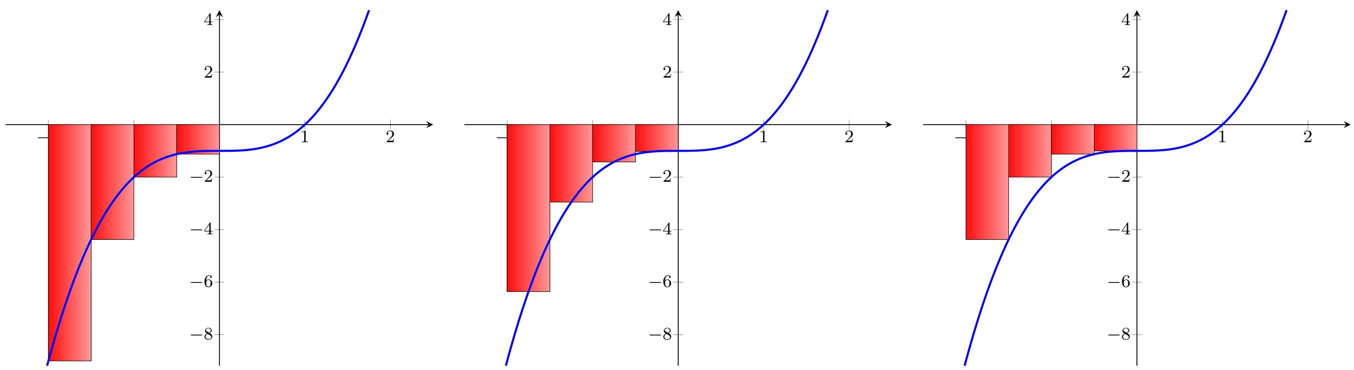
5.2: Definite Integrals

In section 5.1, we assumed f was nonnegative on the interval $[a, b]$.
When $f(x) \leq 0$ on the interval $[a, b]$, then the area between $f(x)$ and the x-axis is negative.

Example. Consider the function $x^3 - 1$ on $[-2, 0]$. Using Riemann sums, we can approximate the area between the curve and the x-axis:



$$L_4 = f(-2)\frac{1}{2} + f(-1.5)\frac{1}{2} + f(-1)\frac{1}{2} + f(-0.5)\frac{1}{2} = -8.25$$

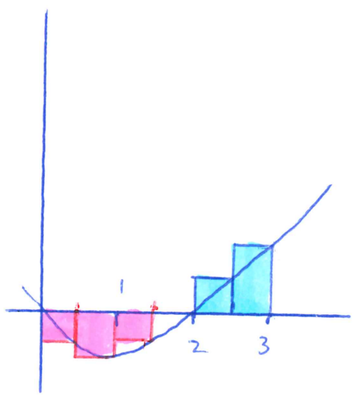
$$M_4 = f(-1.75)\frac{1}{2} + f(-1.25)\frac{1}{2} + f(-0.75)\frac{1}{2} + f(-0.25)\frac{1}{2} = -5.875$$

$$R_4 = f(-1.5)\frac{1}{2} + f(-1)\frac{1}{2} + f(-0.5)\frac{1}{2} + f(0)\frac{1}{2} = -4.25$$

Actual area: $\boxed{-6}$

> **Definition. (Net Area)**
> Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

Example. If $f(x) = x^2 - 2x$, $0 \le x \le 3$, evaluate the Riemann sum with $n = 6$, taking the sample points to be right endpoints.



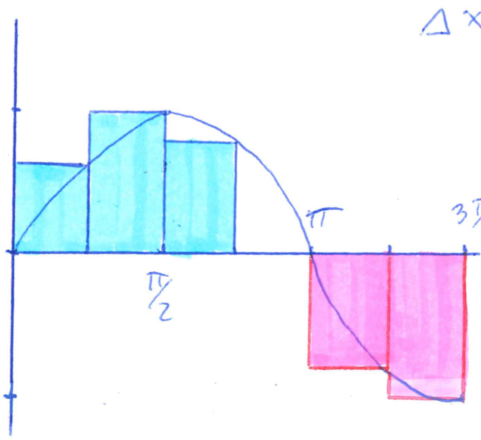
$\Delta x = \frac{3-0}{6} = \frac{1}{2}$

$\sum_{k=1}^{6} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$

$= \frac{1}{2}(-0.75 - 1 - 0.75 + 0 + 1.25 + 3.0)$

$= \frac{1}{2}(1.75) = 0.875 = 7/8$

Example. Find the Riemann sum for $f(x) = \sin(x)$, $0 \le x \le \frac{3\pi}{2}$, with six terms, taking the sample points to be right endpoints.



