

4.6 Linear Approximation and Differentials

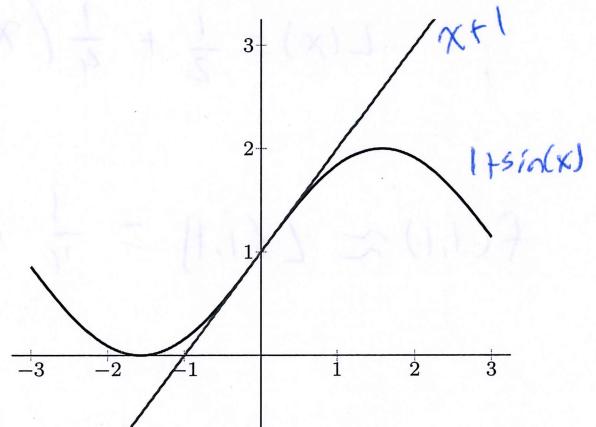
Example. Find the equation of the line tangent to $y = \underline{1 + \sin(x)}$ at $a = 0$. State the line in the slope-intercept form.

$$y - f(a) = f'(a)(x - a)$$

$$\begin{aligned} f(x) &= 1 + \sin(x) \\ f(0) &= 1 \\ f'(x) &= \cos(x) \\ f'(0) &= 1 \end{aligned}$$

$$y - 1 = 1(x - 0)$$

$$\Rightarrow \boxed{y = x + 1}$$



Definition.

Linear Approximation of f at a .

Suppose f is differentiable on an interval I containing the point a . The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a), \text{ for } x \text{ in } I.$$

Example. Consider the function $f(x) = \frac{x}{x+1}$. Find the linearization at $a = 1$. Use the linearization to estimate $f(1.1)$ and compare with the true value of $f(1.1)$.

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) = \frac{x}{x+1}$$

$$f(x) = \frac{x}{x+1}$$

$$f(1) = \frac{1}{2}$$

$$f'(x) = \frac{1}{(x+1)^2}$$

$$f'(1) = \frac{1}{4}$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \boxed{\frac{1}{4} + \frac{x}{4}}$$

$$f(1.1) \approx L(1.1) = \frac{1}{4} + \frac{1.1-1}{4} = \boxed{\frac{21}{40}}$$



$$\text{Definition: } L(x) = f(a) + f'(a)(x-a)$$

Example. Find the linearization $L(x)$ of $f(x) = e^{3x-6}$ at $a = 2$.

$$f'(x) = 3e^{3x-6} \quad f'(2) = 3$$

$$f(2) = 1$$

$$L(x) = f(2) + f'(2)(x-2)$$

$$\boxed{L(x) = 1 + 3(x-2)}$$

Example. Find the linearization $L(x)$ of $f(x) = 9(4x+11)^{\frac{2}{3}}$ at $a = 4$.

$$f'(x) = 6(4x+11)^{-\frac{1}{3}}(4) = \frac{24}{(4x+11)^{\frac{1}{3}}} \quad f'(4) = 8$$

$$f(4) = 81$$

$$L(x) = f(4) + f'(4)(x-4)$$

$$\boxed{L(x) = 8x - 49}$$

Example. Find a linearization over an interval which will contain the given point x_0 . Choose your center at a point *near* x_0 but not at x_0 so that the given function and its derivative are easy to evaluate. Lastly, use the linearization to approximate $f(x_0)$.

a) $f(x) = x^2 + 2x, x_0 = 0.1$

$$f'(x) = 2x + 2$$

$$f(0) = 0$$

$$f'(0) = 2$$

$$L(x) = 0 + 2(x-0) = \boxed{2x}$$

$$f(0.1) \approx L(0.1) = \boxed{0.2}$$

b) $f(x) = \sqrt[3]{x}, x_0 = 8.5$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$f(8) = 2$$

$$f'(8) = \frac{1}{12}$$

$$L(x) = 2 + \frac{1}{12}(x-8)$$

$$\approx \boxed{\frac{x}{12} + \frac{4}{3}}$$

$$f(8.5) \approx L(8.5) = \frac{8.5}{12} + \frac{4}{3} = \boxed{\frac{49}{24}}$$

Example. Use a linear approximation to estimate $\sqrt{146}$.

