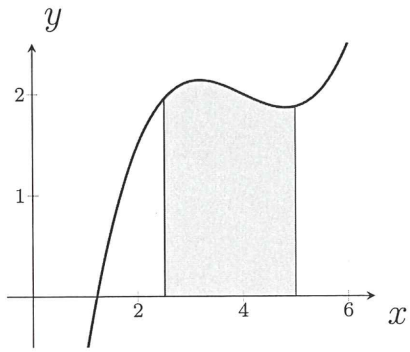
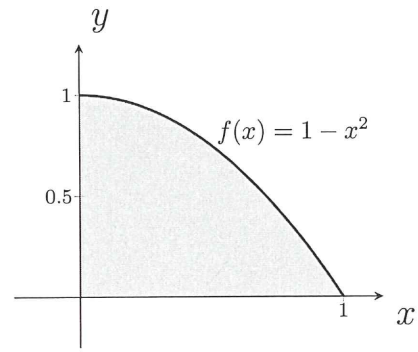
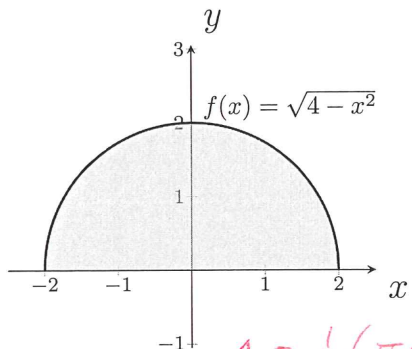
5.1: Approximating Areas under Curves

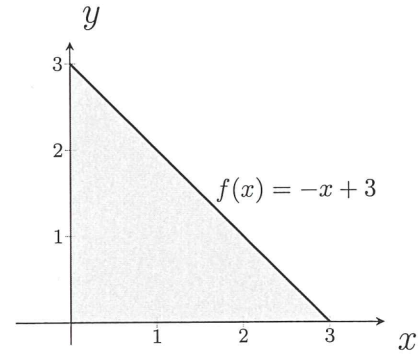




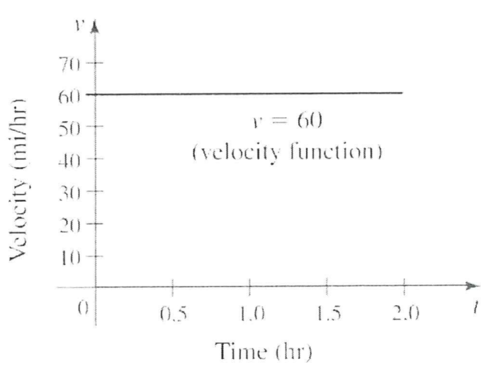
Finding the area under the curve is simple in some cases:

