4.5: Optimization Problems

Note: Please note that these 'word problems' are different from the related rates problems that we did in section 3.11.

Guidelines for Optimization Problems

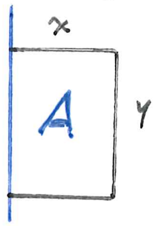
1. Read the problem and identify variables with a diagram. Only put numbers on the diagram if they are constant.
2. Express the function that will be optimized.
3. Identify the constraint(s). Use the constraint(s) to eliminate/rewrite all but the single independent variable of the objective function.
4. Use derivatives to find the absolute max/min of the objective function.

   Make sure you are finding the correct extrema!

   - First derivative test ($f'(x)$ changes signs at $x = c$)
   - Second derivative test ($f'(c) < 0$ (neg) means $f(c)$ is a max).
   - Evaluate at endpoints and critical points.
5. Summarize your result in a sentence.

1. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He decides not to fence along the river. What are the dimensions of the field that has the largest area?

Draw a diagram. Define your variables.



x is width (feet)

y is length (feet)

A is area (feet^2)

Express your constraints and the variable that needs optimizing.

$$\left.\begin{array}{ll} \max & A = xy \\ \text{s.t.} & 2x + y = 2400 \end{array}\right\} \quad y = 2400 - 2x \; ; \quad 0 \le x \le 1200$$

$$\Rightarrow A(x) = x(2400 - 2x) = 2400x - 2x^2$$

Find the local maximum.

$$\left.\begin{array}{l} A'(x) = 2400 - 4x \overset{\text{set}}{=} 0 \\ x = 600 \end{array}\right\} \quad A(600) = 720{,}000 \text{ ft}^2$$

Verify the critical point is the absolute maximum.

$A''(x) = -4$

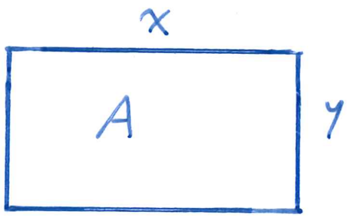
$A''(600) = -4 < 0 \Rightarrow$ concave down $\Rightarrow$ max

x	$A(x)$
0	0
600	720,000
1200	0

State your answer ensuring that you are answering the question asked.

The maximum area is $720{,}000$ ft^2 when $x = 600$ ft.

2. Show among all rectangles with an 8 meter perimeter, the one with the largest area is a square.



x : width (feet)

y : length (feet)

A : area (feet^2)

max $A = xy$

s.t. $2x + 2y = 8$

$\Rightarrow y = 4 - x$

$\Rightarrow A(x) = x(4-x)$

$= 4x - x^2$

$A'(x) = 4 - 2x \overset{\text{set}}{=} 0$

$x = 2$

$A''(x) = -2$

$A''(2) = -2 < 0$ $\rightarrow$ Concave down

$\rightarrow$ maximum

$x = 2 \rightarrow y = 2$

3. Find x and y such that $xy = 50$ but $x + y$ is minimal. $(x > 0,\ y > 0)$

$$\left.\begin{array}{l} \min f = x + y \\ \text{s.t. } xy = 50 \end{array}\right\} \quad y = \frac{50}{x} \Rightarrow x + y = x + \frac{50}{x}$$

$$f'(x) = 1 - \frac{50}{x^2} \overset{\text{set}}{=} 0$$

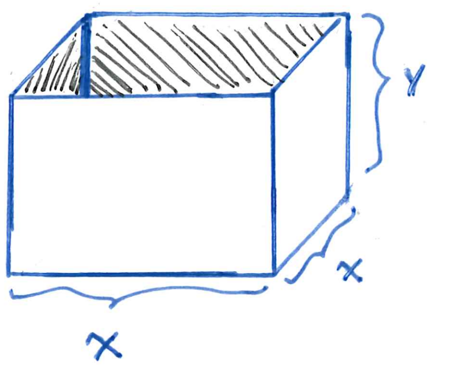
$$x^2 = 50$$

$$x = \sqrt{50} = 5\sqrt{2}$$

$$f''(x) = \frac{100}{x^3} \longrightarrow f''(5\sqrt{2}) = \frac{20}{\sqrt{2}} > 0 \rightarrow \text{Concave up}$$

