2.5 Limits at Infinity

Definition.
Limits at Infinity and Horizontal Asymptotes
If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L$$

We say the limit of $f(x)$ as x approaches infinity is L. In this case, the line $y = L$ is a **horizontal asymptote** of f. The limit at negative infinity,

$$\lim_{x\to-\infty} f(x) = M$$

is defined analogously. When this limit exists, $y = M$ is a horizontal asymptote.

Note: The function *can* cross it's horizontal asymptote (consider $\frac{\sin x}{x}$).

Note: A function can have 0, 1 or 2 horizontal asymptotes.

Example. For each of the following functions, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

a) $f(x) = \dfrac{1}{x^2}$ $\quad \lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

b) $f(x) = \dfrac{1}{x^3}$ $\quad \lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

c) $f(x) = 2 + \dfrac{10}{x^2}$ $\quad \lim_{x\to\infty} 2 + \frac{10}{x^2} = 2$

$\lim_{x\to-\infty} 2 + \frac{10}{x^2} = 2$

d) $f(x) = 5 + \dfrac{\sin x}{\sqrt{x}}$ $\quad \lim_{x\to\infty} 5 + \frac{\sin(x)}{\sqrt{x}} = 5$

$\lim_{x\to-\infty} 5 + \frac{\sin(x)}{\sqrt{x}}$ DNE

e) $f(x) = \left(5 + \dfrac{1}{x} + \dfrac{10}{x^2}\right)$

$\lim_{x\to\pm\infty} f(x) = 5$

f) $f(x) = (3x^{12} - 9x^7)$

$\lim_{x\to\pm\infty} f(x) = \infty$

g) $f(x) = \sin(x)$

$\lim_{x\to\pm\infty} f(x)$ DNE

h) $f(x) = \dfrac{\sin x}{x}$

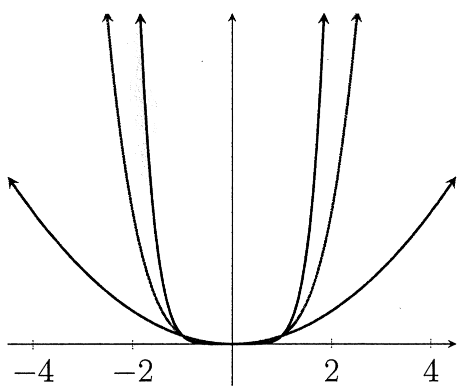
$\lim_{x\to\pm\infty} f(x) = 0$

Definition.
Infinite Limits at Infinity If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

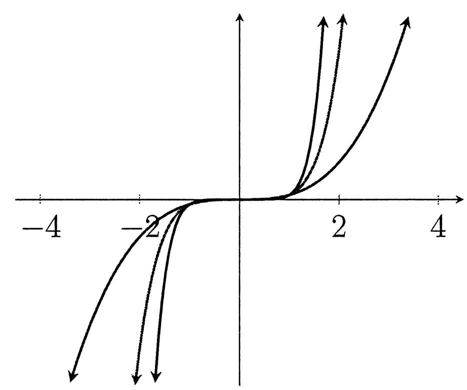
$$\lim_{x\to\infty} f(x) = \infty$$

The limits $\lim\limits_{x\to\infty} = -\infty$, $\lim\limits_{x\to-\infty} = \infty$ and $\lim\limits_{x\to-\infty} = -\infty$ are defined similarly.