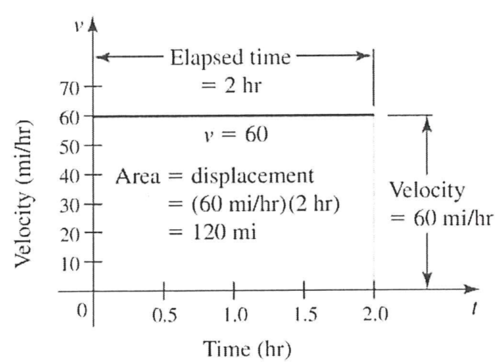


$A = \frac{1}{2}(\pi r^2)$
$= \frac{1}{2}(\pi\, 2^2) = 2\pi$

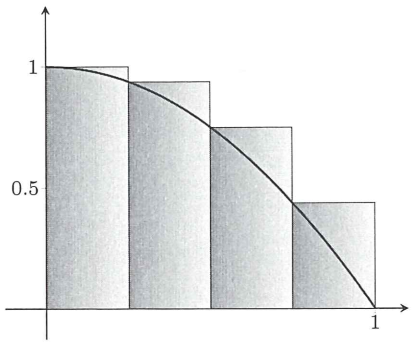


$A = \frac{1}{2} bh = \frac{1}{2}(3)(3) = \frac{9}{2}$

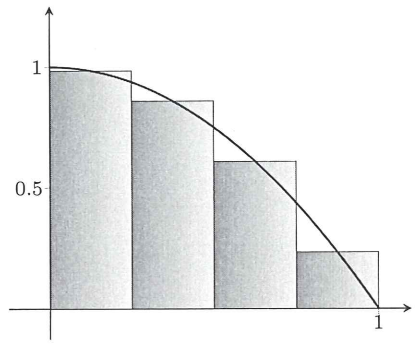




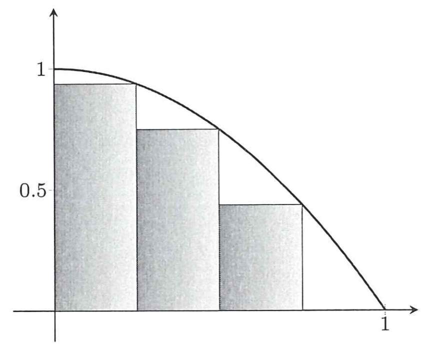
For functions whose curves are irregular shapes, we can approximate the area using rectangles:



Left rectangles

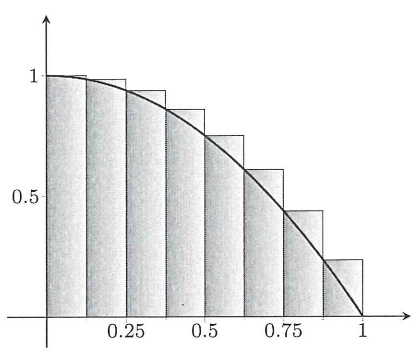


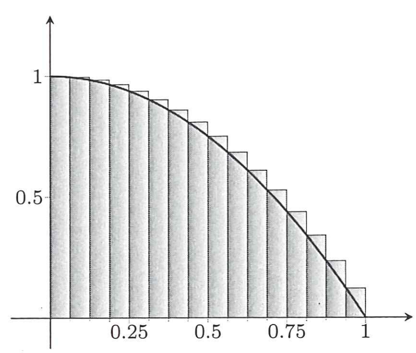
Midpoint rectangles

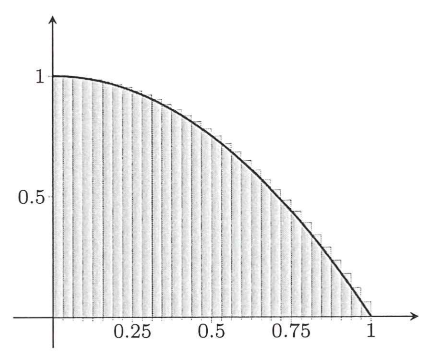


Right rectangles

These approximations are much more accurate when more rectangles are used:







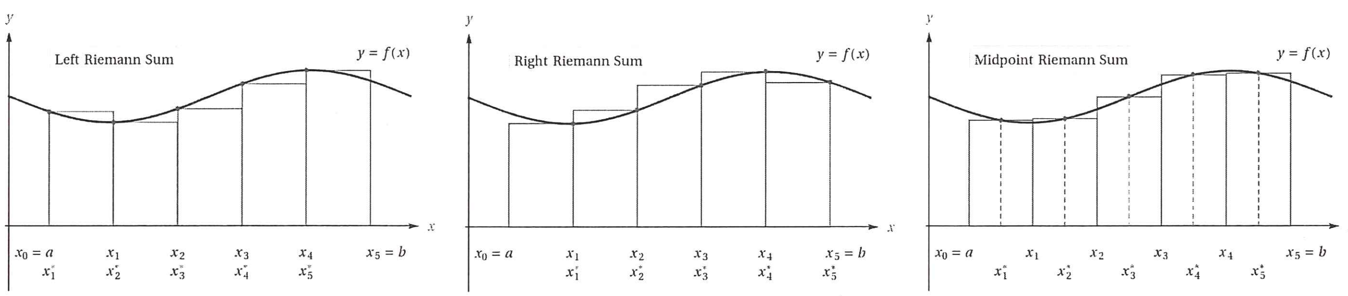
Definition. (Riemann Sum)
Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the k-th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$,
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$, and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

In general, midpoint rectangles give better approximations then left or right rectangles.



