

5.5: Substitution Rule

Theorem 5.6: Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g . On that interval,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

Example. We know

$$\frac{d}{dx} \left[\frac{(2x+1)^4}{4} \right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned} \int 2(2x+1)^3 dx &= \int f(g(x))g'(x) dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C \end{aligned}$$

Procedure: Substitution Rule (Change of Variables)

- Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
- Substitute $u = g(x)$ and $du = g'(x) dx$ in the integral.
- Evaluate the new indefinite integral with respect to u .
- Write the result in terms of x using $u = g(x)$.

Example. Evaluate the following integrals:

a) $\int 2x(x^2 + 3)^4 dx$

$$\begin{aligned} u &= x^2 + 3 \\ du &= 2x dx \\ \int u^4 du &= \frac{u^5}{5} + C \\ &= \boxed{\frac{1}{5}(x^2 + 3)^5 + C} \end{aligned}$$

b) $\int (2x + 1)^3 dx$

$$\begin{aligned} u &= 2x + 1 \\ du &= 2dx \rightarrow \frac{1}{2}du = dx \\ \int u^3 du &= \frac{1}{8}u^4 + C \\ &= \boxed{\frac{1}{8}(2x+1)^4 + C} \end{aligned}$$

c) $\int x^2 \sqrt{x^3 + 1} dx$

$$\begin{aligned} u &= x^3 + 1 \\ du &= 3x^2 dx \rightarrow \frac{1}{3}du = x^2 dx \\ \int u^{1/2} du &= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C \\ &= \boxed{\frac{2}{9} (x^3 + 1)^{3/2} + C} \end{aligned}$$

d) $\int \theta^4 \sqrt{1 - \theta^2} d\theta$

$$\begin{aligned} u &= 1 - \theta^2 \\ du &= -2\theta d\theta \rightarrow -\frac{1}{2}du = \theta d\theta \\ \int u^{1/4} du &= -\frac{1}{2} \cdot \frac{4}{5} u^{5/4} + C \\ &= \boxed{-\frac{2}{5} (1 - \theta^2)^{5/4} + C} \end{aligned}$$

e) $\int \sqrt{4-t} dt$

$$\begin{aligned} u &= 4-t \\ du &= -dt \\ \int u^{1/2} du &= -\frac{2}{3} u^{3/2} + C \\ &= \boxed{-\frac{2}{3} (4-t)^{3/2} + C} \end{aligned}$$

f) $\int (2-x)^6 dx$

$$\begin{aligned} u &= 2-x \\ du &= -dx \\ \int u^6 du &= -\frac{1}{7} u^7 + C \\ &= \boxed{-\frac{1}{7} (2-x)^7 + C} \end{aligned}$$

Example. Evaluate the following integrals:

a) $\int \sec(2\theta) \tan(2\theta) d\theta$

$$u = 2\theta \\ du = 2d\theta \rightarrow \frac{1}{2}du = d\theta$$

$$= \frac{1}{2} \int \sec(u) \tan(u) du = \frac{1}{2} \sec(u) + C \\ = \boxed{\frac{1}{2} \sec(2\theta) + C}$$

b) $\int \csc^2\left(\frac{t}{3}\right) dt$

$$u = \frac{t}{3} \\ du = \frac{1}{3}dt \rightarrow 3du = dt \\ = 3 \int \csc^2(u) du = -3 \cot(u) + C \\ = \boxed{-3 \cot\left(\frac{t}{3}\right) + C}$$

c) $\int \frac{\sin(x)}{1 + \cos^2(x)} dx$

$$u = \cos(x)$$

$$du = -\sin(x) dx \rightarrow -du = \sin(x) dx$$

$$= - \int \frac{1}{1+u^2} du = -\tan^{-1}(u) + C \\ = \boxed{-\tan^{-1}(\cos(x)) + C}$$

d) $\int \frac{\tan^{-1}(x)}{1+x^2} dx$

$$u = \tan^{-1}(x)$$

$$du = \frac{1}{1+x^2} dx$$

$$= \int u du = \frac{u^2}{2} + C = \boxed{(\tan^{-1}(x))^2 + C}$$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2 \cos(\pi t) \text{ m/s}^2$ for all t . If $s = 0$ and $v = 8 \text{ m/s}$ when $t = 0$, find the value of s when $t = 1 \text{ sec}$.

$$v(t) = \int a(t) dt = \int \pi^2 \cos(\pi t) dt = \int \pi \cos(u) du = \pi \sin(\pi t) + C \\ u = \pi t \\ du = \pi dt \\ v(0) = 8 = \pi \sin(0) + C \\ \Rightarrow C = 8$$

$$s(t) = \int v(t) dt = \int (\pi \sin(\pi t) + 8) dt = \int \sin(u) du + \int 8 dt \\ = -\cos(\pi t) + 8t + C \rightarrow s(0) = 0 = -\cos(0) + 0 + C \\ \Rightarrow C = 1$$

