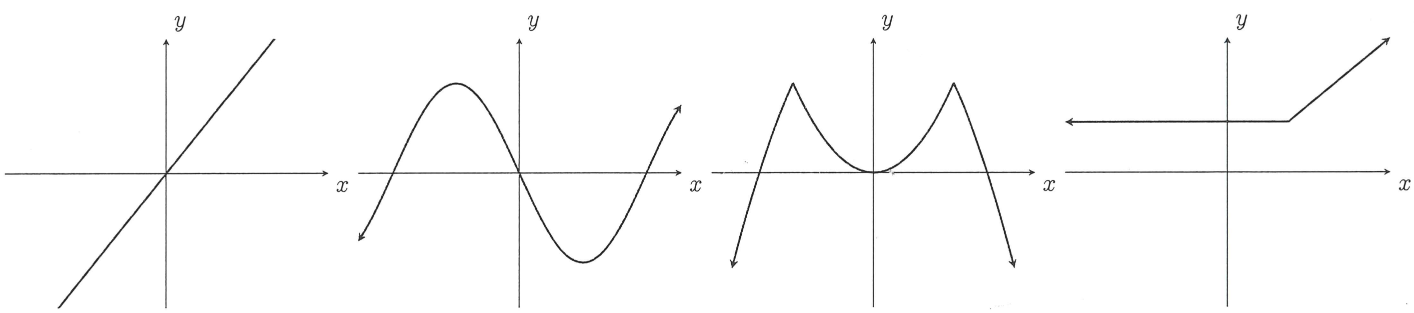
2.6 Continuity

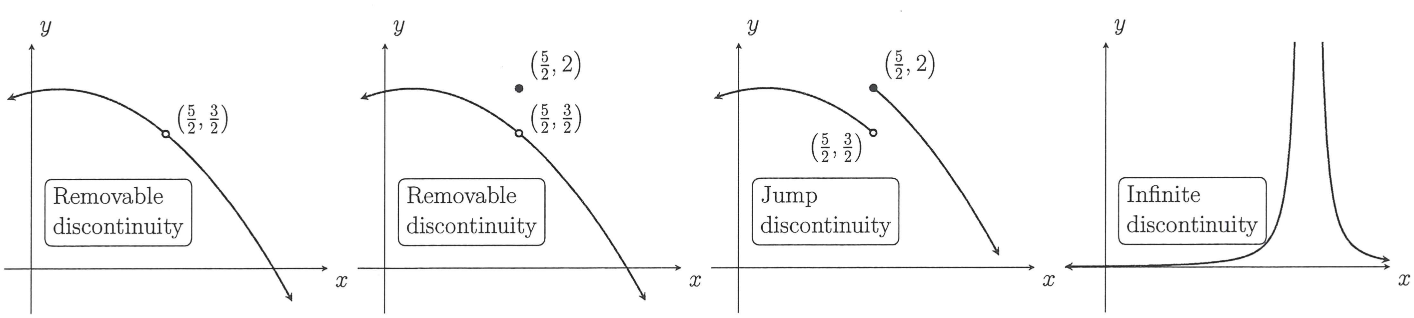
> **Definition. (Continuity at a point)**
> A function f is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$.

> **Continuity Checklist:**
> In order for f to be continuous at a, the following three conditions must hold:
>
> 1. $f(a)$ is defined (a is in the domain of f),
> 2. $\lim_{x \to a} f(x)$ exists,
> 3. $\lim_{x \to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Graphically:



Types of discontinuity:



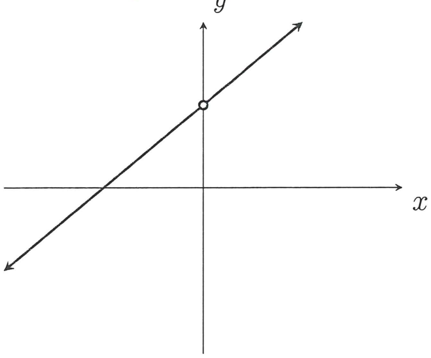
Definition.

A **removable discontinuity** at $x = a$ is one that disappears when the function becomes continuous after defining $f(a) = \lim_{x \to a} f(x)$.