$$f(x) = \sqrt{x} \quad f(144) = 12$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(144) = \frac{1}{24}$$

$$L(x) = 12 + \frac{1}{24}(x - 144) = 6 + \frac{x}{24}$$

$$f(146) \approx L(146) = 6 + \frac{146}{24} = \boxed{\frac{145}{12}} = \boxed{12.083}$$

Example. Use a linear approximation to estimate $(1.999)^4$.

$$f(x) = x^4 \quad f(2) = 16$$

$$f'(x) = 4x^3 \quad f'(2) = 32$$

$$L(x) = 16 + 32(x - 2) = 32x - 96$$

$$f(1.999) \approx L(1.999) = 32(1.999) - 96 = 63.968 - 96 = \boxed{15.968}$$

Example. Use a linear approximation to estimate $\sqrt{\frac{5}{29}}$. $\approx \sqrt{0.172413\dots}$

$$f(x) = \sqrt{x} \quad f(0.16) = \frac{4}{10} = \frac{2}{5}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(0.16) = \frac{5}{4}$$

$$L(x) = \frac{2}{5} + \frac{5}{4}(x - \frac{4}{25}) = \frac{1}{5} + \frac{5}{4}x$$

$$f(\frac{5}{29}) \approx L(\frac{5}{29}) = \frac{1}{5} + \frac{5}{4}(\frac{5}{29}) = \boxed{\frac{241}{380}} \approx 0.4155$$

Example. Find the linearization $L(x)$ of $f(x) = \sqrt{x^2 + 9}$ at $a = -4$. Use $L(x)$ to estimate $f(-4.1)$ and $\sqrt{23.44}$.

$$f(x) = \frac{x}{\sqrt{x^2 + 9}}$$

$$f(-4) = 5$$

$$f'(-4) = -\frac{4}{5}$$

$$L(x) = f(-4) + f'(-4)(x+4)$$

$$= 5 - \frac{4}{5}(x+4) = \boxed{-\frac{4}{5}x + \frac{9}{5}}$$

$$f(-4.1) \approx L(-4.1) = -\frac{4}{5}(-4.1) + \frac{9}{5} = \boxed{\frac{127}{25}}$$

$$\sqrt{23.44} = f(3.8) = -\frac{4}{5}(3.8) + \frac{9}{5} = \boxed{\frac{121}{25}}$$

Example. Use a linearization to show that 0.05 is a good approximation for $\ln(1.05)$.

$$f(x) = \ln(1+x) \quad a=0 \quad \left. \begin{array}{l} f(0) = \ln(1) = 0 \\ f'(0) = \frac{1}{1+0} = 1 \end{array} \right\}$$

$$\rightarrow L(x) = 0 + 1(x - 0) = x$$

$$L(0.05) = 0.05$$

Alternatively

$$f(x) = \ln(x) \quad a=1 \Rightarrow \left. \begin{array}{l} f(1) = 0 \\ f'(1) = 1 \end{array} \right\}$$

Example. Find the linearization of the following functions at the given point and use concavity to identify if the linearization is an overestimate or an underestimate.

$$a) f(x) = \frac{2}{x}; a = 1 \quad \left. \begin{array}{l} f(1) = 2 \\ f'(1) = -2 \end{array} \right\} \Rightarrow L(x) = 2 - 2(x - 1) \\ f'(x) = -\frac{2}{x^2}$$

$$f''(x) = \frac{4}{x^3} \quad f''(1) = 4 > 0 \rightarrow \text{concave up} \\ \rightarrow \text{underestimate}$$

$$b) f(x) = e^{-x}; a = \ln(2) \quad \left. \begin{array}{l} f(\ln(2)) = 2 \\ f'(\ln(2)) = -2 \end{array} \right\} \Rightarrow L(x) = 2 - 2(x - \ln(2))$$

$$f'(x) = -e^{-x} \quad f''(\ln(2)) = 2 > 0 \rightarrow \text{concave up} \\ \rightarrow \text{underestimate}$$

