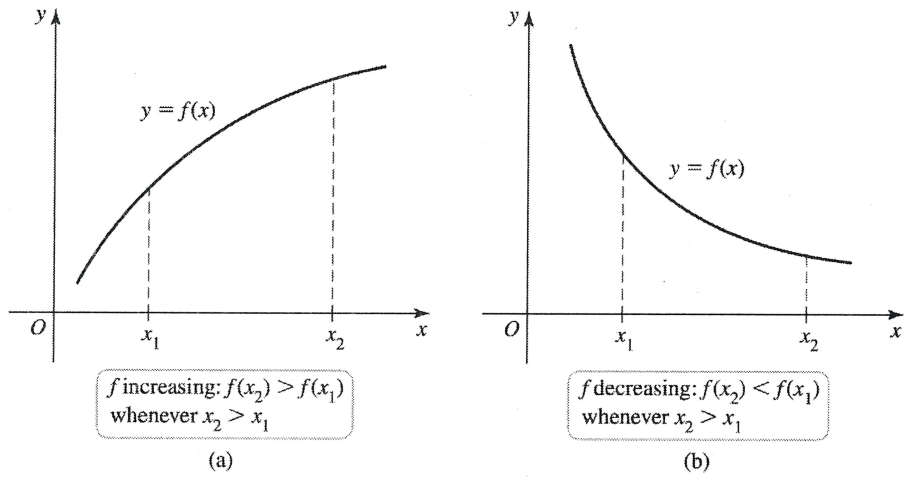
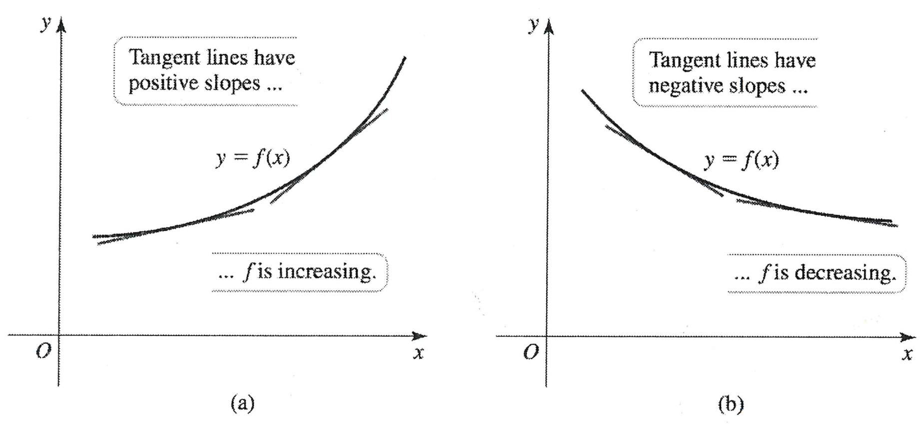
4.3: What Derivatives Tell Us

Definition. (Increasing and Decreasing Functions)
Suppose a function f is defined on an interval I. We say that f is increasing on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is decreasing on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.



Theorem 4.7: Test for Intervals of Increase and Decrease
Suppose f is continuous on an interval I and differentiable at every interior point of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.



Proof. **(Theorem 4.7: Test for Intervals of Increase and Decrease) p258**

Let a and b be any two distinct points in the interval I with $b > a$. By the Mean Value Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c between a and b. Equivalently,

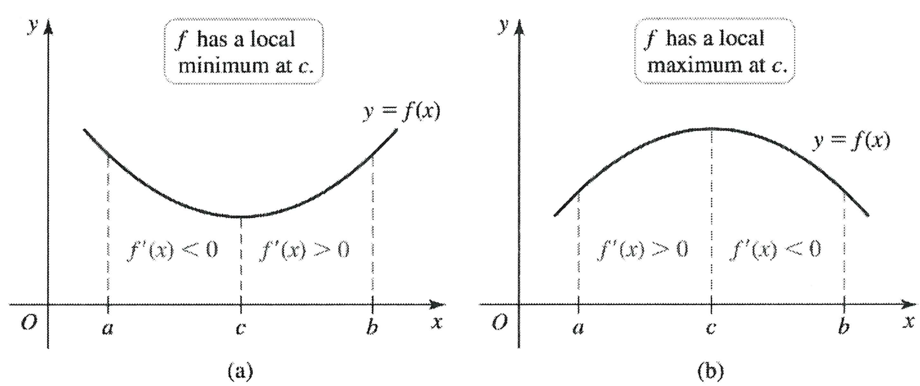
$$f(b) - f(a) = f'(c)(b - a).$$

Notice that $b - a > 0$ by assumption. So if $f'(c) > 0$, then $f(b) - f(a) > 0$. Therefore, for all a and b in I with $b > a$, we have $f(b) > f(a)$, which implies that f is increasing on I. Similarly, if $f'(c) < 0$, then $f(b) - f(a) < 0$ or $f(b) < f(a)$. It follows that f is decreasing on I. $\square$

Theorem 4.8: First Derivative Test

Assume that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a local maximum at c.
- If f' changes sign from negative to positive as x increases through c, then f has a local minimum at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has no local extreme value at c.