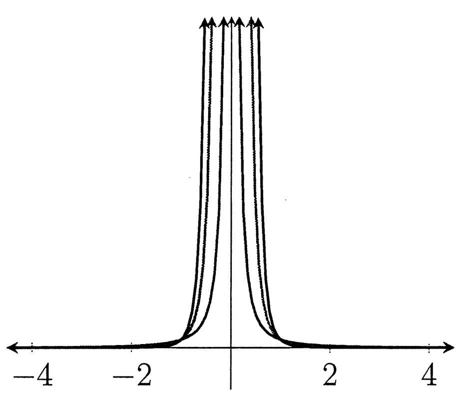
Even functions



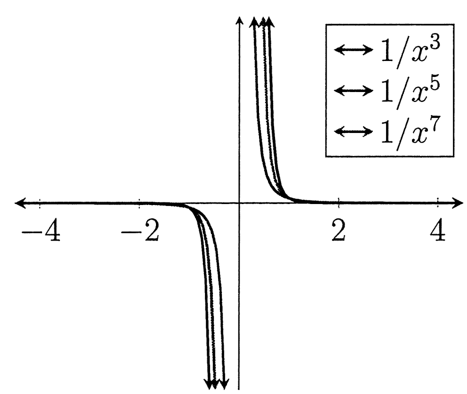
Odd functions



$1/x^n$: n Even



$1/x^n$: n Odd



Theorem. Limits at Infinity of Powers and Polynomials
Let n be a positive integer and let p be the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \text{ where } a_n \neq 0.$$

1. $\lim\limits_{x \to \pm\infty} x^n = \infty$ when n is even.
2. $\lim\limits_{x \to \infty} x^n = \infty$ and $\lim\limits_{x \to -\infty} x^n = -\infty$ when n is odd.
3. $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^n} = \lim\limits_{x \to \pm\infty} x^{-n} = 0.$
4. $\lim\limits_{x \to \pm\infty} p(x) = \lim\limits_{x \to \pm\infty} a_n x^n = \pm\infty$, depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Note: All previous limit laws still apply (e.g. constant multiplier rule)

Note: This theorem *ONLY* applies for $x \to \pm\infty$. When $x \to a$, $|a| < \infty$, we compute the left and right limits and use sm+/sm- (as done in section 2.4).

Example. For the following, find the limits as $x \to -\infty$ and $x \to \infty$:

$f(x) = 2x^{-8}$

$\lim\limits_{x \to \infty} \dfrac{2}{x^8} = 0$

$\lim\limits_{x \to -\infty} \dfrac{2}{x^8} = 0$

$g(x) = -12x^{-5}$

$\lim\limits_{x \to \infty} \dfrac{-12}{x^5} = 0$

$\lim\limits_{x \to -\infty} \dfrac{-12}{x^5} = 0$

$h(x) = 3x^{12} - 9x^7$

$\lim\limits_{x \to \infty} 3x^{12} - 9x^7 = \lim\limits_{x \to \infty} 3x^{12} = \infty$

$\lim\limits_{x \to -\infty} 3x^{12} - 9x^7 = \lim\limits_{x \to -\infty} 3x^{12} = \infty$

$\ell(x) = 2x^{-8} + 4x^3$

$\lim\limits_{x \to \infty} \dfrac{2}{x^8} + 4x^3 = \infty$

$\lim\limits_{x \to -\infty} \dfrac{2}{x^8} + 4x^3 = -\infty$

When finding the limit as $x \to \pm\infty$ of a rational function, $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, we multiply the function by $\frac{1/x^n}{1/x^n}$, where n is the highest degree in the denominator $q(x)$.

Note: To receive full credit for questions of this type, you must show all the fractions in your intermediate steps.

Example.

a) $\lim\limits_{x\to\infty} \frac{1-x}{2x}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to\infty} \frac{\frac{1}{x} - 1}{2} = \frac{0-1}{2} = \boxed{-\frac{1}{2}}$

b) $\lim\limits_{x\to\infty} \frac{1-x}{x^2}\left(\frac{1/x^2}{1/x^2}\right) = \lim\limits_{x\to\infty} \frac{\frac{1}{x^2} - \frac{1}{x}}{1} = \frac{0-0}{1} = \boxed{0}$

c) $\lim\limits_{x\to\infty} \frac{1-x^2}{2x}\left(\frac{1/x}{1/x}\right) = \lim\limits_{x\to\infty} \frac{1/x - x}{2} = \frac{0-\infty}{2} = \boxed{-\infty}$

Theorem. End Behavior and Asymptotes of Rational Functions

Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0$$

with $a_m \neq 0$ and $b_n \neq 0$.