Example. Evaluate the following integrals:

a) $\int (6x^2 + 2) \sin(x^3 + x + 1) dx$

$$u = x^3 + x + 1 \\ du = 3x^2 + 1 dx$$

$$\begin{aligned} &= \int 2 \sin(u) du = -2 \cos(u) + C \\ &= \boxed{-2 \cos(x^3 + x + 1) + C} \end{aligned}$$

b) $\int \frac{\sin(\theta)}{\cos^5(\theta)} d\theta$

$$u = \cos(\theta) \\ du = -\sin(\theta) dx$$

$$= - \int u^{-5} du = \frac{u^{-4}}{4} + C$$

$$= \boxed{\frac{1}{4} \sec^4(x) + C}$$

c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}} dx$$

$$\begin{aligned} &= 2 \int e^u du = 2 e^u + C \\ &= \boxed{2 e^{\sqrt{x}} + C} \end{aligned}$$

d) $\int \frac{2^t}{2^t + 3} dt$

$$u = 2^t + 3 \\ du = \ln(2) 2^t dt$$

$$\begin{aligned} &= \frac{1}{\ln(2)} \int \frac{1}{u} du = \frac{\ln|u|}{\ln(2)} + C \\ &= \boxed{\frac{\ln(2^t + 3)}{\ln(2)} + C} \end{aligned}$$

e) $\int 6x^2 4^{x^3} dx$

$$u = x^3 \\ du = 3x^2 dx$$

$$= 2 \int 4^u du \quad v = 4^u \\ dv = \ln(4) 4^u du$$

$$= \frac{2}{\ln(4)} \int dv$$

$$= \frac{2}{\ln(4)} v + C = \frac{2}{\ln(4)} 4^u + C$$

f) $\int \frac{dx}{\sqrt{36 - 4x^2}} = \int \frac{dx}{2\sqrt{9 - \frac{1}{9}x^2}}$

$$\begin{aligned} &= \int \frac{dx}{6\sqrt{1 - (\frac{x}{3})^2}} \\ u = \frac{x}{3} \quad du = \frac{1}{3} dx \\ &= \int \frac{du}{2\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1}(u) + C \\ &= \boxed{\frac{1}{2} \sin^{-1}\left(\frac{x}{3}\right) + C} \end{aligned}$$

$$= \boxed{\frac{2}{\ln(4)} 4^{x^3} + C}$$

g) $\int \sin(t) \sec^2(\cos(t)) dt$

$$u = \cos(t) \quad du = -\sin(t) dt$$

$$= - \int \sec^2(u) du$$

$$= -\tan(u) + C$$

$$= \boxed{-\tan(\cos(t)) + C}$$

i) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

$$u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}} dx$$

$$= 2 \int \sin(u) du$$

$$= -2 \cos(u) + C$$

$$= \boxed{-2 \cos(\sqrt{x}) + C}$$

k) $\int \frac{3}{\sqrt{1-25x^2}} dx$

$$u = 5x \quad du = 5dx$$

$$= \frac{3}{5} \int \frac{1}{\sqrt{1-u^2}} du$$

$$= \frac{3}{5} \sin^{-1}(u) + C$$

$$= \boxed{\frac{3}{5} \sin^{-1}(5x) + C}$$

h) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

$$u = 1+\sqrt{x} \quad du = \frac{1}{2\sqrt{x}} dx$$

$$= 2 \int \frac{1}{u^2} du$$

$$= -\frac{2}{u} + C$$

$$= \boxed{\frac{-2}{(1+\sqrt{x})^2} + C}$$

j) $\int 5 \cos(7x+5) dx$

$$u = 7x+5 \quad du = 7dx$$

$$= \frac{5}{7} \int \cos(u) du$$

$$= \frac{5}{7} \sin(u) + C$$

$$= \boxed{\frac{5}{7} \sin(7x+5) + C}$$

l) $\int \frac{dx}{\sqrt{1-9x^2}}$

$$u = 3x \quad du = 3dx$$

$$= \frac{1}{3} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{3} \sin^{-1}(u) + C$$

$$= \boxed{\frac{1}{3} \sin^{-1}(3x) + C}$$

Example. Evaluate the following integrals using the recommended substitution:

a) $\int \sec^2(x) \tan(x) dx$

where $u = \tan(x)$.

$$du = \sec^2(x) dx$$

$$= \int u du$$

$$= \frac{u^2}{2} + C$$

$$= \boxed{\frac{1}{2} \tan^2(x) + C}$$

b) $\int \sec^2(x) \tan(x) dx$

where $u = \sec(x)$.

$$du = \sec(x) \tan(x) dx$$

$$= \int u du$$

$$= \frac{u^2}{2} + C$$

$$= \boxed{\frac{1}{2} \sec^2(x) + C}$$

Note:

Equivalent since $\tan^2 \theta + 1 = \sec^2 \theta$

Example. Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2 + 8)^{-1/3}$, $y(0) = 0$.

