New Definite Integrals using Contour Integration involving Branch Points

Sutirtha Sanyal
ssanyal77@gmail.com

Abstract

In this article we evaluate some new (to the best of our knowledge) definite integrals using contour integration method. Our examples involve branch points and associated branch cuts. The corresponding definite integrals over the real line are verified by the Mathematica[1].

I. Complex Integration along a simple closed curve

We will assume complex numbers with argument $0 <= arg(z) < 2\pi$. Contour Integrals evaluated on simple closed curves have been studied extensively and can be evaluated using Cauchy's Residue theorems[2]. Given a meromorphic function $f(z)$, a branch point is any point in the complex plane around which if we make a closed contour, the function does not return to its initial value. At branch point, like essential singularity, the function fails to be analytic, i.e., either the function or its derivatives or both do not exist. But unlike singularities, we cannot encircle a branch point completely because it is not possible to determine residue at that point. Equivalently we cannot cross through branch cuts. In this article we consider principle branch of the function, making the multiple valued function single valued.

II. Branch Points and Branch Cuts

Branch point happens whenever we have expressions of the form $g(z)^a$, where a is non-integer or $(ln(g(z)))^b$ where b is non-zero (we will not consider functions such as $sin^{-1}(z)$ which can be represented in terms of ln). In these cases zeros and poles of $g(z)$ are branch points. In this section we assume g itself does not contain further branch points.

1) For example, if $f(z) = \frac{\sqrt{(z-i)}ln(z)}{\sqrt{(z+i)}(z^2+4)}$, then it has three branch points, namely, $i, -i, 0$.

Consequently we can define branch cuts as shown in Figure 1 in two heavy lines parallel to and one being the $+ve$ x-axis. Also shown is the complete path along which the integration can be evaluated. It is a circle of radius R centred at zero and punctuated by three branch cuts corresponding to three branch points. At each branch point we avoid it by detouring in a circle of very small radius ϵ (not separately mentioned in the figure).

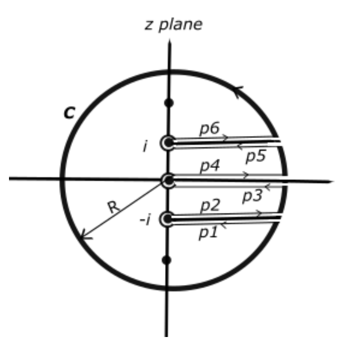

Fig. 1. Contour of Integration for $f(z)$

The integral in $C \to 0$ as $R \to \infty$ (because $R|f(Re^{i\phi})| \to 0$, over the circle and ϕ is the angle between a point on the circle with the $+ve$ x-axis) and so are the integrals along small circles indented at each branch point as $\epsilon \to 0$.

Along $p2$, $\sqrt{(z+i)} = \sqrt{x}$ and along $p1$, $\sqrt{(z+i)} = \sqrt{xe^{2\pi i}} = -\sqrt{x}$. Other terms retain their value. Then we change sign once more in $p1$ to consider the direction of integration. Thus

$$\int_{p1} f(z)dz + \int_{p2} f(z)dz = 2\int_0^\infty \frac{(x^2+4)^{\frac{1}{4}} e^{-i(\pi-(tan^{-1}(\frac{2x}{(x^2+3)})-\frac{1}{2}tan^{-1}(\frac{2}{x})))}(ln(\sqrt{x^2+1})+i(2\pi-tan^{-1}(\frac{1}{x})))}{\sqrt{x}\sqrt{(x^2+9)(x^2+1)}}dx \quad (1)$$

Along $p5$ and $p6$

$$\int_{p5} f(z)dz + \int_{p6} f(z)dz = 2\int_0^\infty \frac{\sqrt{x}e^{-i(\frac{1}{2}tan^{-1}(\frac{2}{x})+tan^{-1}(\frac{2x}{(x^2+3)}))}(ln(\sqrt{x^2+1})+i(tan^{-1}(\frac{1}{x})))}{(x^2+4)^{\frac{1}{4}}\sqrt{(x^2+9)(x^2+1)}}dx \quad (2)$$

Along $p3$ and $p4$

$$\int_{p3} f(z)dz + \int_{p4} f(z)dz = \int_0^\infty \frac{(e^{i(\pi - tan^{-1}(\frac{1}{x}))})(-2i\pi)}{(x^2+4)}dx \tag{3}$$

We have two singularities inside the closed path, $2i$ and $-2i$ where we compute residues.

$$(2\pi i)Res_{2i} + (2\pi i)Res_{-2i} = -\frac{\pi ln(2)}{\sqrt{3}} - i\frac{2\pi^2}{\sqrt{3}} \tag{4}$$

Equating real and imaginary parts of (1)+(2)+(3) with (4):

$$\int_0^\infty \Bigg(2(\frac{(x^2+4)^{\frac{1}{4}}(-ln(\sqrt{(1+x^2)})cos(tan^{-1}(\frac{2x}{(x^2+3)}) - \frac{1}{2}tan^{-1}(\frac{2}{x})) + sin(tan^{-1}(\frac{2x}{(x^2+3)}) - \frac{1}{2}tan^{-1}(\frac{2}{x}))(2\pi - tan^{-1}(\frac{1}{x})))}{\sqrt{x}\sqrt{(x^2+9)(x^2+1)}})$$
$$+ 2(\frac{\sqrt{x}(ln(\sqrt{(1+x^2)})cos(\frac{1}{2}tan^{-1}(\frac{2}{x}) + tan^{-1}(\frac{2x}{(x^2+3)})) + sin(\frac{1}{2}tan^{-1}(\frac{2}{x}) + tan^{-1}(\frac{2x}{(x^2+3)}))(tan^{-1}(\frac{1}{x})))}{(x^2+4)^{\frac{1}{4}}\sqrt{(x^2+9)(x^2+1)}})$$
$$+ \frac{2\pi sin(tan^{-1}(\frac{1}{x}))}{(x^2+4)}\Bigg)dx = -\frac{\pi ln(2)}{\sqrt{3}} \tag{5}$$

and

$$\int_0^\infty \Bigg(2(\frac{(x^2+4)^{\frac{1}{4}}(-ln(\sqrt{(1+x^2)})sin(tan^{-1}(\frac{2x}{(x^2+3)}) - \frac{1}{2}tan^{-1}(\frac{2}{x})) - cos(tan^{-1}(\frac{2x}{(x^2+3)}) - \frac{1}{2}tan^{-1}(\frac{2}{x}))(2\pi - tan^{-1}(\frac{1}{x})))}{\sqrt{x}\sqrt{(x^2+9)(x^2+1)}})$$
$$+ 2(\frac{\sqrt{x}(-ln(\sqrt{(1+x^2)})sin(\frac{1}{2}tan^{-1}(\frac{2}{x}) + tan^{-1}(\frac{2x}{(x^2+3)})) + cos(\frac{1}{2}tan^{-1}(\frac{2}{x}) + tan^{-1}(\frac{2x}{(x^2+3)}))(tan^{-1}(\frac{1}{x})))}{(x^2+4)^{\frac{1}{4}}\sqrt{(x^2+9)(x^2+1)}})$$
$$+ \frac{2\pi cos(tan^{-1}(\frac{1}{x}))}{(x^2+4)}\Bigg)dx = -\frac{2\pi^2}{\sqrt{3}} \tag{6}$$

2) In another example let $f(z) = \frac{\sqrt{(z-i)(z-1-i)}}{z^{\frac{1}{3}}(z-1)^{\frac{2}{3}}}$. We use the contour as shown in Figure 2.