1. **Degree of numerator less than degree of denominator**

   If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of f.

2. **Degree of numerator equals degree of denominator**

   If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m/b_n$ and $y = a_m/b_n$ is a horizontal asymptote of f.

3. **Degree of numerator greater than degree of denominator**

   If $m > n$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$ and f has no horizontal asymptote.

4. **Slant Asymptote**

   If $m = n+1$, then $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$, and f has no horizontal asymptote, but f has a slant asymptote.

5. **Vertical asymptotes**

   Assuming f is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

Example. Evaluate the limits of the following as $x \to -\infty$ and $x \to \infty$. State the equation of the horizontal asymptote.

1. $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$ $\left(\dfrac{1/x^3}{1/x^3}\right) = \dfrac{2 + 7/x^3}{1 - \frac{1}{x} + \frac{1}{x^2} + 7/x^3}$

$\lim_{x\to\infty} f(x) = 2$

$\lim_{x\to -\infty} f(x) = 2 \Rightarrow \boxed{\text{H.A. } y = 2}$

2. $g(x) = \dfrac{1}{x^3-4x+1}$ $\left(\dfrac{1/x^3}{1/x^3}\right) = \dfrac{1/x^3}{1 - 4/x^2 + 1/x^3}$

$\lim_{x\to\infty} g(x) = 0$

$\lim_{x\to -\infty} g(x) = 0 \Rightarrow \boxed{\text{H.A. } y = 0}$

3. $h(x) = \dfrac{3x^5+2x^2-2}{4x^4-3x}$ $\left(\dfrac{1/x^4}{1/x^4}\right) = \dfrac{3x + 2/x^2 - 2/x^4}{4 - 3/x^3}$

$\lim_{x\to\infty} h(x) = \infty$

$\lim_{x\to -\infty} h(x) = -\infty \Rightarrow \boxed{\text{No H.A.}}$

4. $j(x) = \dfrac{4x^2-2x+3}{7x^2-1}$ $\left(\dfrac{1/x^2}{1/x^2}\right) = \dfrac{4 - 2/x + 3/x^2}{7 - 1/x^2}$

$\lim_{x\to\infty} j(x) = 4/7$

$\lim_{x\to -\infty} j(x) = 4/7 \Rightarrow \boxed{\text{H.A. } y = 4/7}$

5. $\ell(x) = \dfrac{1-x^2}{3+2x-x^3}$ $\left(\dfrac{1/x^3}{1/x^3}\right) = \dfrac{1/x^3 - 1/x}{3/x^3 + 2/x^2 - 1}$

$\lim_{x\to\infty} \ell(x) = 0$

$\lim_{x\to -\infty} \ell(x) = 0 \Rightarrow \boxed{\text{H.A. } y = 0}$

Definition.
When the degree of the numerator, m, is greater than the degree of the denominator, n, the function has an oblique asymptote:

$$f(x) = \frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)}$$

where $a(x)$ is the resulting polynomial that we get from polynomial long division and $r(x)$ is the remainder. We are interested in the special case where $m = n + 1$, and $f(x)$ has a **slant asymptote**.

Example. For the following functions, find the vertical asymptotes and the slant asymptotes:

1. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$

① $x^2 + 2x \neq 0$

$x(x+2) \neq 0$

$x \neq 0, \; x \neq -2$

$y(0)$ DNE $\left(\frac{3}{0}\right)$

$y(-2)$ DNE $\left(\frac{-11}{0}\right)$

$\boxed{\text{V.A. } x = 0,\; x = -2}$

②

$$\begin{array}{r|l} & 2x - 3 \\ \hline x^2 + 2x & 2x^3 + x^2 + x + 3 \\ & -(2x^3 + 4x^2) \\ \hline & -3x^2 + x \\ & -(-3x^2 - 6x) \\ \hline & 5x + 3 \end{array}$$