$$\begin{aligned} y(x) &= \int 4x (x^2 + 8)^{-1/3} dx &= 2 \int u^{-1/3} du = -3u^{2/3} + C \\ &\quad u = x^2 + 8 \\ &\quad du = 2x dx \\ &= -3(x^2 + 8)^{2/3} + C \end{aligned}$$

$$y(0) = 0 = -3(0+8)^{2/3} + C \Rightarrow C = 12$$

$$\boxed{y(x) = -3(x^2 + 8)^{2/3} + 12}$$

Example. Evaluate the following integrals:

$$a) \int xe^{-x^2} dx \quad u = -x^2 \\ du = -2x dx \\ = -\frac{1}{2} \int e^u du$$

$$= -\frac{1}{2} e^u + C \\ = \boxed{-\frac{1}{2} e^{-x^2} + C}$$

$$c) \int \frac{dt}{8-3t} \quad u = 8-3t \\ du = -3dt$$

$$= -\frac{1}{3} \int \frac{1}{u} du \\ = -\frac{1}{3} \ln|u| + C = \boxed{-\frac{1}{3} \ln|8-3t| + C}$$

$$e) \int \frac{e^w}{36+e^{2w}} dw \quad u = e^w \\ du = e^w dw$$

$$= \int \frac{du}{36 + (\frac{36}{36})u^2} = \frac{1}{36} \int \frac{du}{1 + (\frac{u}{6})^2} \quad v = \frac{u}{6} \\ dv = \frac{1}{6} du \\ = \frac{1}{6} \int \frac{dv}{1+v^2} \\ = \frac{1}{6} \tan^{-1}(v) + C = \frac{1}{6} \tan^{-1}\left(\frac{u}{6}\right) + C = \boxed{\frac{1}{6} \tan^{-1}\left(\frac{e^w}{6}\right) + C}$$

$$b) \int \frac{e^{1/x}}{x^2} dx \quad u = 1/x \\ du = -\frac{1}{x^2} dx$$

$$= - \int e^u du \\ = -e^u + C = \boxed{-e^{1/x} + C}$$

$$d) \int 5^t \sin(5^t) dt \quad u = 5^t \\ du = 5^t dt$$

$$= \frac{1}{\ln(5)} \int \sin(u) du \\ = -\frac{\cos(u)}{\ln(5)} + C \\ = \boxed{-\frac{\cos(5^t)}{\ln(5)} + C}$$

$$v = \frac{u}{6} \\ dv = \frac{1}{6} du$$

Theorem 5.7: Substitution Rule for Definite Integrals

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g . Then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Example. Evaluate the integrals:

a) $\int_0^1 \frac{x^3}{\sqrt{x^4 + 9}} dx$ $u = x^4 + 9$
 $du = 4x^3 dx$

$$x=0, u=9 \\ x=1, u=10 \\ = \frac{1}{4} \int_9^{10} u^{-\frac{1}{2}} du = \frac{1}{2} u^{\frac{1}{2}} \Big|_9^{10} \\ = \boxed{\frac{1}{2} (\sqrt{10} - 3)}$$

b) $\int_1^3 \frac{dt}{(t-4)^2}$ $u = t-4$
 $du = dt$

$$t=1, u=-3 \\ t=3, u=-1 \\ = \int_{-3}^{-1} u^{-2} du \\ = -u^{-1} \Big|_{-3}^{-1} = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

c) $\int_0^3 \frac{v^2 + 1}{\sqrt{v^3 + 3v + 4}} dv$

$$u = v^3 + 3v + 4 \\ du = 3v^2 + 3 dv$$

$$v=0, u=4 \\ v=3, u=40$$

$$= \frac{1}{3} \int_4^{40} u^{-\frac{1}{2}} du$$

$$= \frac{2}{3} u^{\frac{1}{2}} \Big|_4^{40} = \boxed{\frac{4}{3} (\sqrt{10} - 2)}$$

d) $\int_0^1 2x(4-x^2) dx$ $u = 4-x^2$
 $du = -2x dx$

$$x=0, u=4 \\ x=1, u=3 \\ = \int_4^3 -u du = -\frac{u^2}{2} \Big|_4^3 \\ = -(9-16) \\ = \boxed{7}$$

$$e) \int_2^3 \frac{x}{\sqrt[3]{x^2 - 1}} dx \quad u = x^2 - 1 \\ du = 2x dx$$

$$x = 2, u = 3$$

$$x = 3, u = 8$$

$$= \frac{1}{2} \int_3^8 u^{-1/3} du$$

$$= \frac{3}{4} u^{4/3} \Big|_3^8$$

$$= \boxed{\frac{3}{4} (4 - \sqrt[3]{9})}$$

$$g) \int_0^{\pi/4} \frac{\sin(x)}{\cos^2(x)} dx \quad u = \cos(x) \\ du = -\sin(x) dx$$

$$x = 0, u = 1$$

$$x = \pi/4, u = \sqrt{2}/2$$

$$= - \int_1^{\sqrt{2}/2} u^{-2} du$$

$$= \frac{1}{u} \Big|_1^{\sqrt{2}/2} = \boxed{\sqrt{2} - 1}$$

$$f) \int_0^{\pi/2} \frac{\sin(x)}{1 + \cos(x)} dx \quad u = 1 + \cos(x) \\ du = -\sin(x) dx$$

$$x = 0, u = 2$$

$$x = \pi/2, u = 1$$

$$= - \int_2^1 \frac{1}{u} du$$

$$= - \left[\ln(u) \right]_2^1$$

$$= \boxed{\ln(2)}$$

$$h) \int_{-\pi/12}^{\pi/8} \sec^2(2y) dy \quad u = 2y \\ du = 2 dy$$

$$y = -\pi/3, u = -\pi/6$$

$$y = \pi/8, u = \pi/4$$

$$= \frac{1}{2} \int_{-\pi/6}^{\pi/4} \sec^2(u) du$$

$$= \frac{1}{2} \tan(u) \Big|_{-\pi/6}^{\pi/4}$$

$$= \boxed{\frac{1}{2} \left(1 - \frac{\sqrt{3}}{3} \right)}$$

$$\text{i) } \int_0^1 (1 - 2x^9) dx$$

$$= x - \frac{x^{10}}{5} \Big|_0^1$$

$$= 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$$

$$\text{j) } \int_0^1 (1 - 2x)^9 dx \quad \begin{aligned} u &= 1-2x \\ du &= -2dx \end{aligned}$$