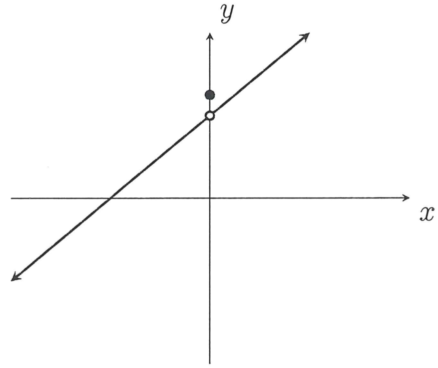
A **jump discontinuity** is one that occurs whenever $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

A **vertical discontinuity** occurs whenever $f(x)$ has a vertical asymptote.

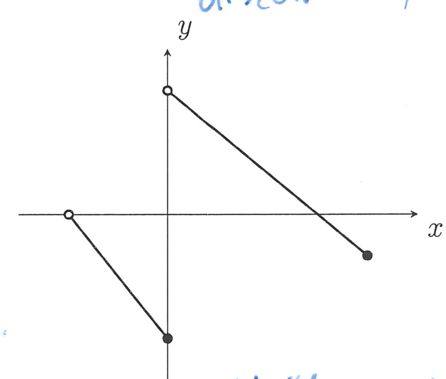
Removable discontinuity



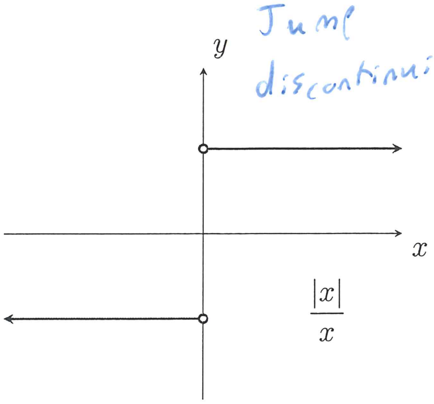
Removable discontinuity



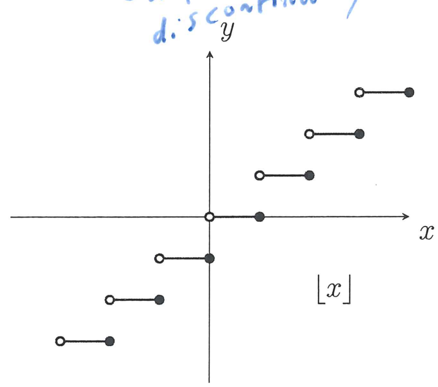
Jump discontinuity



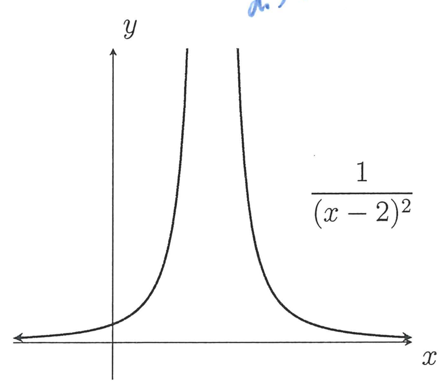
Jump discontinuity

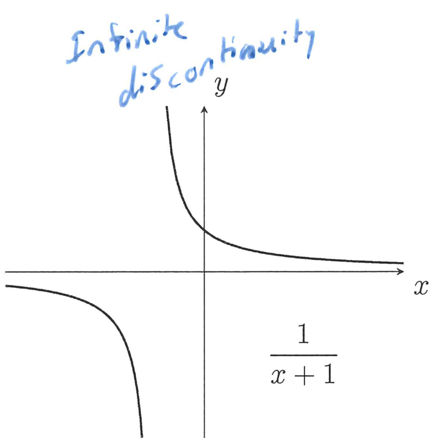


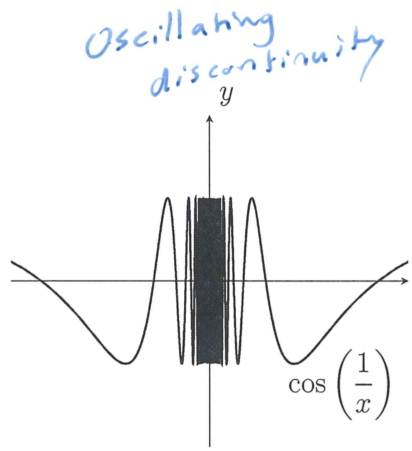