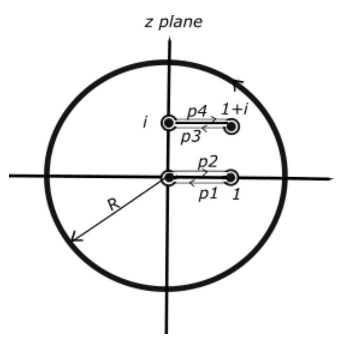

Fig. 2. Contour of Integration for $f(z)$

We have four branch points, $0, 1, i, 1+i$ but two branch cuts as $0, 1$ and $i, 1+i$ lie in the same branch lines. Moreover, beyond 1 and $1+i$ there are no branch cuts. This is because in former case both z and $(z-1)$ (latter case both $(z-i)$ and $(z-1-i)$) changed argument by 2π. The circle is of radius $R > \sqrt{2}$.

$$\int_{p2} f(z)dz + \int_{p1} f(z)dz = \int_0^1 \frac{(x^2+1)^{\frac{1}{4}}(x^2-2x+2)^{\frac{1}{4}}}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}(e^{i(\frac{5\pi}{6} + \frac{1}{2}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{2}tan^{-1}(\frac{1}{x}))} - e^{i(\frac{\pi}{6} + \frac{1}{2}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{2}tan^{-1}(\frac{1}{x}))})dx \tag{7}$$

$$\int_{p4} f(z)dz + \int_{p3} f(z)dz = \int_0^1 \frac{\sqrt{x(1-x)}}{(x^2+1)^{\frac{1}{6}}(x^2-2x+2)^{\frac{1}{3}}}(e^{i(\frac{-\pi}{6} + \frac{2}{3}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{3}tan^{-1}(\frac{1}{x}))} - e^{i(\frac{5\pi}{6} + \frac{2}{3}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{3}tan^{-1}(\frac{1}{x}))})dx \tag{8}$$

Now, $f(z) = (1 - \frac{i}{z})^{\frac{1}{2}}(1 - \frac{(1+i)}{z})^{\frac{1}{2}}(1 - \frac{1}{z})^{\frac{-2}{3}}$. Since $|z| > |1+i|$ along the circle, it can be expanded in powers of $\frac{1}{z}$. Only coefficient of $\frac{1}{z}$ matters.

Thus $\oint f(z)dz = (2\pi i)(\frac{1}{6} - i)$. Since $\oint f(z)dz + \int_{p1} f(z)dz + \int_{p2} f(z)dz + \int_{p3} f(z)dz + \int_{p4} f(z)dz = 0$, we have after equating real and imaginary parts,

$$\int_0^1 \Bigg(\frac{2\sqrt{x(1-x)}}{(x^2+1)^{\frac{1}{6}}(x^2-2x+2)^{\frac{1}{3}}} sin(\frac{\pi}{3} + \frac{2}{3}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{3}tan^{-1}(\frac{1}{x})) - \frac{\sqrt{3}(x^2+1)^{\frac{1}{4}}(x^2-2x+2)^{\frac{1}{4}}}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}} cos(\frac{1}{2}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{2}tan^{-1}(\frac{1}{x})) \Bigg) dx = -2\pi \tag{9}$$

$$\int_0^1 \Bigg(\frac{2\sqrt{x(1-x)}}{(x^2+1)^{\frac{1}{6}}(x^2-2x+2)^{\frac{1}{3}}} sin(\frac{2}{3}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{3}tan^{-1}(\frac{1}{x}) - \frac{\pi}{6}) - \frac{\sqrt{3}(x^2+1)^{\frac{1}{4}}(x^2-2x+2)^{\frac{1}{4}}}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}} sin(\frac{1}{2}tan^{-1}(\frac{1}{(1-x)}) - \frac{1}{2}tan^{-1}(\frac{1}{x})) \Bigg) dx = -\frac{\pi}{3} \tag{10}$$

3) Let $f(z) = \frac{z}{\sqrt{(z^2+1)(z+1)}}$. We integrate f along the contour shown in Figure 3. In this case, integral along the circle does not become zero as the term $\frac{1}{z}$ is present.

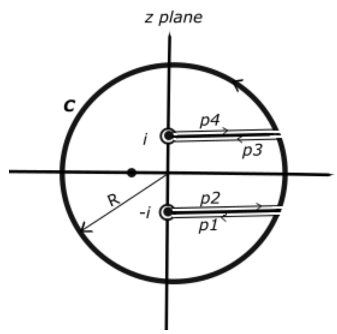

Fig. 3. Contour of Integration for $f(z)$

$$\int_{p2} f(z)dz + \int_{p1} f(z)dz = \int_0^\infty \frac{2\sqrt{(x^2+1)}e^{-i(\pi + tan^{-1}(\frac{1}{x}) - tan^{-1}(\frac{1}{(1+x)}) - \frac{1}{2}tan^{-1}(\frac{2}{x}))}}{\sqrt{(x)}(x^2+4)^{\frac{1}{4}}\sqrt{(x^2+2x+2)}} dx \tag{11}$$

$$\int_{p3} f(z)dz + \int_{p4} f(z)dz = \int_0^\infty \frac{2\sqrt{(x^2+1)}e^{-i(tan^{-1}(\frac{1}{(1+x)}) + \frac{1}{2}tan^{-1}(\frac{2}{x}) - tan^{-1}(\frac{1}{x}))}}{\sqrt{(x)}(x^2+4)^{\frac{1}{4}}\sqrt{(x^2+2x+2)}} dx \tag{12}$$

Now, $f(z) = \frac{1}{z}(1-\frac{i}{z})^{\frac{-1}{2}}(1+\frac{i}{z})^{\frac{-1}{2}}(1+\frac{1}{z})^{-1}$. Since $|z| > 1$ along the circle, it can be expanded in powers of $\frac{1}{z}$. Coefficient of $\frac{1}{z}$ is 1.