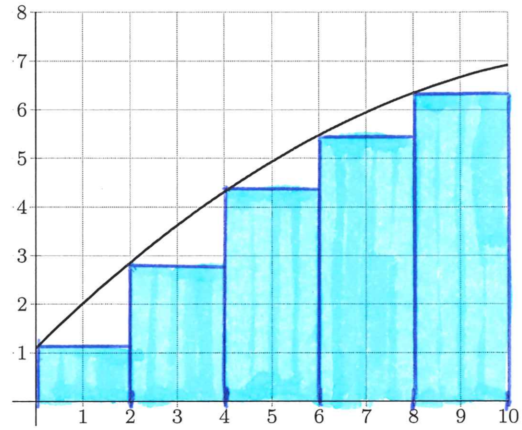
When estimating the area under the graph on the interval $[a, b]$, we define

$$\Delta x = \frac{b-a}{n}$$

to be the width of the rectangles. The height of the rectangles is given by $f(x_i)$, where the x_i's are Δx apart.

Example. Use the plots below to estimate the area under the given graph of $f(x)$:

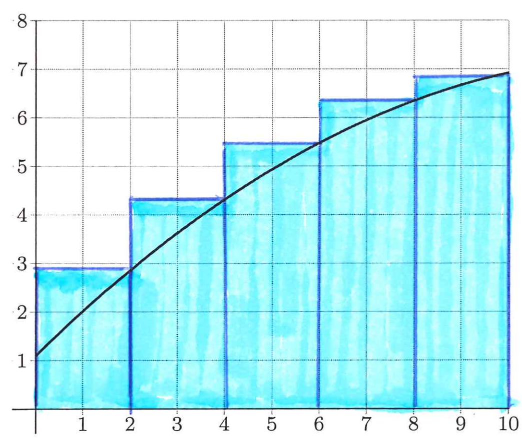
Sketch five rectangles and use them to find a lower estimate for the area under the given graph of $f(x)$ from $x = 0$ to $x = 10$.



$A \approx 2(1) + 2(3) + 2(4.3) + 2(5.5) + 2(6.2)$

$= 2(20) = 40$

Sketch five rectangles and use them to find a upper estimate for the area under the given graph of $f(x)$ from $x = 0$ to $x = 10$.

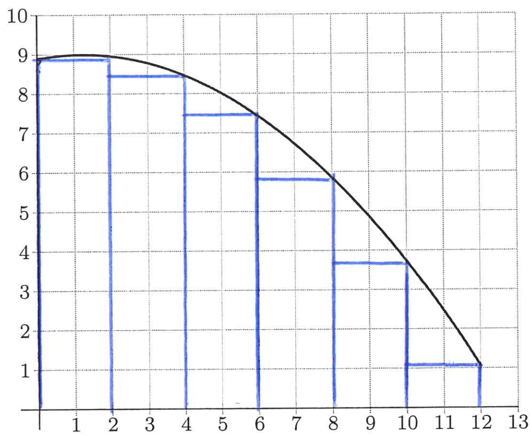


$A \approx 2(3 + 4.3 + 5.5 + 6.2 + 6.8)$

$= 2(25.8) = 51.6$

Use six right rectangles to estimate the area under the given graph of $f(x)$ from $x = 0$ to $x = 12$ .Is the estimate an under-approximation or an over-approximation? Why?

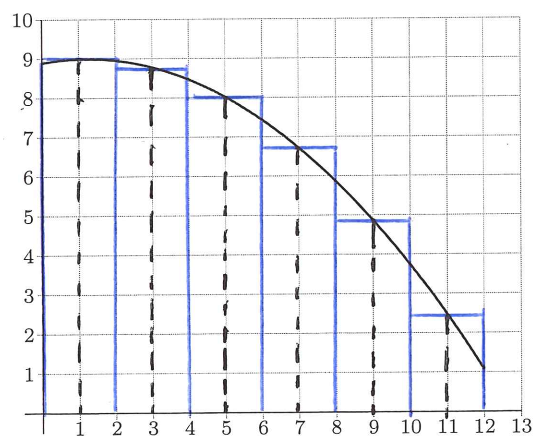
$\Delta x = \frac{12-0}{6} = 2$



$A \approx 2(9) + 2(8.5) + 2(7.5) + 2(5.9) + 2(3.7) + 2(1.1)$
$= 2(35.7) = 71.4$

Use six midpoint rectangles to estimate the area under the given graph of $f(x)$ from $x = 0$ to $x = 12$. Talk about the quality of this estimate.

