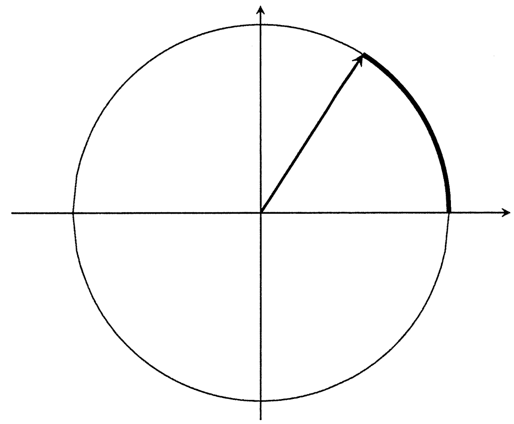
1.4 Trigonometric Functions and Their Inverses

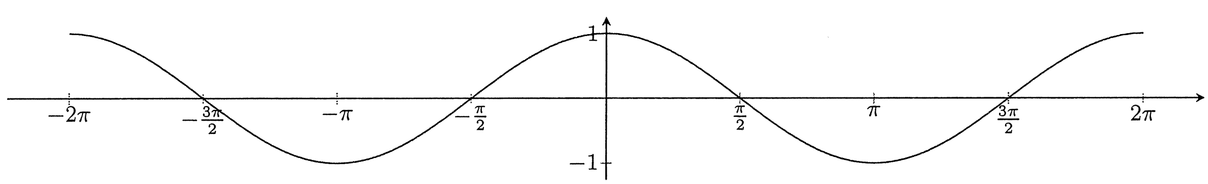
Definition.
The **unit circle** is the circle of radius 1 that is centered at the origin.
The angle corresponding to an arc length of 1 on a unit circle is called a **radian**.



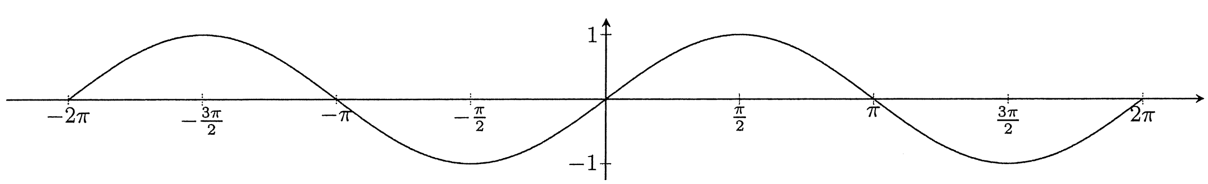
A circle is 2π radians or $360°$. Thus:

$$2\pi = 360° \implies 1 = \frac{180°}{\pi} = \frac{\pi}{180°}$$

Definition.
The coordinates of a unit circle are given by $(\cos(\theta), \sin(\theta))$ for each θ.
The $\sin(\theta)$ and $\cos(\theta)$ functions are **periodic** since these functions repeat themselves over a fixed interval.

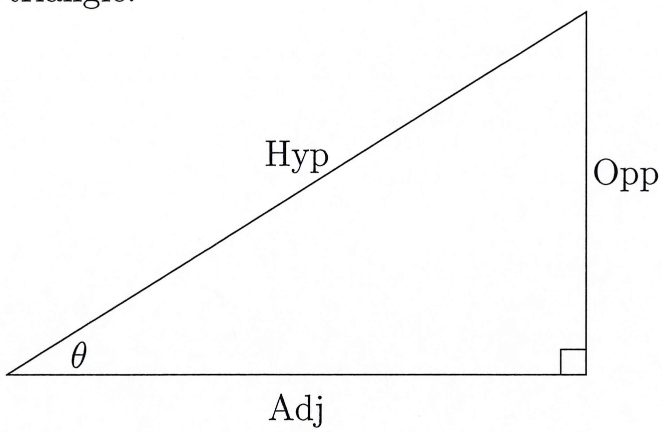


$\cos(\theta)$



$\sin(\theta)$

Definition.
Alternatively, cos(θ) and sin(θ) can be consider the ratio of the sides of a right angle triangle.

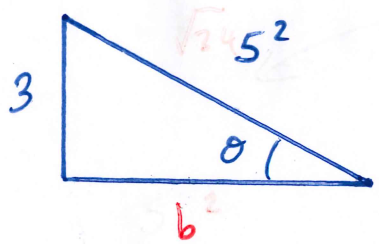


$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}}$$

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}}$$

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}}$$

Example. Find cos(θ) and tan(θ) given that $\sin(\theta) = \frac{3}{5}$ and $\pi/2 \leq \theta \leq \pi$ (2nd quadrant).