Example. Consider the function $f(x) = 6x - x^3$.

a) $f'(x) =$ $6 - 3x^2$

b) Find the intervals on which the function is increasing and decreasing.

crit. pts.

$6 - 3x^2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

$-2 \quad -\sqrt{2} \quad 0 \quad \sqrt{2} \quad 2$

$f'(x) \quad - \quad + \quad -$

$\searrow \quad \nearrow \quad \searrow$

dec: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

inc: $(-\sqrt{2}, \sqrt{2})$

c) Identify the function's local extreme values, if any. (e.g. "local max of __ at $x =$ __")

$f(-\sqrt{2}) = -6\sqrt{2} + 2\sqrt{2} = -4\sqrt{2}$

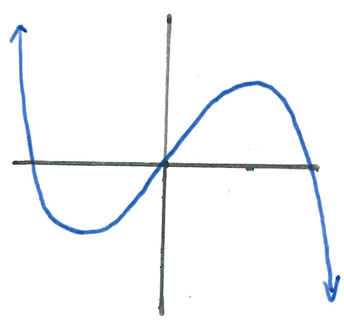
$f(\sqrt{2}) = 6\sqrt{2} - 2\sqrt{2} = 4\sqrt{2}$

local min $(-\sqrt{2}, -4\sqrt{2})$

local max $(\sqrt{2}, 4\sqrt{2})$

d) Which, if any, of the extreme values are absolute?

Neither - unbounded function



Example. Consider the function $f(t) = 12t - t^3$ on $-3 \leq t < \infty$.

a) $f'(t) =$ $12 - 3t^2$

b) Find the intervals on which the function is increasing and decreasing.

Crit pts

$12 - 3t^2 = 0$

$t^2 = 4$

$t = \pm 2$

	-3	-2.5	-2	0	2	10
$f'(x)$		−		+		−
		↘		↗		↘

dec: $(-3,-2) \cup (2,\infty)$

inc: $(-2, 2)$

c) Identify the function's local extreme values, if any. (e.g. "local max of __ at $x =$ __")

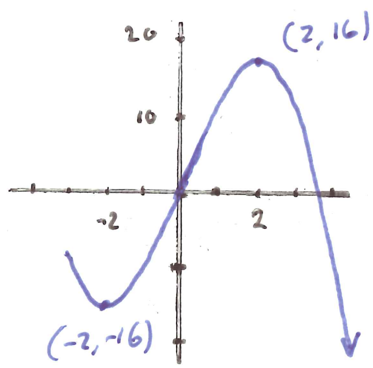
$f(-2) = -16$ local min $(-2,-16)$

$f(2) = 16$ local max $(2, 16)$

d) Which, if any, of the extreme values are absolute?

$f(-3) = -9$

t	$f(t)$
-3	-9
-2	-16
2	16



local max AND abs max $(2, 16)$

No abs min.

Example. Consider the function $f(x) = \cos^2(x)$ on $[-\pi, \pi]$. Find the intervals on which f is increasing and the intervals on which it is decreasing.

