JIT 13.1: Solving Linear Equations Involving Derivatives

Recall that for a function $f(x)$, we can denote the derivative as $\dfrac{df}{dx}$.

Example. Solve the following for $\dfrac{df}{dx}$:

$$2 + 3\frac{dy}{dx} = 1$$

$$3\frac{dy}{dx} = -1$$

$$\boxed{\frac{dy}{dx} = -1/3}$$

$$2x + 3y' = 3x - 5y'$$

Note:

$$\frac{dy}{dx} = y'$$

$$8y' = x$$

$$y' = \frac{1}{8}x$$

$$\boxed{\frac{dy}{dx} = \frac{1}{8}x}$$

$$x + 2y\frac{dy}{dx} = -\frac{dy}{dx} + y$$

$$2y\frac{dy}{dx} + \frac{dy}{dx} = y - x$$

$$\frac{dy}{dx}(2y+1) = y - x$$

$$\boxed{\frac{dy}{dx} = \frac{y-x}{2y+1}}$$

$$5xy + 4\frac{dy}{dx} = 3x^2 - 2xy^2\frac{dy}{dx}$$

$$4\frac{dy}{dx} + 2xy^2\frac{dy}{dx} = 3x^2 - 5xy$$

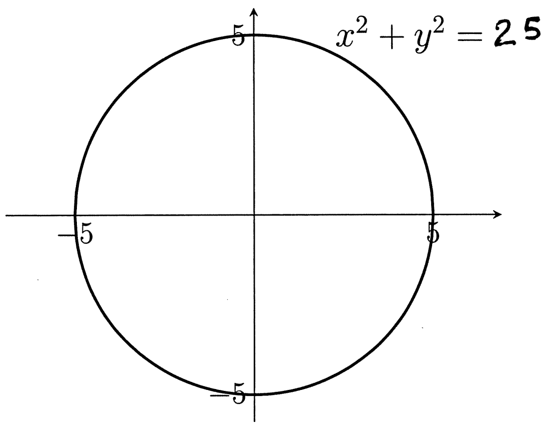
$$\frac{dy}{dx}\left[4 + 2xy^2\right] = 3x^2 - 5xy$$

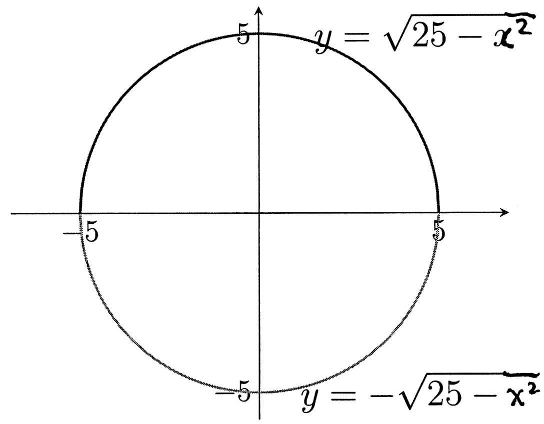
$$\boxed{\frac{dy}{dx} = \frac{3x^2 - 5xy}{4 + 2xy^2}}$$

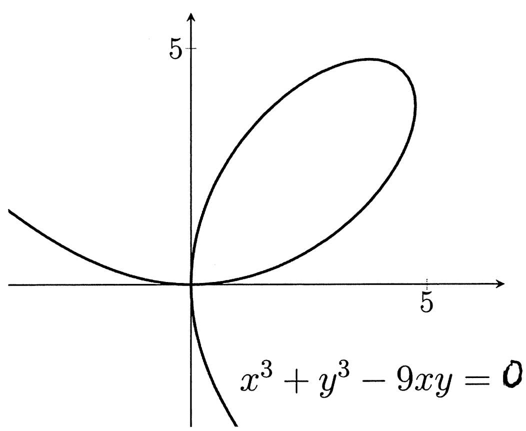
3.8: Implicit Differentiation

Up until now, we have only taken the derivatives of *explicitly* defined functions (functions defined in terms of only x).
An *implicitly* defined function will be written in terms of both x and y:

$$x^2 + y^2 = 25$$







Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example. Find the derivatives of the following by rewriting each function explicitly before taking the derivative, and by using implicit differentiation. Compare the results.