Jump discontinuity

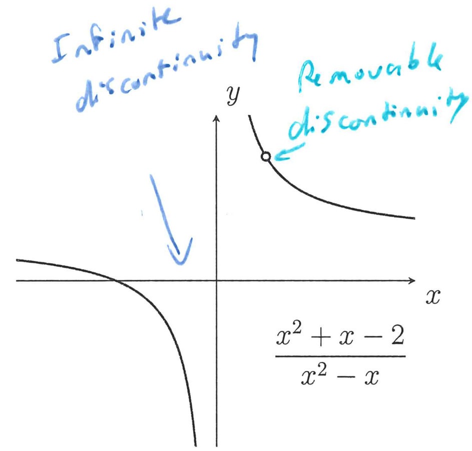


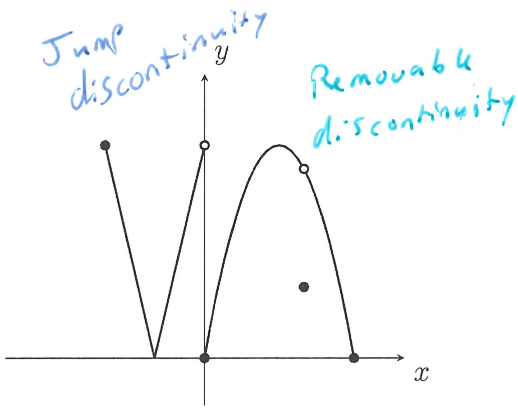
Infinite discontinuity

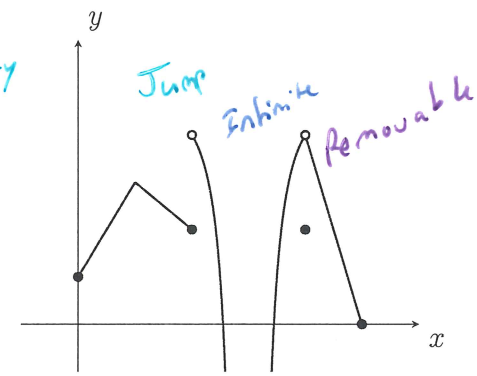


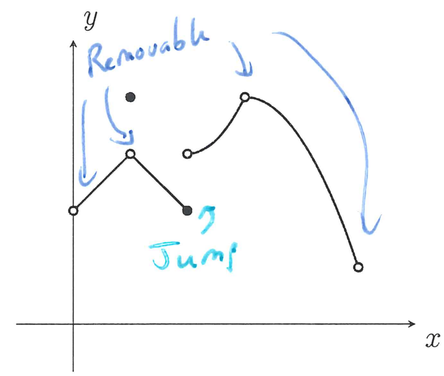








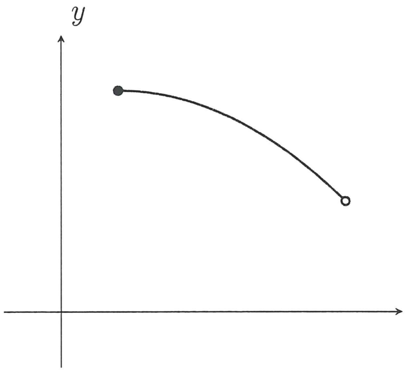




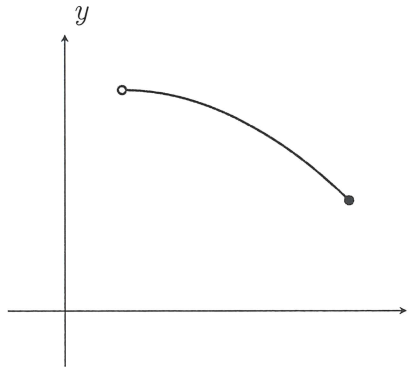
Definition. (Continuity at Endpoints)
A function f is **continuous from the right** (or **right-continuous**) at a if $\lim_{x \to a^+} f(x) = f(a)$, and f is **continuous from the left** (or **left-continuous**) at b if $\lim_{x \to b^-} f(x) = f(b)$.

