4.6 Linear Approximation and Differentials

Example. Find the equation of the line tangent to $y = \underbrace{1 + \sin(x)}_{f(x)}$ at $a = 0$. State the line in the slope–intercept form.

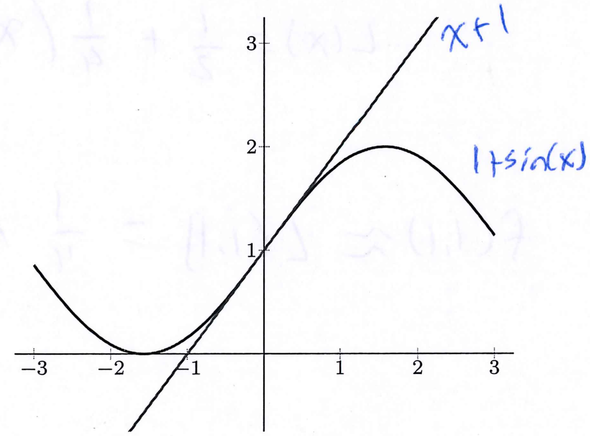
$f(0) = 1$

$f'(x) = \cos(x) \quad f'(0) = 1$

$y - f(a) = f'(a)(x - a)$

$y - 1 = 1(x - 0)$

$\Rightarrow \boxed{y = x + 1}$



> **Definition.**
> **Linear Approximation of f at a.**
> Suppose f if differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function.
>
> $$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

Example. Consider the function $f(x) = \dfrac{x}{x+1}$. Find the linearization at $a = 1$. Use the linearization to estimate $f(1.1)$ and compare with the true value of $f(1.1)$.

$L(x) = f(a) + f'(a)(x-a)$

$f(x) = \dfrac{x}{x+1}$ $\qquad f(1) = \dfrac{1}{2}$

$f'(x) = \dfrac{1}{(x+1)^2}$ $\qquad f'(1) = \dfrac{1}{4}$

$L(x) = \dfrac{1}{2} + \dfrac{1}{4}(x-1) = \boxed{\dfrac{1}{4} + \dfrac{x}{4}}$

$f(1.1) \approx L(1.1) = \dfrac{1}{4} + \dfrac{1.1}{4} = \boxed{\dfrac{21}{40}}$

Example. Find the linearization $L(x)$ of $f(x) = e^{3x-6}$ at $a = 2$.

$f(2) = 1$

$f'(x) = 3e^{3x-6}$ $\quad f'(2) = 3$

$L(x) = f(2) + f'(2)(x-2)$

$= 1 + 3(x-2)$

$\boxed{L(x) = 3x-5}$

Example. Find the linearization $L(x)$ of $f(x) = 9(4x+11)^{\frac{2}{3}}$ at $a = 4$.

$f(4) = 81$

$f'(x) = 6(4x+11)^{-1/3}(4) = \frac{24}{(4x+11)^{1/3}}$ $\quad f'(4) = 8$

$L(x) = f(4) + f'(4)\cdot(x-4)$

$\boxed{L(x) = 8x-49}$

Example. Find a linearization over an interval which will contain the given point x_0. Choose your center at a point *near* x_0 but not at x_0 so that the given function and its derivative are easy to evaluate. Lastly, use the linearization to approximate $f(x_0)$.

a) $f(x) = x^2 + 2x$, $x_0 = 0.1$

$f(0) = 0$

$L(x) = 0 + 2(x-0) = \boxed{2x}$

$f'(x) = 2x+2$ $\quad f'(0) = 2$

$f(0.1) \approx L(0.1) = \boxed{0.2}$

b) $f(x) = \sqrt[3]{x}$, $x_0 = 8.5$.

$f(8) = 2$

$L(x) = 2 + \frac{1}{12}(x-8)$

$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ $\quad f'(8) = \frac{1}{12}$

$= \boxed{\frac{x}{12} + \frac{4}{3}}$

$f(8.5) \approx L(8.5) = \frac{8.5}{12} + \frac{4}{3} = \boxed{\frac{49}{24}}$

Example. Use a linear approximation to estimate $\sqrt{146}$.

$f(x) = \sqrt{x}$ $\quad f(144) = 12$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(144) = \frac{1}{24}$

$L(x) = 12 + \frac{1}{24}(x - 144) = 6 + \frac{x}{24}$

$f(146) \approx L(146) = 6 + \frac{146}{24} = \boxed{\frac{145}{12}}$ $\boxed{\approx 12.08\overline{3}}$

Example. Use a linear approximation to estimate $(1.999)^4$.