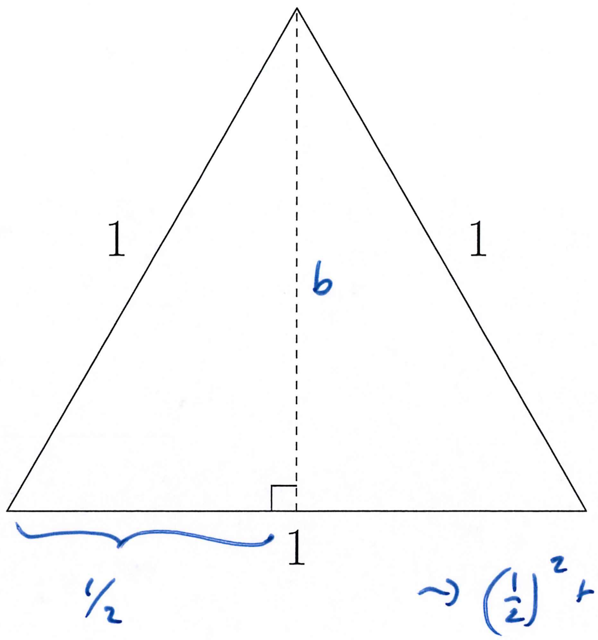


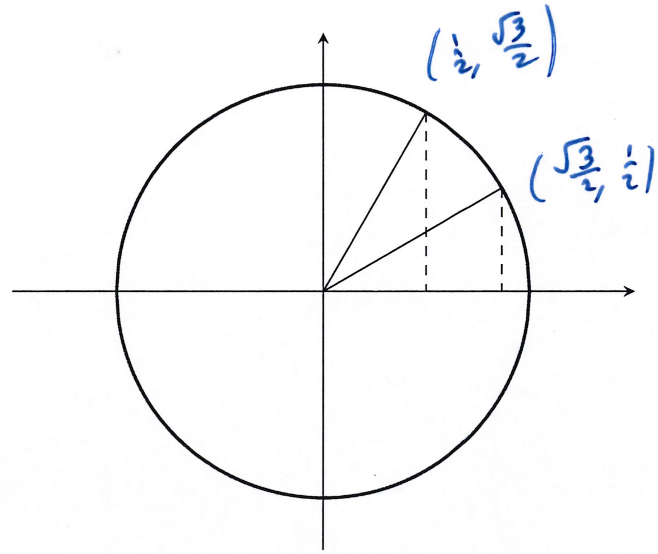
$3^2 + b^2 = 5^2$

$b = -4$

$\Rightarrow \boxed{\cos\theta = -4/5}$

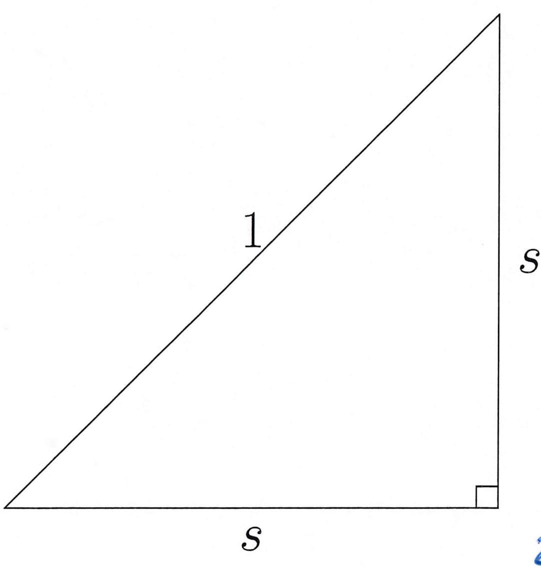
$\boxed{\tan\theta = -3/4}$

