 = \frac{V_B}{R_1}$

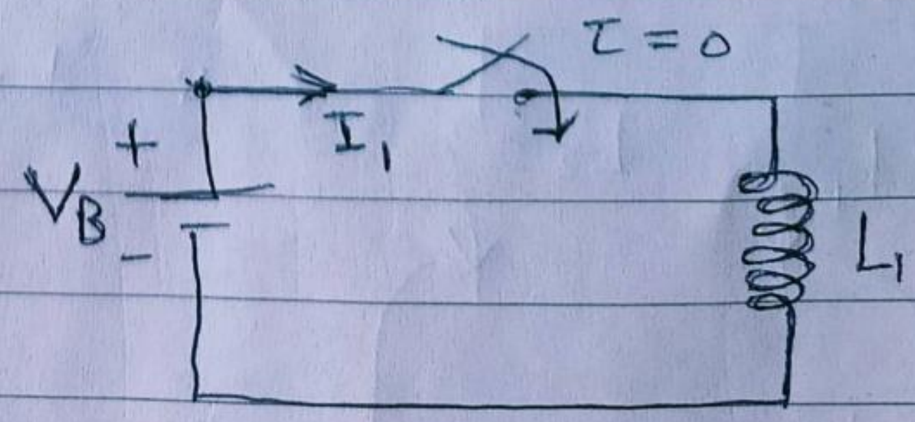
current through L_1 const. $\Rightarrow V_{L_1} = 0 \Rightarrow V_{R_2} = 0 \Rightarrow$ current through $R_2 = 0 \Rightarrow$ all of I_1 must flow through L_1.

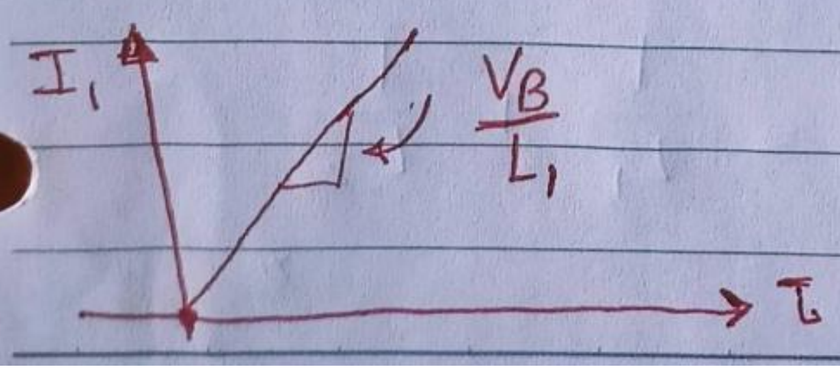
$I_1 = \frac{V_B - 0}{R_1} = \frac{V_B}{R_1}$; current doesn't change.

Example

$\tau < 0$, $I_1 = 0$

$I_1(0^+) = 0$





$I_1 = \frac{1}{L_1}\int V_B \, d\tau = \frac{V_B}{L_1}\tau$

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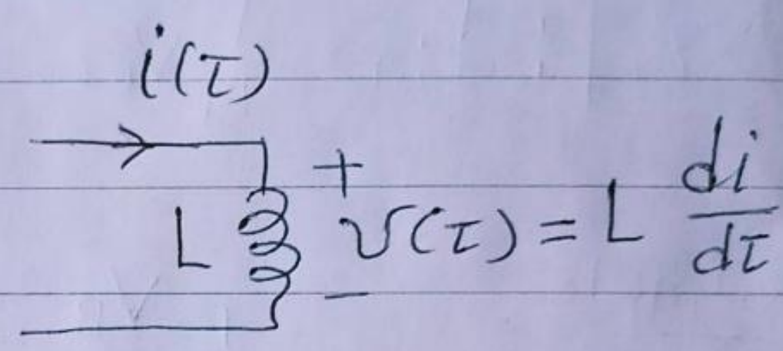
Summary

inductors	capacitors
if $V < \infty \Rightarrow I$ can'T jump , $I = 0 \Rightarrow L \equiv o.c$	If $I < \infty \Rightarrow V$ can'T jump $V = 0 \Rightarrow C \equiv s.c$
If $I = const \Rightarrow L \equiv s.c$	If $V = const \Rightarrow C \equiv o.c$

o.c → open circuit
s.c → short circuit

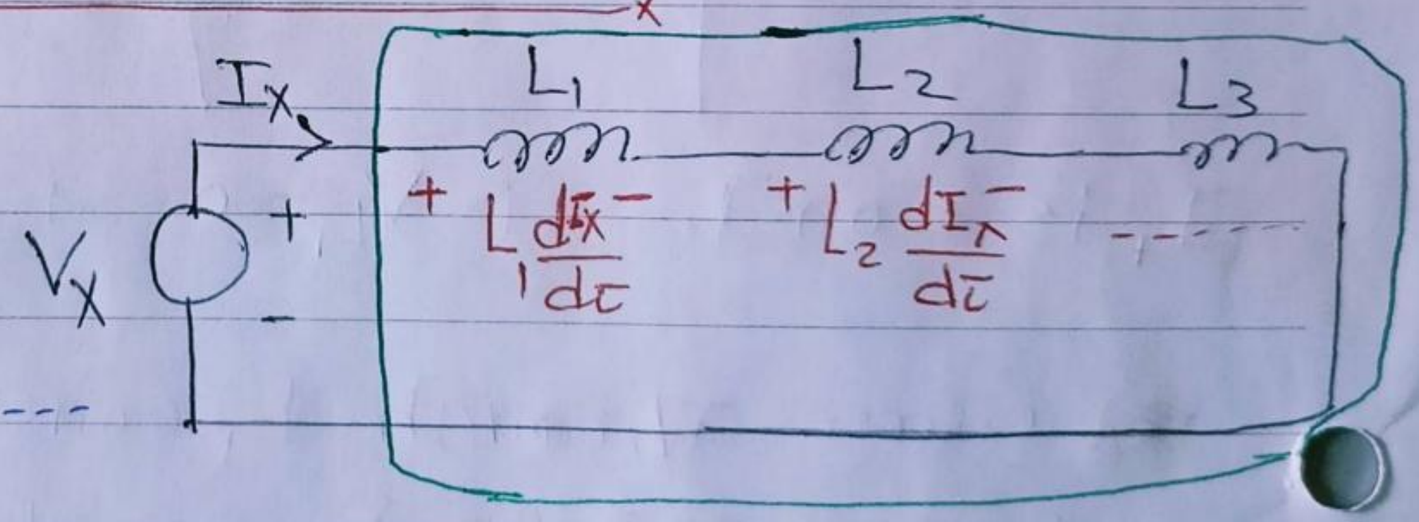
Energy in inductors:



$$P = v(\tau)\, i(\tau)$$

$$\mathcal{E} = \int_0^{\tau} P\, d\tau = \int_0^{\tau} L \frac{di}{d\tau}\, i\, d\tau = \frac{1}{2} L i^2(\tau)$$

inductors in series:

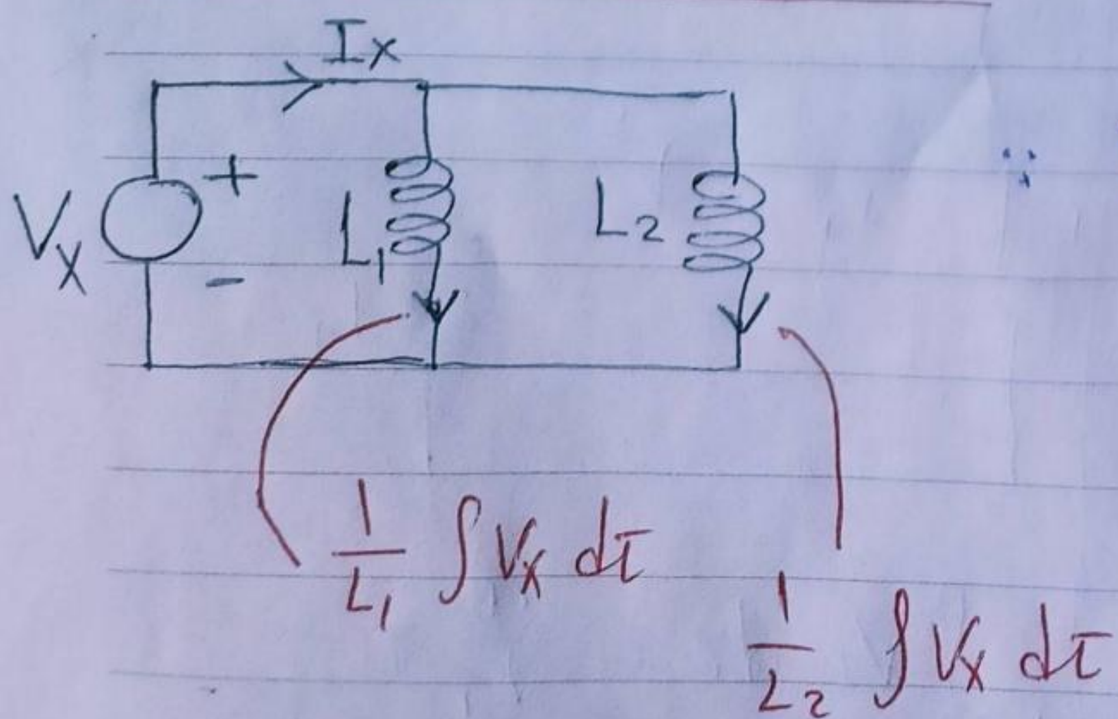


KVL:

$$V_x = L_1 \frac{dI_x}{d\tau} + L_2 \frac{dI_x}{d\tau} + \cdots$$

$$= (L_1 + L_2 + \cdots) \frac{dI_x}{d\tau} \Rightarrow L_{eq} = L_1 + L_2 + \cdots$$

inductors in parallel:



$$\because I_x = \frac{1}{L_1} \int V_x\, d\tau + \frac{1}{L_2} \int V_x\, d\tau$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2}\right) \int V_x\, d\tau$$

$$\because \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$$

#

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Lec 25: RL Circuits

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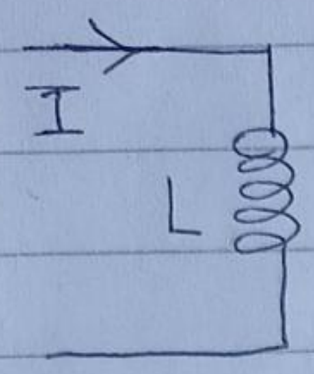
Review of lecture 24



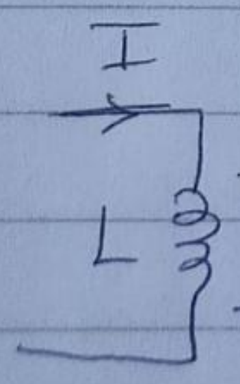
$$V = L\frac{dI}{dt}$$

Inductor	Capacitor
if $V<\infty \Rightarrow I=0$ at $t=0^-$ yields $I=0$ at $t=0^+$ $\Rightarrow L \equiv$ open	if $I<\infty \Rightarrow V=0$ at $t=0^-$ yields $V=0$ at $t=0^+$ $\Rightarrow C \equiv$ short
if $I=$ const. $\Rightarrow V=0$ $\Rightarrow L \equiv$ short	if $V=$ const $\Rightarrow I=0$ $\Rightarrow C \equiv$ open

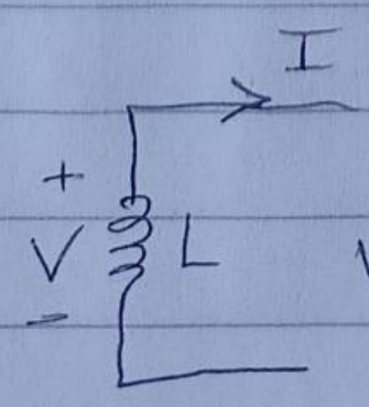
- Energy $= \frac{1}{2} L I^2$
- series & parallel combinations.

Todays lecture

- A simple RL circuit.
- Additional Examples.



$$V = L\frac{dI}{dt}$$



$$V = -L\frac{dI}{dt}$$

$$\alpha\frac{dy}{dt} + y = \text{const}$$

$$\alpha = \tau$$

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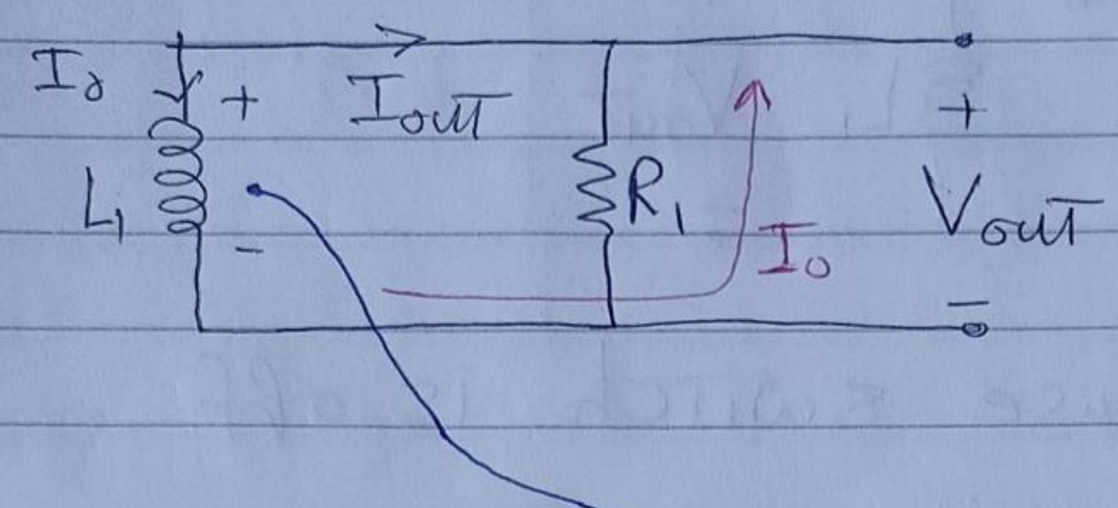
- simple RL circuit
  (sour free or undriven response)



$I_{out}(\infty) = 0$

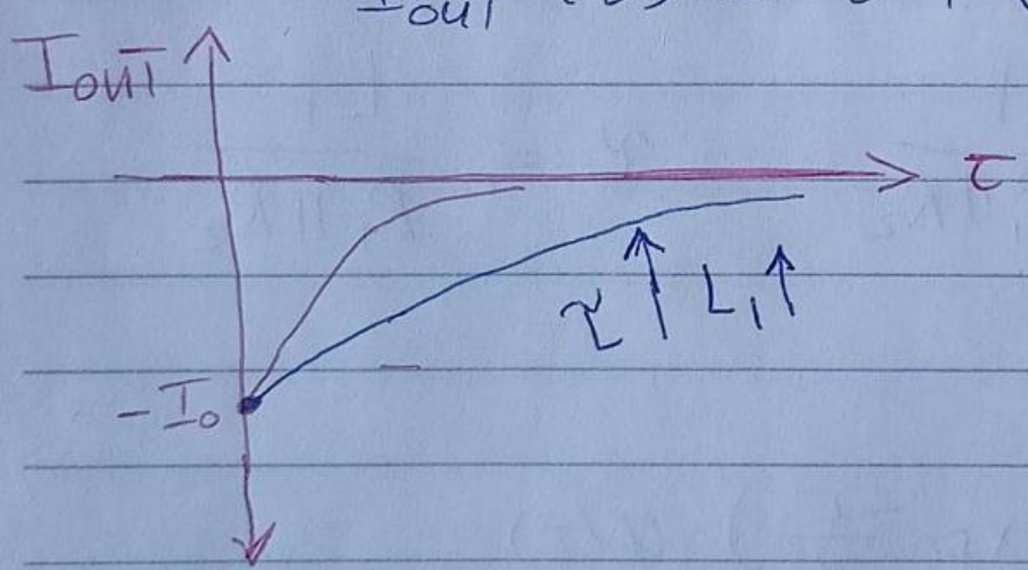
$V_{out}(\infty) = 0$

Find I_{out} first :
$$\begin{cases} -L_1 \dfrac{dI_{out}}{d\tau} = R_1 I_{out} \\ I_{out}(0^+) = -I_0 \end{cases}$$

AT $\tau = 0^-$: $I_{out} = -I_0$

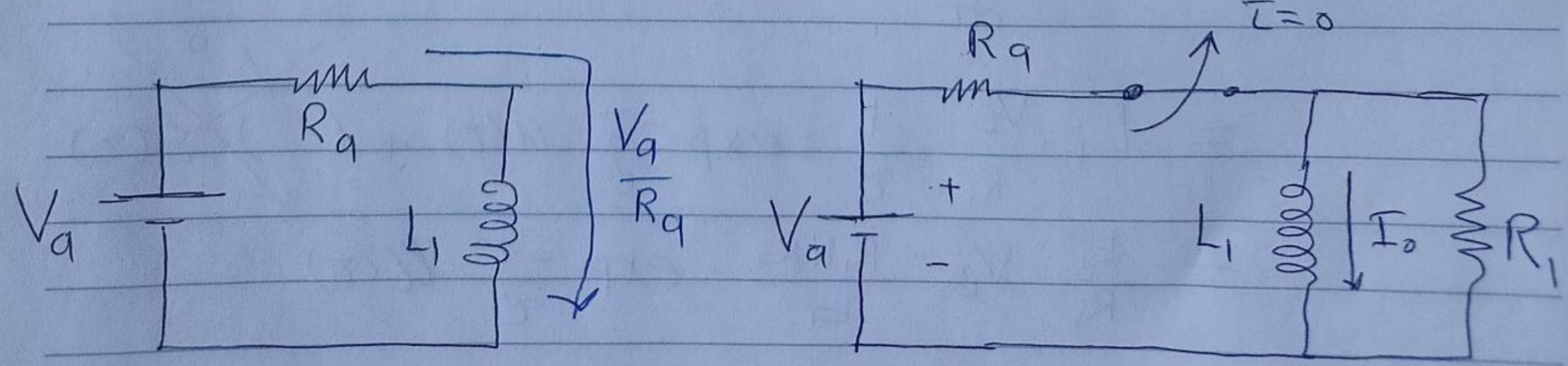
$$\frac{L_1}{R_1}\frac{dI_{out}}{d\tau} + I_{out} = 0 \Rightarrow \tau = \frac{L_1}{R_1}$$

$$I_{out}(\tau) = 0 + (-I_0 - 0)\exp\frac{-\tau}{L_1/R_1} \quad \tau > 0$$



$$V_{out} = R_1 I_{out}$$
$$= -R_1 I_0 \exp\frac{-\tau}{L_1/R_1} \quad \tau > 0$$

To create the initial condition :



$$I_0 = \frac{V_a}{R_a}$$

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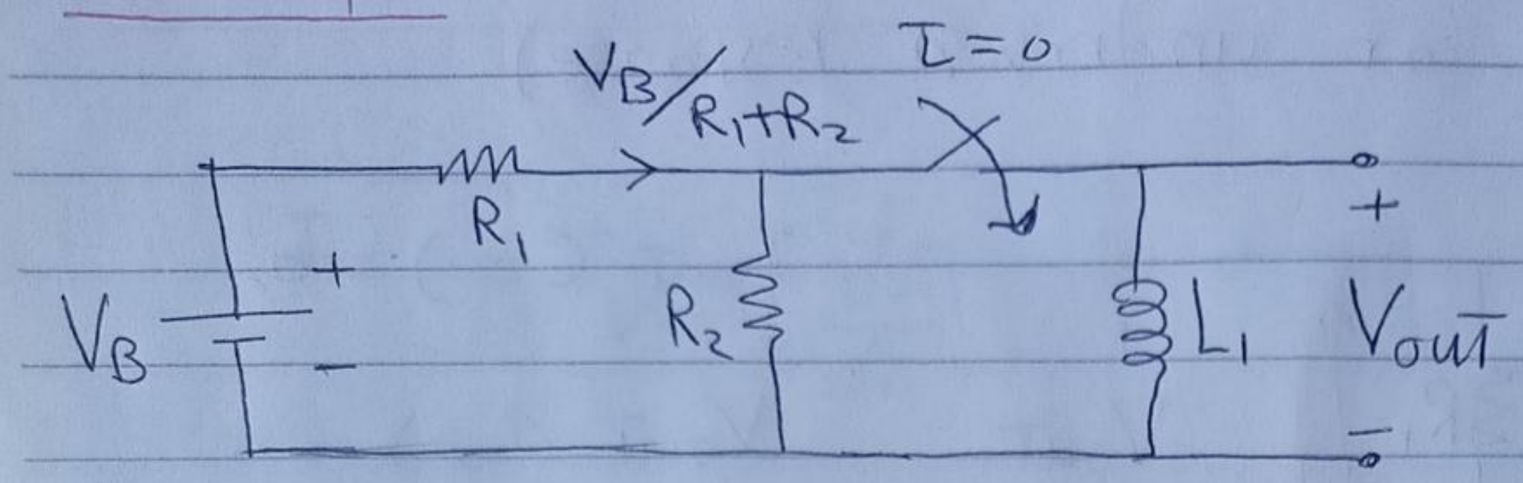
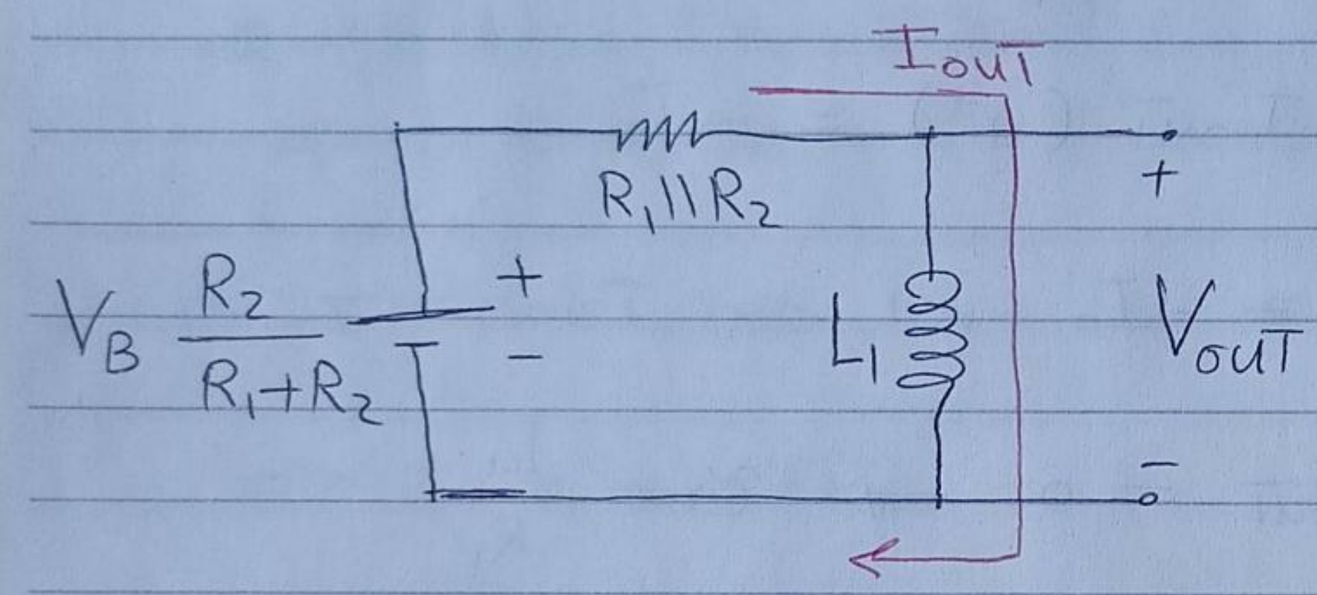
Example



$t<0$, $V_{out}=0$ because switch is off

$$I_{L_1}=0 \quad at \quad t=0^-$$



$$I_{out}(0^+)=0$$

$$I(\infty)=V_B\frac{R_2}{R_1+R_2}\cdot\frac{1}{R_1\|R_2} \qquad \tau=\frac{L_1}{R_1\|R_2}$$

$$=\frac{V_B}{R_1}$$

$$I_{out}(t)=\frac{V_B}{R_1}\left(1-\exp\frac{-t}{\tau}\right)u(t)$$

$$V_{out}=L_1\frac{dI_{out}}{dt}$$

$$=L_1\frac{V_B}{R_1}\frac{1}{\tau}\exp\frac{-t}{\tau}u(t)+\cancel{(\quad)\delta(t)}^{\,0}$$

$$=\frac{\cancel{L_1}}{R_1}V_B\frac{R_1\|R_2}{\cancel{L_1}}\exp\frac{-t}{\tau}u(t)$$

$$=\frac{V_B R_2}{R_1+R_2}\exp\frac{-t}{\tau}u(t)$$

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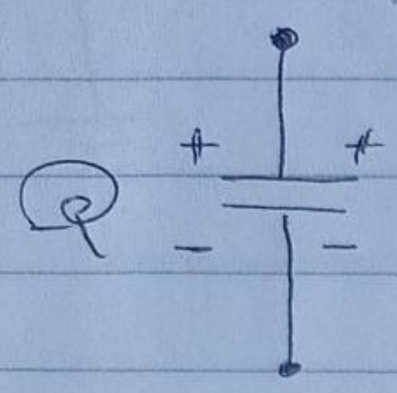
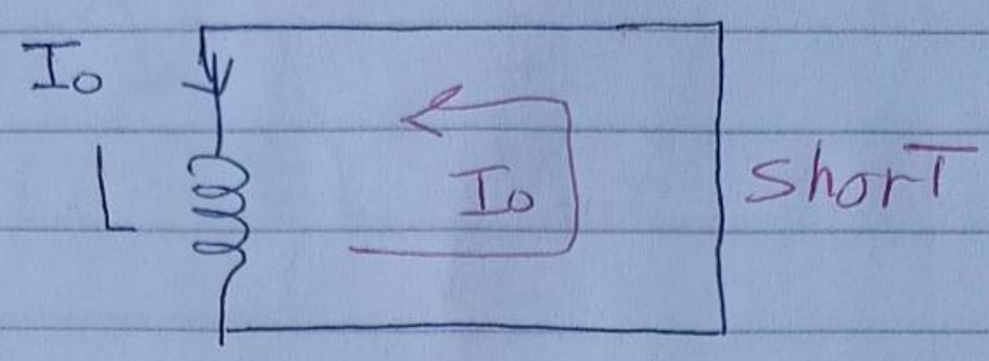
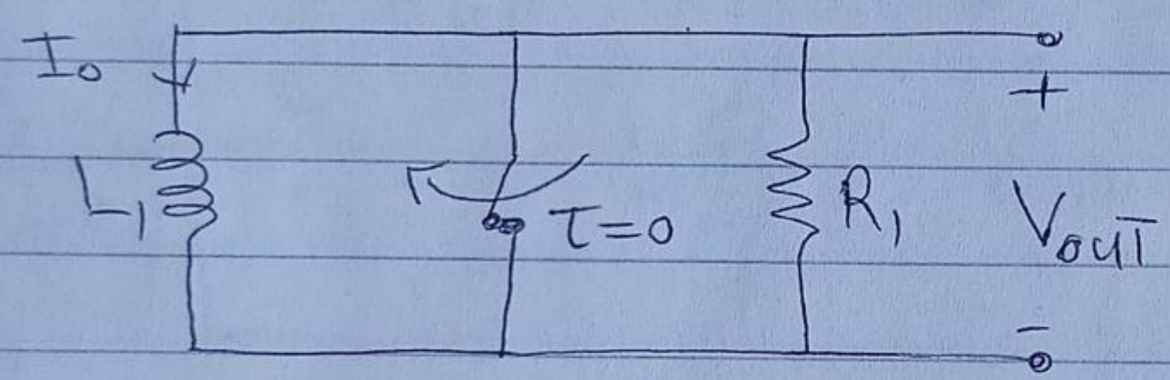
Example

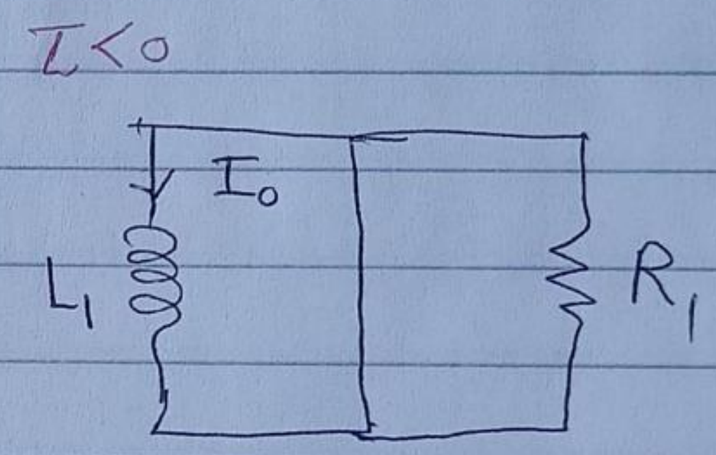




- The inductor maintains this initial condition and the current flows forevers.

Example





ابراهيم الحسن

لا تنسونا من صالح دعائكم

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Lec 26: Source-Free and Driven RL Circuits

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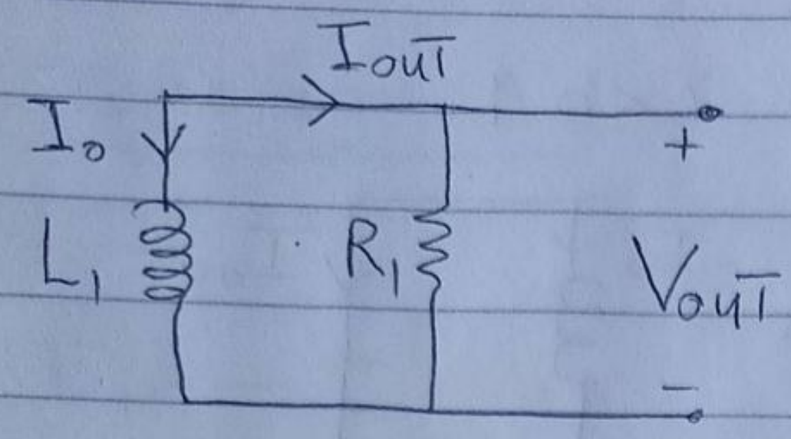
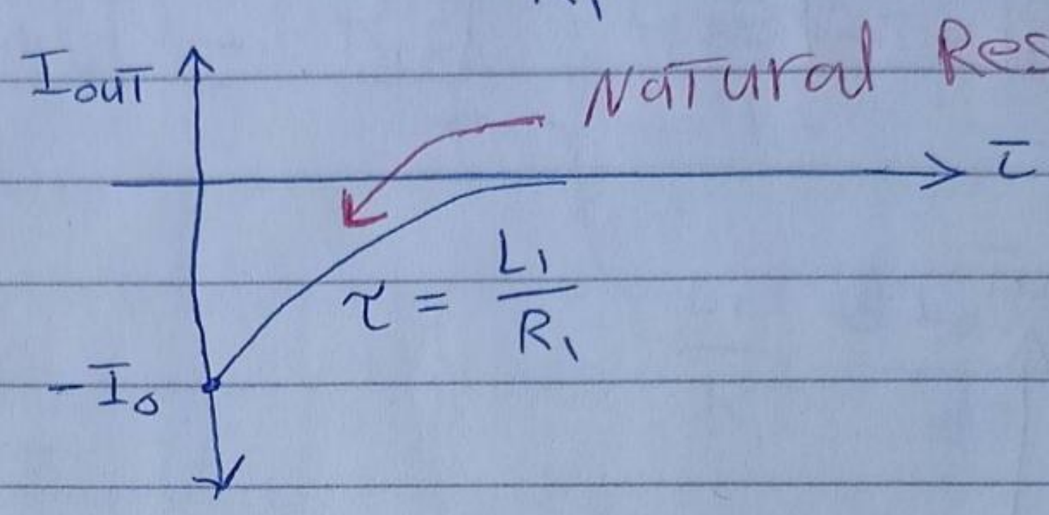
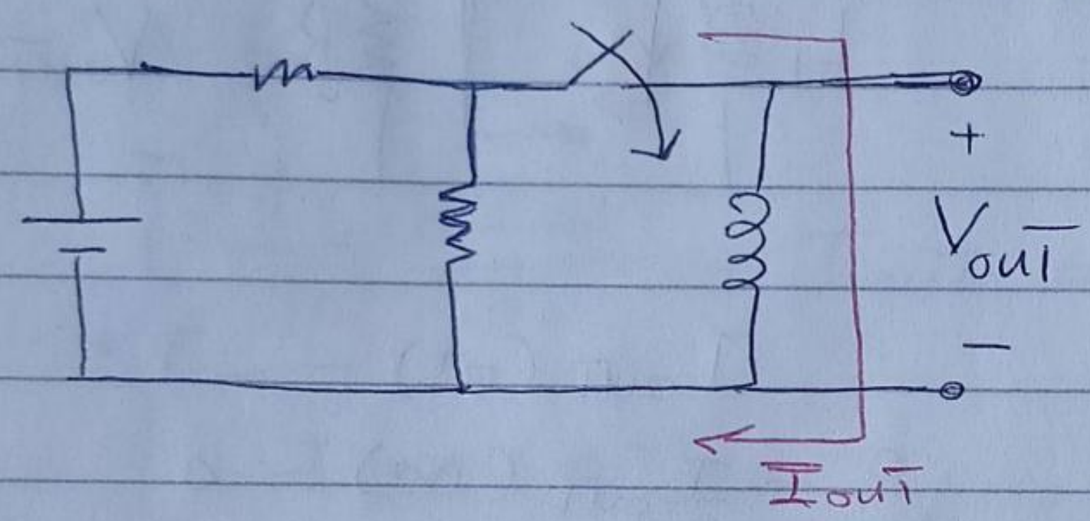
Review of lecture 25



$$I_{out} = -I_0 \exp \frac{-t}{\tau}$$

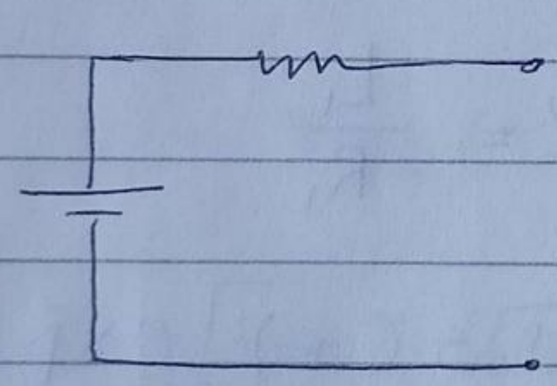
$$\tau = \frac{L_1}{R_1}$$

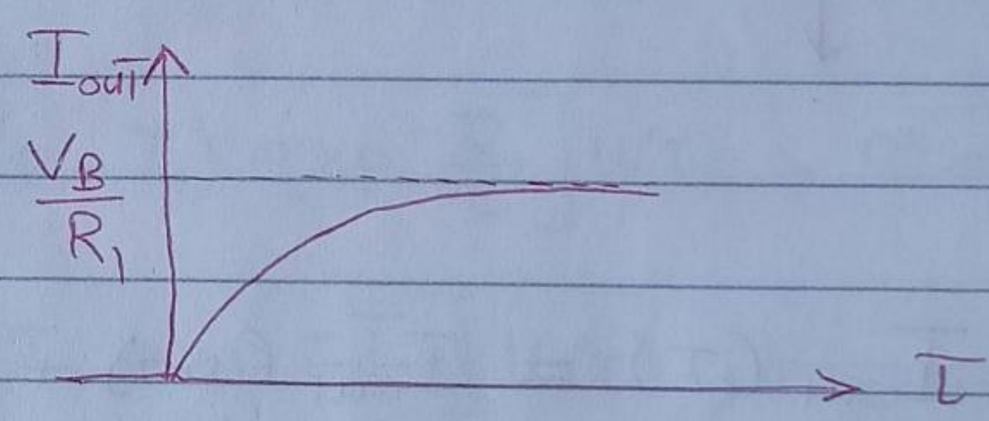




$$I_{out} = \frac{V_B}{R_1}\left(1 - \exp \frac{-t}{\tau}\right)$$

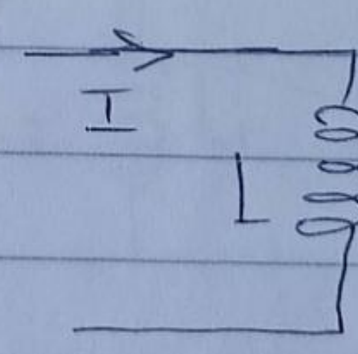
$$\tau = \frac{L_1}{R_1 \parallel R_2}$$





Today's lecture

- undriven RL circuit Example.
- Driven RL circuits



$$V = L \frac{dI}{dt}$$

$-V < \infty \Rightarrow L \equiv$ open if $I_0 = 0$,

$\Rightarrow L \equiv$ short if $I =$ const.

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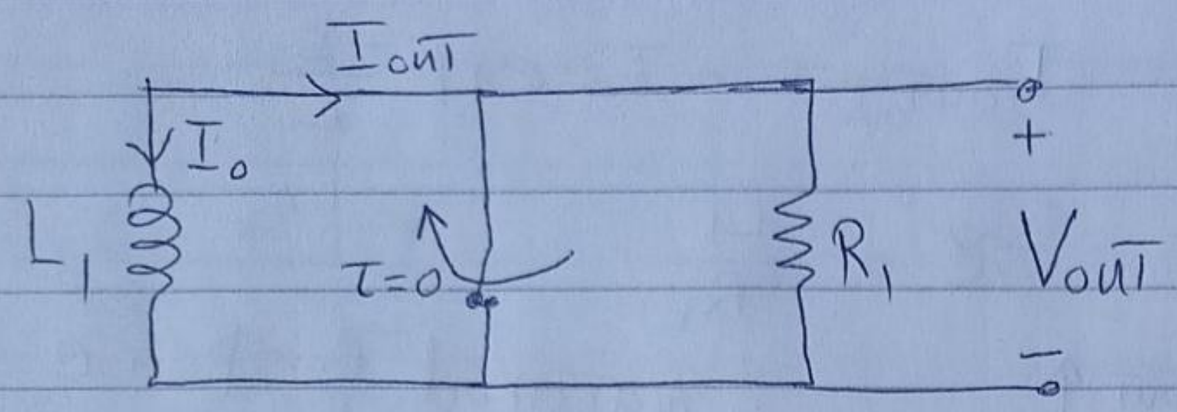
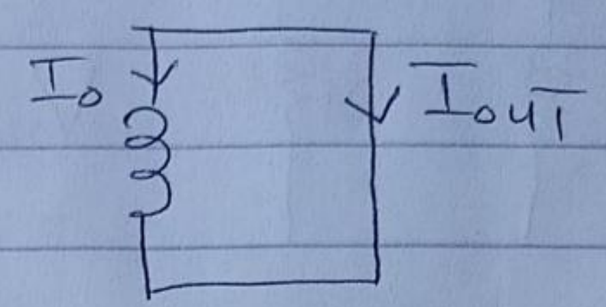
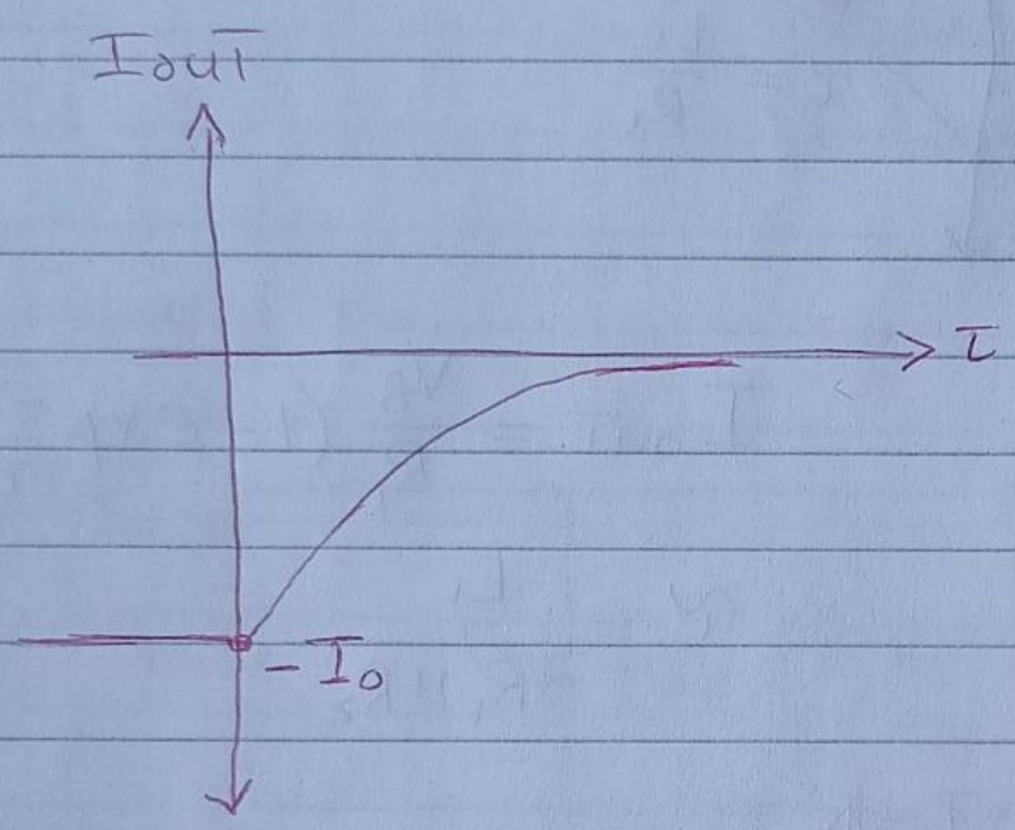
Example

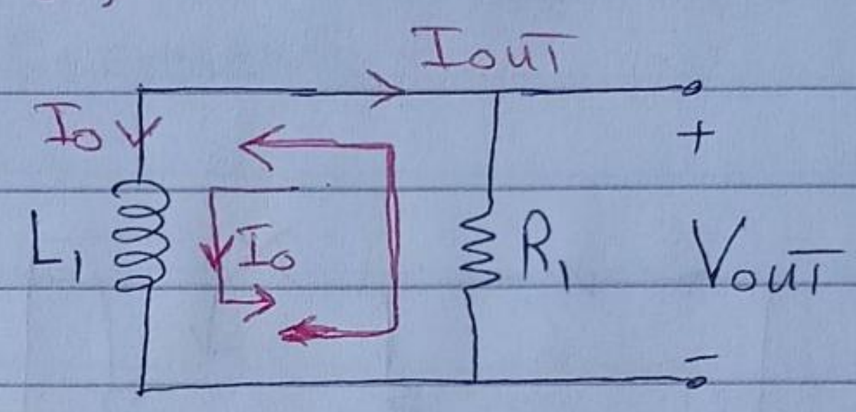


$t<0$;





$t>0$;



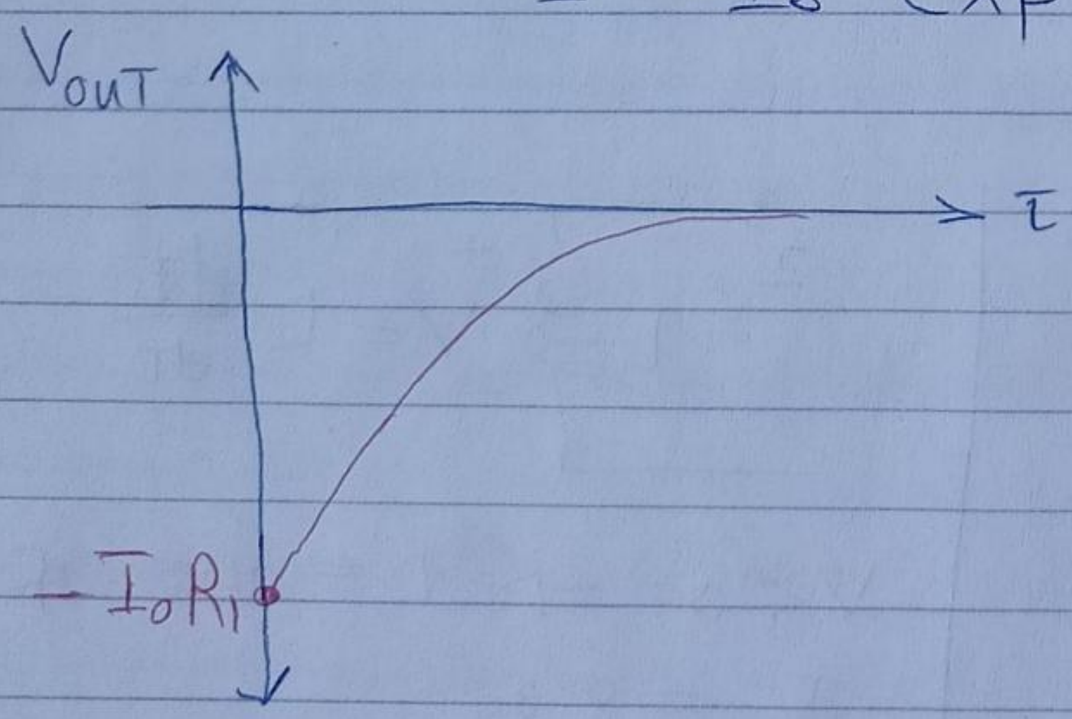
$$I_{OUT}(0^+) = -I_0$$

$$I_{OUT}(\infty) = 0$$

$$\tau = \frac{L_1}{R_1}$$

$$I_{OUT}(t) = I_{OUT}(\infty) + \left[I_{OUT}(0^+) - I_{OUT}(\infty) \right] \exp \frac{-t}{\tau}$$

$$= -I_0 \exp \frac{-t}{\tau} \qquad t>0$$



$$V_{OUT}(0^+) = -I_0 R_1$$

$$V_{OUT}(t) = I_{OUT} R_1$$

$$= I_{OUT}(t) \; R_1$$

$$V_{OUT}(t) = -R_1 I_0 \exp \frac{-t}{\tau} \; u(t)$$

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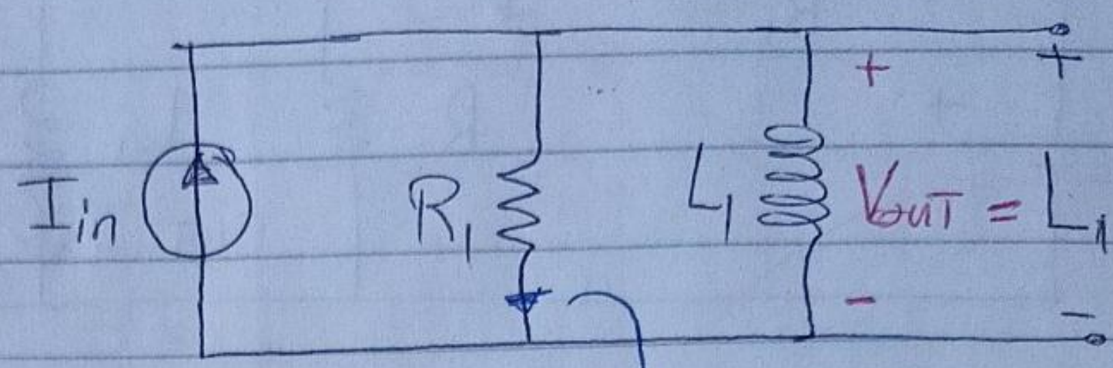
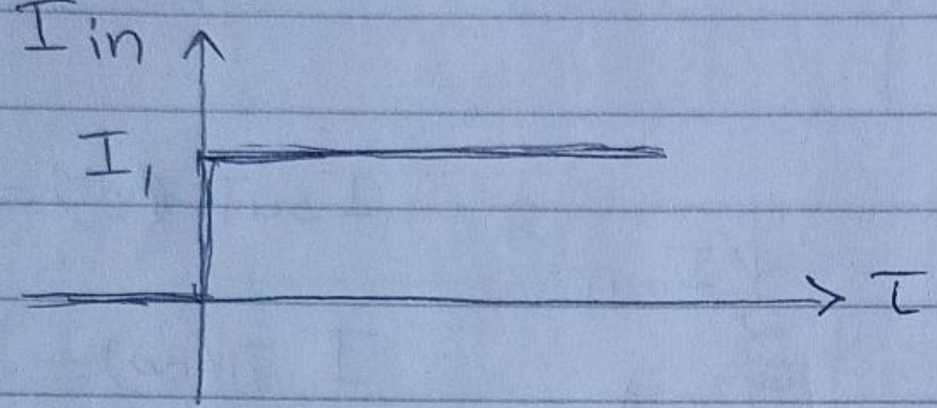
Driven RL circuits

