3.11: Related Rates

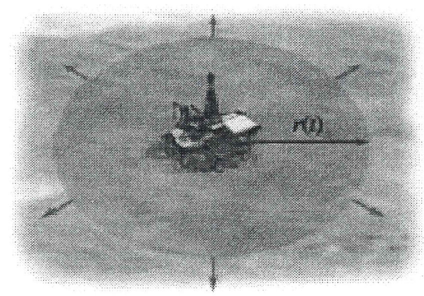
Related rates are problems that use a mathematical relationship between two or more objects under specific constraints. From this, we can differentiate this relationship and examine how each variable changes with respect to time.

Example. An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of $30m/hr$, how fast is the area of the patch increasing when the patch has a radius of $100m$?

Solution:

We are given the radius $r = 100\,m$ which is increasing at a rate of $\frac{dr}{dt} = 30\,m/hr$. First, we note the formula for the area of a circle:

$$A = \pi r^2$$



Now, we differentiate *with respect to time t*:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Note that we want to solve for $\frac{dA}{dt}$ — the rate of change of the area of the oil patch:

$$\frac{dA}{dt} = 2\pi(100\,m)(30\,m/hr) = 6000\pi\,m^2/hr$$

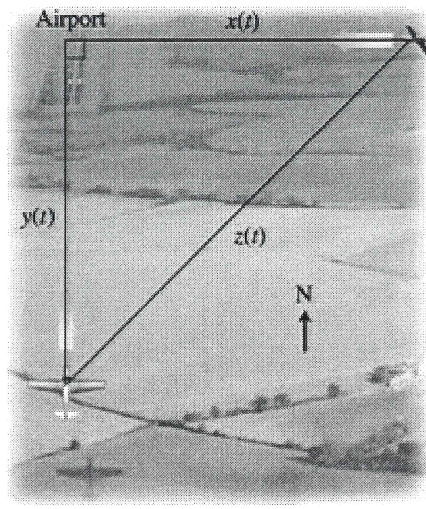
Example. Two small planes approach an airport, one flying due west at $120\,mi/hr$ and the other flying due north at $150\,mi/hr$. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is $180\,mi$ from the airport and the northbound plane is $225\,mi$ from the airport?

Solution:

From the problem statement, we know that

$$x = 180\,mi \qquad\qquad y = 225\,mi$$

$$\frac{dx}{dt} = -120\,mi/hr \qquad\qquad \frac{dy}{dt} = -150\,mi/hr$$



Now we use the Pythagorean theorem to relate the three sides:

$$x^2 + y^2 = z^2$$

Next, we differentiate and get the resulting relationship between the related rates:

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 2z\,\frac{dz}{dt}$$

Before we solve for $\frac{dz}{dt}$, we must find z:

$$z = \sqrt{180^2 + 225^2} = 45\sqrt{4^2 + 5^2} = 45\sqrt{41} \approx 288\,mi$$

This gives us

$$\frac{dz}{dt} = \frac{x\,\frac{dx}{dt} + y\,\frac{dy}{dt}}{z} = \frac{(180\,mi)(-120\,mi/hr) + (225\,mi)(-150\,mi/hr)}{45\sqrt{41}\,mi}$$

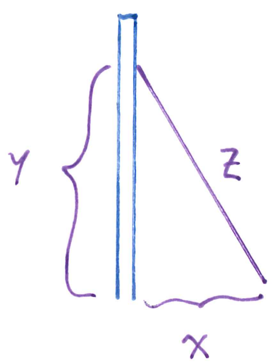
$$= \frac{1230}{\sqrt{41}}\,mi/hr \approx -192\,mi/hr$$

Steps for Related-Rate Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

Note: The *Just-In-Time* book has some examples in chapter 14 that are helpful in setting up the relationships outlined in these types of word problems.

Example. A ladder 13 feet long rests against a vertical wall and is sliding down the wall at a constant rate of 3 ft/s. How fast is the foot of the ladder moving away from the wall when the foot of the ladder is 5 ft from the base of the wall?