Summary: Uses of Linear Approximation

1. To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

2. To approximate the change Δy in the dependent variable when the independent variable x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a)\Delta x.$$

Definition. (Differentials)

Let f be differentiable on an interval containing x . A small change in x is denoted by the **differential** dx . The corresponding change in f is approximated by the **differential** $dy = f'(x) dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x) dx.$$

Example. Find the differential dy .

$$y = \cos(x^2)$$

$$dy = -\sin(x^2) 2x dx$$

$$y = \sqrt{1-x^2}$$

$$dy = \frac{-x}{\sqrt{1-x^2}} dx$$

$$y = 4x^2 - 3x + 2$$

$$dy = (8x-3) dx$$

$$y = x \tan(x^3)$$

$$dy = (\tan(x^3) + x \sec^2(x^3)(3x^2)) dx$$

$$y = \cos^5(x)$$

$$f(x) = \sin^{-1}(x)$$

$$dy = -5 \cos^4(x) \sin(x) dx$$

$$dy = \frac{1}{\sqrt{1-x^2}} dx$$

Example. Let $y = x^2$

- a) Find dy

$$dy = 2x \, dx$$

- b) If $x = 1$ and $dx = 0.01$, find dy .

$$dy = 2(1)(0.01) = 0.02$$

- c) Compare dy and Δy at this point.

$$\Delta y = f(1.01) - f(1) = 1.0201 - 1 = 0.0201$$

Example. Let $y = \sqrt{3 + x^2}$

- a) Find dy

$$dy = \frac{x}{\sqrt{3+x^2}} \, dx$$

- b) If $x = 1$ and $dx = -0.1$, find dy .

$$dy = \frac{1}{\sqrt{3+1}}(-0.1) = -\frac{1}{4} = -0.25$$

- c) Compare dy and Δy at this point.

$$\Delta y = f(0.9) - f(1) \approx 1.7292 - 2 = -0.2708$$

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.01) - f(5) = 0.25$. Use linear approximation to estimate the value of $f'(5)$.

$$\Delta y \approx dy = f'(x) dx \Rightarrow f'(x) \approx \frac{\Delta y}{dx}$$

$$\Delta y = 0.25$$

$$dx = 5.01 - 5 = 0.01$$

$$f'(x) \approx \frac{0.25}{0.01} = 25$$

$$1050.0$$

$$= 1 + 1050.1 = (1) + (10.1) = \Delta$$

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.99) = 7$ and $f(6) = 7.002$. Use linear approximation to estimate the value of $f'(6)$.

$$\Delta y = f(6) - f(5.99) = 0.002$$

$$dx = 6 - 5.99 = 0.01$$

$$f'(6) \approx \frac{0.002}{0.01} = 0.2$$

$$1050.0 = 1 + 1050.1 = (1) + (10.1) = \Delta$$

Example. Compute dy and Δy for $y = e^x$ when $x = 0$ and $\Delta x = 0.5$.

$$\Delta y = e^{0.5} - e^0 \approx 0.648721$$

$$dy = f'(0)(0.5) = 1/2 = 0.5$$

Example. Approximate the change in the area of a circle when its radius increases from 2.00 to 2.02 m.

$$A = \pi r^2$$

$$r = 2.00$$

$$dr = 0.02$$

$$\Delta A \approx dy = [2\pi(2)](0.02) = 0.08\pi \approx 0.251327 \text{ m}^2$$

Example. Approximate the change in the magnitude of the electrostatic force between two charges when the distance between them increases from $r = 20 \text{ m}$ to $r = 21 \text{ m}$ ($F(r) = 0.01/r^2$).

$$y = F'(r) = -\frac{0.02}{r^3}$$

$$dr = 1 \text{ m}$$

$$\Delta y \approx dy = F'(20)(1) = -\frac{0.02}{(20)^3}(1) = \frac{1}{400,000}$$

