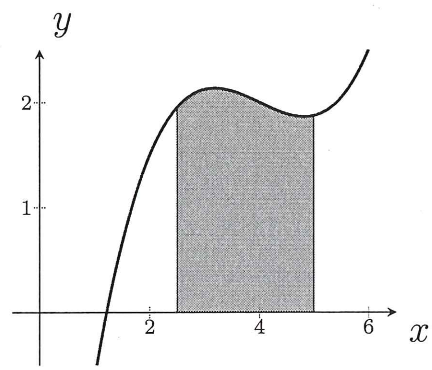
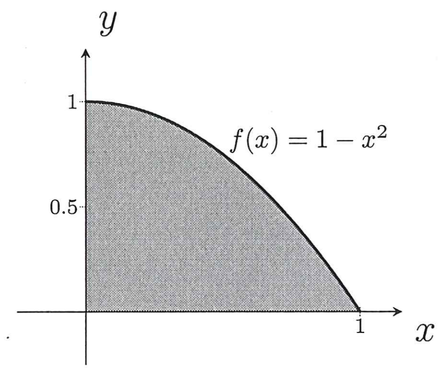
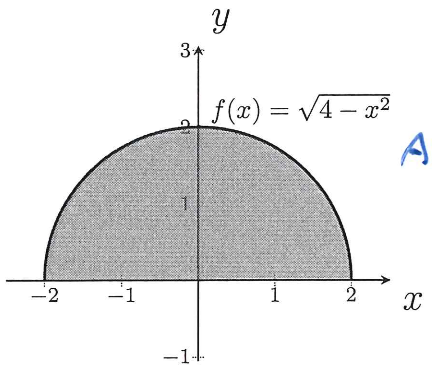
5.1: Approximating Areas under Curves





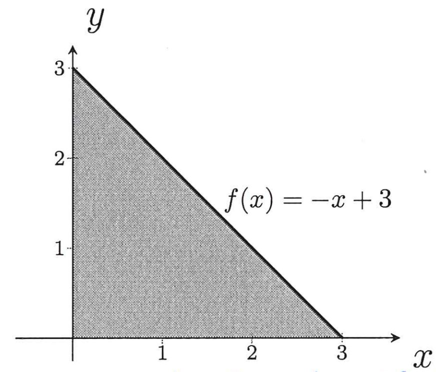
Finding the area under the curve is simple in some cases:



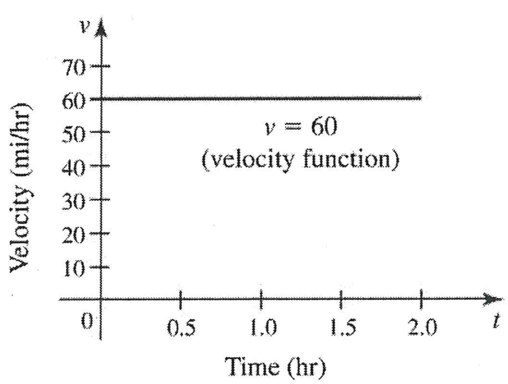
$A = \frac{1}{2}(\pi r^2)$

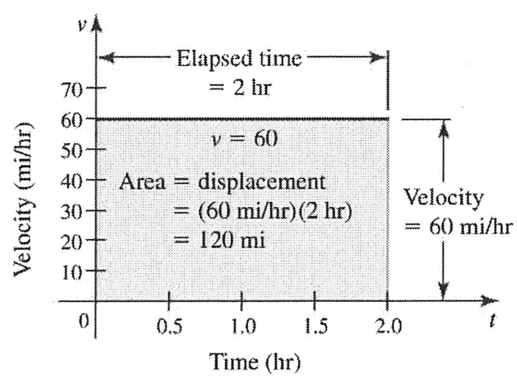
$= \frac{1}{2}(\pi 2^2)$

$= \boxed{2\pi}$

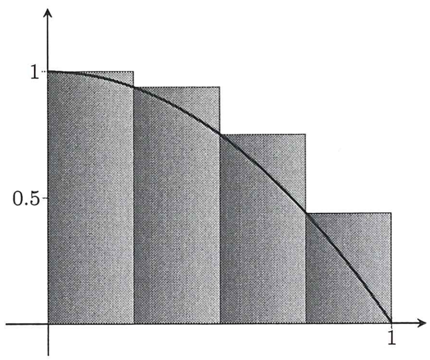


$A = \frac{1}{2} bh = \frac{1}{2}(3)(3) = \boxed{\frac{9}{2}}$

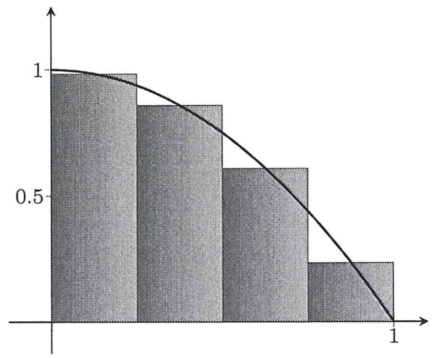




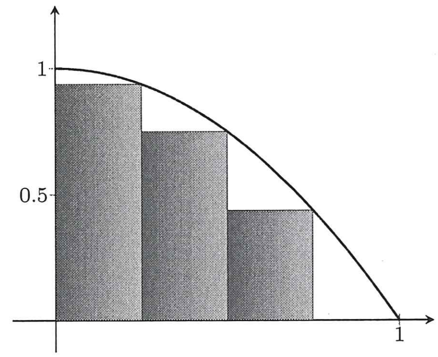
For functions whose curves are irregular shapes, we can approximate the area using rectangles:



Left rectangles

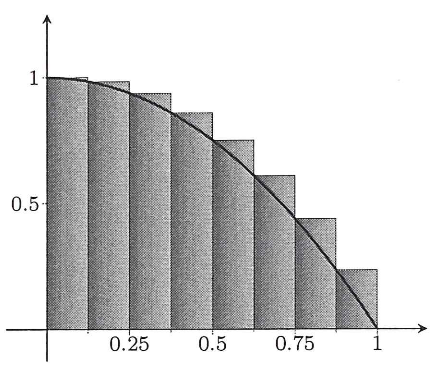


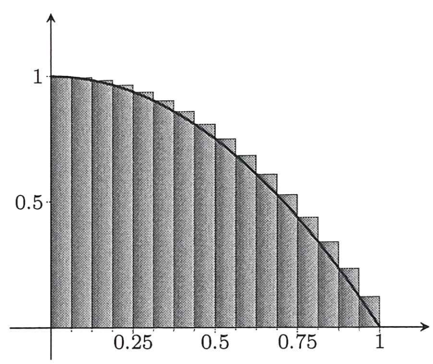
Midpoint rectangles

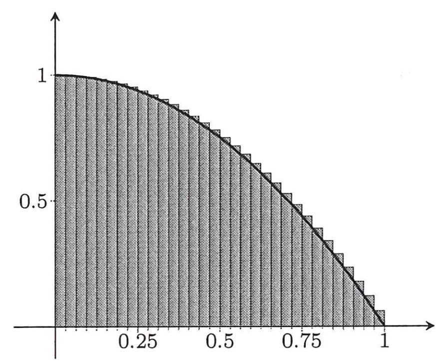


Right rectangles

These approximations are much more accurate when more rectangles are used:







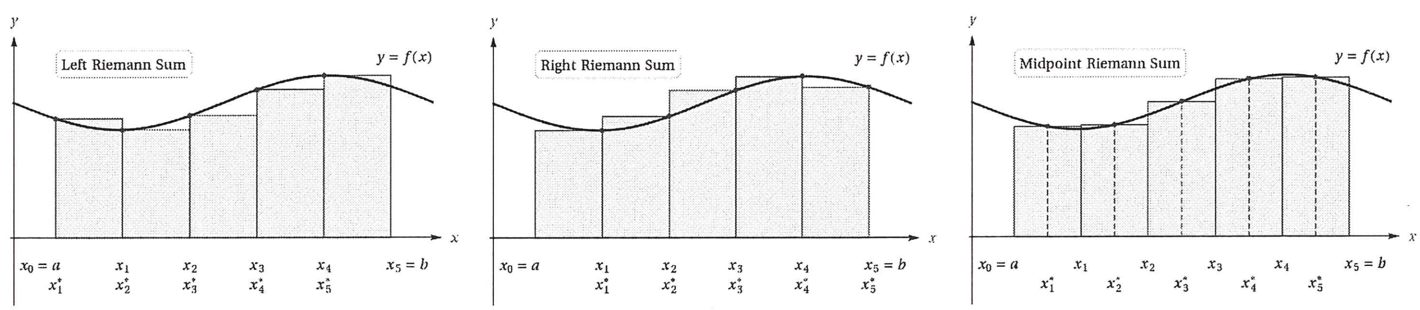
Definition. (Riemann Sum)
Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the k-th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$,
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$, and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

In general, midpoint rectangles give better approximations then left or right rectangles.



When estimating the area under the graph on the interval $[a, b]$, we define

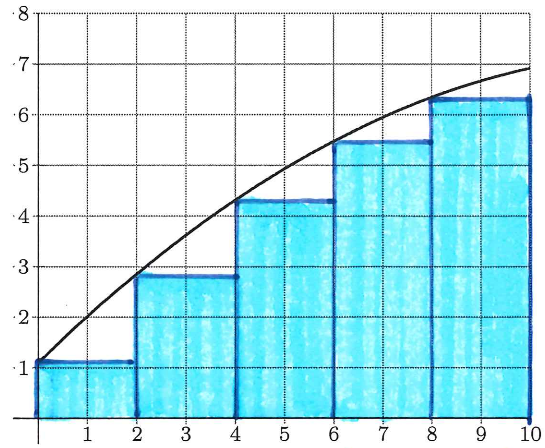
$$\Delta x = \frac{b-a}{n}$$

to be the width of the rectangles. The height of the rectangles is given by $f(x_i)$, where the x_i's are Δx apart.

Example. Use the plots below to estimate the area under the given graph of $f(x)$:

Sketch five rectangles and use them to find a lower estimate for the area under the given graph of $f(x)$ from $x = 0$ to $x = 10$.

$\Delta x = \frac{10-0}{5} = 2$

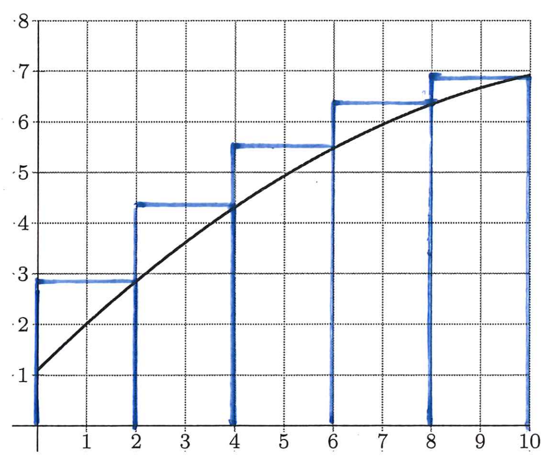


$A \approx 2(1) + 2(3) + 2(4.3) + 2(5.5) + 2(6.2)$

$= 2(20) = \boxed{40}$

Sketch five rectangles and use them to find a upper estimate for the area under the given graph of $f(x)$ from $x = 0$ to $x = 10$.