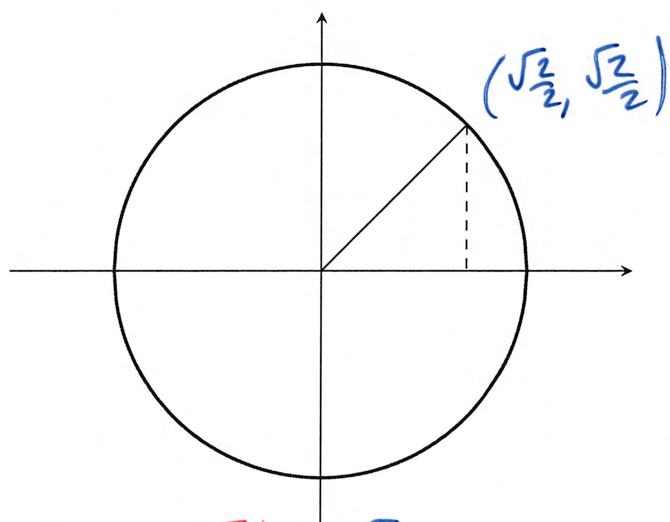




$$\rightsquigarrow \left(\frac{1}{2}\right)^2 + b^2 = 1$$

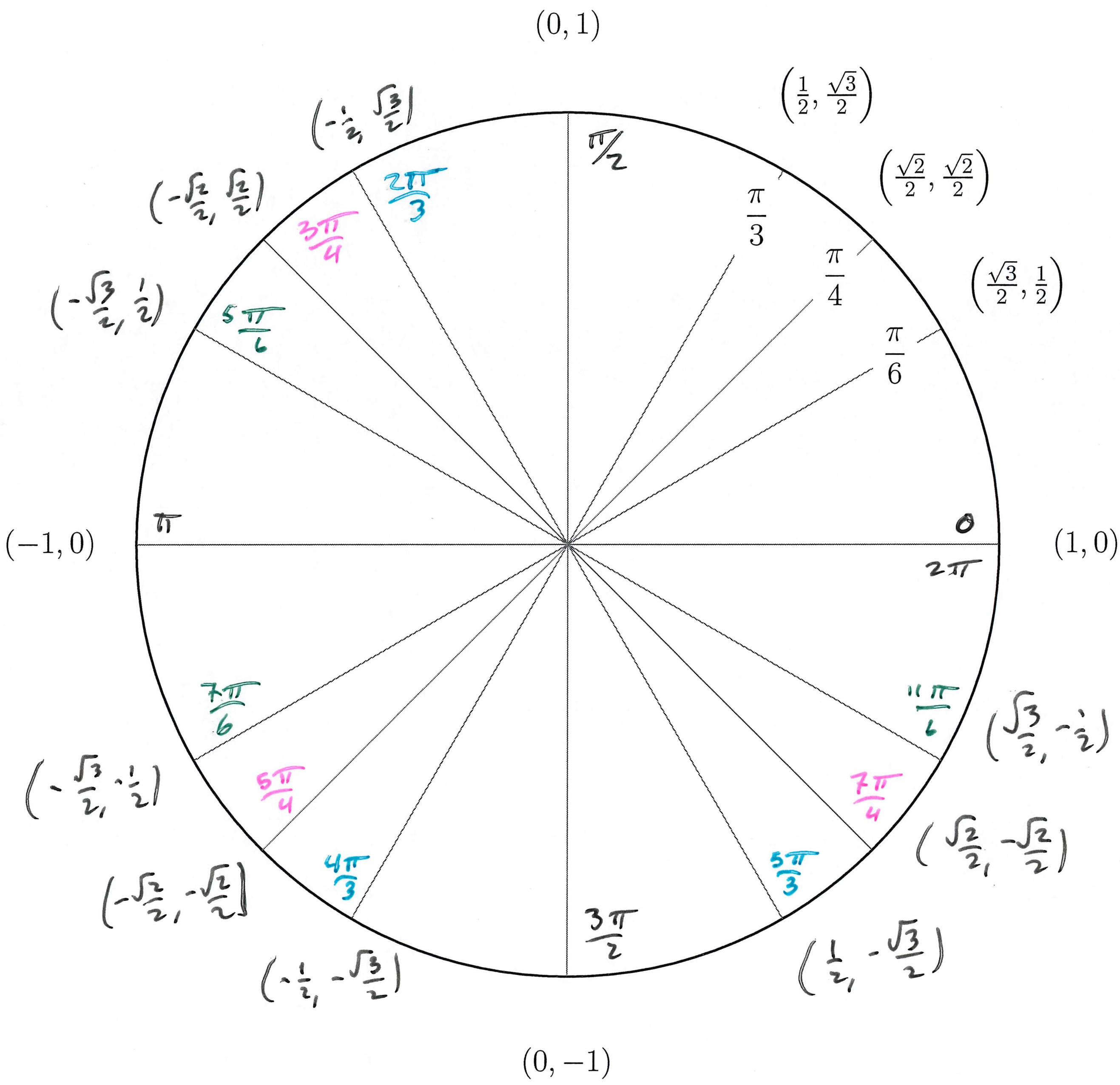
$$b = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$





