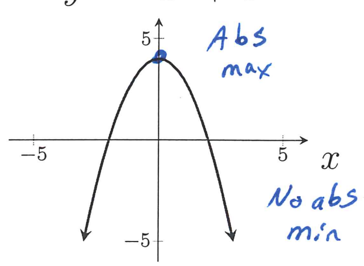
4.1 Maxima and Minima

Definition. (Absolute Maximum and Minimum)

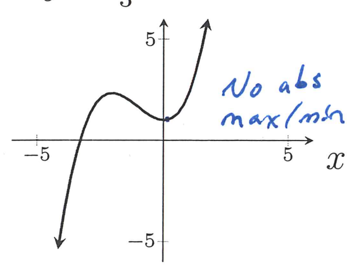
Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

Example. Determine whether the function has any absolute extreme values

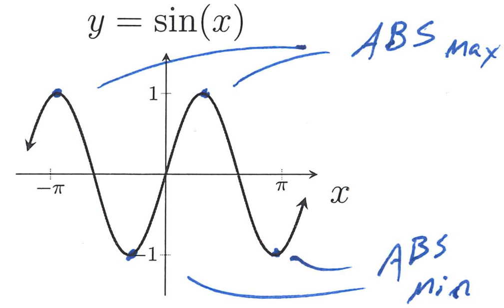
$y = -x^2 + 4$



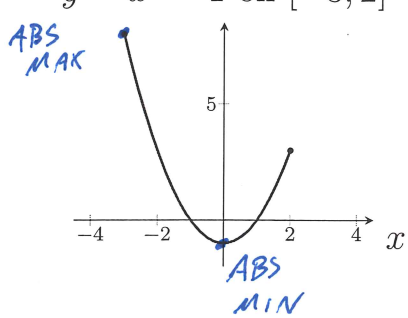
$y = \frac{x^3}{3} + x^2 + 1$



$y = \sin(x)$



$y = x^2 - 1$ on $[-3, 2]$



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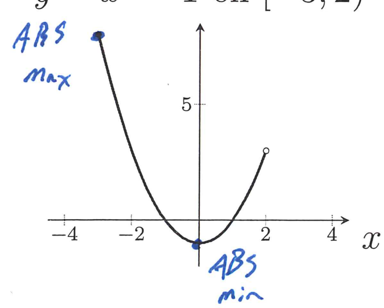
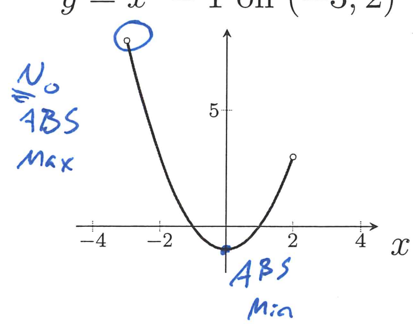
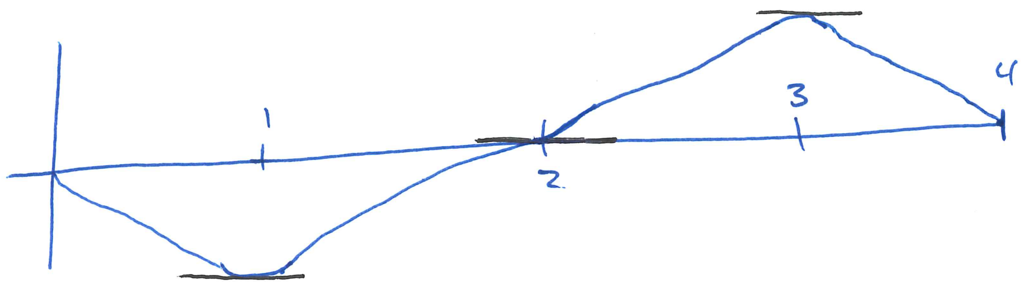


