

4.9 Antiderivatives

Definition. (Antiderivative)

A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for x in I .

Note: we will denote this relationship in the following way:

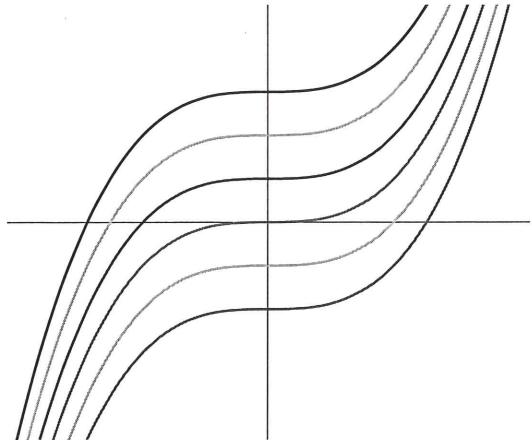
Function	Anti-derivative
$f'(x)$	$f(x)$
$f(x)$	$F(x)$

Example. If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$. In this case, $\tan(x)$ is the *antiderivative* of $\sec^2(x)$.

Theorem 4.15: The Family of Antiderivatives

Let F be any antiderivative of f on an interval I . Then *all* antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

Example. If $f'(x) = x^2$, then $f(x) = \frac{x^3}{3} + C$ is the family of antiderivatives of $f'(x)$.



Example. Find the most general antiderivative of the following functions

$$f(x) = \sin(x)$$

$$k(x) = \frac{1}{1+x^2}$$

$$F(x) = -\cos(x) + C$$

$$K(x) = \tan^{-1}(x) + C$$

$$g(x) = x^n, n \neq -1$$

$$h(x) = 1/x$$

$$G(x) = \frac{x^{n+1}}{n+1} + C$$

$$H(x) = \ln|x| + C$$

$$j(x) = 3x^2$$

$$n(x) = 6x^5$$

$$J(x) = x^3 + C$$

$$N(x) = x^6 + C$$

$$\ell(x) = \frac{41}{x} + 4e^x$$

$$m(x) = \frac{1}{2x^3} = \frac{x^{-3}}{2}$$

$$L(x) = 41 \ln|x| + 4e^x + C$$

$$M(x) = -\frac{x^{-2}}{4} + C$$

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2(x)$	$\tan(x)$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec(x) \tan(x)$	$\sec(x)$
$x^n (n \neq 1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{x}$	$\ln x $	$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
$\cos(x)$	$\sin(x)$	$\sin(x)$	$-\cos(x)$
e^x	e^x		

Note: There are some more ‘complicated’ antiderivatives as well:

$$F(x) = e^{g(x)} \Rightarrow f(x) = \frac{e^{g(x)}}{g'(x)}$$

$$F(x) = k \sec^2(kx) \Rightarrow f(x) = \tan(kx)$$

Focus more on “What can I take the derivative of to get ...” rather than memorizing formulas.

Definition.

Recall that $\frac{d}{dx}[f(x)]$ represents taking the derivative of $f(x)$ with respect to x .

- Finding the antiderivative of f with respect to x is the **indefinite integral**
$$\int f(x) dx.$$
- The **integrand** is the function $f(x)$ we are integrating.
- The **variable of integration**, dx , indicates which variable we are integrating with respect to.

Theorem 4.16: Power Rule for Indefinite Integrals

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Theorem 4.17: Constant Multiple and Sum Rules

Constant Multiple Rule: $\int cf(x) dx = c \int f(x) dx$, for real numbers c .