$\Delta x = \frac{10-0}{5} = 2$



$A \approx 2(3 + 4.3 + 5.5 + 6.2 + 6.8)$

$= 2(25.8) = \boxed{51.6}$

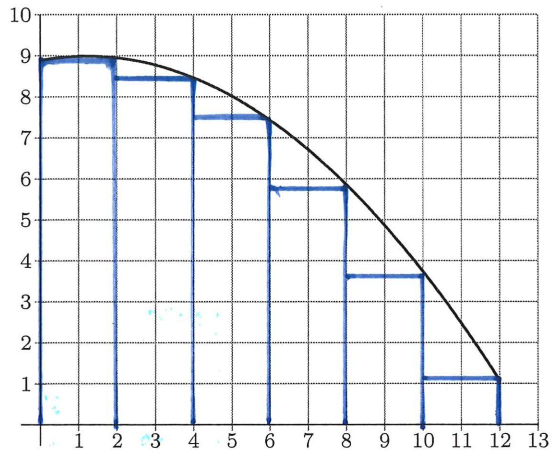
Use six right rectangles to estimate the area under the given graph of $f(x)$ from $x = 0$ to $x = 12$. Is the estimate an under-approximation or an over-approximation? Why?

$\Delta x = \frac{12-0}{6} = 2$

$A \approx 2(9 + 8.5 + 7.5 + 5.9 + 3.7 + 1.1)$

$= 2(35.7) = \boxed{71.4}$

Underestimate since $f(x)$ is decreasing.



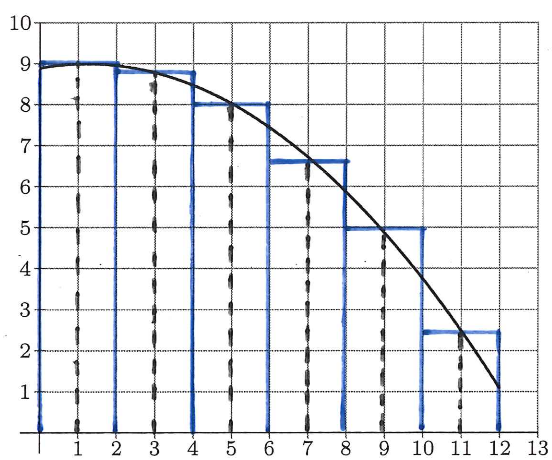
Use six midpoint rectangles to estimate the area under the given graph of $f(x)$ from $x = 0$ to $x = 12$. Talk about the quality of this estimate.

$\Delta x = \frac{12-0}{6} = 2$

$A \approx 2(9 + 8.8 + 8 + 7.6 + 5 + 2.5)$

$= 2(40.9)$

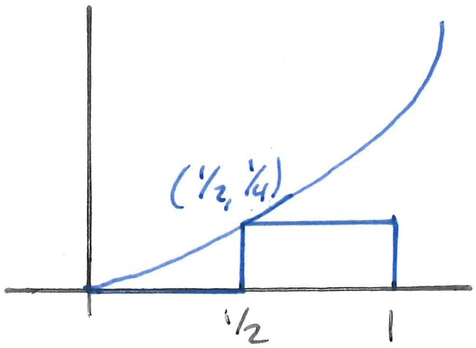
$= \boxed{81.8}$



The midpoint Riemann is a good approximation since the over & under estimates are roughly equal.

Example. Consider $f(x) = x^2$ on the interval $[0, 1]$. True area: $\frac{1}{3} = 0.\overline{3}$

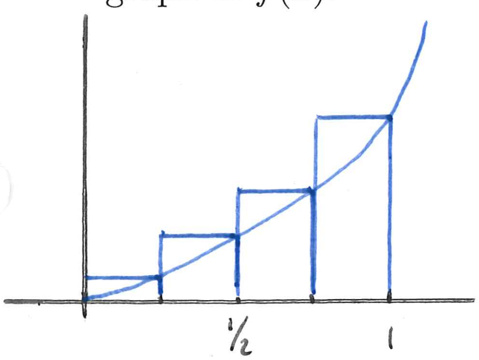
a) Use two rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.



$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

$$A \approx \frac{1}{2}(0) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8} = 0.125$$

b) Use four rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.