Let x = distance between base of wall & foot of ladder

y = height of ladder on wall

z = length of ladder

t = time (in seconds)

Given: $\frac{dy}{dt} = -3$ ft/s, $\frac{dz}{dt} = 0$ ft/s, $z = 13$ ft

Find: $\frac{dx}{dt}$ when $x = 5$ ft

From Pythagorean theorem:

$$x^2 + y^2 = z^2$$

Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\underbrace{\frac{dz}{dt}}_{0}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

Find missing terms:

$$y = \sqrt{z^2 - x^2} = \sqrt{13^2 - 5^2} = 12$$

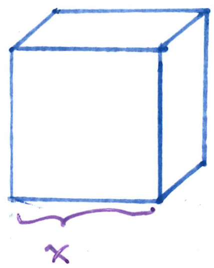
Evaluate

$$\frac{dx}{dt} = -\frac{12\text{ ft}}{5\text{ ft}}(-3\text{ ft/s}) = \boxed{\frac{36}{5}\text{ ft/s}}$$

The foot of the ladder is moving away from the wall at a rate of $\frac{36}{5}$ ft/s.

$\frac{dV}{dt}$ negative

Example. The volume of a cube $\boxed{\text{decreases}}$ at a rate of $0.5\ ft^3/min$. What is the rate of change of the side length when the side lengths are $12\ ft$?



Let x = side length

V = volume of cube

t = time (in minutes)

Given: $\frac{dV}{dt} = -0.5 \frac{ft^3}{min}$

Find: $\frac{dx}{dt}$ when $x = 12\,ft$

From Volume of a cube:

$$V = x^3$$

Differentiate:

$$\frac{dV}{dx} = 3x^2 \frac{dx}{dt}$$

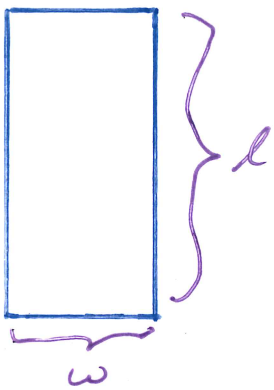
$$\frac{dx}{dt} = \frac{1}{3x^2} \frac{dV}{dt}$$

Evaluate:

$$\frac{dx}{dt} = \frac{1}{3(12\,ft)^2} \cdot (-0.5) \frac{ft^3}{min} = \boxed{-\frac{1}{864}\ ft/min}$$

The sides of the cube are decreasing at a rate of $\frac{1}{864}$ ft/min.

Example. The length of a rectangle is increasing at a rate of $8\,cm/s$ and its width is increasing at a rate of $3\,cm/s$. When the length is $20\,cm$ and the width is $10\,cm$, how fast is the area of the rectangle increasing?



Let ℓ = length of rectangle

w = width of rectangle

A = area of rectangle

t = time

Given:

$$\frac{d\ell}{dt} = 8\ cm/s$$

$$\frac{dw}{dt} = 3\ cm/s$$

Find:

$$\frac{dA}{dt} \quad \text{when } \ell = 20\ cm \ \&\ w = 10\ cm$$

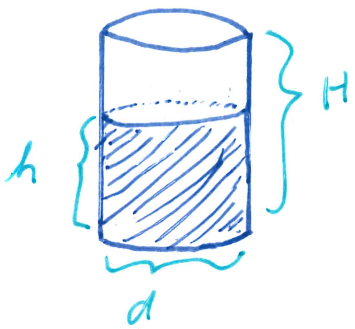
From area of rectangle: $A = \ell w$

Differentiate: $$\frac{dA}{dt} = \frac{d\ell}{dt} w + \ell \frac{dw}{dt}$$

Evaluate: $$\frac{dA}{dt} = (8\ cm/s) \cdot 10\,cm + 20\,cm\,(3\ cm/s)$$
$$= \boxed{140\ cm^2/s}$$

The area of the rectangle is increasing at a rate of $140\ cm^2/s$.

Example. A coffee mug has the shape of a right circular cylinder with inner diameter 4 inches and height 5 inches. If the mug is filled with hot chocolate at a constant rate of $2\, in^3/sec$, how fast is the level of the liquid rising?