$y^2 = x$ $\longrightarrow$ (1) $y = \pm\sqrt{x} = \pm x^{1/2}$

$\Rightarrow \frac{dy}{dx} = \pm \frac{1}{2} x^{-1/2} = \boxed{\pm \frac{1}{2\sqrt{x}}}$

(2) $2y \frac{dy}{dx} = 1 \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2y}}$

Since $y = \pm\sqrt{x}$, these are the same

$\sqrt{x} + \sqrt{y} = 4$

(1) $y = (4 - x^{1/2})^2$

$\frac{dy}{dx} = 2(4 - x^{1/2})^1 (-\frac{1}{2} x^{-1/2})$

$= -\frac{4 - \sqrt{x}}{\sqrt{x}}$

(2) $\frac{1}{2} x^{-1/2} + \frac{1}{2} y^{-1/2} \frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{2\sqrt{y}} \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

Since $\sqrt{y} = 4 - \sqrt{x}$, these are the same.

Example. Find the derivatives of the following equations:

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

$$x^3 + y^3 - 9xy = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left((1)y + x\frac{dy}{dx}\right) = 0$$

$$3y^2\frac{dy}{dx} - x\frac{dy}{dx} = 9y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - x}}$$

$$2y = x^2 + \sin y$$

$$2\frac{dy}{dx} = 2x + \cos(y)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{2 - \cos(y)}}$$

$$x^2y^2 + x\sin y = 4$$

$$2xy^2 + x^2(2y)\frac{dy}{dx} + \sin(y) + x\cos(y)\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\sin(y) + 2xy^2}{2x^2y + x\cos(y)}}$$

$$y^5 + x^2y^3 = 1 + x^4y$$

$$5y^4\frac{dy}{dx} + 2xy^3 + x^2 3y^2\frac{dy}{dx} = 4x^3y + x^4\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[5y^4 + 3x^2y^2 - x^4\right] = 4x^3y - 2xy^3$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}}$$

$$1 + x = \sin\left(xy^2\right)$$

$$1 = \cos(xy^2)\left[y^2 + x2y\frac{dy}{dx}\right]$$

$$\sec(xy^2) = y^2 + 2xy\frac{dy}{dx}$$

$$\boxed{\frac{\sec(xy^2) - y^2}{2xy} = \frac{dy}{dx}}$$

Example. Find the derivatives of the following equations:

$$x^3 - xy + y^3 = 1$$

$$3x^2 - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2 - x) = y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}}$$

$$xe^y = x - y$$

$$e^y + xe^y\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}\left[xe^y + 1\right] = 1 - e^y$$

$$\boxed{\frac{dy}{dx} = \frac{1-e^y}{xe^y+1}}$$

$$\frac{1}{x} + \frac{1}{y} = 1$$

$$x^{-1} + y^{-1} = 1$$

$$-x^{-2} - y^{-2}\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{y^2}{x^2}}$$

$$x^2 - 2x^3y^4 + y^2 = 30y$$

$$2x - 2\left(3x^2y^4 + x^3 4y^3\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 30\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(-8x^3y^3 + 2y - 30\right) = -2x + 6x^2y^4$$

$$\boxed{\frac{dy}{dx} = \frac{x - 3x^2y^4}{4x^3y^3 - y + 15}}$$

$$\tan(xy) = x + y$$

$$\sec^2(xy)\left(y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x\sec^2(xy) - 1\right) = 1 - \sec^2(xy)\,y$$

$$\boxed{\frac{dy}{dx} = \frac{1 - y\sec^2(xy)}{x\sec^2(xy) - 1}}$$

$$x^2 = \frac{x-y}{x+y} \Rightarrow (x+y)x^2 = x - y$$

$$\left(1 + \frac{dy}{dx}\right)x^2 + (x+y)2x = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}(x^2+1) = 1 - x^2 - 2x(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{1 - x^2 - 2x(x+y)}{x^2+1}}$$

Example. Find the second derivative implicitly for the following equations:

$$y^2 - 2x = 1 - 2y$$

$$2y\frac{dy}{dx} - 2 = -2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[2y+2\right] = 2$$

$$\frac{dy}{dx} = \boxed{\frac{1}{y+1}} = (y+1)^{-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -(y+1)^{-2}\frac{dy}{dx}$$

$$= -\frac{1}{(y+1)^2}\cdot\frac{1}{y+1}$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{1}{(y+1)^3}}$$

$$xy = \cot(xy)$$

$$y + x\frac{dy}{dx} = -\csc^2(xy)\left(y + x\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left[x + x\csc^2(xy)\right] = -y\csc^2(xy) - y$$

$$\frac{dy}{dx} = \frac{-y(1+\csc^2(xy))}{x(1+\csc^2(xy))} = \boxed{\frac{-y}{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{x\left(\frac{dy}{dx}\right) - y}{x^2}$$

$$= -\frac{x\left(\frac{-y}{x}\right) - y}{x^2} = \boxed{\frac{2y}{x^2}}$$

$$x^3 + y^3 = 1$$

$$3x^2 + 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \boxed{\frac{-x^2}{y^2}}$$