$$f'(x) = -2\cos(x)\sin(x)$$

Crit pts

$$-2\cos(x)\sin(x) = 0$$

$$x = -\pi/2, \pi/2 \qquad x = -\pi, 0, \pi$$

End points

$-\pi \quad -\pi/2 \quad 0 \quad \pi/2 \quad \pi$

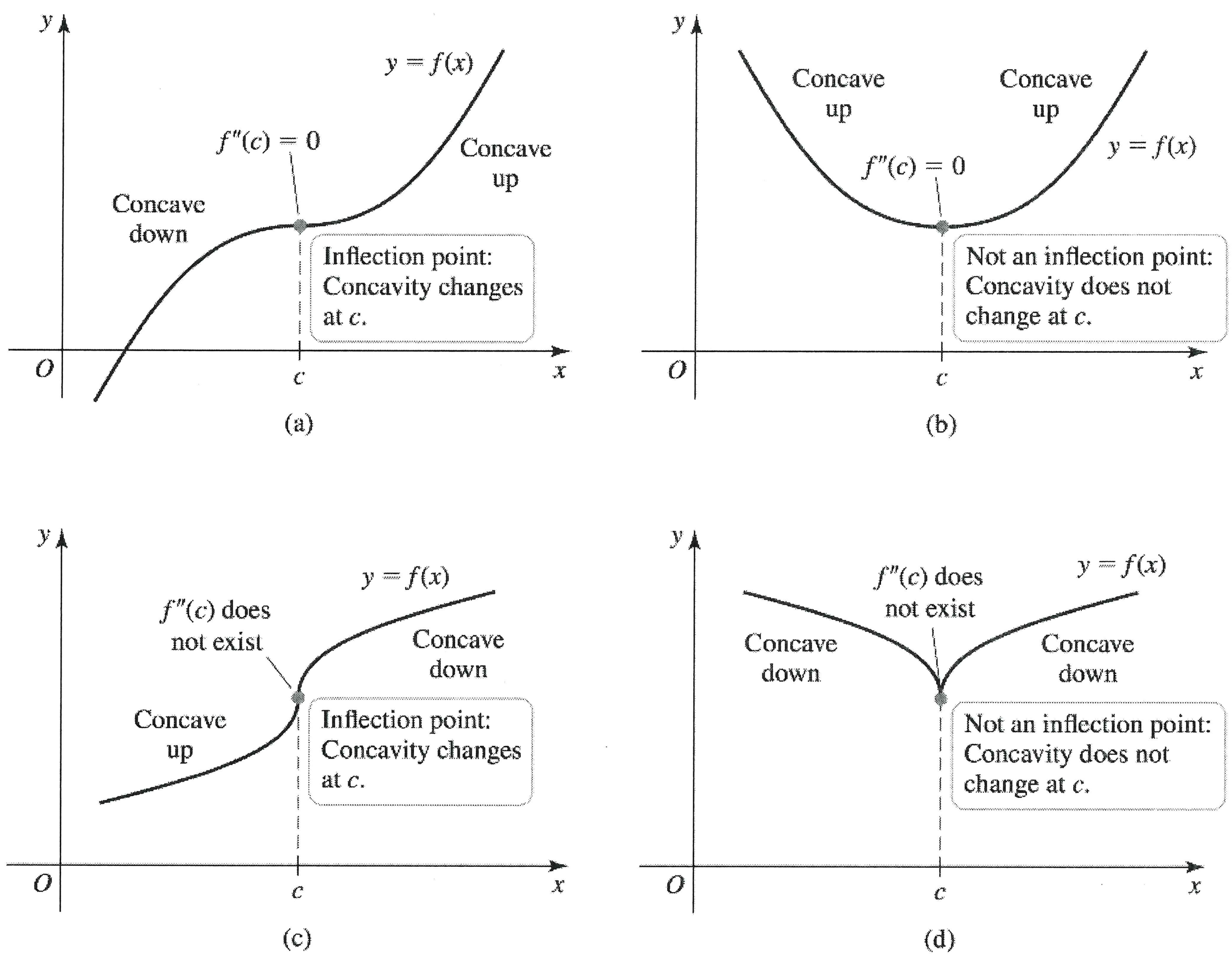
-2	$-$	$-$	$-$	$-$
$\cos(x)$	$-$	$+$	$+$	$-$
$\sin(x)$	$-$	$-$	$+$	$+$
	$-$	$+$	$-$	$+$
	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Dec: $(-\pi, -\pi/2) \cup (0, \pi/2)$

Inc: $(-\pi/2, 0) \cup (\pi/2, \pi)$

Definition. (Concavity and Inflection Point)
Let f be differentiable on an open interval I.

- If f' is increasing on I, then f is *concave up* on I.
- If f' is decreasing on I, then f is *concave down* on I.
- If f is continuous at c and f changes concavity at c, then f has an *inflection point* at c.



Theorem 4.10: Test for Concavity
Suppose f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c, then f has an inflection point at c.

Example. Consider $f(x) = 5 - 3x^2 + x^3$

a) Find the intervals of increasing/decreasing and local max/min.

$f'(x) = -6x + 3x^2 \overset{\text{set}}{=} 0$

$3x(x-2) = 0$

$x = 0$

$x = 2$

	-1	0	1	2	3
$3x$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$
	$\nearrow$		$\searrow$		$\nearrow$

dec: $(0,2)$

inc: $(-\infty,0) \cup (2,\infty)$

local min: $(2,1)$

local max: $(0,5)$

b) Find intervals of concavity and inflection points.

$f''(x) = -6 - 6x \overset{\text{set}}{=} 0$

$x = 1$

	0	1	2
$f''(x)$	$-$		$+$
	$\cap$		$\cup$

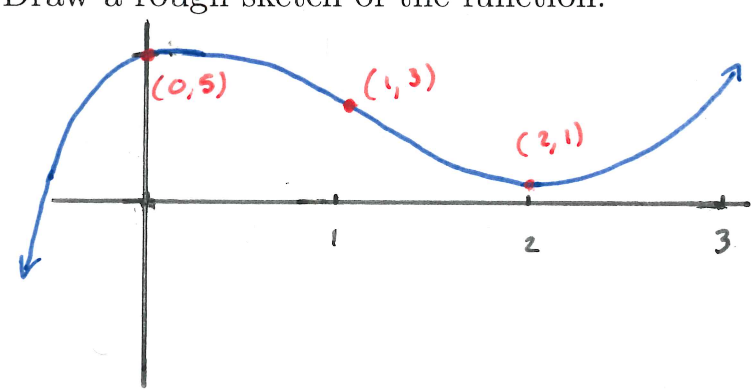
$f(1)$

Inflection pt. $(1,3)$

Concave down $(-\infty,1)$

Concave up $(1,\infty)$

c) Draw a rough sketch of the function.