$f(x) = x^4$ $\quad f(2) = 16$

$f'(x) = 4x^3$ $\quad f'(2) = 32$

$L(x) = 16 + 32(x-2) = 32x - 48$

$f(1.999) \approx L(1.999) = 32(1.999) - 48 = 63.968 - 48 = \boxed{15.968}$

Example. Use a linear approximation to estimate $\sqrt{\frac{5}{29}}$. $\approx \sqrt{0.172413\ldots}$

$f(x) = \sqrt{x}$ $\quad f(0.16) = \frac{4}{10} = \frac{2}{5}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(0.16) = \frac{5}{4}$

$L(x) = \frac{2}{5} + \frac{5}{4}\left(x - \frac{4}{25}\right) = \frac{1}{5} + \frac{5}{4}x$

$f(5/29) \approx L(5/29) = \frac{1}{5} + \frac{5}{4}\left(\frac{5}{29}\right) = \boxed{\frac{241}{580} \approx 0.4155}$

Example. Find the linearization $L(x)$ of $f(x) = \sqrt{x^2+9}$ at $a = -4$. Use $L(x)$ to estimate $f(-4.1)$ and $\sqrt{23.44}$.

$$f'(x) = \frac{x}{\sqrt{x^2+9}}$$

$$f(-4) = 5$$

$$f'(-4) = -\frac{4}{5}$$

$$L(x) = f(-4) + f'(-4)(x+4)$$

$$= 5 - \frac{4}{5}(x+4) = \boxed{-\frac{4}{5}x + \frac{9}{5}}$$

$$f(-4.1) \simeq L(-4.1) = -\frac{4}{5}(-4.1) + \frac{9}{5} = \boxed{\frac{127}{25}}$$

$$\sqrt{23.44} = f(3.8) = -\frac{4}{5}(3.8) + \frac{9}{5} = \boxed{\frac{121}{25}}$$

Example. Use a linearization to show that 0.05 is a good approximation for ln(1.05).

$f(x) = \ln(1+x)$ $a = 0$ $f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+x}$ $f'(0) = \frac{1}{1+0} = 1$

$\longrightarrow \quad L(x) = 0 + 1(x-0) = x$

$\boxed{L(0.05) = 0.05}$

Alternatively

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$ $a = 1 \Rightarrow \begin{cases} f(1) = 0 \\ f'(1) = 1 \end{cases}$

Example. Find the linearization of the following functions at the given point and use concavity to identify if the linearization is an overestimate or an underestimate.

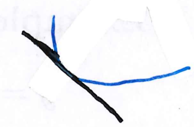
a) $f(x) = \frac{2}{x}$; $a = 1$

$f'(x) = -\frac{2}{x^2}$

$\left.\begin{array}{l} f(1) = 2 \\ f'(1) = -2 \end{array}\right\} \Rightarrow L(x) = 2 - 2(x-1) = 4 - 2x$

$f''(x) = \frac{4}{x^3}$ $\quad f''(1) = 4 > 0 \longrightarrow$ concave up

$\longrightarrow$ underestimate



b) $f(x) = e^{-x}$; $a = \ln(2)$

$f'(x) = -e^{-x}$

$\left.\begin{array}{l} f(\ln(2)) = 2 \\ f'(\ln(2)) = -2 \end{array}\right\} \Rightarrow L(x) = 2 - 2(x - \ln(2))$

$f''(x) = e^{-x}$ $\quad f''(\ln(2)) = 2 > 0 \longrightarrow$ concave up

$\longrightarrow$ underestimate

Summary: Uses of Linear Approximation

1. To approximate f near $x = a$, use

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

2. To approximate the change Δy in the dependent variable when the independent variable x changes from a to $a + \Delta x$, use

$$\Delta y \approx f'(a)\Delta x.$$

Definition. (Differentials)

Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)\,dx.$$

Example. Find the differential dy.

$y = \cos(x^2)$

$dy = -\sin(x^2)\, 2x\, dx$

$y = \sqrt{1 - x^2}$

$dy = \dfrac{-x}{\sqrt{1-x^2}}\, dx$

$y = 4x^2 - 3x + 2$

$dy = (8x-3)\, dx$

$y = x\tan(x^3)$

$dy = \left(\tan(x^3) + x\sec^2(x^3)(3x^2)\right) dx$

$y = \cos^5(x)$

$dy = -5\cos^4(x)\sin(x)\, dx$

$f(x) = \sin^{-1}(x)$

$dy = \dfrac{1}{\sqrt{1-x^2}}\, dx$

Example. Let $y = x^2$

a) Find dy

$dy = 2x\,dx$

b) If $x = 1$ and $dx = 0.01$, find dy.

$dy = 2(1)(0.01) = \boxed{0.02}$

c) Compare dy and Δy at this point.

$\Delta y = f(1.01) - f(1) = 1.0201 - 1 = \boxed{0.0201}$

Example. Let $y = \sqrt{3 + x^2}$

a) Find dy

$dy = \dfrac{x}{\sqrt{3+x^2}}\,dx$

b) If $x = 1$ and $dx = -0.1$, find dy.

$dy = \dfrac{1}{\sqrt{3+1}}(-0.1) = -\dfrac{1}{20} = -0.050$

c) Compare dy and Δy at this point.