$\Delta x = \frac{\frac{3\pi}{2} - 0}{6} = \frac{\pi}{4}$

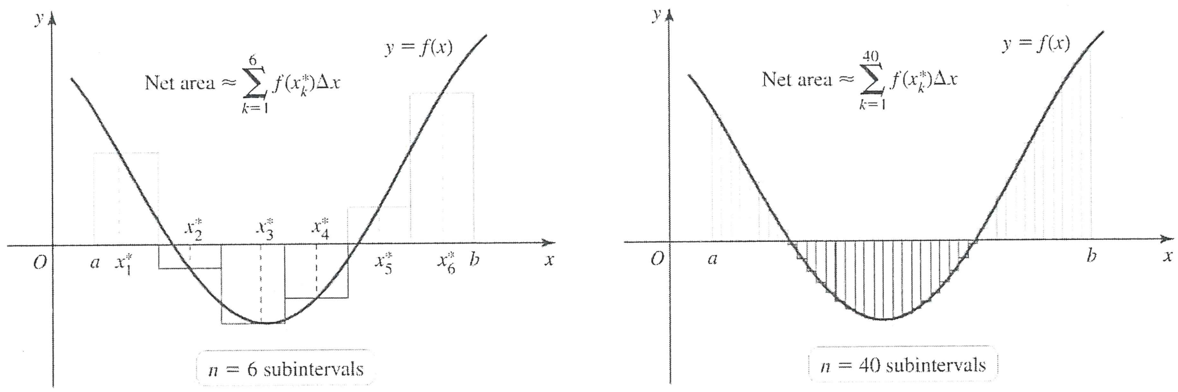
$\sum_{k=1}^{6} f(x_k)\Delta x = \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1\right)$

$= \frac{(\sqrt{2}+2)\pi}{8}$

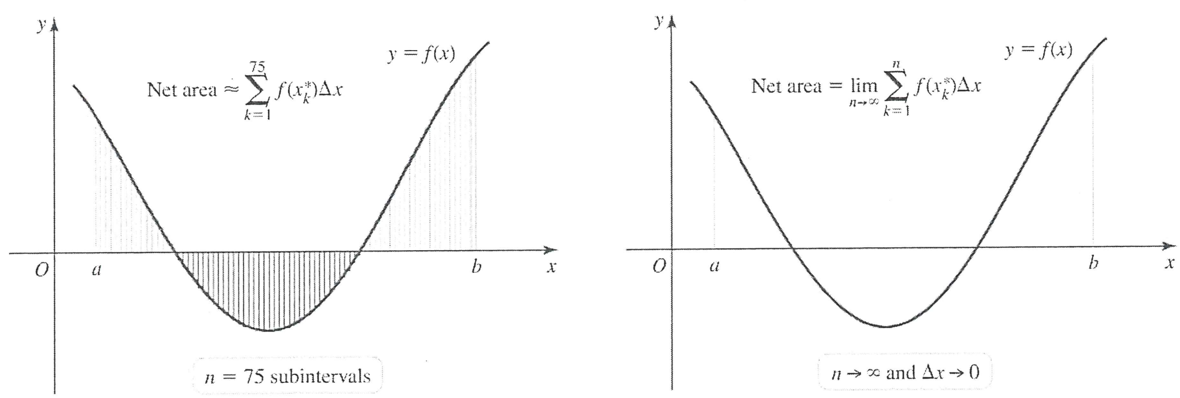
Definition. (Definite Integral)

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$



As the number of subintervals n increases, the Riemann sums approach the net area of the region between the curve $y = f(x)$ and the x-axis on $[a, b]$.