$$x=0, u=1$$

$$x=1, u=-1$$

$$= -\frac{1}{2} \int_1^{-1} u^9 dx$$

$$= -\frac{1}{2} \left. \frac{u^{10}}{10} \right|_1^{-1}$$

$$= -\frac{1}{2} \left(\frac{1}{10} - \frac{1}{10} \right) = \boxed{0}$$

$$\text{k) } \int_0^{\frac{1}{2}} \frac{1}{1+4x^2} dx \quad \begin{aligned} u &= 2x \\ du &= 2dx \end{aligned}$$

$$x=0, u=0$$

$$x=\frac{1}{2}, u=1$$

$$= \frac{1}{2} \int_0^1 \frac{du}{1+u^2}$$

$$= \frac{1}{2} \tan^{-1}(u) \Big|_0^1$$

$$= \boxed{\frac{\pi}{8}}$$

$$\text{l) } \int_0^4 \frac{x}{x^2 + 1} dx \quad \begin{aligned} u &= x^2 + 1 \\ du &= 2x dx \end{aligned}$$

$$x=0, u=1$$

$$x=4, u=17$$

$$= \frac{1}{2} \int_1^{17} \frac{1}{u} du = \frac{1}{2} \ln(u) \Big|_1^{17}$$

$$= \boxed{\frac{1}{2} \ln(17)}$$

$$m) \int_0^\pi 3 \cos^2(x) \sin(x) dx \quad u = \cos(x) \\ du = -\sin(x)$$

$$x=0, u=1$$

$$x=\pi, u=-1$$

$$= -3 \int_1^{-1} u^2 du$$

$$= -u^3 \Big|_1^{-1}$$

$$= \boxed{2}$$

$$n) \int_0^{\frac{\pi}{8}} \sec(2\theta) \tan(2\theta) d\theta \quad u = 2\theta \\ du = 2d\theta$$

$$\theta = 0, u = 0$$

$$\theta = \frac{\pi}{8}, u = \frac{\pi}{4}$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \sec(u) \tan(u) du$$

$$= \frac{1}{2} \sec(u) \Big|_0^{\frac{\pi}{4}}$$

$$= \boxed{\frac{1}{2} (\sqrt{2} - 1)}$$

$$o) \int_0^1 (3t-1)^{50} dt \quad u = 3t-1 \\ du = 3dt$$

$$t=0, u=-1$$

$$t=1, u=2$$

$$= \frac{1}{3} \int_{-1}^2 u du = \frac{1}{3} \frac{u^2}{2} \Big|_{-1}^2$$

$$= \frac{1}{3} (2 - \frac{1}{2})$$

$$= \boxed{\frac{1}{2}}$$

$$p) \int_0^3 \frac{1}{5x+1} dx \quad u = 5x+1 \\ du = 5dx$$

$$x=0, u=1$$

$$x=3, u=16$$

$$= \frac{1}{5} \int_1^{16} \frac{1}{u} du$$

$$= \frac{1}{5} \ln(u) \Big|_1^{16}$$

$$= \boxed{\frac{1}{5} \ln(16)}$$

$$= \frac{4}{5} \ln(2)$$

$$q) \int_0^1 xe^{-x^2} dx \quad u = -x^2 \\ du = -2x dx$$

$$x=0, u=0$$

$$x=1, u=-1$$

$$= -\frac{1}{2} \int_0^{-1} e^u du = -\frac{1}{2} e^u \Big|_0^{-1}$$

$$= -\frac{1}{2} (e^{-1} - e^0) = \boxed{\frac{1}{2}(1-e^{-1})}$$

$$r) \int_e^{e^4} \frac{1}{x \sqrt{\ln(x)}} dx \quad u = \ln(x) \\ du = \frac{1}{x} dx$$

$$x=e, u=1$$

$$x=e^4, u=4$$

$$= \int_1^4 \frac{1}{\sqrt{u}} du = 2\sqrt{u} \Big|_1^4$$

$$= 2(2-1) = \boxed{2}$$

$$s) \int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx \quad u = \sin^{-1}(x) \\ du = \frac{dx}{\sqrt{1-x^2}}$$

$$x=0, u=0$$

$$x=\frac{1}{2}, u=\frac{\pi}{6}$$

$$\int_0^{\pi/6} u du = \frac{u^2}{2} \Big|_0^{\pi/6}$$

$$= \boxed{\frac{\pi^2}{72}}$$

$$t) \int_0^1 \frac{e^z + 1}{e^z + z} dz \quad u = e^z + z \\ du = e^z + 1 dz$$

$$z=0, u=1$$

$$z=1, u=e+1$$

$$\int_1^{e+1} \frac{1}{u} du = \ln(u) \Big|_1^{e+1}$$

$$= \ln(e+1) - \ln(1)$$

$$= \boxed{\ln(e+1)}$$

$$u) \int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} \quad u = 1 + \sqrt{y} \\ du = \frac{1}{2\sqrt{y}} dy$$