Sum Rule: $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$

Table 4.9: Indefinite Integrals of Trigonometric Functions

$\frac{d}{dx}[\sin(x)] = \cos(x)$	\Rightarrow	$\int \cos(x) dx = \sin(x) + C$
$\frac{d}{dx}[\cos(x)] = -\sin(x)$	\Rightarrow	$\int \sin(x) dx = -\cos(x) + C$
$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	\Rightarrow	$\int \sec^2(x) dx = \tan(x) + C$
$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$	\Rightarrow	$\int \csc^2(x) dx = -\cot(x) + C$
$\frac{d}{dx}[\sec(x)] = \sec(x) \tan(x)$	\Rightarrow	$\int \sec(x) \tan(x) dx = \sec(x) + C$
$\frac{d}{dx}[\csc(x)] = -\csc(x) \cot(x)$	\Rightarrow	$\int \csc(x) \cot(x) dx = -\csc(x) + C$

Table 4.10: Other Indefinite Integrals

$\frac{d}{dx}[e^x] = e^x$	\Rightarrow	$\int e^x dx = e^x + C$
$\frac{d}{dx}[\ln x] = \frac{1}{x}$	\Rightarrow	$\int \frac{1}{x} dx = \ln x + C$
$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$	\Rightarrow	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$
$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$	\Rightarrow	$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$
$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{x\sqrt{x^2-1}}$	\Rightarrow	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

Example. Verify the following integration formulas using differentiation.

$$\int \frac{1}{(x+1)^2} dx = \frac{x}{x+1} + C \quad \frac{d}{dx} \left[\frac{x}{x+1} + C \right] = \frac{(x+1)(1) - (x)(1)}{(x+1)^2}$$

$$= \boxed{\frac{1}{(x+1)^2}} \quad \checkmark$$

$$\int \sec^2(5x-1) dx = \frac{1}{5} \tan(5x-1) + C \quad \frac{d}{dx} \left[\frac{1}{5} \tan(5x-1) + C \right] = \frac{5}{5} \sec^2(5x-1)$$

$$= \boxed{\sec^2(5x-1)}$$

$$\int \cos^3(x) dx = \sin(x) - \frac{1}{3} \sin^3(x) + C \quad \frac{d}{dx} \left[\sin(x) - \frac{1}{3} \sin^3(x) + C \right] = \cos(x) - \frac{3}{3} \sin^2(x) \cos(x)$$

$$= \cos(x) \underbrace{\left[1 - \sin^2(x) \right]}_{\cos^2(x)} \quad = \boxed{\cos^3(x)}$$

$$\int \frac{x}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} + C \quad \frac{d}{dx} \left[\sqrt{x^2+1} + C \right] = \frac{1}{2} (x^2+1)^{-1/2} (2x)$$

$$= \frac{2x}{2\sqrt{x^2+1}} = \boxed{\frac{x}{\sqrt{x^2+1}}}$$

Example. Find the most general anti-derivative or indefinite integral

$$\int \left(\frac{t^2}{2} + 4t^3 \right) dt$$

$$= \boxed{\frac{t^3}{6} + t^4 + C}$$

$$\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x \right) dx$$

$$= \boxed{\frac{x^5}{5} + \frac{1}{x^2} + x^2 + C}$$

$$\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) dx$$

$$\int \frac{\csc \theta \cot \theta}{2} d\theta$$

$$= \frac{1}{3/2} \frac{x^{3/2}}{2} + \frac{1}{1/2} 2x^{1/2} + C$$

$$= \boxed{\frac{1}{2} \csc \theta + C}$$

$$= \frac{2}{3} \frac{x^{3/2}}{2} + 4x^{1/2} + C$$

$$= \boxed{\frac{x^{3/2}}{3} + 4x^{1/2} + C}$$

$$\int \left(1 + \frac{1}{4u^3} - (3u)^2\right) du$$

$$\int \left(\frac{7}{\sqrt{1-x^2}} - \frac{3}{\cos^2(x)}\right) dx$$

$$\begin{aligned} & \int 1 + \frac{u^{-3}}{4} - 9u^2 du \\ &= u - \frac{u^{-2}}{8} - 3u^3 + C \\ &= \boxed{u - \frac{1}{8u^2} - 3u^3 + C} \end{aligned}$$