$\rightarrow$ min

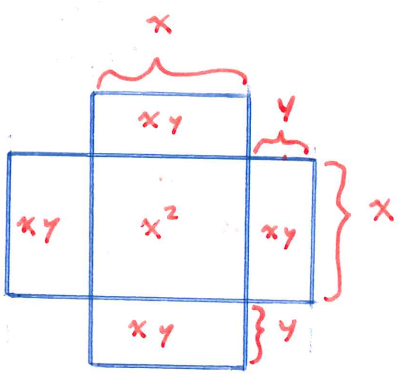
4. A box with a square base and an open top must have a volume of $32,000\,cm^3$. Find the dimensions of the box that minimizes the amount of material used.



x: length & width (cm) $\quad x > 0$

y: height (cm) $\quad y > 0$

V: volume (cm^3)



$$\min \quad S = x^2 + 4xy$$

$$\text{s.t.} \quad x^2 y = 32000$$

$$y = \frac{32000}{x^2} \implies S(x) = x^2 + \frac{128000}{x}$$

Find

$$S'(x) = 2x - \frac{128000}{x^2} \overset{\text{set}}{=} 0$$

$$x^3 = 64000$$

$$\boxed{x = 40}$$

Verify

$$S''(x) = 2 + \frac{256000}{x^3}$$

$$\to S''(40) = 2 + \frac{256000}{40^3} > 0 \to \text{concave up} \to \min$$

The open-top square-based box of volume of $32000\,cm^3$ uses the minimal amount of material when $x = 40$ cm and $y = 20$ cm.

5. If $y = 2x - 89$, what is the minimum value of the product xy?

$$\left.\begin{array}{ll} \min & f = xy \\ \text{s.t.} & y = 2x - 89 \end{array}\right\} \quad f(x) = x(2x-89) = 2x^2 - 89x$$

Find

$$f'(x) = 4x - 89 \overset{\text{set}}{=} 0 \qquad \boxed{x = \frac{89}{4}} \quad \Rightarrow \quad y = -\frac{89}{2}$$

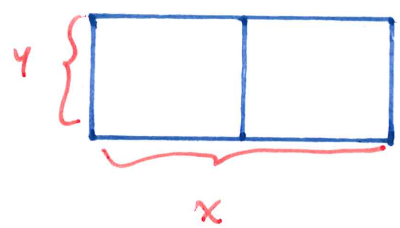
Verify

$$f''(x) = 4 > 0 \quad \rightarrow \quad \text{concave up} \quad \rightarrow \text{min}$$

The product xy is minimized when $x = \frac{89}{4}$ and $y = -\frac{89}{2}$

$$\longrightarrow xy = \left(\frac{89}{4}\right)\left(-\frac{89}{2}\right) = -\frac{89^2}{8}$$

6. A farmer has 900 meters of fencing. The fencing is to be used to enclose a rectangular field and to divide it in half. Find the dimensions of the field that has maximum area.



x: overall length (m)

y: overall width (m)

A: area (m^2)

min $A = xy$

s.t. $2x + 3y = 900$

$\Big\}\; y = 300 - \frac{2}{3}x$

$$\Rightarrow A(x) = x\left(300 - \frac{2}{3}x\right)$$

$$= 300x - \frac{2}{3}x^2$$

Find x, y

$$A'(x) = 300 - \frac{4}{3}x \overset{\text{set}}{=} 0$$

$$300 = \frac{4}{3}x$$

$$\boxed{225 = x}$$

$$y = 300 - \frac{2}{3}(225) = \boxed{150}$$

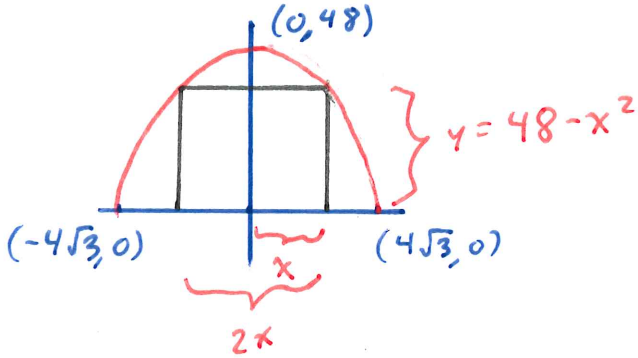
Verify

$A''(x) = -4/3 < 0 \longrightarrow$ concave down $\rightarrow$ max

The area is maximized when $x = 225$ m and $y = 150$m.