$\Delta x = \frac{12-0}{6} = 2$



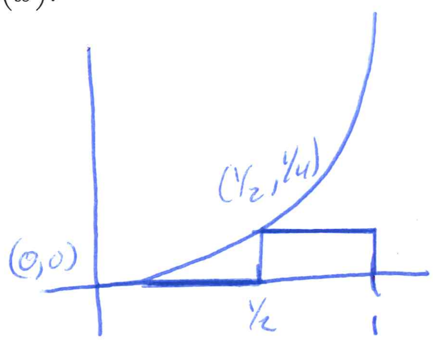
The midpoint Riemann sum is a good approximation. It follows f(x) closely and some of the over/under estimates cancel.

$A \approx 2(9) + 2(8.7) + 2(8) + 2(7.7) + 2(4.9) + 2(2.5)$
$= 2(40.8) = 81.6$

Actual area is $\frac{1}{3} = 0.\overline{3}$

Example. Consider $f(x) = x^2$ on the interval $[0, 1]$.

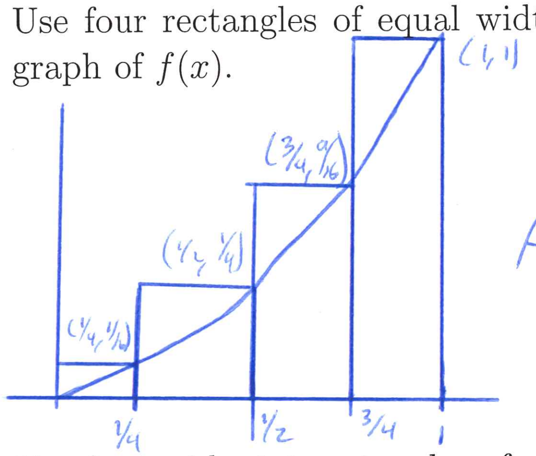
a) Use two rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.



$\Delta x = \frac{1-0}{2} = 1/2$

$A \approx \frac{1}{2}(0) + \frac{1}{2}\left(\frac{1}{4}\right) = \boxed{\frac{1}{8}} = 0.125$

b) Use four rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.

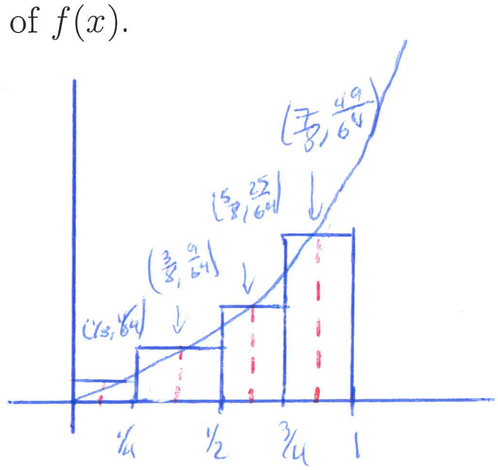


$\Delta x = \frac{1-0}{4} = \frac{1}{4}$

$A \approx \frac{1}{4}(1/16) + \frac{1}{4}(1/4) + \frac{1}{4}(9/16) + \frac{1}{4}(1)$

$= \frac{1}{4}\left(\frac{15}{8}\right) = \frac{15}{32} = 0.46875$

c) Use four midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.



