3.4 The Product and Quotient Rule

Theorem 3.7: Product Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Note: This can also be denoted

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)].$$

Example. For $f(x) = (3x^2)(2x)$, find $f'(x)$ by using the product rule and by

① $f'(x) = \frac{d}{dx}[3x^2](2x) + (3x^2)\frac{d}{dx}[2x] = (6x)(2x) + 3x^2(2) = 12x^2 + 6x^2 = \boxed{18x^2}$

② $f(x) = 6x^3 \rightarrow \boxed{f'(x) = 18x^2}$

Example. For $g(x) = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$, find $g'(x)$.

$g'(x) = \frac{d}{dx}[x + x^{-1}](x - x^{-1} + 1) + (x + x^{-1})\frac{d}{dx}[x - x^{-1} + 1]$

$= (1 - x^{-2})(x - x^{-1} + 1) + (x + x^{-1})(1 + x^{-2})$

$= x - x^{-1} + 1 - x^{-1} + x^{-3} - x^{-2} + x + x^{-1} + x^{-1} + x^{-3}$

$= 2x + 1 - x^{-2} + 2x^{-3} = \boxed{2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}}$

Example. For $h(x) = (x-1)(x^2 + x + 1)$, find $h'(x)$.

$h'(x) = \frac{d}{dx}[x-1](x^2 + x + 1) + (x-1)\frac{d}{dx}[x^2 + x + 1]$

$= (1)(x^2 + x + 1) + (x-1)(2x + 1)$

$= x^2 + x + 1 + 2x^2 + x - 2x - 1 = \boxed{3x^2}$

Example. Use the product rule to find the derivative of $1 - e^{2t}$.

$$\frac{d}{dt}\left[1-e^{2t}\right] = \frac{d}{dt}\left[(1+e^t)(1-e^t)\right]$$
$$= \frac{d}{dt}\left[1+e^t\right](1-e^t) + (1+e^t)\frac{d}{dt}\left[1-e^t\right]$$
$$= e^t(1-e^t) + (1+e^t)(-e^t)$$
$$= e^t - e^{2t} - e^t - e^{2t}$$
$$= \boxed{-2e^{2t}}$$

Theorem 3.8 Quotient Rule
If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of f/g at x exists and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Note: A common phrase for the quotient rule is
"Lo De Hi minus Hi De Lo over Lo squared"

Example. Find the derivative of $y = \dfrac{t^2+1}{3t^2-2t+1}$.

$$y' = \frac{(3t^2-2t+1)\frac{d}{dt}\left[t^2+1\right] - (t^2+1)\frac{d}{dt}\left[3t^2-2t+1\right]}{(3t^2-2t+1)^2}$$
$$= \frac{(3t^2-2t+1)(2t) - (t^2+1)(6t-2)}{(3t^2-2t+1)^2}$$
$$= \frac{6t^3-4t^2+2t-6t^3+2t^2-6t+2}{(3t^2-2t+1)^2} = \frac{-2t^2-4t+2}{(3t^2-2t+1)^2}$$

Example. Find the derivatives of the following functions:

$$f(t) = \frac{2t}{4+t^2}$$

$$f'(t) = \frac{(4+t^2)[2] - 2t[2t]}{(4+t^2)^2}$$

$$= \frac{8+2t^2-4t^2}{(4+t^2)^2}$$

$$= \frac{-2t^2+8}{(t^2+4)^2}$$

$$w = (2x-7)^{-1}(x+5) = \frac{x+5}{2x-7}$$

$$w' = \frac{(2x-7)[1] - (x+5)[2]}{(2x-7)^2}$$

$$= \frac{2x-7-2x-10}{(2x-7)^2}$$

$$= \frac{-17}{(2x-7)^2}$$

$$y = \frac{e^x}{1-e^x}$$

$$y' = \frac{(1-e^x)[e^x] - e^x[-e^x]}{(1-e^x)^2}$$

$$= \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2}$$

$$= \frac{e^x}{(1-e^x)^2}$$

$$h(w) = \frac{w^2-1}{w^2+1}$$

$$h'(w) = \frac{(w^2+1)[2w] - (w^2-1)[2w]}{(w^2+1)^2}$$

$$= \frac{2w^3+2w-2w^3+2w}{(w^2+1)^2}$$

$$= \frac{4w}{(w^2+1)^2}$$

Example. Find the derivative of the following functions. Is using the quotient rule recommended here?