$\boxed{\text{S.A. } y = 2x - 3}$

2. $f(x) = \dfrac{x^2 - 1}{x + 2}$

① $x + 2 \neq 0$

$x \neq -2$

$\left(\frac{3}{0}\right)$ $f(-2)$ DNE

$\boxed{\text{V.A. } x = -2}$

②

$$\begin{array}{r|l} & x - 2 \\ \hline x + 2 & x^2 + 0x - 1 \\ & -(x^2 + 2x) \\ \hline & -2x - 1 \\ & -(-2x - 4) \\ \hline & 3 \end{array}$$

$\boxed{\text{S.A. } y = x - 2}$

3. $g(t) = \dfrac{t^2 - 1}{2t + 4}$

① $2t + 4 \neq 0$

$t \neq -2$

$\left(\frac{3}{0}\right)$ $g(-2)$ DNE

$\boxed{\text{V.A. } t = -2}$

②
$$\begin{array}{r|l} & \tfrac{1}{2}t \quad -1 \\ \hline 2t+4 & t^2 + 0t - 1 \\ & -(t^2 + 2t) \\ \hline & -2t - 1 \\ & -(-2t - 4) \\ \hline & 3 \end{array}$$

$\boxed{\text{S.A. } y = \frac{1}{2}t - 1}$

4. $h(u) = \dfrac{u^2}{u - 1}$

① $u - 1 \neq 0$

$u \neq 1$

$\left(\frac{1}{0}\right)$ $h(1)$ DNE

$\boxed{\text{V.A. } u = 1}$

②
$$\begin{array}{r|l} & u \quad +1 \\ \hline u-1 & u^2 + 0u + 0 \\ & -(u - u) \\ \hline & u + 0 \\ & -(u - 1) \\ \hline & 1 \end{array}$$

$\boxed{\text{S.A. } y = u + 1}$

If the denominator has a square root, we need to change our work depending on if $x \to -\infty$ or $x \to \infty$:

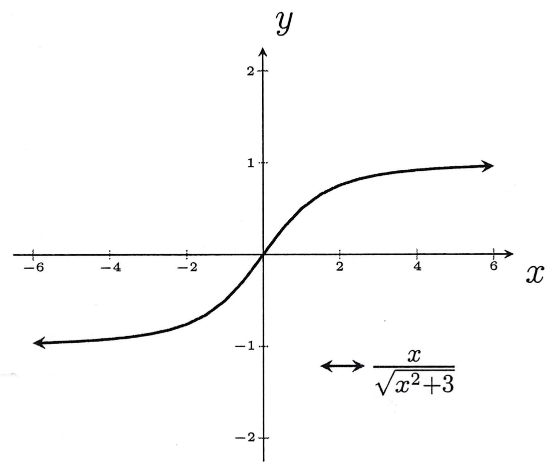
Example. For the following, find the equation of the horizontal asymptotes:

a) $\displaystyle \lim_{x\to\infty} \frac{x}{\sqrt{x^2+3}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}}\sqrt{x^2+3}}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{\sqrt{1+3/x^2}}$

$\boxed{y=1}$

$= \boxed{1}$



b) $\displaystyle \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+3}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to-\infty} \frac{1}{-\sqrt{\frac{1}{x^2}}\sqrt{x^2+3}}$

$\boxed{y=-1}$

$\displaystyle = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+3/x^2}}$

$= \boxed{-1}$

c) $\displaystyle \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}}\sqrt{x^2+x}}$

$\boxed{y=1}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}} = \boxed{1}$

d) $\displaystyle \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to-\infty} \frac{1}{-\sqrt{1/x^2}\sqrt{x^2+x}}$

$\boxed{y=-1}$

$\displaystyle = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x}} = \boxed{-1}$

$$\frac{1}{x^3} > 0 \text{ when } x > 0$$
$$\frac{1}{x^3} < 0 \text{ when } x < 0$$

e) $\displaystyle\lim_{x\to\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6 + 4}} \left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to\infty} \frac{7 - 2/x^3}{-1 + \sqrt{\frac{1}{x^6}}\sqrt{25x^6+4}} = \lim_{x\to\infty} \frac{7 - 2/x^3}{-1 + \sqrt{25 + 4/x^6}}$