$$2s^2 = 1$$

$$s = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$$



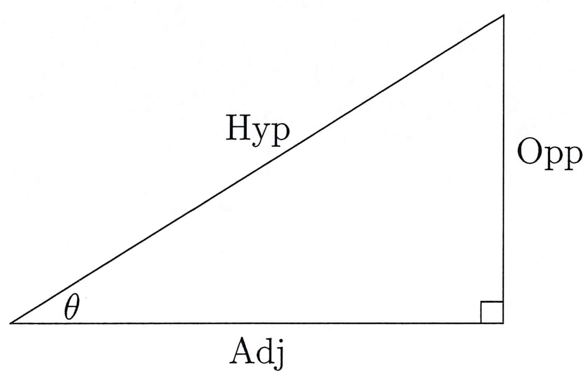
(0, 1)
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{3}$
$\frac{\pi}{4}$
$\frac{\pi}{6}$
$\pi/2$
$\frac{2\pi}{3}$
$\frac{3\pi}{4}$
$\frac{5\pi}{6}$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
(−1, 0)
π
0
2π
(1, 0)
$\frac{7\pi}{6}$
$\frac{5\pi}{4}$
$\frac{4\pi}{3}$
$\frac{3\pi}{2}$
$\frac{5\pi}{3}$
$\frac{7\pi}{4}$
$\frac{11\pi}{6}$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
(0, −1)

There are 6 trig functions, all of which can be written in terms of $\sin(\theta)$ and $\cos(\theta)$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

These functions can also be represented with the sides of a right triangle:



$$\tan\theta = \frac{\text{Opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{Adj}}{\text{Opp}}$$

$$\sec\theta = \frac{\text{Hyp}}{\text{Adj}} \qquad \csc\theta = \frac{\text{Hyp}}{\text{Opp}}$$

Using these functions, we have the following Pythagorean Identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \underline{\csc^2\theta}$$
$$1 + \cot^2(\theta) = \underline{\sec^2\theta}$$

Example. Show the identity: $\tan\theta + \cot\theta = \sec\theta\csc\theta$.

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$$

$$= \sec\theta\csc\theta$$

Definition.
The **Angle Sum Formulas** are

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

Note: Since $\cos(\theta)$ is even and $\sin(\theta)$ is odd, we can derive the difference formula from the sum formula.

Definition.
The **double-angle formulas** are a special case of the angle-sum formulas:

$$\begin{aligned}\sin(2\theta) &= \sin(\theta+\theta)\\ &= \sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta)\\ &= \boxed{2\sin(\theta)\cos(\theta)}\end{aligned} \qquad \begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos(\theta)\cos(\theta)-\sin(\theta)\sin(\theta)\\ &= \boxed{\cos^2(\theta)-\sin^2(\theta)}\end{aligned}$$

Note: Using the Pythagorean Identity, we have 2 additional representations of $\cos(2\theta)$.

Example. Evaluate $\sin\left(\frac{2\pi}{3}\right)$ using the double-angle formula.

$$\begin{aligned}\sin\left(\tfrac{2\pi}{3}\right) &= 2\sin\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{\pi}{3}\right)\\ &= 2\,\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\\ &= \boxed{\frac{\sqrt{3}}{2}}\end{aligned}$$

Definition.
The **half-angle formulas** are derived from the double angle formula:

$$\sin(\theta) = \pm\sqrt{\frac{1-\cos(2\theta)}{2}}$$
$$\cos(\theta) = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

Example. Evaluate $\cos\left(\frac{\pi}{12}\right)$ using the half-angle formula.

$$\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1+\cos(\pi/6)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{\sqrt{3}+2}}{2}$$

$$\boxed{\text{Also } \frac{1+\sqrt{3}}{2\sqrt{2}}}$$

Example. Solve the following equation:

$$\cos(3\theta) = \sin(3\theta), \quad 0 \leq \theta < 2\pi$$

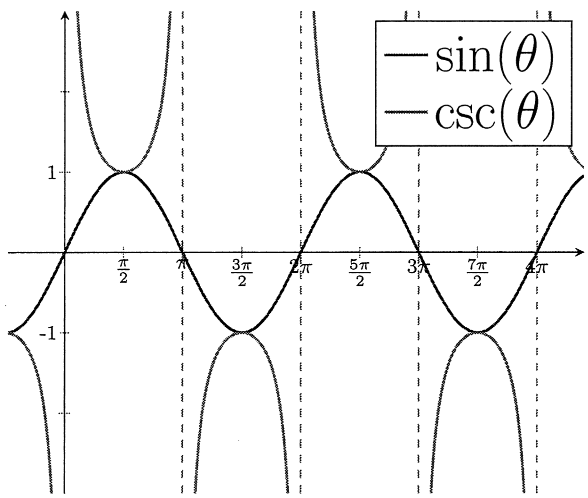
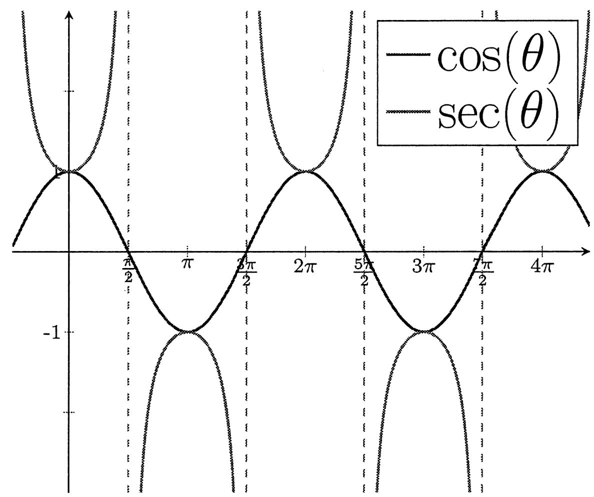
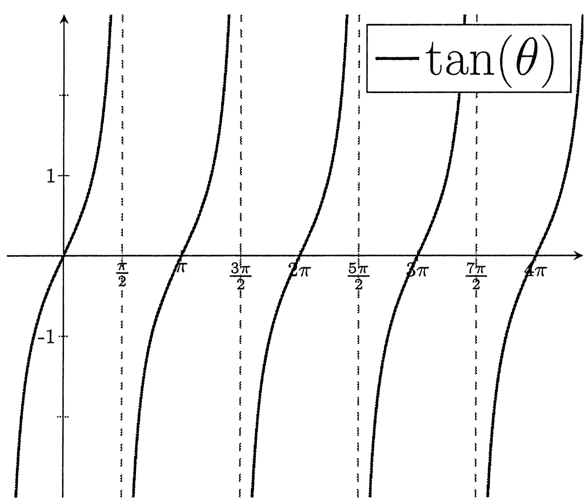
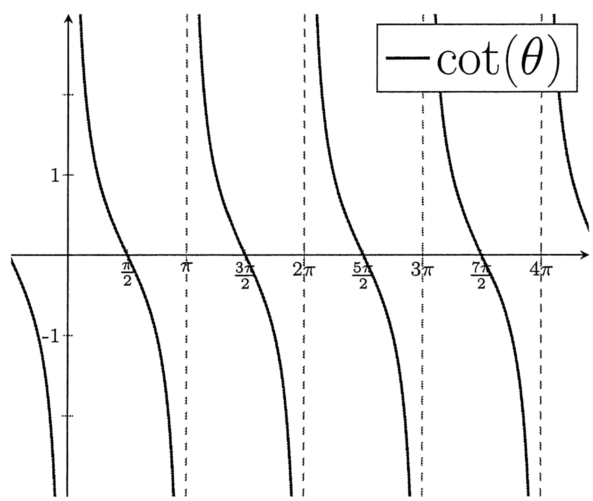
① $\cos(x) = \sin(x)$
at $x = \pi/4 + k\pi;\ k = 0, 1, 2, \ldots$

② $1 = \tan(3\theta)$
$\Rightarrow 3\theta = \pi/4 + k\pi;\ k = 0, 1, 2, \ldots$

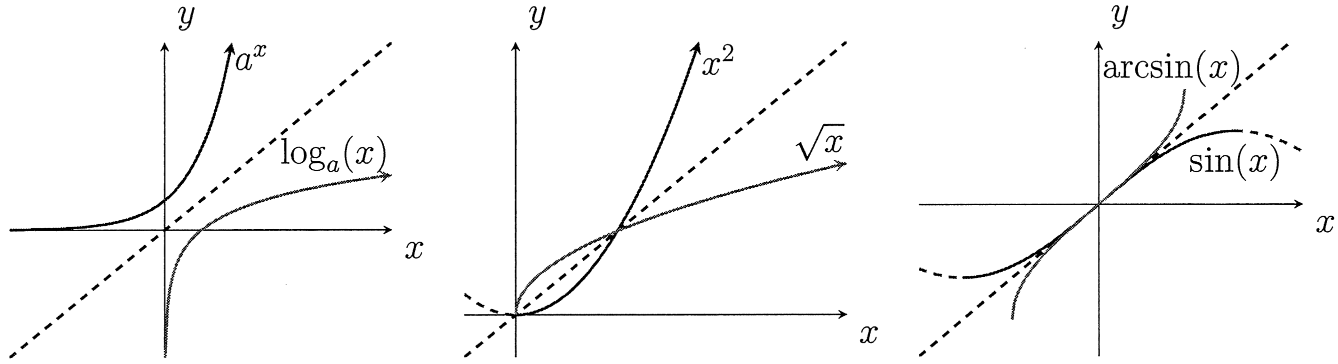
so
$$3\theta = \frac{\pi}{4} + k\pi$$

$$\theta = \frac{\pi}{12} + k\frac{\pi}{3}$$

$$\boxed{\theta = \frac{\pi}{12},\ \theta = \frac{5\pi}{12},\ \theta = \frac{3\pi}{4},\ \theta = \frac{13\pi}{12},\ \theta = \frac{17\pi}{12},\ \frac{7\pi}{4}}$$

