3.2 The Derivative as a Function

Definition. (The Derivative Function)

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is **differentiable** at x. If f is differentiable at every point on an open interval I, we say that f is differentiable on I.

Note: The derivative of f has several notations:

$$f'(x) \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

Note: The derivative of f evaluated at a has several notations:

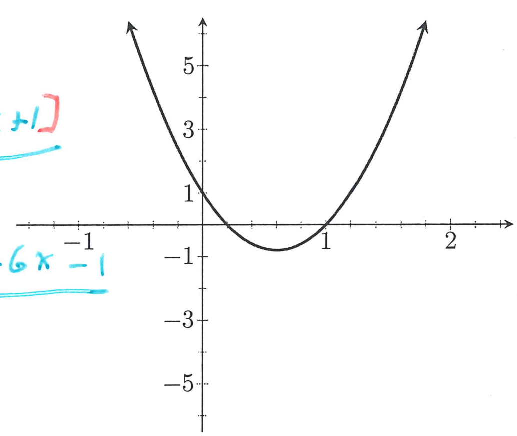
$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Example. Use the limit definition of a derivative to find the derivative function $f'(x)$ for the function $f(x) = 5x^2 - 6x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[5(x+h)^2 - 6(x+h) + 1] - [5x^2 - 6x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 6x - 6h + 1 - 5x^2 + 6x - 1}{h}$$



$$= \lim_{h \to 0} \frac{10xh + 5h^2 - 6h}{h}$$

$$= \lim_{h \to 0} 10x + 5h - 6 = \boxed{10x - 6}$$

Example. Find the derivative of the following functions. If a point is specified, find the tangent line at that point.

$f(w) = \sqrt{4w-3}, \ w = 3$

$$f'(w) = \lim_{h\to 0} \frac{\sqrt{4(w+h)-3} - \sqrt{4w-3}}{h} \left(\frac{\sqrt{4(w+h)-3} + \sqrt{4w-3}}{\sqrt{4(w+h)-3} + \sqrt{4w-3}} \right)$$

$$= \lim_{h\to 0} \frac{4w+4h-3-4w+3}{h\left(\sqrt{4(w+h)-3} + \sqrt{4w-3}\right)}$$

$$= \lim_{h\to 0} \frac{4h}{h\left(\sqrt{4(w+h)-3} + \sqrt{4w-3}\right)}$$

$$= \lim_{h\to 0} \frac{4}{\sqrt{4(w+h)-3} + \sqrt{4w-3}} = \boxed{\frac{4}{2\sqrt{4w-3}}}$$

$$y - f(3) = f'(3)(x-3)$$

$$y = \frac{2}{3}(x-3) + 3$$

$$\boxed{y = \frac{2}{3}x + 1}$$

$g(v) = \dfrac{v}{v+2}, v = 0$

$$g'(v) = \lim_{h\to 0} \frac{\left[\frac{v+h}{v+h+2}\right] - \left[\frac{v}{v+2}\right]}{h} = \lim_{h\to 0} \frac{\frac{(v+h)(v+2) - v(v+h+2)}{(v+h+2)(v+2)}}{h}$$

$$= \lim_{h\to 0} \frac{v^2+hv+2v+2h-v^2-hv-2v}{h(v+h+2)(v+2)}$$

$$= \lim_{h\to 0} \frac{2h}{h(v+h+2)(v+2)}$$

$$= \lim_{h\to 0} \frac{2}{(v+h+2)(v+2)} = \boxed{\frac{2}{(v+2)^2}}$$

$$y - g(0) = g'(0)(x-0)$$

$$\boxed{y = \frac{1}{2}x}$$

$h(m) = 1 + \sqrt{m}, \; m = 1/4, \; m = 1$

$$h'(m) = \lim_{h\to 0} \frac{[1+\sqrt{m+h}\,] - [1+\sqrt{m}\,]}{h} = \lim_{h\to 0} \frac{\sqrt{m+h}-\sqrt{m}}{h}\left(\frac{\sqrt{m+h}+\sqrt{m}}{\sqrt{m+h}+\sqrt{m}}\right)$$

$$= \lim_{h\to 0} \frac{m+h-m}{h(\sqrt{m+h}+\sqrt{m})} = \lim_{h\to 0} \frac{h}{h(\sqrt{m+h}+\sqrt{m})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{m+h}+\sqrt{m}} = \boxed{\frac{1}{2\sqrt{m}}}$$