$\Delta x = \frac{1-0}{4} = \frac{1}{4}$

$A \approx \frac{1}{4}\left(\frac{1}{64}\right) + \frac{1}{4}\left(\frac{9}{64}\right) + \frac{1}{4}\left(\frac{25}{64}\right) + \frac{1}{4}\left(\frac{49}{64}\right)$

$= \frac{1}{4}\left(\frac{84}{64}\right) = \frac{21}{64} = 0.328125$

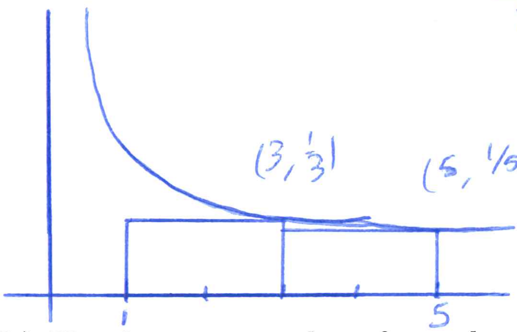
$A = \ln(5) \approx 1.60944$

Example. Consider $f(x) = \dfrac{1}{x}$ on the interval $[1, 5]$.

a) Use two rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.

$\Delta x = \dfrac{5-1}{2} = 2$

$A = 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{5}\right) = \frac{16}{15} \approx 0.5\overline{3}$

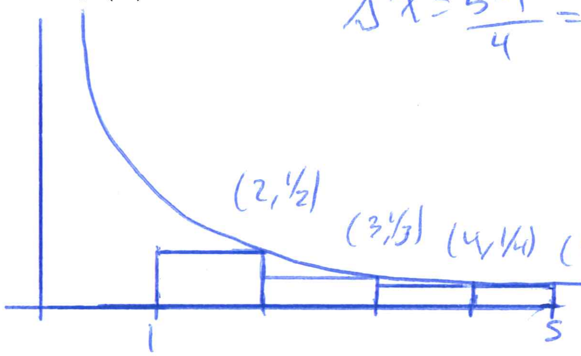


b) Use four rectangles of equal width to find a lower sum for the area under the graph of $f(x)$.

$\Delta x = \dfrac{5-1}{4} = 1$

$A \approx 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right) + 1\left(1/5\right)$

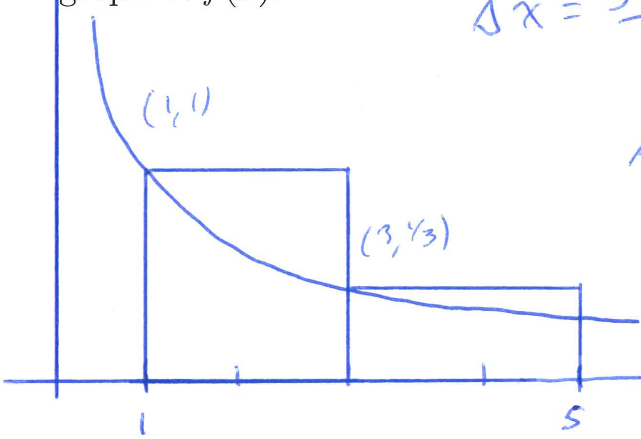
$= \frac{77}{60} \approx 1.28\overline{3}$



c) Use two rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.

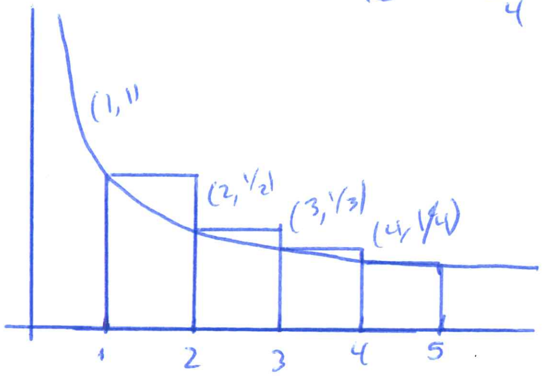
$\Delta x = \dfrac{5-1}{2} = 2$

$A \approx 2(1) + 2\left(\frac{1}{3}\right) = \frac{8}{3} = 2.\overline{66}$



d) Use four rectangles of equal width to find an upper sum for the area under the graph of $f(x)$.