$$y=1, u=2$$

$$y=4, u=3$$

$$= \int_2^3 u^{-2} du = -\frac{1}{u} \Big|_2^3$$

$$= -\frac{1}{3} - (-\frac{1}{2})$$

$$= \boxed{\frac{1}{6}}$$

$$w) \int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}} dx \quad u = 1-16x^2 \\ du = -32x dx$$

$$x=0, u=1$$

$$x=\frac{1}{8}, u=\frac{3}{4}$$

$$= -\frac{1}{32} \int_1^{\frac{3}{4}} \frac{1}{\sqrt{u}} du = -\frac{\sqrt{u}}{64} \Big|_1^{\frac{3}{4}}$$

$$= \boxed{\frac{1-\sqrt{3}/2}{64}}$$

$$v) \int_{\ln(\frac{\pi}{4})}^{\ln(\frac{\pi}{2})} e^w \cos(e^w) dw \quad u = e^w \\ du = e^w dw$$

$$w = \ln(\frac{\pi}{4}), u = \frac{\pi}{4}$$

$$w = \ln(\frac{\pi}{2}), u = \frac{\pi}{2}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(u) du$$

$$= \sin(u) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \boxed{1 - \frac{\sqrt{2}}{2}}$$

$$x) \int_1^{e^2} \frac{\ln(p)}{p} dp \quad u = \ln(p) \\ du = \frac{1}{p} dp$$

$$p=1, u=0$$

$$p=e^2, u=2$$

$$= \int_0^2 u du = \frac{u^2}{2} \Big|_0^2$$

$$= \boxed{2}$$

$$\begin{aligned}
 & y) \int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x) dx \quad u = \sin^2(x) \quad du = 2\sin(x)\cos(x)dx \\
 & \quad x=0, u=0 \quad du = \sin(2x) dx \\
 & \quad x=\frac{\pi}{4}, u=\frac{\sqrt{2}}{2} \\
 & = \int_0^{\frac{\sqrt{2}}{2}} e^u du = e^u \Big|_0^{\frac{\sqrt{2}}{2}} \\
 & = \boxed{e^{\frac{\sqrt{2}}{2}} - 1}
 \end{aligned}
 \qquad
 \begin{aligned}
 & \int_{-\pi}^{\pi} x^2 \sin(7x^3) dx \quad u = 7x^3 \quad du = 21x^2 dx \\
 & \quad x=-\pi, u=-7\pi^3 \\
 & \quad x=\pi, u=7\pi^3 \\
 & = \frac{1}{21} \int_{-7\pi^3}^{7\pi^3} \sin(u) du \\
 & = \boxed{0} \quad \because \text{odd function}
 \end{aligned}$$

Example. Average velocity: An object moves in one dimension with a velocity in m/s given by $v(t) = 8 \sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where t is measured in seconds.

$$\begin{aligned}
 \bar{v} &= \frac{1}{10-0} \int_0^{10} 8 \sin(\pi t) + 2t dt \quad u = \pi t \quad du = \pi dt \\
 &= \frac{8}{10\pi} \int_0^{10\pi} \sin(u) du + \frac{2}{10} \int_0^{10} t dt \quad t=0, u=0 \quad t=10, u=10\pi \\
 &= \frac{-4}{5\pi} \cos(u) \Big|_0^{10\pi} + \frac{1}{10} t^2 \Big|_0^{10} \\
 &= -\frac{4}{5\pi} (1-1) + \frac{1}{10} (100-0) = \boxed{10}
 \end{aligned}$$

Example. Prove $\int \tan(x) dx = \ln |\sec(x)| + C$.

$$\begin{aligned} \int \tan(x) dx &= \int \frac{\sin(x)}{\cos(x)} dx = - \int \frac{1}{u} du = -\ln|u| + C \\ u &= \cos(x) \\ du &= -\sin(x) dx \\ &= -\ln|\cos(x)| + C \\ &= \ln|\sec(x)| + C \end{aligned}$$

Example. Evaluate the integrals:

$$a) \int \frac{x}{(x-2)^3} dx \quad u = x-2 \\ du = dx$$

$$\begin{aligned} &= \int \frac{u+2}{u^3} du \\ &= \int u^{-2} + 2u^{-3} du \\ &= -u^{-1} - u^{-2} + C \\ &= -\frac{u^{-1}}{u^2} + C \\ &= \boxed{-\frac{x-3}{(x-2)^2} + C} \end{aligned}$$

$$b) \int x\sqrt{x-1} dx \quad u = x-1 \\ du = dx$$

$$\begin{aligned} &= \int (u+1) \sqrt{u} du \\ &= \int u^{3/2} + u^{1/2} du \\ &= \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \\ &= 2u^{3/2} \left(\frac{3u+10}{15} \right) + C \\ &= \boxed{2(x-1)^{3/2} \left(\frac{3x+7}{15} \right) + C} \end{aligned}$$

$$c) \int x^3(1+x^2)^{\frac{3}{2}} dx \quad u = x^2 \\ du = 2x dx$$

$$= \frac{1}{2} \int u(1+u)^{\frac{3}{2}} du \\ t = 1+u \\ dt = du$$

$$= \frac{1}{2} \int (t-1) t^{\frac{3}{2}} dt \\ = \frac{1}{2} \int t^{\frac{5}{2}} - t^{\frac{3}{2}} dt \\ = \frac{t^{\frac{7}{2}}}{7} - \frac{t^{\frac{5}{2}}}{5} + C = \boxed{\frac{(1+x^2)^{\frac{7}{2}}}{7} - \frac{(1+x^2)^{\frac{5}{2}}}{5} + C}$$