Let d = diameter of mug $\longrightarrow$ r = radius of mug

H: height of mug

h: height of liquid

V = volume of liquid

t = time (in seconds)

Given:

$d = 4\text{ in} \rightarrow r = 2\text{ in}$

$H = 5\text{ in}$

Find:

$\frac{dh}{dt}$ when $\frac{dV}{dt} = 2\text{ in}^3/s$

Formula of volume of cylinder: $V = \pi r^2 h$

Note: r is constant

Differentiate: $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

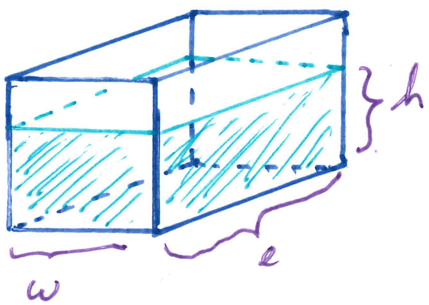
$$\Rightarrow \frac{dh}{dt} = \frac{dV}{dt} \frac{1}{\pi r^2}$$

Evaluate: $\frac{dh}{dt} = (2\text{ in}^3/s) \frac{1}{\pi (2\text{in})^2} = \boxed{\frac{1}{2\pi}\text{ in}/s}$

The level of the fluid is rising at a rate of $\frac{1}{2\pi}$ in/s.

Example. A rectangular swimming pool 10 ft wide by 20 ft long and of uniform depth is being filled with water.

a) At what rate is the volume of the water increasing if the water level is rising at $\frac{1}{4}\, ft/min$?



Let: w = width of pool

l = length of pool

h = height of water

V = volume of water

t = time (in min)

Note: w & l are constant

Given:
$w = 10\,ft$
$l = 20\,ft$

Find: $\frac{dV}{dt}$ when $\frac{dh}{dt} = \frac{1}{4}\,ft/min$

Volume: $V = wlh$

Differentiate: $\frac{dV}{dt} = wl\frac{dh}{dt}$

Evaluate: $\frac{dV}{dt} = (10\,ft)(20\,ft)\left(\frac{1}{4}\,ft/min\right) = \boxed{50\,\frac{ft^3}{min}}$

b) At what rate is the water level rising if the pool is filled at a rate of 10 ft^3/min?

Given: $w = 10\,ft$
$l = 20\,ft$

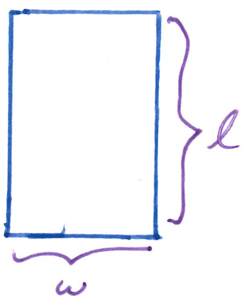
Find: $\frac{dh}{dt}$ when $\frac{dV}{dt} = 10\,ft^3/min$

From before, we have $\frac{dV}{dt} = wl\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV}{dt}\frac{1}{wl}$

$$\frac{dh}{dt} = \frac{(10\,ft^3/min)}{(10\,ft)(20\,ft)} = \boxed{\frac{1}{20}\,ft/min}$$

Example. At all times, the length of a rectangle is twice the width w of the rectangle as the area of the rectangle changes with respect to time t.

a) Find the equation that relates A to w.



Let w = width of rectangle

l = length of rectangle

A = area of rectangle

t = time

$$\left.\begin{aligned} l &= 2w \\ A &= lw \end{aligned}\right\} \quad A = 2w^2$$

b) Find the equation that relates dA/dt to dw/dt.

Differentiate:

$$\frac{dA}{dt} = 4w\frac{dw}{dt}$$

Example. Assume x, y and z are functions of t with $z = x + y^3$. Find dz/dt when $dx/dt = -1$, $dy/dt = 5$ and $y = 2$.

$$z = x + y^3$$

$$\frac{dz}{dt} = \frac{dx}{dt} + 3y^2 \frac{dy}{dt}$$

$$\frac{dz}{dt} = (-1) + 3(2)^2(5) = \boxed{59}$$

Example. Assume $w = x^2y^4$, where x and y are functions of t. Find dw/dt when $x = 3$, $dx/dt = 2$, $dy/dt = 4$, and $y = 1$.

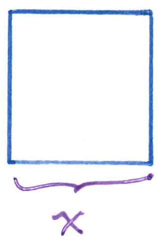
$$w = x^2 y^4$$

$$\frac{dw}{dt} = 2x\left(\frac{dx}{dt}\right)y^4 + x^2\left(4y^3\right)\frac{dy}{dt}$$

$$\frac{dw}{dt} = 2(3)(2)(1)^4 + (3^2)\,4(1)^3(4) = \boxed{156}$$

Example. The sides of a square decrease in length at a rate of $1\,m/s$.

a) At what rate is the area of the square changing when the sides are $5\,m$ long?