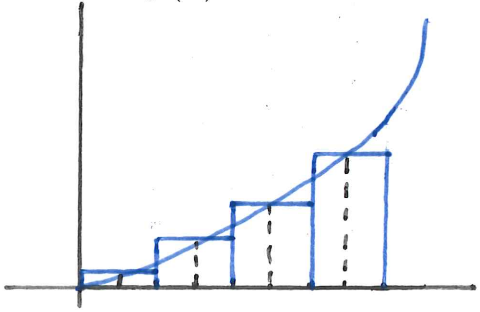
$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$A \approx \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$$

$$= \frac{15}{32} = 0.46875$$

c) Use four midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

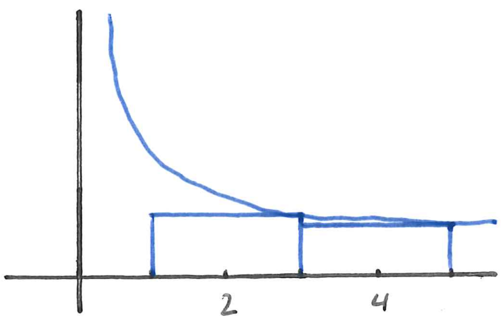


$$A \approx \frac{1}{4}\left(\frac{1}{8}\right)^2 + \frac{1}{4}\left(\frac{3}{8}\right)^2 + \frac{1}{4}\left(\frac{5}{8}\right)^2 + \frac{1}{4}\left(\frac{7}{8}\right)^2$$

$$= \frac{21}{64} = 0.328125$$

Example. Consider $f(x) = \dfrac{1}{x}$ on the interval $[1, 5]$. True area: $\ln(5) \approx 1.609438$

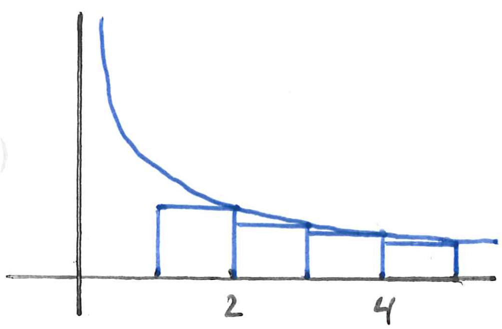
a) Use two rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.



$\Delta x = \frac{5-1}{2} = 2$

$A \approx 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{5}\right) = \frac{16}{15} = 1.0\overline{6}$

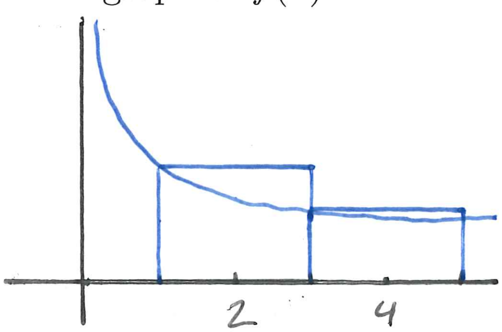
b) Use four rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.



$\Delta x = \frac{5-1}{4} = 1$

$A \approx 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right) + 1\left(\frac{1}{5}\right) = \frac{77}{60} = 1.28\overline{3}$

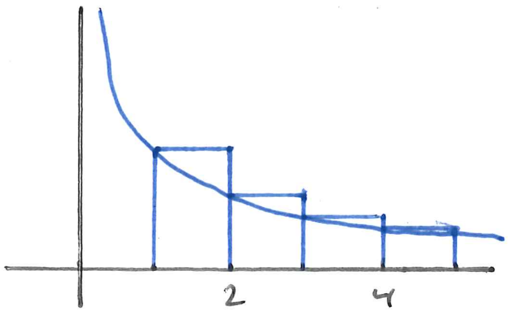
c) Use two rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.