$\Delta y = f(0.9) - f(1) \approx 1.7292 - 2 = -0.2708$

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.01) - f(5) = 0.25$. Use linear approximation to estimate the value of $f'(5)$.

$$\Delta y \approx dy = f'(x)\,dx \Rightarrow f'(x) \approx \frac{\Delta y}{dx}$$

$$\Delta y = 0.25$$

$$dx = 5.01 - 5 = 0.01 \Rightarrow f'(x) \approx \frac{0.25}{0.01} = \boxed{25}$$

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.99) = 7$ and $f(6) = 7.002$. Use linear approximation to estimate the value of $f'(6)$.

$$\Delta y = f(6) - f(5.99) = 0.002$$

$$dx = 6 - 5.99 = 0.01$$

$$f'(6) \approx \frac{0.002}{0.01} = \boxed{0.2}$$

Example. Compute dy and Δy for $y = e^x$ when $x = 0$ and $\Delta x = 0.5$.

$\Delta y = e^{0,5} - e^{0} \approx 0,648721$

$dy = f'(0)(0,5) = \frac{1}{2} = 0,5$

Example. Approximate the change in the area of a circle when its radius increases from 2.00 to 2.02 m.

$A = \pi r^2$

$r = 2,00$

$dr = 0,02$

$\Delta A \approx dA = [2\pi(2)](0.02) = 0.08\pi \approx 0,251327 \ m$

Example. Approximate the change in the magnitude of the electrostatic force between two charges when the distance between them increases from $r = 20\,m$ to $r = 21\,m$ ($F(r) = 0.01/r^2$).

$dr = 1m$

$y = F'(r) = \frac{-0,02}{r^3}$

$\Delta y \approx dy = F'(20)(1) = \frac{-0,02}{(20)^3}(1) = \frac{1}{400,000}$

Example. Approximate the change in the volume of a right circular cylinder of fixed radius $r = 20\,cm$ when its height decreases from $h = 12\,cm$ to $h = 11.9\,cm$ ($V(h) = \pi r^2 h$).

$V = \pi r^2 h$

$h = 12$

$dh = -0.1cm$

$V' = \pi r^2$

$\Delta V \approx dV = [\pi(20)^2](-0.1) = -40\pi \approx -125.663706$

Example. Approximate the change in the volume of a right circular cylinder of a right circular cone of fixed height $h = 4\,m$ when its radius increases from $r = 3\,m$ to $r = 3.05\,m$ ($V(r) = \frac{1}{3}\pi r^2 h$).

$r = 3$

$dr = 0.05$

$V'(r) = \frac{2}{3}\pi r h$

$\Delta V \approx dV = \left[\frac{2}{3}\pi(3)(4)\right](0.05) = \frac{2\pi}{5} \approx 1.256637$

Example. The radius of a sphere is measured and found to be $\underbrace{0.7 \text{ inches}}_{r}$ with a possible error in measurement of at most $\underbrace{0.01 \text{ inches}}_{dr}$.

a) What is the maximum error in using the value of the radius to compute the volume of the sphere?

$$V = \frac{4}{3}\pi r^3$$

$$\Delta V \approx dV = (4\pi r^2)\, dr$$

$$dV = 4\pi \left(\frac{7}{10}\right)^2 \left(\frac{1}{100}\right)$$

$$= \frac{196}{10000}\pi = \frac{49}{2500}\pi \approx \boxed{0.061575 \text{ in}^3}$$

b) Find the relative error of the volume: *relative error* $\dfrac{dV}{V}$

What is the percentage error?

rel error: $\dfrac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = \dfrac{3}{r} dr$

$\Rightarrow$ rel error $= \dfrac{3}{(7/10)}\left(\dfrac{1}{100}\right) = \dfrac{3}{70} \approx \boxed{0.04286}$

Percentage error $=$ 100% $\times$ rel error

$\Rightarrow$ percentage error $= \boxed{4.286\%}$

Example. The radius of a circular disk is given as $24\,cm$ with a maximum error in measurement of $0.2\,cm$.

a) Use differentials to estimate the maximum error in the calculated area of the disk.

$A = \pi r^2$

$\Delta A \approx dA = [2\pi r](dr)$

$= (2\pi \cdot 24)(0.2)$

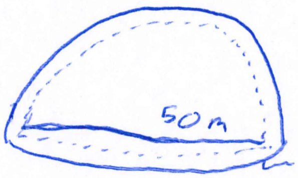
$= 9.6\pi \approx 30.1593$

b) What is the relative error? What is the percentage error?

rel. error: $\dfrac{2\pi r\, dr}{\pi r^2} = \dfrac{2\, dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{1}{60} \approx \boxed{0.01\bar{6}}$

percentage error: $100\% \times$ rel. error $= \boxed{1.\bar{6}\%}$

Example. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.



$= 0.0005\, m$

$r = 25\, m$

$dr = 0.0005\, m$

$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$

$dV = 2\pi r^2\, dr = 2\pi (25^2)(0.0005) = \frac{5\pi}{8} \approx \boxed{1.9635}$