Topology 1:

$I_{in} = I_1 u(t)$





$\frac{L_1}{R_1} \frac{dI_{OUT}}{dt}$

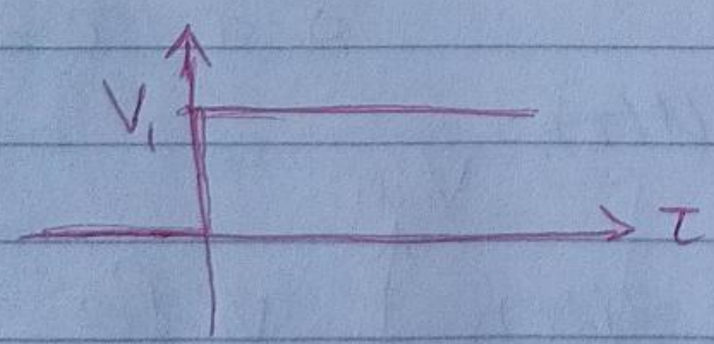
$$\begin{cases} \frac{L_1}{R_1} \frac{dI_{OUT}}{dt} + I_{OUT} = I_1 u(t) \\ I_{OUT}(0^+) = 0 \quad , \quad I_{OUT}(\infty) = I_1 \end{cases}$$

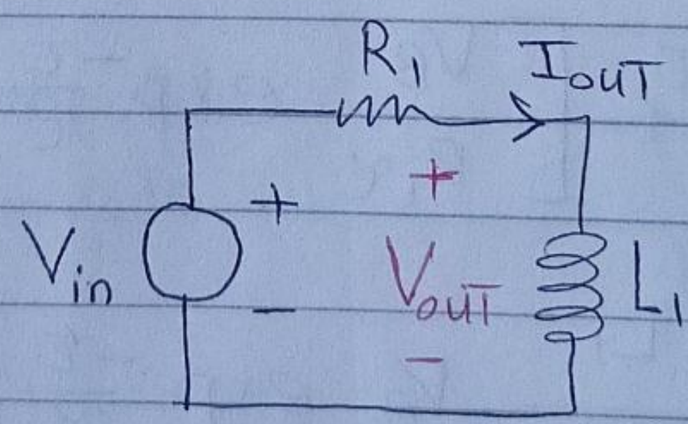
$$\therefore I_{OUT}(t) = \left[I_1 + (0 - I_1) \exp^{-\frac{t}{\tau}} \right] u(t) \quad , \quad \tau = \frac{L_1}{R_1}$$

$$I_{R_1} = I_1 u(t) - \left[I_1 - I_1 \exp^{-\frac{t}{\tau}} \right] u(t) \quad , \quad V_{OUT} = I_{R_1} R_1$$

Topology 2:

$V_{in} = V_1 u(t)$





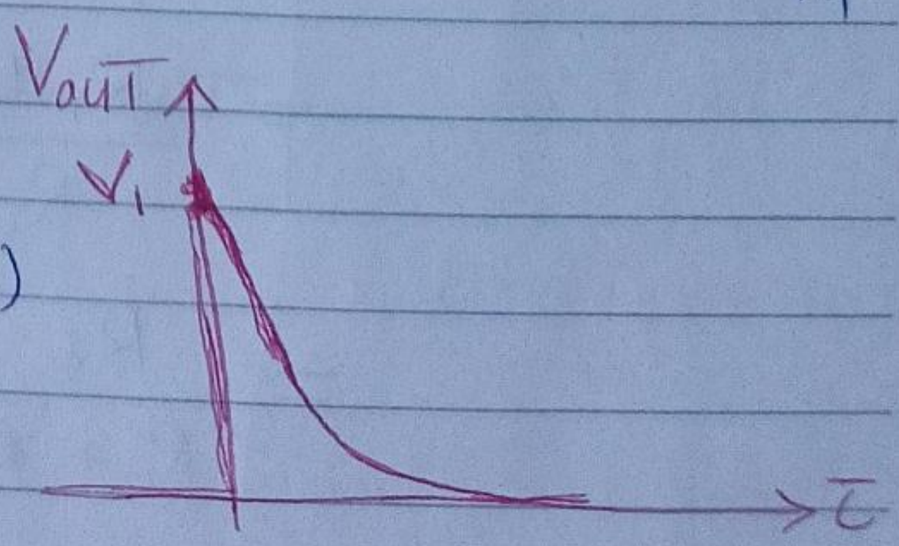
$t < 0$: $I_{OUT}(0^-) = 0$

$I_{OUT}(0^+) = 0$

$I_{OUT}(\infty) = \frac{V_1}{R_1}$

$$I_{OUT} = \left[\frac{V_1}{R_1} + \left(0 - \frac{V_1}{R_1}\right) \exp^{-\frac{t}{\tau}} \right] u(t)$$

$$V_{OUT} = L_1 \frac{dI_{OUT}}{dt} = V_1 \exp^{-\frac{t}{\tau}} u(t)$$



$$\tau = \frac{L_1}{R_1}$$

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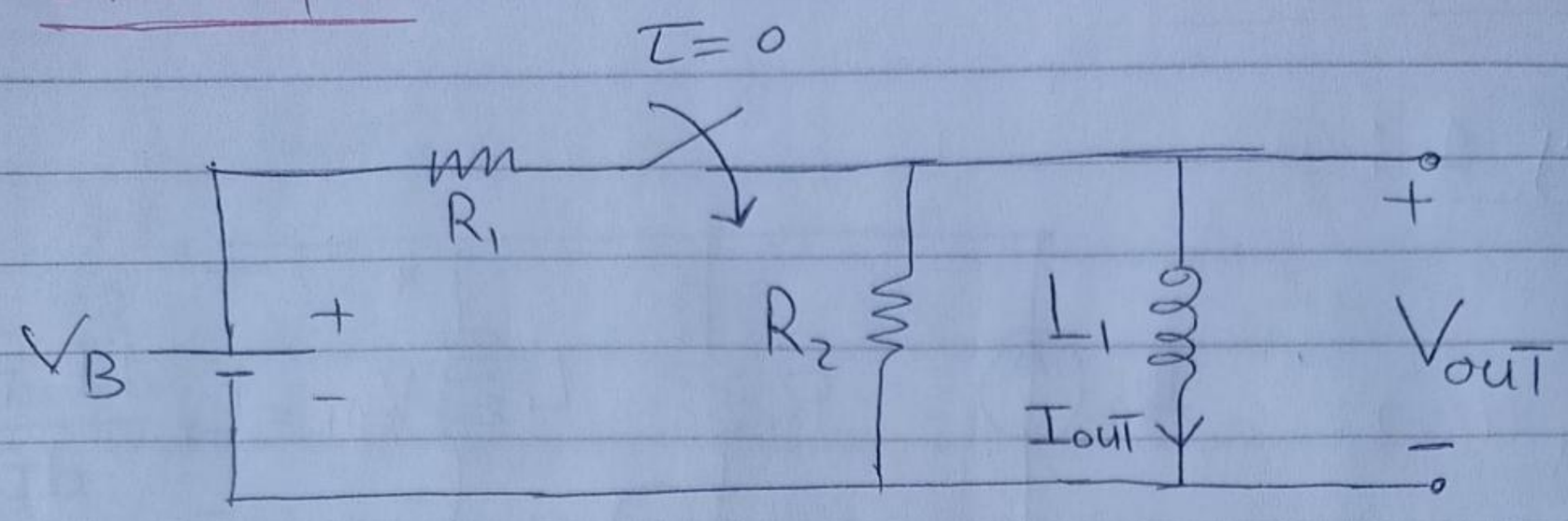
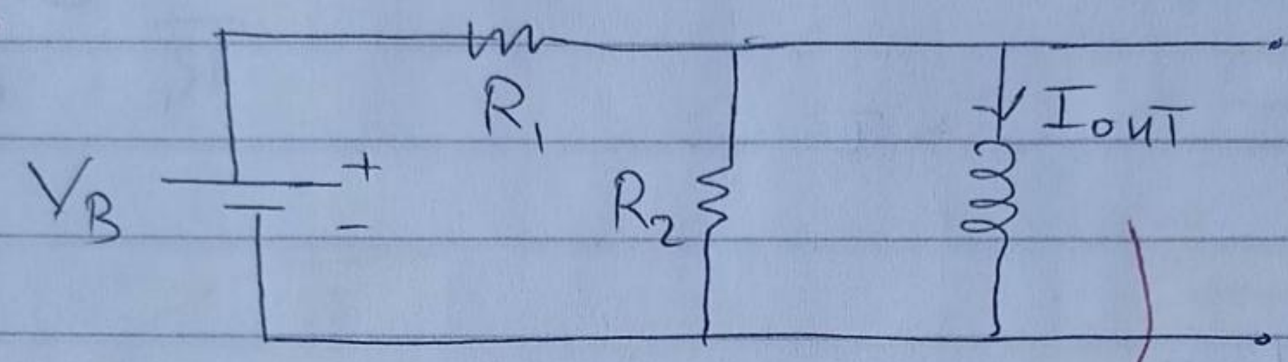
Example



$t > 0$:

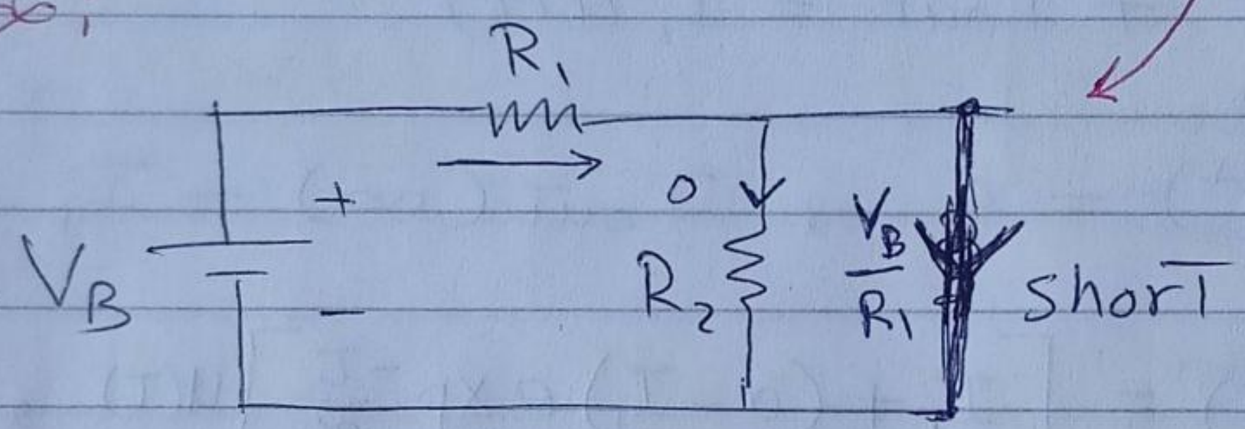


$I_{OUT}(0^+) = 0$

$I_{OUT}(\infty) = \frac{V_B}{R_1}$

$\tau = \frac{L_1}{R_1 \| R_2}$

$t = \infty$:



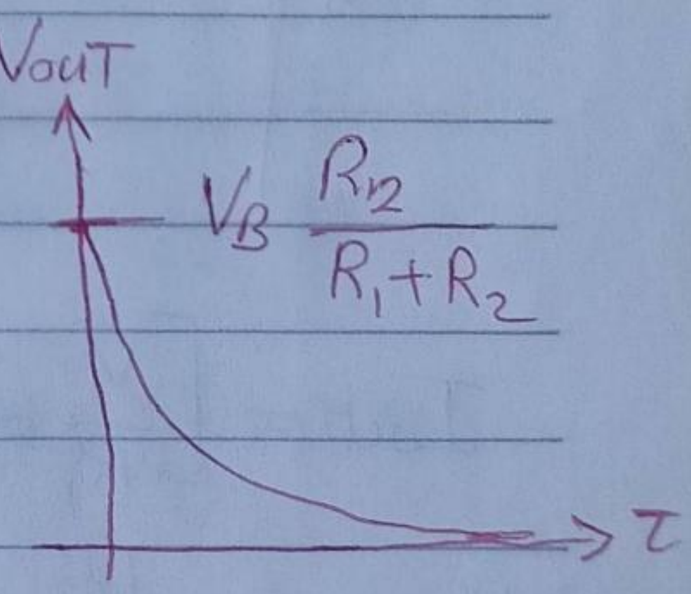
$$I_{OUT}(t) = \left[\frac{V_B}{R_1} + \left(0 - \frac{V_B}{R_1}\right) \exp \frac{-t}{\tau} \right] u(t)$$

$$V_{OUT}(t) = L_1 \left[\frac{V_B}{R_1 \tau} \exp \frac{-t}{\tau} u(t) + \underbrace{\frac{V_B}{R_1}\left(1 - \exp \frac{-t}{\tau}\right)}_{0 \text{ at } t=0} \delta(t) \right]$$

$$= \frac{L_1}{R_1 \tau} V_B \exp \frac{-t}{\tau} u(t)$$

$$= \frac{R_1 \| R_2}{R_1} V_B \exp \frac{-t}{\tau} u(t)$$

$$= \frac{R_2}{R_1 + R_2} V_B \exp \frac{-t}{\tau} u(t)$$



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Lec 27: Introduction to RLC Circuits

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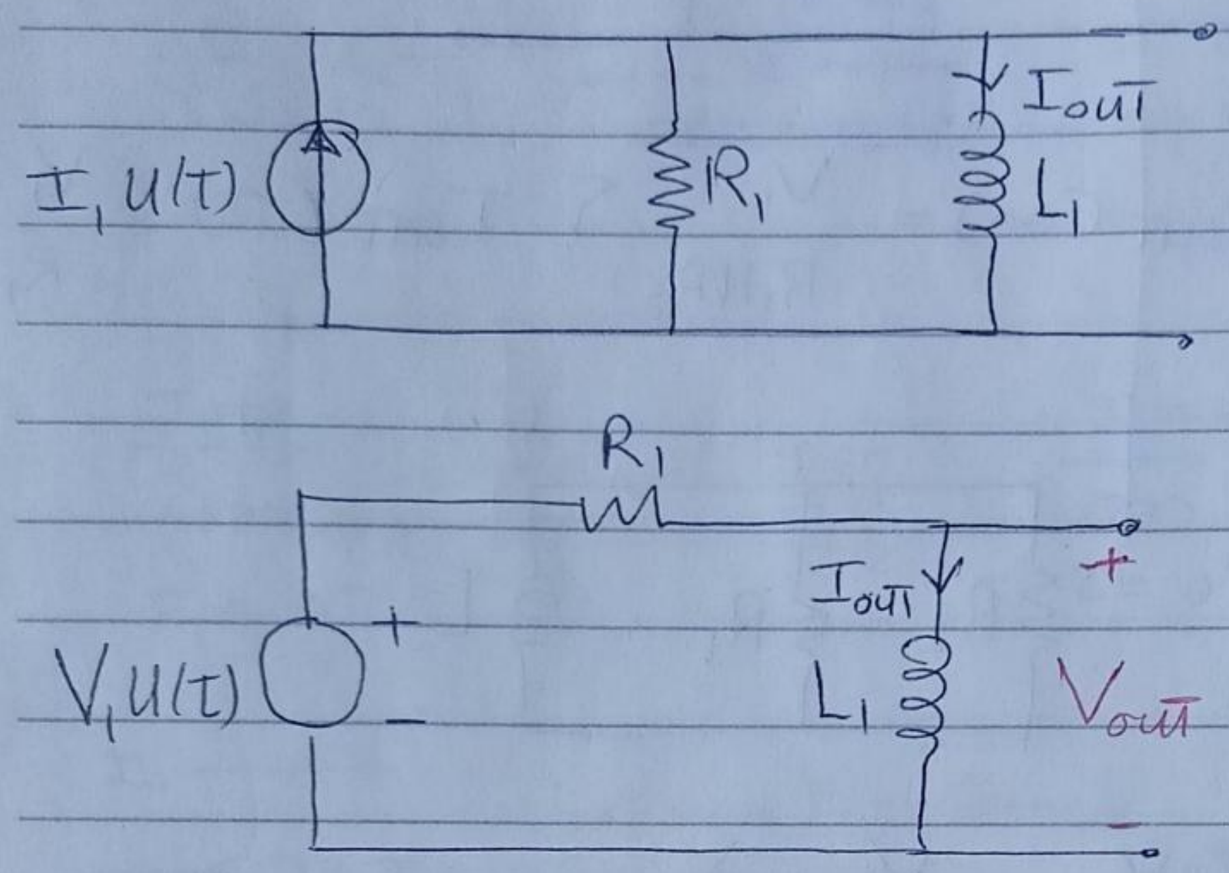
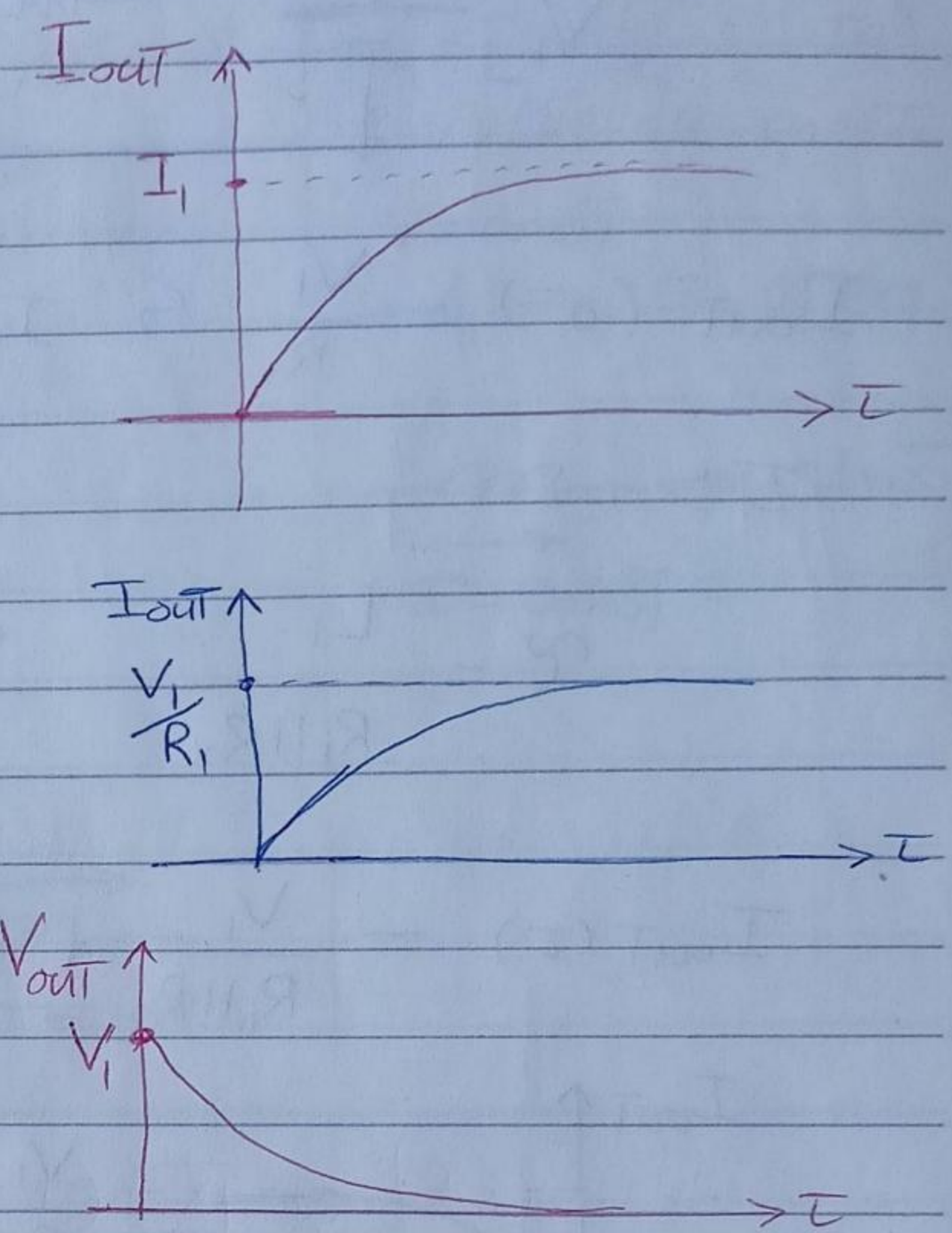
Review of lecture 26

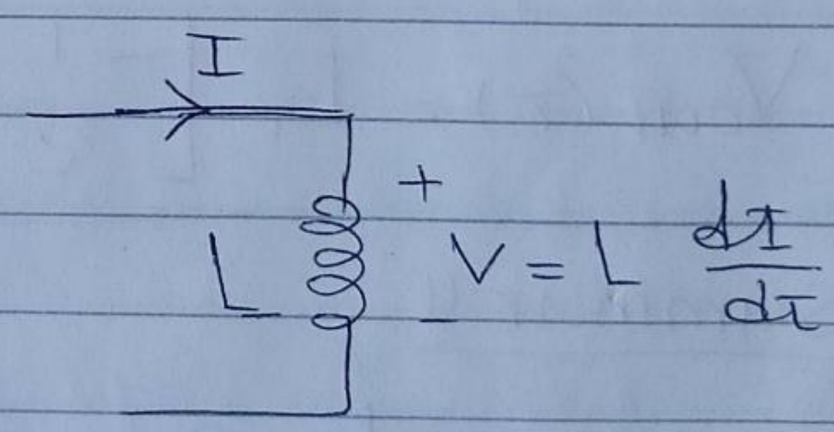
- Driven RL circuit:





Today's lecture

- Another Driven RL circuit Example



- Summary
- introduction to RLC circuit

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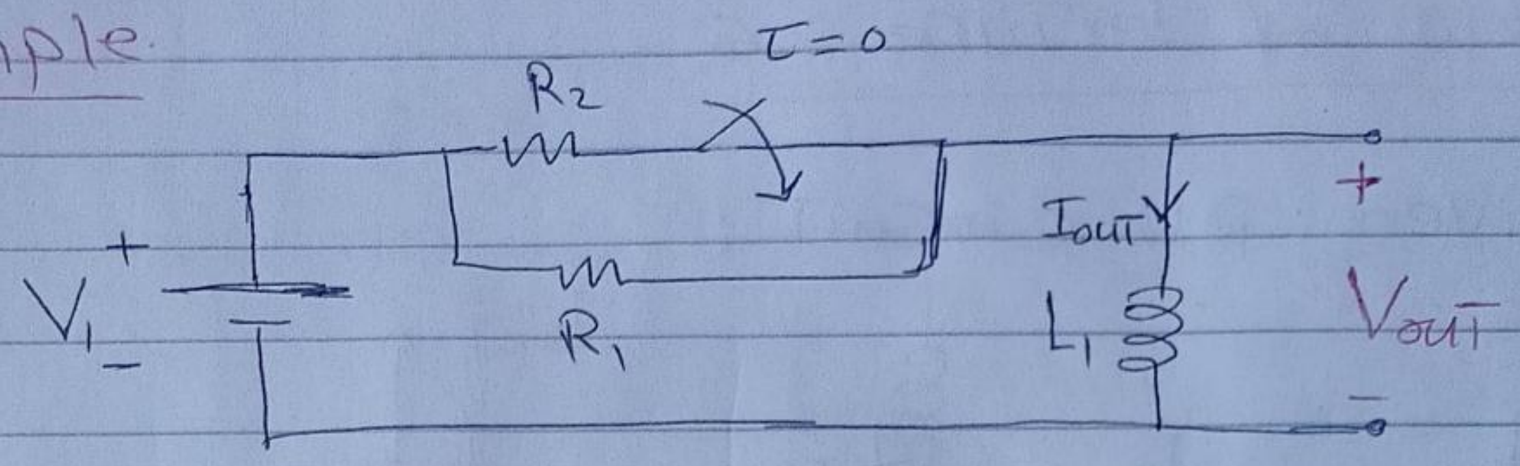
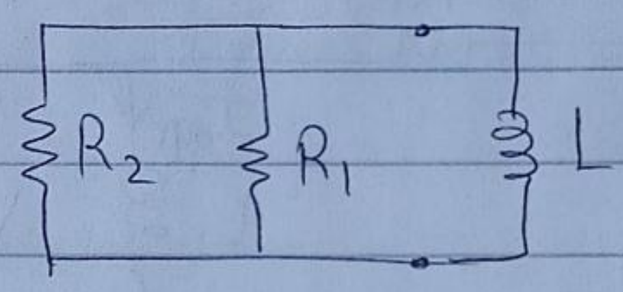
Example



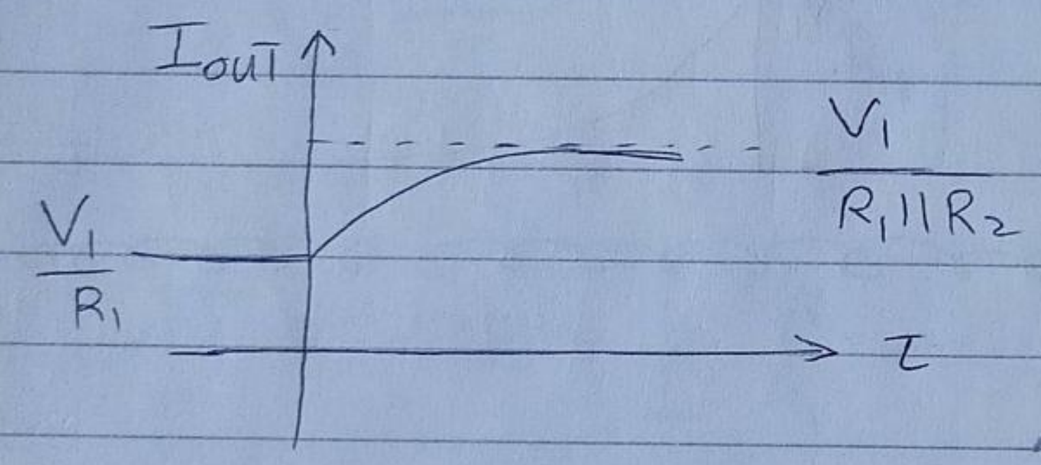
$$I_{out}(0^-) = \frac{V_1}{R_1} \quad I_{out}(\infty) = \frac{V_1}{R_1 \| R_2} \quad I_{out}(0^+) = \frac{V_1}{R_1}$$

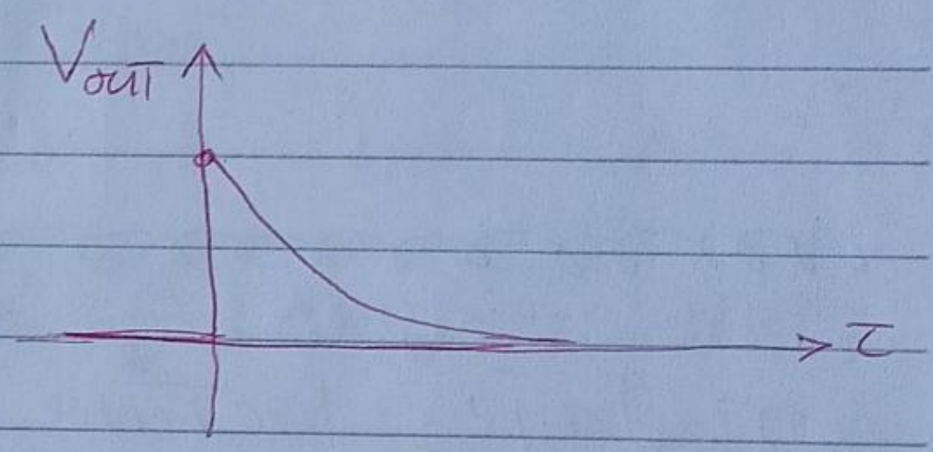
$t > 0$:

$$\tau = \frac{L_1}{R_1 \| R_2}$$



$$I_{out}(t) = \frac{V_1}{R_1 \| R_2} + \left(\frac{V_1}{R_1} - \frac{V_1}{R_1 \| R_2}\right) \exp \frac{-t}{\tau} \quad t > 0$$





$$V_{out}(t) = L_1 \left[\frac{-1}{\tau} \left(\frac{V_1}{R_1} - \frac{V_1}{R_1 \| R_2} \right) \exp \frac{-t}{\tau} \right] u(t)$$

Summary

	R	C	L
Basic eq	$V = IR$	$I = C \frac{dv}{dt}$	$V = L \frac{di}{dt}$
energy stored	—	$\frac{1}{2} C V^2$	$\frac{1}{2} L i^2$
initial condition	—	V_0	I_0
Singularities	—	$I = \infty$ if V jumps	$V = \infty$ if I jumps.

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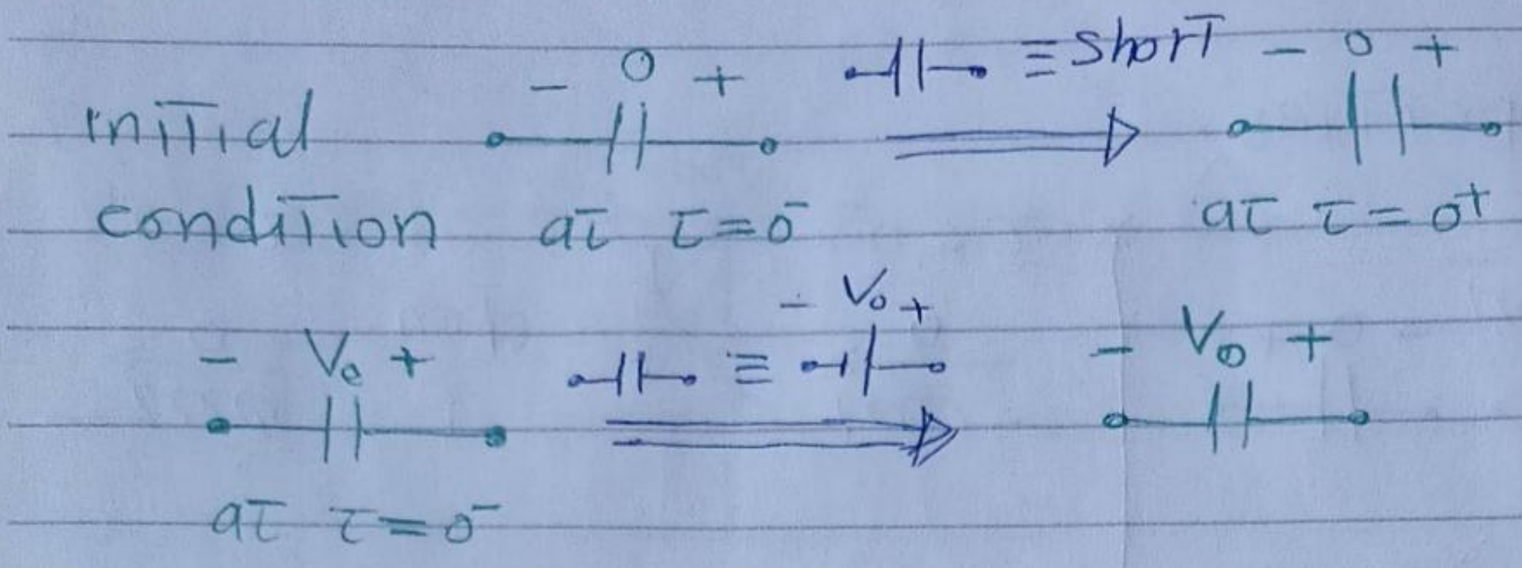
observations

initial condition



- find values (steady-state); value are const

$V = \text{const}$

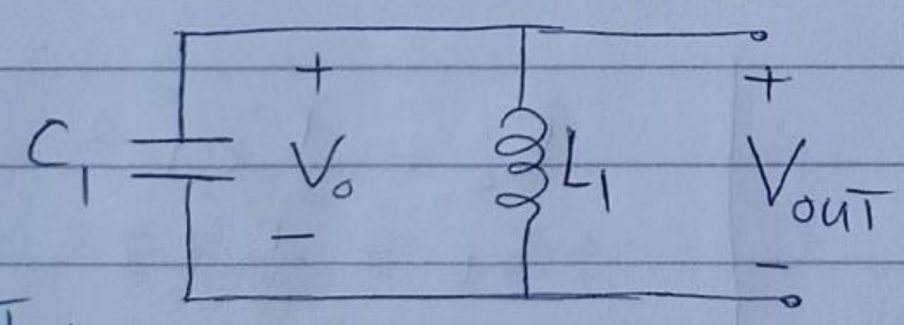
— || — ≡ open

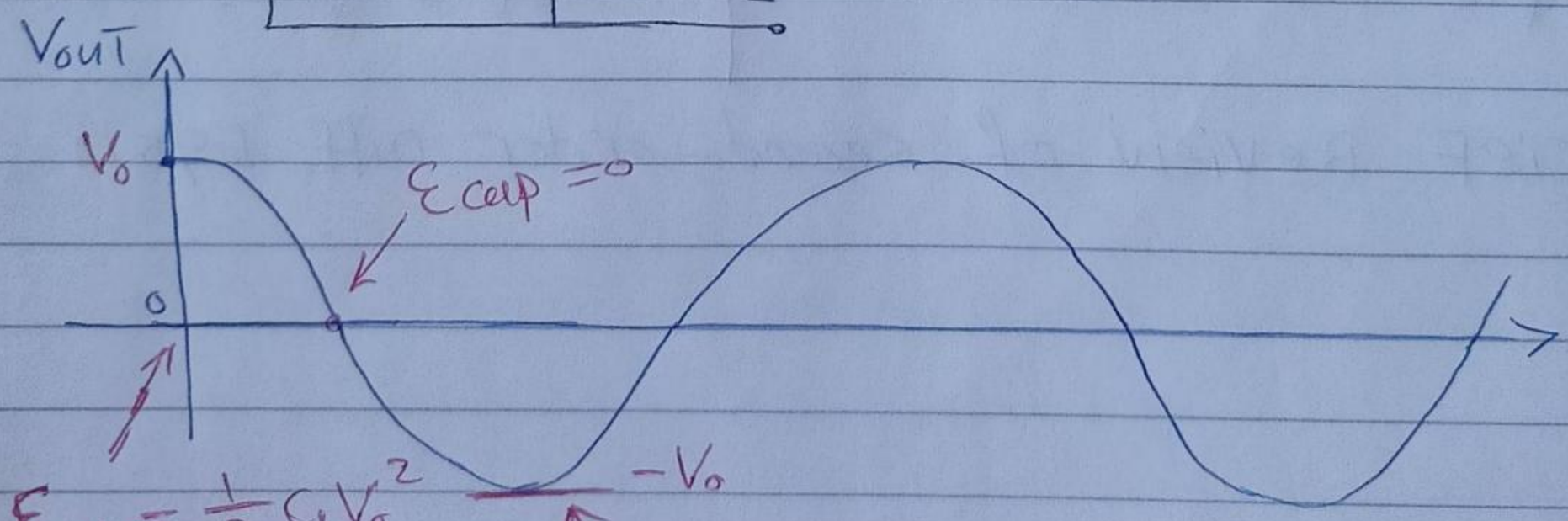
— ∿ — ≡ short

$I = \text{const}$

introduction to RLC circuit.

- A simple LC circuit





$\mathcal{E}_{cap} = \frac{1}{2} C_1 V_0^2$

$\mathcal{E}_{cap} = \frac{1}{2} C_1 V_0^2$

- $\mathcal{E}_{ind} = \mathcal{E}_{cap} = \frac{1}{2} C_1 V_0^2 = \frac{1}{2} L_1 I^2$

$\mathcal{E}_{ind} = 0 \Rightarrow I_{ind} = 0$

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Lec 28: Simple LC Circuit; 2nd-Order Diff. Equations

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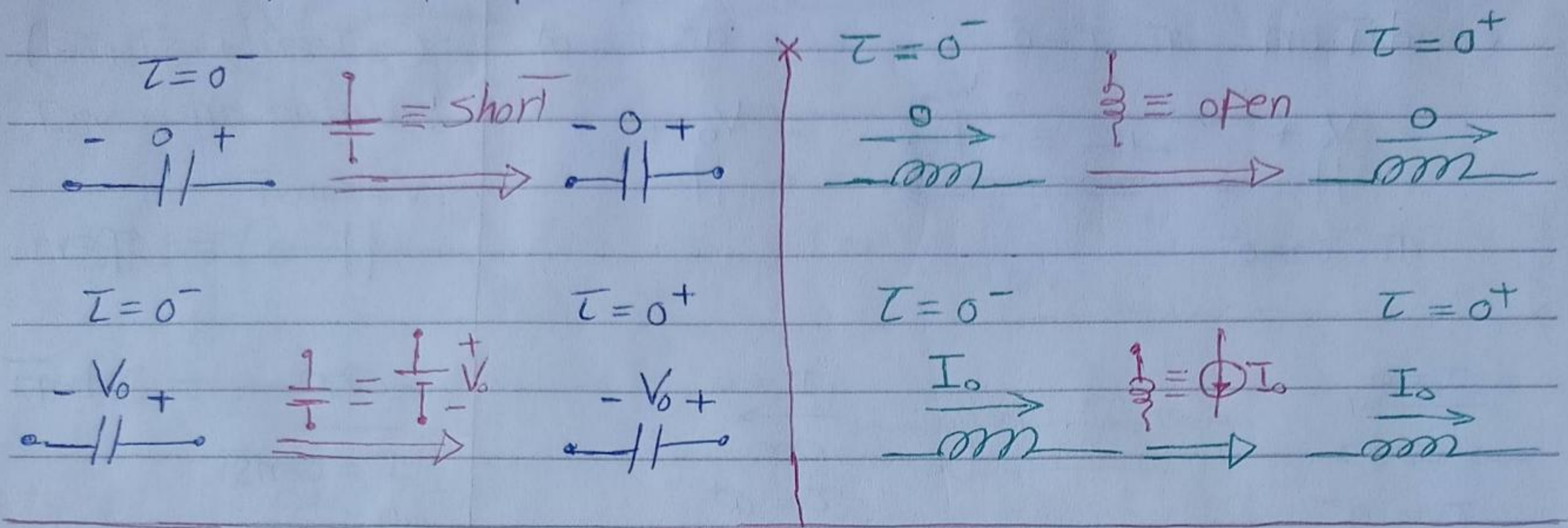
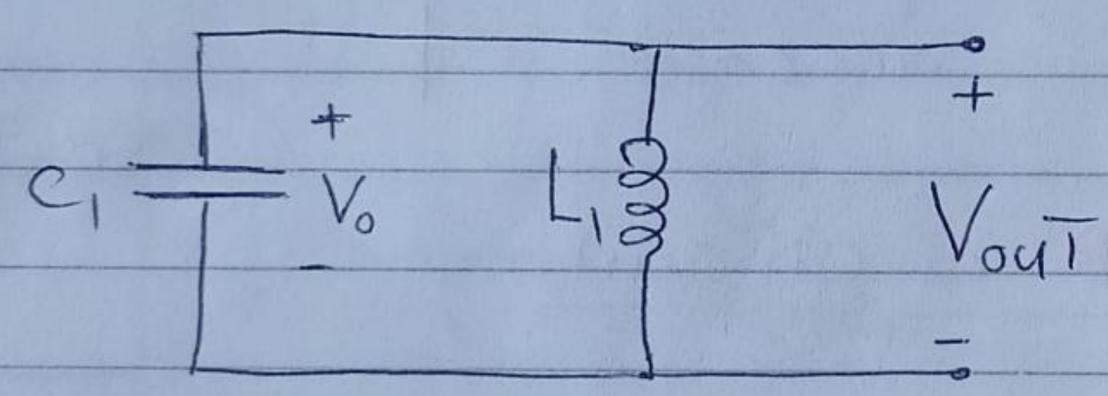
Review of lecture 27

- C & L properties.



- introduction to LC circuits.



Today's lecture:

- simple LC circuit

$E_C = \frac{1}{2} C V^2$

$E_L = \frac{1}{2} L I^2$

- Brief Review of second-order Diff. Eqs.

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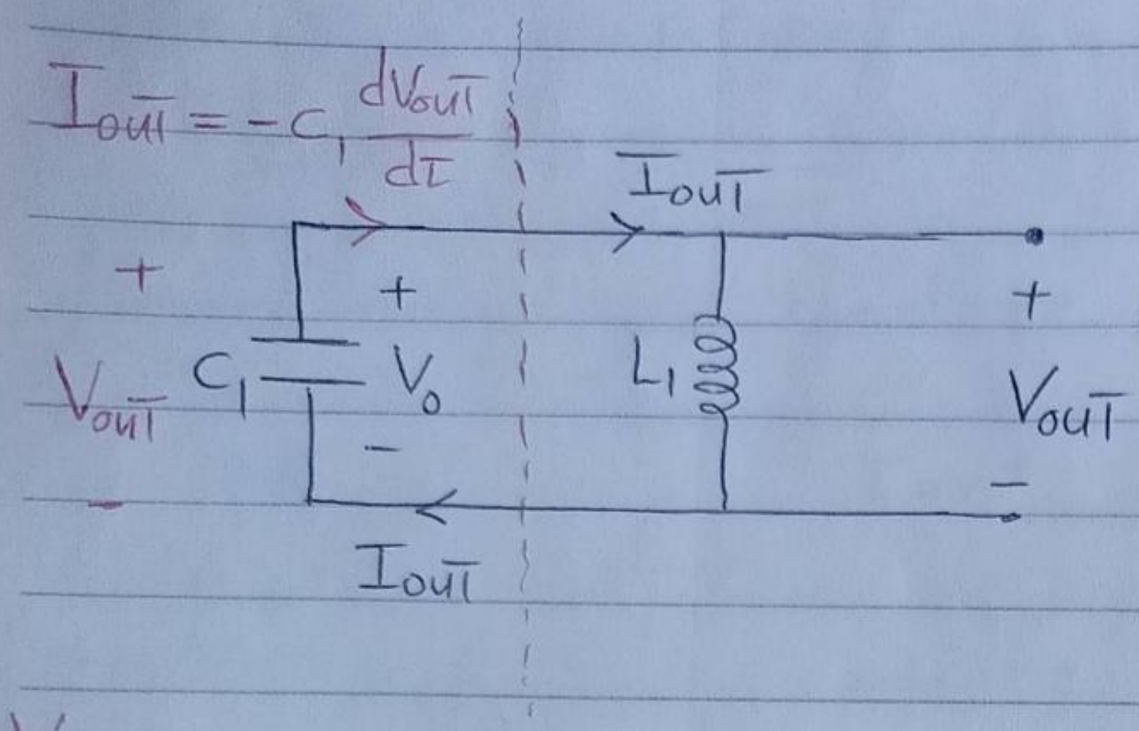
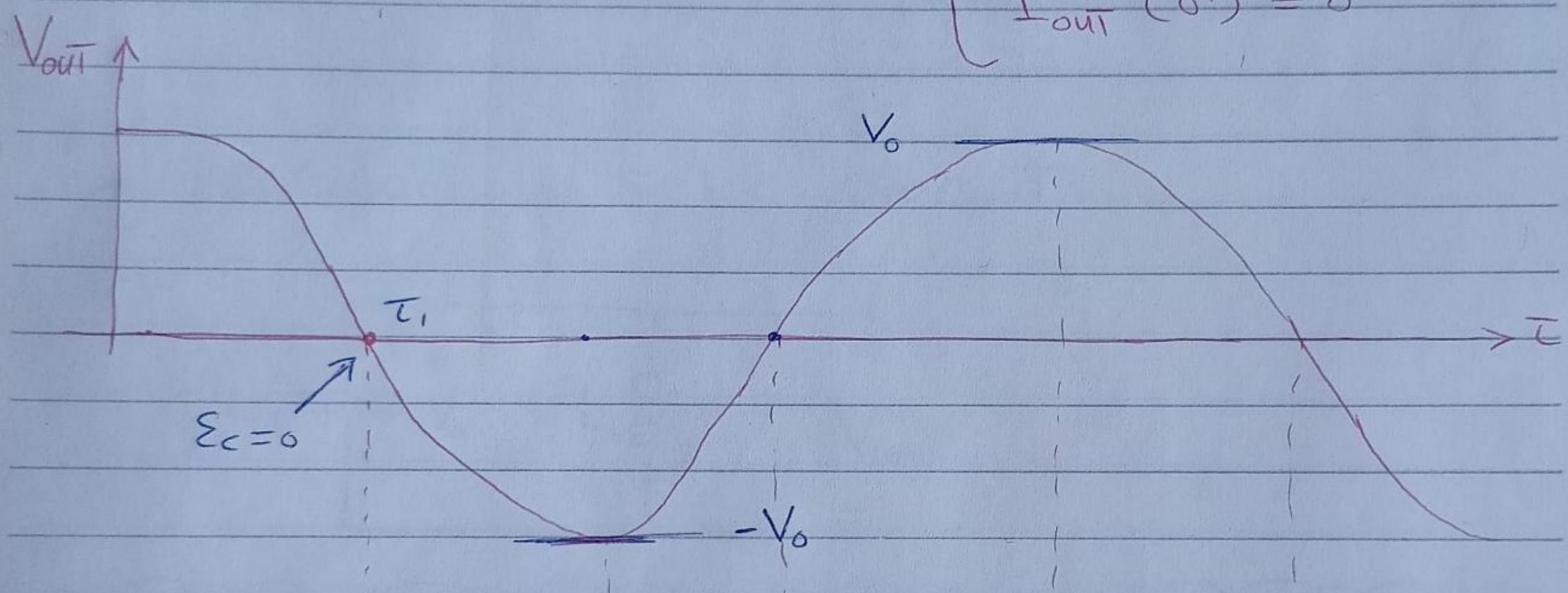
- simple LC circuit (Parallel LC Tank)

$I_{out} = -C_1 \frac{dV_{out}}{d\tau}$

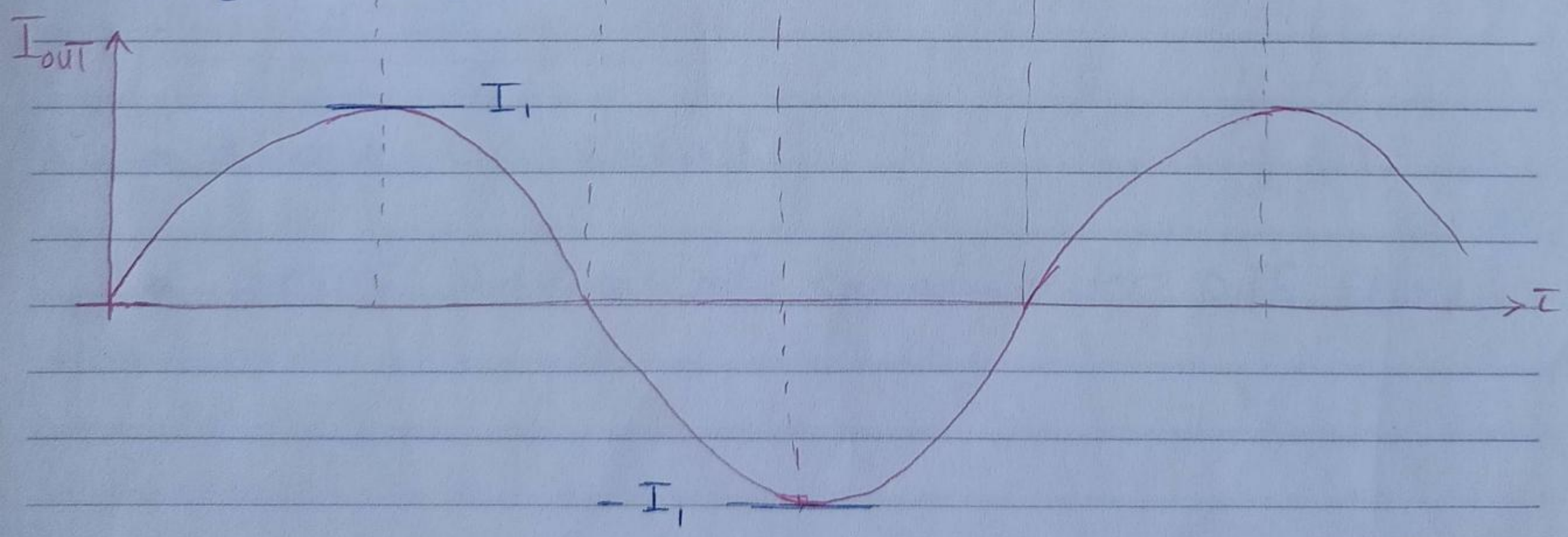


$$V_{out} = L_1 \frac{dI_{out}}{d\tau} = L_1\left(-C_1 \frac{d^2V_{out}}{d\tau^2}\right)$$

$$\begin{cases} L_1 C_1 \dfrac{d^2 V_{out}}{d\tau^2} + V_{out} = 0 \\ V_{out}(0^+) = V_0 \\ I_{out}(0^+) = 0 \end{cases}$$



$$\mathcal{E}_L = \frac{1}{2} L I_1^2 = \frac{1}{2} C_1 V_0^2$$



- <u>observations:</u>

at $\tau = \tau_1$, $dV_{out}/d\tau < 0 \Rightarrow I_{out} > 0$

$dI_{out}/d\tau = 0 \Rightarrow V_{out} = 0$

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- <u>second order Diff. Eqs.:</u>

$$\begin{cases} a\dfrac{d^2y}{d\tau^2} + b\dfrac{dy}{d\tau} + c = 0 \\ y(0) = K_1 \\ \dfrac{dy}{d\tau}(0) = K_2 \end{cases}$$

$$y(\tau) = Ae^{s\tau}$$

$$as^2Ae^{s\tau} + bsAe^{s\tau} + cAe^{s\tau} = 0$$

$$\boxed{as^2 + bs + c = 0}$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Guess:

$$y(\tau) = A_1 e^{s_1\tau} + A_2 e^{s_2\tau}$$

$$y(0) = K_1 = A_1 + A_2$$

$$\frac{dy}{d\tau}(0) = K_2 = A_1 s_1 e^{s_1\tau} + A_2 s_2 e^{s_2\tau} = A_1 s_1 + A_2 s_2$$

$$as^2 + bs + c = 0 \quad , \quad s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three cases:

① $s_1 \neq s_2$, both are real $(b^2 - 4ac > 0)$

② $s_1 = s_2 = \dfrac{-b}{2a}$ $(b^2 = 4ac)$

③ $s_1 \neq s_2$, both are complex $(b^2 - 4ac < 0)$

Solutions for $y(\tau)$:

Case 1: $y(\tau) = A_1 e^{s_1\tau} + A_2 e^{s_2\tau}$, $K_1 = A_1 + A_2$

$K_2 = A_1 s_1 + A_2 s_2$

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Case 2:

$$y(\tau) = A_1 e^{s_1\tau} + A_2 \tau e^{s_1\tau}$$

$$y(0) = K_1 = A_1$$

$$\frac{dy}{d\tau}(0) = K_2 = A_1 s_1 e^{s_1\tau} + A_2 e^{s_1\tau} + A_2 s_1 \tau e^{s_1\tau} \Big|_{\tau=0}$$
$$= A_1 s_1 + A_2$$

Case 3:

$$s_{1,2} = \underbrace{\frac{-b}{2a}}_{\alpha} \pm j \underbrace{\frac{\sqrt{4ac-b^2}}{2a}}_{\omega}$$

$$y(\tau) = A_1 e^{\alpha\tau} \cos \omega\tau + A_2 e^{\alpha\tau} \sin \omega\tau$$

$$y(0) = K_1 = A_1$$

$\frac{dy}{d\tau} = K_2 =$ Take The derivative and find it at $\tau = 0$.

