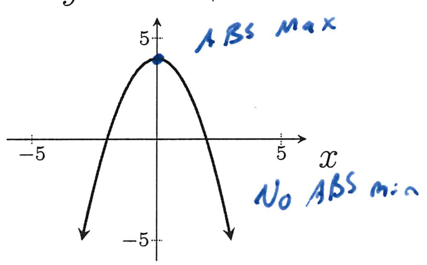
4.1: Maxima and Minima

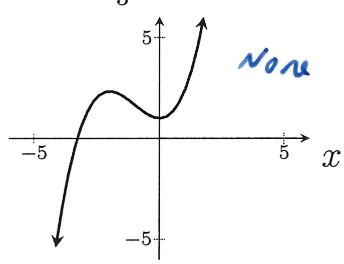
> **Definition. (Absolute Maximum and Minimum)**
> Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

Example. Determine whether the function has any absolute extreme values

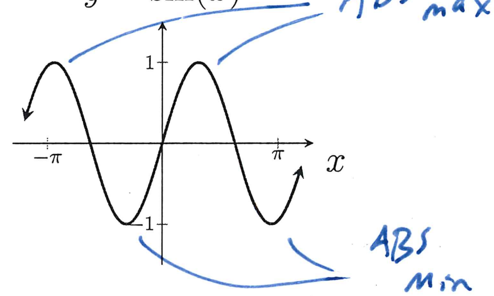
$y = -x^2 + 4$



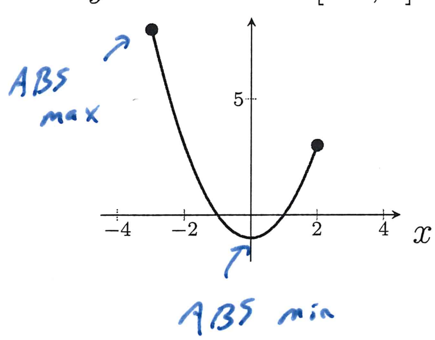
$y = \frac{x^3}{3} + x^2 + 1$



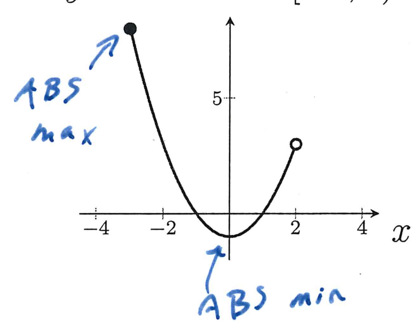
$y = \sin(x)$



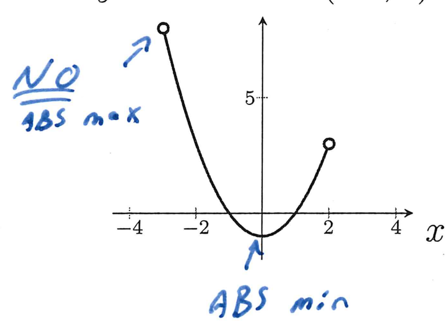
$y = x^2 - 1$ on $[-3, 2]$



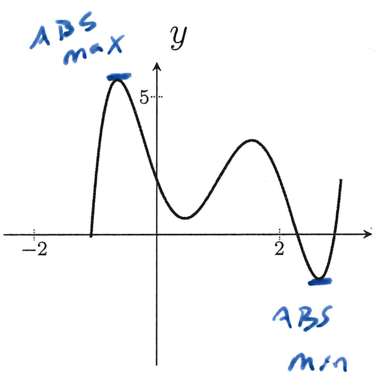
$y = x^2 - 1$ on $[-3, 2)$