$y = x^2 - 1$ on $(-3, 2)$



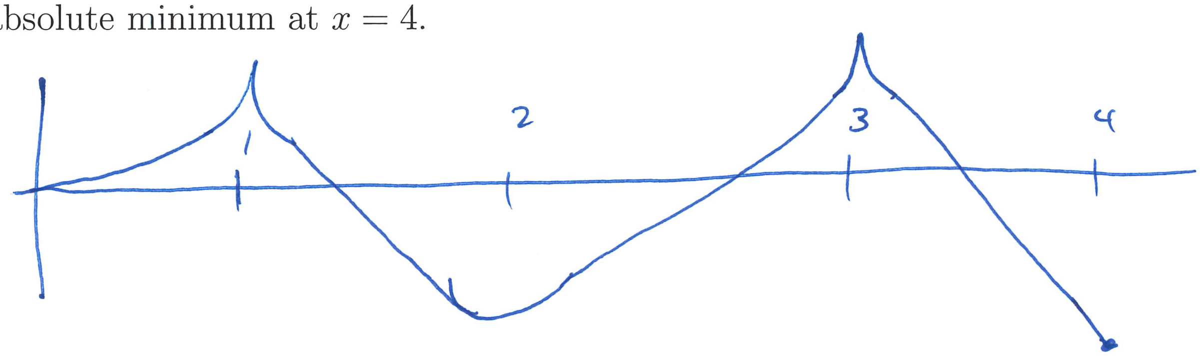
Example. Sketch the graph of a continuous function f on $[0, 4]$ satisfying the given properties:

1. $f'(x) = 0$ for $x = 1, 2$ and 3; f has an absolute minimum at $x = 1$; f has no local extremum at $x = 2$; and f has an absolute maximum at $x = 3$.

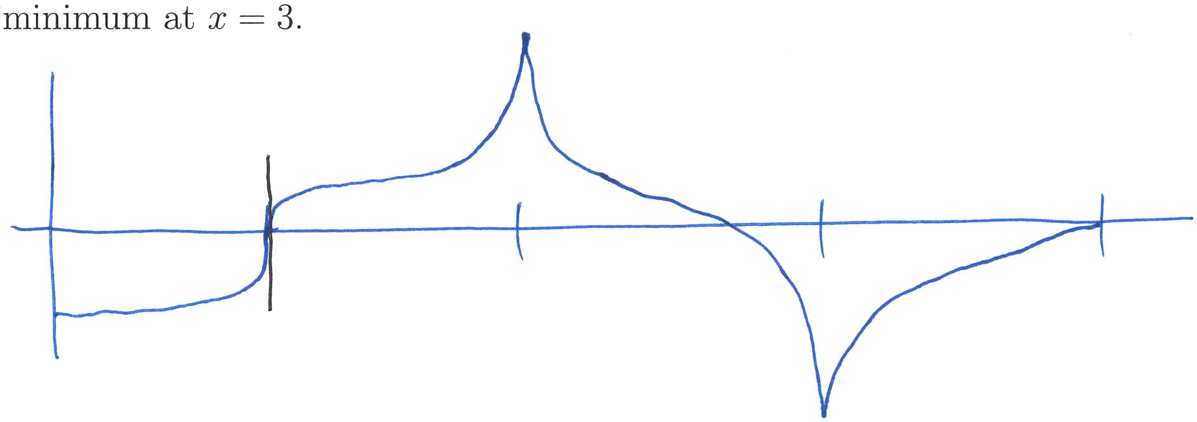


2. $f'(1)$ and $f'(3)$ are undefined; $f'(2) = 0$; f has a local maximum at $x = 1$; f has a local minimum at $x = 2$; f has an absolute maximum at $x = 3$; and f has an absolute minimum at $x = 4$.

1) cusp
2) dis cont
3) vert tan



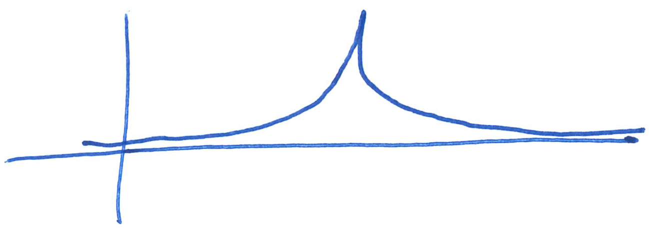
3. $f'(x) = 0$ at $x = 1$ and 3; $f'(2)$ is undefined; f has an absolute maximum at $x = 2$; f has neither a local maximum nor a local minimum at $x = 1$; and f has an absolute minimum at $x = 3$.