$\Delta x = \frac{5-1}{2} = 2$

$A \approx 2(1) + 2\left(\frac{1}{3}\right) = \frac{8}{3} = 2.\overline{6}$

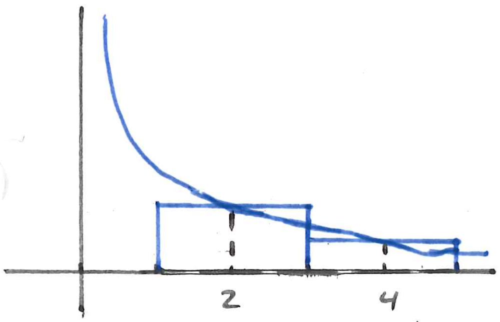
d) Use four rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.



$\Delta x = \frac{5-1}{4} = 1$

$A \approx 1(1) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right) = \frac{25}{12} = 2.08\overline{3}$

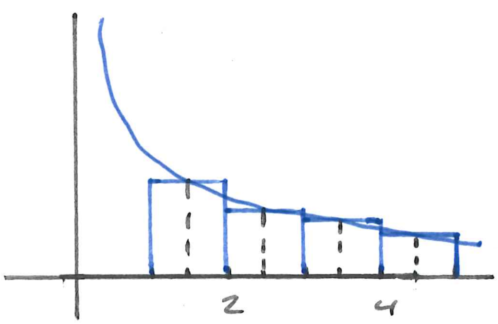
e) Use two midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.



$\Delta x = \frac{5-1}{2} = 2$

$A \approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = \frac{3}{2} = 1.5$

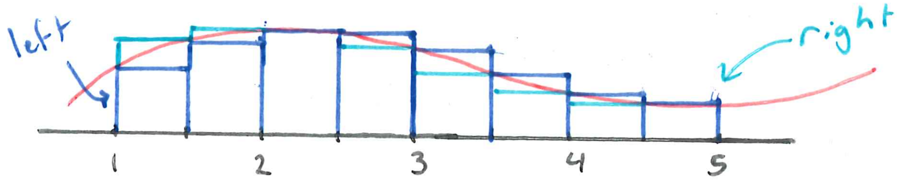
f) Use four midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.



$\Delta x = \frac{5-1}{4} = 1$

$A \approx 1\left(\frac{2}{3}\right) + 1\left(\frac{2}{5}\right) + 1\left(\frac{2}{7}\right) + 1\left(\frac{2}{9}\right)$

$= \frac{496}{315} = 1.5\overline{746031}$



Example. Use the tabulated values of f to evaluate both the left and right Reimann sums. ($n = 8$, $[1, 5]$)

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	0	2	3	2	2	1	0	2	3

$\Delta x = \frac{5-1}{8} = \frac{1}{2}$

$A_L = \frac{1}{2}(0) + \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(2) + \frac{1}{2}(2) + \frac{1}{2}(1) + \frac{1}{2}(0) + \frac{1}{2}(2) = \boxed{6}$

$A_R = \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(2) + \frac{1}{2}(2) + \frac{1}{2}(1) + \frac{1}{2}(0) + \frac{1}{2}(2) + \frac{1}{2}(3) = \boxed{\frac{15}{2}}$

When velocity is a continuous function, the area between the curve and the x-axis gives the displacement.

Example. The velocities (in m/s) of an automobile moving along a straight freeway over a four-second period are given in the following table. Find the midpoint Riemann sum approximation to the displacement on $[0, 4]$ with $n = 2$ and $n = 4$ subintervals.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	20	25	30	35	30	30	35	40	40

$n = 2$

$n = 2$:

$\Delta t = \frac{4-0}{2} = 2$

$s \approx 2(30) + 2(35) = \boxed{130\,m}$

$n = 4$:

$\Delta t = \frac{4-0}{4} = 1$ $\quad s \approx 1(25) + 1(35) + 1(30) + 1(40) = \boxed{130\,m}$

Example. Use the following table of recorded velocities answer the following questions:

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1.0	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

a) Estimate the total displacement using 12 subintervals of $\underbrace{\text{length}}_{\Delta t}$ 5 with left-endpoint values.