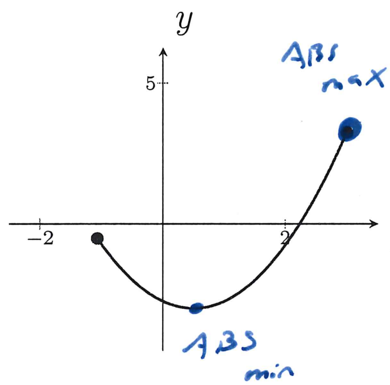


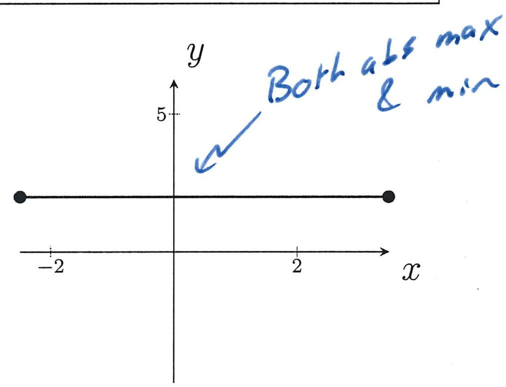
$y = x^2 - 1$ on $(-3, 2)$



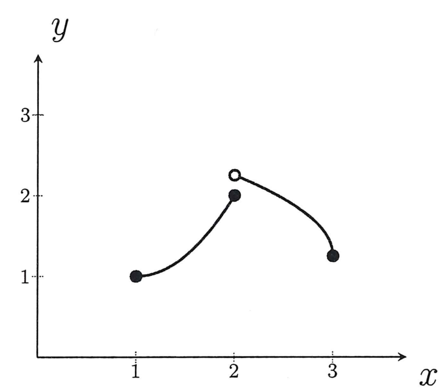
Theorem 4.1: Extreme Value Theorem
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

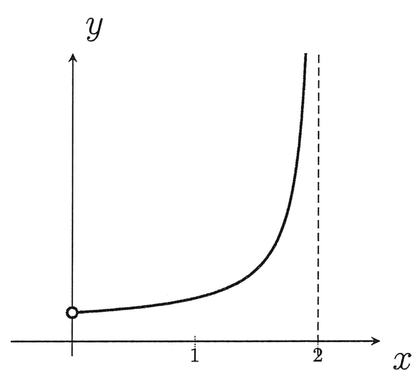




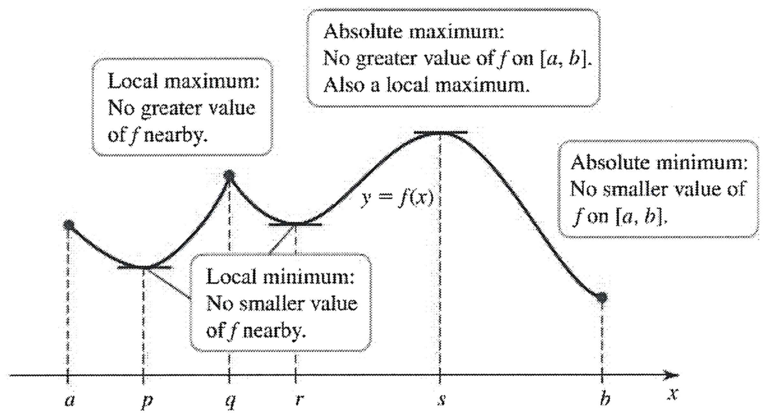


Note: It is important that the function is both continuous *and* the interval is closed:



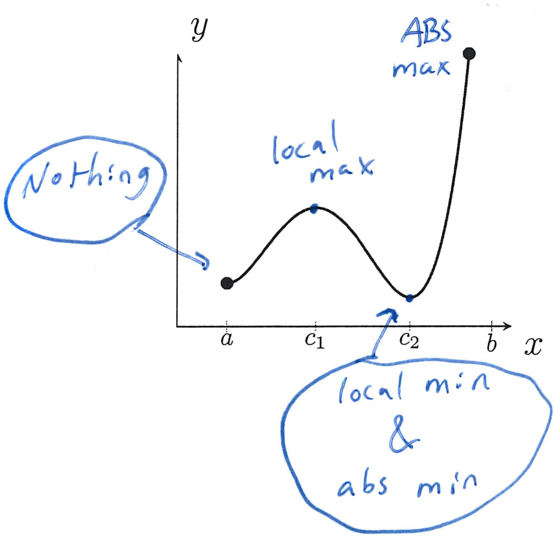


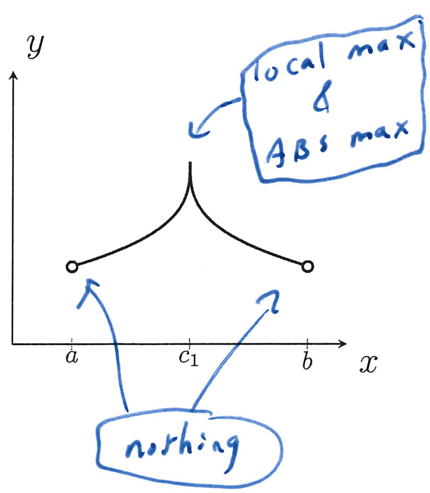
Definition. (Local Maximum and Minimum Values)
Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum value of f.**

