KTH Computer Science
and Communication

# Homework 5
# Helmholtz and Eikonal Equation

**Max 5.0 p**

## Introduction

In this homework you will solve the 2D Helmholtz equation,

$$\Delta u + n(x,y)^2\omega^2 u = f(x,y), \qquad (x,y) \in \Omega \subset \mathbb{R}^2, \tag{1}$$

with various boundary conditions and choices of $n$ (index of refraction) and $f$ (source term). This models time-harmonic waves of frequency $\omega$ triggered by $f$. In the final part you will use the eikonal equation to approximate (1) for high frequencies.

Consider the square domain $\Omega = [0,1]^2$. Discretize it with a uniform grid as described in Lecture notes 5. Your code from Homework 1 should come in handy. Make sure to use `sparse` format in Matlab for your computations. Note that in general the solution $u$ will be complex valued, either because of complex parameters in the boundary conditions (Exercise 2) or by the introduction of a damping parameter (Exercise 1.3).

## 1 Interior problem in a square

We start by testing to solve Helmholtz with the Dirichlet conditions $u = 0$ on the boundary. For the tests below, take a constant $n(x) \equiv 1$ and a Gaussian source of width $w$ centered in $(s_x, s_y)$,

$$f(x,y) = \exp\left(-\frac{(x-s_x)^2 + (y-s_y)^2}{w^2}\right), \qquad s_x = 0.3, \quad s_y = 0.6, \quad w = 0.1.$$

1. Solve the problem for a couple of different frequencies $\omega$. Plot examples of solutions with low, medium and high frequency (say $\omega \approx 30$). **(0.25 p)**

2. Make a frequency sweep, solving the equation for $\omega$ in the range $[1, 15]$. Plot the norm $||u||_2$ as a function of $\omega$ with `semilogy`. Explain the peaks in the plot. What do they signify? Also check the condition number of the matrix with `condest` as a function of $\omega$. **(0.25 p)**

3. Add a damping $\alpha > 0$ to the equation,

$$i\alpha\omega u + \Delta u + n(x,y)^2\omega^2 u = f(x,y),$$

and do the frequency sweep again. How does the result change? Experiment with different $\alpha$. Also show examples of how the solutions close to and away from the peaks change with damping. **(0.25 p)**

4. Let $M$ be the number of grid points per wavelength. Examine numerically how large $M$ you need to take to get a solution with, say, less than 10% error in maxnorm. Try with several different $\omega$. How fast should the required $M$ theoretically grow with $\omega$ for your second order method? **(0.25 p)**

   *Hint:* For each $\omega$, first compute a reference solution with a large $M$ (fine grid). Then compute solutions with smaller $M$ (coarse grid) and use MATLAB's `interp2`-command to evaluate those solutions on the fine grid to be able to compare them with the reference solution.

## 2 Infinite domain problem

The Dirichlet conditions in the previous part model reflecting walls. The waves triggered by $f$ will therefore bounce around inside the domain infinitely long time and the solution becomes very complicated. In this section you should instead use the absorbing condition

$$\frac{\partial u}{\partial n} - i\omega u = 0, \qquad x \in \partial\Omega, \tag{2}$$

which you discretize with a central differencing of the derivative and one layer of ghost points, e.g.,

$$\frac{u_{1,j} - u_{-1,j}}{2h} - i\omega u_{0,j} = 0 \quad \Rightarrow \quad u_{-1,j} = u_{1,j} - 2ih\omega u_{0,j}.$$

We also scale the source with $\omega$

$$f_\omega(x, y) = \omega^{5/2} \exp\left(-\omega^2 \left[(x - s_x)^2 + (y - s_y)^2\right]\right),$$

to inject the same amount of wave energy for each frequency.

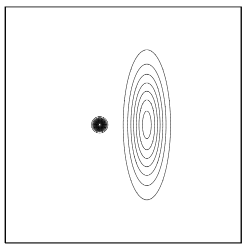
1. Implement the method with the absorbing boundary condition (2). Compute solution with $f$ centered in the middle of the square, $s_x = s_y = 0.5$, $n \equiv 1$ and $\omega = 50$. Your solution should be close the solution to the problem set in the whole $\mathbb{R}^2$, which is a circular wave whose amplitude decays as $1/\sqrt{r}$ where $r$ is the distance to the source. Use `contour` plot and `axis equal` to verify that your solution is (close to) circular. Plot real part and absolute value of $u$. Also check that you have the right decay rate with $r$ in your solution. **(1.0 p)**

2. Try moving the source around. The exact solution in $\mathbb{R}^2$ would be a circular wave wherever $f$ is located, but the absorbing boundary conditions perform differently depending on where $f$ is centered in the grid. When are they good? When are they bad? **(0.5 p)**