Let x = length of square side

A = area of square

t = time (in seconds)

Given: $\dfrac{dx}{dt} = -1\ m/s$

Find: $\dfrac{dA}{dt}$ when $x = 5m$

Area of square

$$A = x^2$$

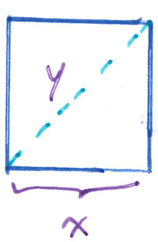
Differentiate

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

Evaluate:

$$\frac{dA}{dt} = 2(5m)(-1\ m/s) = \boxed{-10\ m^2/s}$$

b) At what rate are the lengths of the diagonals of the square changing?



Let x = length of square side

y = length of diagonal

t = time (in seconds)

Given: $\dfrac{dx}{dt} = -1\ m/s$

Find: $\dfrac{dy}{dt}$ when $x = 5m$

Pythagorean theorem

$$x^2 + x^2 = y^2$$

$$y = \sqrt{2x^2} = \sqrt{2}\,x$$

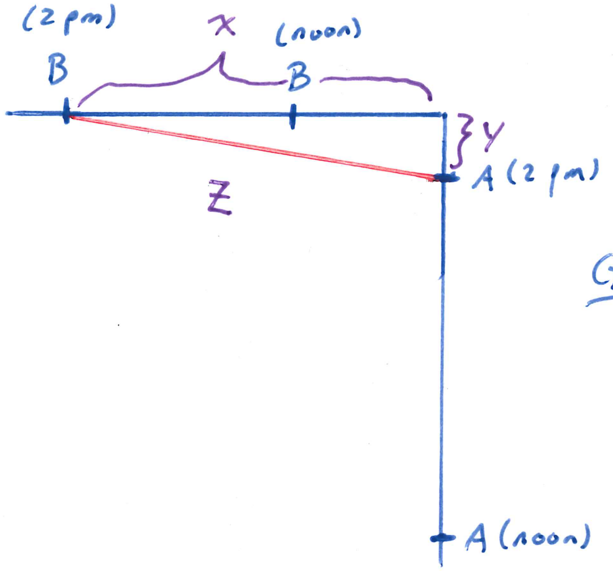
Differentiate:

$$\frac{dy}{dt} = \sqrt{2}\frac{dx}{dt}$$

Evaluate:

$$\frac{dy}{dt} = \sqrt{2}(-1\ m/s) = \boxed{-\sqrt{2}\ m/s}$$

Example. At noon bicyclist A is 25 miles south of an intersection and bicyclist B is 8 miles west of the same intersection. Bicyclist A is traveling north at 11 miles per hour and bicyclist B is traveling 6 miles per hour west of the intersection. How is the distance between riders changing at 2pm?



Let x: distance of A from intersection

y: distance of B from intersection

z: distance between cyclists

t: time (in hours)

Given:

$$\frac{dy}{dt} = -11 \text{ mph} \qquad \frac{dx}{dt} = 6 \text{ mph}$$

Find: $\frac{dz}{dt}$ when $x = 20$ miles and $y = 3$ miles

Pythagorean theorem:

$$x^2 + y^2 = z^2$$

Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$$

Find z:

$$z = \sqrt{x^2+y^2} = \sqrt{20^2+3^2} = \sqrt{409}$$

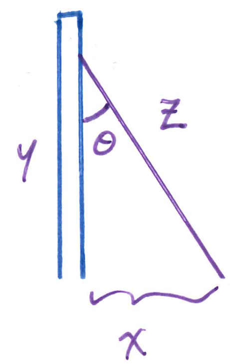
Evaluate:

$$\frac{dz}{dt} = \frac{(20 \text{ mi})\left(6\frac{\text{mi}}{\text{hr}}\right) + (3\text{ mi})\left(-11\frac{\text{mi}}{\text{hr}}\right)}{\sqrt{409}\text{ mi}}$$

$$= \boxed{\frac{87}{\sqrt{409}} \text{ m/hr}} \approx 4.302 \text{ mi/hr}$$

At 2 pm, the distance between the cyclists is increasing at $\frac{87}{\sqrt{409}}$ m/hr.

Example. A ladder 13 feet long rests against a vertical wall and is sliding down the wall at a constant rate of 3 ft/s. How fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ radians?