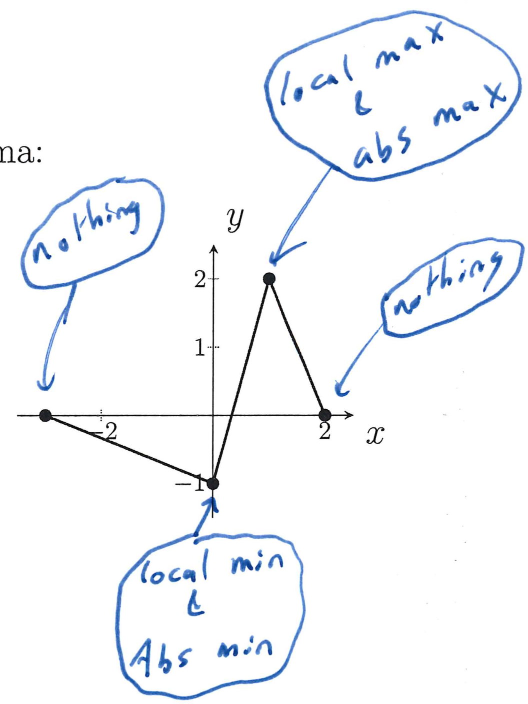


Note: Local extrema **CANNOT** occur at endpoints.

Example. State the absolute extrema and the local extrema:

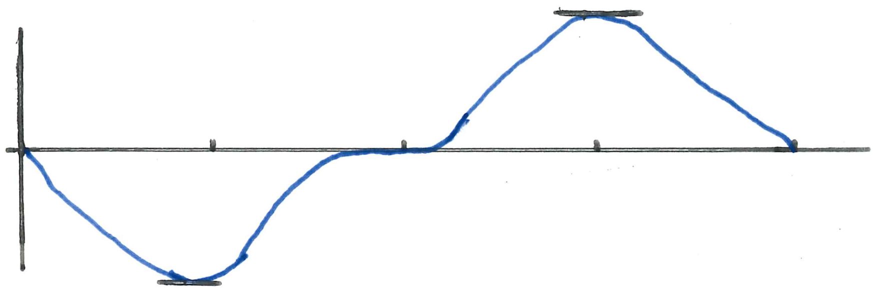




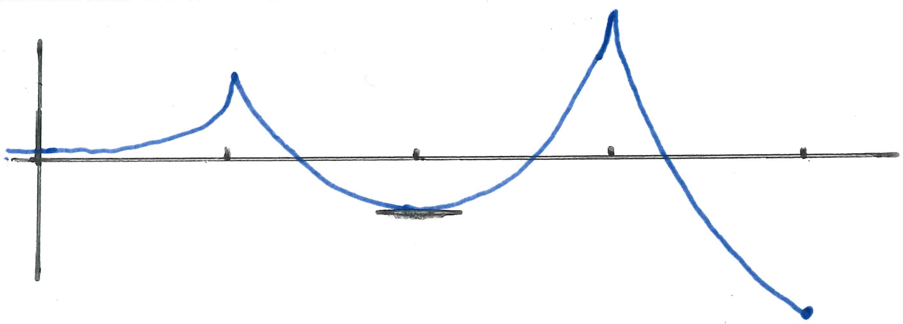


Example. Sketch the graph of a continuous function f on $[0, 4]$ satisfying the given properties:

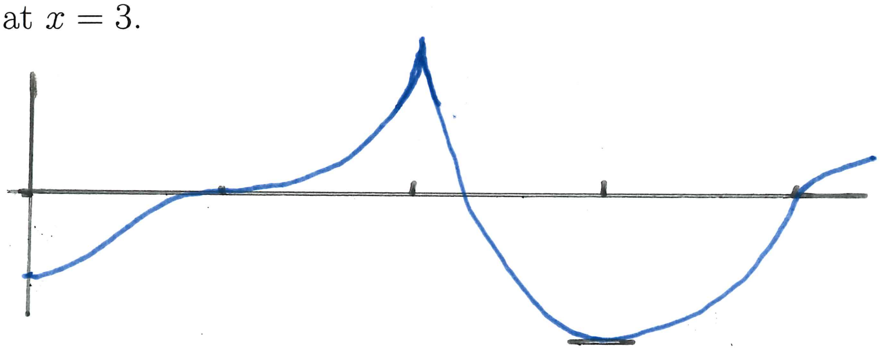
1. $f'(x) = 0$ for $x = 1, 2$ and 3; f has an absolute minimum at $x = 1$; f has no local extremum at $x = 2$; and f has an absolute maximum at $x = 3$.



