4.7 L'Hôpital's Rule

Theorem 4.12: L'Hôpital's Rule
Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Theorem 4.13: L'Hôpital's Rule (∞/∞)
Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Note: Limits of the form $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are called *indeterminant forms.*

Notes on grading:

1. Unless specifically told to use L'Hôpital's Rule, you may use any valid method to evaluate limits.
2. Remember to
   a) keep your limit notation until the direct substitution step
   b) connect each step with equal signs
   c) notate the equal signs where L'Hôpital is used
3. L'Hôpital does NOT replace the quotient rule!

Example. Find the following limits with and without L'Hôpital's Rule:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

w/o $\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = \lim\limits_{x\to 2} x+2 = 4$

w/

$\lim\limits_{x\to 2} \frac{x^2-4}{x-2} \stackrel{LR}{=} \lim\limits_{x\to 2} \frac{2x}{1} = 4$

$$\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1}$$

w/o

$\lim\limits_{x\to\infty} \frac{2x^2+3x}{x^3+x+1}\left(\frac{1/x^3}{1/x^3}\right) = \lim\limits_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = 0$

w/

$\lim\limits_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} \stackrel{LR}{=} \lim\limits_{x\to\infty} \frac{4x+3}{3x^2+1} \stackrel{LR}{=} \lim\limits_{x\to\infty} \frac{4}{6x}$

$= 0$

> *Note:* L'Hôpital's Rule only works for indeterminant forms!

Example. Find the following limit with and without L'H'ôpital's Rule:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin(x)}{1-\cos(x)}$$

$\stackrel{LR}{=} \lim\limits_{x\to\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)} = \frac{0}{1} = 0$

NO!

w/o

$\lim\limits_{x\to\pi/2} \frac{\sin(x)}{1-\cos(x)} = \frac{\sin(\frac{\pi}{2})}{1-\cos(\pi/2)} = \frac{1}{1} = 1$

This limit is NOT an indeterminant form

Example. Find the following limits:

$\frac{0}{0}$ $$\lim_{t\to 1}\frac{t^3-1}{4t^3-t-3}$$

$$\overset{LR}{=} \lim_{t\to 1}\frac{3t^2}{12t^2-1} = \boxed{\frac{3}{11}}$$

$\frac{0}{0}$ $$\lim_{z\to 0}\frac{\tan(4z)}{\tan(7z)}$$

$$\overset{LR}{=} \lim_{z\to 0}\frac{4\sec^2(4z)}{7\sec^2(7z)} = \boxed{\frac{4}{7}}$$

Example. Find the following limits. Repeat L'Hôpital's Rule each time you get an indeterminant form:

$\frac{0}{0}$ $$\lim_{x\to 0}\frac{\sin(x)-x}{x^3} \overset{LR}{=} \lim_{x\to 0}\overset{0/0}{\frac{\cos(x)-1}{3x^2}} \overset{LR}{=} \lim_{x\to 0}\overset{0/0}{\frac{-\sin(x)}{6x}} \overset{LR}{=} \lim_{x\to 0}\frac{-\cos(x)}{6}$$

$$= \boxed{-\frac{1}{6}}$$

$\frac{0}{0}$ $$\lim_{t\to 0}\frac{t\sin(t)}{1-\cos(t)} \overset{LR}{=} \lim_{t\to 0}\overset{0/0}{\frac{\sin(t)+t\cos(t)}{\sin(t)}} \overset{LR}{=} \lim_{t\to 0}\frac{\cos(t)+\cos(t)-t\sin(t)}{\cos(t)}$$

$$= \frac{1+1-0(0)}{1} = \boxed{2}$$

Example. Evaluate:

$\frac{-\infty}{0}$ Not LR

$$\lim_{x\to 0^+} \frac{\ln(x)}{x} \doteq \lim_{x\to 0^+} \frac{1}{x} \lim_{x\to 0^+} \ln(x)$$

$$= (\infty)(-\infty) = \boxed{-\infty}$$

$$\lim_{x\to 3} \frac{2x^2-5x+1}{x^2+x-6} = \frac{18-15+1}{9+3-6}$$

$$= \boxed{\frac{2}{3}}$$

$\frac{0}{0}$

$$\lim_{x\to \frac{1}{2}} \frac{6x^2+5x-4}{4x^2+16x-9}$$

$$\overset{LR}{=} \lim_{x\to\frac{1}{2}} \frac{12x+5}{8x+16}$$

$$= \boxed{\frac{11}{20}}$$

$\frac{-\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x}$$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{1-16x}{24x+5} \quad \frac{-\infty}{\infty}$$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{-16}{24} = \boxed{-\frac{2}{3}}$$