Theorem 4.2: Local Extreme Value Theorem
If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.

Note: If the derivative is zero, then the function **MIGHT** have a max/min.

Example. Sketch a graph of a function $f(x)$ that has a local maximum value at a point c where $f'(c)$ is undefined.



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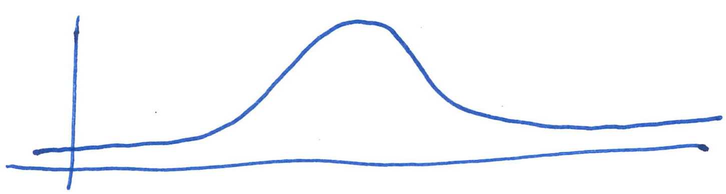
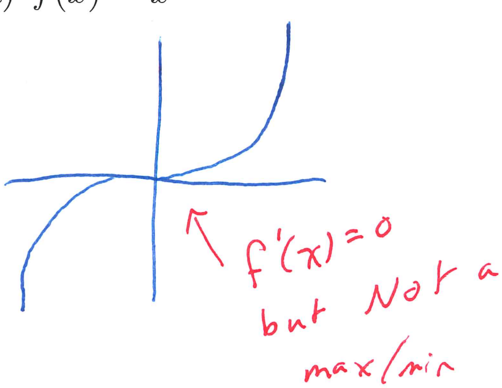
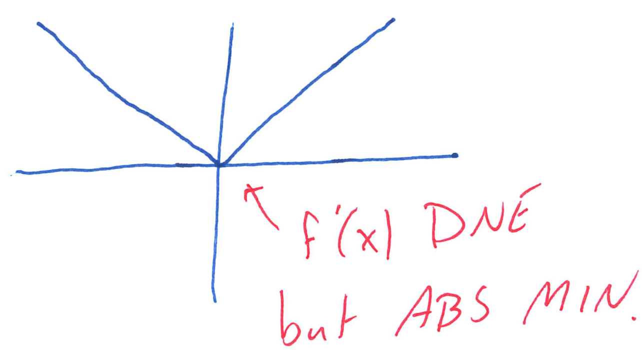


Example. Graph

a) $f(x) = x^3$



b) $f(x) = |x|$



> **Definition. (Critical Point)**
>
> An interior point c of the domain of f at which $f'(c) = 0$ *or* $f'(c)$ fails to exist is called a **critical point** of f.

Example. Find the critical points of

$$f(x) = x^3 + 3x^2 - 24x$$

$$f'(x) = 3x^2 + 6x - 24 \overset{\text{set}}{=} 0$$

$$3(x+4)(x-2) = 0$$

$$\Rightarrow \boxed{\begin{array}{l} x = -4 \\ x = 2 \end{array}}$$

$$g(x) = \sqrt{4 - x^2}$$

$$g'(x) = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}} \overset{\text{set}}{=} 0 \iff \boxed{x = 0}$$

$$\sqrt{4-x^2} \neq 0$$

$$\iff 4 - x^2 \neq 0$$

$$4 \neq x^2$$

$$\pm 2 \neq x$$

critical points unless these are endpoints.

$$h(t) = 3t - \sin^{-1}(t)$$

$$h'(t) = 3 - \frac{1}{\sqrt{1-t^2}} \overset{\text{set}}{=} 0 \iff 3 = \frac{1}{\sqrt{1-t^2}}$$

$$\iff \sqrt{1-t^2} = \frac{1}{3}$$

$$\iff 1 - t^2 = \frac{1}{9}$$

$$\frac{8}{9} = t^2$$

$$\boxed{\pm \frac{2\sqrt{2}}{3} = t}$$

$$\sqrt{1-t^2} \neq 0$$

$$1 \neq t^2$$

$$\boxed{\pm 1 \neq t}$$

crit pts unless they are endpoints.