$\Delta x = \frac{5-1}{4} = 1$

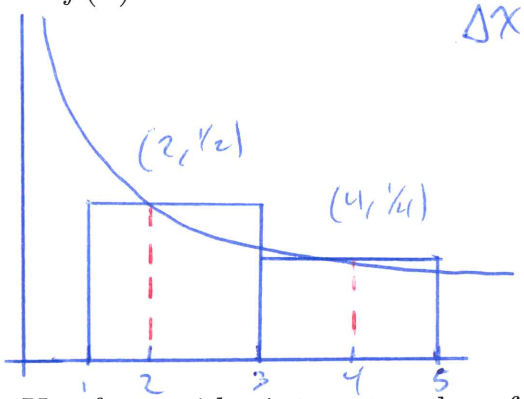


$A \approx 1(1) + 1(\frac{1}{2}) + 1(\frac{1}{3}) + 1(\frac{1}{4}) = \frac{25}{12}$

$= 2.08\overline{3}$

e) Use two midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.

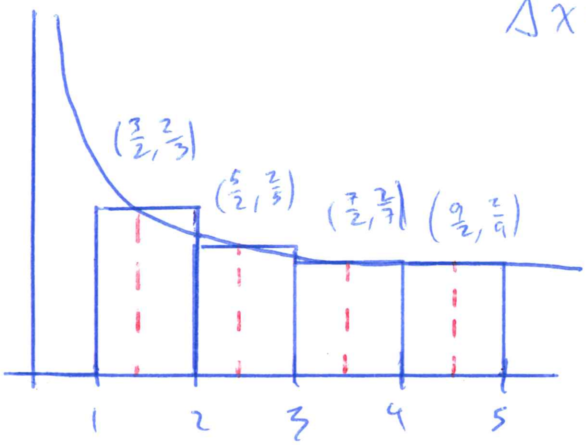
$\Delta x = \frac{5-1}{2} = 2$



$A \approx 2(\frac{1}{2}) + 2(\frac{1}{4}) = \frac{3}{2} = 1.5$

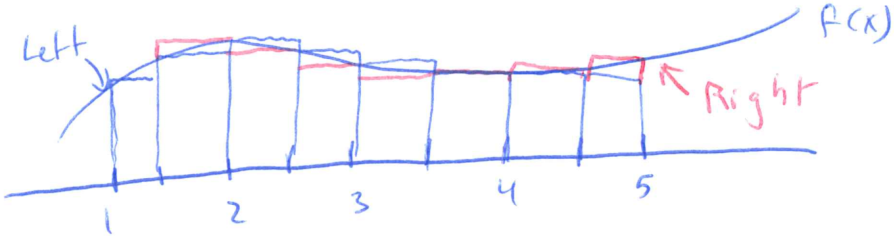
f) Use four midpoint rectangles of equal width to estimate the area under the graph of $f(x)$.

$\Delta x = \frac{5-1}{4} = 1$



$A \approx 1(\frac{2}{3}) + 1(\frac{2}{5}) + 1(\frac{2}{7}) + 1(\frac{2}{9})$

$= \frac{496}{315} \approx 1.5746$



Example. Use the tabulated values of f to evaluate both the left and right Reimann sums. ($n = 8$, $[1, 5]$)

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	0	2	3	2	2	1	0	2	3

$\Delta x = \frac{5-1}{8} = \frac{1}{2}$

$A_L \approx \frac{1}{2}(0) + \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(2) + \frac{1}{2}(2) + \frac{1}{2}(1) + \frac{1}{2}(0) + \frac{1}{2}(2)$

$= 6$

$A_R \approx \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(2) + \frac{1}{2}(2) + \frac{1}{2}(1) + \frac{1}{2}(0) + \frac{1}{2}(2) + \frac{1}{2}(3) = \frac{11}{2}$

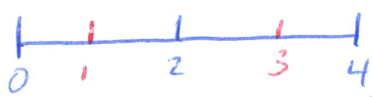
When velocity is a continuous function, the area between the curve and the x-axis gives the displacement.