Definition. (Continuity on an Interval)
A function f is **continuous on an interval I** if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.

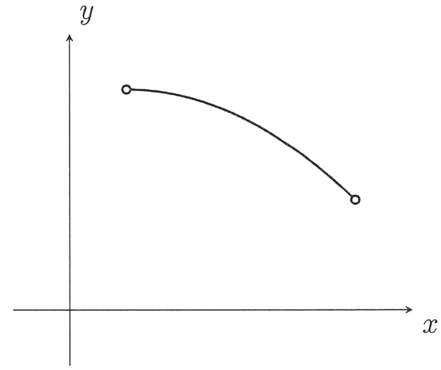
Continuous on $[a, b)$



Continuous on $(a, b]$



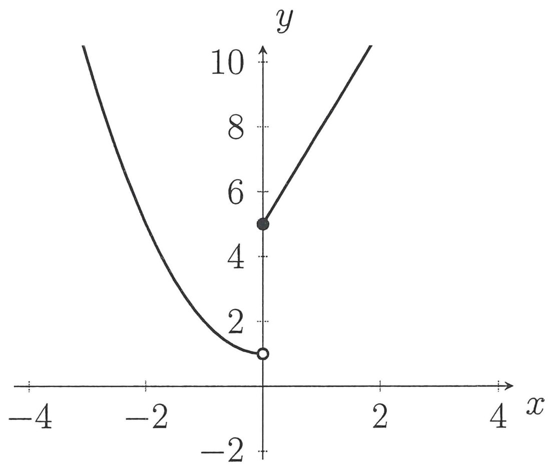
Continuous on (a, b)

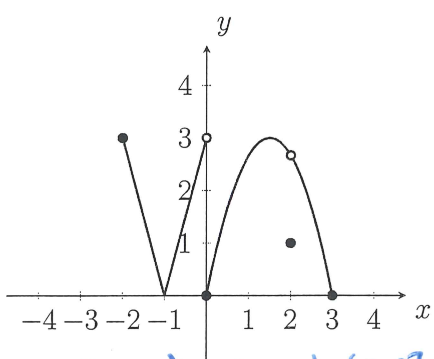


Example. Determine the interval of continuity for the following:

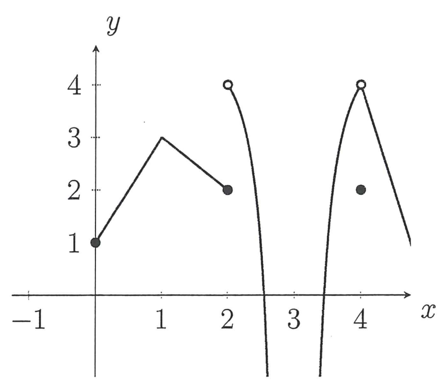
$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ 3x + 5, & x > 0 \end{cases}$$

$(-\infty, 0]$ $\quad (0, \infty)$

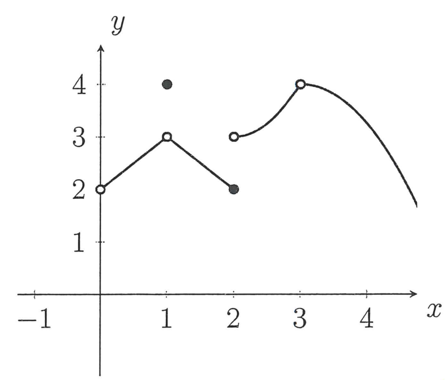




$[-2, 0), [0, 2), (2, 3]$



$[0, 2], (2, 3), (3, 4), (4, \infty)$



$(0, 1), (1, 2], (2, 3), (3, \infty)$

Example. Determine whether the following are continuous at a:

$f(x) = x^2 + \sqrt{7-x},\ a = 4$

$\lim_{x \to 4} f(x) = 4^2 + \sqrt{7-4} = 16 + \sqrt{3}$

$f(4) = 16 + \sqrt{3}$ ✓

$f(4) = \lim_{x \to 4} f(x)$

$g(x) = \dfrac{1}{x-3},\ a = 3$

$\lim_{x \to 3^-} g(x) = \dfrac{1}{\text{sm}^-} = -\infty$ X

$\lim_{x \to 3} g(x)$ DNE

$\lim_{x \to 3^+} g(x) = \dfrac{1}{\text{sm}^+} = \infty$

$h(x) = \begin{cases} \dfrac{x^2 + x}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases},\ a = -1$