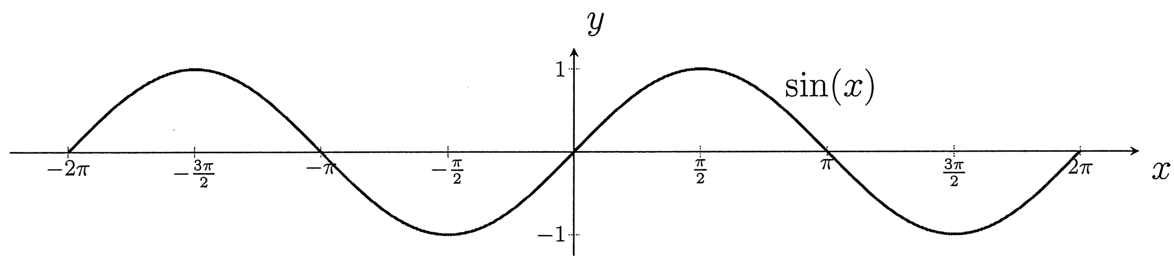

—sin(θ)
—csc(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

—cos(θ)
—sec(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

—tan(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

—cot(θ)
1
-1
π/2
π
3π/2
2π
5π/2
3π
7π/2
4π

Recall that a function has an inverse if it is 1-to-1 (e.g. it passes the horizontal line test).

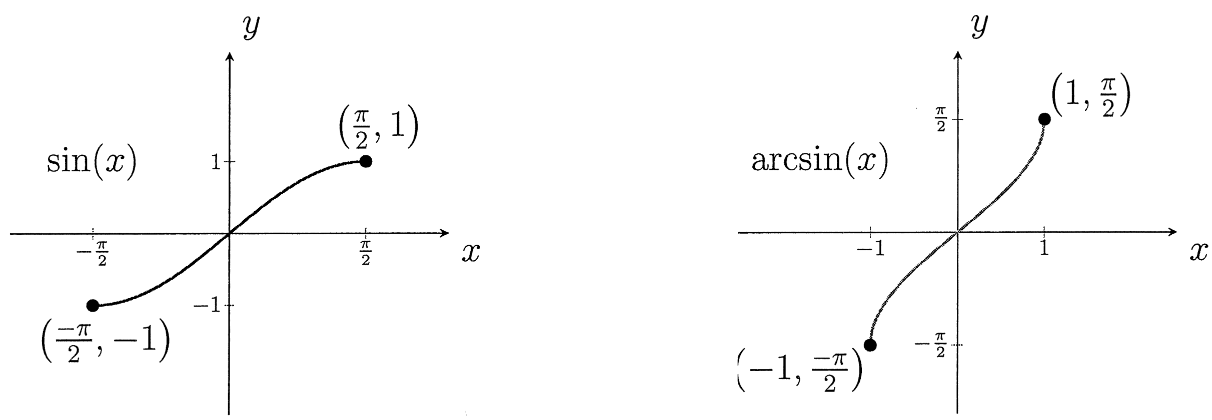


Notice that x^2 and $\sin(x)$ are on restricted domains.

Without restriction on its domain, $\sin(x)$ is NOT 1-to-1:

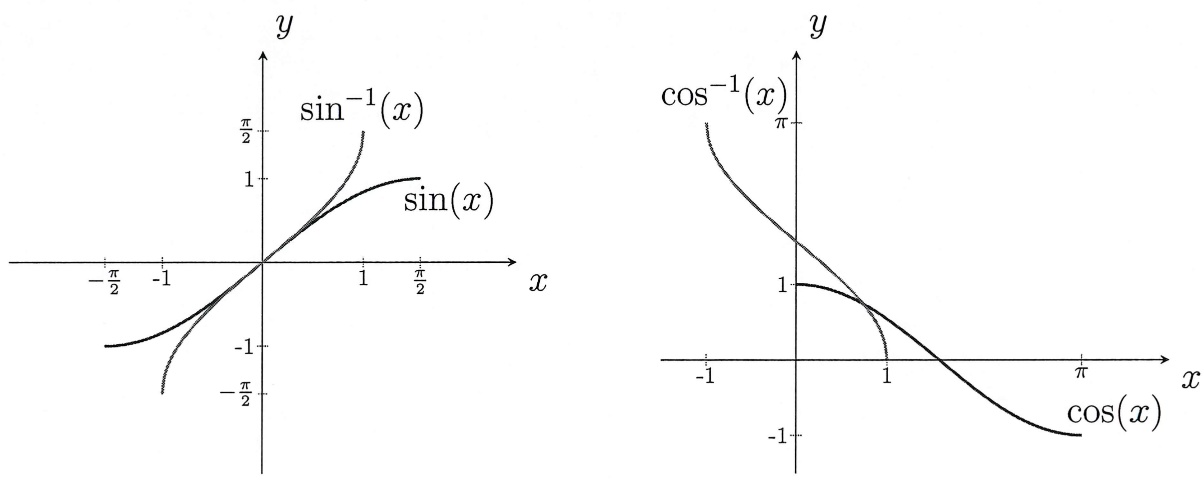


The range of $\sin(x)$ is $[-1, 1]$ and all of these values are attained on a restricted domain of $[-\pi/2, \pi/2]$:



> **Definition. (Inverse Sine and Cosine)**
> $y = \sin^{-1}(x)$ is the value of y such that $x = \sin(y)$, where $-\pi/2 \leq y \leq \pi/2$.
> $y = \cos^{-1}(x)$ is the value of y such that $x = \cos(y)$, where $0 \leq y \leq \pi$.
> The domain of both $\sin^{-1}(x)$ and $\cos^{-1}(x)$ is $\{x \mid -1 \leq x \leq 1\}$.

Note: The inverse sine function can be denoted as $\arcsin(x)$ or $\sin^{-1}(x)$.
This means that $\sin^{-1}(x) \neq \dfrac{1}{\sin(x)}$.
Similarly, $\arccos(x)$ and $\cos^{-1}(x)$ denote the inverse cosine functions.



Example. Solve the following:

$\sin^{-1}(0) = 0$ $\qquad$ $\arcsin(1) = \pi/2$ $\qquad$ $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \pi/3$

$\cos^{-1}(-1) = \pi$ $\qquad$ $\cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$ $\qquad$ $\arccos\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

Definition. (Inverse Tangent and Secant)
$y = \tan^{-1}(x)$ is the value of y such that $x = \tan(y)$, where $-\pi/2 < y < \pi/2$.

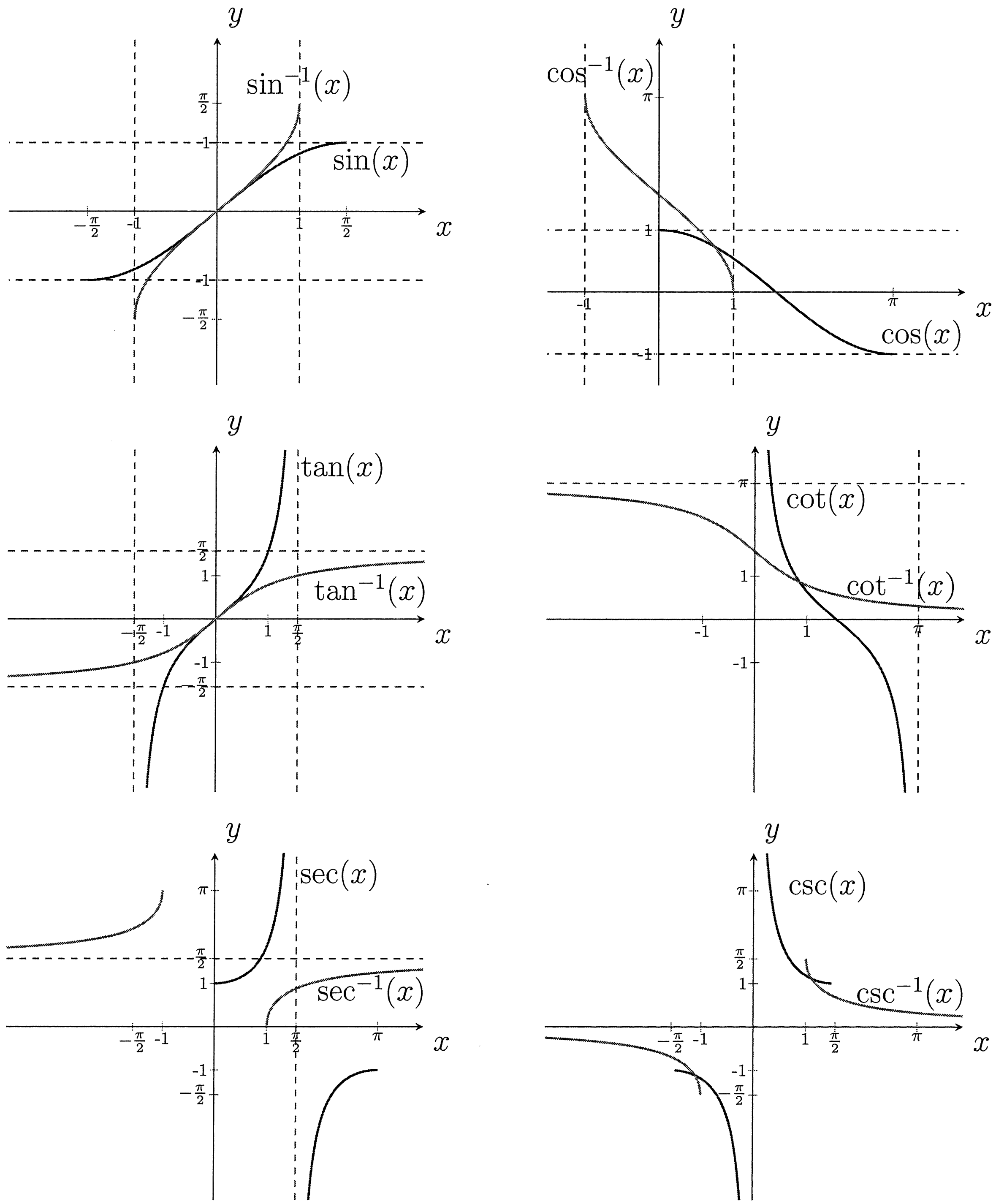
The domain of $\tan^{-1}(x)$ is $\{x \mid -\infty < x < \infty\}$.