$A = xy = 33750\ m^2$

7. A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 48 - x^2$. What are the dimensions of the rectangle with the maximum area? What is the area?



max $A = 2xy$

s.t. $y = 48 - x^2$

$\Rightarrow A(x) = 2x(48 - x^2)$

$= 96x - 2x^3$

Find x, y

$A'(x) = 96 - 6x^2 \overset{\text{set}}{=} 0$

$16 = x^2$

$\pm 4 = x$

$0 \leq x \leq 4\sqrt{3}$

$\Rightarrow x = 4, \ y = 32$

Verify

$A''(x) = -12x$

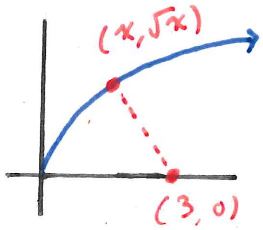
$A''(4) = -48 < 0 \rightarrow$ concave down

$\rightarrow$ max

The rectangle of maximum area has dimensions $x = 4$, $y = 32$. The area is $A = 2(4)32 = 256$.

8. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Distance $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



$$\min \quad d = \sqrt{(3-x)^2 + y^2}$$
$$\text{s.t.} \quad y = \sqrt{x}$$

$$\Rightarrow d(x) = \sqrt{9 - 6x + x^2 + (\sqrt{x})^2} = (x^2 - 5x + 9)^{1/2}$$

Find (x, y)

$$d'(x) = \frac{2x - 5}{2\sqrt{x^2 - 5x + 9}} \overset{\text{set}}{=} 0$$

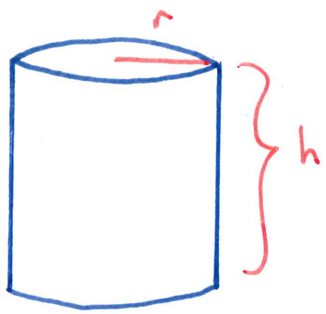
$$2x - 5 = 0$$
$$\boxed{x = \frac{5}{2}} \qquad \boxed{y = \sqrt{\frac{5}{2}}}$$

Verify ($d''(x)$ sucks)

	1	5/2	10
d'	$-$		$+$
	↘		↗

The distance is minimized at $(x, y) = \left(\frac{5}{2}, \sqrt{5/2}\right)$.

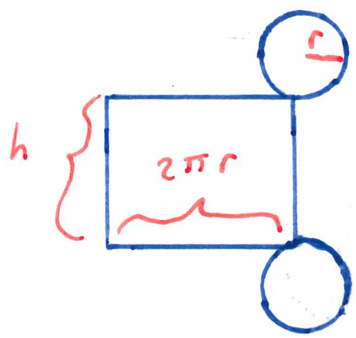
9. A cylindrical can is to be made to hold $1\,L$ ($1000\,cm^3$) of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.



r: radius (cm)

h: height (cm)

S: surface area (cm^2)



$$\min \quad S = 2\pi r h + 2(\pi r^2)$$

$$\text{s.t.} \quad \pi r^2 h = 1000$$

$$\Rightarrow h = \frac{1000}{\pi r^2}$$

$$\Rightarrow S(r) = \frac{2000}{r} + 2\pi r^2$$

Find r, h

$$S'(r) = -\frac{2000}{r^2} + 4\pi r \overset{\text{set}}{=} 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \qquad \Rightarrow h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2}$$

Verify

$$S''(r) = \frac{4000}{r^3} + 4\pi$$

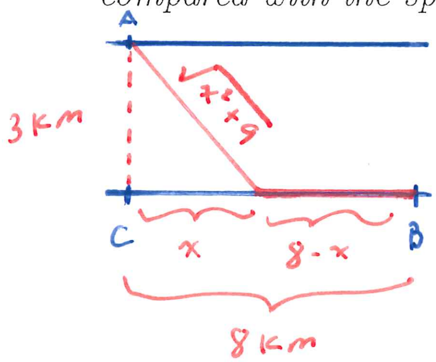
$$S''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0 \longrightarrow \text{concave up}$$

$$\rightarrow \min$$

$$\boxed{\text{The can with } r = \sqrt[3]{\frac{500}{\pi}} \text{ and } h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} \text{ uses the minimum amount of metal.}}$$

A

10. A man launches his boat from point A on a bank of a straight river, $3\,km$ wide, and wants to reach point B, $8\,km$ downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point between point C and point B and then run to B. If he can row $6\,km/h$ and run $8\,km/h$, where should he land to reach B as soon as possible? (*Assume that the speed of the water is negligible compared with the speed at which the man rows.*)



x: distance from point C (km)

