$\sin^2(x) + \cos^2(x) = 1$

MATH 1040 Review

For the following functions, find their derivatives:

$$y = \sqrt[7]{x^3} - \pi e^x + x^e + 3e^{-x}$$

$x^{3/7}$

$$y' = \frac{3}{7}x^{-4/7} - \pi e^x + e x^{e-1} - 3e^{-x}$$

$$f(x) = \left(\frac{1-\sin(x)}{1+\cos(x)}\right)$$

$$f'(x) = \frac{(1+\cos(x))(-\cos(x)) - (1-\sin(x))(-\sin(x))}{(1+\cos(x))^2}$$

$$= \frac{-\cos(x) - \cos^2(x) + \sin(x) - \sin^2(x)}{(1+\cos(x))^2}$$

$$= \frac{\sin(x) - \cos(x) - 1}{(1+\cos(x))^2}$$

$$g(x) = \left(\frac{x^2+3x+1}{e^x}\right)$$

$$g'(x) = \frac{e^x(2x+3) - (x^2+3x+1)e^x}{e^{2x}}$$

$$= \frac{-x^2 - x + 2}{e^x}$$

$$h(y) = -5\cot\left(3e^{4y}\right) + e^{\pi}$$

$$h'(y) = 5\csc^2(3e^{4y})3e^{4y}(4)$$

$$= 60\csc^2(3e^{4y})e^{4y}$$

Find $f''(x)$ for $f(x) = \tan(x)$

$$f'(x) = \sec^2(x)$$

$$f''(x) = 2\sec(x)\left[\sec(x)\tan(x)\right]$$

$$= 2\sec^2(x)\tan(x)$$

$$= x(5-x^2)^{1/2}$$

Find the equation of the line tangent to $\ell(x) = x\sqrt{5-x^2}$ at the point $(1, 2)$.

$$y = f'(a)(x-a) + f(a)$$

$$\ell(1) = 1\sqrt{5-1} = \sqrt{4} = 2$$

$$\ell'(x) = \sqrt{5-x^2} - \frac{x^2}{\sqrt{5-x^2}} \Rightarrow \ell'(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\Rightarrow \boxed{y = \frac{3}{2}(x-1) + 2}$$

Where is the tangent line of $u = \frac{1}{\sqrt{x}} = x^{-1/2}$ parallel to the line $y = -4x - 3$?

Find where $u' = -4$

$$u' = -\frac{1}{2\sqrt{x^3}} \overset{\text{set}}{=} -4$$

$$\Rightarrow \frac{1}{-2} = -4\sqrt{x^3}$$

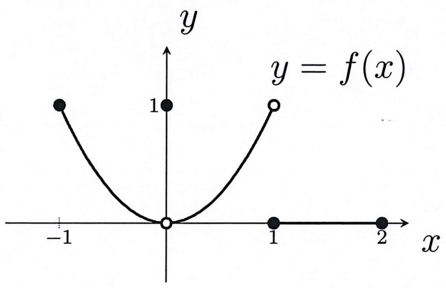
$$\Rightarrow \frac{1}{8} = \sqrt{x^3}$$

$$\Rightarrow \boxed{\frac{1}{2} = x}$$

Note: Limits will be on Test 1 and the final exam.

Example.

Using the graph below, evaluate each limit:



$\lim_{x\to -1^+} f(x) = 1$ $\quad$ $\lim_{x\to 2^-} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$ $\quad$ $\lim_{x\to 0^+} f(x) = 0$ $\quad$ $\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 1^-} f(x) = 1$ $\quad$ $\lim_{x\to 1^+} f(x) = 0$ $\quad$ $\lim_{x\to 1} f(x)$ DNE

State the intervals of continuity on $[-1, 2]$.