2. $f'(1)$ and $f'(3)$ are undefined; $f'(2) = 0$; f has a local maximum at $x = 1$; f has a local minimum at $x = 2$; f has an absolute maximum at $x = 3$; and f has an absolute minimum at $x = 4$.



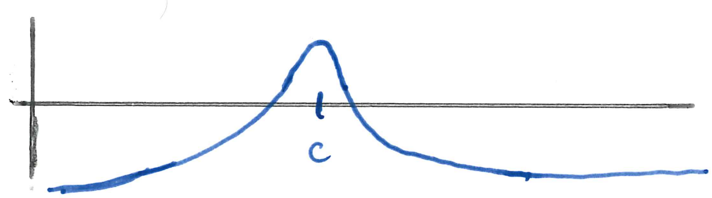
3. $f'(x) = 0$ at $x = 1$ and 3; $f'(2)$ is undefined; f has an absolute maximum at $x = 2$; f has neither a local maximum nor a local minimum at $x = 1$; and f has an absolute minimum at $x = 3$.



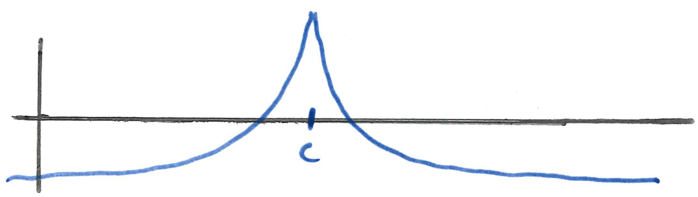
Theorem 4.2: Local Extreme Value Theorem
If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

Note: If the derivative is zero, then the function **MIGHT** have a max/min.

Example. Sketch a graph of a function $f(x)$ that has a local maximum value at a point c where $f'(c)$ is defined.

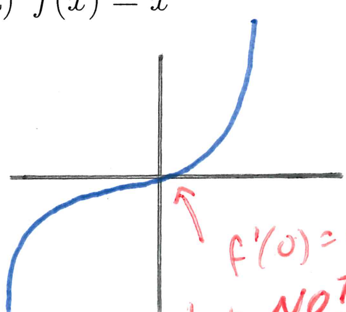


Example. Sketch a graph of a function $f(x)$ that has a local maximum value at a point c where $f'(c)$ is undefined.



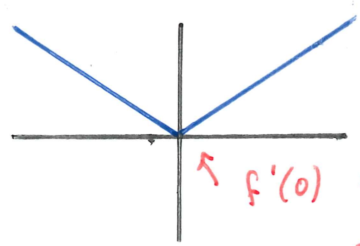
Example. Graph

a) $f(x) = x^3$



$f'(0) = 0$ but NOT a max/min

b) $f(x) = |x|$



$f'(0)$ DNE, but is abs min.

> **Definition. (Critical Point)**
> An interior point c of the domain of f at which $f'(c) = 0$ *or* $f'(c)$ fails to exist is called a **critical point** of f.

Example. Find the critical points of

$$f(x) = x^3 + 3x^2 - 24x$$

$$f'(x) = 3x^2 + 6x - 24 = 3(x+4)(x-2) \overset{\text{set}}{=} 0$$

$$\Rightarrow \boxed{x = -4,\ x = 2}$$

$$g(x) = \sqrt{4 - x^2}$$

$$g'(x) = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}} \overset{\text{set}}{=} 0 \quad \Rightarrow \boxed{x = 0}$$