So, $\oint f(z)dz = 2\pi i$. Now, $\oint f(z)dz + \int_{p1} f(z)dz + \int_{p2} f(z)dz + \int_{p3} f(z)dz + \int_{p4} f(z)dz = (2\pi i)Res_{-1}$. Also $Res_{-1} = \frac{1}{\sqrt{2}}$. So, equating imaginary part,

$$\int_0^\infty \frac{\sqrt{(x^2+1)}sin(tan^{-1}(\frac{1}{(1+x)}) + \frac{1}{2}tan^{-1}(\frac{2}{x}) - tan^{-1}(\frac{1}{x}))}{\sqrt{(x)}(x^2+4)^{\frac{1}{4}}\sqrt{(x^2+2x+2)}} dx = \frac{\pi}{2\sqrt{2}}(\sqrt{2}-1) \tag{13}$$

4) Let $f(z) = \frac{z^p}{(1+z^n)}$ where $p > -1$, n, integer satisfying $n > p+1$.

We integrate f over the contour shown in Figure 4. f has only 1 branch point, the origin and $+ve$ $x-$axis is the branch cut. It includes one singularity at $z = e^{\frac{i\pi}{n}}$. Integral along the arc goes to 0 as radius $\to \infty$. Also the integral along very small arc indented around $z = 0$ of radius ϵ (not mentioned in the figure) goes to 0 as $\epsilon \to 0$ (since $p > -1$).

$$\int_{p1} f(z)dz = \int_0^\infty \frac{x^p}{(1+x^n)} dx \tag{14}$$

Along $p2$, $z = xe^{\frac{2\pi i}{n}}$.

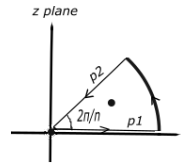

Fig. 4. Contour of Integration for $f(z)$

$$\int_{p2} f(z)dz = -\int_0^\infty \frac{x^p e^{\frac{2i\pi(p+1)}{n}}}{(1+x^n)}dx \tag{15}$$

Residue of f at $z = e^{\frac{i\pi}{n}}$ is $\lim_{z\to e^{\frac{i\pi}{n}}} f(z)(z - e^{\frac{i\pi}{n}}) = \frac{e^{\frac{ip\pi}{n}}}{ne^{\frac{i\pi(n-1)}{n}}}$. Equating imaginary part of (14)+(15) with $(2\pi i)Res_{e^{\frac{i\pi}{n}}} = \frac{-2i\pi e^{\frac{i\pi(p+1)}{n}}}{n}$,

$$\int_0^\infty \frac{x^p}{(1+x^n)}dx = \frac{\pi}{nsin(\frac{\pi(p+1)}{n})} \tag{16}$$

5) Now, let $f(z) = \frac{z^p ln(z)}{(1+z^n)}$ with same p and n.
We integrate f over the contour shown in Figure 4.

$$\int_{p1} f(z)dz = \int_0^\infty \frac{x^p ln(x)}{(1+x^n)}dx \tag{17}$$

Along $p2$, $z = xe^{\frac{2\pi i}{n}}$.

$$\int_{p2} f(z)dz = -\int_0^\infty \frac{x^p e^{\frac{2i\pi(p+1)}{n}}(ln(x) + i\frac{2\pi}{n})}{(1+x^n)}dx \tag{18}$$

Residue of f at $z = e^{\frac{i\pi}{n}}$ is $\lim_{z\to e^{\frac{i\pi}{n}}} f(z)(z - e^{\frac{i\pi}{n}}) = \frac{e^{\frac{ip\pi}{n}}(\frac{i\pi}{n})}{ne^{\frac{i\pi(n-1)}{n}}}$. Equating real part of (17)+(18) with $(2\pi i)Res_{e^{\frac{i\pi}{n}}} = \frac{2\pi^2 e^{\frac{i\pi(p+1)}{n}}}{n^2}$ and using (16),

$$\int_0^\infty \frac{x^p ln(x)}{1+x^n}dx = \frac{-\pi^2 cot(\frac{\pi(p+1)}{n})cosec(\frac{\pi(p+1)}{n})}{n^2} \tag{19}$$

Putting $p = \frac{1}{2}$, $\frac{-1}{2}$, $n = 2$,

$$\int_0^\infty \frac{\sqrt{x}ln(x)}{(1+x^2)}dx = -\int_0^\infty \frac{ln(x)}{\sqrt{x}(1+x^2)}dx = \frac{\pi^2}{2\sqrt{2}} \tag{20}$$

III. More examples of branch points

First we consider functions of type $f(z) = (\sqrt[n]{(z-z_0)} + a)^p$ where p is non-integer and a is positive real. f will still have only one branch point, z_0 if $n >= 2$. This is because to make $\sqrt[n]{(z-z_0)} + a = 0$, $z - z_0$ needs to be $a^n e^{in\pi}$, which is not possible for $n >= 2$.

6) Let, $f(z) = \frac{(\sqrt{z}+1)^{\frac{3}{2}}}{(1+z^2)}$. We integrate f over the contour shown in Figure 5. Integral over the arc goes to 0 as radius $\to \infty$. We avoid the branch point $z = 0$ by detouring over a semicircle of very small radius ϵ centred at $z = 0$, i.e., along it, $z = \epsilon e^{i\phi}$ where ϕ goes from π to 0. The integral over this small semicircle goes to 0 as $\epsilon \to 0$.

$$\int_{p1} f(z)dz = \int_0^\infty \frac{(\sqrt{x}+1)^{\frac{3}{2}}}{(1+x^2)}dx \tag{21}$$

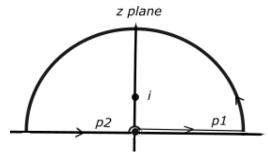

Fig. 5. Contour of Integration for $f(z)$

$$\int_{p2} f(z)dz = \int_0^\infty \frac{(i\sqrt{x}+1)^{\frac{3}{2}}}{(1+x^2)}dx = \int_0^\infty \frac{(x+1)^{\frac{3}{4}} e^{i(\frac{3}{2}tan^{-1}(\sqrt{x}))}}{(1+x^2)}dx \tag{22}$$