$$d) \int \frac{y^2}{(y+1)^4} dy \quad u = y+1 \\ du = dy$$

$$\int \frac{(u-1)^2}{u^4} du = \int \frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4} du \\ = -\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} + C \\ = \boxed{-\frac{1}{y+1} + \frac{1}{(y+1)^2} - \frac{1}{3(y+1)^3} + C}$$

$$e) \int (z+1)\sqrt{3z+2} dz \quad u = 3z+2 \\ du = 3 dz$$

$$= \frac{1}{3} \int \left(\frac{u}{3} + \frac{1}{3} \right) \sqrt{u} du$$

$$= \frac{1}{9} \int u^{\frac{3}{2}} + u^{\frac{1}{2}} du$$

$$= \frac{1}{9} \left(\frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right) + C$$

$$= \frac{2}{9} u^{\frac{3}{2}} \left(\frac{3u+10}{15} \right) + C$$

$$= \boxed{\frac{2}{135} (3z+2)^{\frac{3}{2}} (9z+16) + C}$$

$$f) \int_0^1 \frac{x}{(x+2)^3} dx \quad u = x+2 \\ du = dx \\ x=0, u=2 \\ x=1, u=3$$

$$= \int_2^3 \frac{u-2}{u^3} du$$

$$= \int_2^3 u^{-2} - 2u^{-3} du \\ = -u^{-1} + u^{-2} \Big|_2^3$$

$$= \left(-\frac{1}{3} + \frac{1}{9} \right) - \left(-\frac{1}{2} + \frac{1}{4} \right)$$

$$= \boxed{\frac{1}{36}}$$

Half-Angle Formulas

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

Example. Evaluate the integrals:

a) $\int \cos^2(x) dx$

$$= \int \frac{1 + \cos(2x)}{2} dx$$

$$= \frac{1}{2} \int dx + \frac{1}{4} \int \cos(u) du$$

$$= \frac{x}{2} + \frac{\sin(u)}{4} + C$$

$$= \boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

b) $\int_0^{\frac{\pi}{2}} \cos^2(x) dx$

$$u = 2x \\ du = 2 dx$$

$$x = 0, u = 0$$

$$x = \frac{\pi}{2}, u = \pi$$

$$= \frac{1}{2} \int_0^{\pi/2} dx + \frac{1}{4} \int_0^{\pi} \cos(u) du$$

$$= \frac{\pi}{4} + \frac{1}{4}(0 -)$$

$$= \boxed{\frac{\pi}{4}}$$

$$c) \int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx \quad u = \frac{1}{x} \\ du = -\frac{1}{x^2} dx$$

$$= - \int \cos^2(u) du$$

$$= -\frac{1}{2} \int du - \frac{1}{2} \int \cos\left(\frac{u}{2}\right) du \\ t = \frac{u}{2} \\ dt = \frac{1}{2} du$$

$$= -\frac{u}{2} - \int \cos(t) dt$$

$$= \boxed{-\frac{1}{2x} - \sin\left(\frac{1}{2x}\right) + C}$$

$$e) \int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta \quad u = \theta + \frac{\pi}{6} \\ du = d\theta$$

$$= \int \frac{1 - \cos\left(\frac{u}{2}\right)}{2} du \\ t = \frac{u}{2} \\ dt = \frac{1}{2} du$$

$$= \frac{1}{2} \int du - \int \cos(t) dt$$

$$= \frac{u}{2} - \sin(t) + C$$

$$= \boxed{\frac{\theta}{2} + \frac{\pi}{12} - \sin\left(\frac{\theta}{2} + \frac{\pi}{12}\right) + C}$$

$$d) \int x \sin^2(x^2) dx \quad u = x^2 \\ du = 2x dx$$

$$= \frac{1}{2} \int \frac{1 - \cos(u/2)}{2} du \\ t = \frac{u}{2} \\ dt = \frac{1}{2} du$$

$$= \frac{1}{4} \int du - \frac{1}{2} \int \cos(t) dt$$

$$= \frac{u}{4} - \frac{\sin(t)}{2} + C$$

$$= \boxed{\frac{x^2}{4} - \frac{\sin\left(\frac{x^2}{2}\right)}{2} + C}$$

$$f) \int_0^{\frac{\pi}{4}} \cos^2(8\theta) d\theta \quad u = 8\theta \\ du = 8 d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \cos^2(u) du \quad \begin{matrix} \theta = 0, u = 0 \\ \theta = \frac{\pi}{4}, u = 2\pi \end{matrix}$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{1}{2} du + \frac{1}{8} \int_0^{2\pi} \frac{\cos(u)}{2} du$$

$$t = \frac{u}{2} \quad dt = \frac{1}{2} du$$

$$= \frac{u}{16} \Big|_0^{2\pi} + \frac{1}{16} \int_0^{\pi} \cos(t) dt$$

$$\begin{matrix} u = 0, t = 0 \\ u = 2\pi, t = \pi \end{matrix}$$

$$= \frac{\pi}{8} + \frac{1}{8} \sin(t) \Big|_0^{\pi} = \boxed{\frac{\pi}{8}}$$

Example. If f is continuous and $\int_0^4 f(x) dx = 10$, find $\int_0^2 f(2x) dx$.

