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MATH55 Section 3
Homework 17 (Coloring)
Due Tue. 4/16

52.8 Let G be a graph with n vertices. Prove that $\chi(G) \geq \omega(G)$ and $\chi(G) \geq n/\alpha(G)$.

a. The clique number of a graph $\omega(G)$ tells you about a largest clique in the graph. Let H be a clique subgraph of size $\omega(G)$ in G. We know that the chromatic number of the complete graph H is $\omega(G)$, that is, $\chi(H) = \omega(G)$. Because H is a subgraph of G, we have $\chi(G) \geq \chi(H)$. Therefore $\chi(G) \geq \omega(G)$.

b. Let's consider a k-coloring of graph G where $k = \chi(G)$. We partition the vertices of G by saying that two vertices are equal if they have the same coloring. For each vertex set in the partition, $P_1, P_2, ..., P_k$. Since there cannot be an edge connecting vertices of the same color, the induced graph of the complement of G by any partition forms a clique. That is, $\overline{G}(P_1), \overline{G}(P_2), ..., \overline{G}(P_k)$ each forms a clique. Furthermore, their clique number is equal to the number of vertices. Since the graph $\overline{G}$ is a supergraph of any of them, the clique number $\omega(\overline{G})$ must be greater than or equal to the clique number of any subgraph $\overline{G}(P_1), \overline{G}(P_2), ..., \overline{G}(P_k)$. We have

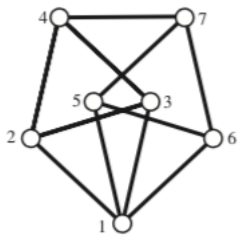
$$
\begin{aligned}
n &= \sum_{i}^{\chi(G)} |P_i| \quad : \text{via the sum principle} \\
&= \sum_{i}^{\chi(G)} |V(\overline{G}(P_i))| \\
&= \sum_{i}^{\chi(G)} |\omega(\overline{G}(P_i))| \quad : \text{each partition froms a clique in } \overline{G} \\
&\leq \chi(G)\omega(\overline{G}) \quad : \text{the size of each clique is less than the size of maximum clique} \\
&= \chi(G)\alpha(G) \quad : \omega(\overline{G}) = \alpha(G) \text{ by Proposition 48.12}
\end{aligned}
$$

Therefore $n \leq \chi(G)\alpha(G)$ so $\chi(G) \geq n/\alpha(G)$.

52.10 Let G be a graph with n vertices. Prove that $\chi(G)\chi(\overline{G}) \geq n$.

From the first problem, we have $\chi(G) \geq \omega(G)$ and $\chi(G) \geq n/\alpha(G)$. Substituting G in the second formula with $\overline{G}$ we get $\chi(\overline{G}) \geq n/\alpha(\overline{G})$. Since we know from Proposition 48.12 that $\alpha(\overline{G}) = \omega(G)$, we then have $\chi(\overline{G}) \geq n/\omega(G)$. Multiplying this formula with $\chi(G) \geq \omega(G)$ gives $\chi(G)\chi(\overline{G}) \geq n$.

52.11 Let G be the seven-vertex graph in the figure. Prove that $\chi(G) = 4$.



In order to show that the minimum color we can use to color this graph is 4, it is suffice to show that there is a 4-coloring for this graph, and that we cannot do 3 or lower coloring.

The 4-coloring is provided in the figure, and we cannot do 3-coloring because there is a K_3 subgraph inside the figure, namely $G(1, 2, 3)$. Furthermore, since we cannot do 3-coloring, we cannot do any lower than 3.

52.15 Let G be a graph with the property that $\delta(H) \leq d$ for all induced subgraph H of G. Prove that $\chi(G) \leq d+1$.

Let's say we have are given a graph G with the property that $\delta(H) \leq d$ for all induced subgraph H of G. We will first prove that if we induce G with a set of one vertex $\{v_1\}$, we will have $\chi(G(\{v_1\})) \leq d+1$. Then, we show that if we induce it with a set of two vertices, we will have $\chi(G(\{v_1, v_2\})) \leq d+1$, and so on. We repeat this, until we get the size of the inducing vertex set to be $|V(G)|$. At which point we will have $\chi(G(V(G))) = \chi(G()) \leq d+1$ which is what we have set out to prove.

Proof by induction on n, the number of vertices we will use to induce on G.

Base Case: Consider the case where we are inducing G with one vertex v_1. We only need one color for the one node, so $\chi(G(\{v_1\})) = 1$. Because there are no edges, so the degree is 0, so this satisfies $\chi(G(\{v_1\})) = 1 \leq d+1$.

Inductive Step: We assume that if we induce G with $n = k$ vertices, let's call the inducing set V_k, we will have $\chi(G(V_k)) \leq d+1$. We will have to show that if we induce G with a bigger set of $n = k+1$ vertices, let's call the inducing set V_{k+1}, we will have $\chi(G(V_{k+1})) \leq d+1$.

We attempt to show that $\chi(G(V_{k+1})) \leq d+1$ by showing that we can color it with a palette $d+1$ color. We will start We take a look at $G(V_{k+1})$, the graph G induced with $k+1$ vertices. Let's look at the property of the graph G that $\delta(H) \leq d$ for all induced subgraph H. From this, we always know we have a vertex with degree less than or equal to d. Let's call that vertex v. Consider the induced subgraph $G(V_{k+1} - v)$. It has k vertices, so $G(V_{k+1} - v) \leq d+1$ by the inductive hypothesis. We take the color of everything except v from the coloring shown to be exist by the inductive hypothesis. Now, we attach v back with all the edges. Since v are only adjacent to at most d things and we have a palette of $d+1$ color, there will always be a color available for v. We color v with the avaiable color and end up with a coloring for $\chi(G(V_{k+1}))$ with $d+1$ color. Thus, we have $\chi(G(V_{k+1})) \leq d+1$.