$(2\pi i)Res_i = \pi(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)^{\frac{3}{2}} = \pi(2+\sqrt{2})^{\frac{3}{4}} e^{i\frac{3}{2}tan^{-1}(\frac{1}{\sqrt{2}+1})}$
Equating real and imaginary parts with (21)+(22),

$$\int_0^\infty \frac{(\sqrt{x}+1)^{\frac{3}{2}} + (x+1)^{\frac{3}{4}} cos(\frac{3}{2}tan^{-1}(\sqrt{x}))}{(1+x^2)}dx = \pi(2+\sqrt{2})^{\frac{3}{4}} cos(\frac{3}{2}tan^{-1}(\frac{1}{\sqrt{2}+1})) \tag{23}$$

$$\int_0^\infty \frac{(x+1)^{\frac{3}{4}} sin(\frac{3}{2}tan^{-1}(\sqrt{x}))}{(1+x^2)}dx = \pi(2+\sqrt{2})^{\frac{3}{4}} sin(\frac{3}{2}tan^{-1}(\frac{1}{\sqrt{2}+1})) \tag{24}$$

7) Let $f(z) = \frac{1}{(1+z^{\frac{4}{3}})^3}$. We integrate over the contour shown in Figure 5. $z = e^{i\frac{3\pi}{4}}$ is a singularity.

$$\int_{p1} f(z)dz = \int_0^\infty \frac{dx}{(1+x^{\frac{4}{3}})^3} \tag{25}$$

$$\int_{p2} f(z)dz = \int_0^{2^{\frac{3}{4}}} \frac{e^{i\left(3tan^{-1}\left(\frac{\sqrt{3}x^{\frac{4}{3}}}{2-x^{\frac{4}{3}}}\right)\right)}}{(x^{\frac{8}{3}}-x^{\frac{4}{3}}+1)^{\frac{3}{2}}}dx - \int_{2^{\frac{3}{4}}}^\infty \frac{e^{-i\left(3tan^{-1}\left(\frac{\sqrt{3}x^{\frac{4}{3}}}{x^{\frac{4}{3}}-2}\right)\right)}}{(x^{\frac{8}{3}}-x^{\frac{4}{3}}+1)^{\frac{3}{2}}}dx \tag{26}$$

$(2\pi i)Res_{e^{i\frac{3\pi}{4}}} = (2\pi i)(\text{coefficient of } \frac{1}{u} \text{ in } \frac{1}{(1+(u+e^{i\frac{3\pi}{4}})^{\frac{4}{3}})^3}) = (2\pi i)(-\frac{27}{64})e^{i\frac{3\pi}{4}}(\frac{5}{18})$.
Equating real and imaginary,

$$\int_0^\infty \frac{dx}{(1+x^{\frac{4}{3}})^3} = \frac{15\pi\sqrt{2}}{128} \tag{27}$$

$$\int_0^{2^{\frac{3}{4}}} \frac{sin\left(3tan^{-1}\left(\frac{\sqrt{3}x^{\frac{4}{3}}}{2-x^{\frac{4}{3}}}\right)\right)}{(x^{\frac{8}{3}}-x^{\frac{4}{3}}+1)^{\frac{3}{2}}}dx + \int_{2^{\frac{3}{4}}}^\infty \frac{sin\left(3tan^{-1}\left(\frac{\sqrt{3}x^{\frac{4}{3}}}{x^{\frac{4}{3}}-2}\right)\right)}{(x^{\frac{8}{3}}-x^{\frac{4}{3}}+1)^{\frac{3}{2}}}dx = \frac{15\pi\sqrt{2}}{128} \tag{28}$$

In (27) we have used the fact that, $\int_0^{2^{\frac{3}{4}}} \frac{cos\left(3tan^{-1}\left(\frac{\sqrt{3}x^{\frac{4}{3}}}{2-x^{\frac{4}{3}}}\right)\right)}{(x^{\frac{8}{3}}-x^{\frac{4}{3}}+1)^{\frac{3}{2}}}dx - \int_{2^{\frac{3}{4}}}^\infty \frac{cos\left(3tan^{-1}\left(\frac{\sqrt{3}x^{\frac{4}{3}}}{x^{\frac{4}{3}}-2}\right)\right)}{(x^{\frac{8}{3}}-x^{\frac{4}{3}}+1)^{\frac{3}{2}}}dx = 0$. This can be proved by integrating f anticlockwise over the quarter-circle (second quadrant).

8) Let $f(z) = \frac{\sqrt{ln(z)}}{(1+z^2)}$. We integrate f over the contour shown in Figure 5. $z = 1$ is another branch point which needs to be avoided like $z = 0$ (by detouring around it in a semicircle of very small radius ϵ centred at $z = 1$, not shown separately in the figure).

$$\int_{p1} f(z)dz = \int_0^1 \frac{i\sqrt{-ln(x)}}{(1+x^2)}dx + \int_1^\infty \frac{\sqrt{ln(x)}}{(1+x^2)}dx \tag{29}$$

Along $p2$, $z = -x$.

$$\int_{p2} f(z)dz = \int_0^1 \frac{e^{i(\frac{\pi}{2}-\frac{1}{2}tan^{-1}(\frac{\pi}{-ln(x)}))}((ln(x))^2+\pi^2)^{\frac{1}{4}}}{(1+x^2)}dx + \int_1^\infty \frac{e^{i(\frac{1}{2}tan^{-1}(\frac{\pi}{ln(x)}))}((ln(x))^2+\pi^2)^{\frac{1}{4}}}{(1+x^2)}dx \tag{30}$$

Equating real and imaginary parts of (29)+(30) with the $(2\pi i)Res_i = \frac{\pi^{\frac{3}{2}}}{\sqrt{2}}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$,

$$\int_0^1 \frac{sin(\frac{1}{2}tan^{-1}(\frac{-\pi}{ln(x)}))((ln(x))^2+\pi^2)^{\frac{1}{4}}}{(1+x^2)}dx + \int_1^\infty \frac{cos(\frac{1}{2}tan^{-1}(\frac{\pi}{ln(x)}))((ln(x))^2+\pi^2)^{\frac{1}{4}} + \sqrt{ln(x)}}{(1+x^2)}dx = \frac{\pi^{\frac{3}{2}}}{2} \tag{31}$$

$$\int_0^1 \frac{cos(\frac{1}{2}tan^{-1}(\frac{-\pi}{ln(x)}))((ln(x))^2+\pi^2)^{\frac{1}{4}} + \sqrt{-ln(x)}}{(1+x^2)}dx + \int_1^\infty \frac{sin(\frac{1}{2}tan^{-1}(\frac{\pi}{ln(x)}))((ln(x))^2+\pi^2)^{\frac{1}{4}}}{(1+x^2)}dx = \frac{\pi^{\frac{3}{2}}}{2} \tag{32}$$

9) Let $f(z) = \frac{ln(ln(z)+i\pi)}{(1+z^2)}$. We use the contour as shown in Figure 5.