Example.

a) For the function $f(x) = 3x + 2x^2$, find a formula for the upper sum obtained by dividing the interval $[0, 1]$ into n equal subintervals.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_k = 0 + k\Delta x = \frac{k}{n}$$

$$f(x_k) = \frac{3k}{n} + \frac{2k^2}{n^2}$$

$$\sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left(\frac{3k}{n} + \frac{2k^2}{n^2}\right)\frac{1}{n}$$

$$= \frac{3}{n^2}\sum_{k=1}^{n} k + \frac{2}{n^3}\sum k^2$$

$$= \frac{3}{n^2}\,\frac{n(n+1)}{2} + \frac{2}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3(n+1)}{2n} + \frac{2n^2+3n+1}{3n^2}$$

b) Take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 3x + 2x^2$ over $[0, 1]$.

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \frac{3(n+1)}{2n} + \frac{2n^2+3n+1}{3n^2}$$

$$= \frac{3}{2} + \frac{2}{3} = \boxed{\frac{13}{6}}$$

Example. Use the limit of the Riemann sum notation to evaluate $\int_0^2 (2-x^2)\,dx$.

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_k = 0 + k\Delta x = \frac{2k}{n}$

$f(x_k) = 2 - \left(\frac{2k}{n}\right)^2 = 2 - \frac{4k^2}{n^2}$

$\sum_{k=1}^{n} \left(2 - \frac{4k^2}{n^2}\right)\frac{2}{n} = \frac{1}{n}\sum_{k=1}^{n} 4 - \frac{8}{n^3}\sum_{k=1}^{n} k^2 = 4 - \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}$

$= 4 - \frac{4(2n^2+3n+1)}{3n^2}$

$\int_0^2 (2-x^2)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} 4 - \frac{4(2n^2+3n+1)}{3n^2} = 4 - \frac{8}{3} = \boxed{\frac{4}{3}}$

Example. Use the the definition of the definite integral to evaluate $\int_1^4 (x^2-1)\,dx$.

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$

$x_k = 1 + k\Delta x = 1 + \frac{3k}{n}$

$f(x_k) = \left(1 + \frac{3k}{n}\right)^2 - 1 = \frac{6k}{n} + \frac{9k^2}{n^2}$

$\sum_{k=1}^{n} \left(\frac{6k}{n} + \frac{9k^2}{n^2}\right)\frac{3}{n} = \sum_{k=1}^{n} \frac{18k}{n^2} + \frac{27k^2}{n^3} = \frac{18}{n^2}\sum_{k=1}^{n} k + \frac{27}{n^3}\sum_{k=1}^{n} k^2$

$= \frac{9(n+1)}{n} + \frac{9(2n^2+3n+1)}{2n^2}$

$\int_1^4 x^2-1\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \frac{9(n+1)}{n} + \frac{9(2n^2+3n+1)}{2n^2}$

$= 9 + 9 = \boxed{18}$

Example. Use the definition of the definite integral to evaluate $\int_1^4 (x^2 - 4x + 2)\, dx$.

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_k = 1 + k\Delta x = 1 + \frac{3k}{n}$$

$$f(x_k) = \left(1 + \frac{3k}{n}\right)^2 - 4\left(1 + \frac{3k}{n}\right) + 2 = \frac{9k^2}{n^2} - \frac{6k}{n} - 1$$

$$\sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left(\frac{9k^2}{n^2} - \frac{6k}{n} - 1\right)\frac{3}{n} = \frac{27}{n^3}\sum_{k=1}^{n} k^2 - \frac{18}{n^2}\sum_{k=1}^{n} k - \frac{3}{n}(n)$$

$$= \frac{9(2n^2+3n+1)}{2n^2} - \frac{18(n+1)}{n} - 3$$

$$\int_1^4 (x^2 - 4x + 2)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \frac{9(2n^2+3n+1)}{2n^2} - \frac{9(n+1)}{n} - 3$$

$$= 9 - 9 - 3 = \boxed{-3}$$

Example. Use the definition of the definite integral to evaluate $\int_0^2 4x^3\, dx$.