$$\sqrt{4-x^2} \neq 0$$

$$4 - x^2 \neq 0$$

$$\pm 2 \neq x \leftarrow$$

critical points unless these are endpoints

$$h(t) = 3t - \sin^{-1}(t)$$

$$h'(t) = 3 - \frac{1}{\sqrt{1-t^2}} \overset{\text{set}}{=} 0$$

$$3 = \frac{1}{\sqrt{1-t^2}}$$

$$1 - t^2 = 1/9$$

$$t^2 = 8/9$$

$$\sqrt{1-t^2} \neq 0$$

$$1 \neq t^2$$

$$\pm 1 \neq t$$

crit pts unless endpoints

$$\boxed{t = \pm \frac{2\sqrt{2}}{3}}$$

Example. Find the critical points of

$$f(x) = \sin(x)\cos(x) \text{ on } [0, 2\pi].$$

$f'(x) = \cos^2(x) - \sin^2(x) \overset{\text{set}}{=} 0$

$\cos(x) = \pm \sin(x)$

$\Rightarrow \boxed{x = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4}$

$$f(t) = t^2 - 2\ln(t^2 + 1)$$

$f'(t) = 2t - \dfrac{2}{t^2+1} 2t \overset{\text{set}}{=} 0 \Rightarrow 2t = \dfrac{4t}{t^2+1}$

$\Rightarrow (t^2+1)t = 2t$

$\Rightarrow t(t^2+1-2) = 0$

$\Rightarrow t(t^2-1) = 0 \Rightarrow \boxed{t = 0,\ t = \pm 1}$

$$f(x) = x\sqrt{x - a}$$

$f'(x) = \sqrt{x-a} + \dfrac{x}{2\sqrt{x-a}} \overset{\text{set}}{=} 0 \Rightarrow 2(x-a) + x = 0$

$3x = 2a$

$\boxed{x = \frac{2}{3}a}$

$x \neq a$

endpoint $\Rightarrow$ Not crit pt.

$$f(x) = \sin^{-1}(x)\cos^{-1}(x)$$

$f'(x) = \dfrac{\cos^{-1}(x)}{\sqrt{1-x^2}} - \dfrac{\sin^{-1}(x)}{\sqrt{1-x^2}} \overset{\text{set}}{=} 0 \Rightarrow \cos^{-1}(x) = \sin^{-1}(x)$

$\Rightarrow \boxed{x = \frac{\sqrt{2}}{2}}$

$x \neq \pm 1$

endpoint $\Rightarrow$ not crit pt

How to find the absolute max/min of $f(x)$ on $[a, b]$:

1. Find $f'(x)$
2. Find all critical points ($f'(x) = 0$ or $f'(x)$ **DNE**)
3. Evaluate $f(x)$ at the critical points within $[a, b]$ and the end points.
4. Identify the absolute max and absolute min using the values found. Include value and location (e.g. ordered pair (x, y))

Example. Find the absolute max and min of $f(x) = 2x - 2x^{\frac{2}{3}}$ on $[-1, 3]$.

① $f'(x) = 2 - \frac{4}{3}x^{-1/3} = 2 - \frac{4}{3\sqrt[3]{x}}$

② $2 - \frac{4}{3\sqrt[3]{x}} = 0 \rightarrow \sqrt[3]{x} = \frac{2}{3} \longrightarrow \boxed{x = \frac{8}{27}, \quad x \neq 0}$

③

x	-1	0	$8/27$	3
$f(x)$	-4	0	$-8/27$	$6 - 2\sqrt[3]{9}$

≈ 1.8398

④ Abs. Min $(-1, -4)$

Abs. Max $(3, 6 - 2\sqrt[3]{9})$

Example. Find the absolute max and min of $f(x) = \frac{x}{x^2+1}$ on $[0, 2]$.

① $f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$

② $\frac{-x^2+1}{(x^2+1)^2} = 0 \rightarrow x = \pm 1$

Note! -1 out of domain

③

x	0	1	2
$f(x)$	0	$1/2$	$2/5$

④ Abs min $(0, 0)$

Abs max $(1, 1/2)$

Example. Find the absolute max and min of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

① $f'(x) = 12x^3 - 12x^2 - 24x$

② $12x(x^2 - x - 2) = 0$

$12x(x-2)(x+1) = 0$

$x = 0, \; x = 2, \; x = -1$

③

x	-2	-1	0	2	3
$f(x)$	33	-4	1	-31	28

④ Abs min $(2, -31)$

Abs max $(-2, 33)$

Example. Find the absolute max and min of $f(x) = \sin(3x)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

① $f'(x) = 3\cos(3x)$

② $3\cos(3x) = 0$

$3x = \pm \pi/2$

$x = \pm \pi/6$

③

x	$-\pi/4$	$-\pi/6$	$\pi/6$	$\pi/3$
$f(x)$	$\sqrt{2}/2$	-1	1	0

④ Abs min $(-\pi/6, -1)$

Abs max $(\pi/6, 1)$

Example. Find the absolute max and min of $f(x) = xe^{1-\frac{x}{2}}$ on $[0, 5]$.