$$\begin{aligned} &= \int 7 \frac{1}{\sqrt{1-x^2}} - 3 \sec^2(x) dx \\ &= \boxed{7 \tan^{-1}(x) - 3 \tan(x) + C} \end{aligned}$$

$$\int \left(\frac{1}{4e^x} - \frac{4}{x} + 4^x\right) dx$$

$$\int \frac{x^2 - 36}{x - 6} dx$$

$$\begin{aligned} &= \int \frac{1}{4} e^{-x} - \frac{4}{x} + 4^x dx \\ &= \boxed{-\frac{1}{4} e^{-x} - 4 \ln|x| + \frac{4^x}{\ln 4} + C} \end{aligned}$$

$$= \int \frac{(x+6)(x-6)}{x-6} dx$$

$$= \int x+6 dx$$

$$= \boxed{\frac{x^2}{2} + 6x + C}$$

$$\int \frac{2 + 3 \cos(y)}{\sin^2(y)} dy$$

$$= \int 2 \csc^2(y) + 3 \cot(y) \csc(y) dy$$

$$= \boxed{-2 \cot(y) - 3 \csc(y) + C}$$

$$\int (u+4)(2u+1) du$$

$$= \int 2u^2 + 9u + 4 du$$

$$= \boxed{u^3 + \frac{9}{2}u^2 + 4u + C}$$

$$\int e^{x+2} dx$$

$$= \boxed{\frac{e^{x+2}}{1} + C}$$

$$\int (\sqrt[4]{x^3} + \sqrt{x^5}) dx$$

$$= \int x^{\frac{3}{4}} + x^{\frac{5}{2}} dx$$

$$= \frac{1}{\frac{7}{4}} x^{\frac{7}{4}} + \frac{1}{\frac{7}{2}} x^{\frac{7}{2}} + C$$

$$= \boxed{\frac{4x^{\frac{7}{4}}}{7} + \frac{2x^{\frac{7}{2}}}{7} + C}$$

$$\int \left(x^2 + 1 + \frac{1}{x^2 + 1} \right) dx$$

$$= \boxed{\frac{x^3}{3} + x + \tan^{-1}(x) + C}$$

$$\int \left(\sin(4x) - \frac{3}{\sin^2(x)} \right) dx$$

$$= \boxed{\int \sin(4x) - 3 \csc^2(x) dx}$$

$$= \boxed{-\frac{\cos(4x)}{4} + 3 \cot(x) + C}$$

$$\int (\csc^2(2t) - 2e^t) dt$$

$$= \boxed{-\frac{\cot(2t)}{2} - 2e^t + C}$$

$$\int x(1 + 2x^4) dx$$

$$= \int x + 2x^5 dx$$

$$= \boxed{\frac{x^2}{2} + \frac{x^6}{3} + C}$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\begin{aligned}
 & \int \frac{\cos \theta}{\sin^2 \theta} d\theta \\
 &= \int \cot \theta \csc \theta d\theta \\
 &= \boxed{-\csc \theta + C} \\
 & \int \frac{\sin(2x)}{\sin(x)} dx \\
 &= \int \frac{2\sin(x)\cos(x)}{\sin(x)} dx \\
 &= \int 2\cos(x) dx \\
 &= \boxed{2\sin(x) + C}
 \end{aligned}$$

$$\begin{aligned}
 & \int \left(\pi + \frac{2}{yt} \right) dt \\
 &= \int \pi + \frac{2}{y} \frac{1}{t} dt \\
 &= \boxed{\pi t + \frac{2}{y} \ln|t| + C} \\
 & \int \frac{t^2 - e^{2t}}{t + e^t} dt \\
 &= \int \frac{(t + e^t)(t - e^t)}{t + e^t} dt \\
 &= \int t - e^t dt \\
 &= \boxed{\frac{t^2}{2} - e^t + C}
 \end{aligned}$$

Definition.