$\frac{0}{0}$

$$\lim_{t\to 0} \frac{e^{2t}-1}{\sin(t)}$$

$$\overset{LR}{=} \lim_{t\to 0} \frac{2e^{2t}}{\cos(t)}$$

$$= \frac{2}{1} = \boxed{2}$$

$\frac{0}{0}$

$$\lim_{t\to 0} \frac{8^t-5^t}{t}$$

$$\overset{LR}{=} \lim_{t\to 0} \frac{\ln(8)\,8^t - \ln(5)\,5^t}{1}$$

$$= \boxed{\ln(8)-\ln(5)}$$

> *Note:* $0 \cdot \infty$ and $\infty - \infty$ are also indeterminant forms. L'Hôpital's Rule can be used after these functions are converted into rational functions of indeterminant form.

Example. Find the following limits. Convert into indeterminant form as needed:

$0 \cdot \infty$ $\quad \lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) = \lim_{x\to 1^-} \frac{1-x}{\cot\left(\frac{\pi}{2}x\right)} \overset{LR}{=} \lim_{x\to 1^-} \frac{-1}{-\csc^2\left(\frac{\pi}{2}\right)\frac{\pi}{2}}$

$\frac{0}{0}$

$= \frac{1}{1\left(\frac{\pi}{2}\right)} = \boxed{\frac{2}{\pi}}$

$\infty \cdot 0$ $\quad \lim_{x\to\infty} x^2 \sin\left(\frac{1}{4x^2}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{4x^2}\right)}{\frac{1}{x^2}} \overset{LR}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{4x^2}\right)\left(\frac{-2}{4x^3}\right)}{\frac{-2}{x^3}}$

$= \lim_{x\to\infty} \frac{1}{4}\cos\left(\frac{1}{4x^2}\right) = \frac{1}{4}(1) = \boxed{\frac{1}{4}}$

$\infty - \infty + 1$ $\quad \lim_{x\to 0^+} (\csc(x) - \cot(x) + \cos(x))$

$= \lim_{x\to 0^+} \frac{1 - \cos(x) + \sin(x)\cdot\cos(x)}{\sin(x)}$

$\overset{LR}{=} \lim_{x\to 0^+} \frac{\sin(x) + \cos^2(x) - \sin^2(x)}{\cos(x)}$

$= \frac{0+1-0}{1} = \boxed{1}$

Indeterminant forms 1^∞, 0^0, and ∞^0.
Assume $\lim_{x\to a} f(x)^{g(x)}$ has the indeterminant form 1^∞, 0^0, or ∞^0.

1. Analyze $L = \lim_{x\to a} g(x)\ln(f(x))$. This limit can be put in the form $0/0$ or ∞/∞, both of which are handled by L'Hôpital's Rule.
2. When L is finite, $\lim_{x\to a} f(x)^{g(x)} = e^L$. If $L=\infty$ or $L=-\infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$ or $\lim_{x\to a} f(x)^{g(x)} = 0$, respectively.

Note: 0^∞ and ∞^∞ are NOT indeterminant forms.

0^0 $y = \lim_{x\to 0^+} x^{-1/\ln(x)}$

$\ln(y) = \lim_{x\to 0^+} \frac{-1}{\ln(x)} \ln(x)$

$= \lim_{x\to 0^+} -1 = -1$

$\Rightarrow \boxed{y = e^{-1}}$

0^0 $y = \lim_{x\to 0^+} x^{x^2}$

$\ln(y) = \lim_{x\to 0^+} x^2 \ln(x)$ $\quad 0\cdot\infty$

$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x^2}$ $\quad \frac{\infty}{\infty}$

$\overset{LR}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \frac{x^2}{-2} = \boxed{0}$

$\Rightarrow \boxed{y = e^0 = 1}$

∞^0 $y = \lim_{x\to\infty} (1+2x)^{1/(2\ln(x))}$

$\ln(y) = \lim_{x\to\infty} \frac{1}{2\ln(x)} \ln(1+2x)$ $\quad \frac{\infty}{\infty}$

$\overset{LR}{=} \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1+2x}$ $\quad \frac{\infty}{\infty}$

$\overset{LR}{=} \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \boxed{y = e^{1/2} = \sqrt{e}}$

0^0 $y = \lim_{x\to 0^+} x^{\sqrt{x}}$

$\ln(y) = \lim_{x\to 0^+} \sqrt{x}\ln(x)$ $\quad 0\cdot\infty$ $\quad = \lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}}$ $\quad \frac{-\infty}{\infty}$

$\overset{LR}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -\frac{x^{1/2}}{2} = 0$

$\Rightarrow \boxed{y = e^0 = 1}$

Note: L'Hôpital does not always work!