① $f'(x) = e^{1-x/2} + xe^{1-x/2}(-1/2)$

② $e^{1-x/2}(1 - x/2) = 0$

$x = 2$

③

x	0	2	5
$f(x)$	0	2	$5e^{-3/2}$ ≈ 1.11565

④ Abs min $(0, 0)$

Abs max $(2, 2)$

Example. Find the absolute max and min of $f(x) = e^x - 2x$ on $[0, 2]$.

① $f'(x) = e^x - 2$

② $e^x - 2 = 0$

$x = \ln(2)$

③

x	0	$\ln(2)$	2
$f(x)$	1	$2 - 2\ln(2)$ 0.6137	$e^2 - 4$ 3.389

④ Abs min $(\ln(2), 2 - 2\ln(2))$

Abs max $(2, e^2 - 4)$

Example. Find the absolute max and min of $f(x) = x^{\frac{1}{3}}(x+4)$ on $[-27, 27]$.

① $f'(x) = \frac{1}{3}x^{-2/3}(x+4) + x^{1/3}$

② $\frac{x+4}{3\sqrt[3]{x^2}} + \sqrt[3]{x} = 0$

$x+4+3x=0$

$x=-1$

$x \neq 0$

③

x	-27	-1	0	27
$f(x)$	69	-3	0	93

④ Abs min : $(-1, -3)$

Abs max : $(27, 93)$

Example. Find the absolute max and min of $y = \sqrt{x^2-1}$.

① $y' = \frac{2x}{2\sqrt{x^2-1}}$

② $\frac{x}{\sqrt{x^2-1}} = 0 \rightarrow x=0$

$x \neq \pm 1$

End points

③

x	-1	0	1
y	0	DNE	0

④ Abs min $(-1,0)$ & $(1,0)$

Abs max N/A

Example. Find the absolute/local max and min of $f(x) = x^2(x^2+4x-8)$ on $[-5, 2]$.

① $f'(x) = 4x^3 + 12x^2 - 16x$

② $4x(x^2+3x-4)=0$

$4x(x+4)(x-1)=0$

$x=0, x=-4, x=1$

③

x	-5	-4	0	1	2
$f(x)$	-75	-128	0	-3	16

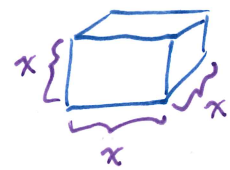
④ Abs min $(-4, -128)$

Abs max $(2, 16)$

Example. Minimum-surface-area box

All boxes with a square base of length x and a volume V have a surface area given by $S(x) = x^2 + \frac{4V}{x}$. Find x such that the box has volume $50\,ft^3$ with minimal surface area.

(1) $S'(x) = 2x - \frac{4V}{x^2}$



(2) $2x - \frac{4V}{x^2} = 0$

$x^3 = 2V$

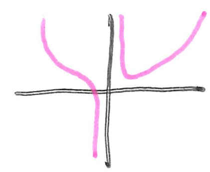
$x = \sqrt[3]{2V} \qquad x \neq 0$

(3)

x	0	$\sqrt[3]{2V} = \sqrt[3]{100}$
$S(x)$	DNE	$3(100)^{2/3} \approx 64.633$

(4) Abs min $\left(\sqrt[3]{100},\ 3\sqrt[3]{100^2}\right)$

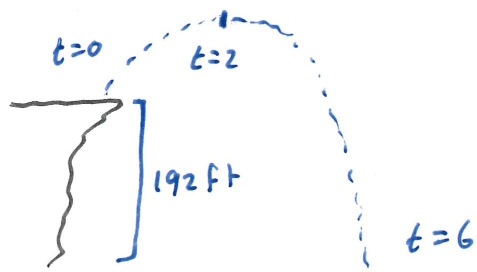
No Abs max



Example. Trajectory high point

A stone is launched vertically upward from a cliff 192 ft above the ground at a speed of 64 ft/s. Its height above the ground t seconds after the launch is given by $s = -16t^2 + 64t + 192$, for $0 \leq t \leq 6$. When does the stone reach its maximum height?