3. Add a variable index of refraction,

   $$n(x, y) = 1 + \sigma \exp\left(-200(x - \ell_x)^2 - 20(y - \ell_y)^2\right).$$

   This models a lens inside which the speed of propagation is lower than outside, causing waves to refract and focus. The number $\sigma$ is a measure of the strength of the lens and $(\ell_x, \ell_y)$ is its center. Take $\ell_x = 0.6$, $\ell_y = 0.5$, $s_x = 0.4$, $s_y = 0.5$ and $\sigma = 1.5$. Compute solutions and plot as before. Try also larger $\omega$ to get sharper results. **(0.5 p)**

## 3 High frequency approximation

In this exercise we consider the high frequency approximation of the problem in Exercise 2 above, given by the eikonal equation,

$$|\nabla\phi| = n(x,y), \qquad \phi(s_x, s_y) = 0.$$

Here $\phi$ is the phase of the simple wave solution to Helmholtz $u(x,y) = A(x,y)e^{i\omega\phi(x,y)}$ when $\omega \gg 1$. To solve the eikonal equation, rewrite it as the time-dependent equation

$$\phi_t + |\nabla\phi| = n(x,y), \qquad \phi(t, s_x, s_y) = 0, \qquad \phi(0,x,y) = 0,$$

and then use the Lax–Friedrichs scheme for time-dependent Hamilton–Jacobi equations to compute its steady state.[1] Discretize the domain $\Omega = [0,1]^2$ as before and let $\phi_{ij}^n$ approximate

$$\phi_{ij}^n \approx \phi(t_n, x_i, y_j), \qquad t_n = n\Delta t.$$

The Lax–Friedrichs scheme is then

$$\phi_{ij}^{n+1} = \frac{1}{4}\left(\phi_{i+1,j}^n + \phi_{i-1,j}^n + \phi_{i,j+1}^n + \phi_{i,j-1}^n\right) - \Delta t H\left(x_i, y_j, \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h}\right),$$

where

$$H(x,y,\phi_x,\phi_y) = \sqrt{\phi_x^2 + \phi_y^2} - n(x,y).$$

Let $(i_*, j_*)$ be the indices of the source point, $s_x = x_{i_*}$, $s_y = y_{j_*}$. Then set

$$\phi_{i_*,j_*}^n = 0, \qquad \forall n,$$

as the source boundary condition. For the outer boundary, just use extrapolation,

$$\phi_{-1,j} = \phi_{0,j}, \qquad \phi_{N+1,j} = \phi_{N,j}, \qquad \phi_{i,-1} = \phi_{i,0}, \qquad \phi_{i,N+1} = \phi_{i,N}.$$

To satisfy the CFL condition, choose your time step such that $\Delta t/\Delta x$ is sufficiently small. The number of steps needed to reach steady state should be of the order (but not equal to) $N$.

1. First take $s_x = s_y = 0.5$, $n(x) \equiv 1$. Run to steady state and plot contour lines of $\phi$. Verify that you get a circular wave. How large CFL number can you use? **(0.5 p)**

2. Add the lens from the Exercise 2, with $\ell_x = 0.6$, $\ell_y = 0.5$, $s_x = 0.4$, $s_y = 0.5$ and $\sigma = 1.5$ as before. Show how the solution change. Compare with the contour plots of the real part in Exercise 2. **(0.25 p)**

3. Trace rays from the source by solving the ray ODEs for $\boldsymbol{x} = (x,y)$ and $\boldsymbol{p} = (p_x, p_y)$,

   $$\frac{d\boldsymbol{x}}{dt} = \frac{\boldsymbol{p}}{n(\boldsymbol{x})^2}, \qquad \boldsymbol{x}(0) = (s_x, s_y),$$
   $$\frac{d\boldsymbol{p}}{dt} = \frac{\nabla n(\boldsymbol{x})}{n(\boldsymbol{x})}, \qquad \boldsymbol{p}(0) = n(s_x, s_y)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix},$$

   where $\theta$ is the initial angle. Use for instance `ode45` in MATLAB to solve the ODEs for a range of $\theta$. Plot the rays superimposed on the contour plots of your solutions for $\phi$. **(0.75 p)**

4. Increase $\sigma$, the strength of the lens. Eventually, the waves are refracted so much that the wave fronts fold over themselves and you get several crossing waves. Check how the different solution methods deal with this: (a) direct solution as in Exercise 2, (b) eikonal solution and (c) ray tracing. **(0.5 p)**

[1] You may of course instead implement the fast marching method, for extra credit.