$f(x)$ is continuous on $(-1, 0)$, $(0, 1)$ and $(1, 2)$

Example. Algebraically, evaluate the following limits

$$\lim_{x\to 0}\left(\sin^2 x + \sec x\right)$$

$$= \lim_{x\to 0} \sin^2(x) + \frac{1}{\cos(x)}$$

$$= 0^2 + 1 = \boxed{1}$$

$$\lim_{y\to 0}\frac{5y^3+8y^2}{3y^4-16y^2}$$

$$= \lim_{y\to 0}\frac{y^2(5y+8)}{y^2(3y^2-16)}$$

$$= \lim_{y\to 0}\frac{5y+8}{3y^2-16} = \frac{8}{-16} = \boxed{-\frac{1}{2}}$$

$$|2x-1| = \begin{cases} 2x-1, & 2x-1 \geq 0 \to x \geq \tfrac{1}{2} \\ -(2x-1), & x < \tfrac{1}{2} \end{cases}$$

$$\lim_{x\to\frac{1}{2}^-} \frac{4x-2}{|2x^3-x^2|}$$

$$= \lim_{x\to\frac{1}{2}^-} \frac{2(2x-1)}{x^2|2x-1|}$$

$$= \lim_{x\to\frac{1}{2}^-} \frac{2(2x-1)}{-x^2(2x-1)}$$

$$= \lim_{x\to\frac{1}{2}^-} \frac{2}{-x^2} = \frac{2}{-1/4} = \boxed{-8}$$

$$\lim_{x\to0} \frac{1-\cos x}{\cos^2 x - 3\cos x + 2}$$

Let $y = \cos(x)$
If $x = 0$, $y = 1$

$$\lim_{y\to1} \frac{1-y}{y^2-3y+2}$$

$$\lim_{y\to1} \frac{1-y}{(y-1)(y-2)} = \lim_{y\to1} \frac{-1}{y-2} = \frac{-1}{-1} = \boxed{1}$$

$$\lim_{x\to0} \frac{x}{\sqrt{5x+1}-1} \left(\frac{\sqrt{5x+1}+1}{\sqrt{5x+1}+1}\right)$$

$$= \lim_{x\to0} \frac{x(\sqrt{5x+1}+1)}{(5x+1)-(1)}$$

$$= \lim_{x\to0} \frac{x(\sqrt{5x+1}+1)}{5x}$$

$$= \lim_{x\to0} \frac{\sqrt{5x+1}+1}{5} = \boxed{\frac{2}{5}}$$

$$\lim_{x\to0} \frac{e^{4x}-1}{e^x-1}$$

$$= \lim_{x\to0} \frac{(e^{2x}-1)(e^{2x}+1)}{e^x-1}$$

$$= \lim_{x\to0} \frac{(e^x-1)(e^x+1)(e^{2x}+1)}{e^x-1}$$

$$= \lim_{x\to0} (e^x+1)(e^{2x}+1) = (2)(2) = \boxed{4}$$

$$\underbrace{\lim_{x\to0} \frac{\sin(x)}{x}} = 1$$

Shown in MATH 1040 using the squeeze theorem.

$$\lim_{x\to0} \frac{\tan(3x)}{5x}$$

$$= \lim_{x\to0} \frac{\sin(3x)}{\cos(3x)} \frac{1}{5x}$$

$$= \frac{1}{5} \lim_{x\to0} \frac{\sin(3x)}{x}\left(\frac{3}{3}\right) \underbrace{\lim_{x\to0} \frac{1}{\cos(3x)}}_{1}$$

$$= \frac{3}{5} \underbrace{\lim_{x\to0} \frac{\sin(3x)}{3x}}_{1}$$

$$= \boxed{\frac{3}{5}}$$

$$\lim_{x\to\infty} \frac{4x^3+1}{2x^3+\sqrt{16x^6+1}} \left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to\infty} \frac{4 + 1/x^3}{2 + \frac{1}{x^3}\sqrt{16x^6+1}}$$