Example. Find the critical points of

$$f(x) = \sin(x)\cos(x) \text{ on } [0, 2\pi].$$

$f'(x) = \cos^2(x) - \sin^2(x) \overset{\text{set}}{=} 0 \iff \cos^2(x) = \sin^2(x)$

$\iff \cos(x) = \pm\sin(x)$

$\iff \boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

$$f(t) = t^2 - 2\ln(t^2 + 1)$$

$f'(t) = 2t - \frac{2}{t^2+1}(2t) \overset{\text{set}}{=} 0 \iff 2t = \frac{2t}{t^2+1}$

$\iff (t^2+1)t = t$

$\iff t((t^2+1)-1) = 0$

$\iff t^3 = 0$

$\iff \boxed{t = 0}$

$$f(x) = x\sqrt{x - a}$$

$f'(x) = \sqrt{x-a} + \frac{x}{2\sqrt{x-a}} \overset{\text{set}}{=} 0 \iff 2(x-a) + x = 0$

$\iff 3x - 2a = 0$

$\iff \boxed{x = \frac{2}{3}a}$

$$f(x) = \sin^{-1}(x)\cos^{-1}(x)$$

$f'(x) = \frac{\cos^{-1}(x)}{\sqrt{1-x^2}} - \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} \overset{\text{set}}{=} 0 \iff \cos^{-1}(x) = \sin^{-1}(x)$

$\iff \boxed{x = \frac{\sqrt{2}}{2}}$

$\sqrt{1-x^2} \neq 0$

$\boxed{\pm 1 \neq x}$

How to find the absolute max/min of $f(x)$ on $[a, b]$:

1. Find $f'(x)$
2. Find all critical points ($f'(x) = 0$ or $f'(x)$ **DNE**)
3. Evaluate $f(x)$ at the critical points within $[a, b]$. & endpoints.
4. Identify the absolute max and absolute min using the values found. Include value and location (e.g. ordered pair (x, y))

Example. Find the absolute max and min of $f(x) = 2x - 2x^{\frac{2}{3}}$ on $[-1, 3]$.

① $f'(x) = 2 - \frac{4}{3} x^{-1/3} = 2 - \frac{4}{3\sqrt[3]{x}}$

② $2 - \frac{4}{3\sqrt[3]{x}} = 0 \iff \sqrt[3]{x} = \frac{2}{3} \iff \boxed{x = \frac{8}{27}, \quad x \neq 0}$

③ $f(0) = 0$, $f(8/27) = -8/27$, $f(-1) = -4$, $f(3) = 6 - 2\sqrt[3]{9} \approx 1.8398$

④ ABS MIN $(-1, -4)$ ABS MAX $(3, 6 - 2\sqrt[3]{9})$

Example. Find the absolute max and min of $f(x) = \frac{x}{x^2+1}$ on $[0, 2]$.

① $f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$

② $\frac{-x^2+1}{(x^2+1)^2} = 0 \iff 1 = x^2 \iff \boxed{x = \pm 1}$

③ $f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$

④ ABS MIN $(0, 0)$ ABS MAX $(1, 1/2)$

Example. Find the absolute max and min of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

① $f'(x) = 12x^3 - 12x^2 - 24x$

② $12x(x^2 - x - 2) = 0 \iff 12x(x-2)(x+1) = 0 \iff x = -1, x = 0, x = 2$

③ $f(-2) = 33$, $f(-1) = -4$, $f(0) = 1$, $f(2) = -31$, $f(3) = 28$

④ ABS MIN: $(2, -31)$ ABS MAX: $(-2, 33)$

Example. Find the absolute max and min of $f(x) = \sin(3x)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$.

① $f'(x) = 3\cos(3x)$

② $3\cos(3x) = 0 \iff 3x = \pi/2,\ 3x = -\pi/2 \iff x = \pi/6,\ x = -\pi/6$

③ $f(-\pi/4) = -\frac{\sqrt{2}}{2},\ f(-\pi/6) = -1,\ f(\pi/6) = 1,\ f(\pi/3) = 0$

④ ABS MIN: $(-\pi/6, -1)$ ABS MAX $(\pi/6, 1)$

Example. Find the absolute max and min of $f(x) = xe^{1-\frac{x}{2}}$ on $[0, 5]$.