Example. Approximate the change in the volume of a right circular cylinder of fixed radius $r = 20 \text{ cm}$ when its height decreases from $h = 12 \text{ cm}$ to $h = 11.9 \text{ cm}$ ($V(h) = \pi r^2 h$).

$$V = \pi r^2 h$$

$$V' = \pi r^2$$

$$h = 12$$

$$dh = -0.1 \text{ cm}$$

$$\Delta V \approx dV = [\pi(20)^2](-0.1) = -40\pi \approx -125.663706$$

Example. Approximate the change in the volume of a right circular cylinder of a right circular cone of fixed height $h = 4 \text{ m}$ when its radius increases from $r = 3 \text{ m}$ to $r = 3.05 \text{ m}$ ($V(r) = \frac{1}{3}\pi r^2 h$).

$$r = 3$$

$$dr = 0.05$$

$$V'(r) = \frac{2}{3}\pi r h$$

$$\Delta V \approx dV = \left[\frac{2}{3}\pi(3)(4) \right](0.05) = \frac{2\pi}{5} \approx 1.256637$$

$$\frac{1}{60.000} = (1) \frac{50.000}{3(0.05)} = (1) (66.666) = 66.666 \times \Delta$$

Example. The radius of a sphere is measured and found to be 0.7 inches with a possible error in measurement of at most 0.01 inches.

- a) What is the maximum error in using the value of the radius to compute the volume of the sphere?

$$V = \frac{4}{3} \pi r^3$$

$$\Delta V \approx dV = (4\pi r^2) dr$$

$$\begin{aligned} dV &= 4\pi \left(\frac{7}{10}\right)^2 \left(\frac{1}{100}\right) \\ &= \frac{196}{10000} \pi = \frac{49}{2500} \pi \approx \boxed{0.061575 \text{ in}^3} \end{aligned}$$

- b) Find the relative error of the volume: relative error $\frac{dV}{V}$

What is the percentage error?

$$\text{rel error: } \frac{4\pi r^2 dr}{\frac{4}{3} \pi r^3} = \frac{3}{r} dr$$

$$\Rightarrow \text{rel error} = \frac{3}{(7/10)} \left(\frac{1}{100}\right) = \frac{3}{70} \approx \boxed{0.04286}$$

$$\text{Percentage error} = 100\% \times \text{rel error}$$

$$\Rightarrow \text{percentage error} = \boxed{4.286\%}$$

Example. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm .

- a) Use differentials to estimate the maximum error in the calculated area of the disk.

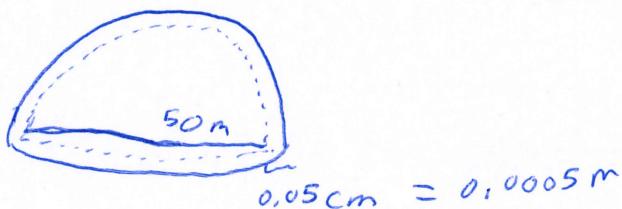
$$\begin{aligned} A &= \pi r^2 \\ \Delta A \approx dA &= [2\pi r](dr) \\ &= (2\pi \cdot 24)(0.2) \\ &= 9.6\pi \approx 30.1593 \end{aligned}$$

- b) What is the relative error? What is the percentage error?

$$\text{rel. error: } \frac{\frac{2\pi r}{\pi r^2} dr}{r} = \frac{2 dr}{r} = \frac{2(0.2)}{24} = \frac{1}{60} \approx 0.016$$

$$\text{percentage error: } 100\% \times \text{rel. error} = 16.6\%$$

Example. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.



$$r = 25 \text{ m}$$

$$dr = 0.0005 \text{ m}$$

$$V = \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right) = \frac{2}{3} \pi r^3$$

$$dV = 2\pi r^2 dr = 2\pi (25^2)(0.0005) = \frac{5\pi}{8} \approx \boxed{1.9635}$$