$$\begin{aligned} u &= 2x \\ du &= 2 dx \\ x = 0, u &= 0 \\ x = 2, u &= 4 \end{aligned}$$

$$\int_0^2 f(2x) dx = \frac{1}{2} \int_0^4 f(u) du = \frac{1}{2}(10) = \boxed{5}$$

Example. If f is continuous and $\int_0^9 f(x) dx = 4$, find $\int_0^3 xf(x^2) dx$.

$$\begin{aligned} u &= x^2 \\ du &= 2x dx \\ x = 0, u &= 0 \\ x = 3, u &= 9 \end{aligned}$$

$$\int_0^3 xf(x^2) dx = \frac{1}{2} \int_0^9 f(u) du = \frac{1}{2}(4) = \boxed{2}$$

Example. Suppose f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the following:

a) $\int_{-1}^1 xf(x^2) dx.$

$$\begin{aligned} u &= x^2 & x = -1, u = 1 \\ du &= 2x dx & x = 1, u = 1 \end{aligned}$$

$$= \frac{1}{2} \int_1^1 f(u) du$$

$$= \boxed{0}$$

b) $\int_{-2}^2 x^2 f(x^3) dx.$

$$\begin{aligned} u &= x^3 & x = -2, u = -8 \\ du &= 3x^2 dx & x = 2, u = 8 \end{aligned}$$

$$= \frac{1}{3} \int_{-8}^8 f(u) du = \frac{2}{3} \int_0^8 f(u) du$$

$$= \frac{2}{3}(9) = \boxed{6}$$

Example. Evaluate the integrals:

$$a) \int \sec^2(10x) dx$$

$$u = 10x \quad du = 10 dx$$

$$= \frac{1}{10} \int \sec^2(u) du$$

$$= \frac{1}{10} \tan(u) + C$$

$$= \boxed{\frac{1}{10} \tan(10x) + C}$$

$$b) \int \tan^{10}(4x) \sec^2(4x) dx$$

$$u = 4x \quad du = 4 dx$$

$$= \frac{1}{4} \int \tan^{10}(u) \sec^2(u) du$$

$$t = \tan(u) \quad dt = \sec^2(u) du$$

$$= \frac{1}{4} \int t^{10} dt = \frac{1}{44} t^{11} + C$$

$$= \boxed{\frac{1}{44} \tan^{11}(4x) + C}$$

$$c) \int (x^{\frac{3}{2}} + 8)^5 \sqrt{x} dx \quad u = x^{\frac{3}{2}}$$

$$du = \frac{3}{2} x^{\frac{1}{2}} dx$$

$$= \frac{2}{3} \int (u+8)^5 du \quad t = u+8$$

$$dt = du$$

$$= \frac{2}{3} \int t^5 dt$$

$$= \frac{1}{9} t^6 + C = \boxed{\frac{1}{9} (x^{\frac{3}{2}} + 8)^6 + C}$$

$$d) \int \frac{2x}{\sqrt{3x+2}} dx \quad u = 3x+2$$

$$du = 3 dx$$

$$= \frac{1}{3} \int \frac{2(u-2)}{3\sqrt{u}} du$$

$$= \frac{2}{9} \int \sqrt{u} - \frac{2}{\sqrt{u}} du$$

$$= \frac{2}{9} \left(\frac{2}{3} u^{\frac{3}{2}} - 4 \sqrt{u} \right) + C$$

$$= \boxed{\frac{4}{27} \sqrt{3x+2} (3x-4) + C}$$

$$e) \int \frac{7x^2 + 2x}{x} dx$$

$$= \int 7x + 2 dx$$

$$= \boxed{\frac{7}{2}x^2 + 2x + C}$$

$$f) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$u = e^x + e^{-x}$
 $du = e^x - e^{-x} dx$

$$\int \frac{1}{u} du = \ln|u| + C$$

$$= \boxed{\ln|e^x + e^{-x}| + C}$$

$$g) \int_0^{\sqrt{3}} \frac{3}{9+x^2} dx$$

$$u = \frac{x}{3}$$

$$= \int_0^{\sqrt{3}} \frac{3}{9 + \frac{9}{9}x^2} dx \quad du = \frac{1}{3} dx$$

$$x=0, u=0$$

$$x=\sqrt{3}, u=\frac{\sqrt{3}}{3}$$

$$= \int_0^{\sqrt{3}/3} \frac{1}{1+u^2} du$$

$$= \tan^{-1}(u) \Big|_0^{\sqrt{3}/3}$$

$$= \boxed{\frac{\pi}{6}}$$

$$h) \int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y) + 2} dy$$

$u = \sin^2(y) + 2$
 $du = 2\sin(y)\cos(y) dy$

$$= \sin(2y) dy$$

$$y=0, u=2$$

$$y=\frac{\pi}{6}, u=\frac{9}{4}$$

$$= \int_2^{9/4} \frac{1}{u} du$$

$$= \left. \ln(u) \right|_2^{9/4} = \ln\left(\frac{9}{4}\right) - \ln(2)$$

$$= \boxed{\ln\left(\frac{9}{8}\right)}$$

$$i) \int \frac{\sec(z) \tan(z)}{\sqrt{\sec(z)}} dz \quad u = \sec(z) \\ du = \sec(z) \tan(z) dz \quad j) \int \frac{1}{\sin^{-1}(x) \sqrt{1-x^2}} dx \quad u = \sin^{-1}(x) \\ du = \frac{1}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{u}} du \\ = 2\sqrt{u} + C \\ = \boxed{2\sqrt{\sec(z)} + C}$$