Example. The velocities (in m/s) of an automobile moving along a straight freeway over a four-second period are given in the following table. Find the midpoint Riemann sum approximation to the displacement on $[0, 4]$ with $n = 2$ and $n = 4$ subintervals.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	20	25	30	35	30	30	35	40	40

$n = 2$

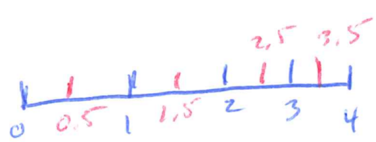
$\Delta t = \frac{4-0}{2} = 2$



$S \approx 2(30) + 2(35) = 130\ m$

$n = 4$

$\Delta t = \frac{4-0}{4} = 1$



$S \approx 1(25) + 1(35) + 1(30) + 1(40) = 130\ m$

Example. Use the following table of recorded velocities answer the following questions:

300 sec

600

etc.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1.0	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

a) Estimate the total displacement using 12 subintervals of length 5 with left-endpoint values.

Δt

$s \approx 300(1.0+1.2+1.7+2.0+1.8+1.6+1.4+1.2+1.0+1.8+1.5+1.2)$

$= 300(17.4) = 5220\text{ m}$

b) Estimate the total displacement using 12 subintervals of length 5 with right-endpoint values.

$s \approx 300(1.2+1.7+2.0+1.8+1.6+1.4+1.2+1.0+1.8+1.5+1.2+0)$

$= 300(16.4) = 4920\text{ m}$

> **Definition.**
> If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then
> $$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

Example. Rewrite the sums without sigma notation and evaluate:

a) $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

$= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

b) $\displaystyle\sum_{k=1}^{4} (-1)^k \cos(k\pi)$

$= -\cos(\pi) + \cos(2\pi) - \cos(3\pi) + \cos(4\pi)$

$= -(-1) + 1 - (-1) + 1 = 4$

Example. Rewrite the following sum without sigma notation. Note that n denotes the number of rectangles and the letters i, j, and k are typically used for indexing.

$$\sum_{k=1}^{n} \frac{1}{n}(k^2+1)$$

$= \frac{1}{n}\left((1+1) + (4+1) + (9+1) + \ldots + (n^2+1)\right)$

$= \frac{1}{n}(1+4+9+\ldots+n^2) + \frac{1}{n}(\underbrace{1+1+\ldots+1}_{n\text{-times}})$

$= \frac{1}{n}(1+4+9+\ldots+n^2) + 1$

Example. Which of the following summations represent the sum $1+2+4+8+16+32$?

a) $\displaystyle\sum_{k=1}^{6} 2^{k-1}$

$= 2^0+2^1+2^2+2^3+2^4+2^5$

$= 1+2+4+8+16+32$

Yes

b) $\displaystyle\sum_{k=0}^{5} 2^{k}$

$= 2^0+2^1+2^2+2^3+2^4+2^5$

$= 1+2+4+8+16+32$

Yes

c) $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

$= 2^0+2^1+2^2+2^3+2^4+2^5$

$= 1+2+4+8+16+32$

Yes

Example. Which of the following summations represent the sum $1-2+4-8+16-32$?

a) $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$

$= (-2)^0+(-2)^1+(-2)^2+(-2)^3+(-2)^4+(-2)^5$

$= 1-2+4-8+16-32$

Yes

b) $\displaystyle\sum_{k=0}^{5} (-1)^{k}2^{k}$

$= (-1)^0 2^0+(-1)^1 2^1+(-1)^2 2^2$
$+(-1)^3 2^3+(-1)^4 2^4+(-1)^5 2^5$

$= 1-2+4-8+16-32$

Yes

c) $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1}2^{k+2}$

$= (-1)^{-1}2^0+(-1)^0 2^1+(-1)^1 2^2$
$+(-1)^2 2^3+(-1)^3 2^4+(-1)^4 2^5$

$= -1+2-4+8-16+32$

No

Example. Express the following sums in sigma notation:

a) $-1+4-9+16-25$

$$= \sum_{k=1}^{5} (-1)^k k^2$$

b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

$$= \sum_{k=1}^{4} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{4} 2^{-k}$$

c) $\frac{5}{2}+\frac{10}{3}+\frac{15}{4}+\frac{20}{5}+\frac{25}{6}+\frac{30}{7}$

$$= \sum_{k=1}^{6} \frac{5k}{k+1}$$

d) $4+9+14+\cdots+44$

$$= \sum_{k=0}^{8} 4+5k$$

Example. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$, evaluate the following