(1) $s' = -32t + 64$



(2) $-32t + 64 = 0$

$t = 2$

(3)

t	0	2	6
s	192	256	0

(4) Abs max $(2, 256)$

Example. Maximizing Revenue

A sales analyst determines that the revenue from sales of fruit smoothies is given by $R(x) = -60x^2 + 300x$, where x is the price in dollars charged per item, for $0 \le x \le 5$.

a) Find the critical points of the revenue function.

b) Determine the absolute maximum value of the revenue function and the price that maximizes the revenue.

a) $R'(x) = -120x + 300 \overset{\text{set}}{=} 0$

$$\boxed{x = \frac{5}{2}}$$

b)

x	$R(x)$
0	0
$5/2$	375
5	0

Abs max: $(5/2, 375)$

Example. Find the absolute and local extreme values of the following

1. $f(x) = |x-3| + |x+2|$ on $[-4, 4]$,

$$f(x) = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases} + \begin{cases} x+2, & x \geq -2 \\ -(x+2), & x < -2 \end{cases} = \begin{cases} -2x+1, & -4 \leq x < -2 \\ 5 & -2 \leq x < 3 \\ 2x-1 & 3 \leq x \leq 4 \end{cases}$$

$$f'(x) = \begin{cases} -2, & -4 \leq x < -2 \\ 0, & -2 < x < 3 \\ 2, & 3 < x \leq 4 \end{cases}$$

crit. pts

$x = -2$, $x = 3$

$\because f'(x)$ DNE

x	$f(x)$
-4	9
-2	5
3	5
4	7

Abs min

$(a, 5)$

where $-2 \leq a \leq 3$

Abs max

$(-4, 9)$

2. $g(x) = |x-3| - 2|x+1|$ on $[-2, 3]$.

$$g(x) = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases} - 2\begin{cases} (x+1), & x \geq -1 \\ -(x-1), & x < -1 \end{cases} = \begin{cases} x+5, & -2 \leq x < -1 \\ -3x+1 & -1 \leq x \leq 3 \end{cases}$$

$$g'(x) = \begin{cases} 1, & -2 \leq x < -1 \\ -3, & -1 < x \leq 3 \end{cases}$$

x	$g(x)$
-2	3
-1	4
3	-8

$g'(x)$ DNE @ $x = -1$

Abs min

$(3, -8)$

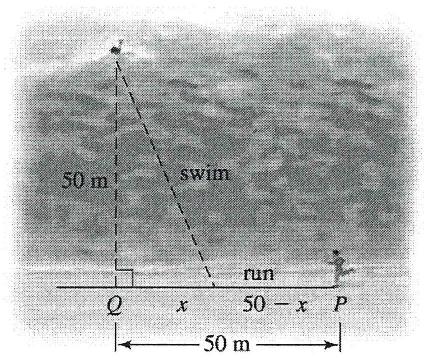
Abs max

$(-1, 4)$

Example. Every second counts (4.01: Q87)

You must get from a point P on the straight shore of a lake to a stranded swimmer who is $50\,m$ from a point Q on the shore that is $50\,m$ from you. Assuming that you can swim at a speed of $2\,m/s$ and run at a speed of $4\,m/s$, the goal of this exercise is to determine the point along the shore, x meters from Q, where you should you stop running and start swimming to reach the swimmer in the minimum time.

a) Find the function T that gives the travel time as a function of x, where $0 \leq x \leq 50$.



b) Find the critical point of T on $(0, 50)$.

c) Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?

d) Graph the function T to check your work.

a) $T(x) = \dfrac{50-x}{4}\dfrac{m}{m/s} + \dfrac{\sqrt{x^2+50^2}}{2}\dfrac{m}{m/s}$

b) $T'(x) = -\dfrac{1}{4} + \dfrac{x}{2\sqrt{x^2+50^2}} \overset{set}{=} 0 \implies \dfrac{2\sqrt{x^2+50^2}}{2} = \dfrac{4x}{2}$

$x^2+50^2 = 4x^2$

$50^2 = 3x^2$

$\sqrt{\dfrac{50^2}{3}} = x$

$\boxed{\dfrac{50}{\sqrt{3}} = x}$

c)

x	$T(x)$	
0	$75/2$	$= 37.5$
$50/\sqrt{3}$	$25/2 + \frac{75}{2\sqrt{3}}$	$\approx 34.1506 \leftarrow$ Abs min
50	$25\sqrt{2}$	≈ 35.355

d)