$$= \lim_{x\to\infty} \frac{4 + 1/x^3}{2 + \sqrt{\frac{1}{x^6}}\sqrt{16x^6+1}}$$

$$= \lim_{x\to\infty} \frac{4 + 1/x^3}{2 + \sqrt{16 + 1/x^6}} = \frac{4}{2+\sqrt{16}} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

$$\lim_{x\to-\infty} \frac{4x^3+1}{2x^3+\sqrt{16x^6+1}} \left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to-\infty} \frac{4 + 1/x^3}{2 + \frac{1}{x^3}\sqrt{16x^6+1}}$$

$$= \lim_{x\to-\infty} \frac{4 + 1/x^3}{2 - \sqrt{\frac{1}{x^6}}\sqrt{16x^6+1}}$$

$$= \lim_{x\to-\infty} \frac{4 + 1/x^3}{2 - \sqrt{16 + 1/x^6}} = \frac{4}{2-\sqrt{16}} = \frac{4}{-2} = \boxed{-2}$$

$$\lim_{x\to-\infty} \left(x + \sqrt{x^2+2x}\right) \left(\frac{x - \sqrt{x^2+2x}}{x - \sqrt{x^2+2x}}\right)$$

$$= \lim_{x\to-\infty} \frac{(x^2) - (x^2+2x)}{x - \sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty} \frac{-2x}{x - \sqrt{x^2+2x}} \left(\frac{1/x}{1/x}\right)$$

$$= \lim_{x\to-\infty} \frac{-2}{1 - \frac{1}{x}\sqrt{x^2+2x}} = \lim_{x\to-\infty} \frac{-2}{1 - \left(-\sqrt{\frac{1}{x^2}}\right)\sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty} \frac{-2}{1 + \sqrt{1 + \frac{2}{x}}} = \frac{-2}{2} = \boxed{-1}$$

$$\lim_{t \to -2^-} \frac{t^3 - 5t^2 + 6t}{t^4 - 4t^2}$$

$$= \lim_{t \to -2^-} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)}$$

$$= \lim_{t \to -2^-} \frac{t(t-3)}{t^2(t+2)} = \frac{-2(-5)}{4(\text{sm} -)}$$

$$= \boxed{-\infty}$$

$$\lim_{t \to -2^+} \frac{t^3 - 5t^2 + 6t}{t^4 - 4t^2}$$

$$= \lim_{t \to -2^+} \frac{t(t-3)}{t^2(t+2)}$$

$$= \frac{-2(-5)}{4(\text{sm} +)} = \boxed{\infty}$$

$$\lim_{x \to -\infty} \frac{3x + 7}{x^2 - 4} \left(\frac{1/x^2}{1/x^2} \right)$$

$$= \lim_{x \to -\infty} \frac{\frac{3}{x} + 7/x^2}{1 - 4/x^2} = \frac{0}{1} = \boxed{0}$$

Find the equation of the slant (oblique) asymptote of $f(x) = \dfrac{3x^5 + x^4 + 2x^2 + 1}{x^4 + 3}$.

$$
\begin{array}{r l}
 & 3x + 1 \\
x^4 + 3 & \overline{)\, 3x^5 + x^4 + 0x^3 + 2x^2 + 0x + 1} \\
 & -(3x^5 \qquad\qquad\qquad\qquad + 9x) \\
\hline
 & x^4 + 0x^3 + 2x^2 - 9x + 1 \\
 & -(x^4 \qquad\qquad\qquad + 3) \\
\hline
 & 2x^2 - 9x - 2
\end{array}
$$

Thus

$$f(x) = \frac{3x^5 + x^4 + 2x^2 + 1}{x^4 + 3} = (3x+1) + \frac{2x^2 - 9x - 2}{x^4 + 3}$$

and since

$$\lim_{x \to \pm\infty} \frac{2x^2 - 9x - 2}{x^4 + 3} = 0$$

then

$$\lim_{x \to \pm\infty} f(x) = 3x + 1$$

so the slant asymptote is

$$\boxed{y = 3x + 1}$$

$f(x)$ is continuous at $x = a$
if $f(a) = \lim_{x \to a} f(x)$.