① $f'(x) = e^{1-x/2} + x e^{1-x/2}(-1/2)$

② $e^{1-x/2} - \frac{x}{2} e^{1-x/2} = 0 \iff 1 = x/2 \iff \boxed{x = 2}$

③ $f(0) = 0,\ f(2) = 2,\ f(5) = 5e^{-3/2} \approx 1.11565$

④ ABS MIN $(0, 0)$ ABS MAX $(2, 2)$

Example. Find the absolute max and min of $f(x) = e^x - 2x$ on $[0, 2]$.

① $f'(x) = e^x - 2$

② $e^x - 2 = 0 \iff x = \ln(2)$

③ $f(0) = 1,\ f(\ln(2)) = 2 - 2\ln(2) \approx 0.6137,\ f(2) = e^2 - 4 \approx 3.389$

④ ABS MIN: $(\ln(2), 2 - 2\ln(2))$ ABS MAX $(2, e^2 - 4)$

Example. Find the absolute max and min of $f(x) = x^{\frac{1}{3}}(x+4)$ on $[-27, 27]$.

① $f'(x) = \dfrac{(x+4)}{3\sqrt[3]{x^2}} + \sqrt[3]{x}$

② $\dfrac{(x+4)}{3\sqrt[3]{x^2}} + \sqrt[3]{x} = 0 \iff x+4+3x=0 \iff x=-1$

$x \neq 0$

③ $f(-27) = 69$, $f(-1) = -3$, $f(0) = 0$, $f(27) = 93$

④ ABS Min $(-1, -3)$ ABS MAX $(27, 93)$

Example. Find the absolute max and min of $y = \sqrt{x^2 - 1}$. $\iff x^2 - 1 \geq 0$

$\iff x^2 \geq 1$

$\iff (-\infty, -1] \cup [1, \infty)$

① $y' = \dfrac{x}{\sqrt{x^2-1}}$

② $x \neq \pm 1$ (endpoints) $\dfrac{x}{\sqrt{x^2-1}} = 0 \iff \underset{\sim}{x=0}$ NOT IN DOMAIN!

③ $f(-1) = 0$, $f(1) = 0$

④ ABS MIN $(-1, 0)$ $(1, 0)$ No ABS MAX (Wait till 4.3)

Example. Find the absolute/local max and min of $f(x) = x^2(x^2 + 4x - 8)$ on $[-5, 2]$.

① $f'(x) = 4x^3 + 12x^2 - 16x$

② $4x(x^2 + 3x - 4) = 0 \iff 4x(x+4)(x-1) \iff x = 0, x = -4, x = 1$

③

x	$f(x)$
-5	-75
-4	-128
0	0
1	-3
2	16

④ ABS MIN: $(-4, -128)$

ABS MAX: $(2, 16)$

Example. Minimum-surface-area box

All boxes with a square base of length x and a volume V have a surface area given by $S(x) = x^2 + \frac{4V}{x}$. Find x such that the box has volume 50 ft^3 with minimal surface area.

$x^2 + 200x^{-1}$

$V = 50$

① $S'(x) = 2x - \frac{200}{x^2}$

② $2x - \frac{200}{x^2} = 0 \iff x^3 = 100 \iff x = \sqrt[3]{100}$

$\underbrace{x \neq 0}$ NOT IN DOMAIN

③

x	$f(x)$
$\sqrt[3]{100}$	$3(100)^{2/3} \approx 64.633$

④ ABS MIN $(\sqrt[3]{100}, 3(100)^{2/3})$

NO ABS MAX

(Think x^2)

Example. Trajectory high point

A stone is launched vertically upward from a cliff 192 ft above the ground at a speed of 64 ft/s. Its height above the ground t seconds after the launch is given by $s = -16t^2 + 64t + 192$, for $0 \leq t \leq 6$. When does the stone reach its maximum height?