Example. Consider $f(x) = xe^{-x^2/2}$

a) Find the intervals of increasing/decreasing and local max/min.

$$f'(x) = e^{-x^2/2} + x e^{-x^2/2}(-x) \overset{\text{set}}{=} 0$$

$$e^{-x^2/2}(1-x^2) = 0$$

$$\boxed{x = \pm 1}$$

	-2	-1	0	1	2
$e^{-x^2/2}$	$+$		$+$		$+$
$1-x^2$	$-$		$+$		$-$
$f'(x)$	$-$		$+$		$-$
	↘		↗		↘

local min $(-1, -e^{-1/2})$

local max $(1, e^{-1/2})$

dec: $(-\infty, -1) \cup (1, \infty)$

inc: $(-1, 1)$

b) Find intervals of concavity and inflection points.

$$f''(x) = -xe^{-x^2/2} - 2xe^{-x^2/2} - x^2e^{-x^2/2}(-x) = xe^{-x^2/2}(x^2-3) \overset{\text{set}}{=} 0$$

$$\Rightarrow x = 0,\ x = \pm\sqrt{3}$$

	$-\sqrt{3}$	0	$\sqrt{3}$	
x	$-$	$-$	$+$	$+$
$e^{-x^2/2}$	$+$	$+$	$+$	$+$
x^2-3	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$+$	$-$	$+$
	$\cap$	$\cup$	$\cap$	$\cup$

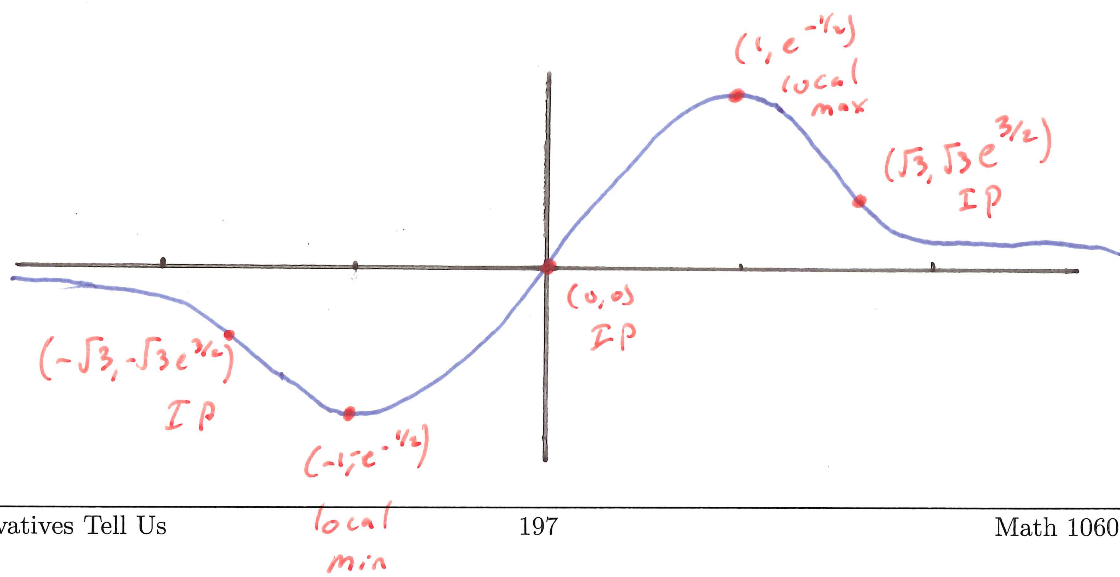
Inflection pts.

$(-\sqrt{3}, -\sqrt{3}e^{-3/2})$; $(0,0)$; $(\sqrt{3}, \sqrt{3}e^{-3/2})$

Concave up $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Concave down $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

c) Draw a rough sketch of the function.