$$= \int \frac{1}{u} du \\ = \ln|u| + C \\ = \boxed{\ln|\sin^{-1}(x)| + C}$$

$$k) \int \frac{x}{\sqrt{4-9x^2}} dx \quad u = 4-9x^2 \\ du = -18x dx$$

$$= -\frac{1}{18} \int \frac{1}{\sqrt{u}} du \\ = -\frac{1}{9} \sqrt{u} + C \\ = \boxed{-\frac{1}{9} \sqrt{4-9x^2} + C}$$

$$l) \int \frac{x}{1+x^4} dx \quad u = x^2 \\ du = 2x dx$$

$$= \frac{1}{2} \int \frac{1}{1+u^2} du \\ = \frac{1}{2} \tan^{-1}(u) + C \\ = \boxed{\frac{1}{2} \tan^{-1}(x^2) + C}$$

$$m) \int \frac{\cos \sqrt{\theta}}{\sqrt{\theta} \sin^2 \sqrt{\theta}} d\theta = 2 \int \frac{1}{u^2} du = -2 \frac{1}{u} + C$$

$$u = \sin(\sqrt{\theta})$$

$$= \boxed{-2 \csc(\sqrt{\theta}) + C}$$

$$du = \frac{\cos(\sqrt{\theta})}{2\sqrt{\theta}} d\theta$$

$$n) \int x^2 \sqrt{2+x} dx = \int (u-2)^2 \sqrt{u} du = \int u^{5/2} - 4u^{3/2} + 4u^{1/2} du$$

$$u = 2+x$$

$$du = dx$$

$$= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{105}(x+2)^{3/2}(15x^2 - 24x + 32) + C}$$

$$o) \int (\sin^5(x) + 3\sin^3(x) - \sin(x)) \cos(x) dx$$

$$u = \sin(x)$$

$$du = \cos(x) dx$$

$$= \int (u^5 + 3u^3 - u) du = \frac{u^6}{6} + \frac{3}{4}u^4 - \frac{u^2}{2} + C$$

$$= \boxed{\frac{\sin^2(x)}{2} \left(\frac{\sin^4(x)}{3} + \frac{3}{2}\sin^2(x) - 1 \right) + C}$$

p) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (x^3 + x^4 \tan(x)) dx$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^4 \tan(x) dx$$

$$= 0 + 0 = \boxed{0}$$

q) $\int_0^{\frac{\pi}{2}} \cos(x) \sin(\sin(x)) dx$

$$u = \sin(x)$$

$$du = \cos(x) dx$$

$$x=0, u=0$$

$$x=\frac{\pi}{2}, u=1$$

$$= \int_0^1 \sin(u) du = -\cos(u) \Big|_0^1$$

$$= \boxed{1 - \cos(1)}$$

r) $\int \frac{1+x}{1+x^2} dx$

$$u = 1+x^2$$

$$du = 2x dx$$

$$= \int \frac{1}{1+x^2} dx + \int \frac{x}{1+x^2} dx$$

$$= \tan^{-1}(x) + \frac{1}{2} \int \frac{1}{u} du$$

$$= \tan^{-1}(x) + \frac{1}{2} \ln|u| + C$$

$$= \boxed{\tan^{-1}(x) + \frac{1}{2} \ln|1+x^2| + C}$$

Example. Evaluate the more challenging integrals:

$$a) \int \frac{dx}{\sqrt{1 + \sqrt{1+x}}} \quad u = 1+x \\ du = dx$$

$$= \int \frac{du}{\sqrt{1+\sqrt{u}}} \quad t = 1+\sqrt{u} \\ dt = \frac{1}{2\sqrt{u}} du \rightarrow 2\sqrt{u} dt = du \\ \rightarrow 2(t-1) dt = du$$

$$= 2 \int \frac{t-1}{\sqrt{t}} dt$$

$$= 2 \int \sqrt{t} - \frac{1}{\sqrt{t}} dt$$

$$= 2 \left(\frac{2}{3} t^{\frac{3}{2}} - 2\sqrt{t} \right) + C$$

$$= \frac{4\sqrt{1+\sqrt{u}}}{3} \left(\sqrt{u} - 2 \right) + C$$

$$= \boxed{\frac{4\sqrt{1+\sqrt{1+x}}}{3} \left(\sqrt{1+x} - 2 \right) + C}$$

$$\text{b) } \int x \sin^4(x^2) \cos(x^2) dx \quad u = x^2$$
$$du = 2x dx$$

$$= \frac{1}{2} \int \sin^4(u) \cos(u) du$$
$$t = \sin(u)$$
$$dt = \cos(u) du$$

$$= \frac{1}{2} \int t^4 dt$$

$$= \frac{1}{10} t^5 + C$$

$$= \frac{1}{10} \sin^5(u) + C$$

$$= \boxed{\frac{1}{10} \sin^5(x^2) + C}$$