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = 0 + k\,\Delta x = \frac{2k}{n}$$

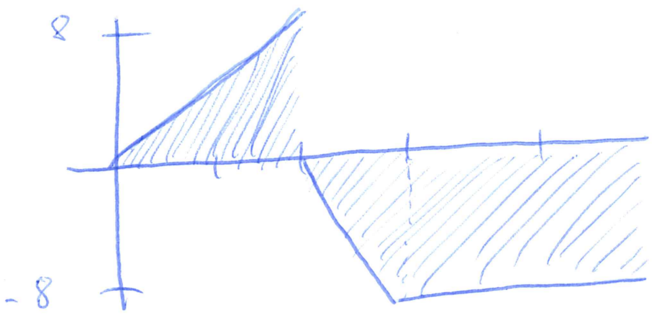
$$f(x_k) = 4\left(\frac{2k}{n}\right)^3 = \frac{32k^3}{n^3}$$

$$\sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\left(\frac{32k^3}{n^3}\right)\left(\frac{2}{n}\right) = \frac{64}{n^4}\sum_{k=1}^{n} k^3 = \frac{64}{n^4}\,\frac{n^2(n+1)^2}{4} = \frac{16(n+1)^2}{n^2}$$

$$\int_0^2 4x^3\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty}\frac{16(n+1)^2}{n^2} = \boxed{16}$$

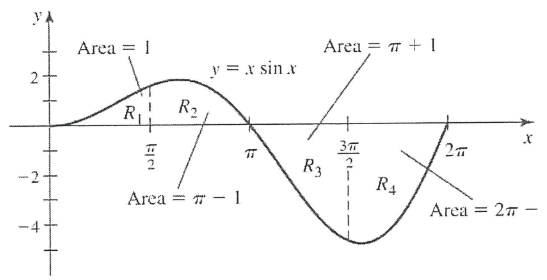
Example. Use a sketch and geometry to evaluate the following integral

$$\int_1^{10} g(x)\,dx, \text{ where } g(x) = \begin{cases} 4x, & \text{if } 0 \le x \le 2 \\ -8x+16, & \text{if } 2 < x \le 3 \\ -8, & \text{if } x > 3 \end{cases}$$



$$\int_1^{10} g(x)\,dx = \tfrac{1}{2}(2+8) + \tfrac{1}{2}(-8) + (7)(-8)$$
$$= \boxed{-55}$$

Example. For the following figure to evaluate the integrals below:



a) $\int_0^{\pi} x\sin(x)\,dx = R_1 + R_2 = 1 + (\pi - 1)$
$= \pi$

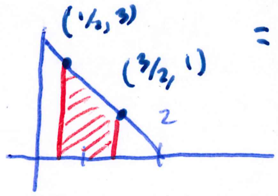
b) $\int_0^{3\pi/2} x\sin(x)\,dx = R_1 + R_2 - R_3$
$= 1 + (\pi - 1) - (\pi + 1)$
$= -1$

c) $\int_0^{2\pi} x\sin(x)\,dx$
$= R_1 + R_2 - R_3 - R_4$
$= 1 + (\pi - 1) - (\pi + 1) - (2\pi - 1)$
$= -2\pi$

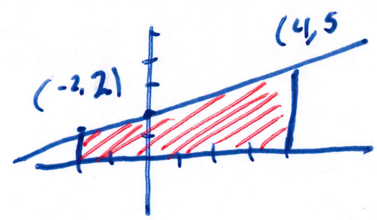
d) $\int_{\pi/2}^{2\pi} x\sin(x)\,dx$
$= R_2 - R_3 - R_4$
$= (\pi - 1) - (\pi + 1) - (2\pi - 1)$
$= -1 - 2\pi$

Example. Graph the following integrands and compute the areas to evaluate the integrals.

a) $\displaystyle\int_{1/2}^{3/2} (-2x+4)\,dx = \frac{1}{2}(3+1)\cdot 1 = \boxed{2}$



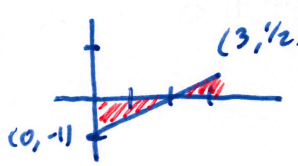
b) $\displaystyle\int_{-2}^{4} \left(\frac{x}{2}+3\right) dx = \frac{1}{2}(2+5)6 = \boxed{21}$



c) $\displaystyle\int_{0}^{3} \left(\frac{1}{2}x-1\right) dx = -\frac{1}{2}(1)(2) + \frac{1}{2}\left(\frac{1}{2}\right)(1)$