Find k such that $f(x)$ is continuous at $x = 1$:

$$f(x) = \begin{cases} k \tan\left(\frac{\pi x}{3}\right), & x \geq 1 \\ x - 2, & x < 1 \end{cases}$$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x - 2 = -1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} k \tan\left(\frac{\pi x}{3}\right) = k \tan\left(\frac{\pi}{3}\right) = k \frac{1}{\sqrt{3}}$

$\Rightarrow -1 = k \frac{1}{\sqrt{3}}$

$\Rightarrow \boxed{-\sqrt{3} = k}$

Find c such that $f(x)$ is continuous:

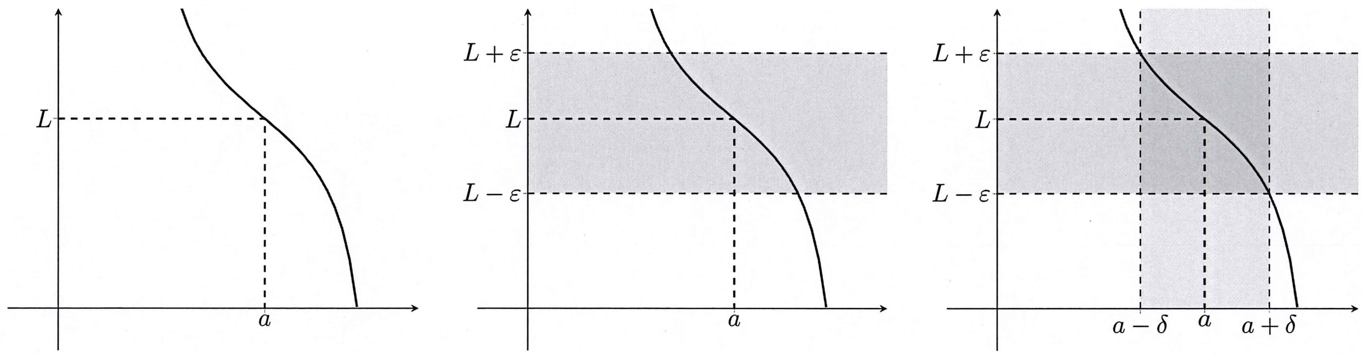
$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

$\lim_{x \to 0} \frac{\sin^2 3x}{x^2} = \lim_{x \to 0} \left(\frac{\sin 3x}{x}\right)^2 \left(\frac{9}{9}\right) = 9 \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)^2$

$= 9 \cdot (1) = 9$

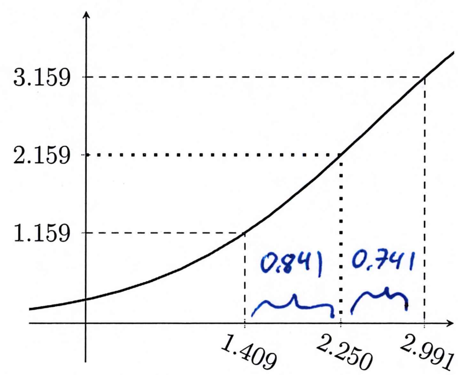
$f(0) = c \qquad \Rightarrow \boxed{c = 9}$

$\delta - \varepsilon$ proofs:



Example. Use the graph of f below to find a number δ such that if $0 < |x - 2.25| < \delta$ then $|f(x) - 2.159| < 1$.