$\frac{0}{0}$ $\displaystyle\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin(x)}}$

$= \displaystyle\lim_{x\to 0^+} \sqrt{\frac{x}{\sin(x)}}$

$= \sqrt{1} = \boxed{1}$

$\frac{\infty}{\infty}$ $\displaystyle\lim_{x\to 0^+} \frac{\cot(x)}{\csc(x)}$

$\overset{LR}{=} \displaystyle\lim_{x\to 0^+} \frac{-\csc^2(x)}{-\csc(x)\cot(x)}$

$= \displaystyle\lim_{x\to 0^+} \frac{\csc(x)}{\cot(x)}$

$\frac{\infty}{\infty}$ $\displaystyle\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

$\overset{LR}{=} \displaystyle\lim_{x\to\infty} \frac{3e^{3x}+3e^{-3x}}{3e^{3x}-3e^{-3x}} = \lim_{x\to\infty} \frac{e^{3x}+e^{-3x}}{e^{3x}-e^{-3x}}$

$\overset{LR}{=} \displaystyle\lim_{x\to\infty} \frac{3e^{3x}-3e^{-3x}}{3e^{3x}+3e^{-3x}} = \boxed{\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}}$

Example. Find the following limits:

$\frac{0}{0}$ $\displaystyle\lim_{x\to 2\pi} \frac{x\sin(x)+x^2-4\pi^2}{x-2\pi} \overset{LR}{=} \lim_{x\to 2\pi} \frac{\sin(x)+x\cos(x)+2x-0}{1}$

$= 0 + 2\pi(1) + 2(2\pi) = \boxed{6\pi}$

$\frac{\infty}{\infty}$ $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{2\tan(x)}{\sec^2(x)} \overset{LR}{=} \lim_{x\to\pi/2} \frac{2\sec^2(x)}{2\sec(x)\cdot\sec(x)\tan(x)} = \lim_{x\to\pi/2} \frac{1}{\tan(x)} = \boxed{0}$

$\frac{0}{0}$ $\displaystyle\lim_{x\to -1} \frac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2} \overset{LR}{=} \lim_{x\to -1} \frac{3x^2-2x-5}{4x^3+6x^2-2x-4}$ $\frac{0}{0}$

$\overset{LR}{=} \displaystyle\lim_{x\to -1} \frac{6x-2}{12x^2+12x-2} = \frac{-8}{-2} = \boxed{4}$

$\frac{\infty}{\infty}$ $\displaystyle\lim_{x\to\infty} \frac{27x^2+3x}{3x^2+x+1}$

$\overset{LR}{=} \displaystyle\lim_{x\to\infty} \frac{54x+3}{6x+1}$ $\frac{\infty}{\infty}$

$\overset{LR}{=} \displaystyle\lim_{x\to\infty} \frac{54}{6} = \boxed{9}$

$\displaystyle\lim_{x\to 0} \frac{x+\sin(x)}{x+\cos(x)} = \frac{0+0}{0+1} = \boxed{0}$

$\frac{0}{0}$ $\displaystyle\lim_{t\to 0} \frac{2t}{\tan(t)}$

$\overset{LR}{=} \displaystyle\lim_{t\to 0} \frac{2}{\sec^2(t)} = \boxed{2}$

$\frac{0}{0}$ $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

$\overset{LR}{=} \displaystyle\lim_{x\to 0} \frac{\frac{1}{\sqrt{1+2x}} + \frac{2}{\sqrt{1-4x}}}{1}$

$= \boxed{3}$

$\frac{0}{0}$ $\lim_{x\to 0} \frac{x^2 - 2x}{x^2 - \sin(x)}$

$\overset{LR}{=} \lim_{x\to 0} \frac{2x-2}{2x-\cos(x)} = \boxed{2}$

$\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin(\theta)}{\csc(\theta)} = \frac{1-1}{1} = \boxed{0}$