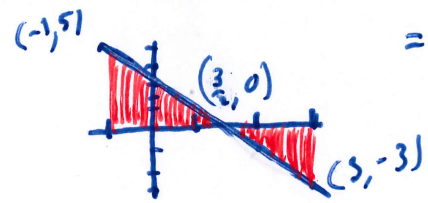
$= -1 + 1/4$

$= \boxed{-3/4}$



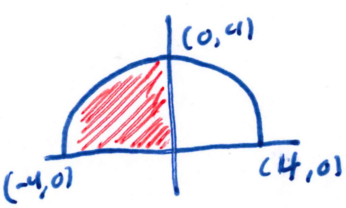
d) $\displaystyle\int_{-1}^{3} (3-2x)\,dx = \frac{1}{2}\left(\frac{5}{2}\right)(5) - \frac{1}{2}\left(\frac{3}{2}\right)(3)$

$= \frac{25}{4} - \frac{9}{4} = \frac{16}{4} = \boxed{4}$



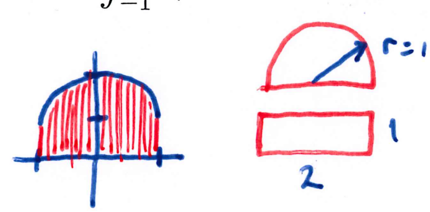
e) $\displaystyle\int_{-4}^{0} \sqrt{16-x^2}\,dx$

$= \frac{1}{4}\pi(4)^2 = \boxed{4\pi}$



f) $\displaystyle\int_{-1}^{1} \left(1+\sqrt{1-x^2}\right) dx = (1)(2) + \frac{1}{2}\pi(1)^2$

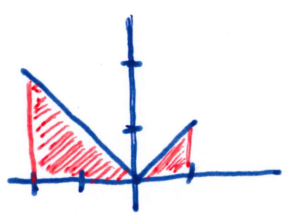
$= \boxed{2 + \frac{\pi}{2}}$



g) $\displaystyle\int_{-2}^{1} |x|\,dx = \frac{1}{2}(2)(2) + \frac{1}{2}(1)(1)$

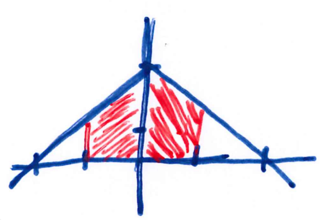
$= 2 + 1/2$

$= \boxed{5/2}$



h) $\displaystyle\int_{-1}^{1} (2-|x|)\,dx = \frac{1}{2}(1+2)(1) + \frac{1}{2}(2+1)(1)$

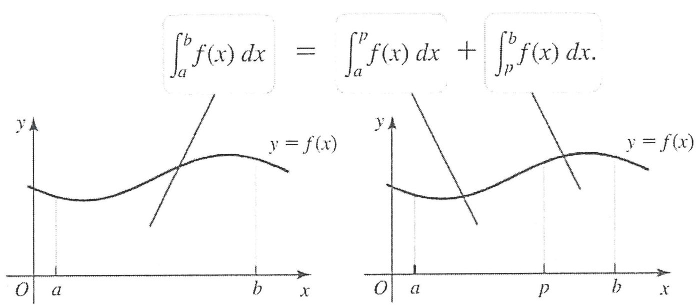
$= \boxed{3}$

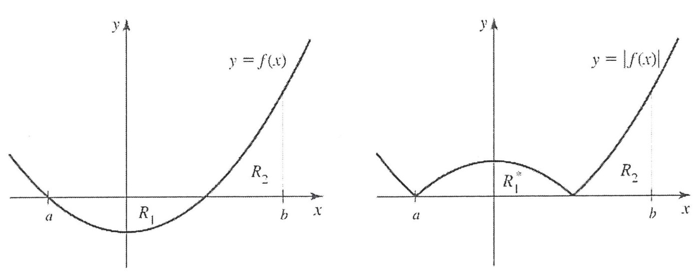


Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and p.

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
5. $\displaystyle\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.