Example. Consider $f(x) = x\left(\sqrt[3]{(x-3)^2}\right)$

a) Find the intervals of increasing/decreasing and local max/min.

$$f'(x) = (x-3)^{2/3} + x\,\tfrac{2}{3}(x-3)^{-1/3} \overset{\text{set}}{=} 0$$

$$(x-3) + \tfrac{2}{3}x = 0 \Rightarrow \overbrace{x = 9/5 \;,\; x \neq 3}^{\text{Crit. pt.}}$$

Test points: 0, 2, 4 (critical points $9/5$, 3)

$f'(x)$: $+$ $\quad -\quad$ $+$

↗ ↘ ↗

Dec: $(9/5, 3)$

local min: $(3, 0)$

Inc: $(-\infty, 9/5) \cup (3, \infty)$

local max: $\left(9/5,\ 9/5\left(\tfrac{6}{5}\right)^{2/3}\right)$

b) Find intervals of concavity and inflection points.

$$f''(x) = \frac{2}{3(x-3)^{1/3}} + \frac{2}{3(x-3)^{1/3}} - \frac{2x}{9(x-3)^{4/3}} \overset{\text{set}}{=} 0 \Rightarrow x = 18/5 \;,\; x \neq 3$$

Test points: 0, $16/5$, 4 (points 3, $18/5$)

$f''(x)$: $-$ $\quad -\quad$ $+$

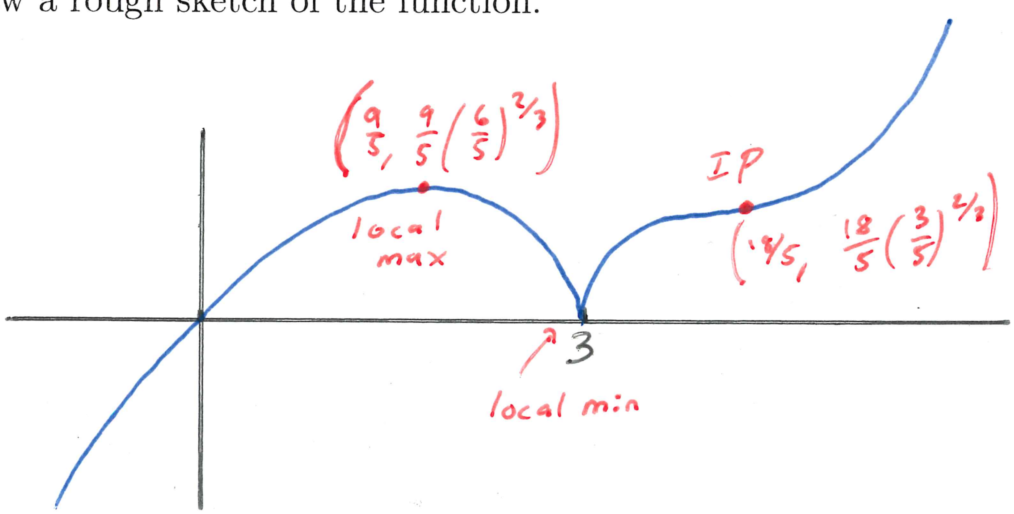
∩ ∩ ∪

Concave down: $(-\infty, 3) \cup (3, 18/5)$

concave up $(18/5, \infty)$

Inflection pt. $\left(18/5,\ \tfrac{18}{5}\left(\tfrac{3}{5}\right)^{2/3}\right)$

c) Draw a rough sketch of the function.



Example. Consider $f(x) = x^2 - x - \ln(x)$

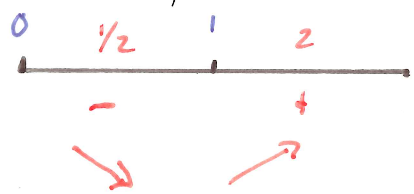
a) Find the intervals of increasing/decreasing and local max/min.

$$f'(x) = 2x - 1 - \tfrac{1}{x} \overset{\text{set}}{=} 0$$

$$2x^2 - x - 1 = 0$$

$$(2x+1)(x-1) = 0$$

$$\underbrace{x = -\tfrac{1}{2}}_{\text{Not in domain}}, \quad x = 1, \quad x \neq 0$$



Dec: $(0, 1)$

Inc: $(1, \infty)$

local min: $(1, 0)$

No local max

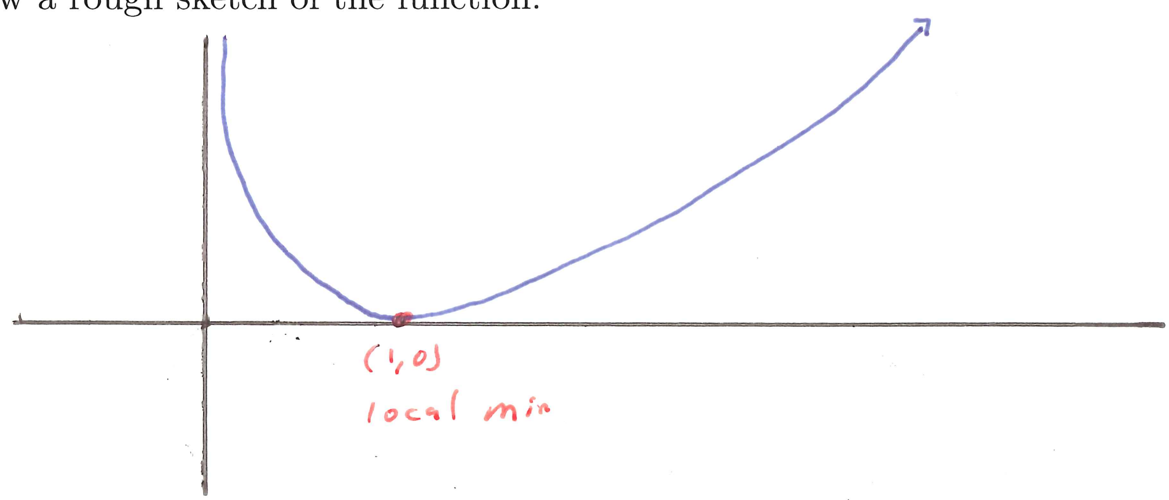
b) Find intervals of concavity and inflection points.