- An equation involving an unknown function and its derivative is called a **differential equation**.
- An **initial condition** allows us to determine the arbitrary constant C .
- A differential equation coupled with an initial condition is called an **initial value problem**.

$$f'(x) = g(x), \text{ where } g \text{ is given, and}$$

$$f(a) = b, \quad \text{where } a \text{ and } b \text{ are given.}$$

Differential equation
Initial condition

Example. Solve the initial value problem:

$$\frac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$$

$$y = 3x^3 - 2x^2 + 5x + C$$

$$y(-1) = 3(-1)^3 - 2(-1)^2 + 5(-1) + C$$

$$= -3 - 2 - 5 + C = 0$$

$$\Rightarrow C = 10$$

$$y = 3x^3 - 2x^2 + 5x + 10$$

$$f'(x) = 1 + 3\sqrt{x}, \quad f(4) = 25$$

$$f(x) = x + 2x^{3/2} + C$$

$$f(4) = 4 + 2(4)^{3/2} + C = 25$$

$$12 + C = 25$$

$$C = 13$$

$$\Rightarrow f(x) = x + 2x^{3/2} + 13$$

$$f'(x) = 8x^3 + 12x + 3, \quad f(1) = 6$$

$$f(x) = 2x^4 + 6x^2 + 3x + C$$

$$f(1) = 11 + C = 6$$

$$\Rightarrow C = -5$$

$$\Rightarrow f(x) = 2x^4 + 6x^2 + 3x - 5$$

$$\frac{dr}{d\theta} = \cos(\pi\theta), \quad r(0) = 1$$

$$r(\theta) = \frac{\sin(\pi\theta)}{\pi} + C$$

$$r(0) = 0 + C = 1$$

$$C = 1$$

$$\Rightarrow r(\theta) = \frac{\sin(\pi\theta)}{\pi} + 1$$

$$f'(t) = 2\cos(t) + \sec^2(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}, \quad f\left(\frac{\pi}{3}\right) = 4$$

$$f(t) = 2\sin(t) + \tan(t) + C$$

$$f\left(\frac{\pi}{3}\right) = \sqrt{3} + \sqrt{3} + C = 4$$

$$C = 4 - 2\sqrt{3}$$

$$\boxed{f(t) = 2\sin(t) + \tan(t) + 4 - 2\sqrt{3}}$$

$$g'(x) = 7x(x^6 - \frac{1}{7}); \quad g(1) = 2$$

$$g'(x) = 7x^7 - x$$

$$g(x) = \frac{7}{8}x^8 - \frac{1}{2}x^2 + C$$

$$g(1) = \frac{3}{8} + C = 2$$

$$\Rightarrow C = \frac{13}{8}$$

$$\Rightarrow \boxed{g(x) = \frac{7}{8}x^8 - \frac{7}{2}x^2 + \frac{13}{8}}$$

$$f'''(x) = \sin(x), \quad f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1$$

$$f''(x) = -\cos(x) + C$$

$$f''(0) = -\cos(0) + C = 1 \Rightarrow C = 0$$

$$\underline{f'(x) = -\sin(x) + C}$$

$$f'(0) = -\sin(0) + C = 1 \Rightarrow C = 1$$

$$\underline{f'(x) = 1 - \sin(x)}$$

$$f(x) = x + \cos(x) + C$$

$$f(0) = 0 + \cos(0) + C = 1 \Rightarrow C = 0$$

$$\boxed{f(x) = x + \cos(x)}$$

$$f'(x) = \frac{4}{\sqrt{1-x^2}}, f\left(\frac{1}{2}\right) = 1$$

$$f(x) = 4 \sin^{-1}(x) + c \rightarrow f\left(\frac{1}{2}\right) = \frac{2\pi}{3} + c = 1$$