Example. Suppose that $\int_{-3}^{0} g(t)\,dt = \sqrt{2}$. Evaluate the following:

a) $\int_{-3}^{0} g(u)\,du$ $= \boxed{\sqrt{2}}$

b) $\int_{0}^{-3} g(t)\,dt$ $= -\int_{-3}^{0} g(t)\,dt = \boxed{-\sqrt{2}}$

c) $\int_{0}^{-3} [-g(x)]\,dx$

$= -\int_{0}^{-3} g(x)\,dx$

$= \int_{-3}^{0} g(x)\,dx = \boxed{\sqrt{2}}$

d) $\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\,dr$ $= \frac{1}{\sqrt{2}} \int_{-3}^{0} g(r)\,dr$

$= \frac{1}{\sqrt{2}} \cdot \sqrt{2} = \boxed{1}$

Example. Suppose that $\int_{0}^{3} f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Evaluate the following:

a) $\int_{3}^{4} f(z)\,dz$

$= \int_{0}^{4} f(z)\,dz - \int_{0}^{3} f(z)\,dz$

$= 7 - 3 = \boxed{4}$

b) $\int_{4}^{3} f(z)\,dz$ $= -\int_{3}^{4} f(z)\,dz$

$= \boxed{-3}$

Example. Use the fact that $\int_0^{\pi/2} (\cos(\theta) - 2\sin(\theta))\,d\theta = -1$ to evaluate the following

a) $\int_0^{\pi/2} (2\sin(\theta) - \cos(\theta))\,d\theta$

$= \int_0^{\pi/2} -(\cos(\theta) - 2\sin(\theta))\,d\theta$

$= -(-1) = \boxed{1}$

b) $\int_{\pi/2}^{0} (4\cos(\theta) - 8\sin(\theta))\,d\theta$

$= -\int_0^{\pi/2} 4(\cos(\theta) - 2\sin(\theta))\,d\theta$

$= -4\int_0^{\pi/2} \cos(\theta) - 2\sin(\theta)\,d\theta = -4(-1)$

$= \boxed{4}$

Example. Suppose that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$ and $\int_7^9 h(x)\,dx = 4$. Evaluate the following:

a) $\int_1^9 -2f(x)\,dx) = -2\int_1^9 f(x)\,dx$

$= -2(-1)$

$= 2$

b) $\int_7^9 [f(x) + h(x)]\,dx$

$= \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 4 + 5 = \boxed{9}$

c) $\int_7^9 [2f(x) - 3h(x)]\,dx$

$= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx$

$= 2(5) - 3(4) = \boxed{-2}$

d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx$

$= -(-1) = \boxed{1}$

e) $\int_1^7 f(x)\,dx$

$= \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx$

$= -1 - 5 = \boxed{-6}$

f) $\int_7^9 [h(x) - f(x)]\,dx$

$= \int_7^9 h(x)\,dx - \int_7^9 f(x)\,dx$

$= 4 - 5 = \boxed{-1}$

Example. Given $\int_1^3 e^x\,dx = e^3 - e$, find $\int_1^3 (2e^x - 1)\,dx$

$$\int_1^3 2e^x - 1\,dx = 2\int_1^3 e^x\,dx - \int_1^3 dx$$

$$= 2(e^3 - e) - (3-1) = \boxed{2e^3 - 2e - 2}$$

Example. Suppose that $f(x) \geq 0$ on $[0,2]$ and $f(x) \leq 0$ on $[2,5]$ where $\int_0^2 f(x)\,dx = 6$ and $\int_2^5 f(x)\,dx = -8$. Evaluate the following:

a) $\int_0^5 f(x)\,dx$

$$= \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx$$

$$= 6 + (-8) = -2$$

b) $\int_0^5 |f(x)|\,dx$

$$= \int_0^2 |f(x)|\,dx + \int_2^5 |f(x)|\,dx$$

$$= |6| + |-8| = \boxed{14}$$

c) $\int_2^5 4|f(x)|\,dx$

$$= 4\int_2^5 |f(x)|\,dx$$

$$= 4|-8| = \boxed{32}$$

d) $\int_0^5 (f(x) + |f(x)|)\,dx$

$$= \int_0^5 f(x)\,dx + \int_0^5 |f(x)|\,dx$$

$$= -2 + 14$$

$$= \boxed{12}$$