$$\int_{p1} f(z)dz = \int_0^1 \frac{\frac{1}{2}ln((ln(x))^2+\pi^2) + i(\pi - tan^{-1}(\frac{-\pi}{ln(x)}))}{(1+x^2)}dx + \int_1^\infty \frac{\frac{1}{2}ln((ln(x))^2+\pi^2) + i(tan^{-1}(\frac{\pi}{ln(x)}))}{(1+x^2)}dx \tag{33}$$

Along $p2$, $z=-x$.

$$\int_{p2} f(z)dz = \int_0^1 \frac{\frac{1}{2}ln((ln(x))^2+4\pi^2) + i(\pi - tan^{-1}(\frac{-2\pi}{ln(x)}))}{(1+x^2)}dx + \int_1^\infty \frac{\frac{1}{2}ln((ln(x))^2+4\pi^2) + i(tan^{-1}(\frac{2\pi}{ln(x)}))}{(1+x^2)}dx \tag{34}$$

Equating real and imaginary parts with the $(2\pi i)Res_i = \pi(ln(\frac{3\pi}{2}) + i\frac{\pi}{2})$,

$$\int_0^\infty \frac{ln((ln(x))^2+\pi^2) + ln((ln(x))^2+4\pi^2)}{(1+x^2)}dx = 2\pi ln(\frac{3\pi}{2}) \tag{35}$$

$$\int_0^1 \frac{tan^{-1}(\frac{\pi}{ln(x)}) + tan^{-1}(\frac{2\pi}{ln(x)})}{(1+x^2)}dx = -\int_1^\infty \frac{tan^{-1}(\frac{\pi}{ln(x)}) + tan^{-1}(\frac{2\pi}{ln(x)})}{(1+x^2)}dx \tag{36}$$

10) Let $f(z) = \frac{ln(\sqrt{(z-i)}+1)}{(z^2+1)}$. We evaluate the integral over the contour as shown in Figure 6.

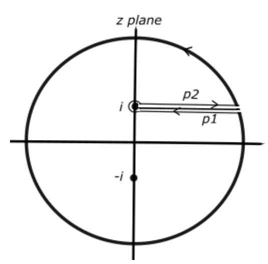

Fig. 6. Contour of Integration for $f(z)$

$$\begin{aligned}\int_{p2} f(z)dz + \int_{p1} f(z)dz = &\int_0^1 \frac{ln(\frac{1+\sqrt{x}}{1-\sqrt{x}})e^{-itan^{-1}(\frac{2}{x})}}{x\sqrt{(x^2+4)}}dx \\ &+ \int_1^\infty \frac{(ln(1+\sqrt{x}) - ln(\sqrt{x}-1) - i\pi)e^{-itan^{-1}(\frac{2}{x})}}{x\sqrt{(x^2+4)}}dx\end{aligned} \tag{37}$$

$(2\pi i)Res_{-i} = -\pi ln(\sqrt{-2i}+1) = -i\frac{\pi^2}{2}$
Equating real and imaginary parts,

$$\int_0^1 \frac{ln(\frac{1+\sqrt{x}}{1-\sqrt{x}})cos(tan^{-1}(\frac{2}{x}))}{x\sqrt{(x^2+4)}}dx + \int_1^\infty \frac{ln(\frac{\sqrt{x}+1}{\sqrt{x}-1})cos(tan^{-1}(\frac{2}{x})) - \pi sin(tan^{-1}(\frac{2}{x}))}{x\sqrt{(x^2+4)}}dx = 0 \tag{38}$$

$$\int_0^1 \frac{ln(\frac{1+\sqrt{x}}{1-\sqrt{x}})sin(tan^{-1}(\frac{2}{x}))}{x\sqrt{(x^2+4)}}dx + \int_1^\infty \frac{ln(\frac{\sqrt{x}+1}{\sqrt{x}-1})sin(tan^{-1}(\frac{2}{x})) + \pi cos(tan^{-1}(\frac{2}{x}))}{x\sqrt{(x^2+4)}}dx = \frac{\pi^2}{2} \tag{39}$$

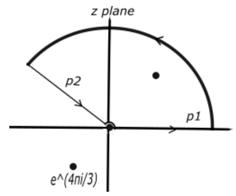

Fig. 7. Contour of Integration for $f(z)$

11) Let $f(z) = \frac{1}{\sqrt{(1+z^{\frac{3}{4}})(1+z^3)}}$. $z = 0$ and $z = e^{\frac{4i\pi}{3}}$ are branch points. We avoid both by integrating over the contour as shown in Figure 7. It includes one singularity at $e^{\frac{i\pi}{3}}$. Along $p2$, $z = xe^{\frac{2\pi i}{3}}$.

$$\int_{p1} f(z)dz + \int_{p2} f(z)dz = \int_0^\infty \left(\frac{1}{\sqrt{(1+x^{\frac{3}{4}})(1+x^3)}} - \frac{e^{\frac{2\pi i}{3}}}{\sqrt{(1+ix^{\frac{3}{4}})(1+x^3)}} \right) dx \tag{40}$$

$(2\pi i)Res_{e^{\frac{i\pi}{3}}} = \frac{2\pi e^{-i(\frac{\pi}{6}+\frac{1}{2}tan^{-1}(\frac{1}{\sqrt{2}+1}))}}{3(2+\sqrt{2})^{\frac{1}{4}}}$.
Equating real and imaginary,

$$\int_0^\infty \left(\frac{1}{\sqrt{(1+x^{\frac{3}{4}})(1+x^3)}} - \frac{cos(\frac{2\pi}{3} - \frac{1}{2}tan^{-1}(x^{\frac{3}{4}}))}{(1+x^{\frac{3}{2}})^{\frac{1}{4}}(1+x^3)} \right) dx = \frac{2\pi cos(\frac{\pi}{6} + \frac{1}{2}tan^{-1}(\frac{1}{\sqrt{2}+1}))}{3(2+\sqrt{2})^{\frac{1}{4}}} \tag{41}$$

$$\int_0^\infty \frac{sin(\frac{2\pi}{3} - \frac{1}{2}tan^{-1}(x^{\frac{3}{4}}))}{(1+x^{\frac{3}{2}})^{\frac{1}{4}}(1+x^3)} dx = \frac{2\pi sin(\frac{\pi}{6} + \frac{1}{2}tan^{-1}(\frac{1}{\sqrt{2}+1}))}{3(2+\sqrt{2})^{\frac{1}{4}}} \tag{42}$$

12) Let $f(z) = \frac{1}{\sqrt{ln(z)}(1+z^3)}$. We integrate using the contour shown in Figure 7.

$$\int_{p1} f(z)dz = -i\int_0^1 \frac{dx}{\sqrt{-ln(x)}(1+x^3)} + \int_1^\infty \frac{dx}{\sqrt{ln(x)}(1+x^3)} \tag{43}$$

Along $p2$, $z = xe^{i\frac{2\pi}{3}}$.