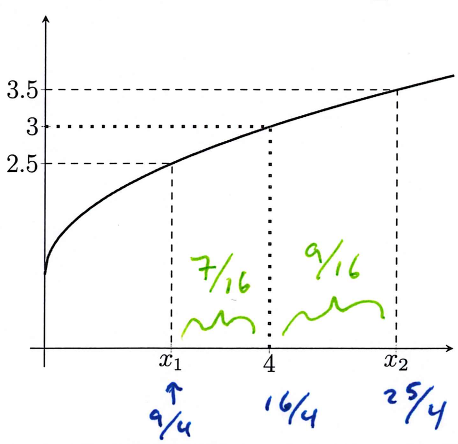
$$\begin{array}{r} 2.991 \\ -\ 2.250 \\ \hline 0.741 \end{array} \qquad \begin{array}{r} 2.250 \\ -\ 1.409 \\ \hline 0.841 \end{array}$$



pick $\delta = \min\{0.741, 0.841\}$

$\Rightarrow \boxed{\delta = 0.741}$

Example. Use the graph of $g(x) = \sqrt{x}+1$ to help find a number δ such that if $|x - 4| < \delta$ then $\left|(\sqrt{x}+1) - 3\right| < \dfrac{1}{2}$.



$$|(\sqrt{x}+1) - 3| < \tfrac{1}{2}$$

$$-\tfrac{1}{2} < \sqrt{x} + 1 - 3 < \tfrac{1}{2}$$

$$\frac{3}{2} < \sqrt{x} < \frac{5}{2}$$

$$\frac{9}{4} < x < \frac{25}{4}$$

$\Rightarrow \boxed{\delta = 7/16}$

Example. Algebraically, prove the following limits:

$$\lim_{x\to 3}(10-3x)=1$$

Want $|(10-3x)-1|<\varepsilon$

$\Rightarrow |9-3x|<\varepsilon$

$|-3(x-3)|<\varepsilon$

$3|x-3|<\varepsilon$

$|x-3|<\varepsilon/3 \rightarrow \boxed{\delta=\varepsilon/3}$

Let $\delta=\varepsilon/3$, then for $\varepsilon>0$

$|x-3|<\delta$ means

$|(10-3x)-1| = |9-3x|$

$= 3|x-3|$

$< 3(\varepsilon/3) = \varepsilon.$

$$\lim_{x\to 14}\left(2-\frac{2}{7}x\right)=-2$$

Want $|(2-\frac{2}{7}x)-(-2)|<\varepsilon$

$|4-\frac{2}{7}x|<\varepsilon$

$|(-\frac{7}{2})(x-14)|<\varepsilon$

$\frac{7}{2}|x-14|<\varepsilon \rightarrow$ $|x-14|<\varepsilon\cdot\frac{2}{7}$

Let $\delta=\frac{2\varepsilon}{7}$, then for $\varepsilon>0$

$|x-14|<\delta$ means

$|(2-\frac{2}{7}x)-(-2)| = |4-\frac{2}{7}x|$

$=\frac{7}{2}|x-14|$

$<\frac{7}{2}\delta$

$=\frac{7}{2}\left(\frac{2\varepsilon}{7}\right)=\varepsilon.$

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3}=7$$

Want $\left|\left(\frac{x^2+x-12}{x-3}\right)-7\right|<\varepsilon$

$\left|\frac{(x-3)(x+4)}{x-3}-7\right|<\varepsilon$

$|x+4-7|<\varepsilon$

$|x-3|<\varepsilon \rightarrow \delta=\frac{\varepsilon}{1}$

Let $\delta=\varepsilon$, then for $\varepsilon>0$

$|x-3|<\delta$ means

$\left|\frac{x^2+x-12}{x-3}-7\right| = \left|\frac{(x-3)(x+4)}{x-3}-7\right|$

$= |x+4-7|$

$= |x-3|$

$<\delta=\varepsilon.$

Rates of change

Example. Find the average rate of change of $f(x) = 3x^2 - 4x$ over the interval $[-1, 4]$ and the instantaneous rate of change at $x = 3$.