$$f''(x) = 2 + \underbrace{\tfrac{1}{x^2}}_{x \neq 0} \overset{\text{set}}{=} 0 \;\rightarrow\; x^2 = -\tfrac{1}{2} \;\Rightarrow\; \text{No such } x$$

$\Rightarrow$ $f''(x) > 0$ everywhere $\rightarrow$ concave up $(0, \infty)$

$\Rightarrow$ No inflection point

c) Draw a rough sketch of the function.



Example. Given the first derivative $y' = (x-2)^{-1/3}$

a) Find the intervals of increasing/decreasing and local max/min.

Solve $y' = 0 \rightarrow \frac{1}{(x-2)^{1/3}} = 0 \rightarrow$ No such x

$x \neq 2$

	0	2	4
y'	$-$		$+$
	↘		↗

Dec: $(-\infty, 2)$

Inc: $(2, \infty)$

Local min @ $x = 2$

b) Find intervals of concavity and inflection points.

$y'' = -\frac{1}{3}(x-2)^{-4/3} \longrightarrow y'' \neq 0$ for any x

$x \neq 2$

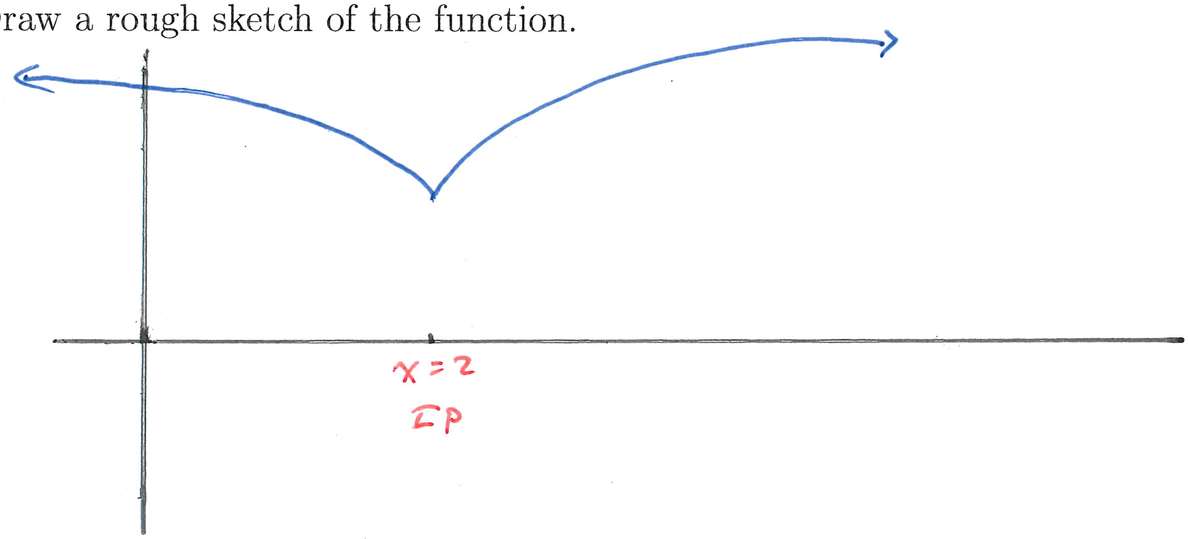
	0	2	4
y''	$-$		$-$
	$\cap$		$\cap$

Concave down: $(-\infty, 2) \cup (2, \infty)$

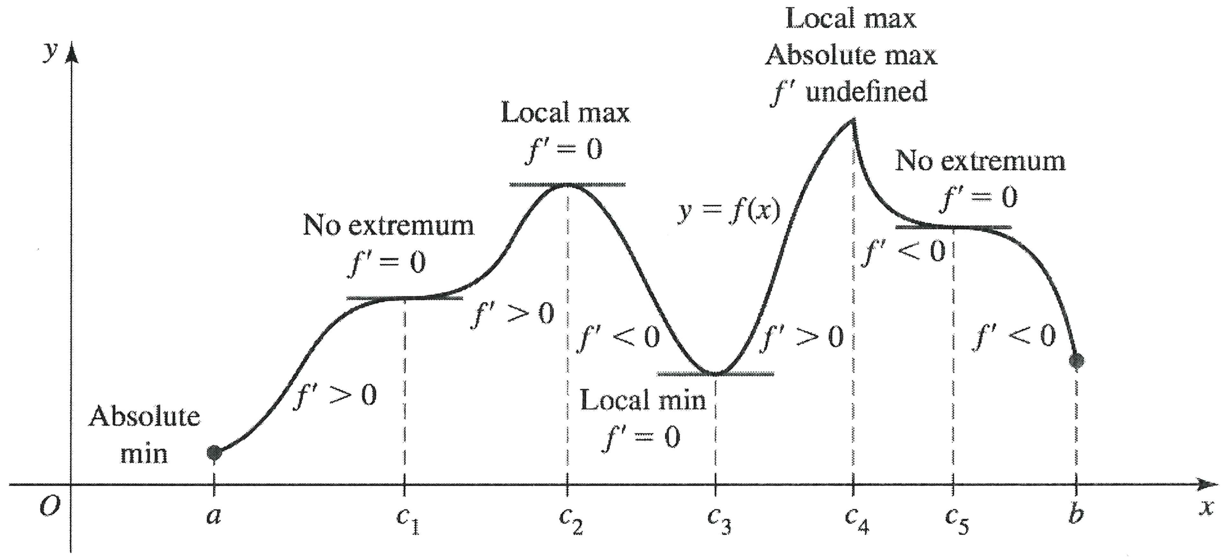
Never concave up

No inflection pt

c) Draw a rough sketch of the function.



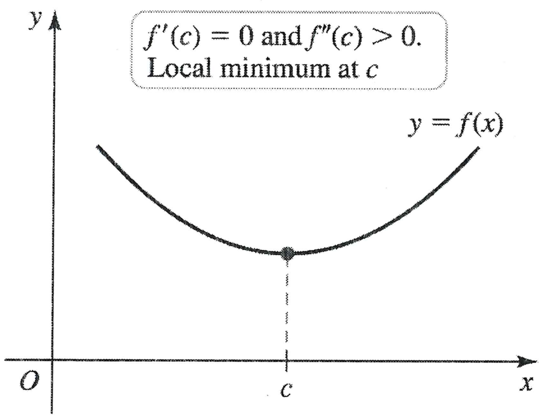
As a visual summary, here is figure 4.27 from Briggs:



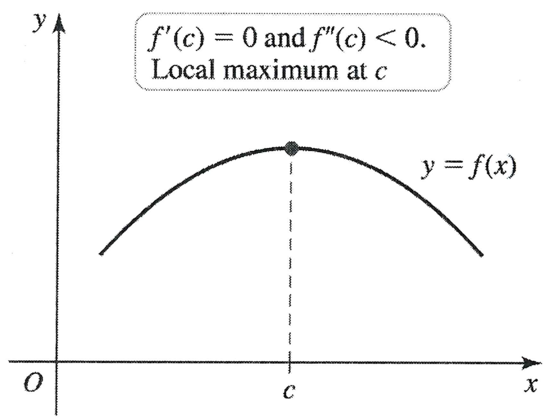
$f(x)$	$f'(x)$	$f''(x)$
increasing	positive	—
decreasing	negative	—
max	zero (pos to neg)	—
min	zero (neg to pos)	—
concave up	increasing	positive
concave down	decreasing	negative
Inflection point	max/min	changes sign

Theorem 4.11: Second Derivative Test for Local Extrema
Suppose f" is continuous on an open interval containing c with f'(c)=0.

- If $f''(c) > 0$, then f has a local minimum at c (Figure 4.40a).
- If $f''(c) < 0$, then f has a local maximum at c (Figure 4.40b).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.



(a)



(b)

Example. For each of the following, use the Second Derivative Test for Extrema to determine local max/mins:

a) $f(x) = x^5 - 5x + 3$

$f'(x) = 5x^4 - 5 \overset{\text{set}}{=} 0$

$\boxed{x = \pm 1}$

$f''(x) = 20x^3$

$f''(-1) = -20 < 0$
$\rightarrow$ local max $(-1, 7)$

$f''(1) = 20 > 0$
$\rightarrow$ local min $(1, -1)$

b) $f(x) = 2x^3 - 3x^2 + 12$

$f'(x) = 6x^2 - 6x \overset{\text{set}}{=} 0$

$\boxed{x = 0, 1}$

$f''(x) = 12x - 6$

x	$f''(x)$	
0	-6	$\rightarrow$ local max $(0, 12)$
1	6	$\rightarrow$ local min $(1, 11)$