Let x = distance between base of wall & foot of ladder

y = height of ladder on wall

z = length of ladder (constant)

θ = angle between wall and ladder

t = time (in seconds)

Given:

$z = 13$ ft

$\frac{dy}{dt} = -3$ ft/s

Find

$\frac{d\theta}{dt}$ when $\theta = \pi/4$

Using trig

$\cos\theta = \frac{y}{z}$

differentiate

$$-\sin\theta \frac{d\theta}{dt} = \frac{1}{z}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{z\sin\theta}\frac{dy}{dt}$$

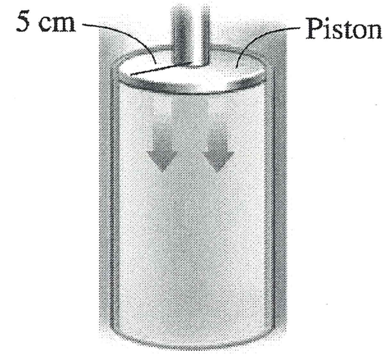
Evaluate:

$$\frac{d\theta}{dt} = \frac{-3 \text{ ft/s}}{(13\text{ ft})(-\sin \pi/4)} = \boxed{\frac{3\sqrt{2}}{13} \text{ rad/sec}}$$

Note: radians are unitless

When $\theta = \pi/4$, the angle between the ladder and the wall is increasing at a rate of $\frac{3\sqrt{2}}{13}$ rad/sec.

Example. Piston compression A piston is seated at the top of a cylindrical chamber with radius 5 cm when it starts moving into the chamber at a constant speed of 3 cm/s. What is the rate of change of the volume of the cylinder when the piston is 2 cm from the base of the chamber?



Let r = radius of cylinder (constant)

h = height of piston

V = volume of cylinder

t = time (in seconds)

Given:

$r = 5\,cm$

$\frac{dh}{dt} = -3\,cm/s$

Find: $\frac{dV}{dt}$ when $h = 2\,cm$

Volume of cylinder

$$V = \pi r^2 h$$

Differentiate

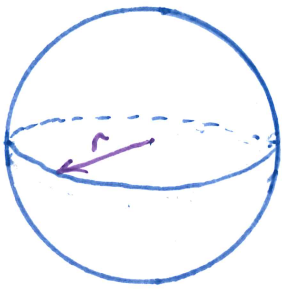
$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

Evaluate

$$\frac{dV}{dt} = \pi (5\,cm)^2 (-3\,cm/s) = -75\pi\ cm^3/s$$

The volume of cylinder is decreasing at a rate of $75\pi\ cm^3/s$.

Example. Suppose we have a snowball that is a perfect sphere. If the snowball is melting at a rate of $5\,in^3/min$, how fast is the radius changing when the radius is 10 inches? How fast is the surface area changing?



Let r = radius of sphere
V = volume of sphere
SA = surface area of sphere
t = time (in minutes)

Given:

$$\frac{dV}{dt} = -5\ in^3/min$$

Find:

$$\frac{dr}{dt} \text{ when } r = 10\ in; \quad \frac{dSA}{dt} \text{ when } r = 10\ in$$

Volume of sphere:

$$V = \frac{4}{3}\pi r^3$$

Differentiate:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

Evaluate:

$$\frac{dr}{dt} = \frac{1}{4\pi(10 in)^2}(-5\ in^3/min) = \boxed{-\frac{1}{80\pi}\ in/min}$$

Surface area of sphere:

$$SA = 4\pi r^2$$

Differentiate:

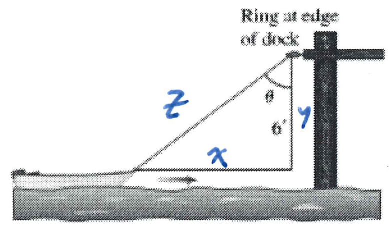
$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt}$$

Evaluate:

$$\frac{dSA}{dt} = 8\pi(10 in)\left(-\frac{1}{80\pi}\ in/min\right) = \boxed{-1\ in^2/min}$$

The radius of the sphere is decreasing at a rate of 1/80 in/min and the surface area is decreasing at a rate of 1 in²/min when the radius is 10 in.

Example. A dingy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at a rate of 2 ft/sec. At what rate is the angle θ changing when 10 ft of rope is out?



Let: x = horizontal distance from dock

y: vertical distance from dock (constant)

z: length of rope

θ = Angle between rope & vertical

t = time (in seconds)

Given:

$y = 6$ ft

$\frac{dz}{dt} = -2$ ft/sec

From trig

$\cos\theta = \frac{y}{z} = yz^{-1}$

Differentiate

$-\sin\theta \frac{d\theta}{dt} = -yz^{-2}\frac{dz}{dt}$

$\frac{d\theta}{dt} = \frac{y\frac{dz}{dt}}{z^2\sin\theta}$

Find:

$\frac{d\theta}{dt}$ when $z = 10$ ft.