$$w(z) = \frac{4}{z^3} = 4z^{-3}$$

$$w'(z) = \frac{z^3[0] - 4[3z^2]}{(z^3)^2}$$

$$= \frac{-12z^2}{z^6} = \boxed{\frac{-12}{z^4}}$$

$\Updownarrow$

— or —

$$w'(z) = \frac{d}{dz}\left[4z^{-3}\right] = \boxed{-12z^{-4}}$$

$$f(x) = \frac{x^2 - 2ax + a^2}{x - a} = \frac{(x-a)^2}{x-a}$$

$$f'(x) = \frac{(x-a)[2x-2a] - (x^2-2ax+a^2)[1]}{(x-a)^2}$$

$$= \frac{2x^2 - 2ax - 2ax + 2a^2 - x^2 + 2ax - a^2}{(x-a)^2}$$

$$= \frac{x^2 - 2ax + a^2}{(x-a)^2} = \frac{(x-a)^2}{(x-a)^2} = \boxed{1}$$

— or —

$$\frac{d}{dx}\left[\frac{(x-a)^2}{x-a}\right] = \frac{d}{dx}\left[x-a\right] = \boxed{1}$$

Example. Find the second derivative of the following functions.

$$f(x) = x^{\frac{5}{2}}e^x$$

$$f'(x) = \left[\frac{5}{2}x^{3/2}\right]e^x + x^{5/2}\left[e^x\right]$$

$$= \boxed{\left(x^{5/2} + \frac{5}{2}x^{3/2}\right)e^x}$$

$$f''(x) = \left[\frac{5}{2}x^{3/2} + \frac{15}{4}x^{1/2}\right]e^x$$

$$+ \left(x^{5/2} + \frac{5}{2}x^{3/2}\right)\left[e^x\right]$$

$$= \boxed{\left(x^{5/2} + 5x^{3/2} + \frac{15}{4}x^{1/2}\right)e^x}$$

$$y(t) = \frac{t}{t+2}$$

$$y'(t) = \frac{(t+2)[1] - t[1]}{(t+2)^2}$$

$$= \frac{2}{(t+2)^2} = \boxed{\frac{2}{t^2+4t+4}}$$

$$y''(t) = \frac{(t^2+4t+4)[0] - 2[2t+4]}{(t+2)^4}$$

$$= \frac{-4(t+2)}{(t+2)^4} = \boxed{\frac{-4}{(t+2)^3}}$$

Example. Use the table below to evaluate the following

x	1	2	3	4	5
$f(x)$	5	4	3	2	1
$f'(x)$	3	5	2	1	4
$g(x)$	4	2	5	3	1
$g'(x)$	2	4	3	1	5

$$\frac{d}{dx}[f(x)\cdot g(x)]\bigg|_{x=5}$$

$= f'(5)\cdot g(5) + f(5)\cdot g'(5)$

$= 4\cdot 1 + 1\cdot 5$

$= \boxed{9}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\bigg|_{x=3} = \frac{g(3)\cdot f'(3) - f(3)g'(3)}{[g(3)]^2}$$

$= \dfrac{5\cdot 2 - 3\cdot 3}{5^2}$

$= \boxed{\dfrac{1}{25}}$

$$\frac{d}{dx}[x\cdot g(x)]\bigg|_{x=2}$$

$= [1]\cdot g(2) + 2\cdot g'(2)$

$= 2 + 2\cdot 4$

$= \boxed{10}$

$$\frac{d}{dx}\left[\frac{x\cdot f(x)}{g(x)}\right]\bigg|_{x=4}$$

$= \dfrac{g(4)[1\cdot f(4) + 4\cdot f'(4)] - 4\cdot f(4)\, g'(4)}{[g(4)]^2}$

$= \dfrac{3[1\cdot 2 + 4\cdot 1] - 4\cdot 2\cdot 1}{3^2} = \boxed{\dfrac{10}{9}}$

$h(x) = (-2x^3)\cdot f(x)$, find $h'(4)$.

$h'(x) = -6x^2 f(x) - 2x^3 f'(x)$

$h'(4) = -6(4)^2 f(4) - 2(4)^3 f'(4)$

$= -92(2) - 128(1)$

$= \boxed{-224}$

$r(x) = \dfrac{2g(x)}{-3\sqrt[4]{x}}$, find $r'(1)$.

$r'(x) = \dfrac{-3x^{1/4}[2g'(x)] - 2g'(x)[-\frac{3}{4}x^{-3/4}]}{(3x^{1/4})^2}$

$= \dfrac{-2\sqrt[4]{x}\, g'(x) + \frac{1}{2}\frac{g'(x)}{\sqrt[4]{x^3}}}{3\sqrt{x}}$

$r'(1) = \dfrac{-2(1)\cdot g'(1) + \frac{g(1)}{2(1)}}{3\cdot(1)}$

$= \boxed{-\dfrac{2}{3}}$