$0 \le x \le 8$

T: time spent traveling (h)

min $T(x) = \dfrac{\sqrt{x^2+9}}{6} + \dfrac{(8-x)}{8} = \dfrac{1}{6}(x^2+9)^{1/2} + 1 - \dfrac{x}{8}$

$\left(\dfrac{km}{km/h}\right)$ $\left(\dfrac{km}{km/h}\right)$

Find x

$T'(x) = \dfrac{2x}{12\sqrt{x^2+9}} - \dfrac{1}{8} \overset{\text{set}}{=} 0 \quad \Rightarrow \quad \dfrac{4}{3}x = \sqrt{x^2+9}$

$\dfrac{16}{9}x^2 = x^2+9$

$\dfrac{7}{9}x^2 = 9$

$\boxed{x = 9/\sqrt{7}}$

Verify

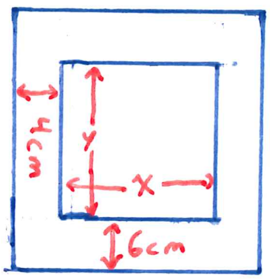
$1 \quad 9/\sqrt{7} \quad 7$

$T'(x)$: $-$ $+$

↘ ↗

The time traveled is minimized when the man rows to the point that is $x = 9/\sqrt{7}$ km away from point C.

11. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.



$$\min \; A = (x + 2(4))(y + 2(6))$$

$$\text{s.t.} \quad xy = 384$$

$$\rightarrow y = \frac{384}{x} \qquad \Rightarrow A(x) = (x+8)\left(\frac{384}{x} + 12\right)$$

$$= 12x + \frac{8(384)}{x} + 480$$

Find x, y

$$A'(x) = 12 - \frac{8(384)}{x^2} \overset{\text{set}}{=} 0$$

$$x^2 = \frac{2}{3}(384) = 256$$

$$\boxed{x = 16} \rightarrow \boxed{y = \frac{384}{16} = 24}$$

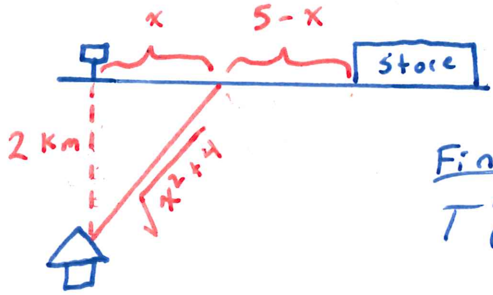
Verify:

$$A''(x) = \frac{24(384)}{x^3}$$

$$A''(16) = \frac{24(384)}{16^3} > 0 \longrightarrow \text{concave up} \rightarrow \text{minimum}$$

$\rightarrow$ The poster with minimal area has $x = 16$ and $y = 24$, or dimensions of 24 cm by 36 cm.

12. A cabin is located 2 km directly into the woods from a mailbox on a straight road. A store is located on the road 5 km from the mailbox. A woman wished to walk from the cabin to the store. She can walk 3 km/h through the woods and 4 km/h along the road. Find the point on the road toward which she would walk in order to minimize the total time for her walk.



$$\min \; T(x) = \frac{\sqrt{x^2+4}}{3}\left(\frac{km}{km/h}\right) + \frac{5-x}{4}\left(\frac{km}{km/h}\right)$$