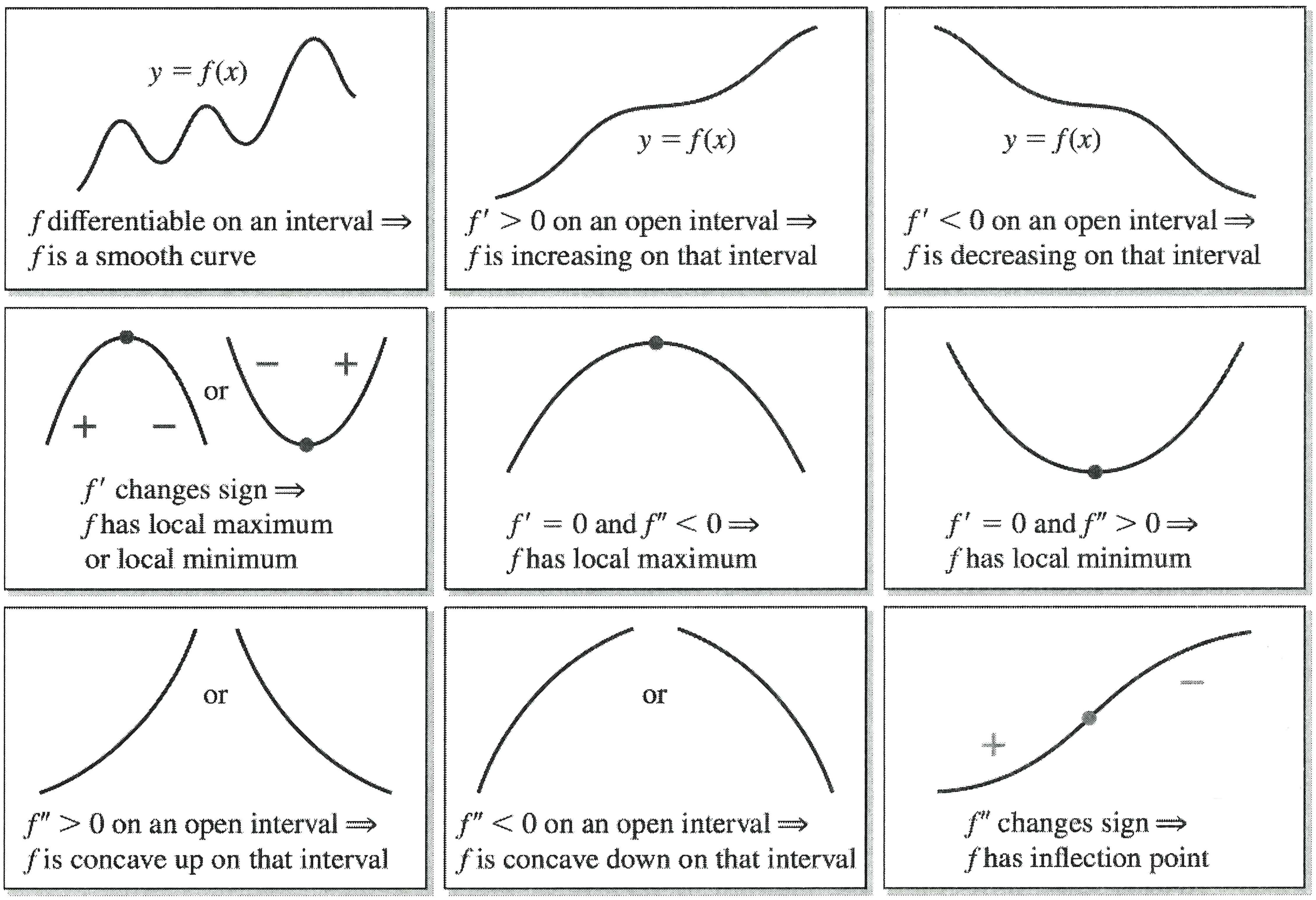
c) $f(x) = x^3 - 6$

$f'(x) = 3x^2 \overset{\text{set}}{=} 0$

$\boxed{x = 0}$

$f''(x) = 6x$

x	$f''(x)$	
0	0	$\rightarrow$ Inconclusive.


$y = f(x)$
f differentiable on an interval $\Rightarrow$
f is a smooth curve
$y = f(x)$
$f' > 0$ on an open interval $\Rightarrow$
f is increasing on that interval
$y = f(x)$
$f' < 0$ on an open interval $\Rightarrow$
f is decreasing on that interval
+
−
or
−
+
f' changes sign $\Rightarrow$
f has local maximum
or local minimum
$f' = 0$ and $f'' < 0 \Rightarrow$
f has local maximum
$f' = 0$ and $f'' > 0 \Rightarrow$
f has local minimum
or
$f'' > 0$ on an open interval $\Rightarrow$
f is concave up on that interval
or
$f'' < 0$ on an open interval $\Rightarrow$
f is concave down on that interval
−
+
f'' changes sign $\Rightarrow$
f has inflection point