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Lec 29: Ideal and Lossy LC Tanks

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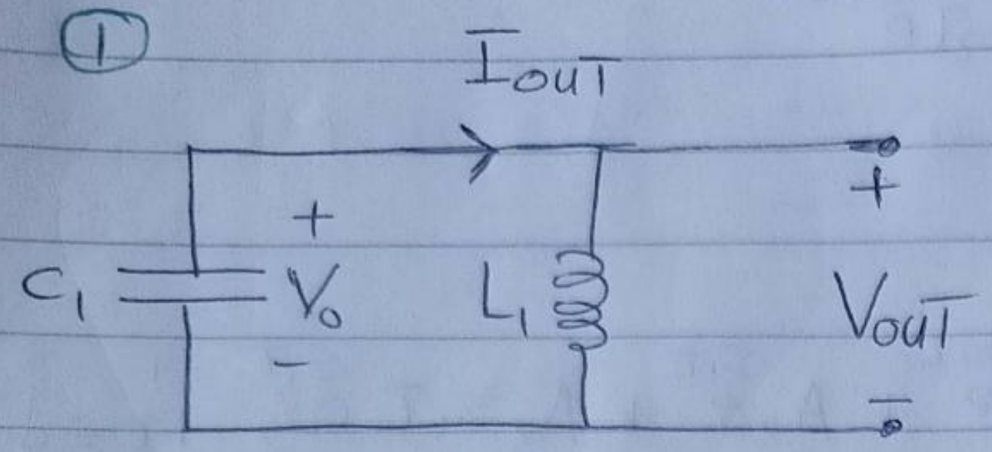
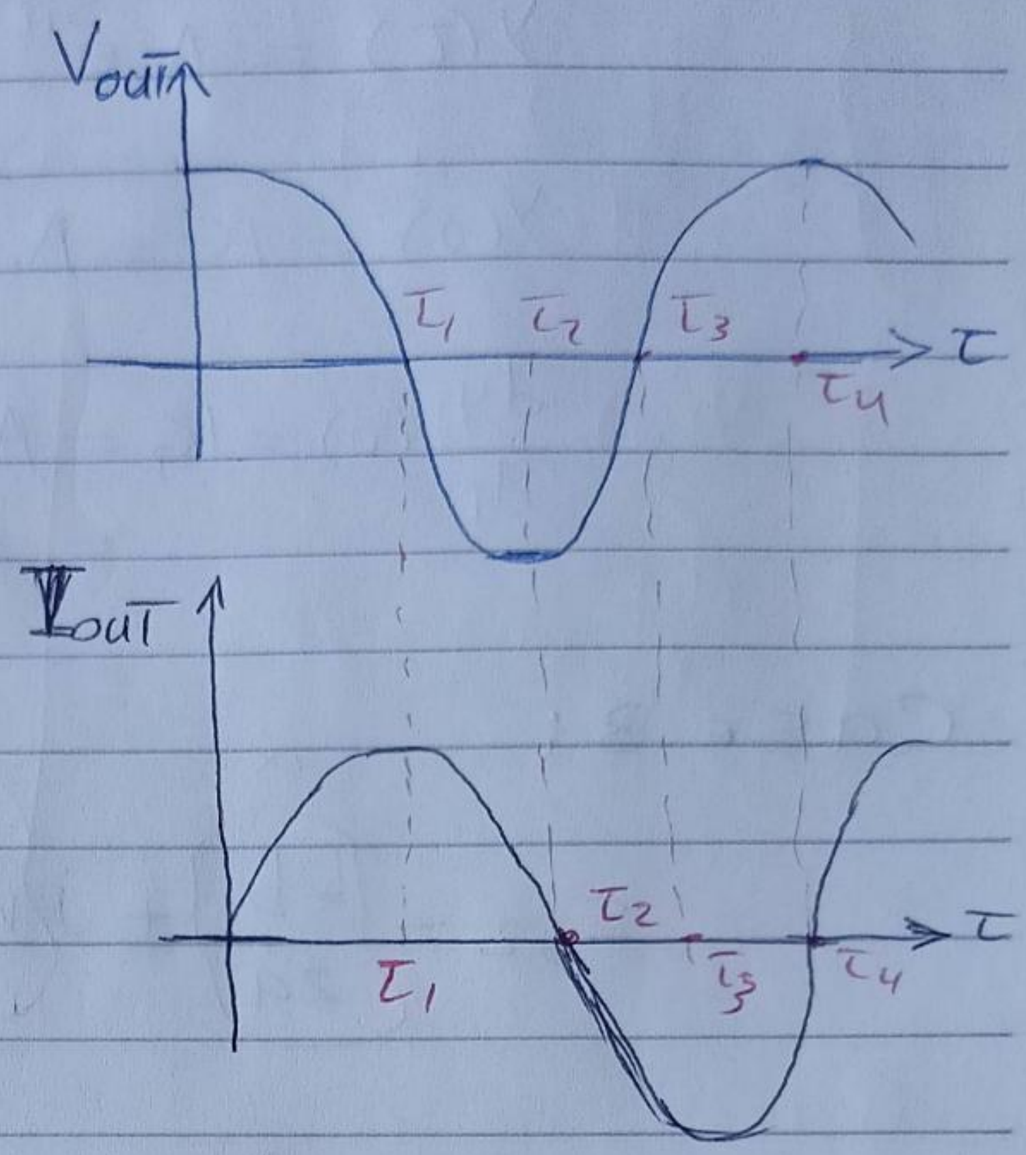
Review of lecture 27

①





② second order Diff. Eqs.

$$\begin{cases} a\dfrac{d^2y}{d\tau^2} + b\dfrac{dy}{d\tau} + cy = 0 \Rightarrow as^2 + bs + c = 0 \\ y(0) = K_1 \\ y'(0) = K_2 \end{cases}$$

$$\begin{cases} b^2 - 4ac > 0 \Rightarrow y(\tau) = A_1 e^{s_1\tau} + A_2 e^{s_2\tau} \\ b^2 - 4ac = 0 \Rightarrow y(\tau) = A_1 e^{s_1\tau} + A_2 \tau e^{s_1\tau} \\ b^2 - 4ac < 0 \Rightarrow y(\tau) = A_1 e^{\alpha\tau}\cos w\tau + A_2 e^{\alpha\tau}\sin w\tau \end{cases}$$

$$s_{1,2} = \underbrace{\frac{-b}{2a}}_{\alpha} \pm j\underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{w_1}$$

Today's Lecture:

- Ideal LC Tanks
- Lossy LC Tanks

$$\begin{cases} A_1 e^{s_1\tau} + A_2 e^{s_2\tau} \\ A_1 e^{s_1\tau} + A_2 \tau e^{s_1\tau} \\ A_1 e^{\alpha\tau}\cos w_1\tau + A_2 e^{\alpha\tau}\sin w_1\tau \end{cases}$$

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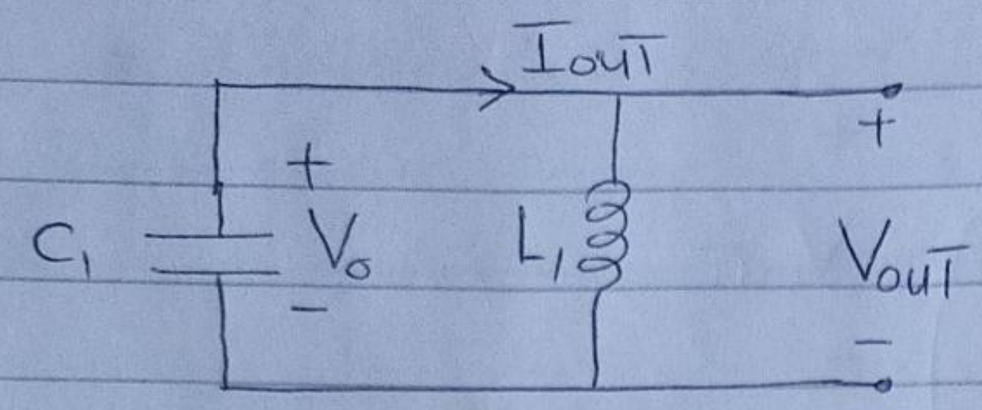
• Ideal LC Tank:



$V_{OUT}(0^+) = V_0$

$I_L(0^+) = 0$

$$I_{OUT} = -C_1 \frac{dV_{OUT}}{d\tau} \quad , \quad V_{OUT} = L_1 \frac{dI_{OUT}}{d\tau} \Rightarrow$$

$$V_{OUT} = L_1 \left(-C_1 \frac{d^2 V_{OUT}}{d\tau^2}\right) \Rightarrow L_1 C_1 \frac{d^2 V_{OUT}}{d\tau^2} + V_{OUT} = 0$$

$$, \; L_1 C_1 s^2 + 1 = 0 \Rightarrow s_{1,2} = \pm\sqrt{\frac{-1}{L_1 C_1}} \Rightarrow s_{1,2} = \pm \frac{j}{\sqrt{L_1 C_1}}$$

$$\Rightarrow V_{OUT}(\tau) = A_1 \cos W_1 \tau + A_2 \sin W_1 \tau$$

To find A_1, A_2:

$$V_{OUT}(0) = A_1 = V_0$$

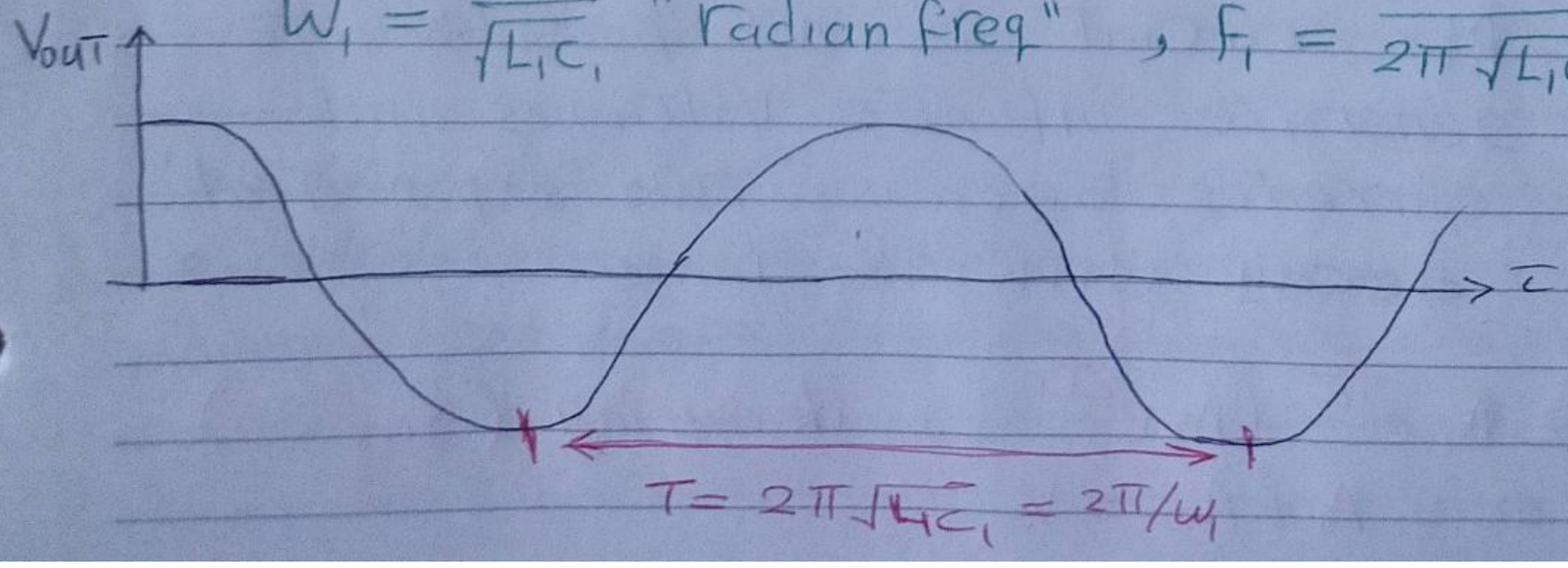
$$I_L = -I_C = -C_1 \frac{dV_{OUT}}{d\tau} = 0 \quad \text{at } \tau = 0$$

$$\because \frac{dV_{OUT}}{d\tau} = 0 \Rightarrow -A_1 W_1 \sin W_1 \tau + A_2 W_2 \sin W_2 \tau = 0$$

$$\boxed{\text{at } \tau = 0^+} \Rightarrow A_2 W_1 = 0 \Rightarrow A_2 = 0$$

$$V_{OUT}(\tau) = V_0 \cos W_1 \tau = V_0 \cos \frac{1}{\sqrt{L_1 C_1}} \tau \qquad T = \frac{2\pi}{W_1} = 2\pi\sqrt{L_1 C_1}$$

$$W_1 = \frac{1}{\sqrt{L_1 C_1}} \text{ "radian freq"} \quad , \quad f_1 = \frac{1}{2\pi\sqrt{L_1 C_1}} \text{ (Hz)}$$



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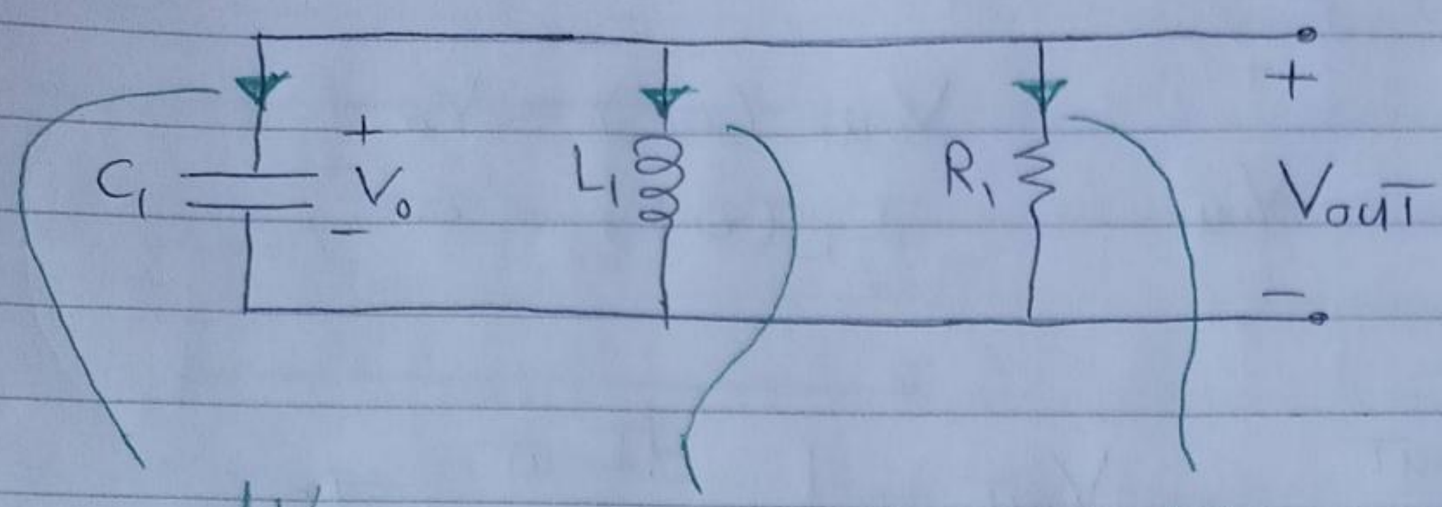
• Lossy LC Tank:



$$C_1 \frac{dV_{OUT}}{d\tau} + \frac{1}{L}\int V_{OUT}\, d\tau + \frac{V_{OUT}}{R_1} = 0$$

$$C_1 \frac{d^2 V_{OUT}}{d\tau^2} + \frac{1}{R_1}\frac{dV_{OUT}}{d\tau} + \frac{1}{L_1} V_{OUT} = 0$$

$$\Downarrow$$

$$\begin{cases} V_{OUT}(0^+) = V_0 \\ I_L(0^+) = 0 \end{cases}$$

$$C_1 s^2 + \frac{1}{R_1} s + \frac{1}{L_1} = 0$$

$$s^2 + \frac{1}{R_1 C_1} s + \frac{1}{L_1 C_1} = 0$$

$$\therefore s_{1,2} = \frac{-1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1^2C_1^2}}$$

Case 1:

$$\frac{1}{R_1^2 C_1^2} > \frac{4}{L_1^2 C_1^2} \Rightarrow V_{OUT} = A_1 e^{s_1\tau} + A_2 e^{s_2\tau}$$

$s_1, s_2 < 0 \Rightarrow$ Two exponentials decay

Note: RLC circuit is not possible to generate infinite voltages and currents because two exponential decay with the time.

$V_{OUT} = A_1 e^{s_1\tau} + A_2 e^{s_2\tau}$, decaying $(s_{1,2} < 0)$

$V_{OUT}(0) = A_1 + A_2 = V_0$

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Lec 30: Overdamped, Critically-Damped, and Underdamped Circuits

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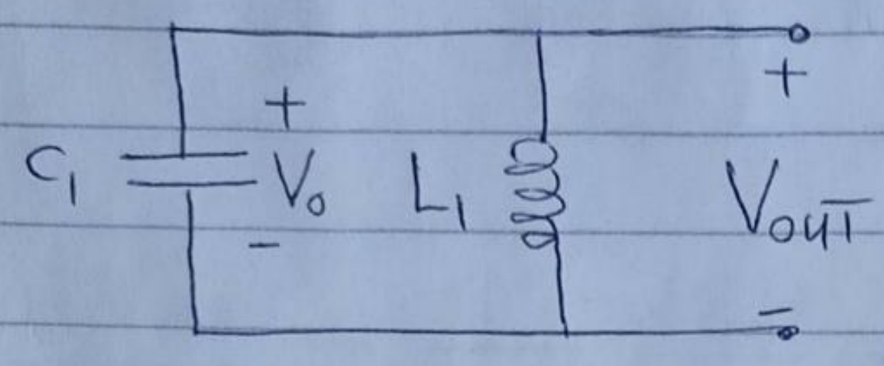
Review of Lecture 29

- Ideal LC Tank:

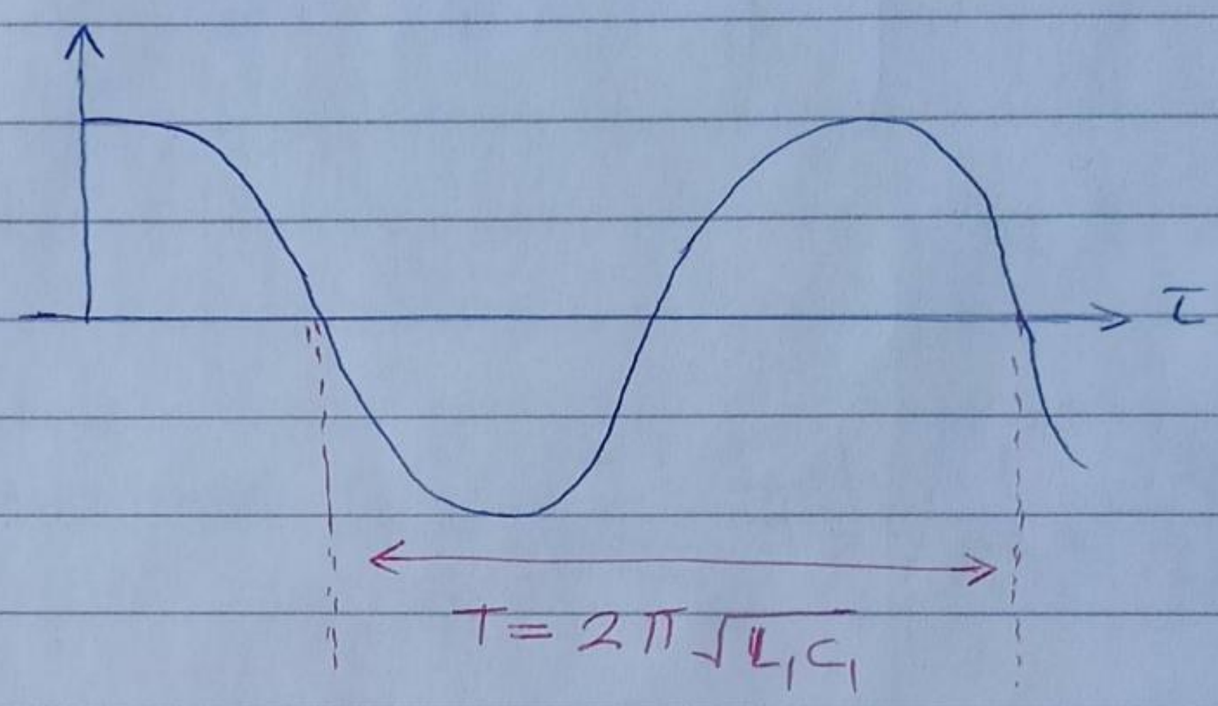


$$\begin{cases} L_1 C_1 \dfrac{dV_{OUT}}{d\tau} + V_{OUT} = 0 \\ V_{OUT}(0^+) = V_0 \\ I_L(0^+) = 0 \end{cases}$$

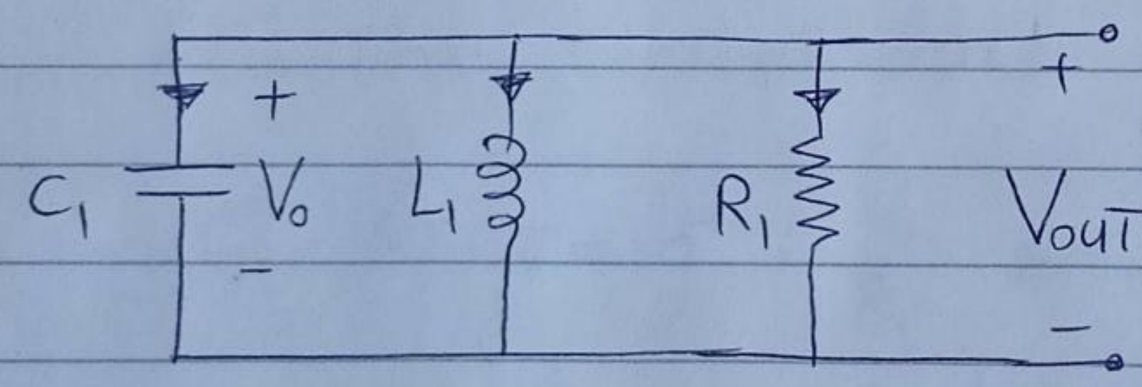
$$L_1 C_1 s^2 + 1 = 0$$

$$V_{OUT} = V_0 \cos \omega_1 \tau$$

$$\omega_1 = \frac{1}{\sqrt{L_1 C_1}}$$



- Lossy LC Tank:



$$C_1 \frac{d^2 V_{OUT}}{d\tau^2} + \frac{1}{R_1}\frac{dV_{OUT}}{d\tau} + \frac{1}{L_1} V_{OUT} = 0$$

$$s^2 + \frac{1}{R_1 C_1} s + \frac{1}{L_1 C_1} = 0$$

$$s_{1,2} = \frac{-1}{2R_1C_1} \pm \sqrt{\frac{1}{R_1^2 C_1^2} - \frac{4}{L_1^2 C_1^2}}$$

$$\boxed{s_{1,2} < 0}$$

Todays lecture

- Lossy LC Tank
- overdamped, crit. damped and overdamped response

① $A_1 e^{s_1 \tau} + A_2 e^{s_2 \tau}$

② $A_1 e^{s_1 \tau} + A_2 \tau e^{s_2 \tau}$

③ $A_1 e^{\alpha \tau} \cos \omega_1 \tau + A_2 e^{\alpha \tau} \sin \omega_1 \tau$

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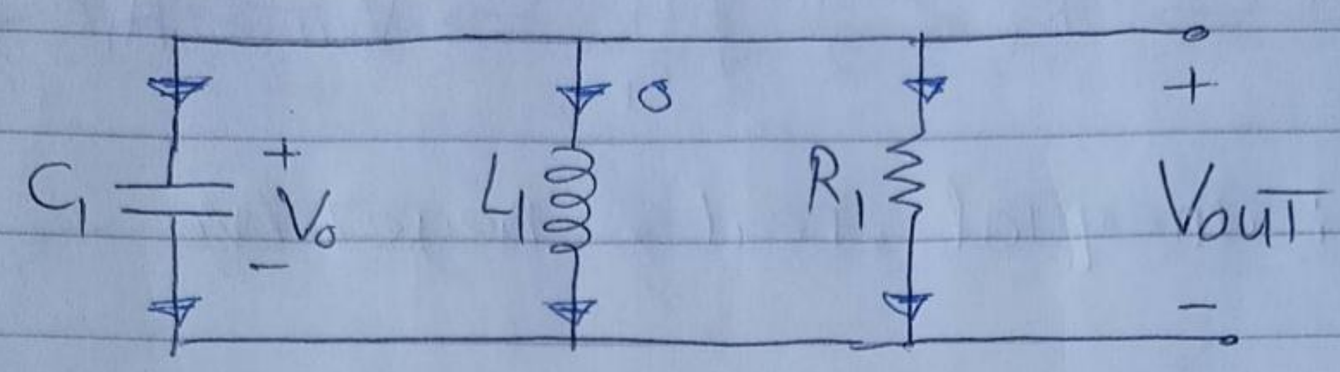
• Lossy LC Tank:



$$\begin{cases} \dfrac{d^2 V_{OUT}}{d\tau^2} + \dfrac{1}{R_1 C_1}\dfrac{dV_{OUT}}{d\tau} + \dfrac{1}{L_1 C_1} V_{OUT} = 0 \\ V_{OUT}(0^+) = V_0 \\ I_{L_1}(0^+) = 0 \end{cases}$$

$$s^2 + \frac{1}{R_1 C_1} s + \frac{1}{L_1 C_1} = 0 \Rightarrow S_{1,2} = \frac{-1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2 C_1^2} - \frac{4}{L_1 C_1}}$$

observations:

① The circuit has an order of 2.

② AT $\tau = 0^+$: $\dfrac{V_0}{R_1} + C_1 \left.\dfrac{dV_{OUT}}{d\tau}\right|_{\tau=0^+} = 0$

$$\Rightarrow \left.\frac{dV_{OUT}}{d\tau}\right|_{0^+} = \frac{-V_0}{R_1 C_1}$$

③ unit of S_1, S_2 : $\dfrac{1}{\text{time}}$ $\Rightarrow$ The larger $|S|$ is, The smaller corresponding "time" is

④ The exponentials in all responses decay

$\Rightarrow V_{OUT}$ always $\rightarrow 0$

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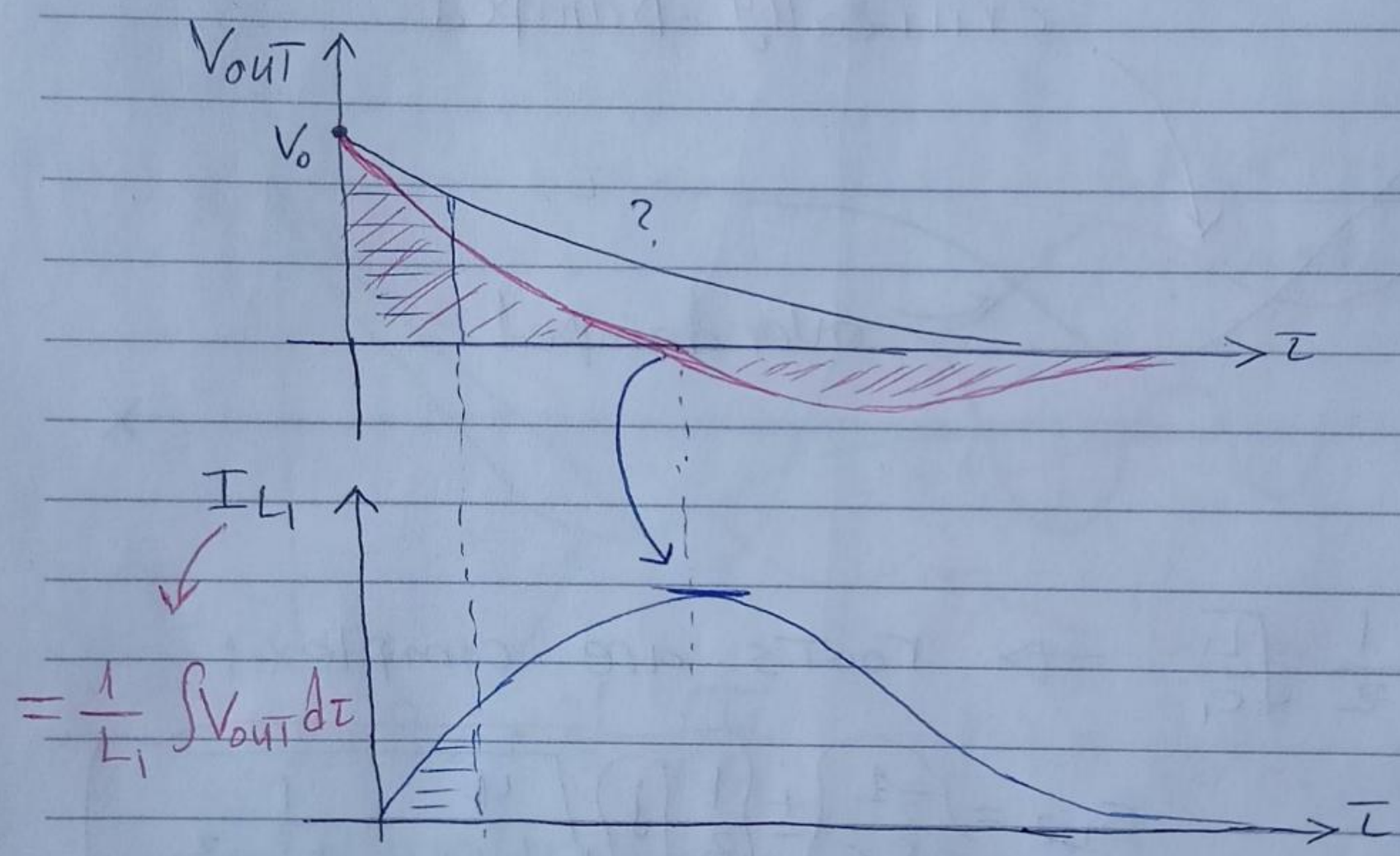
Case 1 :

$$\frac{1}{R_1^2 C_1^2} - \frac{4}{L_1 C_1} > 0 \Rightarrow R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Rightarrow V_{OUT} = A_1 e^{s_1 \tau} + A_2 e^{s_2 \tau}$$

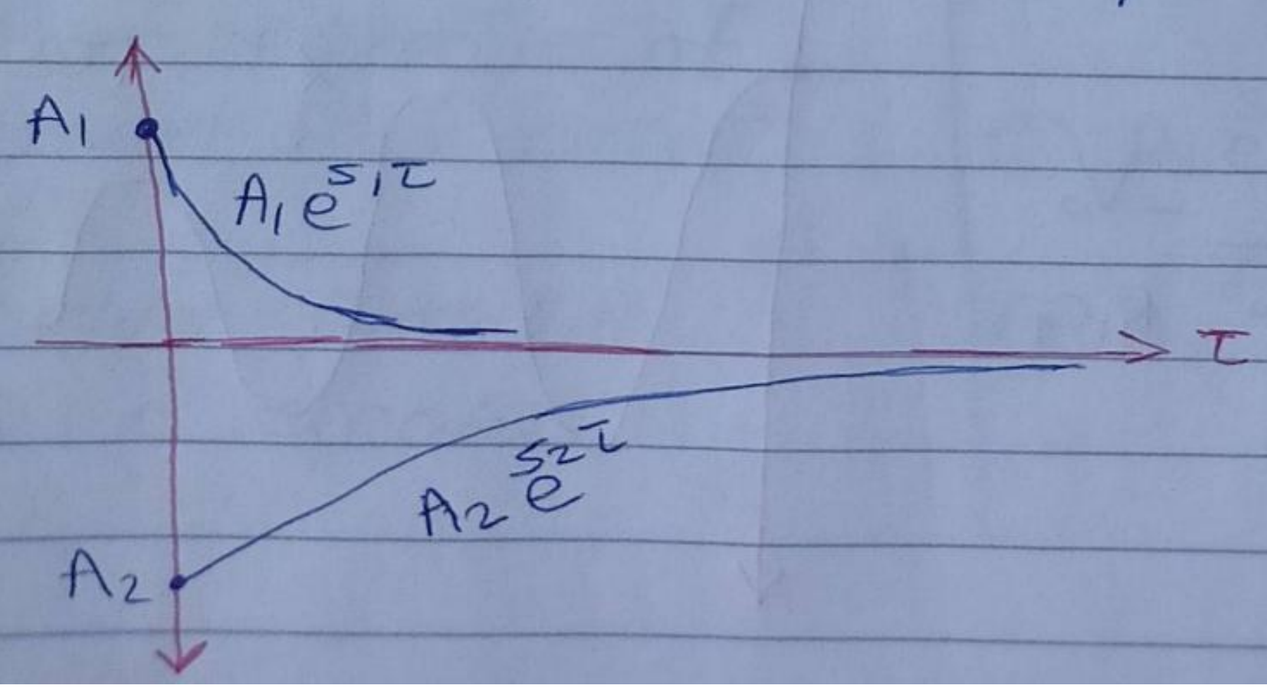
, s_1 & s_2 : unequal, real and negetive.

$$\begin{cases} A_1 + A_2 = V_0 \\ A_1 s_1 + A_2 s_2 = \dfrac{-V_0}{R_1 C_1} \end{cases} \Rightarrow \text{we can Find } A_1 \text{ \& } A_2$$



- interesting point : $A_1 > 0$ & $A_2 < 0$

$e^{s_1 \tau}$: is a "fast" exponential.
$e^{s_2 \tau}$: is a "slow" exponential.



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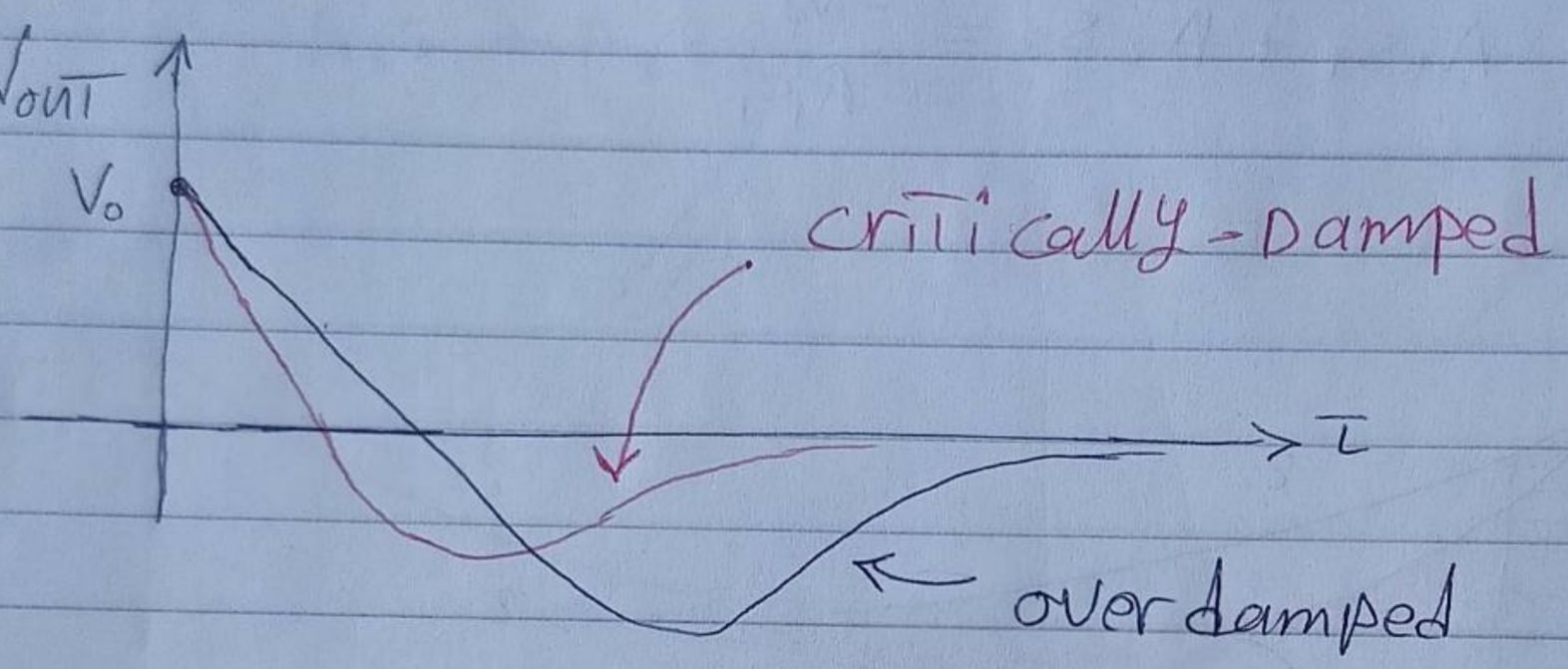
Case 2:

$$R_1 = \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Rightarrow V_{OUT} = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \quad s_{1,2} = \frac{-1}{R_1 C_1}$$

crit. damped

$$\begin{cases} A_1 = V_0 \\ A_1 s_1 + A_2 = 0 \end{cases}$$



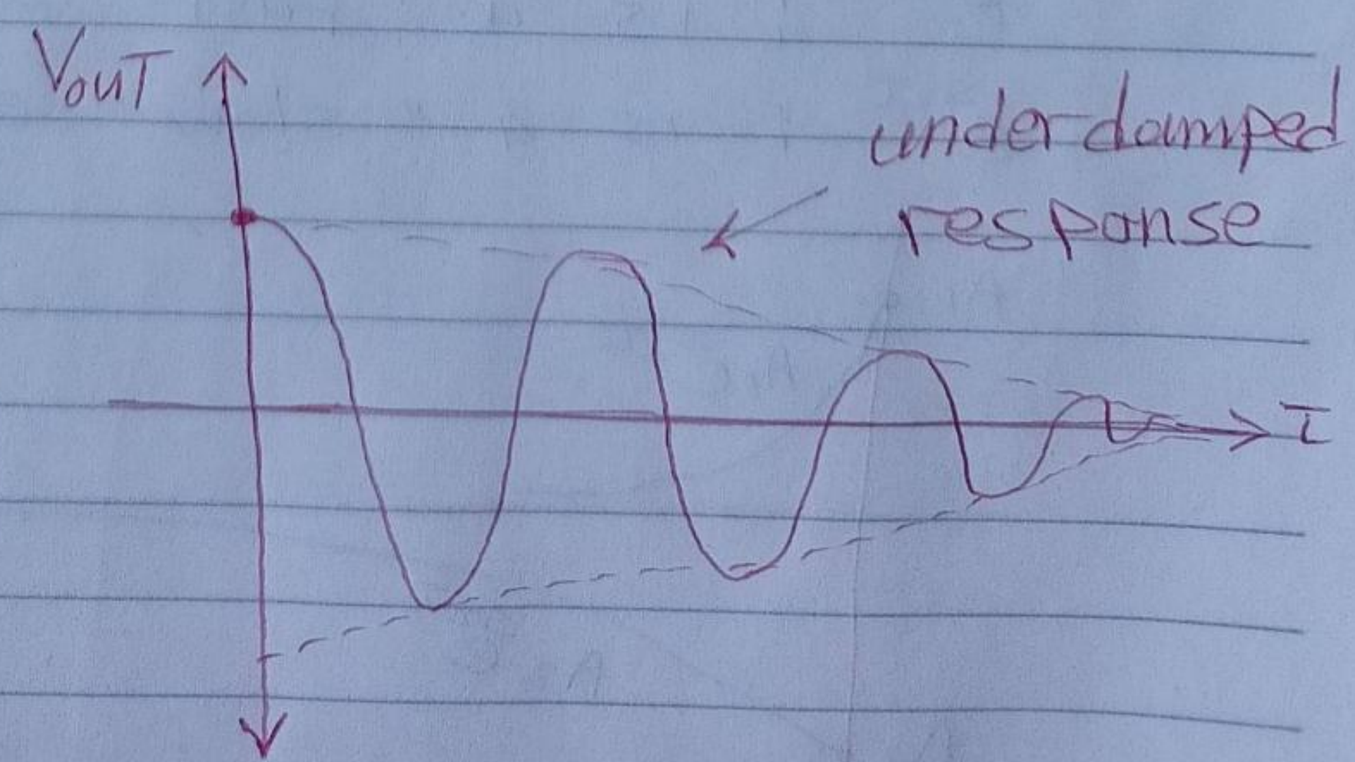
Case 3:

$$R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Rightarrow \text{roots are complex:}$$

$$s_{1,2} = \underbrace{\frac{-1}{2R_1C_1}}_{\alpha} \pm \frac{1}{2} j \underbrace{\sqrt{\frac{4}{L_1C_1} - \frac{1}{R_1^2C_1^2}}}_{\omega_1}$$

$$V_{OUT}(t) = A_1 e^{\alpha t}\cos\omega_1 t + A_2 e^{\alpha t}\sin\omega_1 t$$

$$\begin{cases} A_1 = V_0 \\ \left.\frac{dV_{OUT}}{dt}\right|_{t=0^+} = \frac{-V_0}{R_1C_1} \end{cases}$$



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Lec 31: Driven Parallel RLC Circuit (Part 1)

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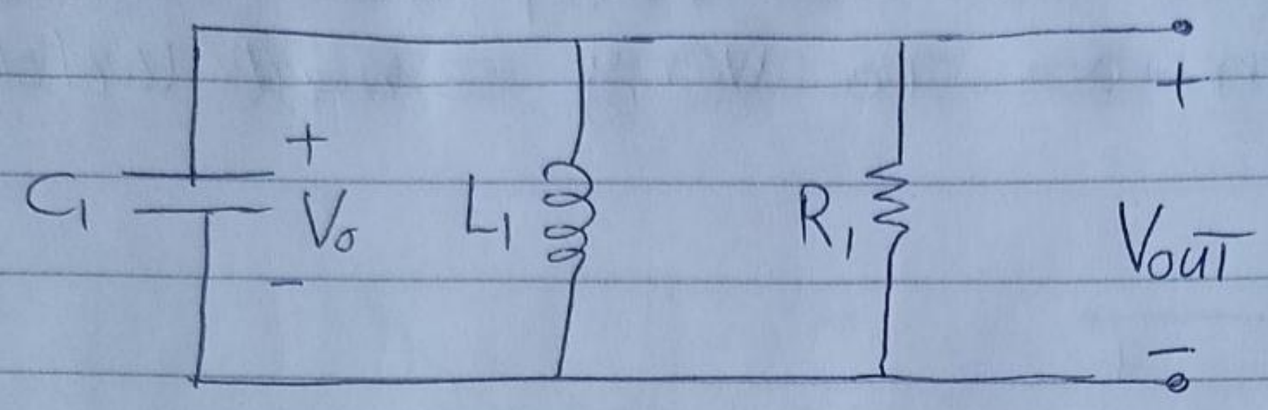
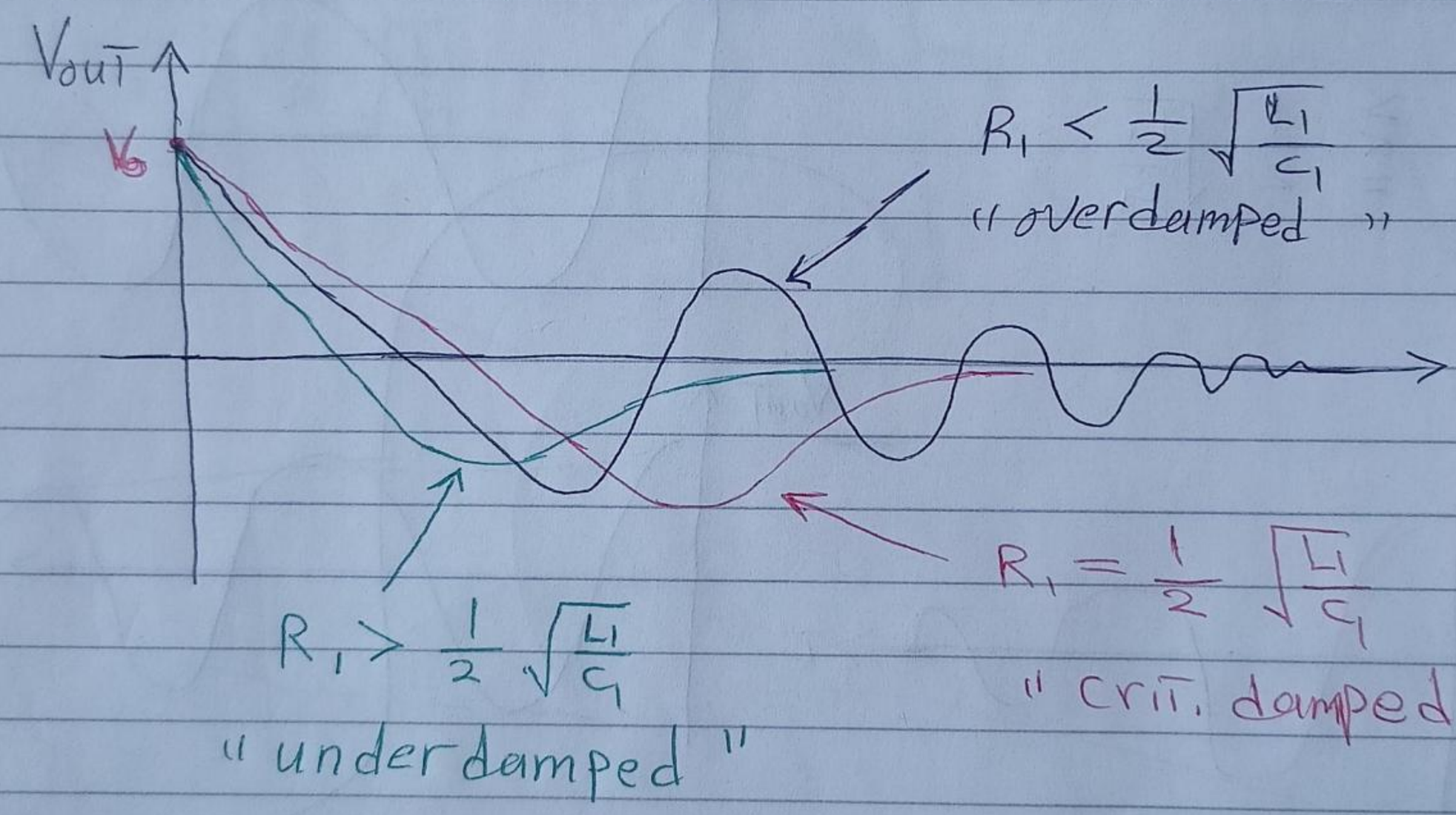
Review of lecture 30

- Lossy Tank:



$$s_{1,2} = \frac{-1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1C_1}}$$



Today's lecture:

- other properties of parallel RLC circuit.
- Driven parallel RLC circuit.