$$= \frac{7}{-1+5} = \boxed{\frac{7}{4}}$$

$$\lim_{x\to-\infty} \frac{7x^3 - 2}{-x^3 + \sqrt{25x^6+4}} \left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to-\infty} \frac{7 - 2/x^3}{-1 - \sqrt{1/x^6}\sqrt{25x^6+4}}$$

$$= \lim_{x\to\infty} \frac{7 - 2/x^3}{-1 - \sqrt{25 + 4/x^6}} = \boxed{-\frac{7}{6}}$$

$$\boxed{\begin{array}{l} \text{H.A. } y = 7/4 \\ \quad\; y = -7/6 \end{array}}$$

f) $\displaystyle\lim_{x\to\infty} \frac{\sqrt[3]{x^6 + 8}}{4x^2 + \sqrt{3x^4 + 1}} \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to\infty} \frac{\sqrt[3]{1/x^6}\sqrt[3]{x^6+8}}{4 + \sqrt{1/x^4}\sqrt{3x^4+1}} = \lim_{x\to\infty} \frac{\sqrt[3]{1 + 8/x^6}}{4 + \sqrt{3 + 1/x^4}}$

$$= \boxed{\frac{1}{4+\sqrt{3}}}$$

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x^6+8}}{4x^2 + \sqrt{3x^4+1}} \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x\to-\infty} \frac{\sqrt[3]{1/x^6}\sqrt[3]{x^6+8}}{4 + \sqrt{1/x^4}\sqrt{3x^4+1}}$$

$$\boxed{\text{H.A. } y = \frac{1}{4+\sqrt{3}}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt[3]{1 + 8/x^6}}{4 + \sqrt{3 + 1/x^4}} = \boxed{\frac{1}{4+\sqrt{3}}}$$

$$\boxed{\frac{1}{x^2} > 0 \text{ for all } x}$$

g) $\displaystyle\lim_{x\to\infty} \frac{2x}{\sqrt{x^2 - x - 2}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{2}{\sqrt{\frac{1}{x^2}}\sqrt{x^2-x-2}} = \lim_{x\to\infty} \frac{2}{\sqrt{1 - 1/x - 2/x^2}} = \boxed{2}$

$$\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2-x-2}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to-\infty} \frac{2}{-\sqrt{\frac{1}{x^2}}\sqrt{x^2-x-2}} = \lim_{x\to-\infty} \frac{2}{-\sqrt{1 - 1/x - 2/x^2}} = \boxed{-2}$$

Example. For the following, sketch a graph with the following properties:

1. $\lim_{x \to 0} f(x) = -\infty$

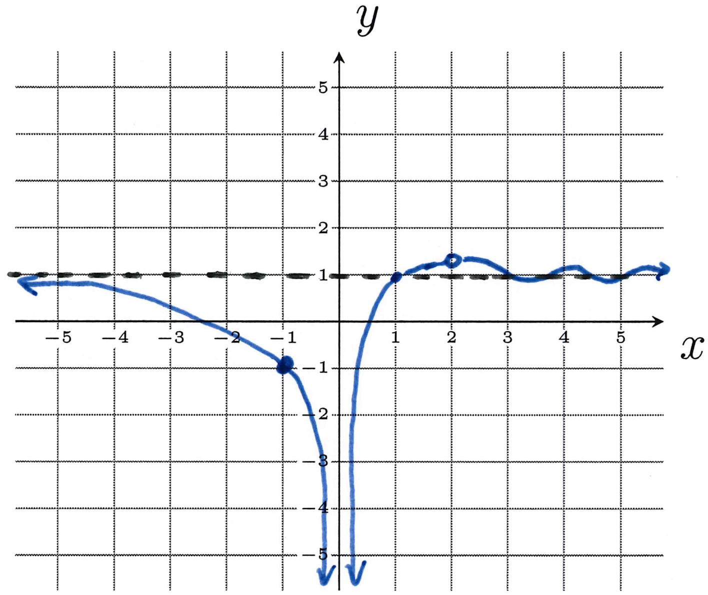
   $\lim_{x \to 2} f(x) = \dfrac{5}{4}$

   $\lim_{x \to \pm\infty} f(x) = 1$

   $f(2)$ DNE

   $f(1) = 1$

   $f(-1) = -1$



2. $\lim_{x \to -1^-} f(x) = \infty$

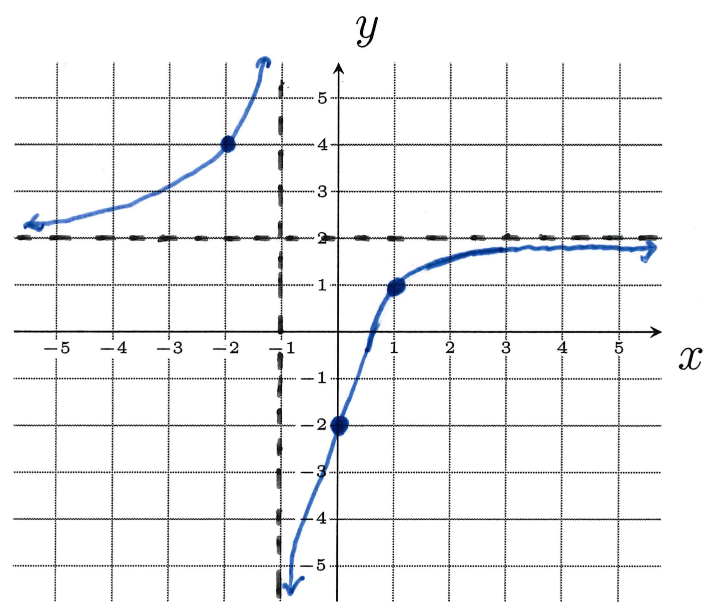
   $\lim_{x \to -1^+} f(x) = -\infty$

   $\lim_{x \to \pm\infty} f(x) = 2$

   $f(0) = -2$

   $f(1) = 1$

   $f(-2) = 4$



Example. Find *all* asymptotes (vertical, horizontal, slant)

1. $\dfrac{x^3 - 10x^2 + 16x}{x^2 - 8x} = \dfrac{x(x-2)(x-8)}{x(x-8)} = x-2, \; x \neq 0, \; x \neq 8$

$\therefore$ No vertical asymptotes $\left(\frac{c}{0}, \; c \neq 0\right)$

$\therefore$ No horizontal asymptotes

$\therefore$ slant asymptote is $x-2$

$$\begin{array}{r|l} & \quad\quad\quad\quad x \; - 2 \\ x^2 - 8x & x^3 - 10x^2 + 16x + 0 \\ & -(x^3 - 8x^2) \\ & \quad\quad -2x^2 + 16x \\ & \quad\quad -(-2x^2 + 16x) \\ & \quad\quad\quad\quad\quad 0 \end{array}$$