$$\Rightarrow c = 1 - \frac{2\pi}{3}$$

$$\boxed{f(x) = 4 \sin^{-1}(x) + 1 - \frac{2\pi}{3}}$$

$$\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left. \frac{dr}{dt} \right|_{t=1} = 1, \quad r(1) = 1$$

$$r'(t) = -\frac{1}{t^2} + c \rightarrow r'(1) = -\frac{1}{1} + c = 0 \Rightarrow c = 1$$

$$r'(t) = 2 - \frac{1}{t^2}$$

$$r(t) = 2t + \frac{1}{t} + c \rightarrow r(1) = 2(1) + \frac{1}{1} + c = 1 \Rightarrow c = -2$$

$$\boxed{r(t) = 2t + \frac{1}{t} - 2}$$

$$f''(x) = 4 + 6x + 4x^2, \quad f(0) = 3, \quad f(1) = 10$$

$$f'(x) = 4x + 3x^2 + \frac{4}{3}x^3 + c$$

$$f(x) = 2x^2 + x^3 + \frac{1}{3}x^4 + cx + D$$

$$f(0) = D = 3 \Rightarrow D = 3$$

$$f(1) = 2 + 1 + \frac{1}{3} + c + 3 = 10 \Rightarrow c = \frac{11}{3}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{3}x^4 + x^3 + 2x^2 + \frac{17}{3}x + 3}$$

$$f'(x) = \frac{4}{\sqrt{1-x^2}}, f\left(\frac{1}{2}\right) = 1$$

$$f(x) = 4 \sin^{-1}(x) + c \rightarrow f\left(\frac{1}{2}\right) = \frac{2\pi}{3} + c = 1$$

$$\Rightarrow c = 1 - \frac{2\pi}{3}$$

$$\boxed{f(x) = 4 \sin^{-1}(x) + 1 - \frac{2\pi}{3}}$$

$$\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left. \frac{dr}{dt} \right|_{t=1} = 1, \quad r(1) = 1$$

$$r'(t) = -\frac{1}{t^2} + c \rightarrow r'(1) = -\frac{1}{1} + c = 0 \Rightarrow c = 1$$

$$r'(t) = 2 - \frac{1}{t^2}$$

$$r(t) = 2t + \frac{1}{t} + c \rightarrow r(1) = 2(1) + \frac{1}{1} + c = 1 \Rightarrow c = -2$$

$$\boxed{r(t) = 2t + \frac{1}{t} - 2}$$

$$f''(x) = 4 + 6x + 4x^2, \quad f(0) = 3, \quad f(1) = 10$$

$$f'(x) = 4x + 3x^2 + \frac{4}{3}x^3 + C$$

$$f(x) = 2x^2 + x^3 + \frac{1}{3}x^4 + cx + D$$

$$f(0) = D = 3 \Rightarrow D = 3$$

$$f(1) = 2 + 1 + \frac{1}{3} + c + 1 = 10 \Rightarrow c = \frac{17}{3}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{3}x^4 + x^3 + 2x^2 + \frac{17}{3}x + 3}$$

Initial Value Problems for Velocity and Position

Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem.

$$s'(t) = v(t), \quad s(0) = s_0, \text{ where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), \quad v(0) = v_0, \text{ where } v_0 \text{ is the (known) initial velocity.}$$

Recall:

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

Example. Solve the following velocity and position initial value problems

$$v(t) = \sin(t) + 3\cos(t); \quad s(0) = 4$$

$$s(t) = -\cos(t) + 3\sin(t) + C \rightarrow s(0) = -1 + C = 4 \rightarrow C = 5$$

$$\Rightarrow \boxed{s(t) = -\cos(t) + 3\sin(t) + 5}$$

$$a(t) = 2e^t - 12, \quad v(0) = 1, \quad s(0) = 0$$

$$v(t) = 2e^t - 12x + C \rightarrow v(0) = 2 + C = 1 \Rightarrow C = -1$$

$$\Rightarrow v(t) = 2e^t - 12x - 1$$

$$s(t) = 2e^t - 6x^2 - x + C \rightarrow s(0) = 2 + C = 0 \Rightarrow C = -2$$

$$\Rightarrow \boxed{s(t) = 2e^t - 6x^2 - x - 2}$$

Example. The acceleration of gravity near the surface of Mars is 3.72 m/s^2 . A rock is thrown straight up from the surface with an initial velocity of 23 m/s . How high does the rock go?