① $A_1 e^{s_1 t} + A_2 e^{s_2 t}$

② $A_1 e^{s_1 t} + A_2 t e^{s_2 t}$

③ $A_1 e^{\alpha t}\cos\omega_1 t + A_2 e^{\alpha t}\sin\omega_1 t$

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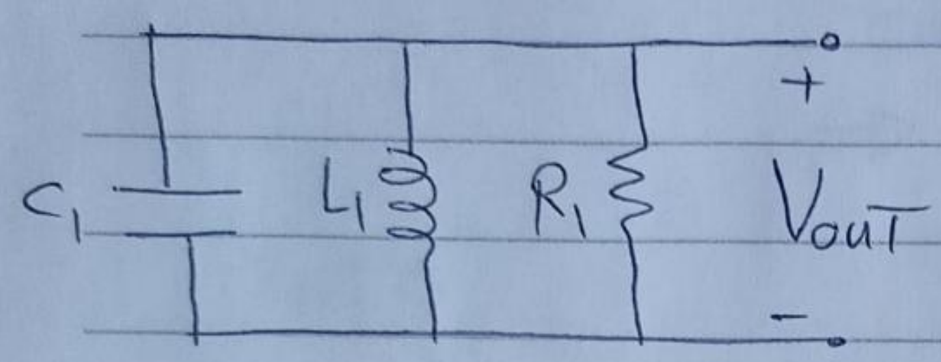
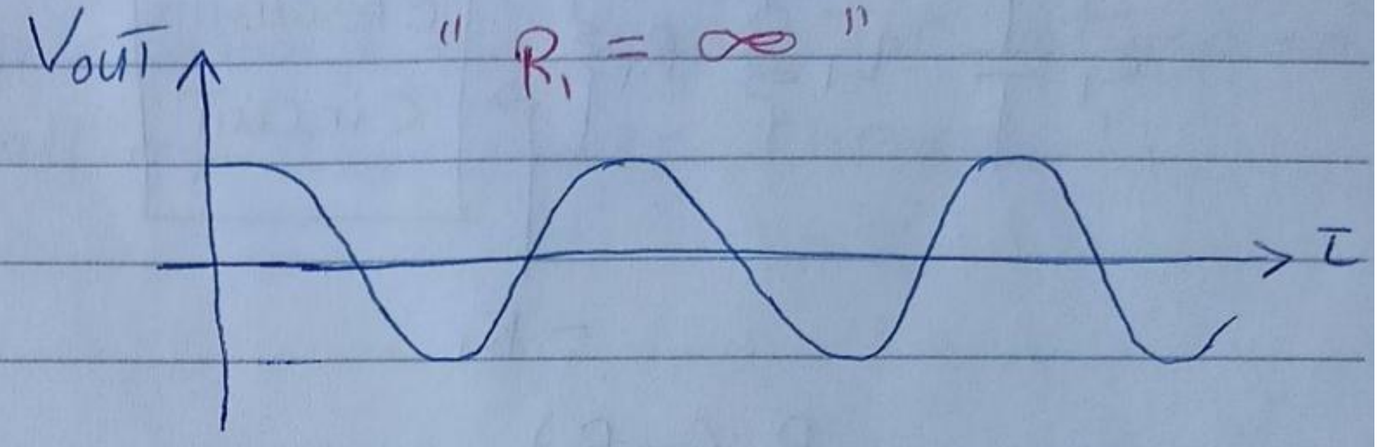
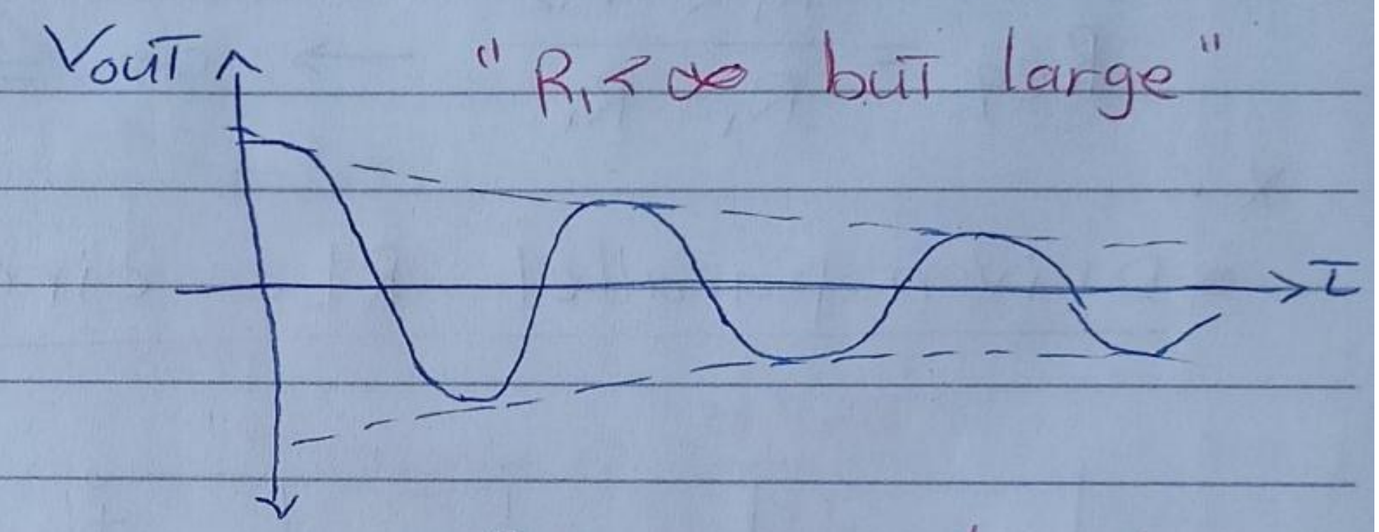
- other properties of parallel RLC circuit:

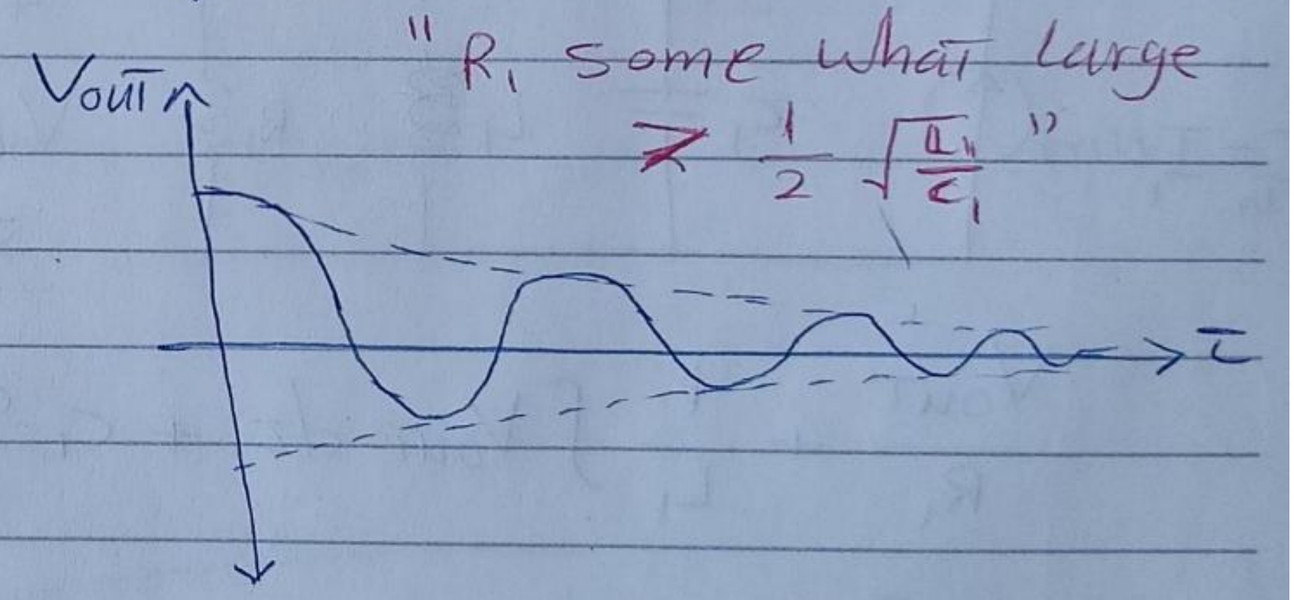
Q: what happens to the response as R_1 goes from ∞ to very small values?

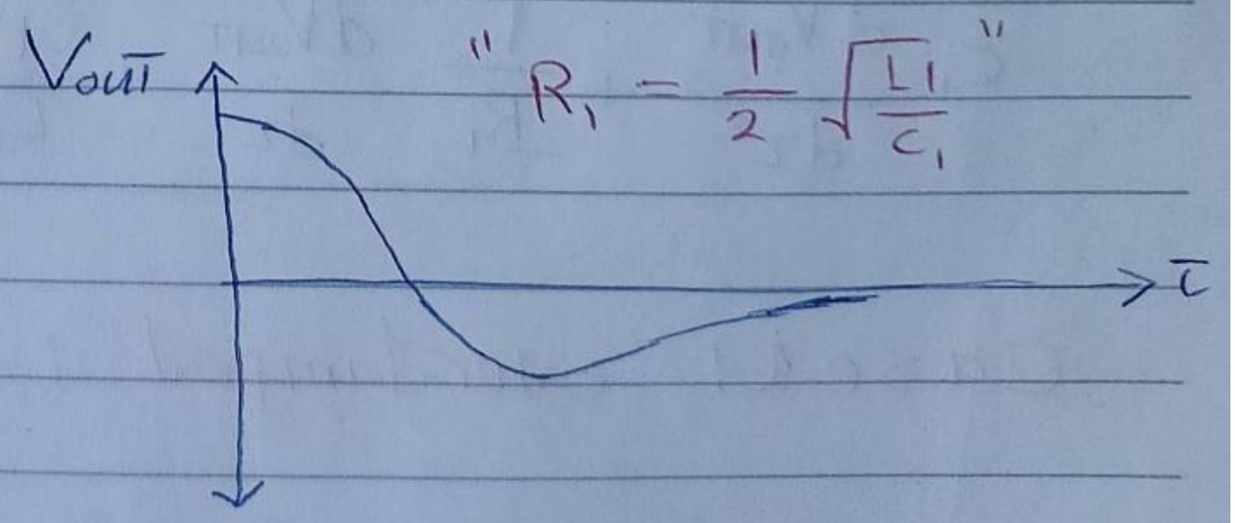


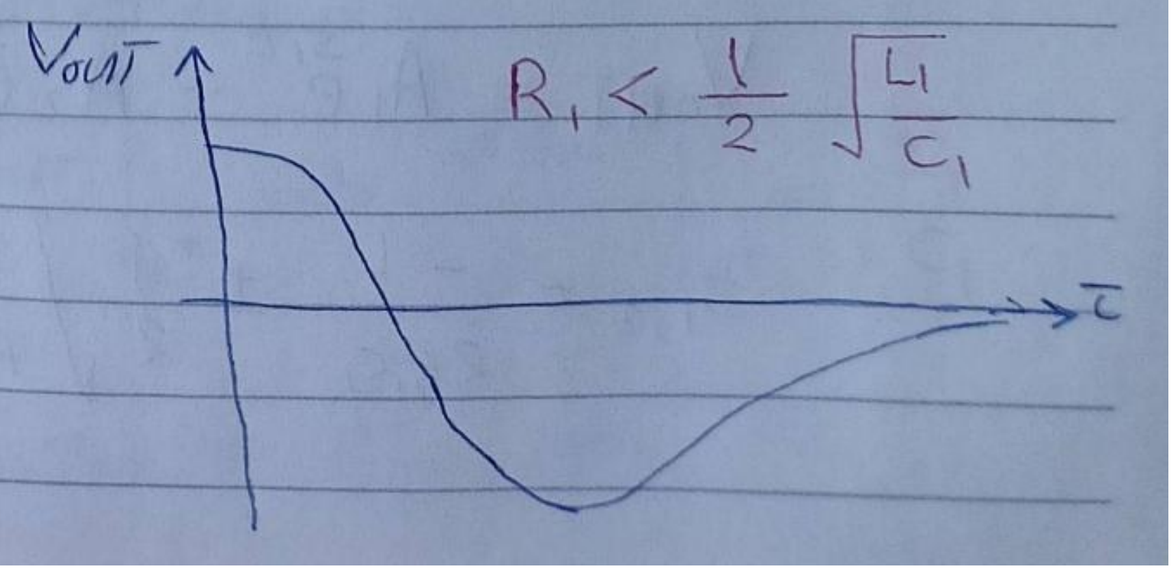


$$R_1 \gtreqless \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$$









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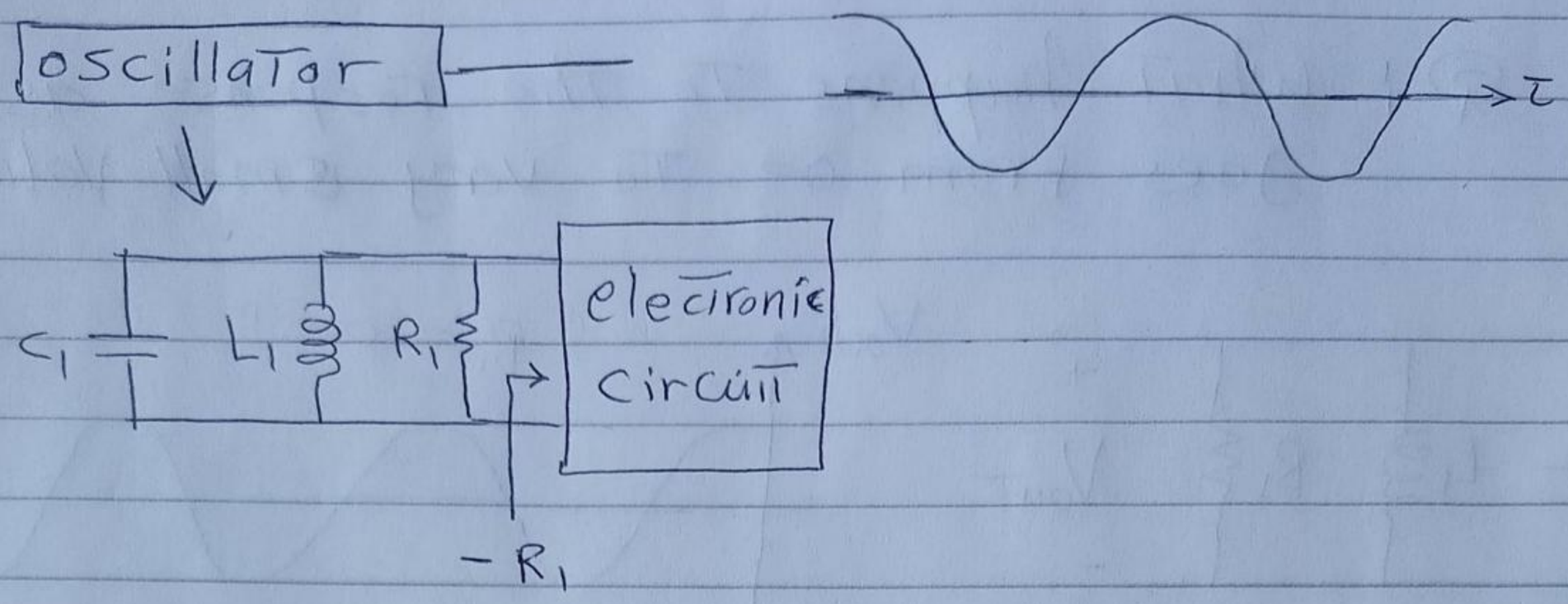
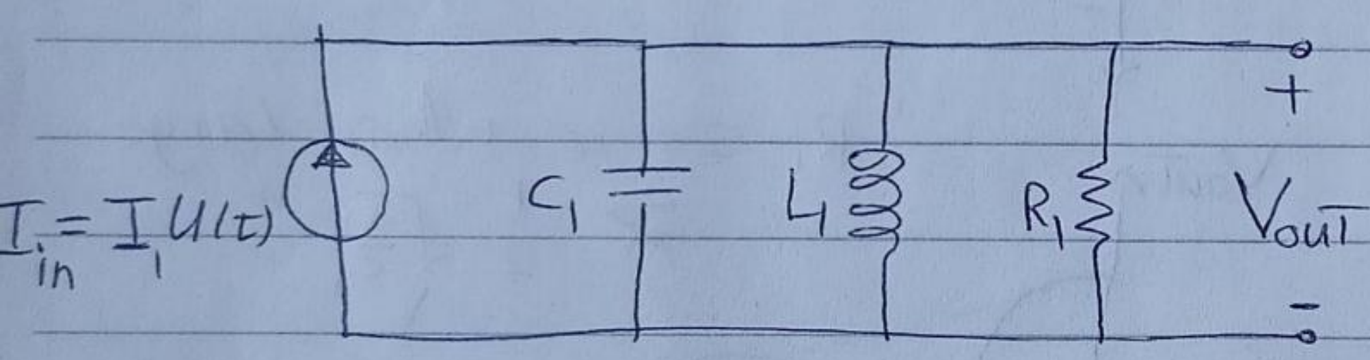
Example



$$R_{eq} = \frac{R_1(-R_1)}{R_1 - R_1} \longrightarrow \infty \Rightarrow \text{open circuit}$$

- **Driven parallel RLC circuit:**



$$I_{in} = I_1 u(\tau)$$

$$\frac{V_{out}}{R_1} + \frac{1}{L_1}\int V_{out}\, d\tau + C_1 \frac{dV_{out}}{d\tau} = I_{in}$$

$$C_1 \frac{d^2 V_{out}}{d\tau^2} + \frac{1}{R_1}\frac{dV_{out}}{d\tau} + \frac{1}{L_1} V_{out} = \frac{dI_{in}}{d\tau} = I_1\, \delta(\tau)$$

Case 1: overdamped : $R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$

$$V_{out} = A_1 e^{s_1 \tau} + A_2 e^{s_2 \tau}$$

$$s_{1,2} = \frac{-1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2 C_1^2} - \frac{4}{L_1 C_1}}$$

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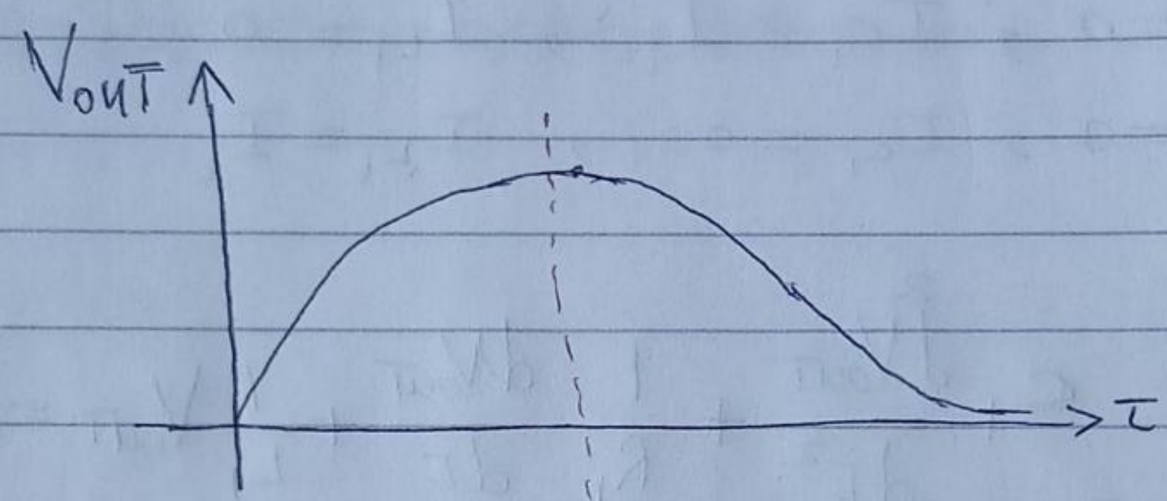
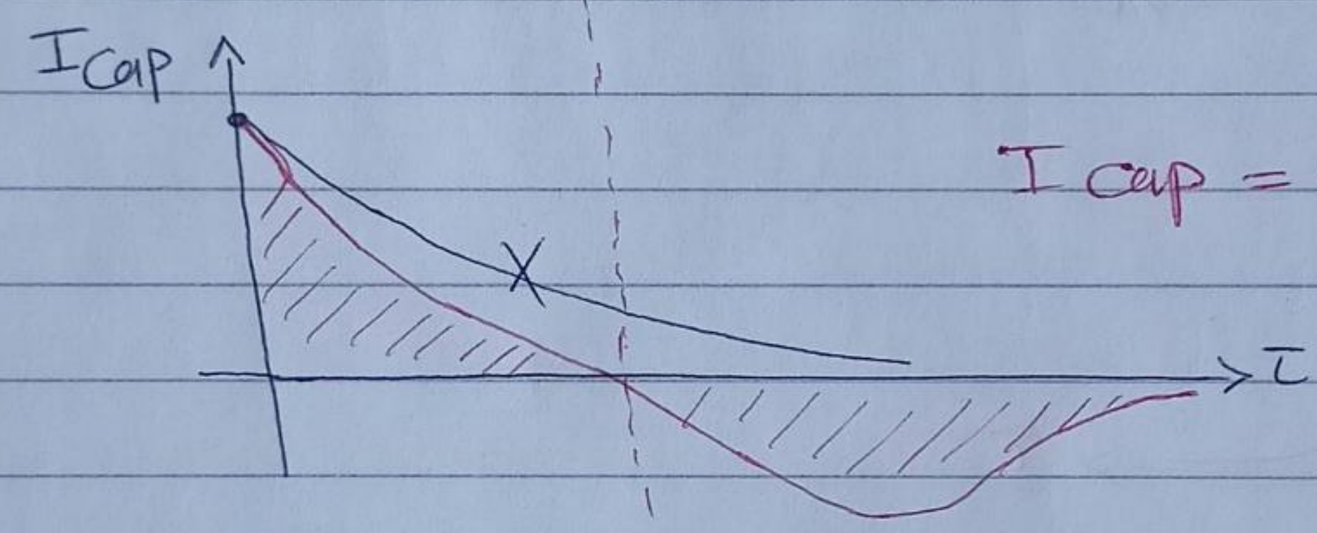
- observations:

① $V_{out}(0^+) = 0$
$I_L(0^+) = 0$

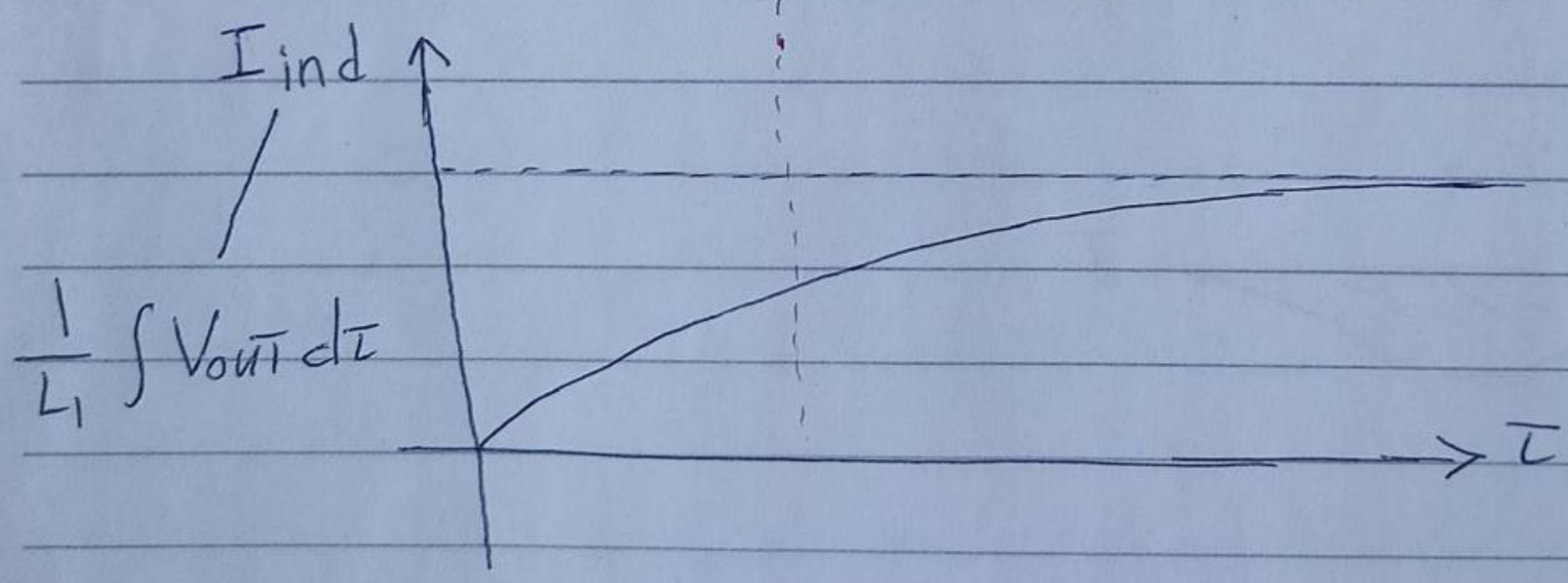
② All of $I_1 u(t)$ Flows Through C_1 at $t = 0^+$

③ At $t = \infty$, all currents and V_{out} are const $\Rightarrow$ cap current $= 0$, inductor Voltage $= 0$
$\Rightarrow$ all of I_{in} flows Through L_1





$$I_{cap} = C_1 \frac{dV_{out}}{dt}$$



$$V_{out}(0^+) = 0 \Rightarrow A_1 + A_2 = 0$$

At $t = 0^+$: $I_1 = C_1 \left.\frac{dV_{out}}{dt}\right|_{0^+} \Rightarrow \left.\frac{dV_{out}}{dt}\right|_{0^+} = \frac{I_1}{C_1}$

$$\Rightarrow A_1 S_1 + A_2 S_2 = \frac{I_1}{C_1} \quad \#$$

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Lec 32: Driven Parallel RLC Circuit (Part 2)

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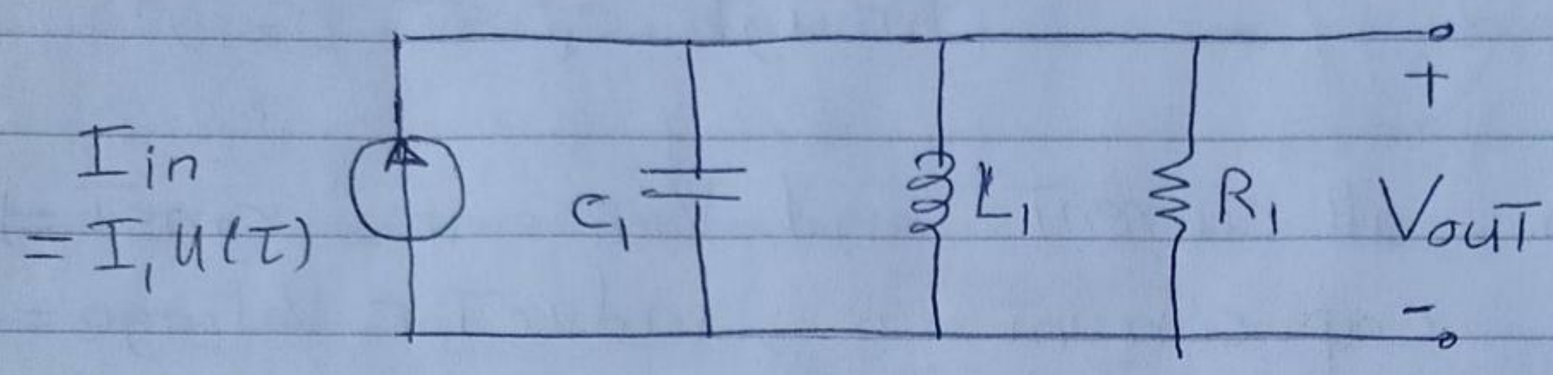
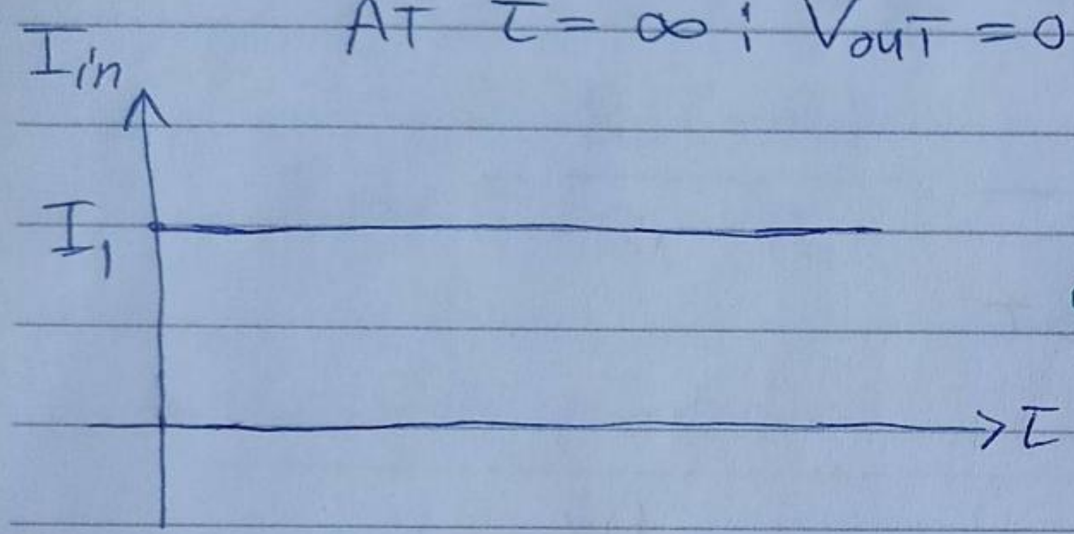
Review of Lecture 31

- Driven parallel RLC circuit.

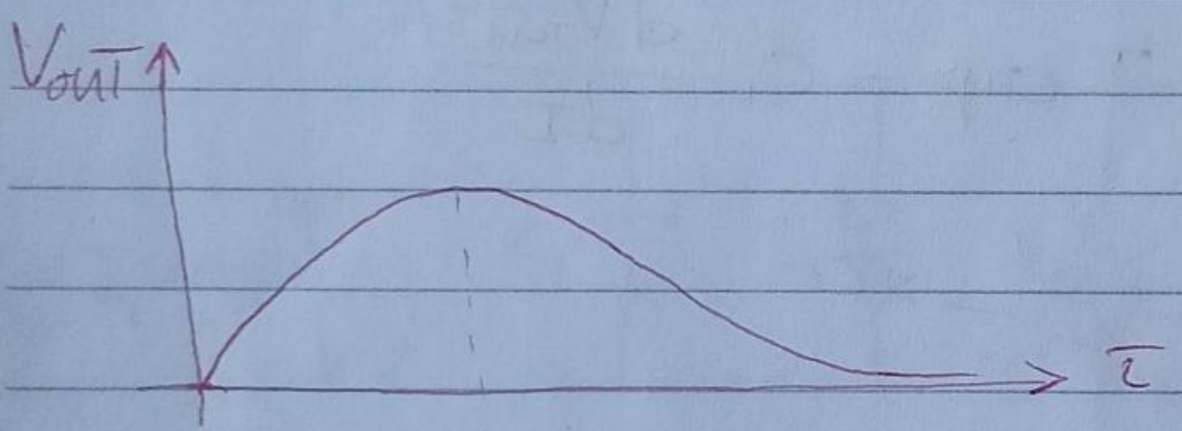


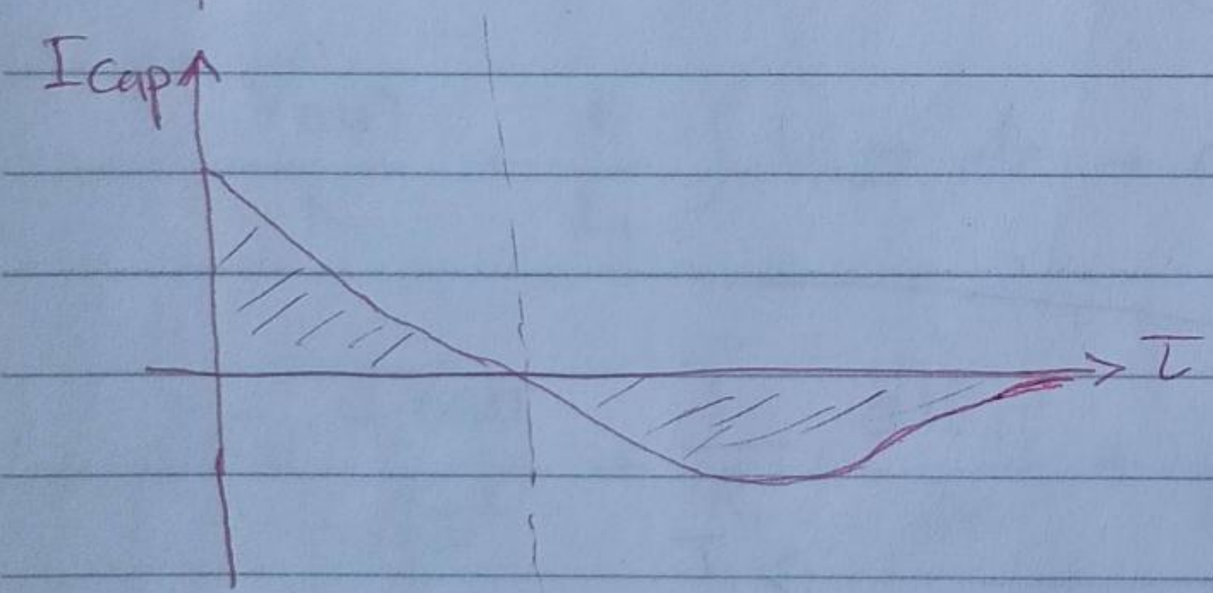
AT $t = 0^+$: $V_{OUT} = 0$, $I_{C_1} = I_1$, $I_{L_1} = 0$

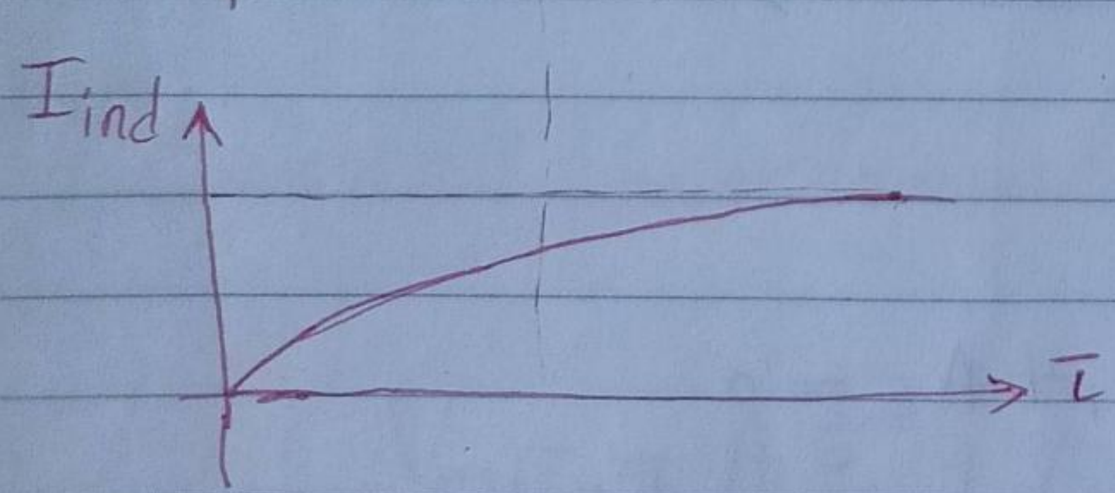
AT $t = \infty$: $V_{OUT} = 0$, $I_{C_1} = 0$, $I_{L_1} = I_1$



$$C_1 \frac{d^2 V_{OUT}}{dt^2} + \frac{1}{R_1} \frac{dV_{OUT}}{dt} + \frac{1}{L_1} V_{OUT} = I_{in}$$







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Todays Lecture

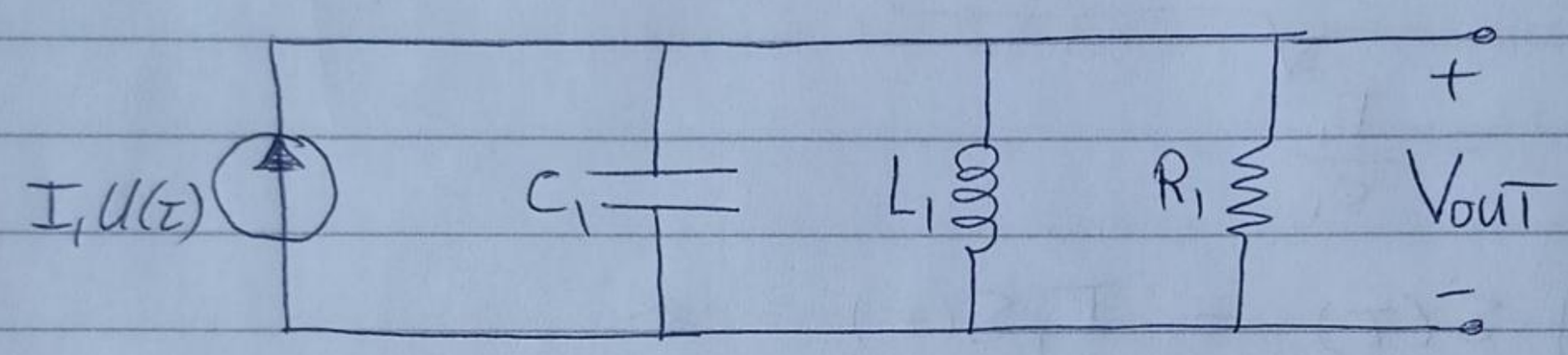
- Driven parallel RLC circuit ; overdamped, crit. damped, and underdamped responses

① $A_1 e^{s_1 t} + A_2 e^{s_2 t}$

② $A_1 e^{s_1 t} + A_2 t e^{s_1 t}$

③ $A_1 e^{\alpha t} \cos \omega_1 t + A_2 e^{\alpha t} \sin \omega_1 t$

- Driven parallel RLC circuit



① overdamped: $R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$

$$s_{1,2} = \frac{-1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1C_1}}$$

$$\begin{cases} V_{OUT} = (A_1 e^{s_1 t} + A_2 e^{s_2 t})\, u(t) \\ V_{OUT}(0^+) = 0 \implies \begin{cases} A_1 + A_2 = 0 \\ A_1 s_1 + A_2 s_2 = \frac{I_1}{C_1} \end{cases} \\ C_1 \left.\frac{dV_{OUT}}{dt}\right|_{0^+} = I_1 \end{cases}$$

verify Diff. eq. :

$$\frac{dV_{OUT}}{dt} = (A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t})\, u(t) + (A_1 s_1 e^{s_1 t} + A_2 e^{s_2 t})\, \delta(t)$$

$$\frac{d^2V_{OUT}}{dt^2} = (A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t})\, u(t) + (A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t})\, \delta(t)$$

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Lec 32, 3

$$C_1 \frac{d^2 V_{OUT}}{d\tau^2} + \frac{1}{R_1}\frac{dV_{OUT}}{d\tau} + \frac{1}{L_1} V_{OUT} =$$

$$= C_1\left[\cancel{A_1 S_1^2 e^{S_1\tau} + A_2 S_2^2 e^{S_2\tau}}\right]u(\tau) + C_1\left[A_1 S_1 e^{S_1\tau} + A_2 S_2 e^{S_2\tau}\right]\delta(\tau)$$

$$+ \frac{1}{R_1}\left[\cancel{A_1 S_1 e^{S_1\tau} + A_2 S_2 e^{S_2\tau}}\right]u(\tau) + \frac{1}{L_1}\left[\cancel{A_1 e^{S_1\tau} + A_2 e^{S_2\tau}}\right]u(\tau)$$

$$= C_1 \underbrace{\left[A_1 S_1 e^{S_1\tau} + A_2 S_2 e^{S_2\tau}\right]}_{=\frac{I_1}{C_1}} \delta(\tau) \overset{?}{=} I_1 \delta(\tau)$$

$$= C_1\left(\frac{I_1}{C_1}\right)\delta(\tau) = I_1\delta(\tau) \quad \#$$

Note: The other parts that have $u(\tau)$ will add up to zero because they add up the characteristic equation of circuit, so you should add up to zero.

$$I_{cap} = C_1\left[A_1 S_1 e^{S_1\tau} + A_2 S_2 e^{S_2\tau}\right]u(\tau)$$

$$I_{ind} = \frac{1}{L_1}\int V_{OUT}\, d\tau = \frac{1}{L_1}\left[\frac{A_1}{S_1} e^{S_1\tau} + \frac{A_2}{S_2} e^{S_2\tau}\right]u(\tau) + K$$

Inductor current $= 0$ at $\tau = 0^+$

$$\frac{1}{L_1}\left[\frac{A_1}{S_1} + \frac{A_2}{S_2}\right] + K = 0 \Rightarrow K = \frac{-1}{L_1}\left(\frac{A_1}{S_1} + \frac{A_2}{S_2}\right) \overset{?}{=} I_1$$

$$K = -\frac{1}{L_1}\left(\frac{A_1}{S_1} - \frac{A_2}{S_2}\right) = \frac{-A_1}{L_1}\left(\frac{S_2 - S_1}{S_1 \cdot S_2}\right) \leftarrow \frac{I_1}{C_1}$$