$y = \sec^{-1}(x)$ is the value of y such that $x = \sec(y)$, where $0 \leq y \leq \pi,\ y \neq \pi/2$.

The domain of $\sec^{-1}(x)$ is $(-\infty, -1] \cup [1, \infty)$

Function	Restricted Domain	Range
$\sin(x)$	$[-\pi/2, \pi/2]$	$[-1, 1]$
$\cos(x)$	$[0, \pi]$	$[-1, 1]$
$\tan(x)$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
$\cot(x)$	$(0, \pi)$	$(-\infty, \infty)$
$\sec(x)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc(x)$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$

Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\csc^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$


sin⁻¹(x)
sin(x)
cos⁻¹(x)
cos(x)
tan(x)
tan⁻¹(x)
cot(x)
cot⁻¹(x)
sec(x)
sec⁻¹(x)
csc(x)
csc⁻¹(x)
x
y

Example. Solve the following:

$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\qquad \sec^{-1}(2) = \pi/3 \qquad \cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$

$= -\pi/4$

While $\sin(x)$ and $\sin^{-1}(x)$ are inverse functions, the inverse relationship only holds when working in the correct domains:

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0 \neq \pi \qquad \sin\left(\sin^{-1}(-1)\right) = \sin\left(-\pi/2\right) = -1$$

Example. Solve the following:

$\tan(\tan^{-1}(5)) = \boxed{5}$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}(-1)$

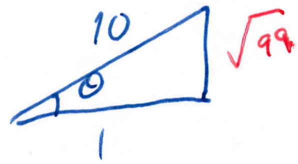
$= \boxed{-\pi/4}$

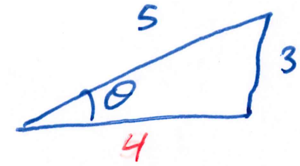
$\cos\left(\arcsin\frac{1}{2}\right) = \cos\left(\frac{\pi}{6}\right)$

$= \boxed{\frac{\sqrt{3}}{2}}$

$\cos^{-1}(\cos(5\pi)) = \cos^{-1}(-1)$

$= \boxed{\pi}$





$\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right)$ $= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $= \boxed{\pi/3}$

$\tan\left(\underbrace{\sec^{-1}(10)}_{\theta}\right)$ $\boxed{\tan\theta = \sqrt{99}}$

$\sin\left(2\underbrace{\sin^{-1}\left(\frac{3}{5}\right)}_{\theta}\right)$

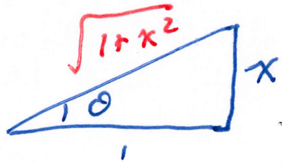
$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

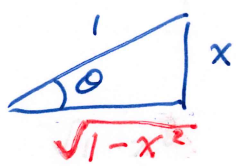
$= \boxed{\frac{24}{25}}$

Example. Simplify the following using triangles.

$\cos\left(\tan^{-1}(x)\right) = \cos(\theta) = \boxed{\frac{1}{\sqrt{1+x^2}}}$



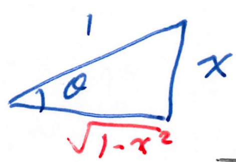
$\sec\left(\sin^{-1}(x)\right) = \sec(\theta) = \boxed{\sqrt{1-x^2}}$



$\cos\left(2\sin^{-1}(x)\right) = \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$= \left(\sqrt{1-x^2}\right)^2 - x^2$

$= \boxed{1-2x^2}$



$\sin\left(2\tan^{-1}(x)\right) = \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$= 2\,\frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}$

$= \boxed{\frac{2x}{1+x^2}}$