$$\Delta t = \frac{60-0}{12} = 5\,min = 300\ sec$$

$$S \approx 300(1.0+1.2+1.7+2.0+1.8+1.6+1.4+1.2+1.0+1.8+1.5+1.2)$$

$$= 300(17.4)$$

$$= \boxed{5220\ m}$$

b) Estimate the total displacement using 12 subintervals of length 5 with right-endpoint values.

$$S \approx 300(1.2+1.7+2.0+1.8+1.6+1.4+1.2+1.0+1.8+1.5+1.2+0)$$

$$= 300(16.4)$$

$$= \boxed{4920\ M}$$

Definition.
If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

Example. Rewrite the sums without sigma notation and evaluate:

a) $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

$= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} = \boxed{\frac{7}{6}}$

b) $\displaystyle\sum_{k=1}^{4} (-1)^k \cos(k\pi)$

$= -\cos(\pi) + \cos(2\pi) - \cos(3\pi) + \cos(4\pi)$

$= -(-1) + 1 - (-1) + 1 = \boxed{4}$

Example. Rewrite the following sum without sigma notation. Note that n denotes the number of rectangles and the letters i, j, and k are typically used for indexing.

$$\sum_{k=1}^{n} \frac{1}{n}(k^2 + 1)$$

$= \frac{1}{n}\left((1^2+1) + (2^2+1) + (3^2+1) + \ldots + (n^2+1)\right)$

$= \frac{1}{n}(1 + 4 + 9 + \ldots + n^2) + \frac{1}{n}(\underbrace{1 + 1 + \ldots + 1}_{n\text{-times}})$

$= \frac{1}{n}(1 + 4 + 9 + \ldots + n^2) + 1$

Example. Which of the following summations represent the sum $1+2+4+8+16+32$?

a) $\sum_{k=1}^{6} 2^{k-1}$

$= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$= 1 + 2 + 4 + 8 + 16 + 32$

b) $\sum_{k=0}^{5} 2^k$

$= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$= 1 + 2 + 4 + 8 + 16 + 32$

c) $\sum_{k=-1}^{4} 2^{k+1}$

$= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$= 1 + 2 + 4 + 8 + 16 + 32$

All equivalent

Example. Which of the following summations represent the sum $1-2+4-8+16-32$?

a) $\sum_{k=1}^{6} (-2)^{k-1}$

$= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 + (-2)^5$

$= 1 - 2 + 4 - 8 + 16 - 32$

b) $\sum_{k=0}^{5} (-1)^k 2^k$

$= (-1)^0 2^0 + (-1)^1 2^1 + (-1)^2 2^2 + (-1)^3 2^3 + (-1)^4 2^4 + (-1)^5 2^5$

$= 1 - 2 + 4 - 8 + 16 - 32$

c) $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

$= (-1)^{-1} 2^0 + (-1)^0 2^1 + (-1)^1 2^2 + (-1)^2 2^3 + (-1)^3 2^4 + (-1)^4 2^5$

$= -1 + 2 - 4 + 8 - 16 + 32$

Equivalent

Not equivalent

Example. Express the following sums in sigma notation:

$(-1)^1\ (-1)^2\ (-1)^3\ (-1)^4\ (-1)^5$

a) $-1 + 4 - 9 + 16 - 25$

$1^2\ 2^2\ 3^2\ 4^2\ 5^2$

$= \boxed{\sum_{k=1}^{5} (-1)^k k^2}$

b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

$= \sum_{k=1}^{4} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{4} 2^{-k}$

$5\cdot1\ \ 5\cdot2\ \ 5\cdot3\ \ 5\cdot4\ \ 5\cdot5\ \ 5\cdot6$

c) $\frac{5}{2} + \frac{10}{3} + \frac{15}{4} + \frac{20}{5} + \frac{25}{6} + \frac{30}{7}$