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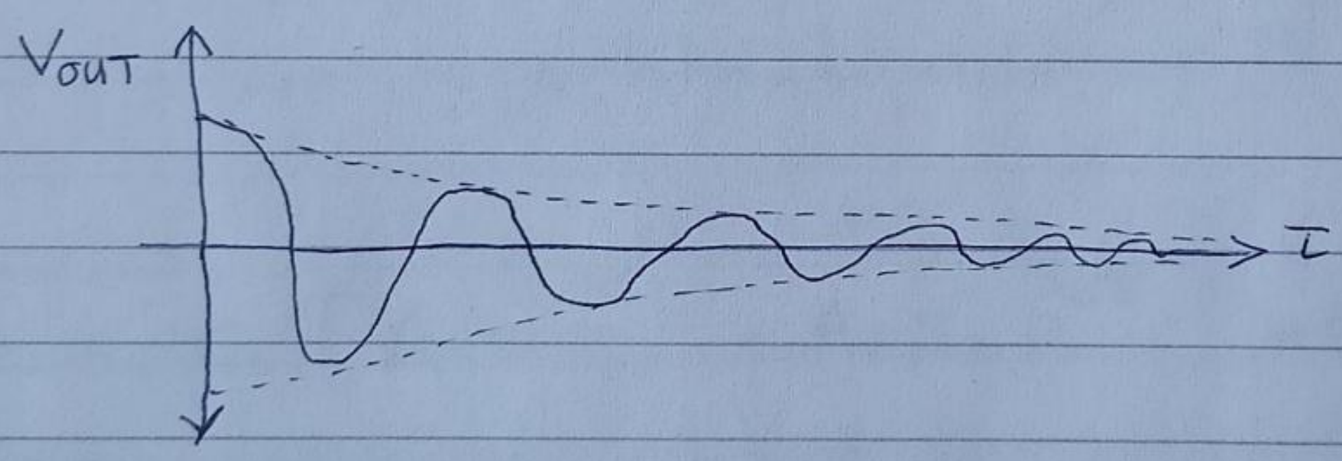
② critically damped Response

$$\begin{cases} V_{OUT} = \left[A_1 e^{s_1 t} + A_2 e^{s_2 t}\right] u(t) \\ V_{OUT}(0^+) = 0 \Rightarrow A_1 = 0 \\ C_1 \left.\dfrac{dV_{OUT}}{dt}\right|_{0^+} \Rightarrow A_1 s_1 + A_2 = \dfrac{I_1}{C_1} \Rightarrow A_2 = \dfrac{I_1}{C_1} \end{cases}$$

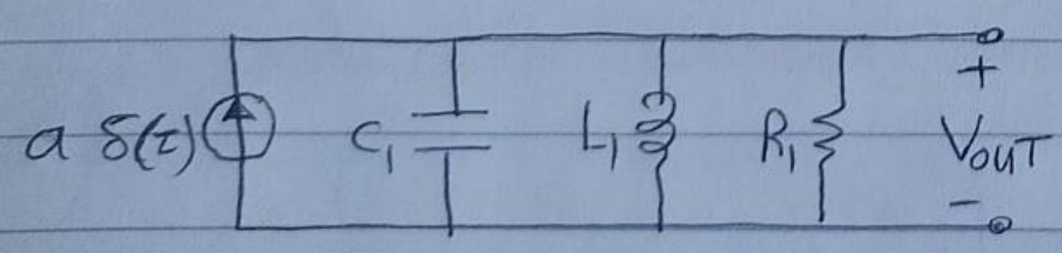
③ underdamped Response. $R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$

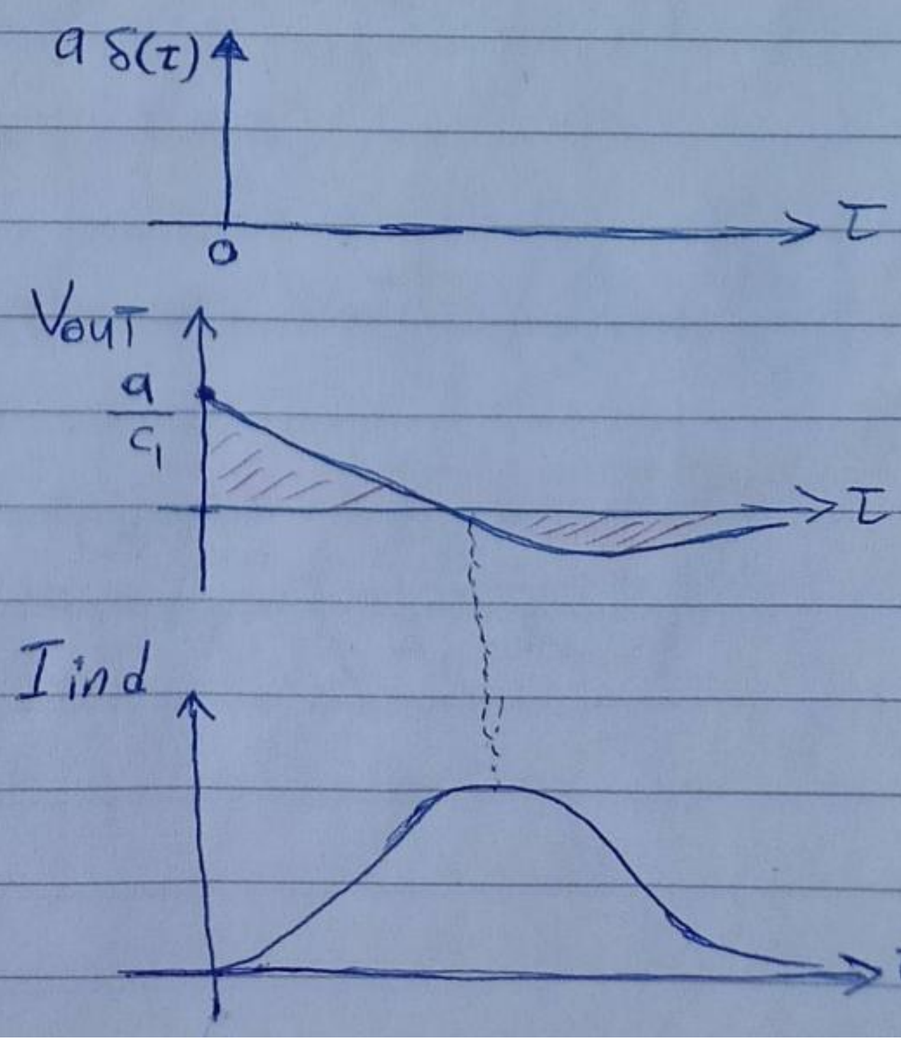
$$\begin{cases} V_{OUT} = \left(A_1 e^{\alpha t} \cos w_1 t + A_2 e^{\alpha t} \sin w_1 t\right) u(t) \\ V_{OUT}(0) = 0 \\ \left.\dfrac{dV_{OUT}}{dt}\right|_{0^+} = \dfrac{I_1}{C_1} \end{cases}$$

$$s_{1,2} = \underbrace{\frac{-1}{2R_1C_1}}_{\alpha} \pm \frac{1}{2} j \underbrace{\sqrt{\frac{4}{L_1C_1} - \frac{1}{R_1^2C_1^2}}}_{w_1}$$



- Impulse response





$$V_{OUT} = L_1 \frac{dI_L}{dt}, \quad I_{cap} = C_1 \frac{dV_{OUT}}{dt}$$

$$V_{OUT} = \frac{1}{C_1}\int I_{cap}\, dt$$

$$= \frac{1}{C_1}\int a\,\delta(t)\,dt = \frac{a}{C_1}$$

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Lec 33: Series RLC Circuit

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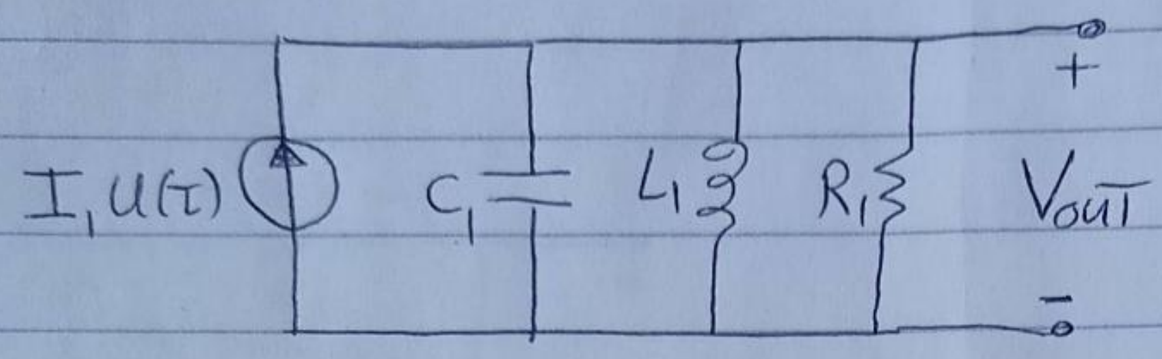
Review of lecture 32

- Driven RLC Tank



$V_{out}(0^+) = 0$

$I_{ind}(0^+) = 0 \Rightarrow$

$$C_1 \left.\frac{dV_{out}}{dt}\right|_{0^+} = I_1$$

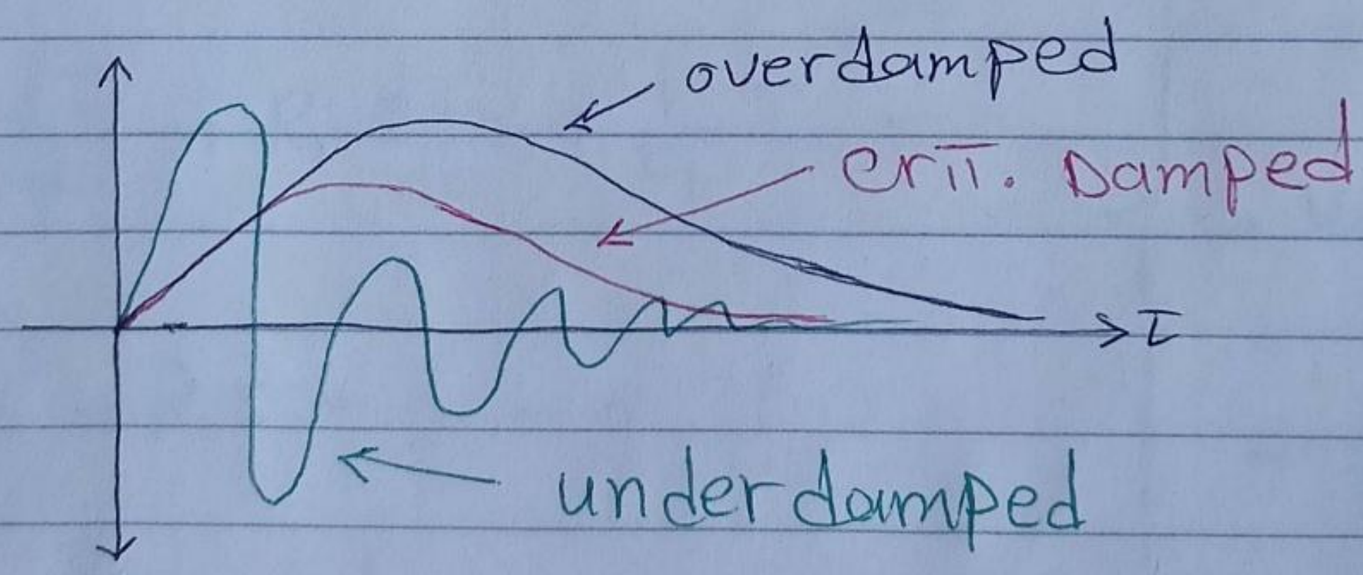
- overdamped: $V_{out}(t) = (A_1 e^{s_1 t} + A_2 e^{s_2 t})\, u(t)$

$$A_1 + A_2 = 0 \quad , \quad A_1 s_1 + A_2 s_2 = \frac{I_1}{C_1}$$

- crit. Damped: $V_{out}(t) = (A_1 e^{s_1 t} + A_2 t e^{s_1 t})\, u(t)$

$$A_1 = 0 \quad , \quad A_1 s_1 + A_2 = \frac{I_1}{C_1}$$

- underdamped: $V_{out}(t) = (A_1 e^{\alpha t} \cos w_1 t + A_2 e^{\alpha t} \sin w_1 t)\, u(t)$



Todays lecture

- series RLC circuit
  - Quantitative study
  - Natural response

① $A_1 e^{s_1 t} + A_2 e^{s_2 t}$

② $A_1 e^{s_1 t} + A_2 t e^{s_1 t}$

③ $A_1 e^{\alpha t} \cos w_1 t + A_2 e^{\alpha t} \sin w_1 t$

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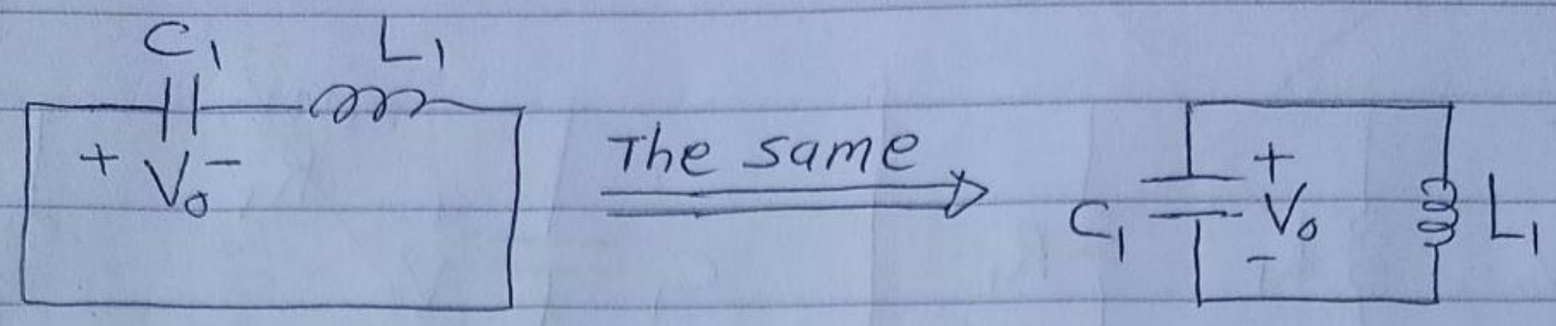
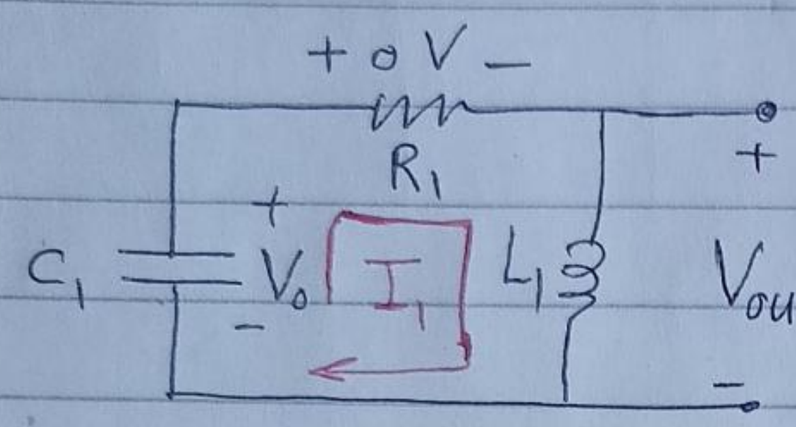
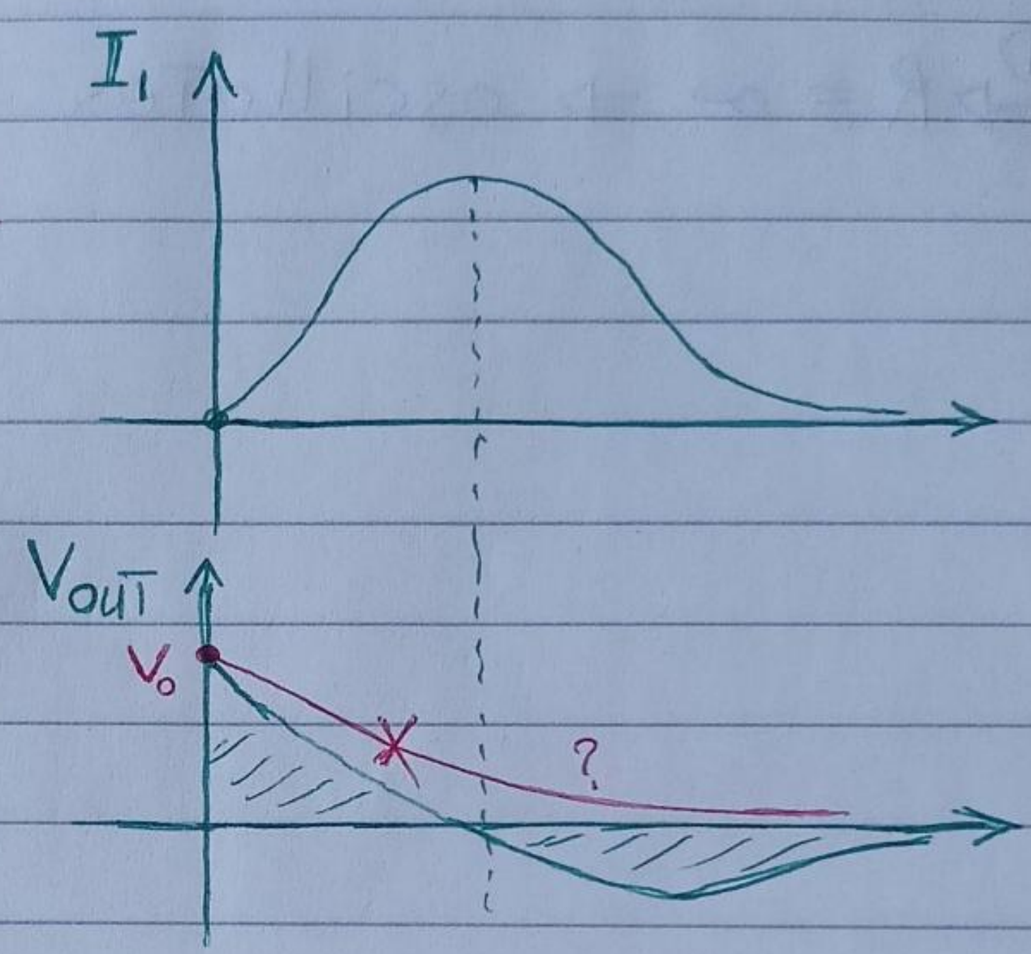
• Series RLC circuit





$$V_{OUT} = L_1 \frac{dI_1}{d\tau}$$



$$\begin{cases} I_{L_1}(0^+) = 0 \\ I_1(\infty) = 0 \\ V_{OUT}(0^+) = V_0 \end{cases}$$

• Quantitative Analysis

$$-\frac{1}{C_1}\int I_1\, d\tau = I_1 R_1 + L_1 \frac{dI_1}{d\tau}$$

$$L_1 \frac{dI_1}{d\tau} + R_1 \frac{dI_1}{d\tau} + \frac{1}{C_1} I_1 = 0$$

$$L_1 s^2 + R_1 s + \frac{1}{C_1} = 0$$

$$s^2 + \frac{R_1}{L_1} s + \frac{1}{L_1 C_1} = 0$$

$$\begin{cases} \bullet\ V_{OUT}(0^+) = V_{cap}(0^+) = V_{ind}(0^+) = V_0 \\ \bullet\ V_{ind}(0^+) = L_1 \left.\frac{dI_1}{d\tau}\right|_{0^+} = V_0 \\ \quad \Rightarrow \left.\frac{dI_1}{d\tau}\right|_{0^+} = \frac{V_0}{L_1} \\ \bullet\ I_{L_1} = I_1(0^+) = 0 \end{cases}$$

$$s_{1,2} = \frac{-R_1}{2L_1} \pm \frac{1}{2}\sqrt{\frac{R_1^2}{L_1^2} - \frac{4}{L_1 C_1}}$$

$$\frac{R_1^2}{L_1^2} \gtreqless \frac{4}{L_1 C_1} \Rightarrow R_1 \gtreqless 2\sqrt{\frac{L_1}{C_1}}$$

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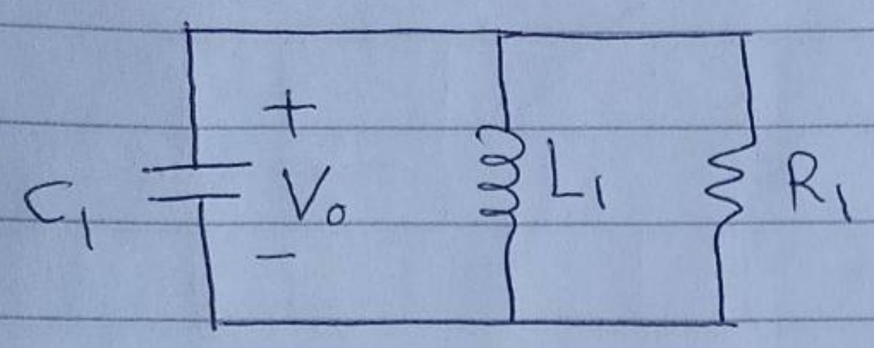
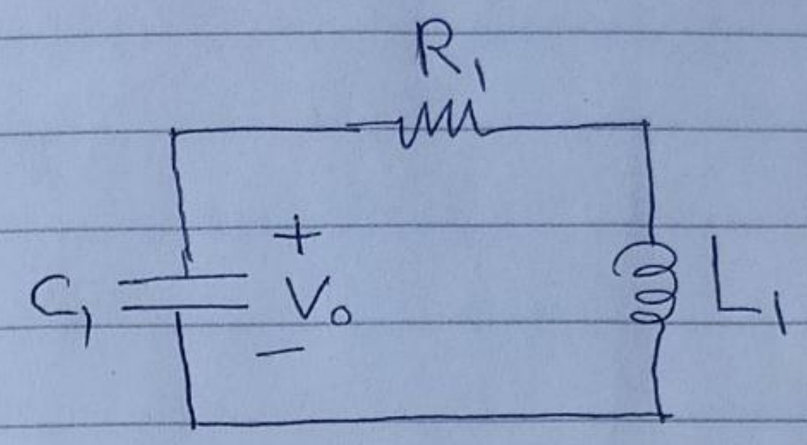
- Comparison of parallel & series RLC circuits:





- $R_1 = \infty \Rightarrow$ oscillates
- $R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Rightarrow$ dampens The oscillation

- $R_1 = 0 \Rightarrow$ oscillates
- $R_1 > 2\sqrt{\frac{L_1}{C_1}} \Rightarrow$ dampens The oscillation

---

- Three Natural Response:

① If $R_1 > 2\sqrt{\frac{L_1}{C_1}} \Rightarrow$ overdamped:

$$\begin{cases} I_1(t) = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t}\right) u(t) \\ I_1(0^+) = 0 \Rightarrow A_1 + A_2 = 0 \\ \left.\frac{dI_1}{dt}\right|_{0^+} = \frac{V_0}{L_1} = A_1 s_1 + A_2 s_2 \end{cases}$$

② if $R_1 = 2\sqrt{\frac{L_1}{C_1}} \Rightarrow$ critically-Damped:

$$\begin{cases} I_1(t) = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t}\right) u(t) \\ A_1 = 0 \quad , \quad A_1 s_1 + A_2 = \frac{V_0}{L_1} \end{cases}$$

③ If $R_1 < 2\sqrt{\frac{L_1}{C_1}} \Rightarrow$ underdamped:

$$\begin{cases} I_1(t) = \left(A_1 e^{\alpha t} \cos \omega_1 t + A_2 e^{\alpha t} \sin \omega_1 t\right) u(t) \\ A_1 = 0 \quad , \quad \left.\frac{dI_1}{dt}\right|_{0^+} = \frac{V_0}{L_1} \end{cases}$$

$$s_{1,2} = -\frac{R_1}{2L_1} \pm \frac{1}{2}\sqrt{\frac{R_1^2}{L_1^2} - \frac{4}{L_1 C_1}} = \underbrace{\frac{-R_1}{2L_1}}_{\alpha} \pm \underbrace{\frac{1}{2} j \sqrt{\frac{4}{L_1 C_1} - \frac{R_1^2}{L_1^2}}}_{\omega_1}$$

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Lec 34: Source-Free and Driven RLC Circuits

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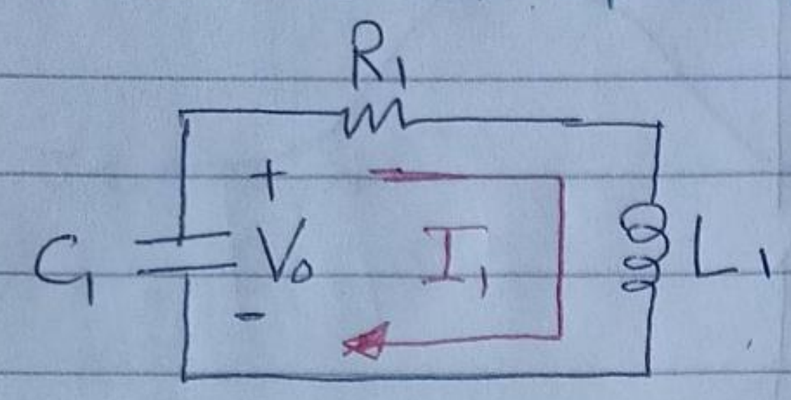
Review of lecture 33

- Natural Resp. of series RLC circuit



$$\frac{d^2 I_1}{d\tau^2} + \frac{R_1}{L_1}\frac{dI_1}{d\tau} + \frac{1}{L_1 C_1} I_1 = 0$$

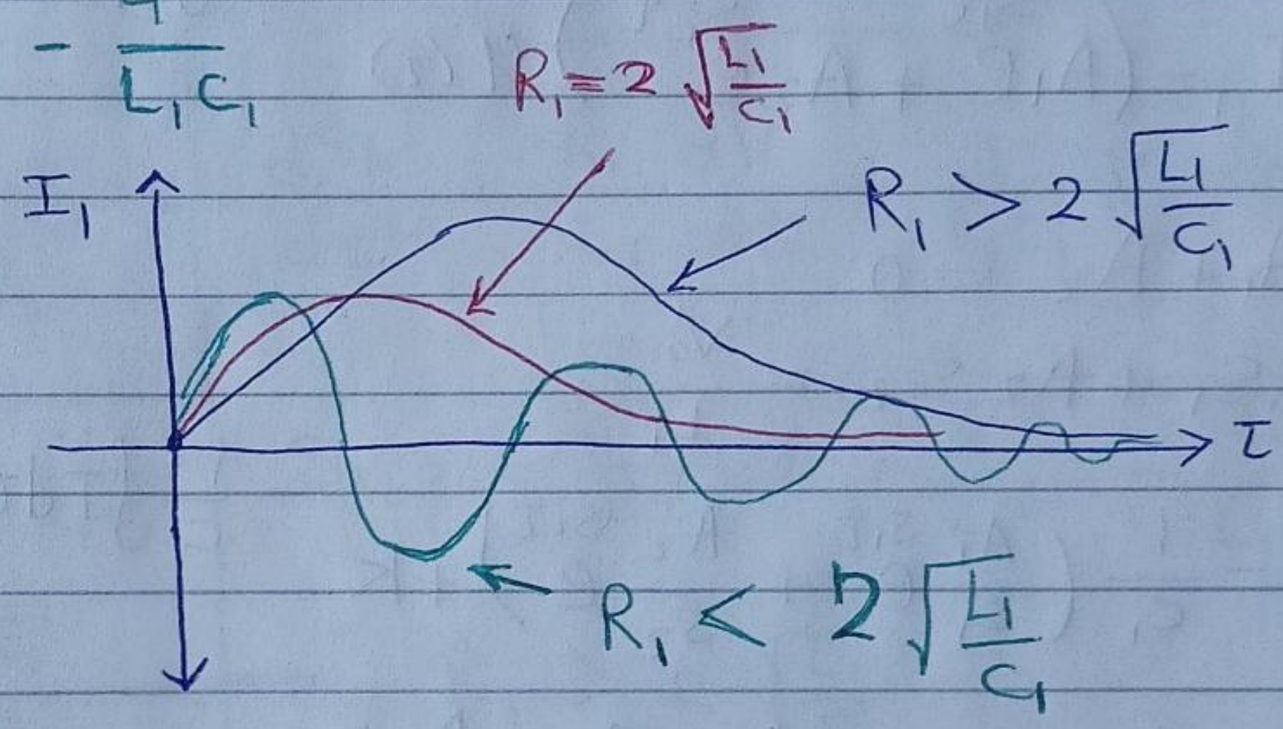
$$\begin{cases} I_1(0^+) = 0 \\ \left.\frac{dI_1}{d\tau}\right|_{0^+} = \frac{V_0}{L_1} \end{cases}$$

$$S_{1,2} = \frac{-R_1}{2L_1} \pm \frac{1}{2}\sqrt{\frac{R_1^2}{L_1^2} - \frac{4}{L_1 C_1}}$$

① $R_1 > 2\sqrt{\frac{L_1}{C_1}}$

② " = "

③ " < "



Today's lecture:

- Natural Resp. of series RLC circuit.
- A Note on Diff. Eqs.
- Driven series RLC circuit

① $A_1 e^{s_1\tau} + A_2 e^{s_2\tau}$

② $A_1 e^{s_1\tau} + A_2 \tau e^{s_1\tau}$

③ $A_1 e^{\alpha\tau}\cos\omega_1\tau + A_2 e^{\alpha\tau}\sin\omega_1\tau$

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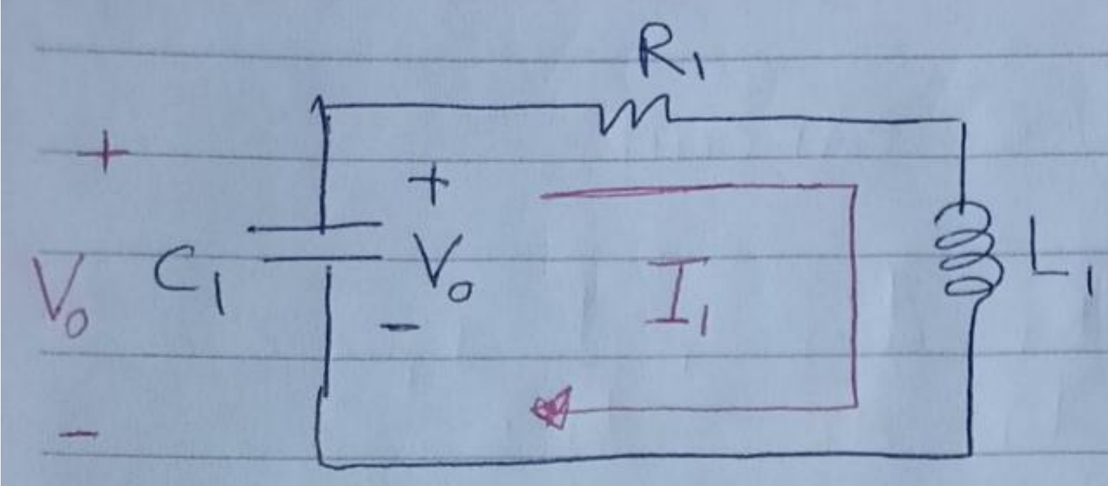
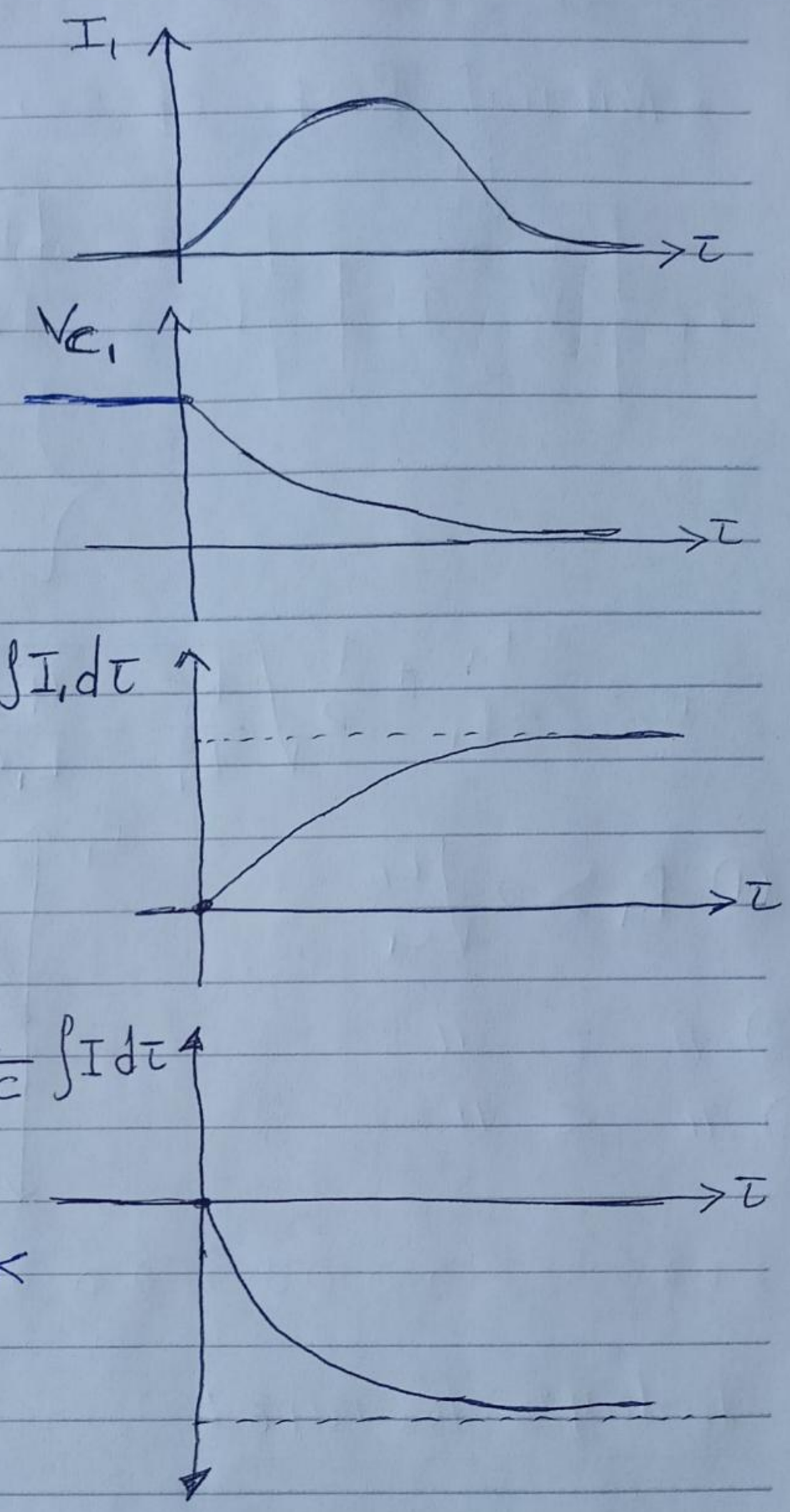
- Cap Voltage





$$V_{C_1} = -\frac{1}{C_1}\int I_1(\tau)\, d\tau + K$$

For Example :

$$I_1 = \left(A_1 e^{s_1\tau} + A_2 e^{s_2\tau}\right) u(\tau)$$

$$A_1 + A_2 = 0$$

$$A_1 s_1 + A_2 s_2 = \frac{V_0}{L_1}$$

$$V_{C_1} = -\frac{1}{C_1}\left(\frac{A_1}{s_1} e^{s_1\tau} + \frac{A_2}{s_2} e^{s_2\tau}\right) + K$$

AT $\tau = 0^+$; $V_0 = -\frac{1}{C_1}\left(\frac{A_1}{s_1} + \frac{A_2}{s_2}\right) + K$

$$K = V_0 + \frac{1}{C_1}\left(\frac{A_1}{s_1} + \frac{A_2}{s_2}\right)$$

$$\therefore V_{C_1}(\tau) = -\frac{1}{C_1}\left(\frac{A_1}{s_1} e^{s_1\tau} + \frac{A_2}{s_2} e^{s_2\tau}\right) + V_0 + \frac{1}{C_1}\left(\frac{A_1}{s_1} + \frac{A_2}{s_2}\right), \quad \tau > 0$$

AT $\tau = \infty$; $V_{C_1} = V_0 + \frac{1}{C_1}\left(\frac{A_1}{s_1} + \frac{A_2}{s_2}\right) \overset{?}{=} 0$

Alternative Method : Find V_{C_1} directly

Current Through C_1 : $C_1 \frac{dV_{C_1}}{d\tau}$, Voltage on R_1 : $R_1 C_1 \frac{dV_{C_1}}{d\tau}$

Voltage on L_1 : $L_1 \frac{d}{d\tau}\left(C_1 \frac{dV_{C_1}}{d\tau}\right)$

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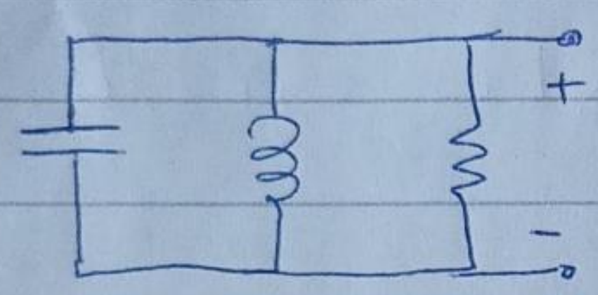
KVL:

$$+V_{C_1} + R_1 C_1 \frac{dV_{C_1}}{d\tau} + L_1 C_1 \frac{d^2 V_{C_1}}{d\tau^2} = 0$$

$$V_{C_1}(0^+) = V_0$$

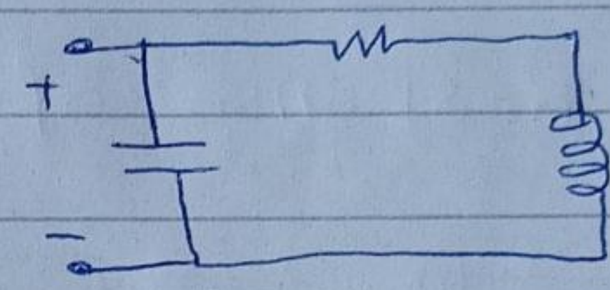
$$I_1(0^+) = 0 \Rightarrow I_{C_1}(0^+) = 0 = C_1 \frac{dV_{C_1}}{d\tau}\Big|_{0^+} = 0$$

- summary of parallel & series RLC circuits.



$$\frac{d^2V}{d\tau^2} + \frac{1}{R_1C_1}\frac{dV}{d\tau} + \frac{1}{L_1C_1}V = 0$$

$$S_{1,2} = -\frac{1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1C_1}}$$



$$\frac{d^2V}{d\tau^2} + \frac{R_1}{L_1}\frac{dV}{d\tau} + \frac{1}{L_1C_1}V = 0$$

$$S_{1,2} = -\frac{R_1}{2L_1} \pm \sqrt{\frac{R_1^2}{L_1^2} - \frac{4}{L_1C_1}}$$

overdamped: $R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Longleftrightarrow R_1 > 2\sqrt{\frac{L_1}{C_1}}$

crit. damped: $R_1 = \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Longleftrightarrow R_1 = 2\sqrt{\frac{L_1}{C_1}}$

underdamped: $R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Longleftrightarrow R_1 < 2\sqrt{\frac{L_1}{C_1}}$

- A Note on diff. eqs.

- first-order Eq: $\frac{dy}{d\tau} + \alpha y = \beta \qquad y(\infty) = \frac{\beta}{\alpha}$ (if y becomes const.)

- second-order Eq:

$$\frac{d^2y}{d\tau^2} + \alpha\frac{dy}{d\tau} + \beta y = \gamma \qquad y(\infty) = \frac{\gamma}{\beta} \text{ (if y becomes const)}$$

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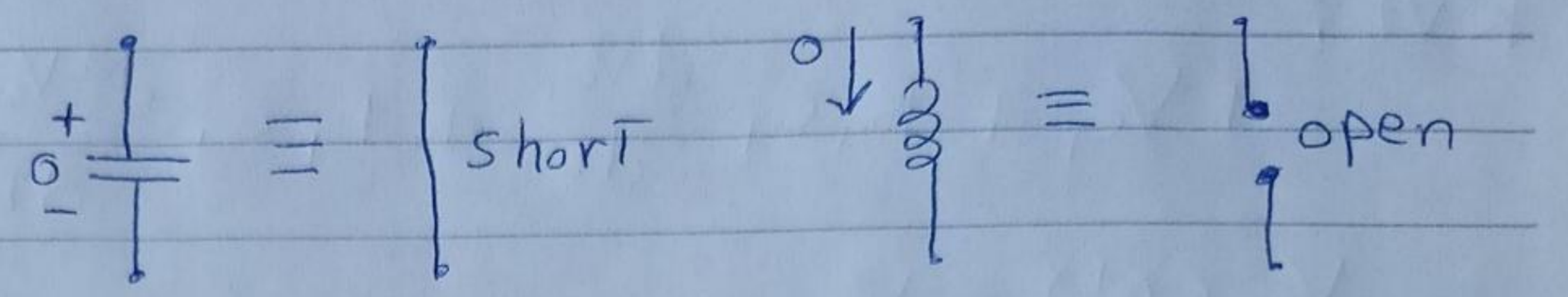
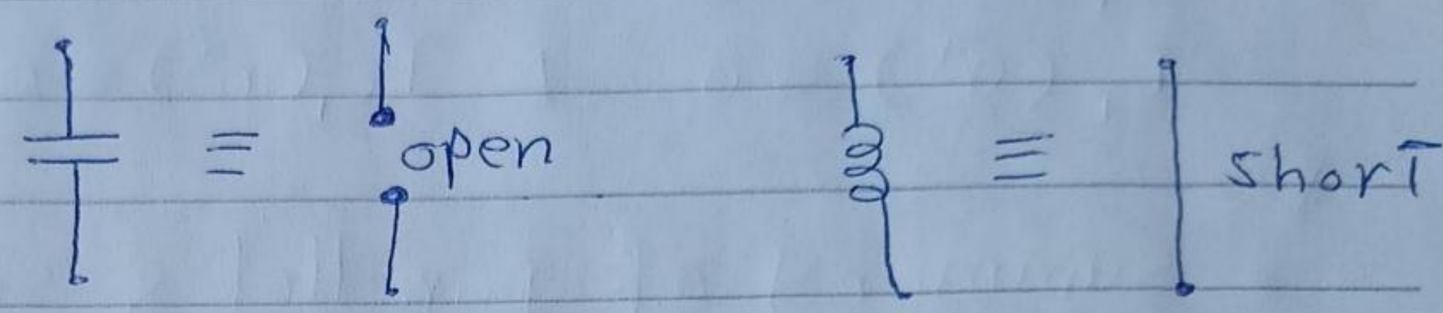
Reminder:

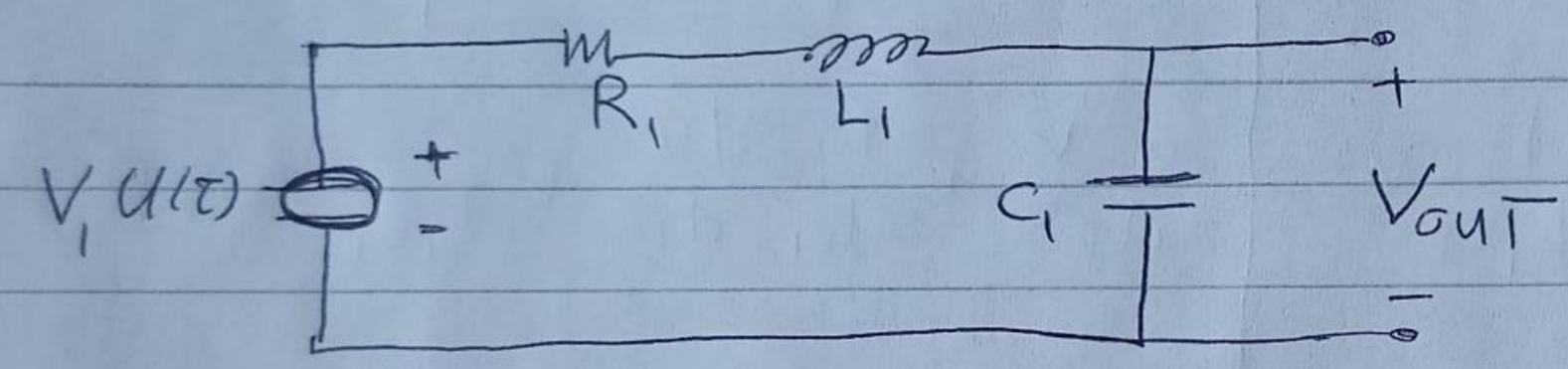
AT $t=0^+$:



AT $t=\infty$:
(iF all values become const)



- Driven Response of series RLC circuit.



