2.3 Techniques for Computing Limits

Example.

a) $\lim_{x \to 3} \frac{1}{2}x - 7$ $= \frac{1}{2}(3) - 7$

$= \boxed{-11/2}$

b) $\lim_{x \to 2} 6$ $= 6$

Definition. (Briggs)
Limit Laws: Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. The following properties hold, where c is a real number, and $n > 0$ is an integer.

1. **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. **Difference:** $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. **Constant multiple:** $\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$

4. **Product:** $\lim_{x \to a} (f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

5. **Quotient:** $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

6. **Power:** $\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n$

7. **Root:** $\lim_{x \to a} (f(x))^{1/n} = (\lim_{x \to a} f(x))^{1/n}$

Example. Suppose $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = 5$ and $\lim_{x\to 2} h(x) = 8$.

a) $\lim_{x\to 2} \dfrac{f(x) - g(x)}{h(x)} = \dfrac{4-5}{8} = \boxed{-\dfrac{1}{8}}$

b) $\lim_{x\to 2} (6f(x)g(x) + h(x)) = 6(4)(5) + 8 = \boxed{128}$

c) $\lim_{x\to 2} (g(x))^3 = 5^3 = \boxed{125}$

Example. For $g(x) = \dfrac{x+6}{x^2-36}$, find

1. $\lim_{x\to 0} g(x) = \dfrac{0+6}{0-36} = \boxed{-1/6}$

2. $\lim_{x\to -6} g(x) = \lim_{x\to -6} \dfrac{x+6}{(x+6)(x-6)} = \lim_{x\to -6} \dfrac{1}{x-6} = \boxed{-\dfrac{1}{12}}$

Example. $\lim_{x\to 2} \frac{\sqrt{2x^3+9}+3x-1}{4x+1}$ $= \frac{\sqrt{2(2)^3+9}+3(2)-1}{4(2)+1}$

$= \frac{\sqrt{25}+5}{9}$

$= \boxed{\frac{10}{9}}$

Example. $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} -2x+4 & \text{if } x \leq 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -2x+4 = -2(1)+4 = 2$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x-1} = 0$

$\Longrightarrow \lim_{x\to 1} f(x)$ DNE

Example. $\lim\limits_{x\to 2} \dfrac{x^2-6x+8}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(x-2)(x-4)}{(x-2)(x+2)} = \lim\limits_{x\to 2} \dfrac{x-4}{x+2} = \dfrac{-2}{4} = \boxed{-\dfrac{1}{2}}$

Example. $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} = \lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x}+1} = \boxed{\dfrac{1}{2}}$

– or –

$\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}\left(\dfrac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim\limits_{x\to 1} \dfrac{x-1}{(x-1)(\sqrt{x}+1)} = \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x}+1} = \boxed{\dfrac{1}{2}}$

Example. $\lim\limits_{x\to -4} \sqrt{16-x^2}$

$\lim\limits_{x\to -4^-} \sqrt{16-x^2}$ DNE $\because$ $-4 \le x \le 4$

$\lim\limits_{x\to -4^+} \sqrt{16-x^2} = \boxed{0}$

$\Rightarrow \lim\limits_{x\to -4} \sqrt{16-x^2}$ DNE

Example. $\lim_{x\to 2} \dfrac{x^3 - 6x^2 + 8x}{\sqrt{x-2}} \left(\dfrac{\sqrt{x-2}}{\sqrt{x-2}}\right) = \lim_{x\to 2} \dfrac{x(x^2-6x+8)\sqrt{x-2}}{x-2}$

$$= \lim_{x\to 2} \frac{x(x-4)(x-2)\sqrt{x-2}}{x-2}$$

$$= \lim_{x\to 2} x(x-4)\sqrt{x-2} \qquad \text{DNE}$$

$\lim_{x\to 2^-} x(x-4)\sqrt{x-2}$ DNE $\because$ $x \geq 2$

$\lim_{x\to 2^+} x(x-4)\sqrt{x-2} = 2(2-4)\sqrt{2-2} = 0$

Example. $\lim_{y\to a} \dfrac{(y-a)^{12} + 6y - 6a}{y-a} = \lim_{y\to a} \dfrac{(y-a)\left[(y-a)^{11} + 6\right]}{y-a}$

$$= \lim_{y\to a} (y-a)^{11} + 6$$

$$= \boxed{6}$$

The Squeeze Theorem: Assume the functions f, g and h satisfy $f(x) \le g(x) \le h(x)$ for all values of x near a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Example. Consider the function $f(x) = x^2 \sin(1/x)$. What is $\lim\limits_{x\to 0} f(x)$?

Since $-1 \le \sin(1/x) \le 1$, then $-x^2 \le x^2 \sin(1/x) \le x^2$

so $\lim\limits_{x\to 0} -x^2 \le \lim\limits_{x\to 0} x^2 \sin(1/x) \le \lim\limits_{x\to 0} x^2$

$\Rightarrow \quad 0 \le \lim\limits_{x\to 0} x^2 \sin(1/x) \le 0$

$\boxed{\text{Thus, by the squeeze theorem, } \lim\limits_{x\to 0} x^2 \sin(1/x) = 0}$

Example. Use the squeeze theorem on $-|x| \le x \sin \frac{1}{x} \le |x|$.

$-1 \le \sin(1/x) \le 1$

$-|x| \le x \sin(1/x) \le |x|$

$\Rightarrow \lim\limits_{x\to 0} -|x| \le \lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right) \le \lim\limits_{x\to 0} |x|$

$\Rightarrow \quad 0 \le \lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right) \le 0$

$\boxed{\text{Thus, by the squeeze theorem, } \lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0}$

Example. $\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}$ $= \lim_{x\to 0} \frac{1-\cos^2(x)}{1-\cos(x)} = \lim_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{1-\cos(x)}$

$= \lim_{x\to 0} 1+\cos(0) = 1+1 = \boxed{2}$

Example. $\lim_{x\to 0} \frac{1-\cos 2x}{\sin x}$ $= \lim_{x\to 0} \frac{1-[\cos^2(x)-\sin^2(x)]}{\sin x}$

$= \lim_{x\to 0} \frac{1-\cos^2(x)+\sin^2(x)}{\sin(x)}$

$= \lim_{x\to 0} \frac{\sin^2(x)+\sin^2(x)}{\sin(x)}$

$= \lim_{x\to 0} 2\sin(x) = \boxed{0}$