2. $\dfrac{\cos x + 2\sqrt{x}}{\sqrt{x}}$

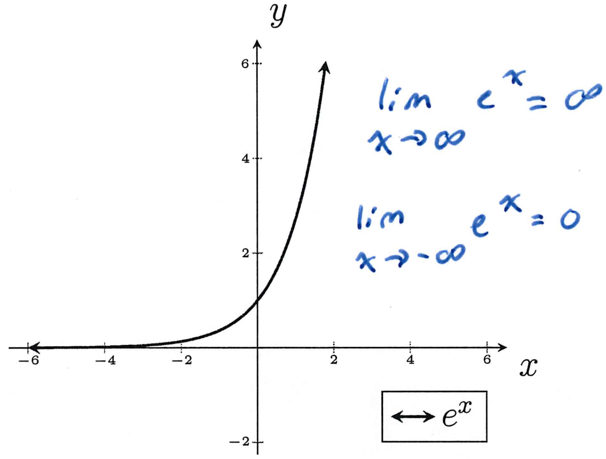
$\sqrt{x} \neq 0 \rightarrow x \neq 0$

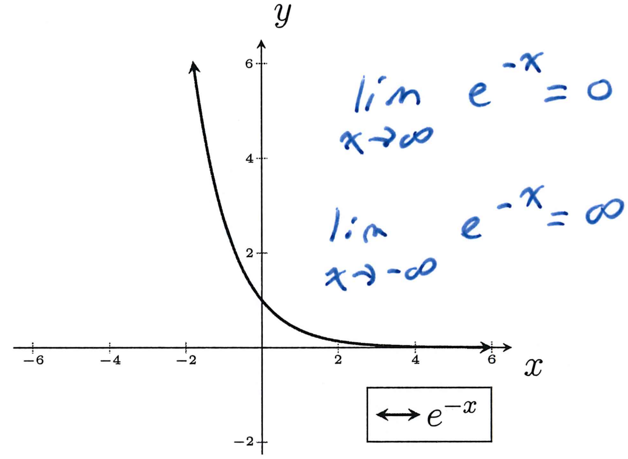
$\dfrac{\cos(0) + 2\sqrt{0}}{\sqrt{0}} \quad \left(\frac{1}{0}\right) \Rightarrow$ vertical asymptote $\boxed{x = 0}$

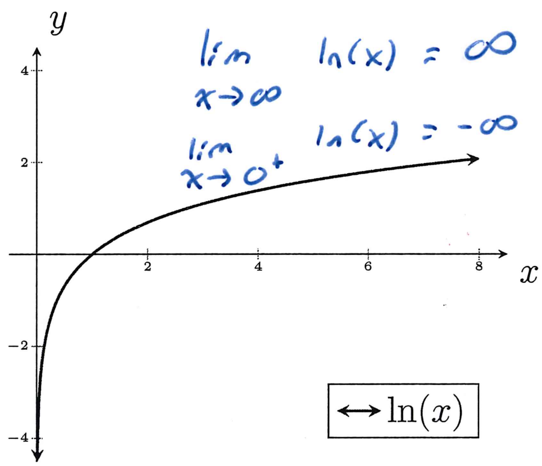
$\displaystyle \lim_{x \to \infty} \frac{\cos(x) + 2\sqrt{x}}{\sqrt{x}} = \lim_{x \to \infty} \underbrace{\frac{\cos(x)}{\sqrt{x}}}_{0} + 2 = 2 \Rightarrow \boxed{H.A. \; y = 2}$

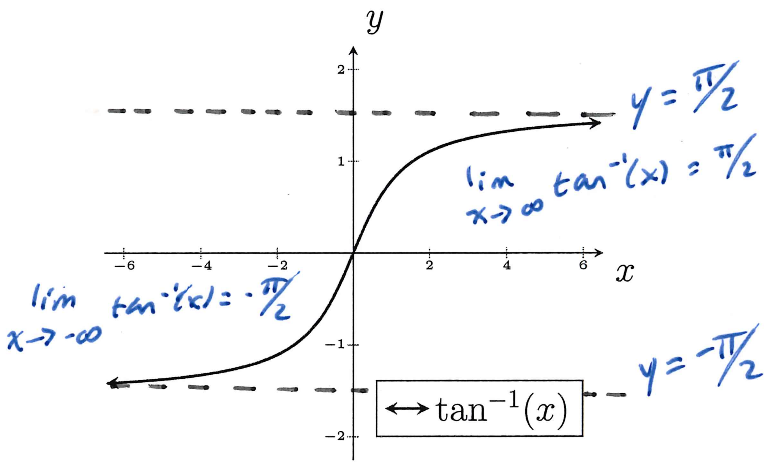
$\displaystyle \lim_{x \to -\infty} \frac{\cos(x) + 2\sqrt{x}}{\sqrt{x}}$ DNE $\quad (x > 0)$

Other function end behavior to consider include $e^x, e^{-x}, \ln(x)$ and $\tan^{-1}(x)$:









a) $\lim\limits_{x\to-\infty} \sin x$ DNE

b) $\lim\limits_{x\to\infty} \sin x$ DNE

c) $\lim\limits_{x\to-\infty} \cos x$ DNE

d) $\lim\limits_{x\to\infty} \cos x$ DNE