$1+1\ \ 2+1\ \ 3+1\ \ 4+1\ \ 5+1\ \ 6+1$

$= \boxed{\sum_{k=1}^{6} \frac{5k}{k+1}}$

d) $4 + 9 + 14 + \cdots + 44$

$4+5\ \ 4+10\ \ 4+40$

$= \boxed{\sum_{k=0}^{8} 4+5k}$

Example. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$, evaluate the following

a) $\sum_{k=1}^{n} 8a_k = 8\sum_{k=1}^{n} a_k$

$= 8(0)$

$= \boxed{0}$

b) $\sum_{k=1}^{n} 250b_k = 250\sum_{k=1}^{n} b_k$

$= 250(1)$

$= \boxed{250}$

c) $\sum_{k=1}^{n} (a_k + 1) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1$

$= 0 + n(1)$

$= \boxed{n}$

d) $\sum_{k=1}^{n} (b_k - 1) = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1$

$= 1 - n(1)$

$= \boxed{1-n}$

Theorem 5.1: Sums of Powers of Integers
Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Example. Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = 1 + 2 + \dots + (n-1) + n$$

$$+ \quad \sum_{k=1}^{n} k = n + (n-1) + \dots + 2 + 1$$

$$2\sum_{k=1}^{n} k = \underbrace{(n+1) + (n+1) + \dots + (n+1) + (n+1)}_{n\text{-times}}$$

$$2\sum_{k=1}^{n} k = n(n+1)$$

$$\Rightarrow \boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

Example. Evaluate the following sums

a) $\sum_{k=1}^{10} k = \frac{10(11)}{2} = \boxed{55}$

b) $\sum_{k=1}^{10} (1+k^2) = \sum_{k=1}^{10} 1 + \sum_{k=1}^{10} k^2 = 10 + \frac{10(11)(21)}{6} = \boxed{395}$

c) $\sum_{k=1}^{10} k^3 = \frac{10^2(11)^2}{4} = 25(121) = \boxed{3025}$

d) $\sum_{k=1}^{7} (-2k-4) = -2\sum_{k=1}^{7} k - \sum_{k=1}^{7} 4 = (-2)\frac{7(8)}{2} - 4(7) = -56 - 28 = \boxed{-84}$

e) $\sum_{k=1}^{6} (3-k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = 3(6) - \frac{6(7)(13)}{6} = 18 - 91 = \boxed{-73}$

f) $\sum_{m=1}^{3} \frac{2m+2}{3} = \frac{2}{3}\sum_{m=1}^{3} m + \sum_{m=1}^{3} \frac{2}{3} = \frac{2}{3}\cdot\frac{3(4)}{2} + \frac{2}{3}(3) = 4 + 2 = \boxed{6}$

Definition. (Left, Right and Midpoint Riemann Sums in Sigma Notation) Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** for f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- $\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$.
- $\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **midpoint Riemann sum** if $x_k^* = a + (k - 1/2)\Delta x$.

Example. For the function $f(x) = 3x^2$, find a formula for the upper sum obtained by dividing the interval $[0, 1]$ into n equal subintervals.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_k = 0 + k\Delta x = \frac{k}{n}$$

$$f(x_k) = 3\left(\frac{k}{n}\right)^2$$

$$\sum_{k=1}^{n} 3\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \frac{3}{n^3}\sum_{k=1}^{n} k^2 = \frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \boxed{\frac{2n^2+3n+1}{2n^2}}$$

Now take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 3x^2$ over $[0, 1]$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} 3\left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{2n^2+3n+1}{2n^2} = \boxed{1}$$

Example. For the function $f(x) = 2x$, find a formula for the upper sum obtained by dividing the interval $[0, 3]$ into n equal subintervals.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$x_k = 0 + k\Delta x = \frac{3k}{n}$$

$$f(x_k) = 2\left(\frac{3k}{n}\right)$$

$$\sum_{k=1}^{n} 2\left(\frac{3k}{n}\right)\frac{3}{n} = \frac{18}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{18}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{9n+9}{n}$$

Now take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 2x$ over $[0, 3]$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{3k}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \frac{9n+9}{n} = \boxed{9}$$