$$\frac{d^2y}{dx^2} = -\frac{y^2(2x) - x^2\left(2y\frac{dy}{dx}\right)}{y^4}$$

$$= -\frac{\left(\frac{2xy^3}{y} + \frac{2x^4}{y}\right)}{y^4} = \frac{-2x}{y^5}\underbrace{(y^3+x^3)}_{1}$$

$$= \boxed{\frac{-2x}{y^5}}$$

$$x = e^y$$

$$1 = e^y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \boxed{e^{-y}}$$

$$\frac{d^2y}{dx^2} = -e^{-y}\frac{dy}{dx}$$

$$= \boxed{-e^{-2y}}$$

Example. Find the equation of all lines tangent to the curve $x + y^3 - y = 1$ at $x = 1$.

$$1 + 3y^2 \frac{dy}{dx} - \frac{dy}{dx} = 0$$

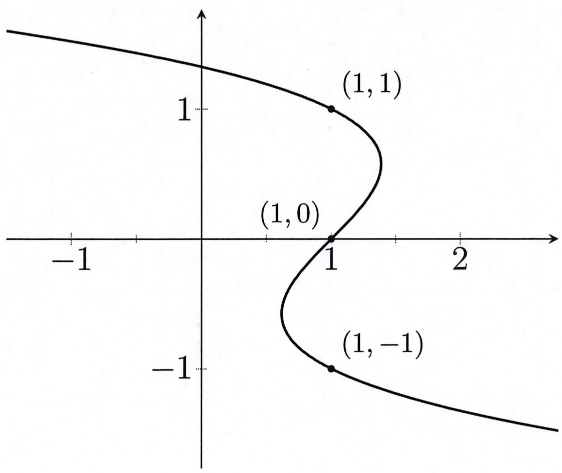
$$\frac{dy}{dx} = \frac{1}{1-3y^2}$$

$x = 1$

$\Rightarrow 1 + y^3 - y = 1$

$y(y^2-1) = 0$

$y = -1, 0, 1$



$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{1}{-2}$$

$$y + 1 = -\frac{1}{2}(x-1)$$

$$\left.\frac{dy}{dx}\right|_{(1,0)} = 1$$

$$y - 0 = 1(x-1)$$

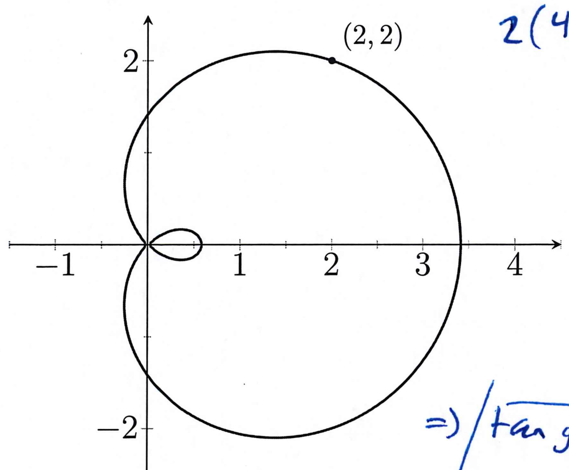
$$\left.\frac{dy}{dx}\right|_{(1,1)} = -\tfrac{1}{2}$$

$$y - 1 = -\tfrac{1}{2}(x-1)$$

Example. Find the equation of the tangent line and normal line for $\left(x^2 + y^2 - 2x\right)^2 = 2\left(x^2 + y^2\right)$ at $(x, y) = (2, 2)$.

$$2(x^2+y^2-2x)\left(2x + 2y\frac{dy}{dx} - 2\right) = 2\left(2x + 2y\frac{dy}{dx}\right)$$

- we can evaluate at $(2,2)$



$$2(4+4-4)\left(4 + 4\left.\frac{dy}{dx}\right|_{(2,2)} - 2\right) = 2\left(4 + 4\left.\frac{dy}{dx}\right|_{(2,2)}\right)$$

$$16 + 32\left.\frac{dy}{dx}\right|_{(2,2)} = 8 + 8\left.\frac{dy}{dx}\right|_{(2,2)}$$

$$24\left.\frac{dy}{dx}\right|_{(2,2)} = -8$$

$$\left.\frac{dy}{dx}\right|_{(2,2)} = -\tfrac{1}{3}$$

$\Rightarrow$ tangent line $\boxed{y - 2 = -\tfrac{1}{3}(x-2)}$

$\Rightarrow$ normal line $\boxed{y - 2 = 3(x-2)}$