① $s'(t) = -32t + 64$

② $-32t + 64 = 0 \iff t = 2$

③ $s(2) = 256$ ft

④ ABS MAX $(2, 256)$

Example. Maximizing Revenue

A sales analyst determines that the revenue from sales of fruit smoothies is given by $R(x) = -60x^2 + 300x$, where x is the price in dollars charged per item, for $0 \leq x \leq 5$.

a) Find the critical points of the revenue function.

b) Determine the absolute maximum value of the revenue function and the price that maximizes the revenue.

a) $R'(x) = -120x + 300 \overset{\text{set}}{=} 0$

$\boxed{x = \frac{5}{2}}$

b) $R\left(\frac{5}{2}\right) = 375$

x	$R(x)$
0	0
5/2	375
5	0

ABS MAX $\left(\frac{5}{2}, 375\right)$

Example. Find the absolute and local extreme values of the following

1. $f(x) = |x-3| + |x+2|$ on $[-4, 4]$,

$$f(x) = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases} + \begin{cases} x+2, & x \geq -2 \\ -(x+2), & x < -2 \end{cases} = \begin{cases} -2x+1, & -4 \leq x < -2 \\ 5, & -2 \leq x < 3 \\ 2x-1 & 3 \leq x \leq 4 \end{cases}$$

$$f'(x) = \begin{cases} -2, & -4 \leq x < -2 \\ 0, & -2 < x < 3 \\ 2, & 3 < x \leq 4 \end{cases} \Rightarrow$$

$\boxed{f'(x) \text{ DNE at } x = -2, 3}$ $\rightarrow$

$\boxed{\text{ABS MIN} = 5 \text{ when } -2 \leq x \leq 3. \text{ Also local min}}$

$\boxed{\text{ABS MAX} = 9 \text{ when } x = -4}$

2. $g(x) = |x-3| - 2|x+1|$ on $[-2, 3]$.

$$g(x) = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases} - 2\begin{cases} (x+1), & x \geq -1 \\ -(x+1), & x < -1 \end{cases} = \begin{cases} x+5 & -2 \leq x < -1 \\ -3x+1 & -1 \leq x \leq 3 \end{cases}$$

$$g'(x) = \begin{cases} 1, & -2 \leq x < -1 \\ -3, & -1 < x < 3 \end{cases}$$

$g(-1) = 4$ ABS MAX

$g'(x)$ DNE at $x = -1$

$g(3) = -8$

$g'(x) \neq 0$ anywhere

(see Sec 4.3)

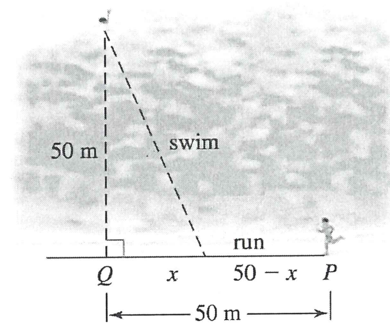
Example. Every second counts (4.01: Q87)

You must get from a point P on the straight shore of a lake to a stranded swimmer who is $50\,m$ from a point Q on the shore that is $50\,m$ from you. Assuming that you can swim at a speed of $2\,m/s$ and run at a speed of $4\,m/s$, the goal of this exercise is to determine the point along the shore, x meters from Q, where you should you stop running and start swimming to reach the swimmer in the minimum time.

a) Find the function T that gives the travel time as a function of x, where $0 \le x \le 50$.

b) Find the critical point of T on $(0, 50)$.

c) Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?



d) Graph the function T to check your work.

a) $T(x) = \dfrac{50-x}{4} + \dfrac{\sqrt{x^2+50^2}}{2}$

b) $T'(x) = -\dfrac{1}{4} + \dfrac{x}{2\sqrt{x^2+50^2}} \overset{\text{set}}{=} 0$

$\iff \dfrac{x}{\sqrt{x^2+50^2}} = \dfrac{1}{2}$

$\iff 4x^2 = x^2 + 2500$

$\iff x = \sqrt{\dfrac{2500}{3}}$

$= \dfrac{50}{\sqrt{3}}$

c)

x	$T(x)$
0	$75/2 = 37.5$
$\sqrt{\dfrac{2500}{3}}$	$\dfrac{25}{2} + \dfrac{75}{2\sqrt{3}} \approx \boxed{34.1506}$
50	$25\sqrt{2} \approx 35.355$

ABS Min →

d)