$$\int_{p2} f(z)dz = -\int_0^1 \frac{e^{i(\frac{\pi}{6}+\frac{1}{2}tan^{-1}(\frac{2\pi}{-3ln(x)}))}}{((ln(x))^2 + \frac{4\pi^2}{9})^{\frac{1}{4}}(1+x^3)} dx - \int_1^\infty \frac{e^{i(\frac{2\pi}{3}-\frac{1}{2}tan^{-1}(\frac{2\pi}{3ln(x)}))}}{((ln(x))^2 + \frac{4\pi^2}{9})^{\frac{1}{4}}(1+x^3)} dx \tag{44}$$

$(2\pi i)Res_{e^{i\frac{\pi}{3}}} = 2\sqrt{\frac{\pi}{3}}e^{-i\frac{5\pi}{12}}$.
Equating real and imaginary,

$$\int_0^1 \frac{cos(\frac{5\pi}{6} + \frac{1}{2}tan^{-1}(\frac{2\pi}{3ln(x)}))}{((ln(x))^2 + \frac{4\pi^2}{9})^{\frac{1}{4}}(1+x^3)} dx + \int_1^\infty \left(\frac{1}{\sqrt{ln(x)}(1+x^3)} - \frac{cos(\frac{2\pi}{3} - \frac{1}{2}tan^{-1}(\frac{2\pi}{3ln(x)}))}{((ln(x))^2 + \frac{4\pi^2}{9})^{\frac{1}{4}}(1+x^3)} \right) dx = 2\sqrt{\frac{\pi}{3}}cos(\frac{5\pi}{12}) \tag{45}$$

$$\int_0^1 \left(\frac{sin(\frac{\pi}{6} - \frac{1}{2}tan^{-1}(\frac{2\pi}{3ln(x)}))}{((ln(x))^2 + \frac{4\pi^2}{9})^{\frac{1}{4}}(1+x^3)} + \frac{1}{\sqrt{-ln(x)}(1+x^3)} \right) dx + \int_1^\infty \frac{sin(\frac{2\pi}{3} - \frac{1}{2}tan^{-1}(\frac{2\pi}{3ln(x)}))}{((ln(x))^2 + \frac{4\pi^2}{9})^{\frac{1}{4}}(1+x^3)} dx = 2\sqrt{\frac{\pi}{3}}sin(\frac{5\pi}{12}) \tag{46}$$

IV. Definite Integral as sum of a convergent infinite series

Sometimes we may not be able to determine the coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ in simple form. In such cases it is still possible to integrate $f(z)$ over a circle. However, the result will be represented as a sum of a convergent infinite series.

We will denote coefficient of z^n in the expansion of $(1+z)^p$ (p real) as ${}^pC_n$.

13) Let $f(z) = \frac{1}{\sqrt{z(z-1)(z-2)}}$. Between $x = 1$ and $x = 2$ there is no branch cut. From $x = 2$ there is a branch cut. We evaluate the integral over the circle of radius $1 < R < 2$, as shown in Figure 8.

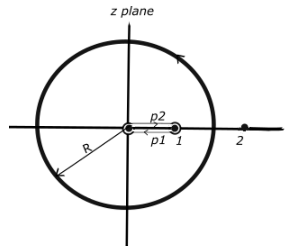

Fig. 8. Contour of Integration for $f(z)$

$$\int_{p2} f(z)dz + \int_{p1} f(z)dz = \int_0^1 \Big(\frac{1}{\sqrt{x(1-x)(2-x)}}\Big)(e^{-i\pi} - 1)dx \tag{47}$$

Now, $\oint f(z)dz + \int_{p2} f(z)dz + \int_{p1} f(z)dz = 0$. Also, over the circle, $f(z) = -\frac{i}{z\sqrt{2}}(1-\frac{1}{z})^{\frac{-1}{2}}(1-\frac{z}{2})^{\frac{-1}{2}}$.

Coefficient of $\dfrac{1}{z}, A_{-1} = \dfrac{-i}{\sqrt{2}} \displaystyle\sum_{n=0}^{n=\infty} ({}^{\frac{-1}{2}}C_n)^2 \frac{1}{2^n} = \frac{-i}{\sqrt{2}}(1 + \sum_{n=1}^{n=\infty} (\frac{(2n-1)!!}{n!})^2 \frac{1}{2^{3n}})$.

$\oint f(z)dz = (2\pi i)A_{-1}$. Therefore,

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)(2-x)}} = \frac{\pi}{\sqrt{2}}(1 + \sum_{n=1}^{n=\infty} (\frac{(2n-1)!!}{n!})^2 \frac{1}{2^{3n}}) \tag{48}$$

Assuming we have factors of the form $(z)^\alpha(z-a)^{1-\alpha}$ in $f(z)$ (where a is positive real and α is positive real < 1), then further we can choose any number of factors for $f(z)$ of the form $(z-b)^\beta$ from the z-plane satisfying i) b should not be within or on the circle of radius a, centred at 0 and ii) points need to be chosen from the negative real axis in such a manner that there is no branch cut from $x = -a$ to $x = 0$.

14) Let $f(z) = \frac{1}{z^\alpha(z-1)^{1-\alpha}(z+\frac{3}{2}+\frac{1}{2}i)^\beta}$ where $0 < \alpha < 1$. We can use the same contour as shown in Figure 8 where $1 < R < \frac{\sqrt{10}}{2}$.

$$\int_{p2} f(z)dz + \int_{p1} f(z)dz = \int_0^1 \Big(\frac{1}{x^\alpha(1-x)^{1-\alpha}(x^2+3x+\frac{10}{4})^{\frac{\beta}{2}}}\Big)\Big(e^{-i(\pi(1-\alpha)+\beta tan^{-1}(\frac{1}{(2x+3)}))} - e^{-i(\pi(1+\alpha)+\beta tan^{-1}(\frac{1}{(2x+3)}))}\Big)dx \tag{49}$$

Over the circle, $f(z) = \frac{1}{(\frac{3}{2}+i\frac{1}{2})^\beta}\frac{1}{z}(1-\frac{1}{z})^{-(1-\alpha)}(1+\frac{z}{\frac{3}{2}+i\frac{1}{2}})^{-\beta}$.

Coefficient of $\frac{1}{z}$ is

$A_{-1} = (\dfrac{2}{\sqrt{10}})^\beta e^{-i\beta tan^{-1}(\frac{1}{3})} \displaystyle\sum_{n=0}^{n=\infty} (-1)^n ({}^{-(1-\alpha)}C_n)({}^{-\beta}C_n)(\frac{2}{\sqrt{10}})^n e^{-intan^{-1}(\frac{1}{3})}$.