- a) Write the initial value problem

$$a(t) = 3.72 \quad v(0) = 23 \quad s(0) = 0$$

- b) Find the velocity and position functions.

$$v(t) = -3.72t + c \rightarrow v(0) = c = 23 \Rightarrow v(t) = -3.72t + 23$$

$$s(t) = -1.86t^2 + 23t + c \rightarrow s(0) = c = 0 \Rightarrow s(t) = -1.86t^2 + 23t$$

- c) Maximum height is reached when velocity is 0. Find the time when this happens and the maximum height.

$$\text{Solve } v(x) = 0$$

$$-3.72t + 23 = 0 \\ t = \frac{23}{3.72} \approx 6.183 \text{ sec}$$

Find height

$$s\left(\frac{23}{3.72}\right) = -1.86\left(\frac{23}{3.72}\right)^2 + 23\left(\frac{23}{3.72}\right) \approx 71.102 \text{ m}$$

Example. A ball is thrown vertically upward from a height of 48 feet above ground at a speed of 32 ft/s .

a) How high above the ground will it get?

$$\begin{cases} a(t) = -32 \text{ ft/s}^2 \\ v(0) = 32 \text{ ft/s} \\ s(0) = 48 \text{ ft} \end{cases}$$

$$v(t) = -32t + c$$

$$v(0) = 0 + c = 32 \rightarrow c = 32$$

$$\underline{v(t) = -32t + 32}$$

$$s(t) = -16t^2 + 32t + c$$

$$s(0) = 0 + 0 + c = 48 \rightarrow c = 48$$

$$s(t) = -16t^2 + 32t + 48$$

Velocity = 0 at peak height

- Solve $v(t) = 0$

$$-32t + 32 = 0 \\ t = 1$$

- Evaluate $s(1)$

$$s(1) = -16 + 32 + 48 = \boxed{64 \text{ m}}$$

b) How long after it is thrown will it hit the ground?

The ball hits the ground when $s(t) = 0$.

Solve $s(t) = 0$

$$-16t^2 + 32t + 48 = 0$$

$$-16(t+1)(t-3) = 0$$

$$\Rightarrow t = -1, \boxed{t = 3}$$

Example. A stone was dropped off a cliff and hits the ground with a speed of 120 ft/s . What is the height of the cliff (assuming $a(t) = -32 \text{ ft/s}^2$).

$$a(t) = -32 \text{ ft/s}^2 \quad v(0) = 0$$

$$v(t) = -32t + c \quad v(0) = 0 + c = 0 \Rightarrow c = 0$$

$$\boxed{v(t) = -32t \text{ ft/s}}$$

Let t_f be the final time
 $v(t_f) = -32t_f = -120 \rightarrow t_f = \frac{120}{32} = 3.75 \text{ sec}$
 $\Rightarrow s(3.75) = 0$

$$s(t) = -16t^2 + c$$

$$s(3.75) = -16\left(\frac{120}{32}\right)^2 + c = 0 \Rightarrow \boxed{c = 225 \text{ ft}}$$

$$\boxed{s(t) = -16t^2 + 225 \text{ ft}}$$

Example. Find the antiderivative of $f(x) = \frac{2+x^2}{1+x^2}$

$$\int \frac{2+x^2}{1+x^2} dx = \int \frac{1}{1+x^2} + \frac{1+x^2}{1+x^2} dx$$

$$= \boxed{\tan^{-1}(x) + x + C}$$