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Lec 35: Step Response of Series RLC Circuit

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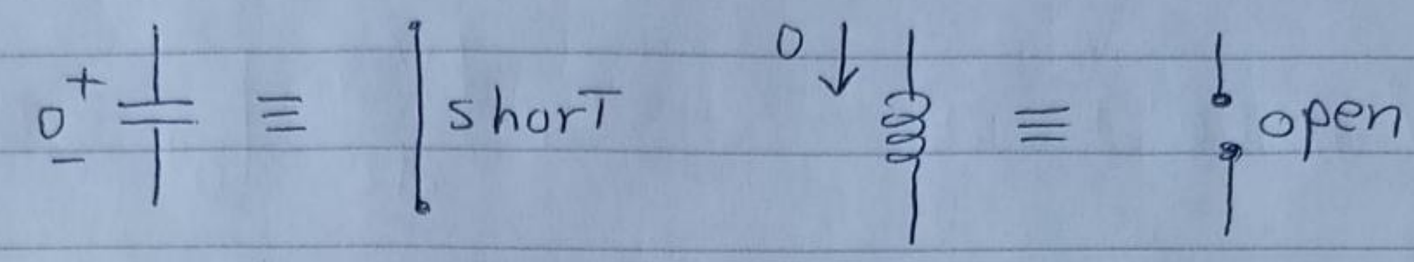
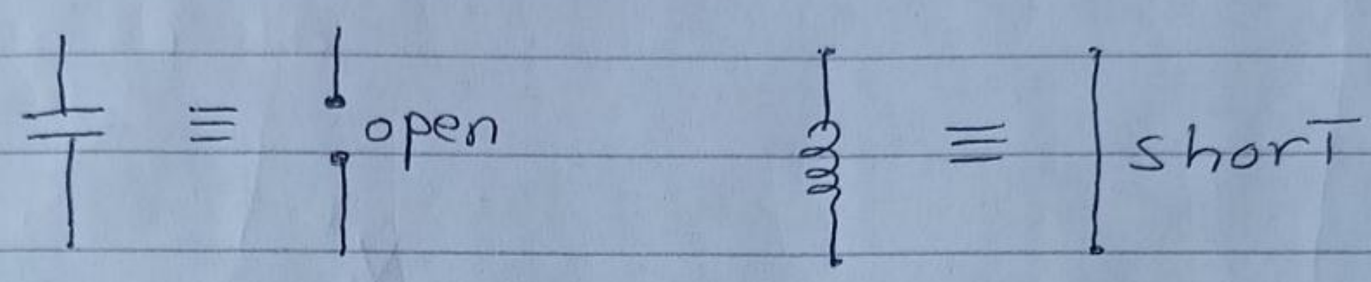
Review of lecture 34

① $\frac{d^2y}{d\tau^2} + \alpha \frac{dy}{d\tau} + \beta y = \gamma \Rightarrow y(\infty) = \frac{\gamma}{\beta}$

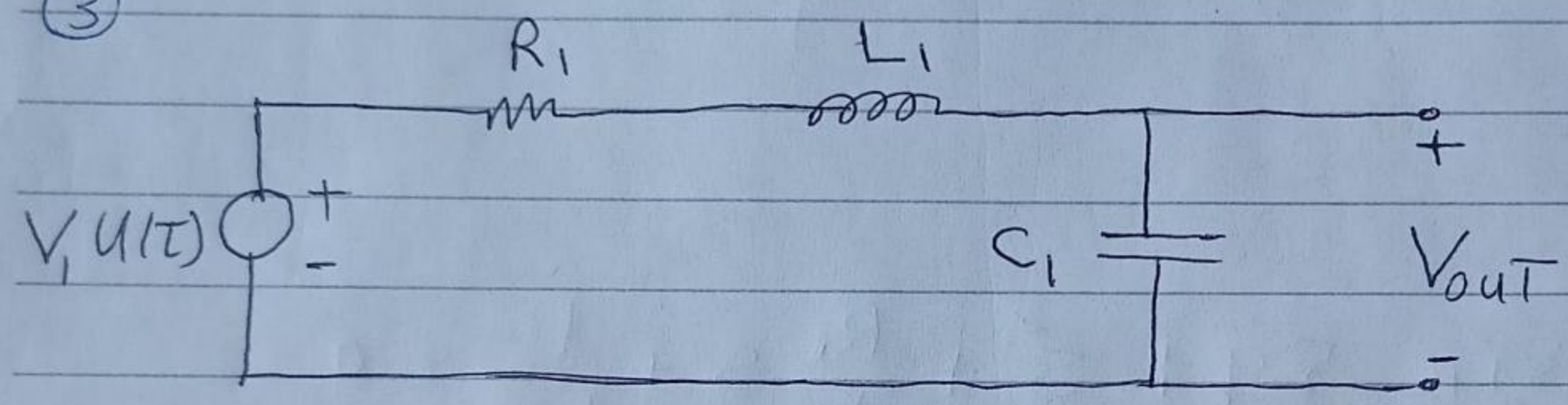
② AT $\tau = 0^+$:



$\tau = \infty$:



③



Todays Lecture:

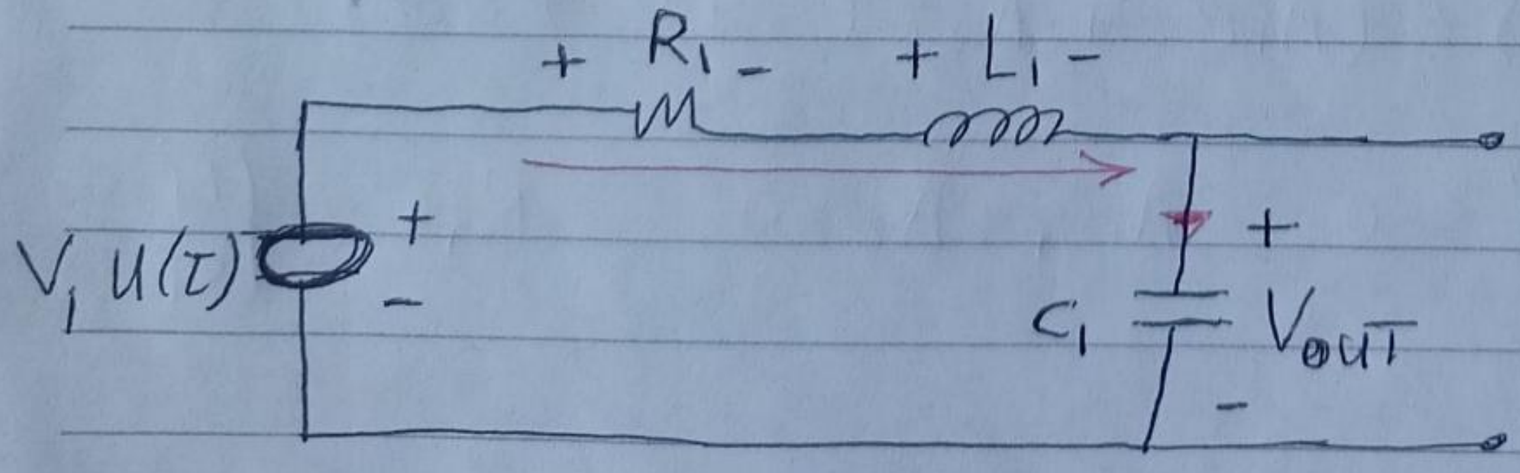
- Step Response of series RLC circuit.
- impulse Response.

① $A_1 e^{s_1\tau} + A_2 e^{s_2\tau}$

② $A_1 e^{s_1\tau} + A_2 \tau e^{s_1\tau}$

③ $A_1 e^{\alpha\tau} \cos\omega_1\tau + A_2 e^{\alpha\tau} \sin\omega_1\tau$

- Step response



- Current Through C_1:	- Voltage on R_1:	- Voltage on L_1:
$C_1 \frac{dV_{OUT}}{d\tau}$	$R_1 C_1 \frac{dV_{OUT}}{d\tau}$	$L_1 \frac{d}{d\tau}\left(C_1 \frac{dV_{OUT}}{d\tau}\right)$

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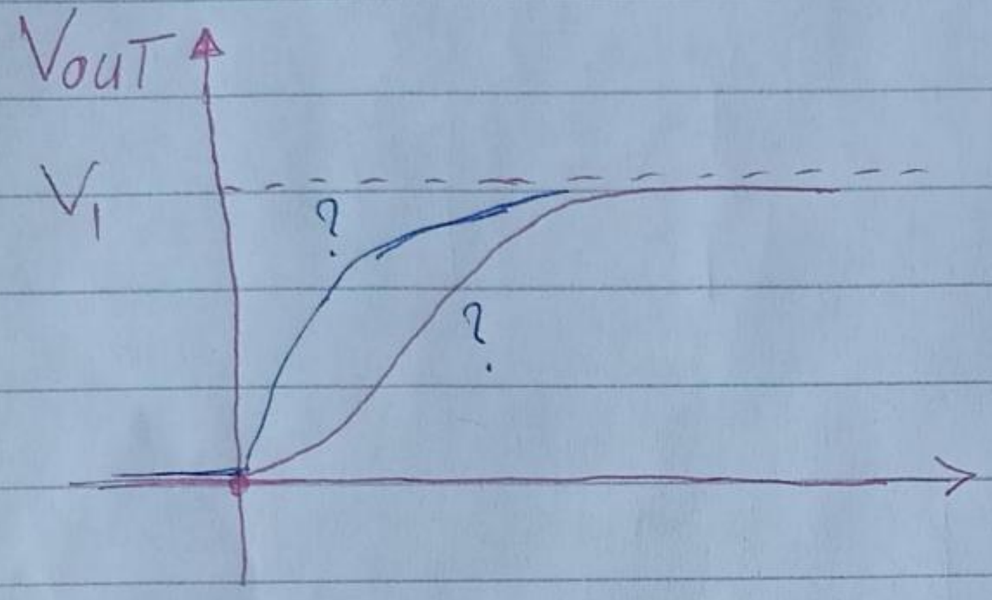
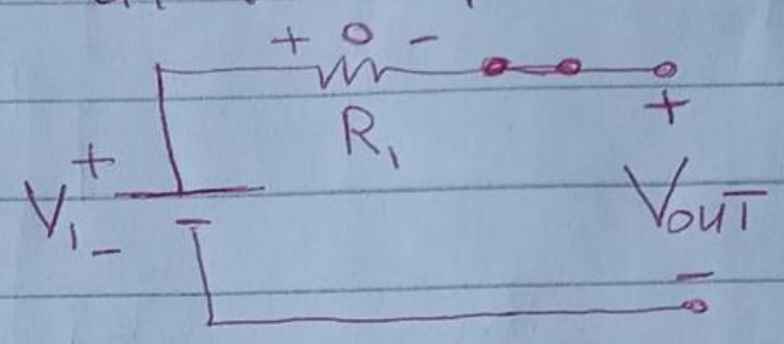
$$+V_1 u(\tau) - R_1 C_1 \frac{dV_{OUT}}{d\tau} - L_1 C_1 \frac{d^2 V_{OUT}}{d\tau^2} - V_{OUT} = 0$$

$$L_1 C_1 \frac{d^2 V_{OUT}}{d\tau^2} + R_1 C_1 \frac{dV_{OUT}}{d\tau} + V_{OUT} = V_1 u(\tau)$$

$$V_{OUT}(0^+) = 0 \qquad C_1 \frac{dV_{OUT}}{d\tau}\Big|_{0^+} = 0$$



at $\tau = \infty$:



characteristic Eq : $s^2 + \frac{R_1}{L_1} s + \frac{1}{L_1 C_1} = 0$

⇒ Case 1 : overdamped : $R_1 > 2\sqrt{\frac{L_1}{C_1}}$

$$V_{OUT} = \left(A_1 e^{s_1 \tau} + A_2 e^{s_2 \tau} + V_1\right) u(\tau) \qquad A_1 + A_2 = 0$$

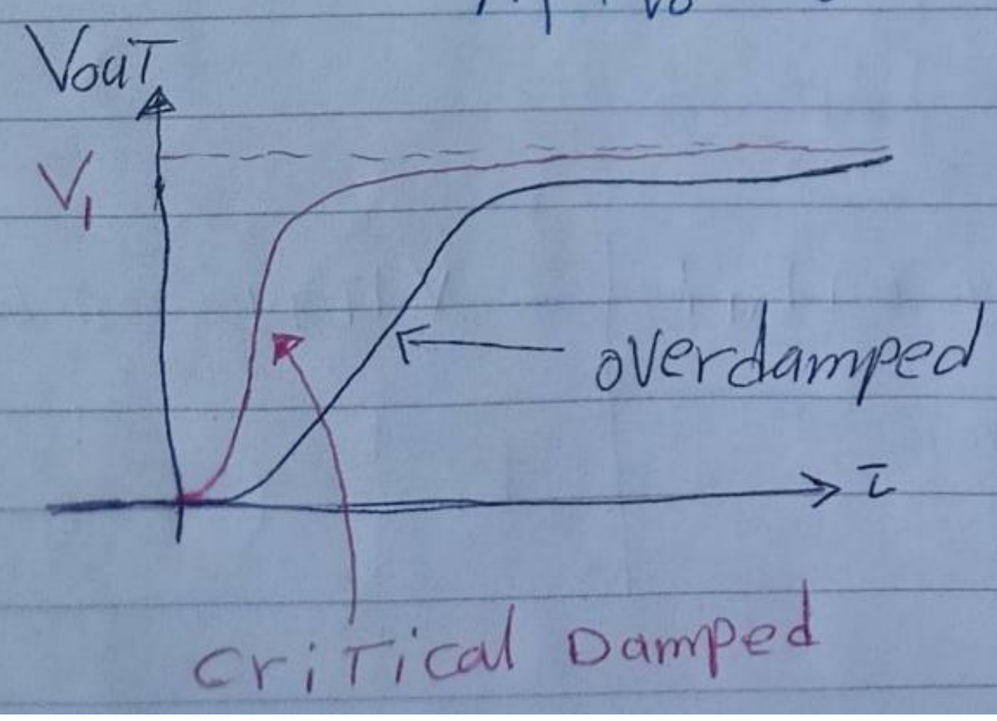
$$A_1 s_1 + A_2 s_2 = 0$$

- since $\frac{dV_{OUT}}{d\tau}\Big|_{0^+} = 0$, V_{OUT} takes off slowly.

⇒ Case 2 :

$$V_{OUT}(\tau) = \left(A_1 e^{s_1 \tau} + A_2 \tau e^{s_1 \tau} + V_0\right) u(\tau)$$

$$A_1 + V_0 = 0 \qquad A_1 s_1 + A_2 = 0 \qquad R_1 = 2\sqrt{\frac{L_1}{C_1}}$$



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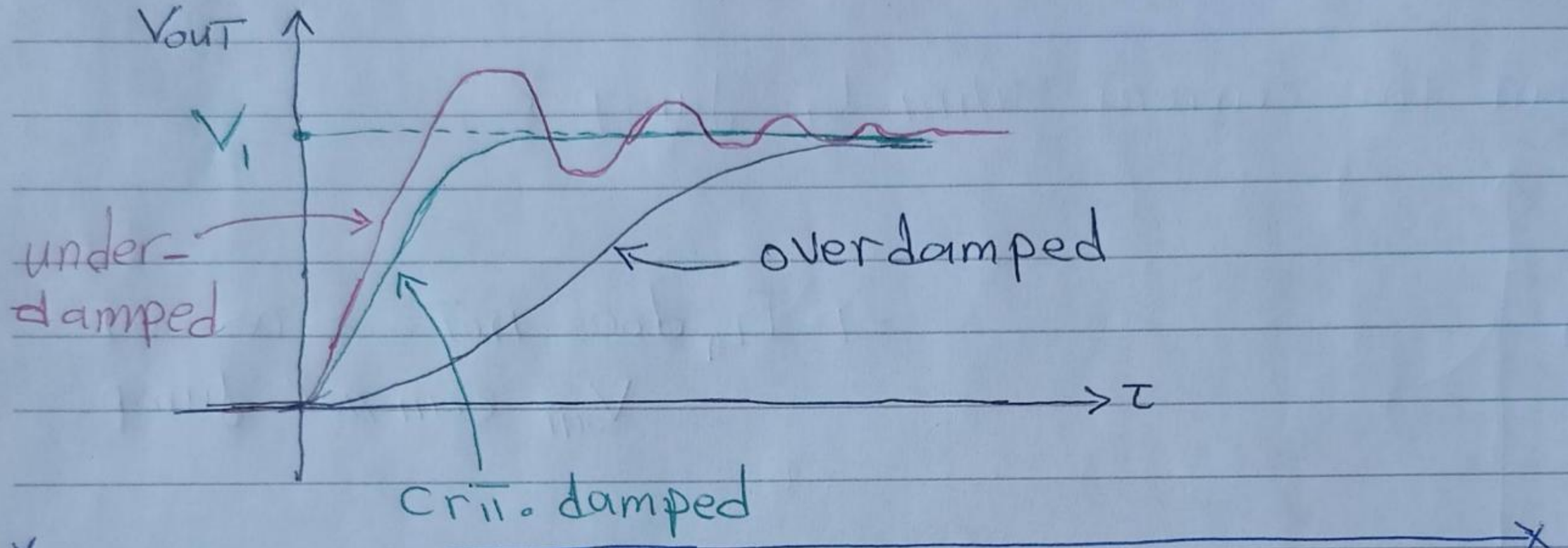
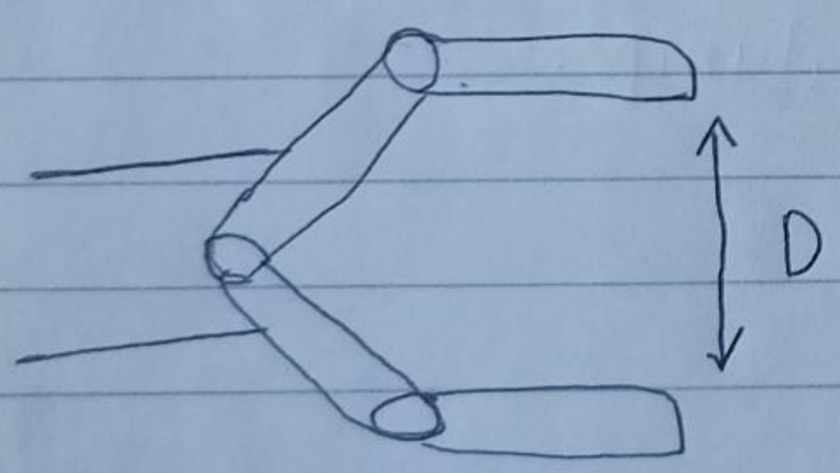
Case 3: underdamped

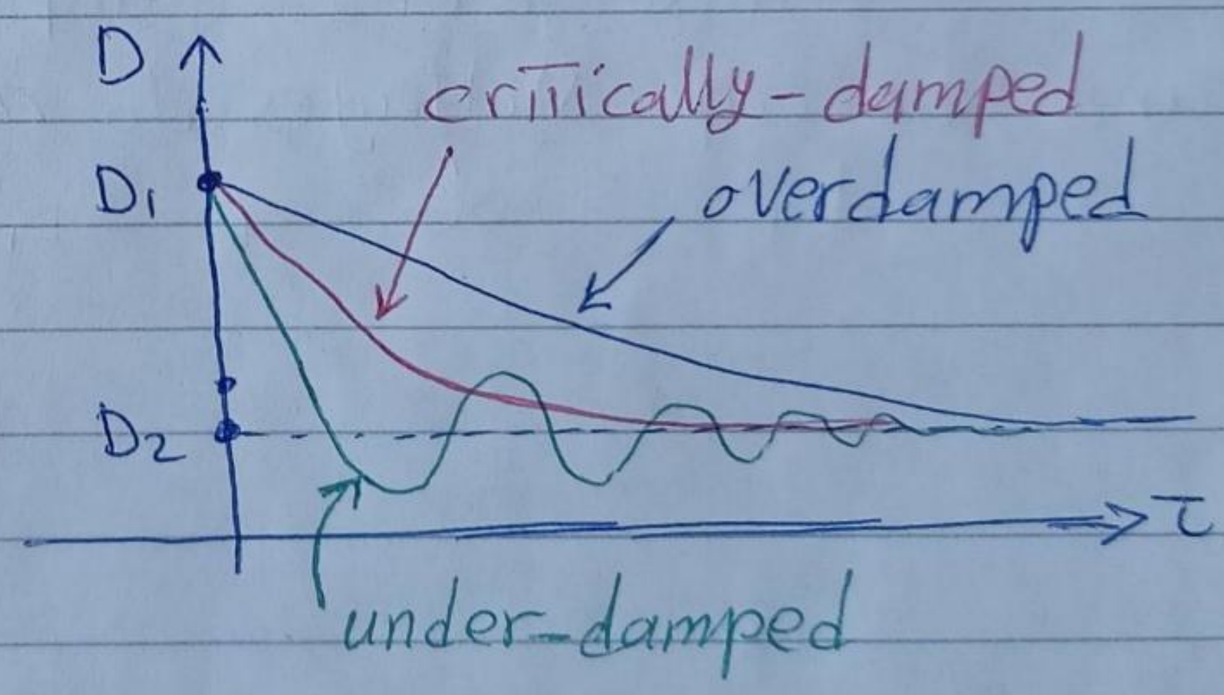
$$V_{OUT} = \left(A_1 e^{\alpha\tau} \cos \omega_1 \tau + A_2 e^{\alpha\tau} \sin \omega_1 \tau + V_0\right) u(\tau)$$

$$S_{1,2} = \underbrace{-\frac{R_1}{2L_1}}_{\alpha} \pm \underbrace{\frac{1}{2} j \sqrt{\frac{4}{L_1 C_1} - \frac{R_1^2}{L_1^2}}}_{\omega_1}$$



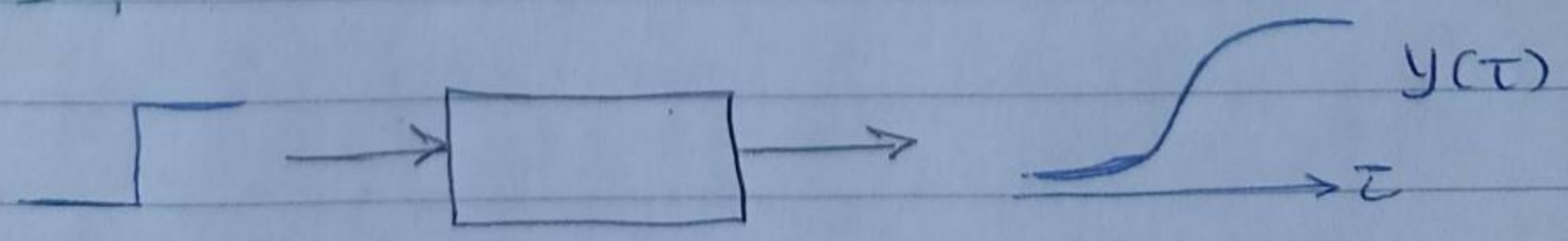
Robotics Analogy

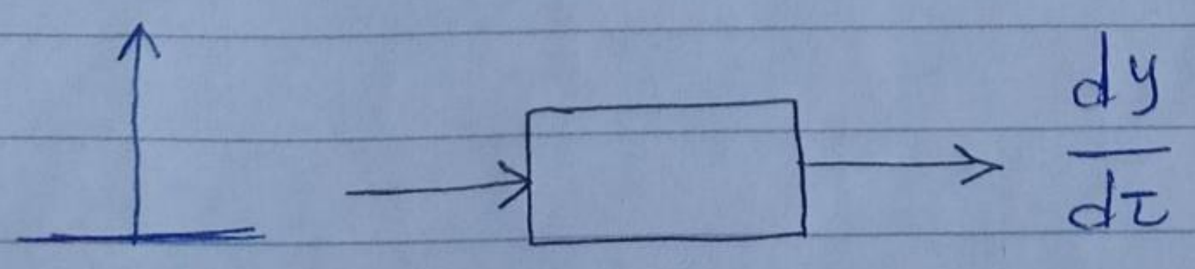




- impulse Response of series RLC circuit.

Approch 1:





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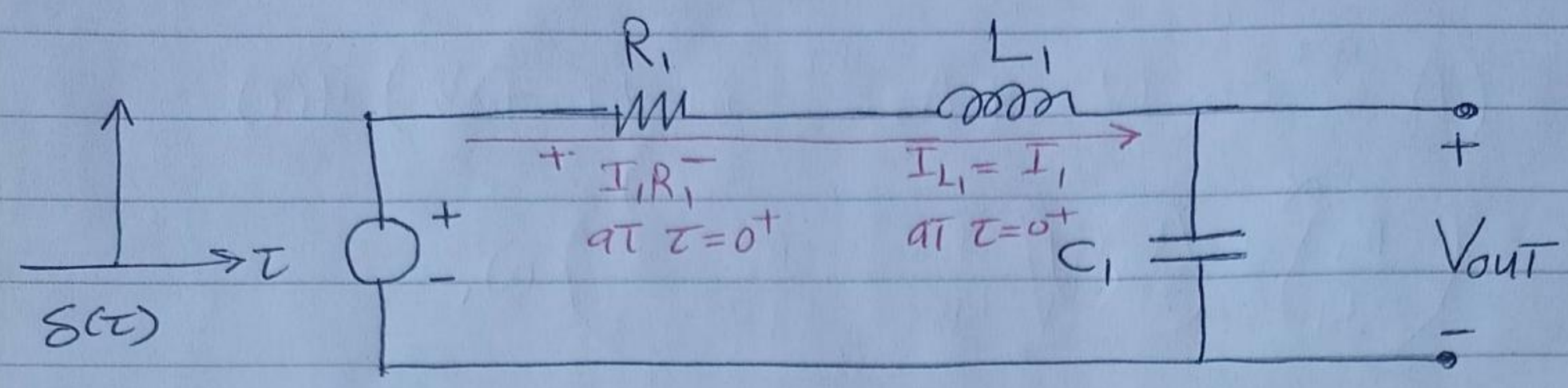
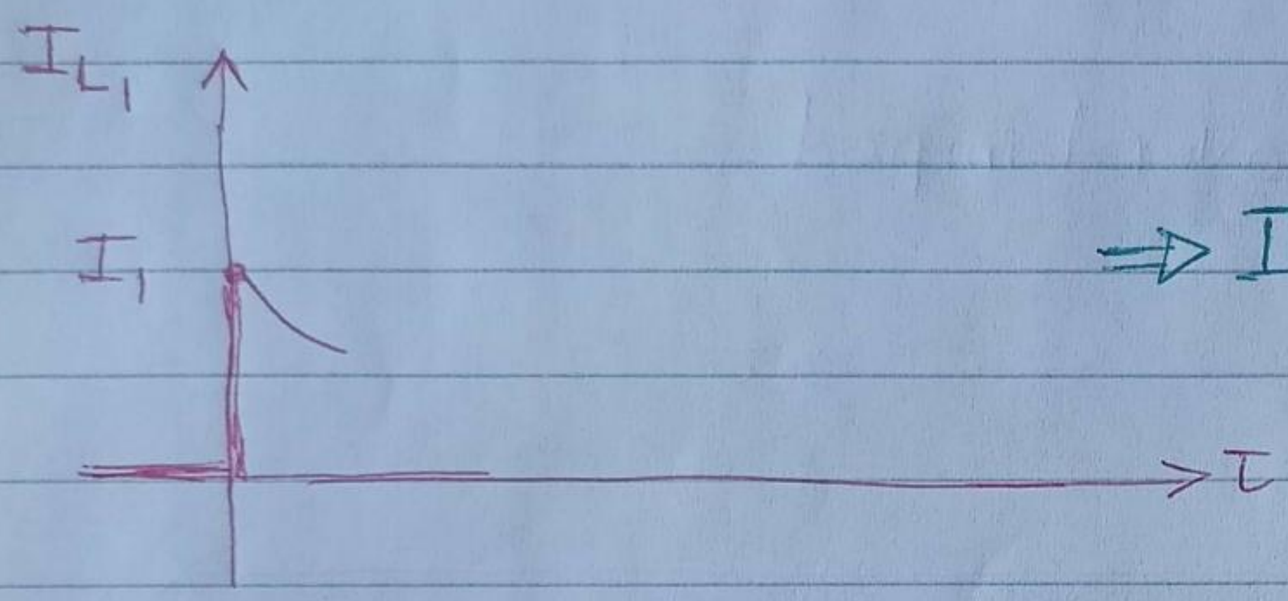
Approach:



Can the current thru L_1 jump?



$\Rightarrow I_{L_1}$ does not $\rightarrow \infty \Rightarrow$
V_{OUT} can not jump.

since $\delta(t)$ jumps to ∞ but the resistor and the cap voltages do not, the voltage across $L_1 \rightarrow \infty \Rightarrow I_{L_1}$ can jump.

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Lec 36: Impulses and Doublets; Impulse Response of Series RLC Circuit

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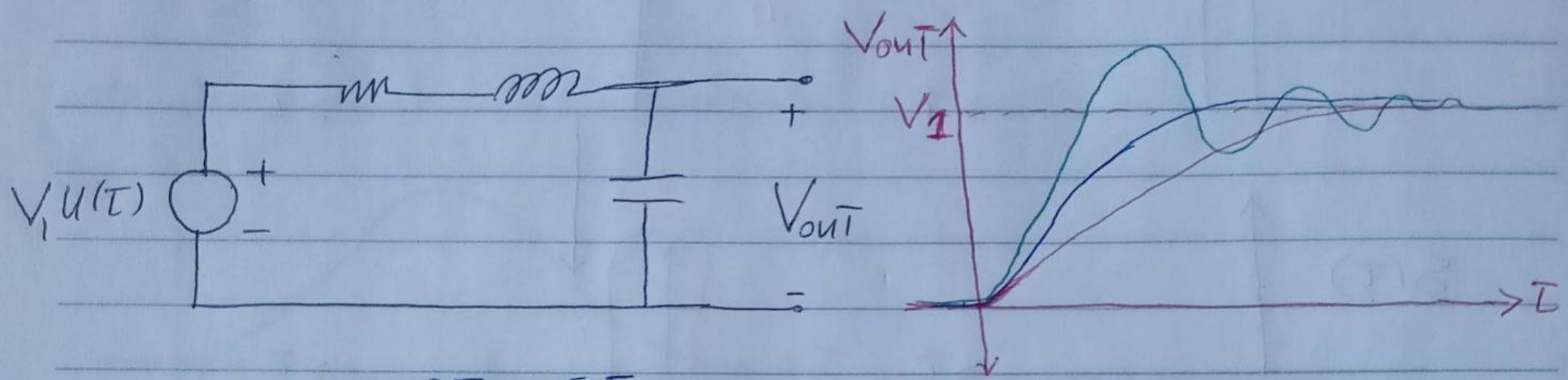
Review of lecture 35

- Step Response of series RLC circuit.

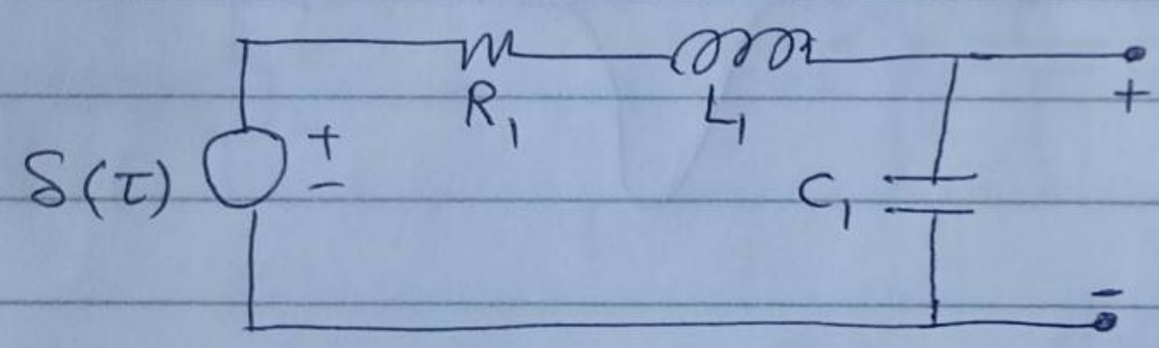


$V_{out}(t) = (A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_1)\, u(t)$ "overdamped"

$V_{out}(t) = (A_1 e^{s_1 t} + A_2 t e^{s_1 t} + V_1)\, u(t)$ "crit. damped"

$V_{out}(t) = (A_1 e^{\alpha t} \cos \omega_1 t + A_2 e^{\alpha t} \sin \omega_1 t + V_1)\, u(t)$ "underdamped"

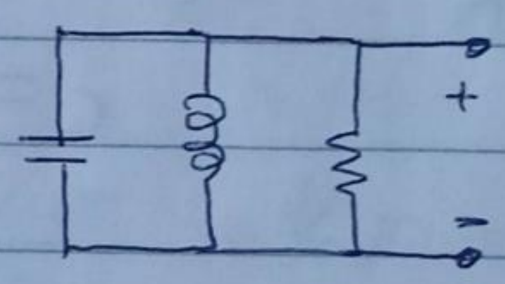
- impulse Response



- Current Through L_1 may jump.
- Voltage on C_1 cannot jump

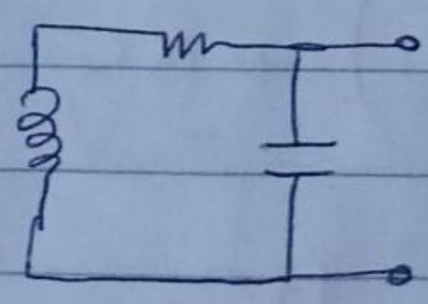
Today's lecture

- Note on Impulse & Doublet Functions



$$\frac{dv^2}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC} = 0$$

- Impulse Response of series RLC circuit



$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC} = 0$$

- other Examples

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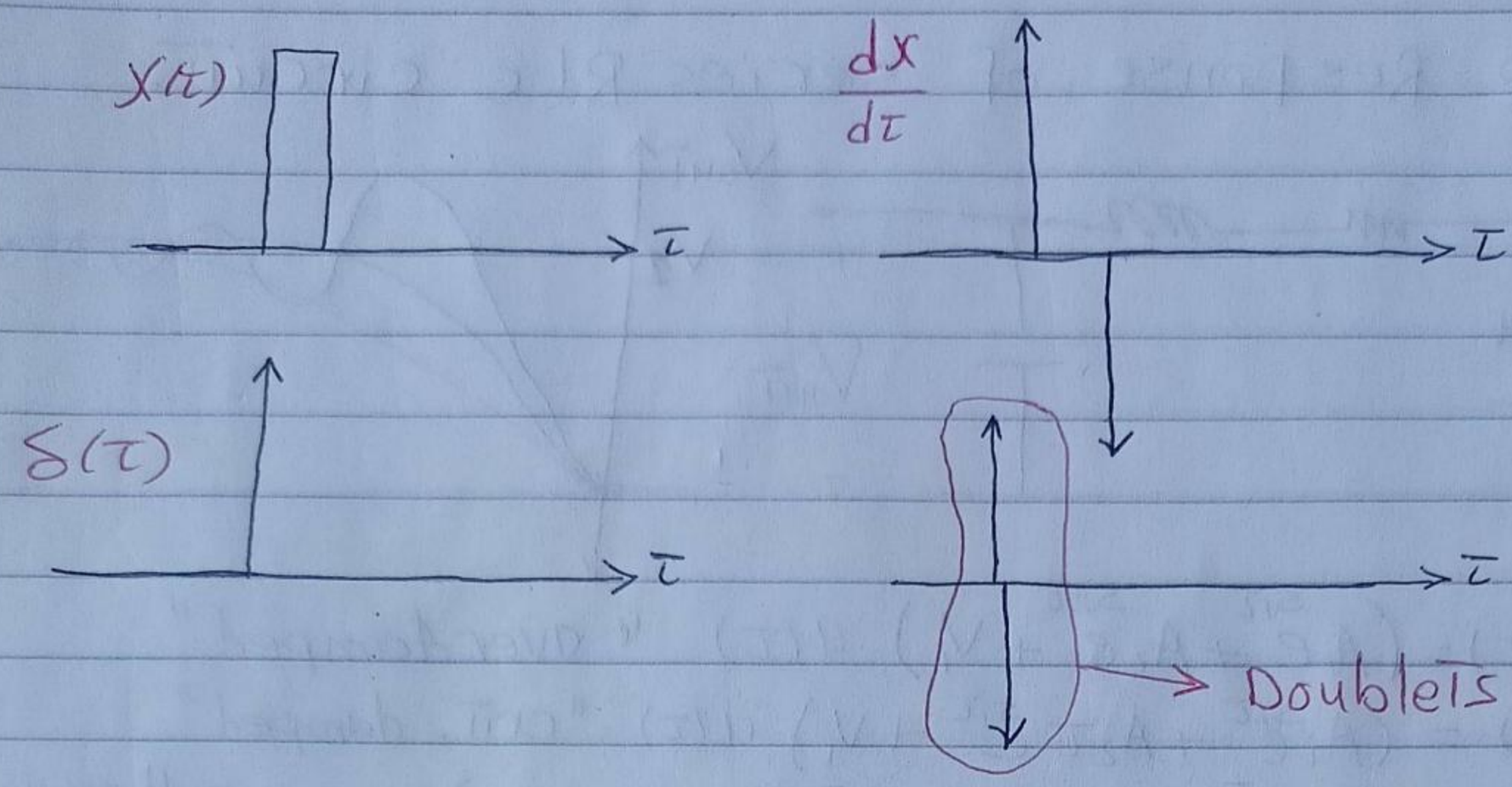
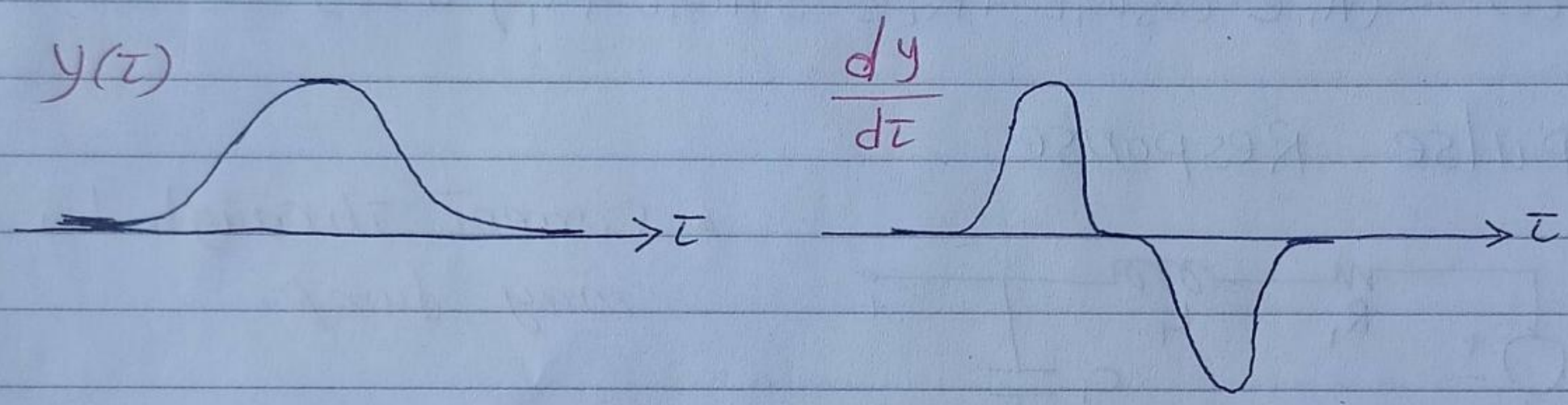
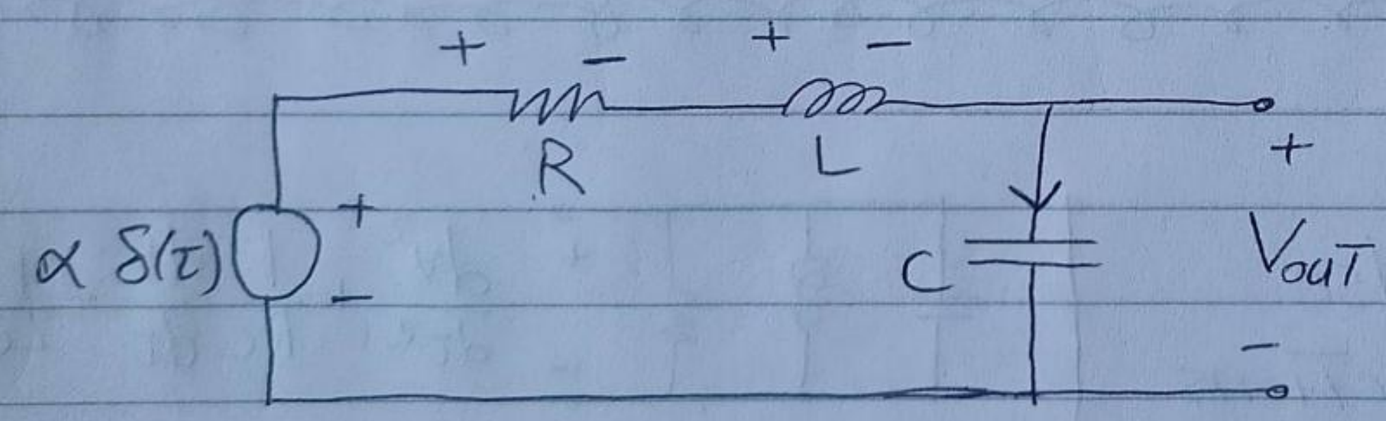
- <u>impulse and Doublets</u>





- Impulse Response of series RLC circuit.



$$L\frac{d}{d\tau}\left(C\frac{dV_{OUT}}{d\tau}\right)+RC\frac{dV_{OUT}}{d\tau}+V_{OUT}=\alpha\,\delta(\tau)$$

$$\frac{d^2V_{OUT}}{d\tau^2}+\frac{R}{L}\frac{dV_{OUT}}{d\tau}+\frac{1}{LC}V_{OUT}=\frac{\alpha}{LC}\delta(\tau)$$

IF $V_{OUT}\propto\delta(\tau)\Rightarrow\frac{dV_{OUT}}{d\tau}\propto$ doublet $\Rightarrow$ NOT possible

IF $\frac{dV_{OUT}}{d\tau}\propto\delta(\tau)\Rightarrow\frac{d^2V_{OUT}}{d\tau^2}\propto$ doublets $\Rightarrow$ NOT Possible

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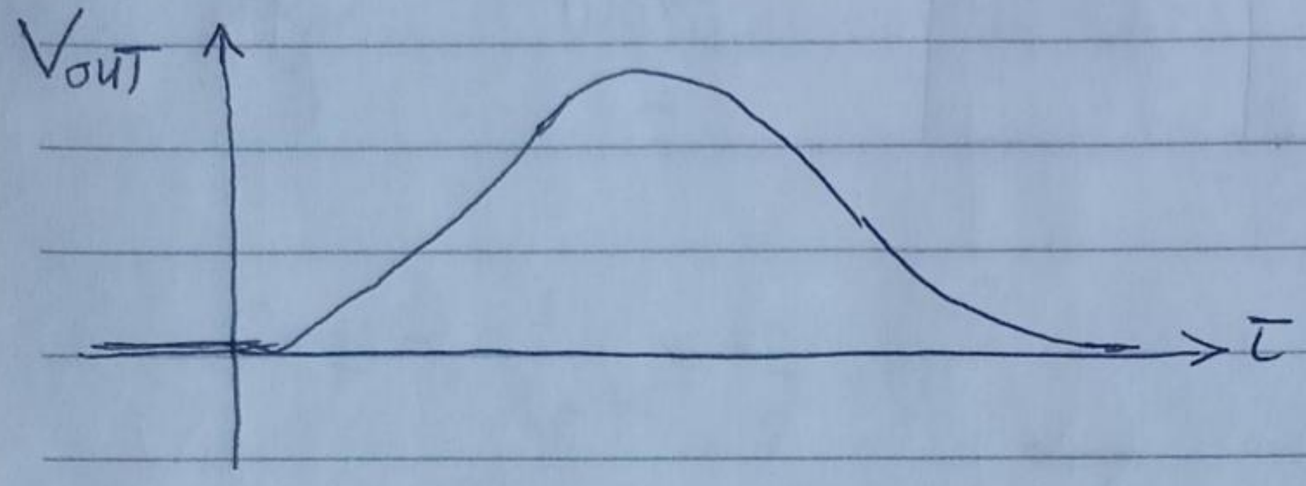
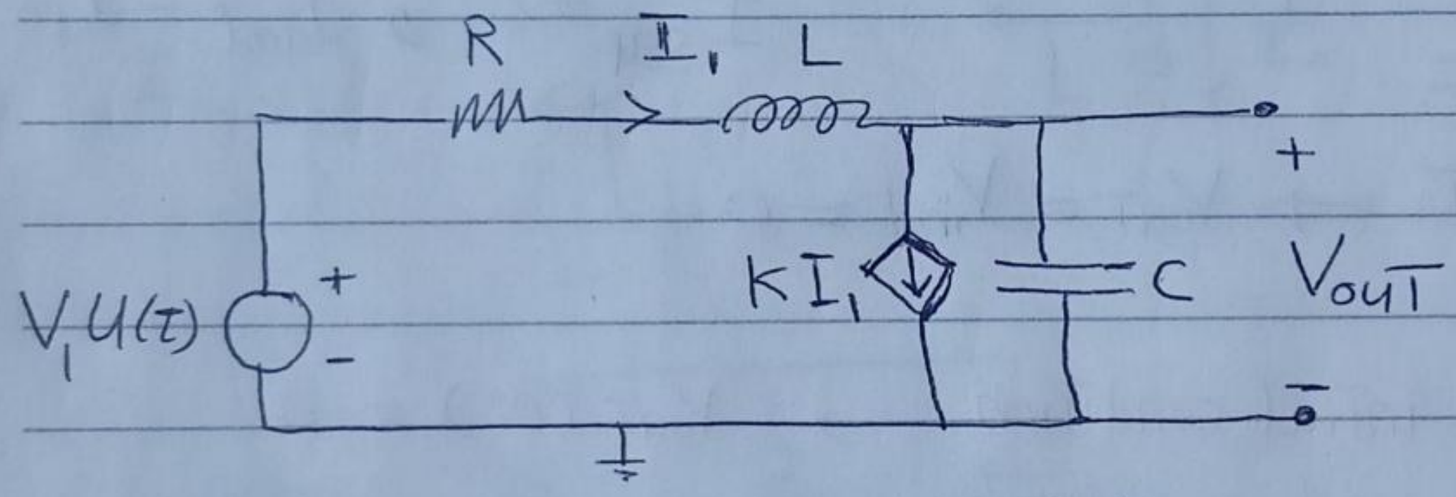
$\Rightarrow \frac{d^2 V_{out}}{d\tau^2}$ must have $\delta(\tau)$

$\Rightarrow \frac{dV_{out}}{d\tau}$ has a step

$\Rightarrow V_{out}$ is contineous



Example



KCL: $I_1 = KI_1 + C\frac{dV_{out}}{d\tau} \Rightarrow I_1 = \frac{C}{1-K}\frac{dV_{out}}{d\tau}$

$$L\frac{d}{d\tau}\left(\frac{C}{1-K}\frac{dV_{out}}{d\tau}\right) + \frac{RC}{1-K}\frac{dV_{out}}{d\tau} + V_{out} = V_1 u(\tau)$$

$$\frac{LC}{1-K}\frac{d^2V_{out}}{d\tau^2} + \frac{RC}{1-K}\frac{dV_{out}}{d\tau} + V_{out} = V_1 u(\tau)$$

$$\frac{d^2V_{out}}{d\tau^2} + \frac{R}{L}\frac{dV_{out}}{d\tau} + \frac{1}{L\frac{C}{1-K}}V_{out} = \frac{V_1}{\frac{LC}{1-K}}u(\tau)$$

what if $K=1$? $\Rightarrow V_{out}=0$ because all of I_1 must flow through the dependent current source.