Rewrite $\sin\theta$ in terms of sides:

$\sin\theta = \frac{x}{z} \Rightarrow$ Find x.

$x = \sqrt{z^2 - y^2} = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$

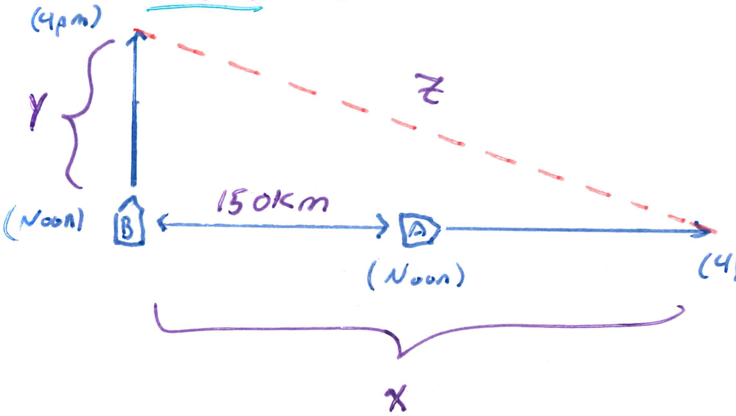
$\Rightarrow \sin\theta = 8/10 = 4/5$

Evaluate:

$\frac{d\theta}{dt} = \frac{(6\text{ ft})(-2\text{ ft/sec})}{(10\text{ ft})^2(4/5)} = \boxed{-\frac{3}{2}\text{ radians/sec}}$

When there is 10 feet of rope out, the angle is decreasing at a rate of 3/2 radians/sec.

Example. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4pm?



Let:
x = distance of ship A (east/west)
y = distance of ship B (North/south)
z = distance between ships A & B
t = time (in hours)

Given:

$\frac{dx}{dt} = 35$ km/h

$\frac{dy}{dt} = 25$ km/h

Find

$\frac{dz}{dt}$ @ 4pm

$x = 150 + 4(35) = 290$ km

$y = 4(25) = 100$ km

Pythagorean thm

$$z^2 = x^2 + y^2$$

Differentiate

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$$

Find z:

$$z = \sqrt{x^2+y^2} = \sqrt{(290)^2 + (100)^2} = 10\sqrt{941} \text{ km}$$

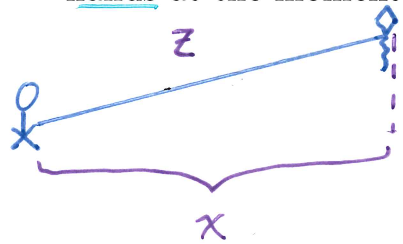
Evaluate

$$\frac{dz}{dt} = \frac{(290\text{ km})(35\text{ km/h}) + (100\text{ km})(25\text{ km/h})}{10\sqrt{941}\text{ km}}$$

$$= \boxed{\frac{1265}{\sqrt{941}}\text{ km/h}} \approx 41.238 \text{ km/h}$$

The distance between the boats is increasing at a rate of $\frac{1265}{\sqrt{941}}$ km/h at 4 pm.

Example. Once Kate's kite reaches a height of 50 ft (above her hands), it rises no higher but drifts due east in a wind blowing 5 ft/s. How fast is the string running through Kate's hands at the moment that she has released 120 ft of string?



Let:

x = horizontal distance from kite

y = vertical distance from kite (constant)

z = length of string

t = time (in seconds)

Given

$y = 50$ ft (constant)

$\frac{dx}{dt} = 5$ ft/s

Find:

$\frac{dz}{dt}$ when $z = 120$ ft.