We can equate real and imaginary parts of (49) with $-(2\pi i)A_{-1}$. Let $\alpha = \frac{1}{2}, \beta = \frac{-1}{3}$.

$$\int_0^1 \frac{(x^2+3x+\frac{10}{4})^{\frac{1}{6}}}{\sqrt{x(1-x)}} sin(\frac{1}{3}tan^{-1}(\frac{1}{(2x+3)}))dx = \pi(\frac{\sqrt{10}}{2})^{\frac{1}{3}}(\sum_{n=0}^{n=\infty}(-1)^n({}^{\frac{-1}{2}}C_n)({}^{\frac{1}{3}}C_n)(\frac{2}{\sqrt{10}})^n sin((\frac{1}{3}-n)tan^{-1}(\frac{1}{3}))) \tag{50}$$

$$\int_0^1 \frac{(x^2+3x+\frac{10}{4})^{\frac{1}{6}}}{\sqrt{x(1-x)}} cos(\frac{1}{3}tan^{-1}(\frac{1}{(2x+3)}))dx = \pi(\frac{\sqrt{10}}{2})^{\frac{1}{3}}(\sum_{n=0}^{n=\infty}(-1)^n({}^{\frac{-1}{2}}C_n)({}^{\frac{1}{3}}C_n)(\frac{2}{\sqrt{10}})^n cos((\frac{1}{3}-n)tan^{-1}(\frac{1}{3}))) \tag{51}$$

15) Let $f(z) = \frac{\sqrt{z(z-1)}}{\sqrt{(z^2+4)}}$. We use the contour shown in Figure 8 with $1 < R < 2$.

$$\int_{p2} f(z)dz + \int_{p1} f(z)dz = \int_0^1 \frac{\sqrt{x(1-x)}}{\sqrt{(x^2+4)}}(e^{-i\frac{\pi}{2}} - e^{i\frac{\pi}{2}})dx \tag{52}$$

Over the circle, $f(z)=\frac{z(1-\frac{1}{z})^{\frac{1}{2}}}{\sqrt{(2i+z)}\sqrt{(2i-z)}e^{i\frac{\pi}{2}}}=(\frac{-1}{2})z(1-\frac{1}{z})^{\frac{1}{2}}(1+\frac{z^2}{4})^{\frac{-1}{2}}$.
Coefficient of $\frac{1}{z}$ is
$A_{-1}=\frac{-1}{2}(\sum_{n=1}^{n=\infty}({}^{\frac{1}{2}}C_{2n})(\frac{1}{4^{n-1}})({}^{\frac{-1}{2}}C_{n-1}))$.
Equating imaginary part of (52) with $-2\pi iA_{-1}$,

$$\int_0^1\frac{\sqrt{x(1-x)}}{\sqrt{(x^2+4)}}dx=-\frac{\pi}{2}(\sum_{n=1}^{n=\infty}({}^{\frac{1}{2}}C_{2n})(\frac{1}{4^{n-1}})({}^{\frac{-1}{2}}C_{n-1})) \tag{53}$$

16) Let $f(z)=\sqrt{\frac{z}{(z-1)}}(z+2)^{\frac{1}{3}}(z+3)^{\frac{2}{3}}$. This is permissible as $-1<=x<=0$ has no branch cut.
We use the contour shown in Figure 8 with $1<R<2$.

$$\int_{p2}f(z)dz+\int_{p1}f(z)dz=\int_0^1\sqrt{\frac{x}{(1-x)}}(x+2)^{\frac{1}{3}}(x+3)^{\frac{2}{3}}(e^{-i\frac{\pi}{2}}-e^{i\frac{\pi}{2}})dx \tag{54}$$

Over the circle, $f(z)=2^{\frac{1}{3}}3^{\frac{2}{3}}(1-\frac{1}{z})^{\frac{-1}{2}}(1+\frac{z}{2})^{\frac{1}{3}}(1+\frac{z}{3})^{\frac{2}{3}}$.
Coefficient of $\frac{1}{z}$ is
$A_{-1}=2^{\frac{1}{3}}3^{\frac{2}{3}}\left(\sum_{n=1}^{n=\infty}(-1)^n({}^{\frac{-1}{2}}C_n)\left(\sum_{m=0}^{m=n-1}({}^{\frac{1}{3}}C_m)({}^{\frac{2}{3}}C_{n-1-m})(\frac{1}{2})^m(\frac{1}{3})^{n-1-m}\right)\right)$.
Equating imaginary part of (54) with $-2\pi iA_{-1}$,

$$\int_0^1\sqrt{\frac{x}{(1-x)}}(x+2)^{\frac{1}{3}}(x+3)^{\frac{2}{3}}dx=\pi 2^{\frac{1}{3}}3^{\frac{2}{3}}\left(\sum_{n=1}^{n=\infty}(-1)^n({}^{\frac{-1}{2}}C_n)\left(\sum_{m=0}^{m=n-1}({}^{\frac{1}{3}}C_m)({}^{\frac{2}{3}}C_{n-1-m})(\frac{1}{2})^m(\frac{1}{3})^{n-1-m}\right)\right) \tag{55}$$

17) Let $f(z)=\frac{z^{\frac{1}{4}}(z-1)^{\frac{3}{4}}}{(z-3)^3}$. First we use the contour of Figure 8 with $1<R<3$.

$$\int_{p2}f(z)dz+\int_{p1}f(z)dz=\int_0^1\frac{x^{\frac{1}{4}}(1-x)^{\frac{3}{4}}}{(3-x)^3}(e^{-i\frac{9\pi}{4}}-e^{-i\frac{7\pi}{4}})dx \tag{56}$$

Over the circle, $f(z)=-\frac{1}{3^3}z(1-\frac{1}{z})^{\frac{3}{4}}(1-\frac{z}{3})^{-3}$.
Coefficient of $\frac{1}{z}$ is
$A_{-1}=-\frac{1}{3^3}\sum_{n=2}^{n=\infty}(({}^{\frac{3}{4}}C_n)({}^{-3}C_{n-2})(\frac{1}{3})^{n-2})$.
Equating imaginary part of (56) with $-2\pi iA_{-1}$,

$$\int_0^1\frac{x^{\frac{1}{4}}(1-x)^{\frac{3}{4}}}{(3-x)^3}dx=-\frac{\pi\sqrt{2}}{3^3}\sum_{n=2}^{n=\infty}(({}^{\frac{3}{4}}C_n)({}^{-3}C_{n-2})(\frac{1}{3})^{n-2}) \tag{57}$$