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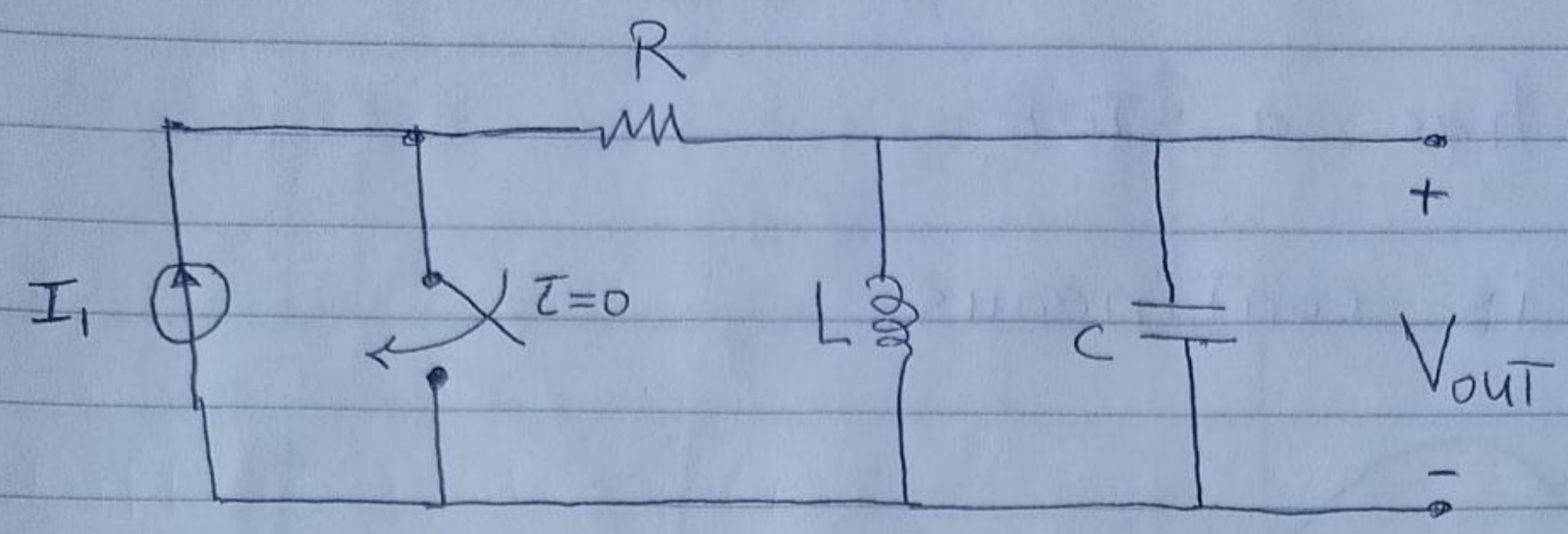
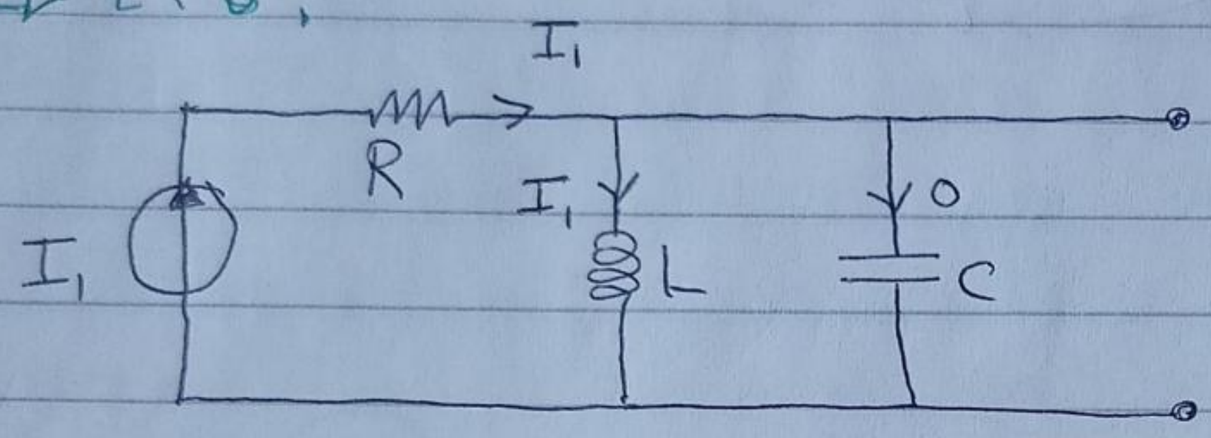
Example



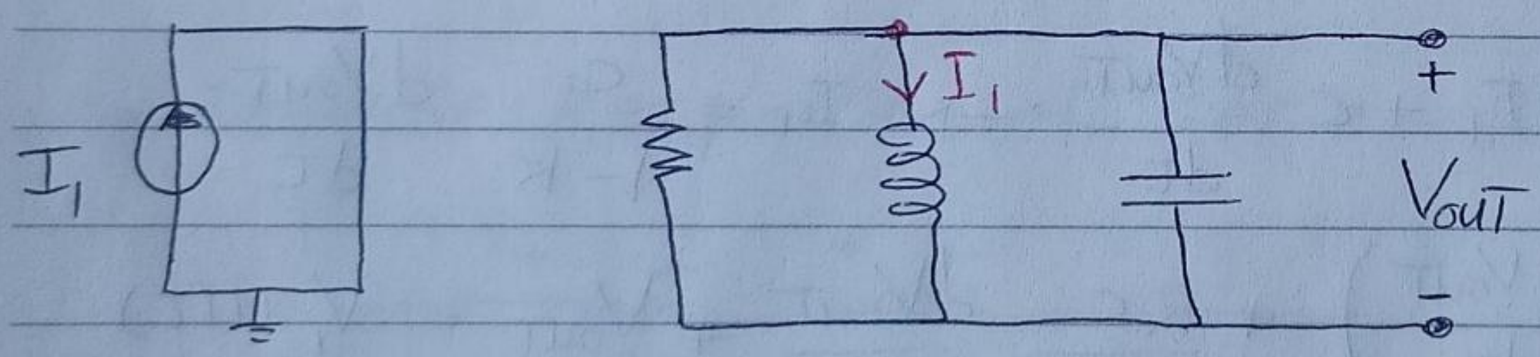
$\Rightarrow t<0$:



Since all quantities are const. $\Rightarrow$ $I_{cap}=0$, $I_{ind}=I_1$

Since I_{ind} is const. $\Rightarrow V_{OUT}=V_{ind}=0$

$\Rightarrow t>0$: I_1: initial condition, $V_{OUT}(0^+)=0$

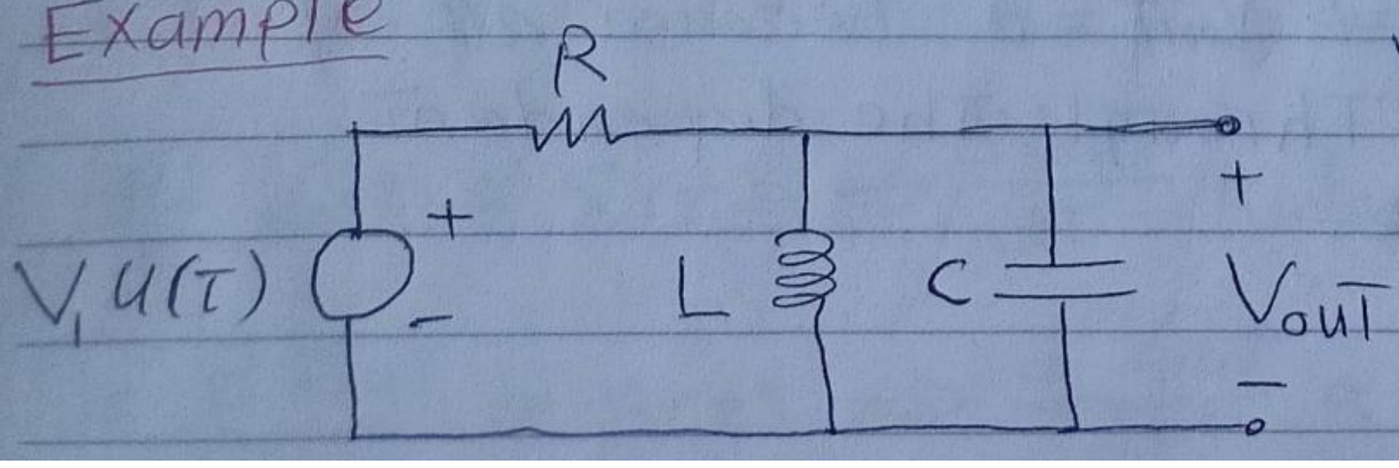


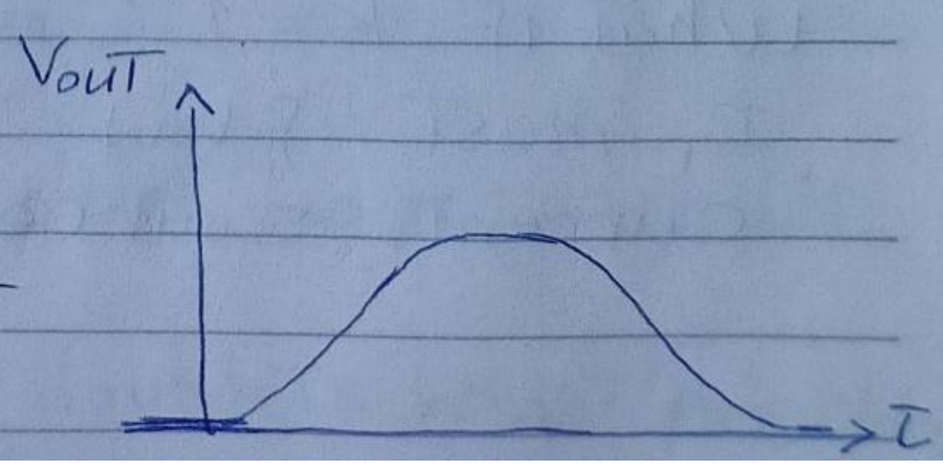
$$\begin{cases} \dfrac{d^2V_{OUT}}{dt^2}+\dfrac{1}{RC}\dfrac{dV_{OUT}}{dt}+V_{OUT}=0 \\ V_{OUT}(0)=0 \\ \left.\dfrac{dV_{OUT}}{dt}\right|_{0^+}=\dfrac{-I_1}{C} \end{cases}$$

KCL at $t=0^+$: $I_R=0 \xrightarrow{\text{due to}} V_{OUT}(0^+)=0$

$I_1+C\left.\dfrac{dV_{OUT}}{dt}\right|_{0^+}=0 \Rightarrow \left.\dfrac{dV_{OUT}}{dt}\right|_{0^+}=\dfrac{-I_1}{C}$

Example





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Lec 37: Diff. Equations with Impulse Inputs; More RLC Circuits Examples

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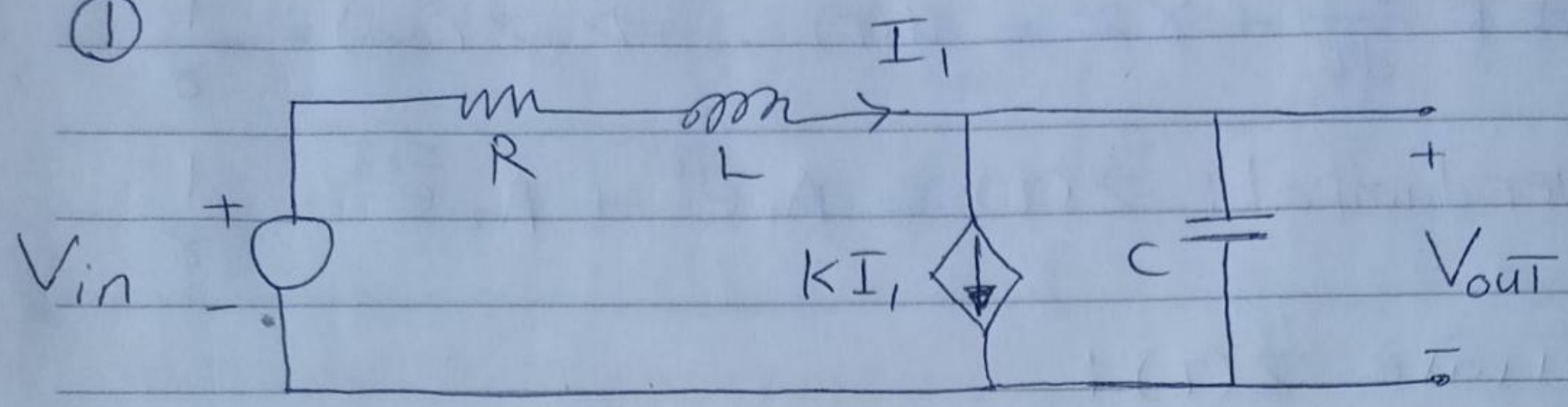
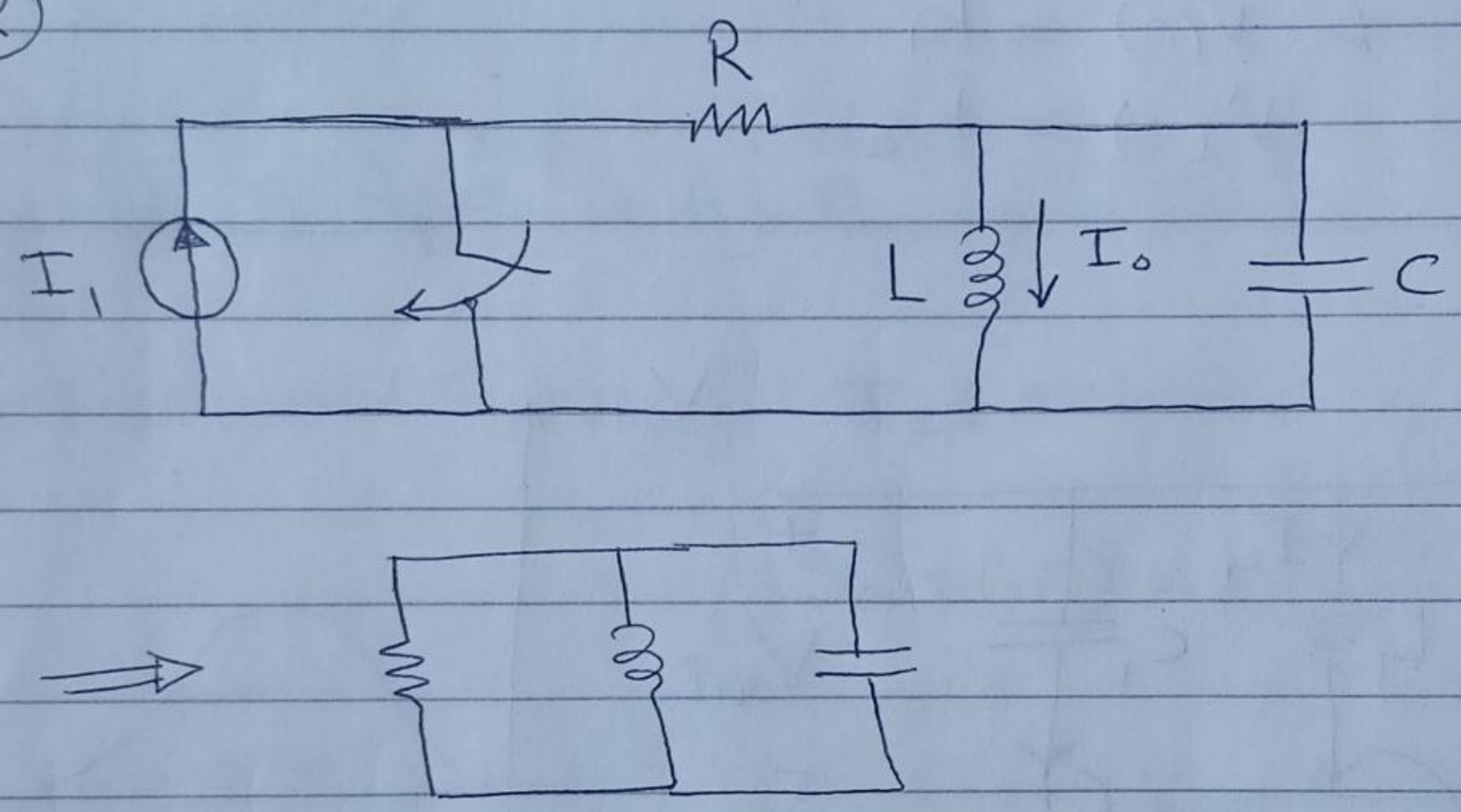
Review of lecture 36

①



$$C_{eq} = \frac{1}{1-K} C$$

②



Today's lecture:

- Diff. Eqs. with Impulses.
- other RLC circuit Examples.

① $A_1 e^{s_1 t} + A_2 e^{s_2 t}$

② $A_1 e^{s_1 t} + A_2 t e^{s_1 t}$

③ $A_1 e^{\alpha t} \cos \omega_1 t + A_2 e^{\alpha t} \sin \omega_1 t$

Diff. Eqs. with Impulses.

$$\alpha \frac{d^2 y}{dt} + \beta \frac{dy}{dt} + \gamma y = \delta(t)$$

Procedure:

① Replace impulse with a step.

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② Create a new diff. Eq.:

$$\alpha \frac{d^2 z}{d\tau^2} + \beta \frac{dz}{d\tau} + \gamma z = u(\tau) \quad \Rightarrow z(\infty) = \frac{1}{\gamma}$$

e.g., overdamped: $z(\tau) = A_1 e^{s_1 \tau} + A_2 e^{s_2 \tau} + \frac{1}{\gamma}$

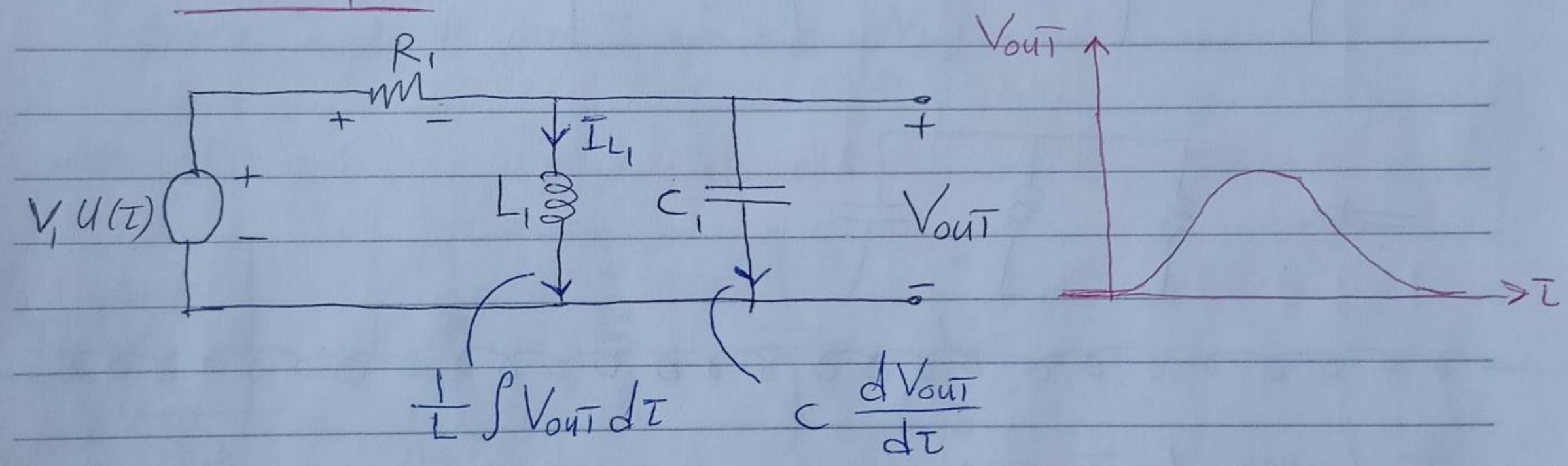
③ Differentiate $z(\tau)$:

$$\frac{dz}{d\tau} = A_1 s_1 e^{s_1 \tau} + A_2 s_2 e^{s_2 \tau} = y(\tau)$$

at $\tau = 0 \Rightarrow y(0) = K_1$

$y'(0) = K_2$

<u>Example:</u>



$$R_1\left(\frac{1}{L_1}\int V_{out}\, d\tau + C_1 \frac{dV_{out}}{d\tau}\right) + V_{out} = V_1 u(\tau)$$

$$R_1 C_1 \frac{d^2 V_{out}}{d\tau^2} + \frac{dV_{out}}{d\tau} + \frac{R_1}{L_1} V_{out} = V_1 \delta(\tau)$$

$$R_1 C_1 \frac{d^2 z}{d\tau^2} + \frac{dz}{d\tau} + \frac{R_1}{L} z = V_1 u(\tau)$$

$$z(\tau) = \left(A_1 e^{s_1 \tau} + A_2 e^{s_2 \tau} + \frac{V_1 L}{R_1}\right) u(\tau)$$

$$\frac{dz}{d\tau} = V_{out}(\tau) = \left(s_1 A_1 e^{s_1 \tau} + s_2 A_2 e^{s_2 \tau}\right) u(\tau)$$

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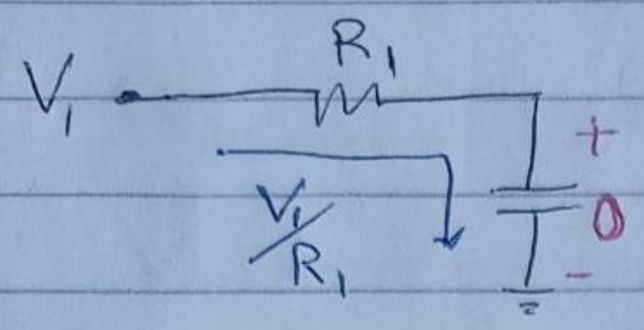
Find initial conditions: $V_{OUT}(0^+) = 0$



$$C \left.\frac{dV_{OUT}}{d\tau}\right|_{0^+} = \frac{V_1}{R_1}$$

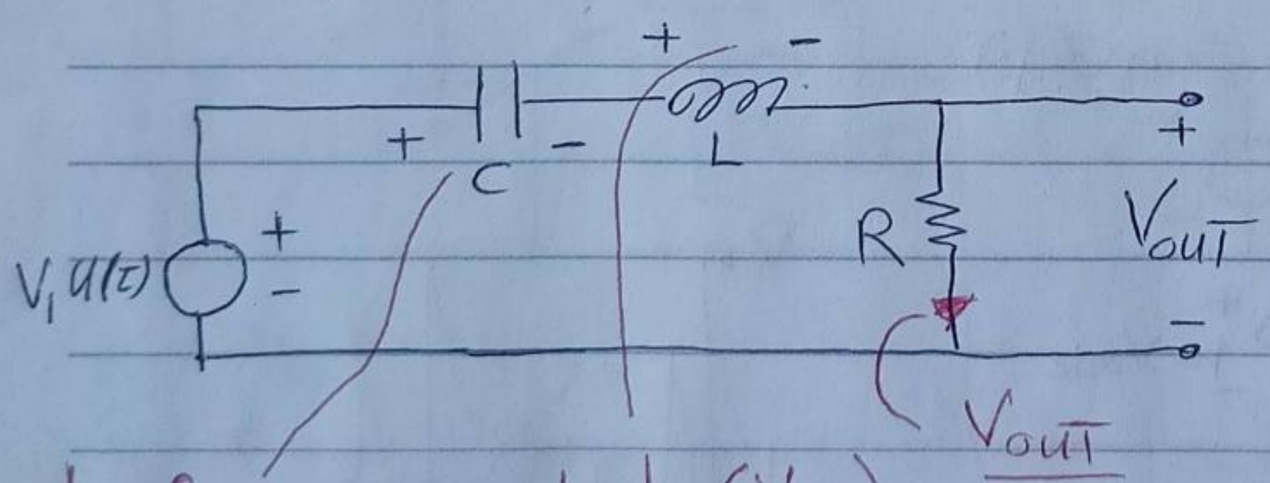
Alternative Approach:

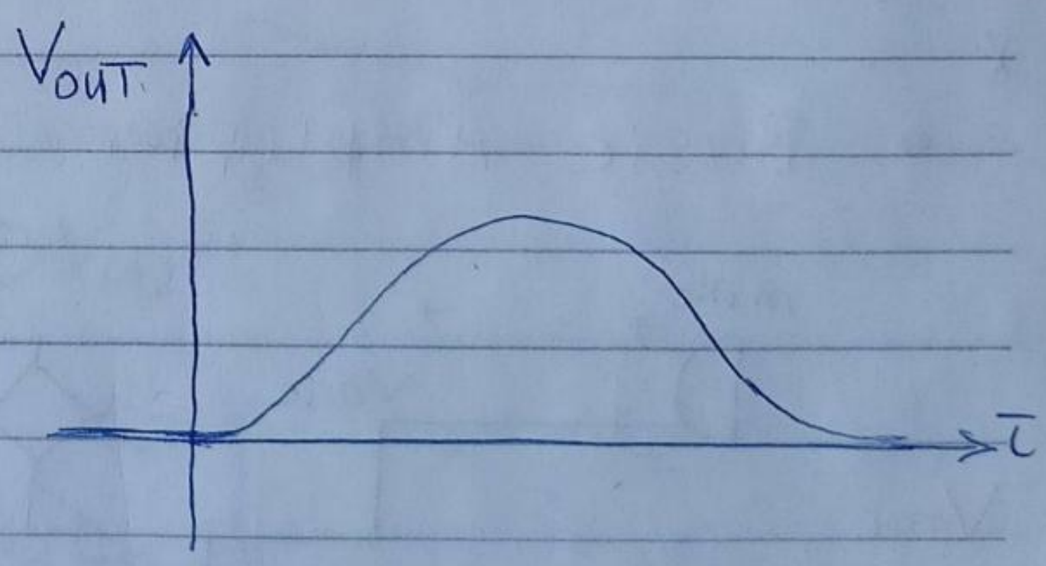
$$R_1\left[C_1 \frac{d}{d\tau}\left(L_1 \frac{dI_1}{d\tau}\right) + I_{L_1}\right] + L_1 \frac{dI_1}{d\tau} = V_1\, u(\tau)$$

$$I_{L_1} = A_1 e^{s_1\tau} + A_2 e^{s_2\tau} + I_{L_1}(\infty) \quad , \; I_{L_1}(\infty) = \frac{V_1}{R_1}$$

initial conditions
$$\begin{cases} I_{L_1}(0^+) = 0 \\ V_{OUT}(0^+) = L_1 \dfrac{dI_{L_1}}{d\tau} = 0 \Rightarrow \left.\dfrac{dI_{L_1}}{d\tau}\right|_{0^+} = 0 \end{cases}$$

Example





$$\frac{1}{C}\int \frac{V_{OUT}}{R} d\tau + \frac{L}{R}\frac{dV_{OUT}}{d\tau} + V_{OUT} = V_1\, u(\tau)$$

$$\frac{L}{R}\frac{d^2 V_{OUT}}{d\tau^2} + \frac{dV_{OUT}}{d\tau} + \frac{1}{RC} V_{OUT} = V_1\, \delta(\tau)$$

$$\frac{L}{R}\frac{d^2 Z}{d\tau^2} + \frac{dZ}{d\tau} + \frac{1}{RC} Z = V_1\, u(\tau)$$

$$Z(\tau) = A_1 e^{s_1\tau} + A_2 e^{s_2\tau} + V_1 RC$$

$$\frac{dZ}{d\tau} = V_{OUT}(\tau) = s_1 A_1 e^{s_1\tau} + s_2 A_2 e^{s_2\tau}$$

$$\begin{cases} V_{OUT}(0^+) = 0 \\ V_{L_1} = V_1 \text{ at } 0^+ \end{cases}$$

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Lec 38: Introduction to Op Amps

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Review of lecture 37

$$\alpha \frac{d^2y}{d\tau^2} + \beta \frac{dy}{d\tau} + \gamma y = \delta(\tau)$$

$$\alpha \frac{d^2Z}{d\tau^2} + \beta \frac{dZ}{d\tau} + \gamma Z = u(\tau)$$

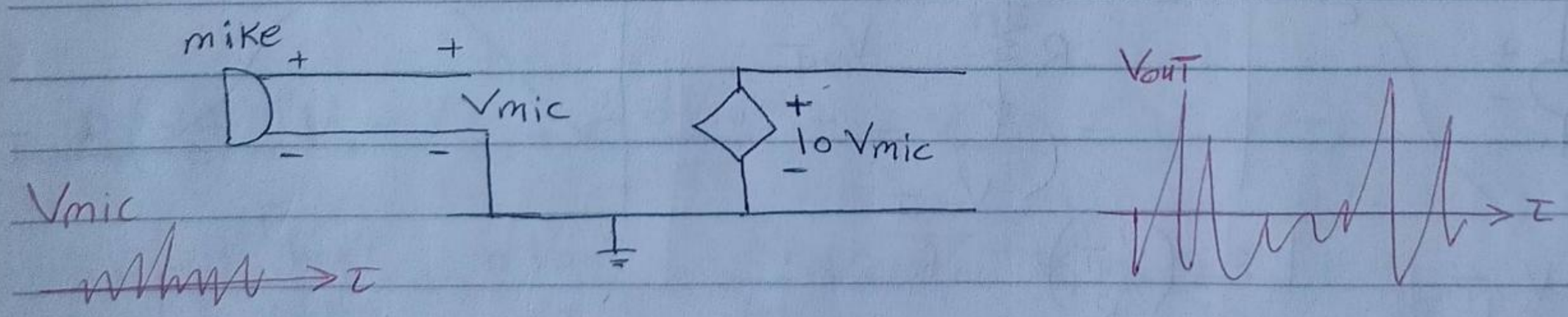
$$Z(\tau) = \left(A_1 e^{s_1\tau} + A_2 e^{s_2\tau} + \frac{1}{\gamma}\right) u(\tau)$$

$$\Rightarrow \frac{dZ}{d\tau} = y(\tau)$$

Today's lecture:

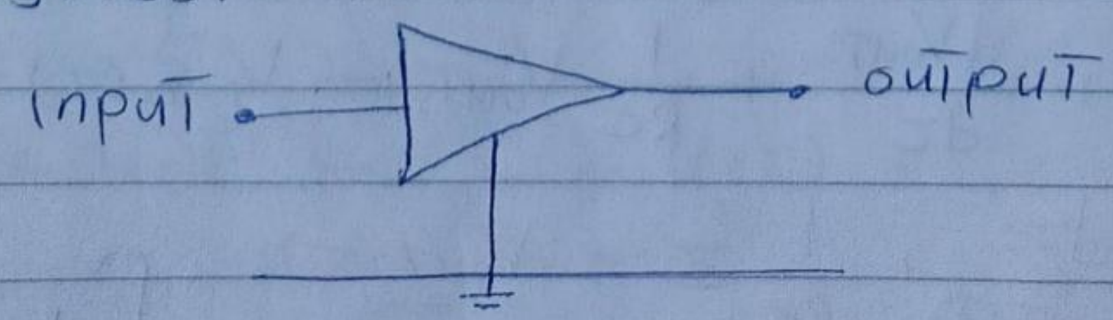
- Basic Amplifier concepts
- operational Amplifier
- circuit Example

Basic Amplifier concepts.



NOTES: ① Voltage Gain (A_o): output / input

② Symbol



③ Amplifier has a zero input current.

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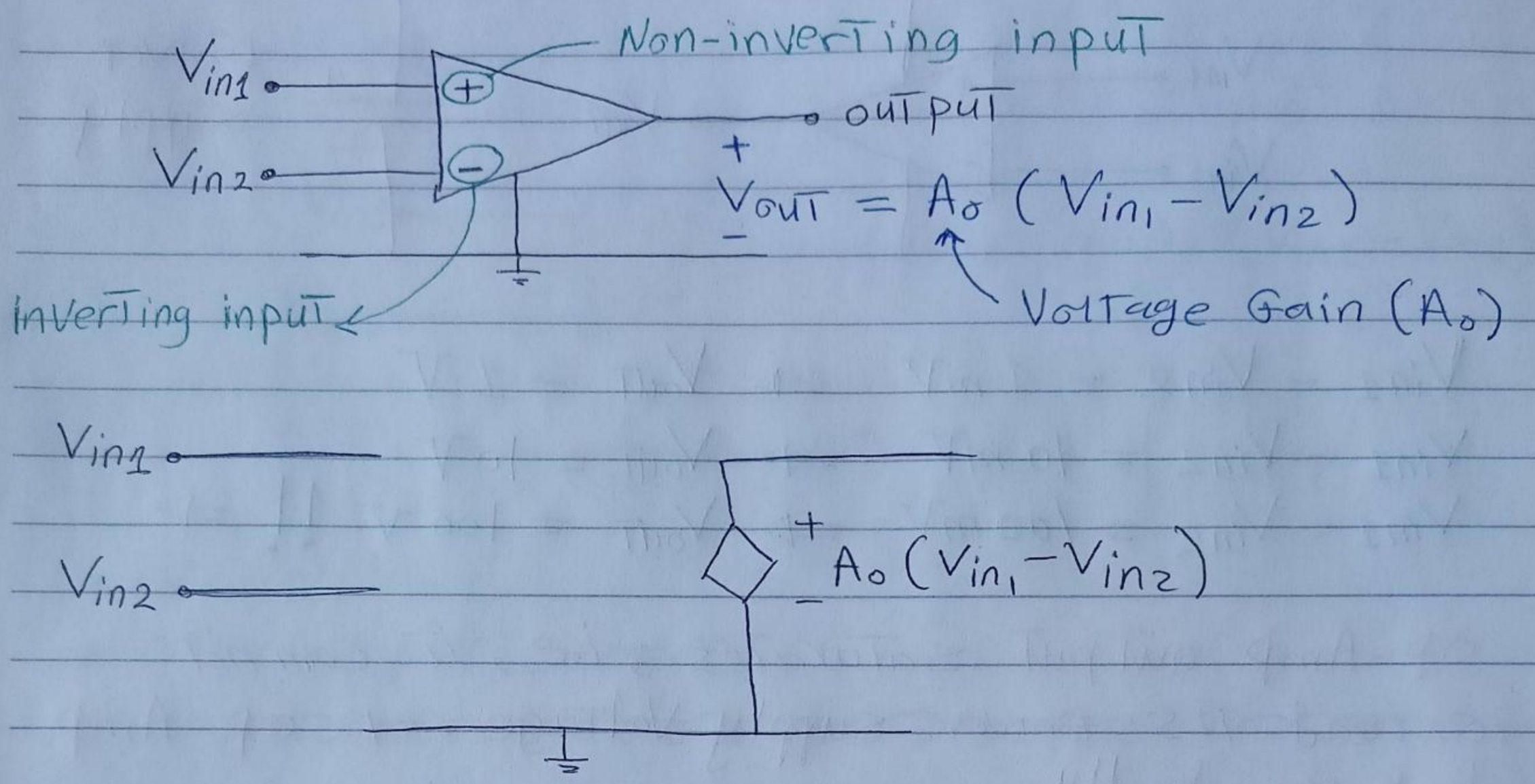
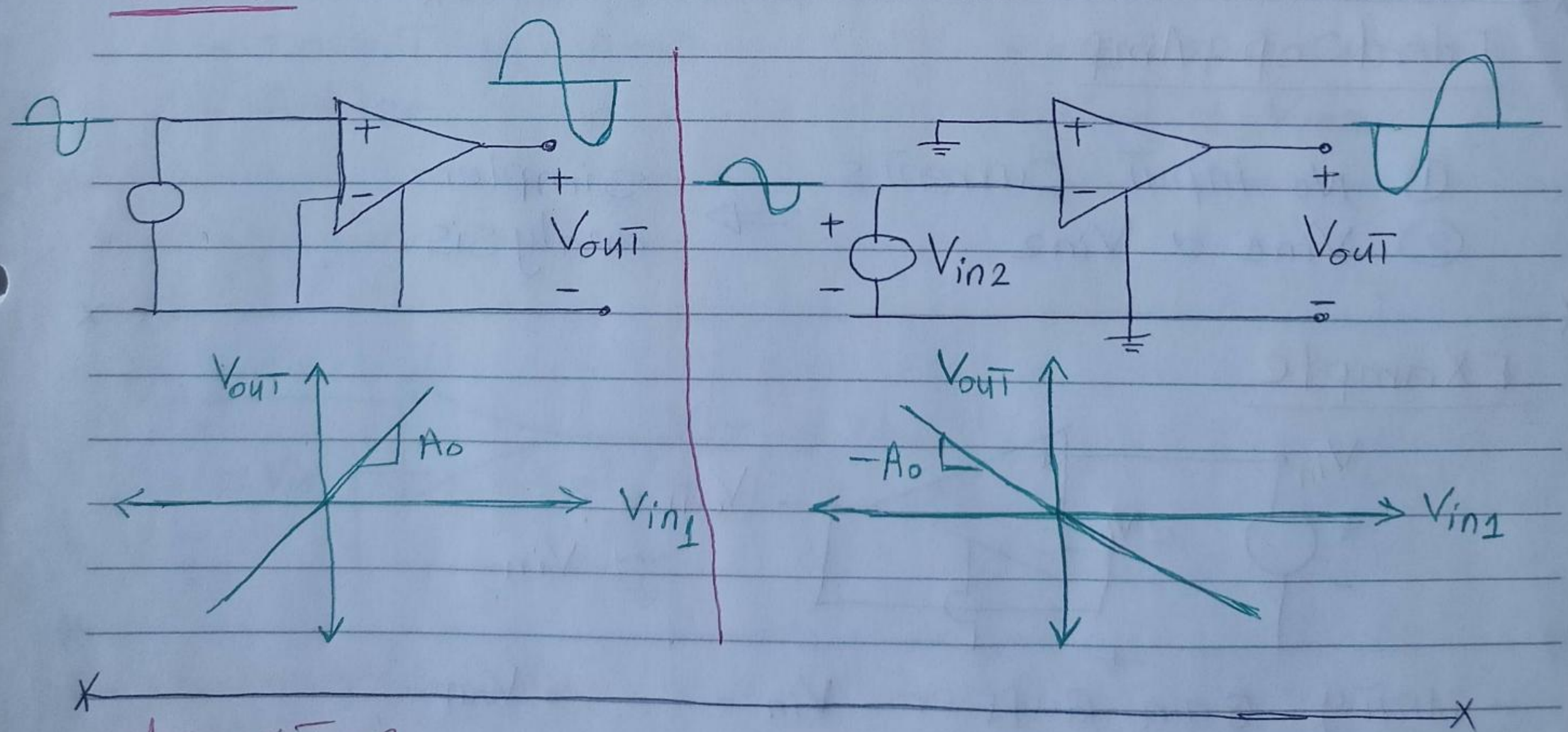
- operational Amplifier



- op-Amp has zero input currents.

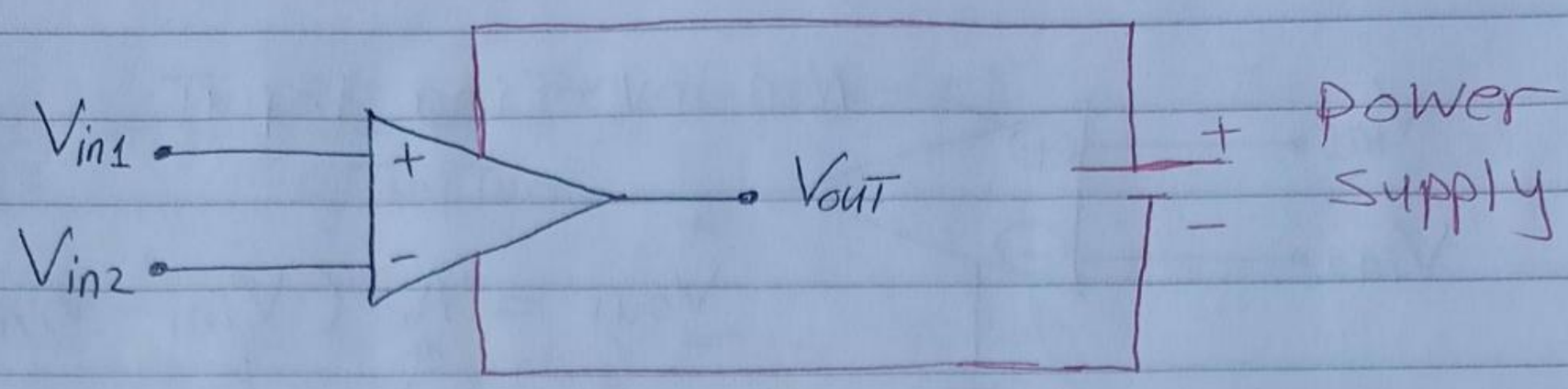
Study



- observation

op-Amps Typically have high gain
: 100 — 100,000

Example



$V_{in1} - V_{in2} = 1\,mV \Rightarrow V_{OUT} = 1\,V$

$V_{in1} - V_{in2} = 10\,mV \Rightarrow V_{OUT} = 10\,V$

$V_{in1} - V_{in2} = 100\,mV \Rightarrow V_{OUT} = 100\,V$!! $\Rightarrow$

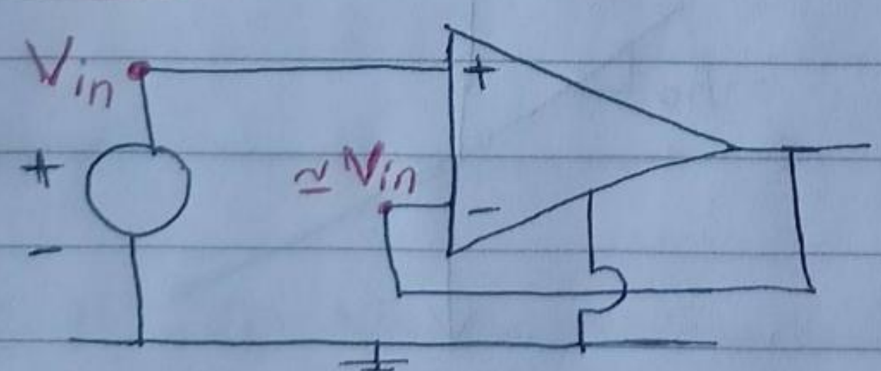
op-Amp output saturates, i.e, it can not exceed its power supply voltage. $\Rightarrow$ op-Amp behaves badly.

In a well-designed circuit, $|V_{in1} - V_{in2}|$ is small.

Ideal op-Amp

① No input currents
② $V_{in1} \simeq V_{in2}$ $\Rightarrow$ simpler analysis.

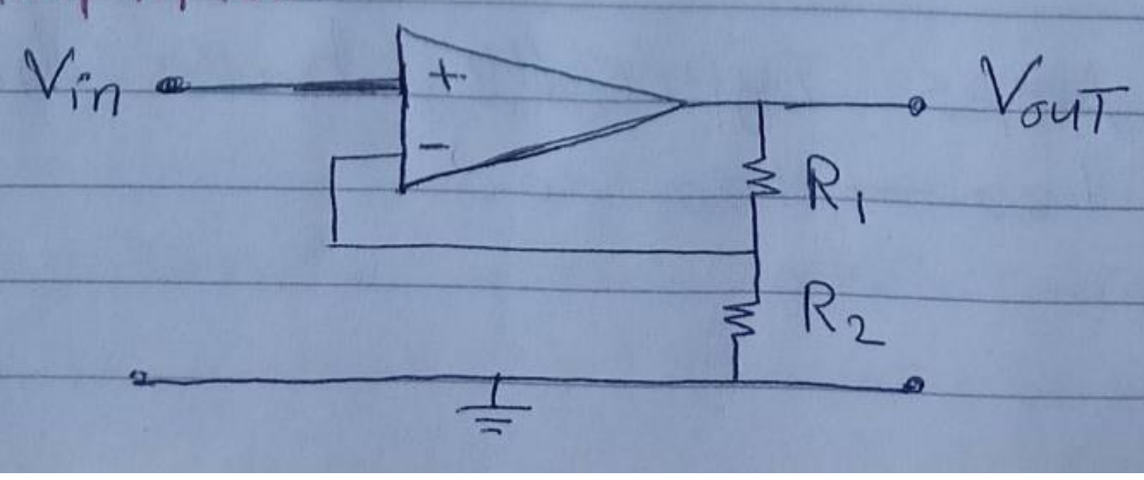
Example



$V_{OUT} = ?$
$\approx V_{in}$

- Unity-Gain Buffer

- Non-inverting Amplifier.



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Lec 39: Noninverting and Inverting Amplifiers

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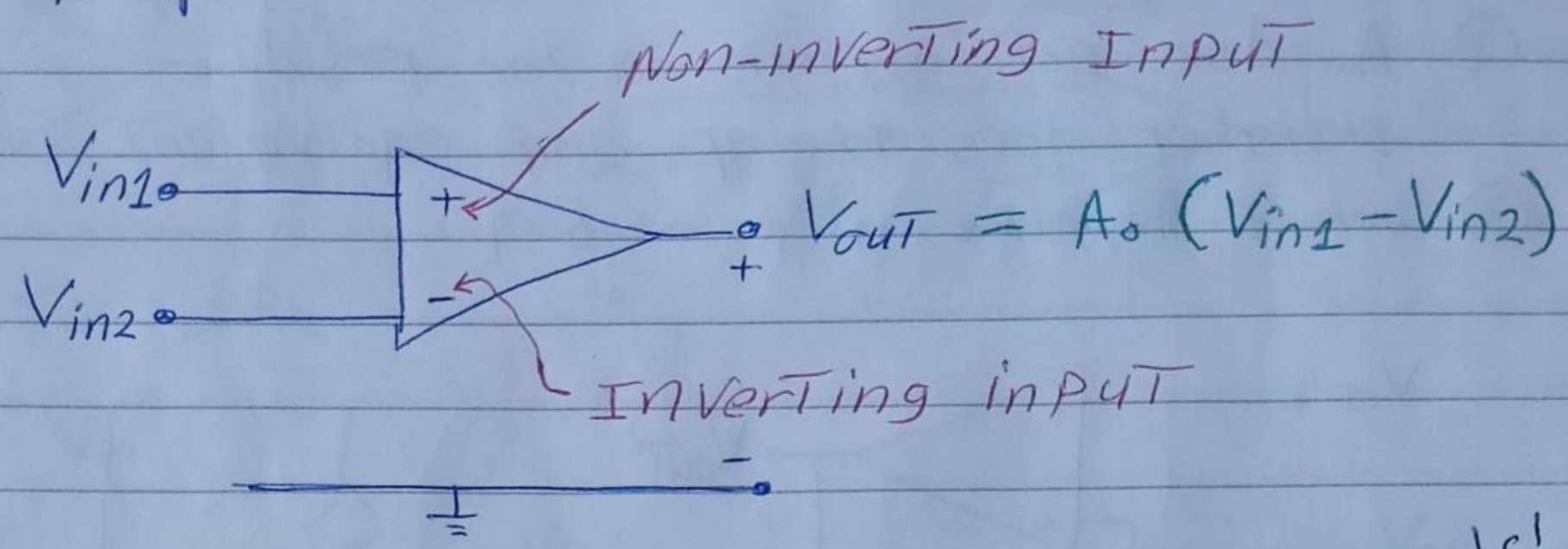
Review of lecture 38

- op - Amp

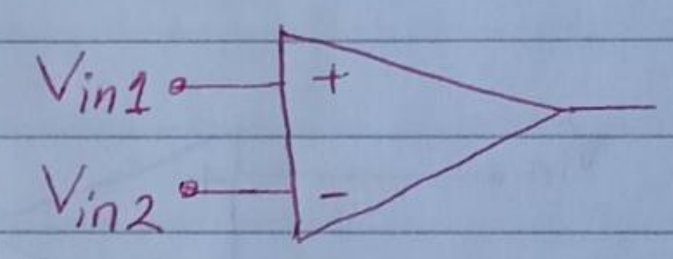


- $V_{in1} \approx V_{in2}$.
- zero input currents.