Pythagorean theorem:

$$z^2 = x^2 + y^2$$

y is constant

Differentiate

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 0$$

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

Find x:

$$x = \sqrt{z^2 - y^2} = \sqrt{120^2 - 50^2}$$

$$= 10\sqrt{144 - 25}$$

$$= 10\sqrt{119} \text{ ft}$$

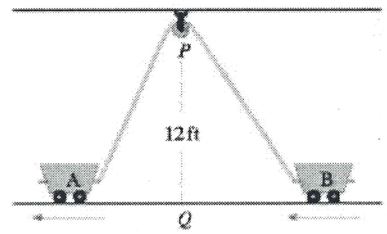
Evaluate

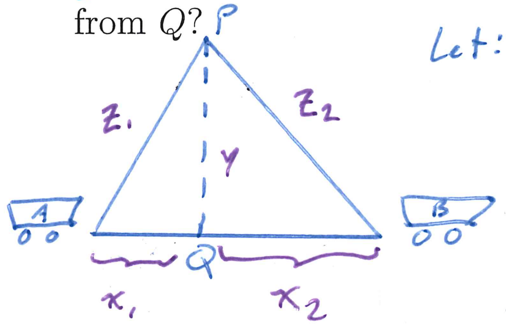
$$\frac{dz}{dt} = \frac{10\sqrt{119} \text{ ft}}{120 \text{ ft}}(5 \text{ ft/s})$$

$$= \boxed{\frac{5\sqrt{119}}{12} \text{ ft/s}} \approx 4.5453 \text{ ft/s}$$

When Kate has released 120 ft of string, the string is running through her hands at a rate of $\frac{5\sqrt{119}}{12}$ ft/s.

Example. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P. The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?





Let:

x_1 = distance from cart A to midpoint Q

x_2 = distance from cart B to midpoint Q

y = height (distance from Q to P) (constant)

z_1 = distance from cart A to pulley P

z_2 = distance from cart B to pulley P

Given:

$y = 12$ ft (constant)

$\frac{dx_1}{dt} = 2$ ft/s

$z_1 + z_2 = 39$ ft

Find

$\frac{dx_2}{dt}$ when $x_1 = 5$ ft

From Pythagorean theorem, we have

$$z_2^2 = x_2^2 + y^2 \Rightarrow x_2 = \sqrt{z_2^2 + y^2}$$

Differentiating, we have

$$\frac{dx_2}{dt} = \frac{1}{2}\left(z_2^2 + y^2\right)^{-1/2}\left(2z_2\frac{dz_2}{dt}\right) = \frac{z_2\frac{dz_2}{dt}}{\sqrt{z_2^2 + y^2}}$$

To solve this we must know z_2 and $\frac{dz_2}{dt}$:

$$z_1 + z_2 = 39 \Rightarrow z_2 = 39 - z_1$$

$$\frac{dz_2}{dt} = -\frac{dz_1}{dt}$$

Rewriting in terms of x_1:

$$z_1 = \sqrt{x_1^2 + y^2} = \sqrt{5^2 + 12^2} = 13 \text{ ft} \Rightarrow z_2 = 26 \text{ ft}$$

$$\frac{dz_1}{dt} = \frac{1}{2}\left(x_1^2 + y^2\right)^{-1/2}\left(2x_1\frac{dx_1}{dt}\right) = \frac{x_1\frac{dx_1}{dt}}{\sqrt{x_1^2 + y^2}}$$

Substituting into $\frac{dx_2}{dt}$:

$$\frac{dx_2}{dt} = \frac{z_2 \frac{dz_2}{dt}}{\sqrt{z_2^2 + y^2}} = \frac{z_2 \left(-\frac{dz_1}{dt}\right)}{\sqrt{z_2^2 + y^2}}$$

$$= \frac{-z_2 \frac{x_1 \frac{dx_1}{dt}}{\sqrt{x_1^2 + y^2}}}{\sqrt{z_2^2 + y^2}}$$

Evaluate:

$$\frac{dx_2}{dt} = \frac{-26\text{ ft} \frac{(5\text{ ft})(2\text{ ft/s})}{13\text{ ft}}}{\sqrt{26^2 + 12^2}\text{ ft}}$$

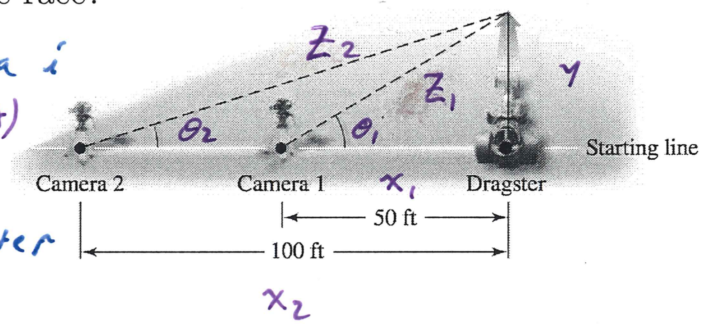
$$= \boxed{\frac{-10}{\sqrt{133}}\text{ ft/s}} \approx -0.867\text{ ft/s}$$

When cart A is 5 ft from Q, cart B is moving towards Q at a rate of $\frac{10}{\sqrt{133}}$ ft/s.