$h(-1) = 0$, $h(-1) \neq \lim_{x \to -1} h(x)$ X

$\lim_{x \to -1} \dfrac{x(x+1)}{x+1} = \lim_{x \to -1} x = -1$

$j(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases},\ a = 0$

$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$

$j(0) = 0$ $\Rightarrow j(0) = \lim_{x \to 0} j(x) = 0$ ✓

$k(x) = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - x}, & x \neq 2 \\ -1, & x = 2 \end{cases},\ a = 2$

$\lim_{x \to 2} K(x) = \lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - x} = \dfrac{4+2-6}{2} = 0$

$K(2) = -1$

$\Rightarrow K(2) \neq \lim_{x \to 2} K(x)$ X

Theorem 2.9: Continuity Rules
If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

a) $f + g$

b) $f - g$

c) cf

d) fg

e) f/g, provided that $g(a) \neq 0$.

f) $(f(x))^n$

Theorem 2.10: Polynomial and Rational Functions

a) A polynomial function is continuous for all x.

b) A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$.

Theorem 2.11: Continuity of Composite Functions at a Point
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Theorem 2.12: Limits of Composite Functions

1. If g is continuous at a and f is continuous at $g(a)$, then
$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$
2. If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then
$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$

Theorem 2.13: Continuity of Functions with Roots
Assume n is a positive integer. If n is an odd integer, then $(f(x))^{1/n}$ is continuous at all points at which f is continuous.
If n is even, then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous at $f(a) > 0$.

Theorem 2.14: Continuity of Inverse Functions
If a function f is continuous on an interval I and has an inverse on I, then its inverse f^{-1} is also continuous (on the interval consisting of the points $f(x)$, where x is in I).

Theorem 2.15: Continuity of Transcendental Functions
The following functions are continuous at all points of their domains.

Trigonometric		**Inverse Trigonometric**		**Exponential**	
$\sin x$	$\cos x$	$\sin^{-1} x$	$\cos^{-1} x$	b^x	e^x
$\tan x$	$\cot x$	$\tan^{-1} x$	$\cot^{-1} x$	**Logarithmic**	
$\sec x$	$\csc x$	$\sec^{-1} x$	$\csc^{-1} x$	$\log_b x$	$\ln x$

Example. Determine the intervals of continuity for the following functions:

a) $g(x) = \dfrac{3x^2 - 6x + 7}{x^2 + x + 1}$ $\boxed{D: (-\infty, \infty)}$

$x^2 + x + 1 \neq 0$

$x \neq \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

b) $h(x) = \dfrac{3x^2 - 6x + 7}{x^2 - x - 1}$ $\boxed{D: \left(-\infty, \frac{1-\sqrt{5}}{2}\right), \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right), \left(\frac{1+\sqrt{5}}{2}, \infty\right)}$

$x^2 - x - 1$

$x \neq \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

c) $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$

$x^2 - 1 \neq 0$

$x \neq \pm 1$

$\boxed{D: (-\infty, -1), (-1, 1), (1, \infty)}$

d) $t(x) = \dfrac{x^2 - 4x + 3}{x^2 + 1}$ $\boxed{D: (-\infty, \infty)}$

$x^2 + 1 \neq 0$

$x \neq \pm i$

e) $q(x) = \sqrt[3]{x^2 - 2x - 3}$

$D: (-\infty, \infty)$

f) $r(x) = \sqrt{x^2 - 2x - 3}$

$x^2 - 2x - 3 \geq 0$

$(x-3)(x+1) \geq 0$

	-1		3	
$(x+1)$	$---$	$+++$		$+++$
$(x-3)$	$---$	$---$		$+++$
	$+++$	$---$		$+++$

$D: (-\infty, -1], [3, \infty)$

g) $a(x) = \sec x$

$D: \quad x \neq \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

h) $b(x) = \sqrt{\sin x}$

$\sin(x) \geq 0$

$D: \ldots, (-2\pi, -\pi), (0, \pi), (2\pi, 3\pi), \ldots$

i) $\ell(x) = \begin{cases} x^3 + 4x + 1, & x \leq 0 \\ 2x^3, & x > 0 \end{cases}$