A.R.O.C. $\frac{f(4) - f(-1)}{4 - (-1)} = \frac{32 - 7}{5} = \frac{25}{5} = \boxed{5}$

IROC $f'(3) = 6(3) - 4 = \boxed{14}$

Limit definition of the derivative Recall the following definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example. Use the limit definition of the derivative to find $f'(x)$ when $f(x) = -\frac{1}{x^2}$ and then evaluate $f'(3)$.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{x^2}{x^2}\right)\left[-\frac{1}{(x+h)^2}\right] - \left[-\frac{1}{x^2}\right]\left(\frac{(x+h)^2}{(x+h)^2}\right)}{h}$

$= \lim_{h \to 0} \frac{-x^2 + (x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-x^2 + x^2 + 2xh + h^2}{hx^2(x+h)^2}$

$= \lim_{h \to 0} \frac{2xh + h^2}{hx^2(x+h^2)} = \lim_{h \to 0} \frac{2x + h}{x^2(x+h)^2} = \frac{2x}{x^4} = \boxed{\frac{2}{x^3}}$

$f'(3) = \frac{2}{3^3} = \boxed{\frac{2}{27}}$

Example. Use the limit definition of the derivative to find $f'(x)$ when $f(x) = \frac{1-x}{2x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{x}{x}\right)\left[\frac{1-(x+h)}{2(x+h)}\right] - \left[\frac{1-x}{2x}\right]\left(\frac{(x+h)}{(x+h)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{x - x(x+h) - (x+h) + x(x+h)}{2hx(x+h)} = \lim_{h \to 0} \frac{-h}{2hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{2x(x+h)} = \boxed{\frac{-1}{2x^2}}$$

Function	Derivative
Increasing	Positive
Decreasing	Negative
Max/Min	Zero
Inflection point	max/min
Constant	0
Linear	constant
Quadratic	linear

A function is not differentiable wherever it has a

1. discontinuity
2. sharp corner/cusp
3. vertical tangent

The Chain Rule and Product Rule

Example. Find the derivatives of the following functions

$y = \cos\left(2x^5 + 7x\right)$

$y' = -\sin\left(2x^5+7x\right)\left(10x^4+7\right)$

$p(x) = \sqrt{2}x + \sqrt{3x}$

$(3x)^{1/2}$

$p'(x) = \sqrt{2} + \frac{1}{2}(3x)^{-1/2}(3)$

$= \sqrt{2} + \dfrac{3}{2\sqrt{3x}}$

$= \sqrt{2} + \dfrac{\sqrt{3}}{2\sqrt{x}}$

$y = x^{2e} - e^{\frac{3x-2}{x^2-3x}}$

$y' = 2e\,x^{2e-1} - e^{\frac{3x-2}{x^2-3x}}\left(\dfrac{(x^2-3x)(3)-(3x-2)(2x-3)}{(x^2-3x)^2}\right)$

$= 2e\,x^{2e-1} - \dfrac{e^{\frac{3x-2}{x^2-3x}}}{(x^2-3x)^2}\left(3x^2-9x-6x^2+13x-6\right)$

$= 2e\,x^{2e-1} - \dfrac{e^{\frac{3x-2}{x^2-3x}}}{(x^2-3x)^2}\left(-3x^2+4x-6\right)$

$y = f\left(\sqrt[3]{g(4x^3)}\right)$

$y' = f'\left(\sqrt[3]{g(4x^3)}\right)\cdot\dfrac{d}{dx}\left[\sqrt[3]{g(4x^3)}\right]$

$= f'\left(\sqrt[3]{g(4x^3)}\right)\cdot\dfrac{1}{3}\left[g(4x^3)\right]^{-2/3}\dfrac{d}{dx}\left[g(4x^3)\right]$

$= f'\left(\sqrt[3]{g(4x^3)}\right)\cdot\dfrac{1}{3}\left[g(4x^3)\right]^{-2/3}\cdot g'(4x^3)\cdot 12x^2$