ابراهيم محمد

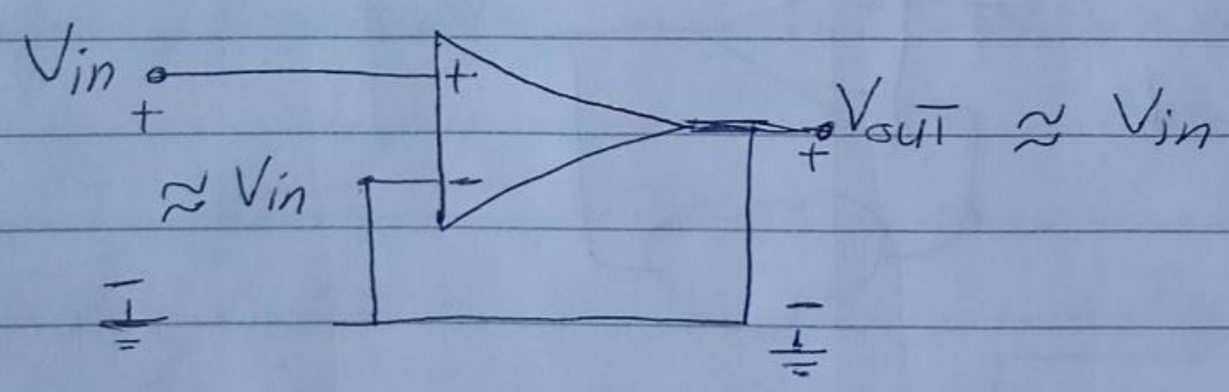
Today's lecture:

- Non-inverting Amp.
- Inverting Amp
- Adder

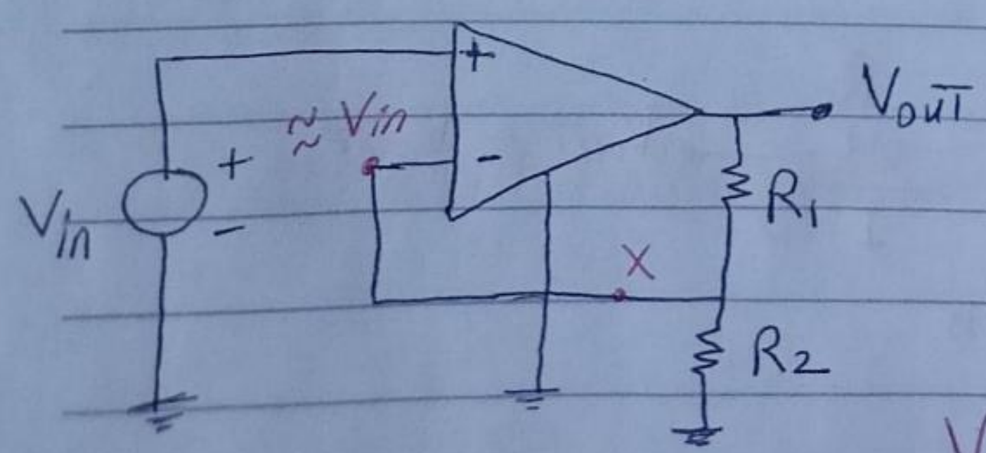


- No input currents
- $V_{in1} \approx V_{in2}$

- Unity - Gain buffer



- Non-inverting Amp.



$$V_x = \frac{R_2}{R_1 + R_2} V_{out} \approx V_{in}$$

$$V_{out} = \left(1 + \frac{R_1}{R_2}\right) V_{in} \quad \& \quad Gain = 1 + \frac{R_1}{R_2}$$

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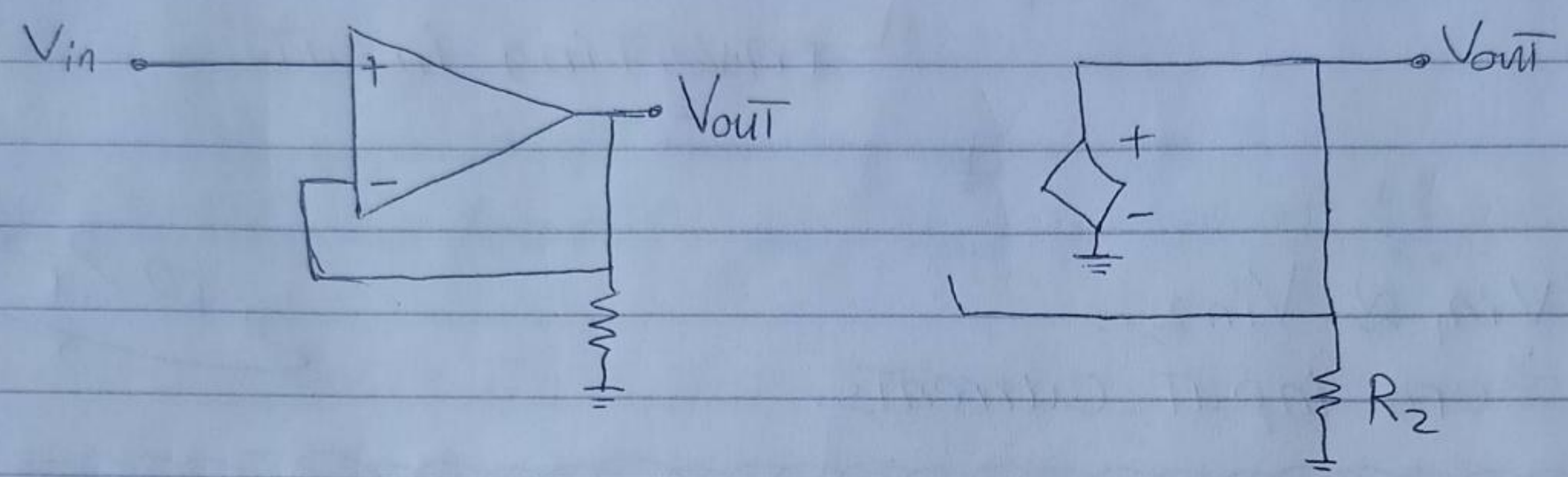
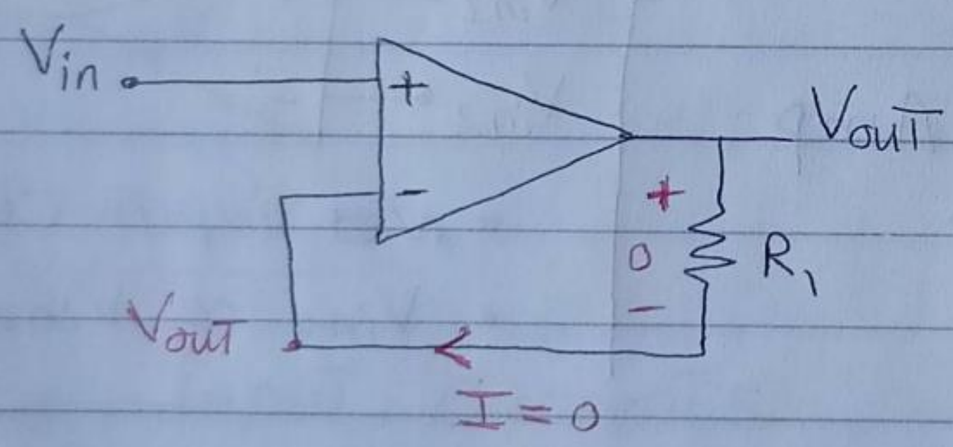
observations

① V_{out} has the same sign as V_{in} ⇒ Non-inverting
② A_o does not appear in the result.
Neigher does the op amp equivalent circuit.
③ what happens if $R_1 \to 0$?

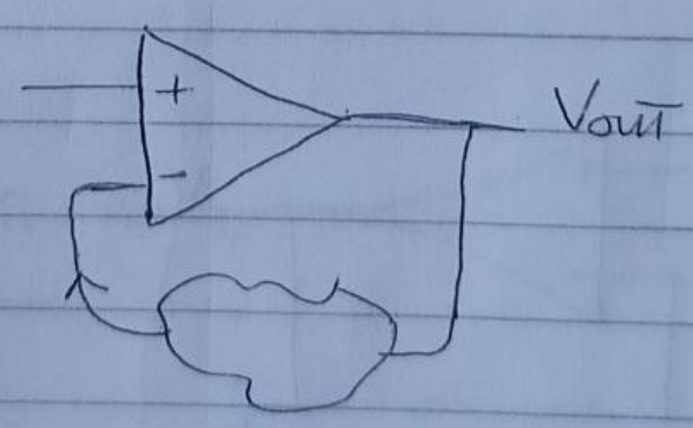


④ what happens if $R_2 \to \infty$?

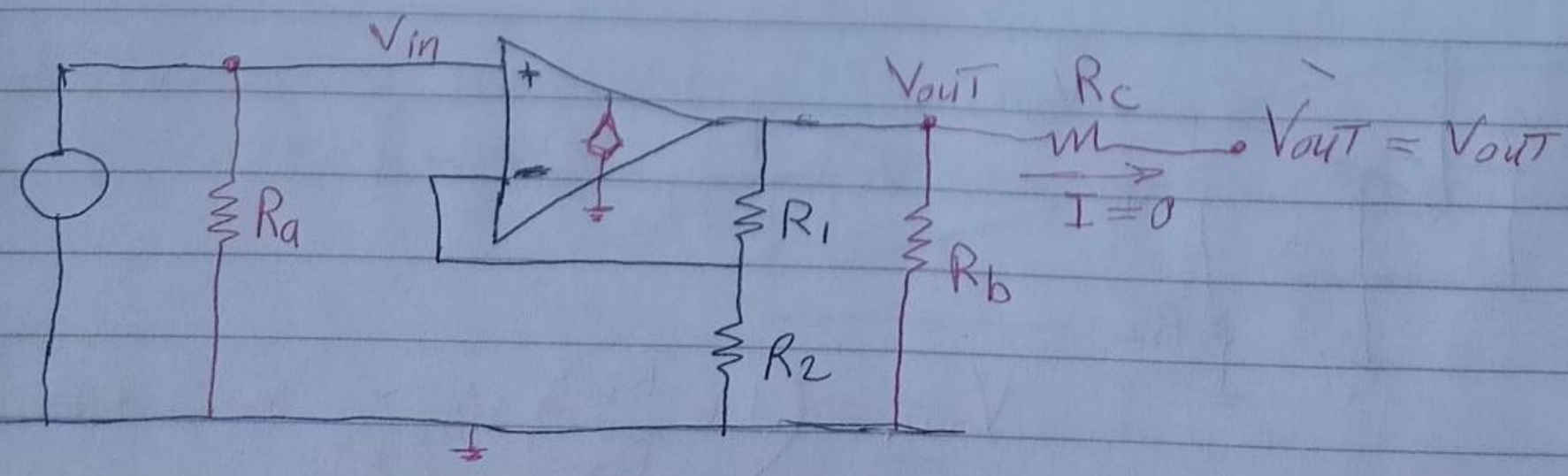


⑤ "feedback" :

⇒ Negetive feedback



• Twists on the Non-inverting Amp.



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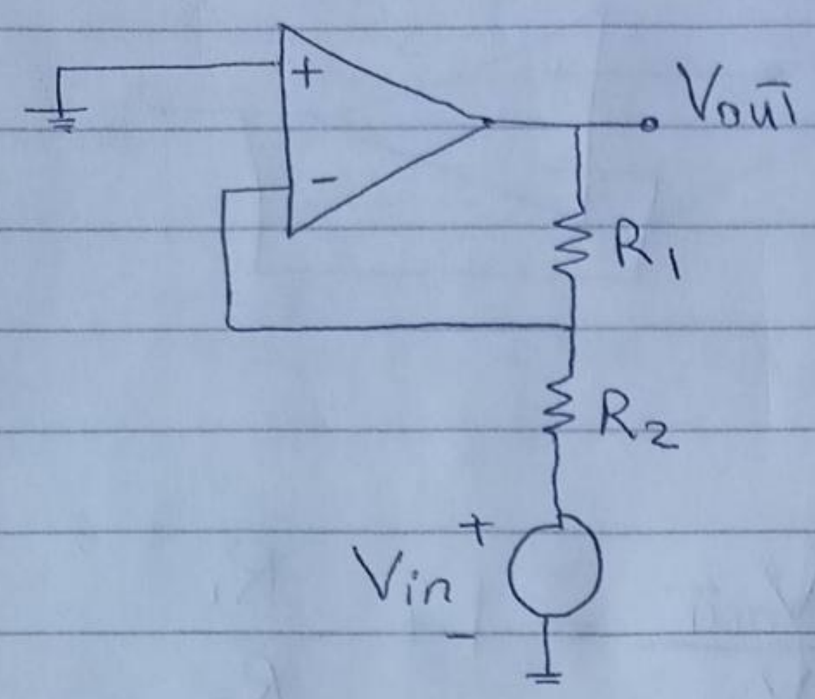
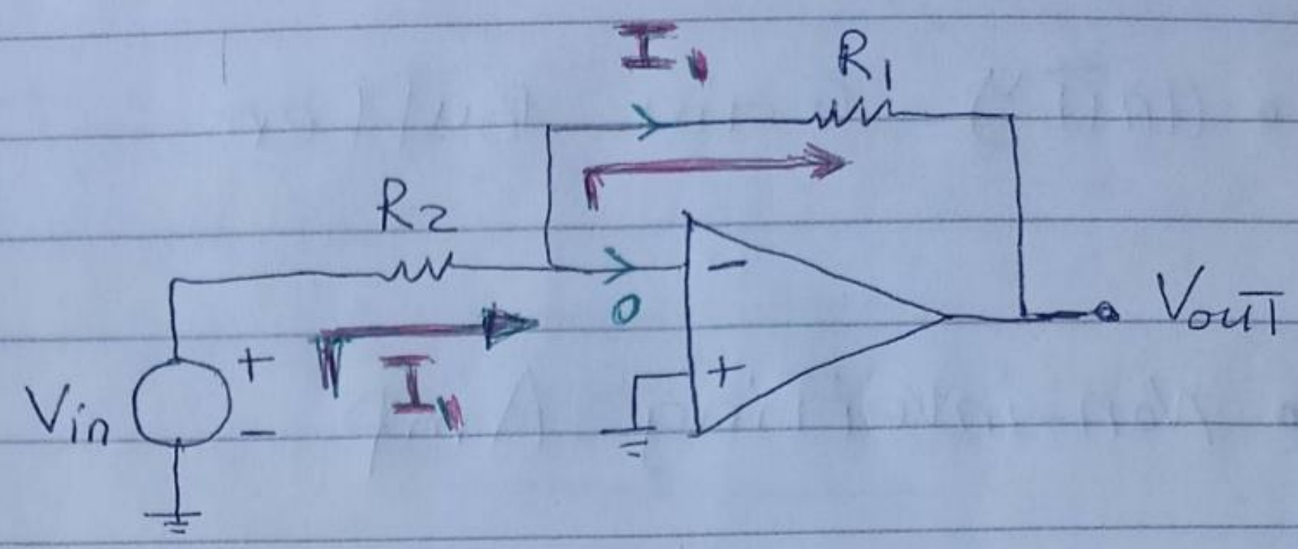
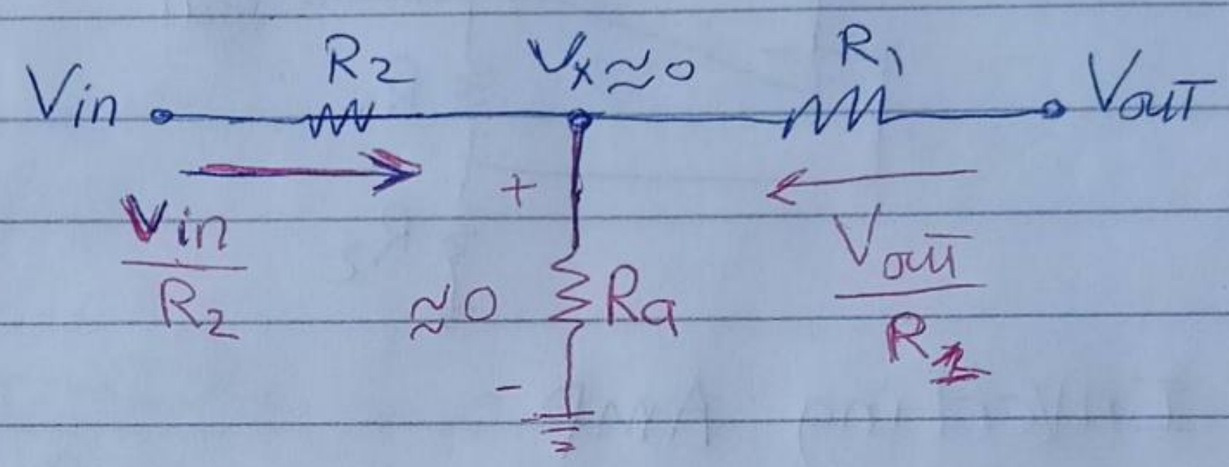
• Inverting Amp.





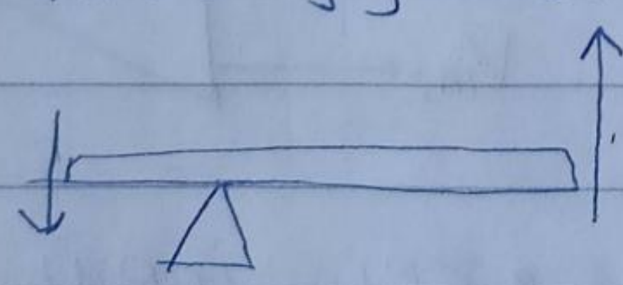


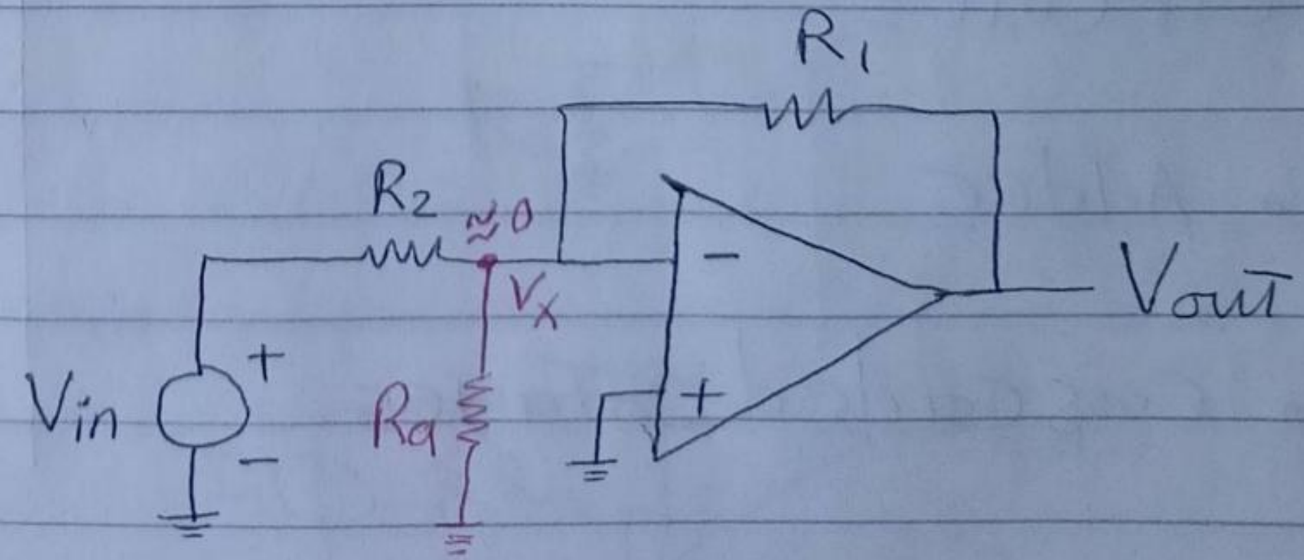
$$\frac{V_{in}}{R_2} + \frac{V_{out}}{R_1} = 0 \Rightarrow \boxed{V_{out} = \frac{-R_1}{R_2} V_{in}}$$

observations:

① Circuit has negetive feedback.
② Node X has a voltage close to 0.
Node x ; a "virtual ground.

③ Analogy with seesaw:





KCL still holds because $\frac{V_x}{R_a}$ is small

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Lec 40: Adder (Summer) Circuit, Cascaded Op Amp Stages

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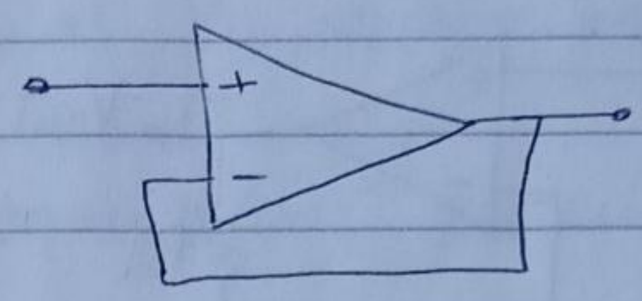
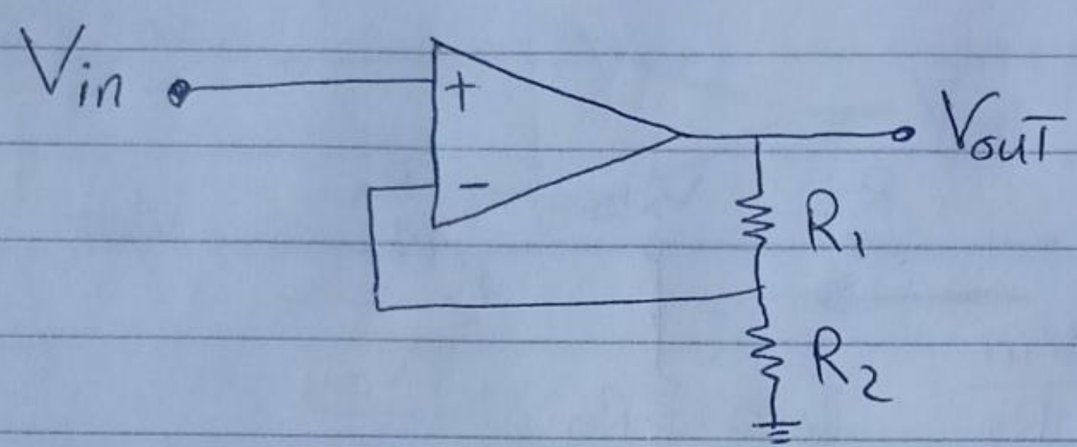
Review of lecture 39

- Unity - Gain Buffer

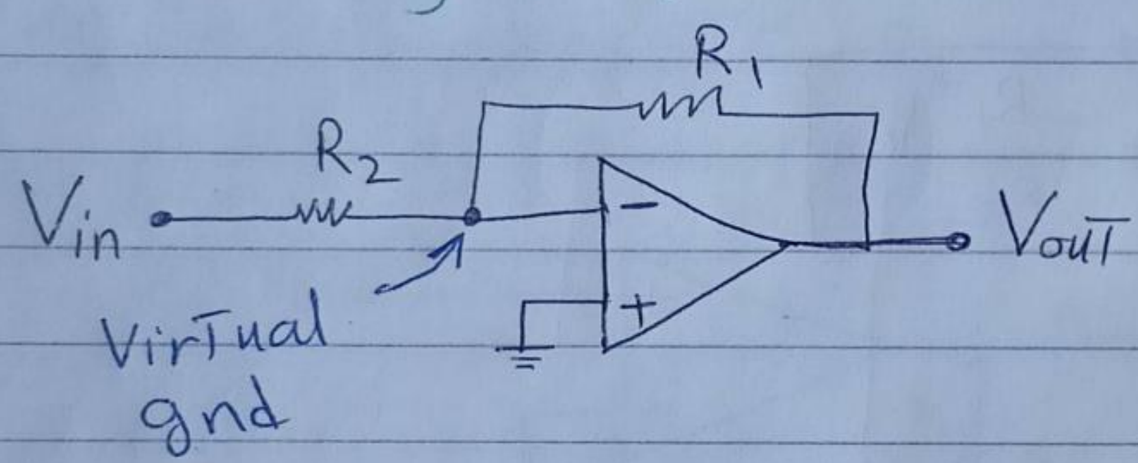


- Non-inverting Amp



$$\frac{V_{out}}{V_{in}} = 1 + \frac{R_1}{R_2}$$

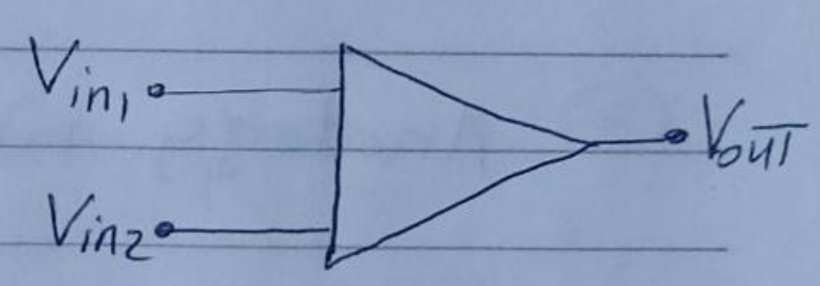
- Inverting Amp



$$\frac{V_{out}}{V_{in}} = -\frac{R_1}{R_2}$$

Today's lecture

- practice with op-Amp circuits
- Adder
- cascaded stages



- zero input currents
- $V_{in1} \approx V_{in2}$

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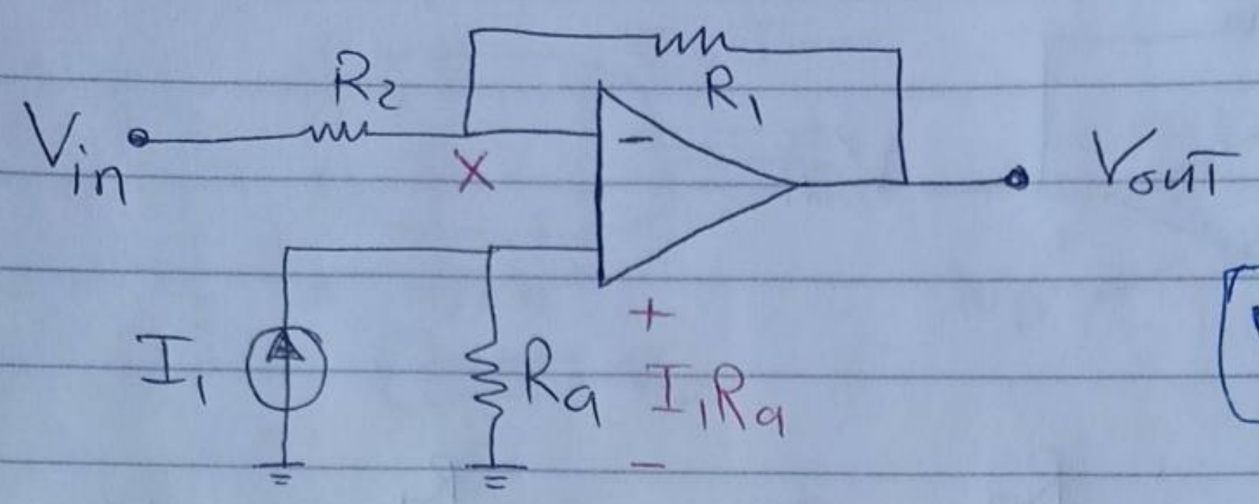
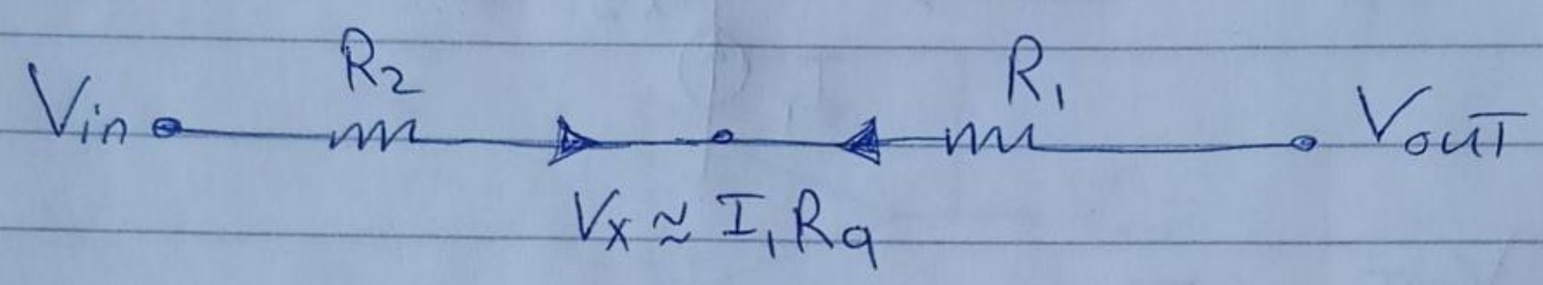
• Other Examples



$$\boxed{V_X \approx I_1 R_a}$$



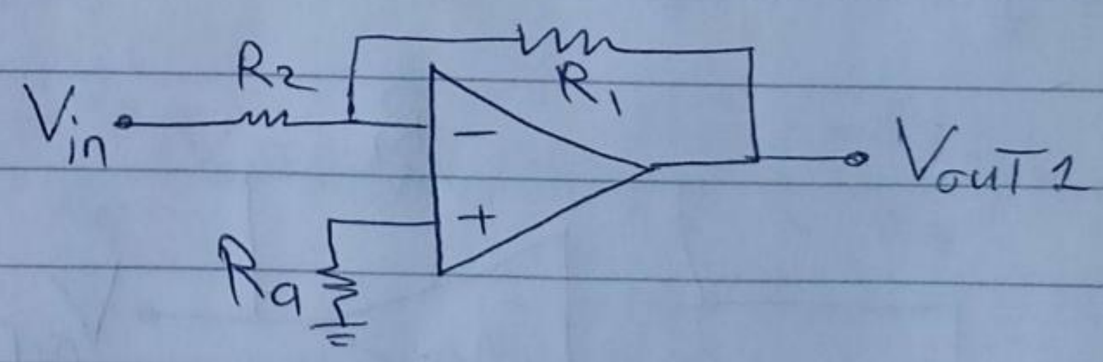
$$\frac{V_{in} - I_1 R_a}{R_2} + \frac{V_{OUT} - I_1 R_a}{R_1} = 0$$

$$\frac{V_{in}}{R_2} - \frac{I_1 R_a}{R_2} - \frac{I_1 R_a}{R_1} = -\frac{V_{OUT}}{R_1}$$

$$V_{OUT} = -R_1 \left(\frac{V_{in}}{R_2} - \frac{I_1 R_a}{R_2} - \frac{I_1 R_a}{R_1} \right)$$

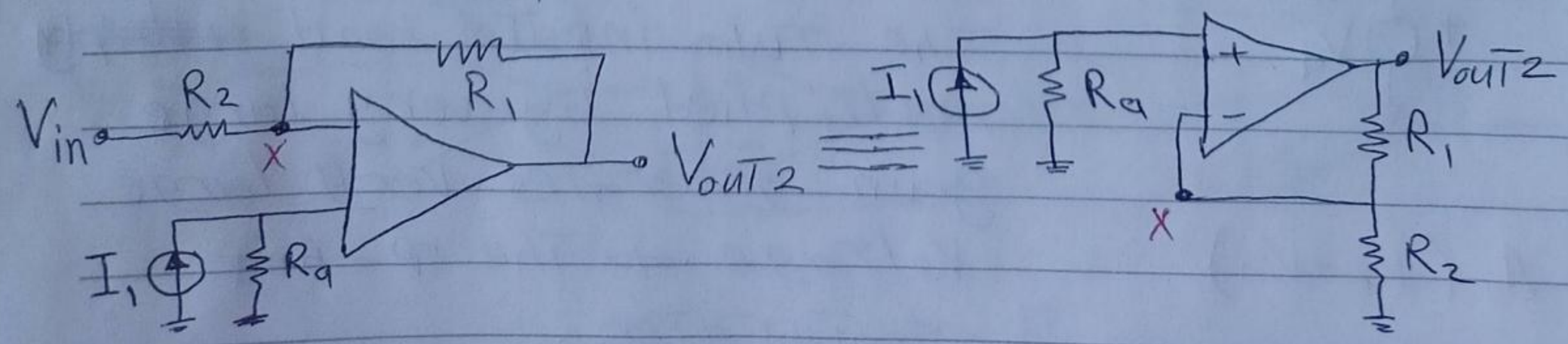
Alternative method : superposition

→ V_{in} not zero, $I_1 = 0$:



$$V_{OUT1} = -\frac{R_1}{R_2} V_{in}$$

→ $V_{in} = 0$, I_1 not zero :



$$V_{OUT2} = \left(1 + \frac{R_1}{R_2}\right) I_1 R_a$$

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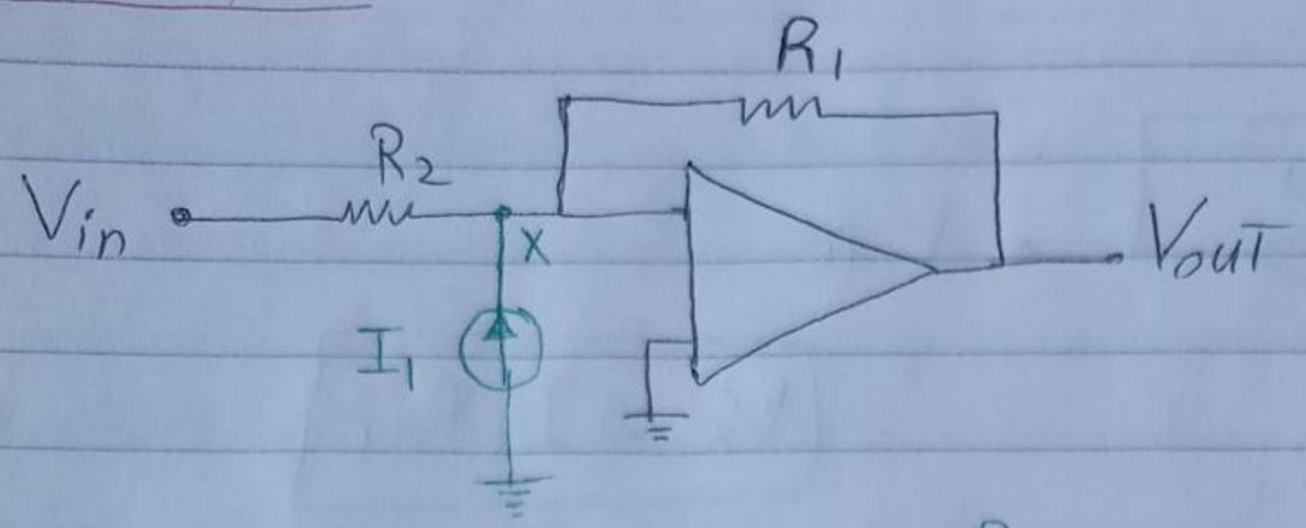
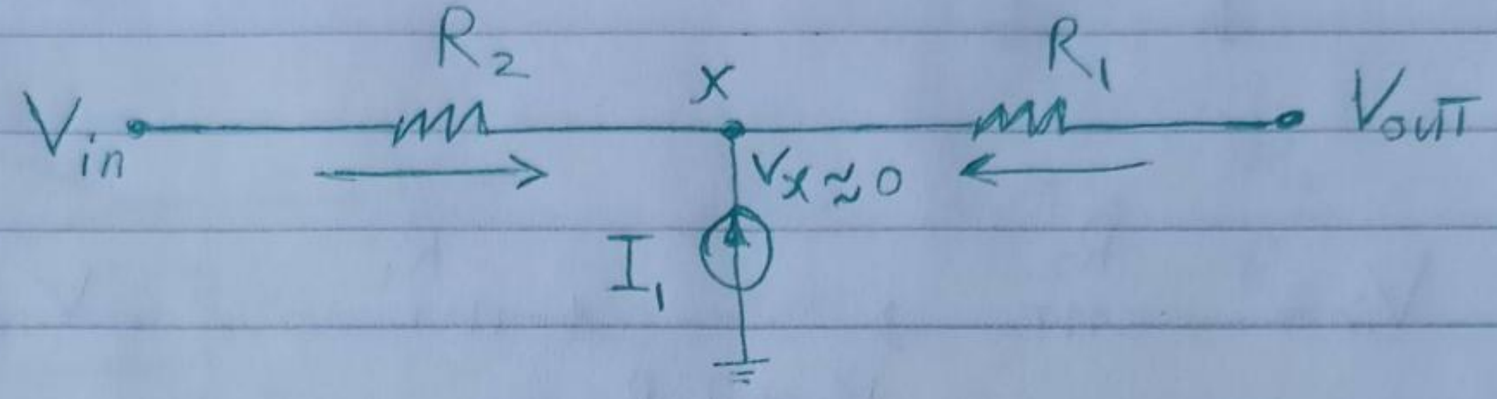
Example

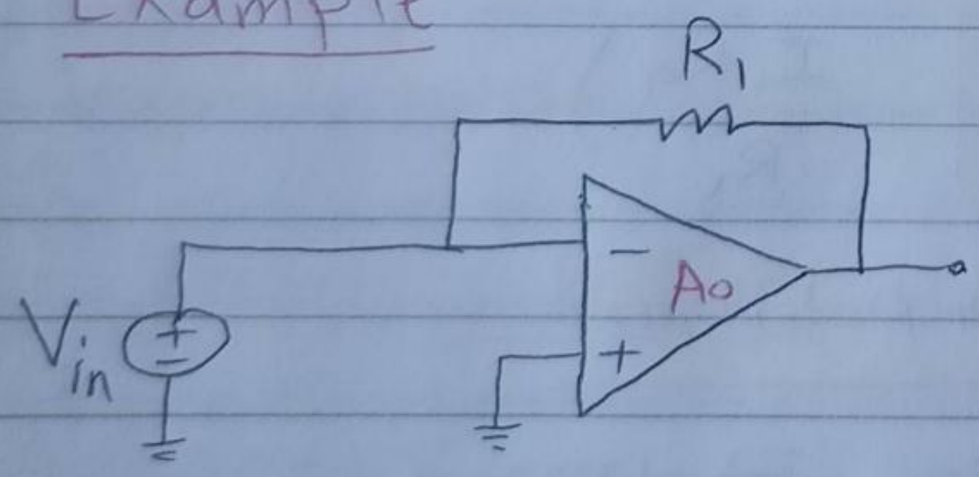




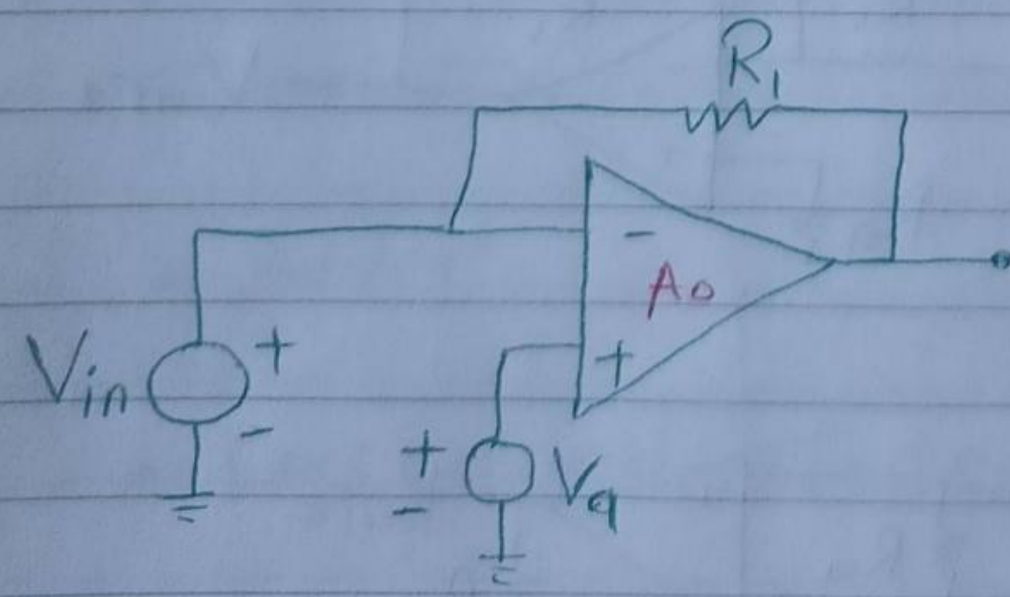
$$\frac{V_{in} - 0}{R_2} + \frac{V_{OUT}}{R_1} + I_1 = 0$$

$$\frac{V_{OUT}}{R_1} = -\left(\frac{V_{in}}{R_2} + I_1\right) \Rightarrow V_{OUT} = R_1\left(-\frac{V_{in}}{R_2} - I_1\right)$$

Example



$V_{OUT} = -A_o\,(V_{in} - 0) \Rightarrow$ op-Amp saturates.



$V_{OUT} \Rightarrow$ The op-Amp measures the difference between the two inputs and ~~multiply~~ multiplied by very large gain generates very large voltage $\Rightarrow$ The op-Amp saturates.

$V_{OUT} = A_o\,(V_a - V_{in})$

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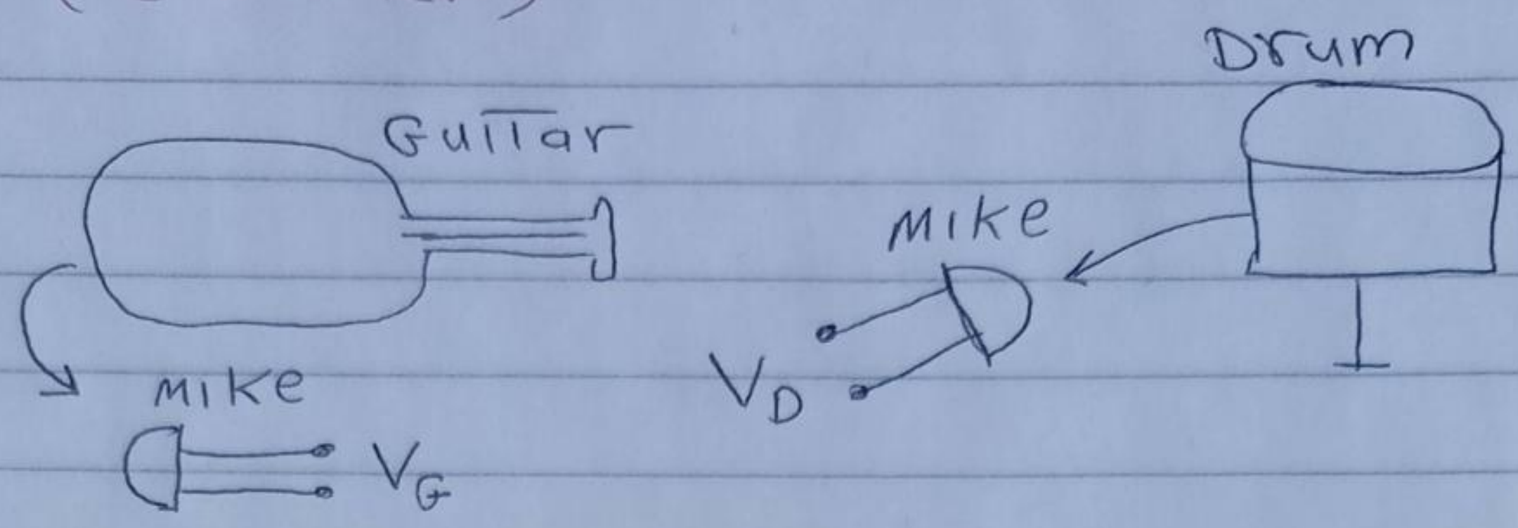
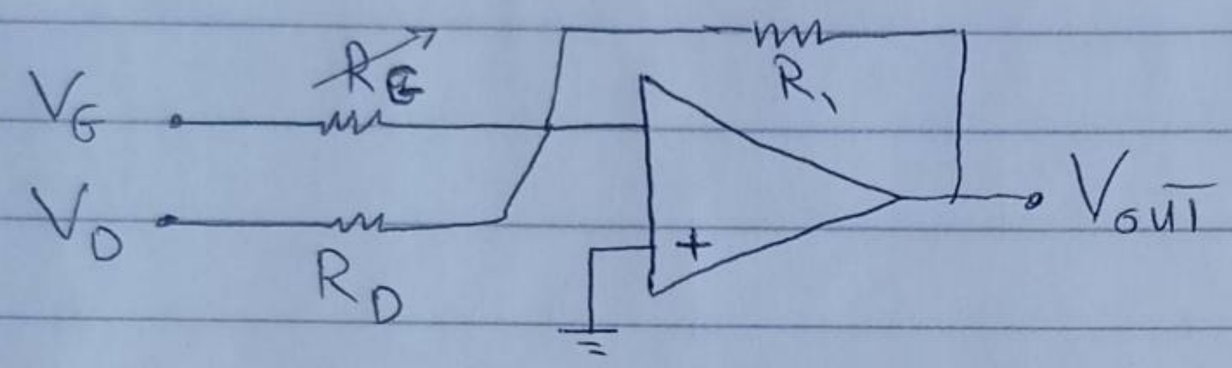
- Adder (Summer)

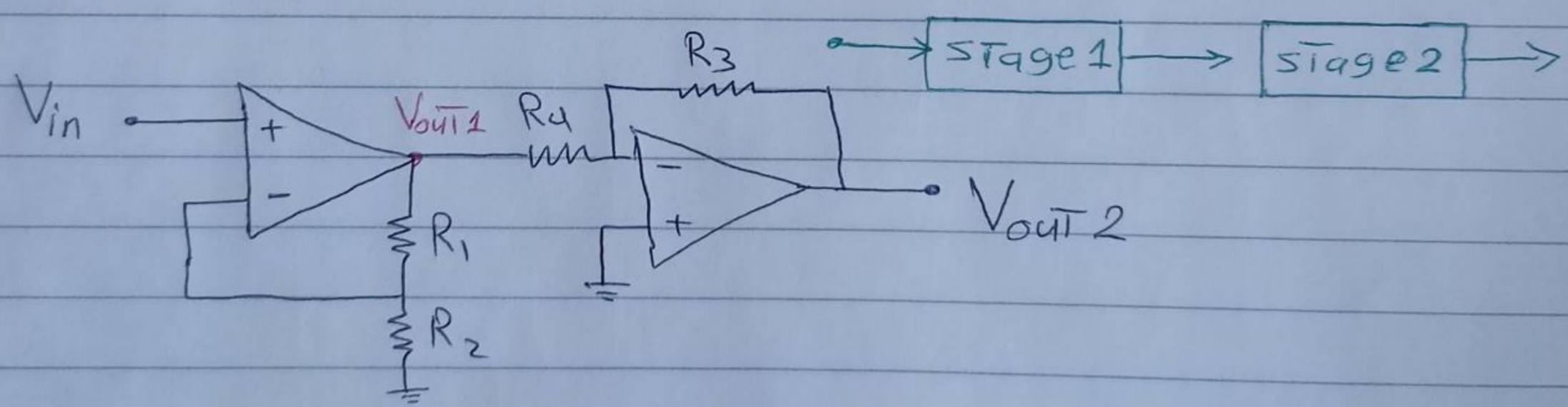


How do we "mix" V_G and V_D ? $(V_G + V_D)$



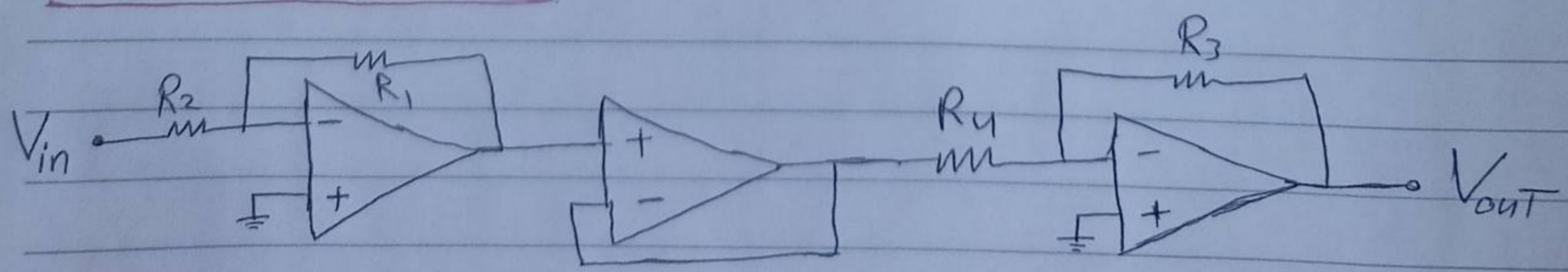
$$V_{OUT} = -\frac{R_1}{R_G} V_G - \frac{R_1}{R_D} V_D = -R_1 \left(\frac{V_G}{R_G} + \frac{V_D}{R_D} \right)$$

- Cascaded stages



$$V_{OUT1} = \left(1 + \frac{R_1}{R_2}\right) V_{in} \quad , \quad V_{OUT2} = \frac{-R_3}{R_4} V_{OUT1} = \left(1 + \frac{R_1}{R_2}\right)\left(\frac{-R_3}{R_4}\right) V_{in}$$

Another Example



$$V_{OUT} = V_{in} \frac{-R_1}{R_2} \times 1 \times \frac{-R_3}{R_4}$$

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