$\frac{0}{0}$ $\lim_{x\to 2} \frac{\sqrt[3]{3x+2}-2}{x-2}$

$\overset{LR}{=} \lim_{x\to 2} \frac{\frac{1}{(3x+2)^{2/3}}}{1}$

$= \boxed{\frac{1}{4}}$

$\frac{\infty}{\infty}$ $\lim_{x\to\infty} \frac{100x^3 - 3}{x^4 - 2}$

$\overset{LR}{=} \lim_{x\to\infty} \frac{300x^2}{4x^3} = \lim_{x\to\infty} \frac{75}{x}$

$= \boxed{0}$

$\frac{0}{0}$ $\displaystyle\lim_{x \to 0} \frac{\sin^2(3x)}{x^2}$

$\overset{LR}{=} \displaystyle\lim_{x \to 0} \frac{6\sin(3x)\cos(3x)}{2x}$

$\frac{0}{0}$ $= \displaystyle\lim_{x \to 0} 3\,\frac{\sin(3x)\cos(3x)}{x}$

$\overset{LR}{=} \displaystyle\lim_{x \to 0} 9\cos^2(3x) - 9\sin^2(3x)$

$= \boxed{9}$

$\displaystyle\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \sin(\theta)}{\csc(\theta)} = \frac{1-1}{1} = \boxed{0}$

$\infty \cdot 0$ $\displaystyle\lim_{x \to 0^+} \cot(2x)\sin(6x)$

$\frac{0}{0}$ $= \displaystyle\lim_{x \to 0^+} \frac{\sin(6x)}{\tan(2x)}$

$\overset{LR}{=} \displaystyle\lim_{x \to 0^+} \frac{6\cos(6x)}{2\sec^2(2x)} = \frac{6}{2} = \boxed{3}$

$\infty - \infty$ $\displaystyle\lim_{x \to 0} \left(\cot(x) - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x\cos(x) - \sin(x)}{x\sin(x)}$ $0/0$

$\overset{LR}{=} \displaystyle\lim_{x \to 0} \frac{\cos(x) - x\sin(x) - \cos(x)}{\sin(x) + x\cos(x)}$

$= \displaystyle\lim_{x \to 0} \frac{-x\sin(x)}{\sin(x) + x\cos(x)}$ $\frac{0}{0}$

$= \displaystyle\lim_{x \to 0} \frac{-\sin(x) - x\cos(x)}{\cos(x) + \cos(x) - x\sin(x)}$

$= \frac{0-0}{1+1-0} = \boxed{0}$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$

common denom $\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$

$\infty - \infty$

$$\lim_{\theta\to 0}\left(\frac{1}{1-\cos(\theta)} - \frac{2}{\sin^2(\theta)}\right)$$

$$= \lim_{\theta\to 0} \frac{1+\cos\theta}{1-\cos^2\theta} - \frac{2}{\sin^2\theta}$$

$\frac{0}{0}$ $$= \lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin^2\theta}$$

$$\overset{LR}{=} \lim_{\theta\to 0} \frac{-\sin\theta}{2\sin\theta\cos\theta} = \lim_{\theta\to 0} \frac{-1}{2\cos\theta}$$

$$= \boxed{-\frac{1}{2}}$$

1^∞

$$y = \lim_{x\to 0}(1-2x)^{\frac{1}{x}}$$

$$\ln(y) = \lim_{x\to 0} \frac{1}{x} \ln(1-2x) \quad \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0} \frac{1}{1} \frac{-2}{1-2x} = -2$$

$$\Rightarrow y = e^{-2} = \boxed{\frac{1}{e^2}}$$

0^0

$$y = \lim_{\theta\to 0^+}(\sin(\theta))^{\tan(\theta)}$$

$$\ln(y) = \lim_{\theta\to 0^+} \tan\theta \ln(\sin\theta) \quad 0\cdot\infty$$

$$= \lim_{\theta\to 0^+} \frac{\ln(\sin\theta)}{\cot\theta} \quad \frac{-\infty}{\infty}$$

$$\overset{LR}{=} \lim_{\theta\to 0^+} \frac{\frac{\cos\theta}{\sin\theta}}{-\csc^2\theta} = \lim_{\theta\to 0^+} -\sin\theta\cos\theta$$