Find x

$$T'(x) = \frac{x}{3\sqrt{x^2+4}} - \frac{1}{4} \overset{set}{=} 0$$

$$\frac{4}{3}x = \sqrt{x^2+4}$$

$$\frac{16}{9}x^2 = x^2+4$$

$$\frac{7}{9}x^2 = 2$$

$$x = \frac{6}{\sqrt{7}}$$

Verify

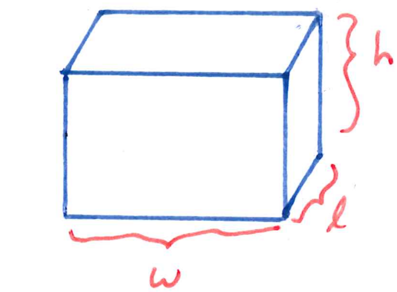
$0 \quad 6/\sqrt{7} \quad 2$

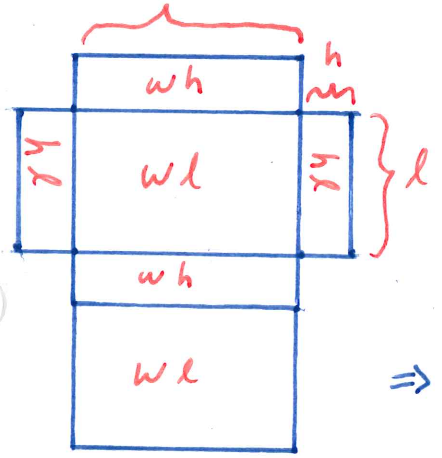
$T'(x)$ $-$ $+$

$\searrow \quad \nearrow$

The time spent traveling is minimized if she walks to the point that is $x = 6/\sqrt{7}$ away from the mailbox.

13. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides costs \$6/ft^2. If the box must have a volume of 50 ft^3 determine the dimensions that will minimize the cost to build the box.





$w \geq 0$: width (ft)

$l \geq 0$: length (ft)

$h \geq 0$: height (ft)

C : cost (\$)

$$\min\ C = \underbrace{10(2wl)}_{(\$/ft^2)(ft^2)} + \underbrace{6(2wh + 2hl)}_{(\$/ft^2)\cdot(ft^2)}$$

$$\text{s.t.}\quad \left.\begin{aligned} wlh &= 50 \\ l &= 3w \end{aligned}\right\} \quad h = \frac{50}{3w^2}$$

$$\Rightarrow C(w) = 20w(3w) + 12w\left(\frac{50}{3w^2}\right) + 12\left(\frac{50}{3w^2}\right)(3w)$$

$$= 60w^2 + \frac{800}{w}$$

Find w, l, h:

$$C'(w) = 120w - 800/w^2 \overset{\text{set}}{=} 0 \quad \Rightarrow \quad w = \sqrt[3]{\frac{20}{3}},\ l = 3\sqrt[3]{\frac{20}{3}},\ h = \frac{50}{3\left(\sqrt[3]{\frac{20}{3}}\right)^2}$$

Verify

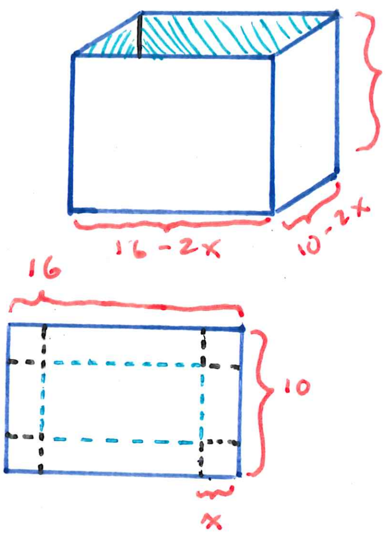
$$C''(w) = 120 + 1600/w^3$$

$$C''\left(\sqrt[3]{\frac{20}{3}}\right) > 0 \quad \rightarrow \text{concave up}$$

$\rightarrow$ min.

The cost of the box is minimized when $w = \sqrt[3]{20/3}$, $l = 3\sqrt[3]{20/3}$ and $h = \dfrac{50}{3\left(\sqrt[3]{20/3}\right)^2}$.

14. An open-top rectangular box is constructed from a 10 *in.* by 16 *in.* piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find the dimensions of the box of largest volume and the maximum volume.



x: side length of square removed

max $V(x) = (16-2x)(10-2x)\,x$

$= 4x^3 - 52x^2 + 160x$

$V'(x) = 12x^2 - 104x + 160 \overset{\text{set}}{=} 0$

$4(3x-20)(x-2) = 0$

$\Rightarrow x = \frac{20}{3}, \quad x = 2$

$\frac{20}{3} = 6 + \frac{2}{3}$

Domain

lengths must be non-negative

$\left.\begin{array}{l} x \geq 0 \\ 10-2x \geq 0 \\ 16-2x \geq 0 \end{array}\right\} \left.\begin{array}{l} x \geq 0 \\ x \leq 5 \\ x \leq 8 \end{array}\right\} \; 0 \leq x \leq 5 \quad \Rightarrow \quad \boxed{x = 2}$

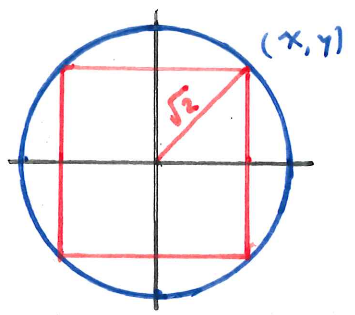
Verify

$V''(x) = 24x - 104$

$V''(2) = 48 - 104 < 0 \quad \rightarrow$ concave down $\rightarrow$ max

The dimensions of the box of largest volume are $6 \times 10 \times 2$ in.

15. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius $\sqrt{2}\,cm$.



x : ½ width of rectangle (cm) $0 \le x \le \sqrt{2}$

y : ½ height of rectangle (cm) $0 \le y \le \sqrt{2}$

A : area of rectangle (cm^2)

$$\left.\begin{array}{ll} \max & A = xy \\ \text{s.t.} & x^2 + y^2 = \sqrt{2}^{\,2} \end{array}\right\} \quad y = \sqrt{2-x^2} \quad \Rightarrow \quad A(x) = x\sqrt{2-x^2}$$

Find (x, y)

$$A'(x) = \sqrt{2-x^2} - \frac{x^2}{\sqrt{2-x^2}} \overset{\text{set}}{=} 0 \rightarrow \quad 2 - x^2 = x^2$$

$$2 = 2x^2$$

$$\pm 1 = x \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

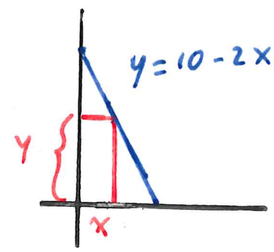
Verify

$$\begin{array}{c|cc} & x=1 & \\ \hline A'(x) & + \nearrow & - \searrow \end{array}$$

max at $x = 1$

The inscribed rectangle of largest area is a square with side length 2 cm.

16. A rectangle is constructed with one side on the positive x-axis, one side on the positive y-axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?



$x \geq 0$: width of rectangle

$y \geq 0$: height of rectangle

A : area of rectangle

$$\left.\begin{array}{l} \max\ A = xy \\ \text{s.t.}\ \ y = 10-2x \end{array}\right\} \quad A(x) = x(10-2x) = 10x - 2x^2$$

Find x, y

$$A'(x) = 10 - 4x \stackrel{\text{set}}{=} 0 \implies \boxed{x = \frac{5}{2}}, \ \boxed{y = 5}$$

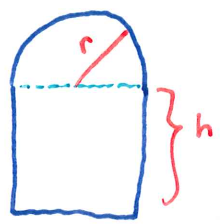
Verify

$$A''(x) = -4$$

$$A''\left(\tfrac{5}{2}\right) = -4 < 0 \rightarrow \text{concave down} \rightarrow \max$$

The rectangle has a maximum area of $A = \frac{25}{2}$ when $x = \frac{5}{2}$ and $y = 5$.

17. A Norman window has the shape of a rectangle surmounted by a semi-circle. (Thus, the diameter of the semi-circle is equal to the width of the rectangle.) If the perimeter of the window is 20 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.



$r \geq 0$: radius of semi-circle (ft)

$h \geq 0$: height of rectangle (ft)

A : area of window (ft^2)

$$\left.\begin{array}{ll} \text{max} & A = \frac{1}{2}\pi r^2 + 2rh \\ \text{s.t.} & \pi r + 2r + 2h = 20 \end{array}\right\} \quad h = \frac{20 - r(2+\pi)}{2}$$

$$\Rightarrow A(r) = \frac{1}{2}\pi r^2 + 2r\left(\frac{20 - r(2+\pi)}{2}\right)$$

$$= -\left(\frac{1}{2}\pi + 2\right)r^2 + 20r$$

Find r, h

$$A'(r) = -2\left(\frac{1}{2}\pi + 2\right)r + 20 \overset{\text{set}}{=} 0$$

$$r = \frac{10}{\frac{1}{2}\pi + 2}, \quad h = \frac{20}{\pi + 4}$$

Verify

$$A''(r) = -2\left(\frac{1}{2}\pi + 2\right) < 0 \rightarrow \text{concave down} \rightarrow \text{maximum.}$$

The Norman window with a 20 ft perimeter has the maximum area when it has a radius of $r = \frac{10}{\frac{1}{2}\pi + 2}$ and height of $h = \frac{20}{\pi + 4}$.

Note: $\left(\frac{2}{2}\right)r = \frac{20}{\pi + 4}$