$\lim_{x \to 0^-} \ell(x) = 0$

$\lim_{x \to 0^+} \ell(x) = 1$

Not cont

$D: (-\infty, 0], (0, \infty)$

j) $m(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x \geq \frac{\pi}{4} \end{cases}$

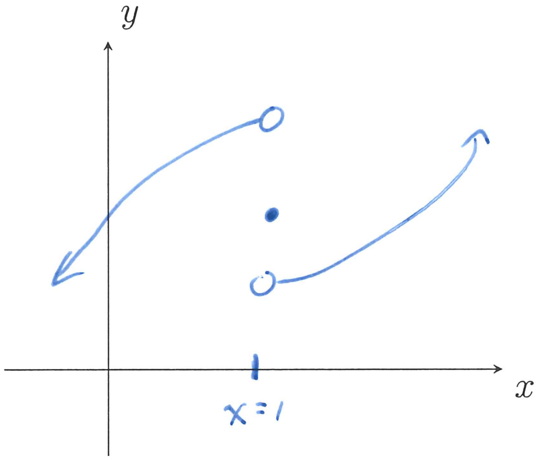
$\lim_{x \to \pi/4^-} m(x) = \frac{\sqrt{2}}{2}$

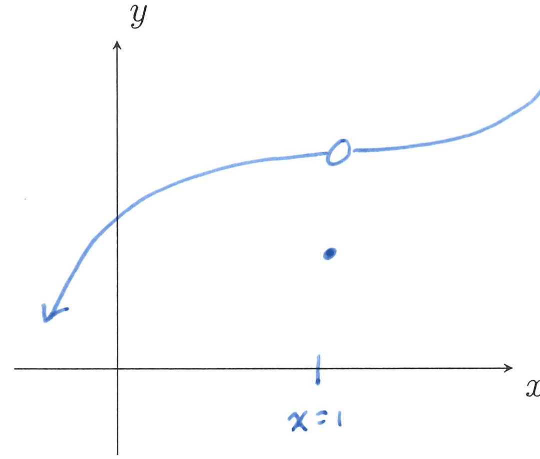
$\lim_{x \to \pi/4^+} m(x) = \frac{\sqrt{2}}{2}$

$D: (-\infty, \infty)$

Example. Sketch a function that:

Is defined, but not continuous at $x = 1$, | Has a limit, but not continuous at $x = 1$.





Example. Determine the value of a for which $f(x)$ is continuous:

1. $f(x) = \begin{cases} \dfrac{x^3-1}{x-1}, & x \neq 1 \\ a, & x = 1 \end{cases}$

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 + 0x^2 + 0x - 1 \\ & -(x^3 - x^2) \\ \hline & x^2 + 0x \\ & -(x^2 - x) \\ \hline & x - 1 \\ & -(x-1) \\ \hline & 0 \end{array}$$

$\lim_{x\to 1} f(x) = \lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} x^2+x+1 = 3$

Want $f(1) = 3 \Rightarrow a = 3$

2. $f(x) = \begin{cases} \dfrac{t^2+3t-10}{t-2}, & t \neq 2 \\ a, & t = 2 \end{cases}$

$\lim_{t\to 2} f(x) = \lim_{t\to 2} \frac{(t-2)(t+5)}{t-2} = \lim_{t\to 2} t+5 = 7$

Want $f(2) = 7 \Rightarrow a = 7$

3. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}$

$$① \begin{cases} \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^-} x+2 = 4 \\ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3 \end{cases}$$

$$② \begin{cases} \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3 \\ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b = 6 - a + b \end{cases}$$

$① \Rightarrow 4a - 2b + 3 = 4$

$② \Rightarrow 9a - 3b + 3 = 6 - a + b$

$$\Rightarrow \begin{cases} 4a - 2b = 1 \\ 10a - 4b = 3 \end{cases} \Rightarrow a = b = \tfrac{1}{2}$$

Example. Redefine the following functions so that they are continuous everywhere:

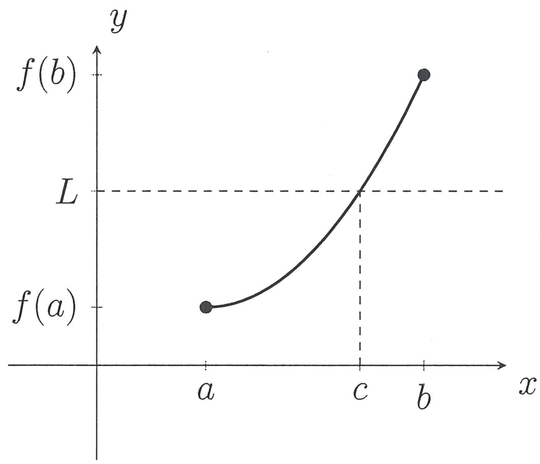
1. $g(x) = \dfrac{x^3 - x^2 - 2x}{x-2} = \dfrac{x(x+1)(x-2)}{x-2} = x(x+1), \; x \neq 2$

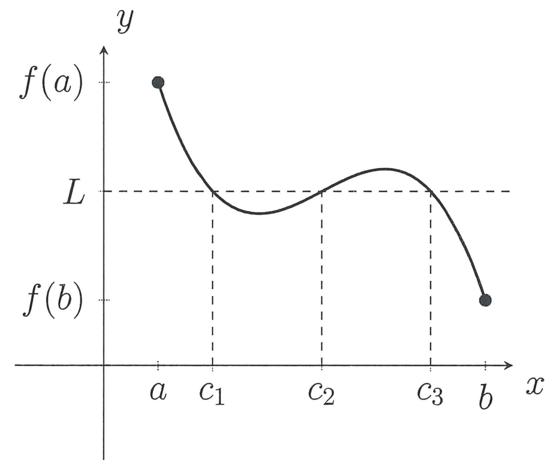
Let $g(x) = x(x+1)$

2. $g(x) = \dfrac{x^2 + x - 6}{x-2} = \dfrac{(x+3)(x-2)}{x-2} = x+3, \; x \neq 2$

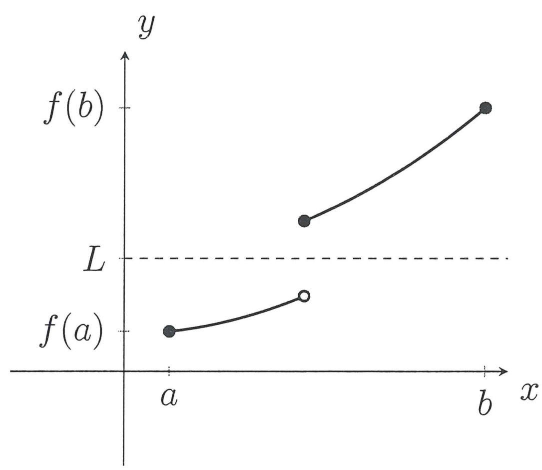
Let $g(x) = x+3$

Theorem 2.16: Intermediate Value Theorem
Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.





Note: It is important that the function be continuous on the interval $[a, b]$:



Example. Show that $f(x)$ has a root using the IVT: $f(x) = x^3 + 4x + 4$

x	$f(x)$
0	4
1	9
-1	-1

Since $f(-1) < 0 < f(0)$, then there exist c such that $-1 < c < 0$ and $f(c) = 0$.

Example. Show that $\sqrt{x^4 + 25x^3 + 10} = 5$ on the interval $(0, 1)$.

$\sqrt{0^4 + 25(0) + 10} = \sqrt{10}$

$\sqrt{1^4 + 25(1) + 10} = \sqrt{36} = 6$

Since $\sqrt{10} < 5 < 6$, then there exists c such that $0 < c < 1$ and $\sqrt{c^4 + 25c^3 + 10} = 5$.

Example. Show that $-x^5 - 4x^2 + 2\sqrt{x} + 5 = 0$ on $(0, 3)$.

$-0^5 - 4(0)^2 + 2\sqrt{0} + 5 = 5$

$-3^5 - 4(3)^2 + 2\sqrt{3} + 5 = -274 + 2\sqrt{3}$

Since $-274 + 2\sqrt{3} < 0 < 5$ then there exist c such that $0 < c < 3$ and

$-c^5 - 4c^2 + 2\sqrt{}$