$$= 0$$

$$\Rightarrow y = e^0 = \boxed{1}$$

∞^0

$$y = \lim_{x\to 0^+}(\tan x)^x$$

$$\ln(y) = \lim_{x\to 0^+} x \ln(\tan(x)) \quad 0\cdot -\infty$$

$$= \lim_{x\to 0^+} \frac{\ln(\tan x)}{1/x} \quad \frac{-\infty}{\infty}$$

$$\overset{LR}{=} \lim_{x\to 0^+} \frac{\frac{\sec^2(x)}{\tan(x)}}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2}{\sin x \cos x} \quad \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0^+} \frac{-2x}{-\sin^2(x) + \cos^2(x)} = \frac{0}{0+1} = 0$$

$$\Rightarrow y = e^0 = \boxed{1}$$

$y = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x$ $\quad 1^\infty \longrightarrow \ln(y) = \lim_{x\to\infty} x \ln\left(1 + \frac{a}{x}\right) \quad \infty \cdot 0$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{1/x} \quad \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}\left(\frac{-a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ax}{x+a} \quad \frac{\infty}{\infty}$$

$$\overset{LR}{=} \lim_{x\to\infty} \frac{a}{1} = a$$

$\Rightarrow y = \boxed{e^a}$

$y = \lim_{x\to 0} (e^{ax} + x)^{\frac{1}{x}}$ $\quad 1^\infty \longrightarrow \ln(y) = \lim_{x\to 0} \frac{\ln(e^{ax}+x)}{x} \quad \frac{0}{0}$

$$\overset{LR}{=} \lim_{x\to 0} \frac{ae^{ax}+1}{e^{ax}+x} = \frac{a+1}{1} = a+1$$

$\Rightarrow y = \boxed{e^{a+1}}$

$y = \lim_{x\to 0^+} \frac{x^x - 1}{\ln(x) + x - 1}$ $\quad \frac{0^0 - 1}{-\infty + 0 - 1}$

$w = \lim_{x\to 0^+} x^x \longrightarrow \ln(w) = \lim_{x\to 0^+} x\ln(x) \quad 0(-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \frac{-\infty}{\infty}$$

$$\overset{LR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to 0^+} -x = 0$$

$\Rightarrow w = e^0 = 1 = \lim_{x\to 0} x^x$

$$y = \frac{\lim_{x\to 0^+} x^x - 1}{\lim_{x\to 0^+} \ln(x) + x - 1} = \boxed{0}$$

$\frac{1-1}{-\infty}$

Definition. (Growth Rates of Functions (as $x \to \infty$))

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then f *grows faster than* g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, equivalently, if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

The functions f and g have *comparable growth rates* if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

where $0 < M < \infty$ (M is positive and finite)

Theorem 4.14: Ranking Growth Rates as $x \to \infty$

Let $f \ll g$ mean that g grows faster than f as $x \to \infty$. With positive real numbers p, q, r, and s and with $b > 1$,

$$(\ln(x))^q \ll x^p \ll x^p(\ln(x))^r \ll x^{p+s} \ll b^x \ll x^x$$

Example. Rank the functions in order of increasing growth rates as $x \to \infty$:

x^3, $\ln(x)$, x^x, and 2^x

$\ln(x) << x^3 << 2^x << x^x$

x^{100}, $\ln\left(x^{10}\right)$, x^x, and 10^x

$\ln(x^{10}) << x^{100} << 10^x << x^x$

Example. Use limits to compare and rank growth ranks of the following functions:

$\ln(x^{20})$; $\ln(x)$ $\quad \lim_{x\to\infty} \frac{\ln(x^{20})}{\ln(x)} = \lim_{x\to\infty} 20\frac{\ln(x)}{\ln(x)} = 20$

$\Rightarrow$ comparable.

$\ln(x)$; $\ln(\ln(x))$ $\quad \lim_{x\to\infty} \frac{\ln(\ln(x))}{\ln(x)} \overset{LR}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln(x)}\cdot\frac{1}{x}}{1/x} = 0$

$\Rightarrow \ln(\ln(x)) << \ln(x)$

100^x; x^x ________________

$100^x << x^x$
Via Thm 4.14

e^{x^2}; $x^{x/10}$ ________________

$\lim_{x\to\infty} \left(\frac{x^{1/10}}{e^x}\right)^x \to \lim_{x\to\infty} \frac{x^{1/10}}{e^x} \overset{LR}{=} \lim_{x\to\infty} \frac{\frac{1}{10}x^{-9/10}}{e^x} = 0$

$\Rightarrow x^{1/10} << e^x$

$\Rightarrow x^{x/10} << e^{x^2}$