But $z=3$ is not a branch point. We can enclose it by integrating over a circle of $R>3$.
Then, over the circle, $f(z)=\frac{1}{z^2}(1-\frac{1}{z})^{\frac{3}{4}}(1-\frac{3}{z})^{-3}$. Coefficient of $\frac{1}{z}$ is 0.
$(2\pi i)Res_3=(2\pi i)($ coefficient of $\frac{1}{u}$ in $\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}(1+\frac{u}{3})^{\frac{1}{4}}(1+\frac{u}{2})^{\frac{3}{4}}}{u^3})=(2\pi i)3^{\frac{1}{4}}2^{\frac{3}{4}}(\frac{-1}{384})$.
Equating imaginary part of (56) with $(2\pi i)Res_3$,

$$\int_0^1\frac{x^{\frac{1}{4}}(1-x)^{\frac{3}{4}}}{(3-x)^3}dx=\frac{\pi 3^{\frac{1}{4}}2^{\frac{5}{4}}}{384} \tag{58}$$

From (57) and (58) we get,

$$\sum_{n=2}^{n=\infty}(({}^{\frac{3}{4}}C_n)({}^{-3}C_{n-2})(\frac{1}{3})^{n-2})=-\frac{3^{\frac{13}{4}}2^{\frac{3}{4}}}{384} \tag{59}$$

18) Let $f(z)=\frac{(z-2)^2}{z^{\frac{1}{3}}(z-1)^{\frac{2}{3}}(z-3)^2}$. First we use the contour of Figure 8 with $1<R<2$.

$$\int_{p2} f(z)dz + \int_{p1} f(z)dz = \int_0^1 \frac{(2-x)^2}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}(3-x)^2}(e^{-i\frac{2\pi}{3}} - e^{-i\frac{4\pi}{3}})dx \tag{60}$$

Over the circle, $f(z) = \frac{4}{9z}(1-\frac{1}{z})^{\frac{-2}{3}}(1-\frac{z}{2})^2(1-\frac{z}{3})^{-2}$. Coefficient of $\frac{1}{z}$ is
$A_{-1} = \frac{4}{9}\Big(\sum_{n=0}^{n=\infty}({}^{\frac{-2}{3}}C_n)\Big(({}^{-2}C_n)(\frac{1}{3})^n + ({}^{-2}C_{n-1})(\frac{1}{3})^{n-1} + ({}^{-2}C_{n-2})(\frac{1}{3})^{n-2}(\frac{1}{4})\Big)\Big)$.
Equating imaginary part with $-2\pi i A_{-1}$,

$$\int_0^1 \frac{(2-x)^2}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}(3-x)^2}dx = \frac{8\pi}{9\sqrt{3}}\Big(\sum_{n=0}^{n=\infty}({}^{\frac{-2}{3}}C_n)\Big(({}^{-2}C_n)(\frac{1}{3})^n + ({}^{-2}C_{n-1})(\frac{1}{3})^{n-1} + ({}^{-2}C_{n-2})(\frac{1}{3})^{n-2}(\frac{1}{4})\Big)\Big) \tag{61}$$

When, $2 < R < 3$, over the circle, $f(z) = \frac{1}{9}z(1-\frac{1}{z})^{\frac{-2}{3}}(1-\frac{2}{z})^2(1-\frac{z}{3})^{-2}$. Coefficient of $\frac{1}{z}$ is
$A_{-1} = \frac{1}{9}\Big(\sum_{n=0}^{n=\infty}({}^{-2}C_n)(\frac{1}{3})^n\Big(({}^{\frac{-2}{3}}C_{n+2}) + 4({}^{\frac{-2}{3}}C_{n+1}) + 4({}^{\frac{-2}{3}}C_n)\Big)\Big)$. Equating imaginary part with $-2\pi i A_{-1}$,

$$\int_0^1 \frac{(2-x)^2}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}(3-x)^2}dx = \frac{2\pi}{9\sqrt{3}}\Big(\sum_{n=0}^{n=\infty}({}^{-2}C_n)(\frac{1}{3})^n\Big(({}^{\frac{-2}{3}}C_{n+2}) + 4({}^{\frac{-2}{3}}C_{n+1}) + 4({}^{\frac{-2}{3}}C_n)\Big)\Big) \tag{62}$$

When, $3 < R$, over the circle, $f(z) = \frac{1}{z}(1-\frac{1}{z})^{\frac{-2}{3}}(1-\frac{2}{z})^2(1-\frac{3}{z})^{-2}$. Coefficient of $\frac{1}{z}$ is 1.
$(2\pi i)Res_3 = (2\pi i)($ coefficient of $\frac{1}{u}$ in $\frac{(1+u)^2(1+\frac{u}{3})^{\frac{-1}{3}}(1+\frac{u}{2})^{\frac{-2}{3}}}{3^{\frac{1}{3}}2^{\frac{2}{3}}u^2}) = (2\pi i)\frac{14}{3^{\frac{7}{3}}2^{\frac{2}{3}}}$.
So,

$$\int_0^1 \frac{(2-x)^2}{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}(3-x)^2}dx = \frac{2\pi}{\sqrt{3}}(1 - \frac{14}{3^{\frac{7}{3}}2^{\frac{2}{3}}}) \tag{63}$$

From (61), (62) and (63) we get,

$$\begin{aligned} &4\Big(\sum_{n=0}^{n=\infty}({}^{\frac{-2}{3}}C_n)\Big(({}^{-2}C_n)(\frac{1}{3})^n + ({}^{-2}C_{n-1})(\frac{1}{3})^{n-1} + ({}^{-2}C_{n-2})(\frac{1}{3})^{n-2}(\frac{1}{4})\Big)\Big) \\ &= \Big(\sum_{n=0}^{n=\infty}({}^{-2}C_n)(\frac{1}{3})^n\Big(({}^{\frac{-2}{3}}C_{n+2}) + 4({}^{\frac{-2}{3}}C_{n+1}) + 4({}^{\frac{-2}{3}}C_n)\Big)\Big) \\ &= 9(1 - \frac{14}{3^{\frac{7}{3}}2^{\frac{2}{3}}}) \end{aligned} \tag{64}$$

V. Conclusion

In this article we have discussed some new definite integrals evaluated using Contour Integration technique. We exclusively address cases where branch points are present in the integrand. The resulting integrals over the real line are independently verified using the Mathematica (http://www.wolframalpha.com/widgets/view.jsp?id=8ab70731b1553f17c11a3bbc87e0b605). Sage[3] is used for miscellaneous other verifications.

References

[1] S. Wolfram, "The mathematica book 5th edition; wolfram research," 2003.
[2] M. R. Spiegel, S. Lipschutz, J. J. Schiller, and D. Spellman, "Schaum's outlines: Complex variables, 2nd edition," 2009.
[3] W. Stein *et al.*, "Sage: Software for algebra and geometry experimentation," 2006.