$= \dfrac{4x^2 f'\left(\sqrt[3]{g(4x^3)}\right)g'(4x^3)}{\left[g(4x^3)\right]^{2/3}}$

$$\frac{d}{dx}\left[\frac{f(x)-3g(x)}{2}\right] = \frac{1}{2} f(x) - \frac{3}{2} g(x)$$

$$= \frac{1}{2} f'(x) - \frac{3}{2} g'(x)$$

$$\frac{d}{dx}\left[\frac{x[g(x)]^2}{h(x)}\right]$$

$$= \frac{h(x)\frac{d}{dx}\left[x[g(x)]^2\right] - x[g(x)]^2\frac{d}{dx}\left[h(x)\right]}{(h(x))^2}$$

$$= \frac{h(x)\left[[g(x)]^2 + 2x[g(x)]g'(x)\right] - x[g(x)]^2 h'(x)}{[h(x)]^2}$$

$$h(\theta) = \sqrt[3]{-\theta + \cot(9+2\theta)}$$

$$h'(\theta) = \frac{1}{3}\left(-\theta + \cot(9+2\theta)\right)^{-2/3}\left(-1 - \csc^2(9+2\theta)\,2\right)$$

$$= \frac{-(1+2\csc^2(9+2\theta))}{3\left(-\theta+\cot(9+2\theta)\right)^{2/3}}$$

$$y(\theta) = \tan^2(\cot(3\theta))$$

$$y'(\theta) = 2\tan(\cot(3\theta))\frac{d}{d\theta}[\ ]$$

$$= 2\tan(\cot(3\theta))\sec^2(\cot(3\theta))\frac{d}{d\theta}[\ ]$$

$$= -2\tan(\cot(3\theta))\sec^2(\cot(3\theta))\csc^2(3\theta)(3)$$

$$\frac{d}{dx}[f\cdot g\cdot h] = f'gh + fg'h + fgh'$$

$$y = 3x^2(e^{-x}+2)^4\tan(3x+2)$$

$$y' = 6x(e^{-x}+2)^4\tan(3x+2) + 3x^2\,4(e^{-x}+2)^3 e^{-x}(-1)\tan(3x+2) + 3x^2(e^{-x}+2)^4\sec^2(3x+2)(3)$$

$$h(x) = \frac{-1}{2\sqrt[5]{\csc^2(4x)}} = -\frac{1}{2}(\csc(4x))^{-2/5}$$

$$h'(x) = \frac{1}{5}(\csc(4x))^{-7/5}(-\csc(4x)\cot(4x))(4)$$

$$= -\frac{4\csc(4x)\cot(4x)}{5(\csc(4x))^{7/5}}$$

$$= -\frac{4\cot(4x)}{5(\csc(4x))^{2/5}}$$

Example. Let $f(1) = 3$, $f'(1) = 4$, $g(1) = 2$, $g'(1) = 6$, and $g(3) = 5$, $g'(3) = 2$.
Now, let $H(x) = (g\circ f))(x) = g(f(x))$ and find $H'(1)$.

$$H'(x) = \frac{d}{dx}[g(f(x))] = g'(f(x))\cdot f'(x)$$

$$\Rightarrow H'(1) = g'(f(1))\cdot f'(1)$$

$$= g'(3)\cdot 4 = 2\cdot 4 = 8$$

Example. Find $\dfrac{d^2}{d\theta^2}\left[\sin^2(3\theta)\right]$.

$$\frac{d}{d\theta}[\sin^2(3\theta)] = 2\sin(3\theta)\cos(3\theta)\,3$$

$$\frac{d^2}{d\theta^2}[\sin^2(3\theta)] = \frac{d}{d\theta}[6\sin(3\theta)\cos(3\theta)]$$

$$= 6\cos(3\theta)\cos(3\theta) + 6\sin(3\theta)[-\sin(3\theta)]$$

$$= 6[\cos^2(3\theta) - \sin^2(3\theta)]$$