a) $\sum_{k=1}^{n} 8a_k$ $= 8\underbrace{\sum_{k=1}^{n} a_k}_{0} = 8(0) = 0$

b) $\sum_{k=1}^{n} 250b_k$ $= 250\underbrace{\sum_{k=1}^{n} b_k}_{1} = 250$

c) $\sum_{k=1}^{n} (a_k+1)$

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1$$

$$= 0 + n = n$$

d) $\sum_{k=1}^{n} (b_k-1)$ $= \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1$

$$= 1-n$$

Theorem 5.1: Sums of Powers of Integers
Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Example. Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = 1+2+\dots+(n-1)+n$$

$$+\quad \sum_{k=1}^{n} k = n+(n-1)+\dots+2+1$$

$$2\sum_{k=1}^{n} k = \underbrace{(n+1)+(n+1)+\dots+(n+1)+(n+1)}_{n\text{-times}} = n(n+1)$$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

Example. Evaluate the following sums

a) $\sum_{k=1}^{10} k = \frac{10(11)}{2} = 55$

b) $\sum_{k=1}^{10} (1+k^2) = \sum_{k=1}^{10} 1 + \sum_{k=1}^{10} k^2 = 10 + \frac{10(11)(21)}{6} = 395$

c) $\sum_{k=1}^{10} k^3 = \frac{10^2(11)^2}{4} = \frac{100(121)}{4} = 25(121) = 3025$

d) $\sum_{k=1}^{7} (-2k-4) = -2\sum_{k=1}^{7} k - \sum_{k=1}^{7} 4 = (-2)\frac{7(8)}{2} - 4(7)$ $= -56 - 28 = -84$

e) $\sum_{k=1}^{6} (3-k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = 3(6) - \frac{6(7)(13)}{6}$ $= 18 - 91 = -73$

f) $\sum_{m=1}^{3} \frac{2m+2}{3} = \frac{2}{3}\sum_{m=1}^{3} m + \sum_{m=1}^{3} \frac{2}{3} = \frac{2}{3}\cdot\frac{3(4)}{2} + \frac{2}{3}(3)$ $= 4+2 = \boxed{6}$

Definition. (Left, Right and Midpoint Riemann Sums in Sigma Notation) Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** for f on $[a, b]$ is $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$.
- $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$.
- $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta$ is a **midpoint Riemann sum** if $x_k^* = a + (k - 1/2)\Delta x$.

Example. For the function $f(x) = 3x^2$, find a formula for the upper sum obtained by dividing the interval $[0, 1]$ into n equal subintervals.

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$ $\quad x_i = 0 + i\left(\frac{1}{n}\right) = i/n$ $\quad f(x_i) = 3(x_i)^2 = 3\, i^2/n^2$

$$\Rightarrow \sum_{i=1}^{n} \frac{3i^2}{n^2}\cdot\frac{1}{n} = \sum_{i=1}^{n} \frac{3i^2}{n^3} = \frac{3}{n^3}\sum_{i=1}^{n} i^2 = \frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \boxed{\frac{2n^2+3n+1}{2n^2}}$$

Now take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 3x^2$ over $[0, 1]$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3i^2}{n^3} = \lim_{n\to\infty} \frac{2n^2+3n+1}{2n^2} = \boxed{1}$$

Example. For the function $f(x) = 2x$, find a formula for the upper sum obtained by dividing the interval $[0, 3]$ into n equal subintervals.

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$ $\quad x_i = 0 + i\,\Delta x = \frac{3i}{n}$ $\quad f(x_i) = \frac{6i}{n}$

$$\Rightarrow \sum_{i=1}^{n} \frac{6i}{n} \cdot \frac{3}{n} = \sum_{i=1}^{n} \frac{18i}{n^2} = \frac{18}{n^2} \sum_{i=1}^{n} i = \frac{18}{n^2} \frac{n(n+1)}{2} = \boxed{\frac{9n+9}{n}}$$

Now take the limit of the sum as $n \to \infty$ to calculate the area under $f(x) = 2x$ over $[0, 3]$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{18i}{n^2} = \lim_{n\to\infty} \frac{9n+9}{n} = \boxed{9}$$