The second eigenvectors of the Google matrix and their relation to link spamming

SIAM Student Chapter

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September 19, 2014

Outline

Scientific fame and glory

What determines scientific fame?

- Number of publications?
- Number of citations?
- h-index?
- Erdös number?

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PageRank and link spamming

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The research:

- What is the relation between link spamming and the second eigenvectors of the Google matrix?
- Develop an algorithm to compute a complete set of independent eigenvectors for the second eigenvalue.

A model of the web surfer

Web pages are connected through hyperlinks.

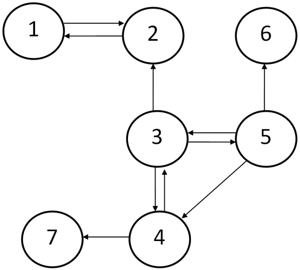

Figure: Model of part of the web

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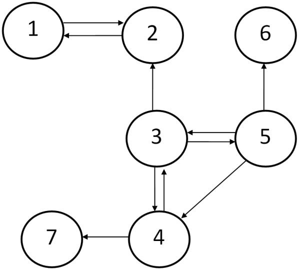

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Mathematical model:

- Binary matrix $\mathbf{G}$, with $\mathbf{G}_{i,j} = 1$ if page j links to page i.
- Row-stochastic transition matrix $\mathbf{P}$.

The matrices G and P

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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$$\mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1/7 & 1/7 \\ 1 & 0 & 1/3 & 0 & 0 & 1/7 & 1/7 \\ 0 & 0 & 0 & 1/2 & 1/3 & 1/7 & 1/7 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/7 & 1/7 \\ 0 & 0 & 1/3 & 0 & 0 & 1/7 & 1/7 \\ 0 & 0 & 0 & 0 & 1/3 & 1/7 & 1/7 \\ 0 & 0 & 0 & 1/2 & 0 & 1/7 & 1/7 \end{pmatrix}$$

Teleportation

Teleportation is jumping to a web page without following a link.

$$\mathbf{A} = p\mathbf{P}^T + \frac{1-p}{N}\mathbf{e}\mathbf{e}^T$$

Here, $\mathbf{e} = [1, 1, \ldots, 1]^T$.

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Probability p:	Follow an outlink,
Probability $1-p$:	Teleport to any web page.

The PageRank

- PageRank vector $\mathbf{x}$ is the probability vector after infinitely long surfing.
- Google matrix $\mathbf{A}$ has a unique largest eigenvalue $\lambda_1 = 1$ (by Perron-Frobenius).
- PageRank vector $\mathbf{x} > 0$ is the eigenvector of $\mathbf{A}$ corresponding to $\lambda_1 = 1$.
- Compute the PageRank vector by the Power method.

Link spamming

How can you increase your PageRank by using link structures?

Irreducible closed subchains

An irreducible closed subchain:

- Once entered a closed subchain, you cannot leave;
- Every node in an irreducible subchain can be reached.
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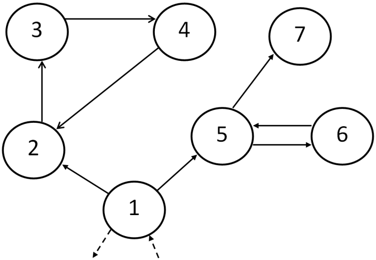
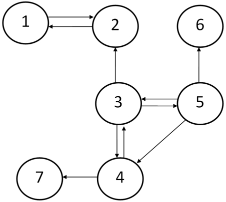
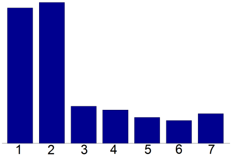


Figure: Irreducible closed subchains?

Link spamming: node 4



with $\mathbf{x}^{(1)} =$ 

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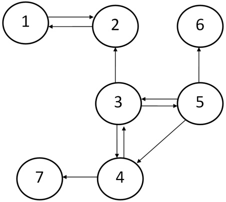
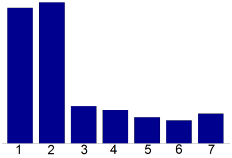


with $\mathbf{x}^{(1)} =$ 

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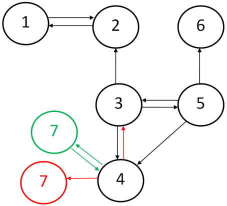

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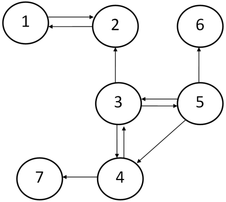

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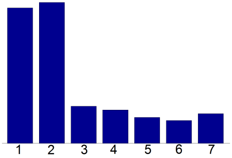

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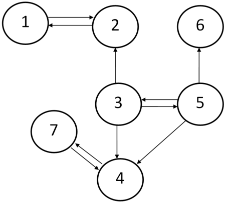

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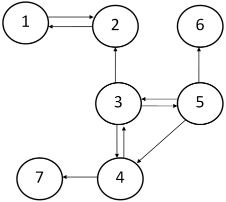

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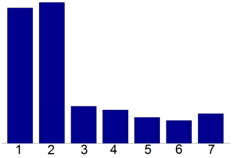

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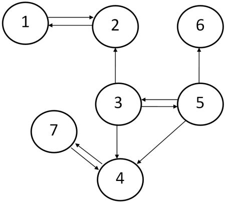

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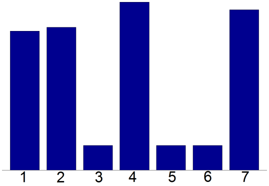

Tarjan's algorithm

Detection of link spamming:

- Find the irreducible subchains of the graph.
- An irreducible subchain consisting of a group of nodes, without outlinks, is an irreducible closed subchain.

The second eigenvectors of $\mathbf{A}$

$$\mathbf{A}\mathbf{x}^{(\mathbf{2})} = \lambda \mathbf{x}^{(\mathbf{2})},$$

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Computation of all the second eigenvectors

To solve $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, we can solve

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The nonzero-elements in $\mathbf{x}$ correspond to nodes in an irreducible closed subchain. Thereafter, apply Tarjan's algorithm to find the *different* irreducible closed subchains.

Detection of link spamming

Two link spamming detection algorithms:

1. Search the web for irreducible closed subchains (Tarjan's algorithm)
2. - Compute a first eigenvector of $\mathbf{P^T}$,
   - Determine the nonzero entries,
   - Use Tarjan's algorithm only on the nodes corresponding to nonzero entries.

Results

Test problem	Size	Closed subchains	CPU-time Tarjan	CPU-time Eigenvector
wb-cs-stanford	9914	113	0.3	1.4
flickr	820878	5394	399.3	160.8
wikipedia-20051105	1634989	68	1515.3	140.2
wikipedia-20060925	2983494	63	5077.1	166.6
wikipedia-20061104	3148440	59	5696.9	155.1
wikipedia-20070206	3566907	58	7462.7	313.6
wb-edu	9845725	49573	75703.2	2825.6*

Computing time for web crawls by Gleich

Note: For `wb-edu` the eigenvector algorithm found 41606 subchains.

Conclusions

The relation between link spamming and the second eigenvector of the Google matrix:

- The second eigenvectors are combinations of the dominant eigenvectors of the matrix $\mathbf{P}^T$
- The dominant eigenvectors of $\mathbf{P}^T$ correspond to irreducible closed subchains
- Nodes in these subchains get artificially high PageRanks
- We have proposed an efficient algorithm to compute all second eigenvectors of the Google matrix and to detect this kind of link spamming