Example. A camera is set up at the starting line of a drag race 50ft from a dragster at the starting line (camera 1 in the figure). Two seconds after the start of the race, the dragster has traveled 100ft and the camera is turning at 0.75rad/s while filming the dragster.

a) What is the speed of the dragster at this point?

b) A second camera (camera 2 in the figure) filming the dragster is located on the starting line 100 ft away from the dragster at the start of the race. How fast is this camera turning 2 s after the start of the race?



Let x_i = distance between camera i and dragster (constant)

y = distance between dragster and start line

z_i = distance between camera i and dragster

θ_i = angle between camera i and start line

Given:	Find:
$x_1 = 50$ ft $x_2 = 100$ ft $y = 100$ ft $\frac{d\theta_1}{dt} = 0.75$ radians/s	a) $\frac{dy}{dt}$ b) $\frac{d\theta_2}{dt}$

a) From trig: $\tan\theta_1 = \frac{y}{x_1}$

Differentiate: $\sec^2\theta_1 \frac{d\theta_1}{dt} = \frac{1}{x_1}\frac{dy}{dt}$

$z_1 = \sqrt{x_1^2 + y^2} = 50\sqrt{5}$

$$\frac{dy}{dt} = x_1\left(\frac{z_1}{x_1}\right)^2\frac{d\theta_1}{dt} = (50\text{ ft})(\sqrt{5})^2\, 0.75\,\frac{\text{rad}}{s} = \boxed{\frac{375}{2}\text{ ft/s}}$$

$z_2 = \sqrt{x_2^2 + y^2} = 100\sqrt{2}$

b) $\tan\theta_2 = \frac{y}{x_2}$

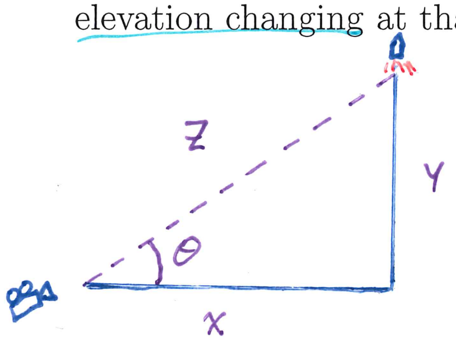
Differentiate:

$\sec^2(\theta_2)\frac{d\theta_2}{dt} = \frac{1}{x_2}\frac{dy}{dt}$

$$\frac{d\theta_2}{dt} = \frac{\cos^2(\theta_2)}{x_2}\frac{dy}{dt}$$

$$= \frac{(\sqrt{2})^2}{100\text{ ft}}\,\frac{375}{2}\text{ ft/s} = \boxed{\frac{15}{4}\text{ rad/s}}$$

Example. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft. How fast is the distance from the television camera to the rocket changing at that moment? Also, if the television camera is always kept aimed at the rocket, how fast is the cameras angle of elevation changing at that same moment?



Let:

x = distance from camera to launch pad

y = height of rocket

z = distance between rocket and camera

Given:

$x = 4000$ ft (constant)

$y = 3000$ ft

$\frac{dy}{dt} = 600$ ft/s

Find:

① $\frac{dz}{dt}$

② $\frac{d\theta}{dt}$

① Pythagorean theorem:

$$z^2 = x^2 + y^2$$

Differentiate:

$$2z\frac{dz}{dt} = 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{y}{z}\frac{dy}{dt}$$

$$z = \sqrt{x^2+y^2} = 5000 \text{ ft}$$

Evaluate:

$$\frac{dz}{dt} = \frac{3000 \text{ ft}}{5000 \text{ ft}} 600 \text{ ft/s} = \boxed{360 \text{ ft/s}}$$

② Trig:

$$\tan\theta = \frac{y}{x}$$

Differentiate:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{x}\frac{dy}{dt}$$

Evaluate:

$$\frac{d\theta}{dt} = \frac{(4/5)^2}{4000 \text{ ft}} 600 \text{ ft/s}$$

$$= \boxed{\frac{12}{125} \text{ rad/s}} = 0.096 \text{ rad/s}$$