$m = 1/4$: $y - h(1/4) = h'(1/4)(x - 1/4)$ $\boxed{y = x + 5/4}$

$m = 1$: $y - h(1) = h'(1)(x-1)$ $\boxed{y = \frac{x}{2} + \frac{3}{2}}$

$\frac{d}{dx}(\sqrt{ax+b})$. Then find $\frac{d}{dx}(f(x))$ where $f(x) = \sqrt{5x+9}$ and find $f'(-1)$.

$$\lim_{h\to 0} \frac{\sqrt{a(x+h)+b} - \sqrt{ax+b}}{h}\left(\frac{\sqrt{a(x+h)+b}+\sqrt{ax+b}}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}\right)$$

$$= \lim_{h\to 0} \frac{ax+ah+b-ax-b}{h(\sqrt{a(x+h)+b}+\sqrt{ax+b})}$$

$$= \lim_{h\to 0} \frac{a}{\sqrt{a(x+h)+b}+\sqrt{ax+b}}$$

$$= \boxed{\frac{a}{2\sqrt{ax+b}}} \Rightarrow \boxed{f'(x) = \frac{5}{2\sqrt{5x+9}}} \quad \boxed{f'(-1) = \frac{5}{4}}$$

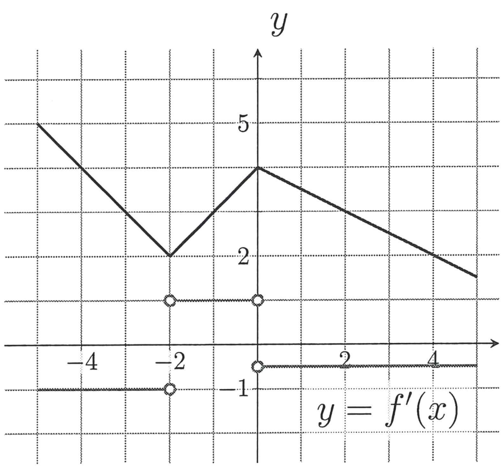
$$\frac{d}{dx}(ax^2+bx+c) = \lim_{h\to 0} \frac{[a(x+h)^2+b(x+h)+c] - [ax^2+bx+c]}{h}$$

$$= \lim_{h\to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h}$$

$$= \lim_{h\to 0} \frac{2axh + ah^2 + bh}{h}$$

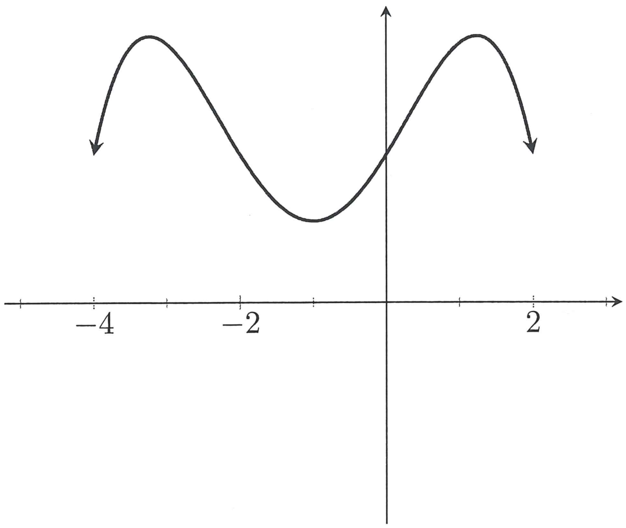
$$= \lim_{h\to 0} 2ax + ah + b$$

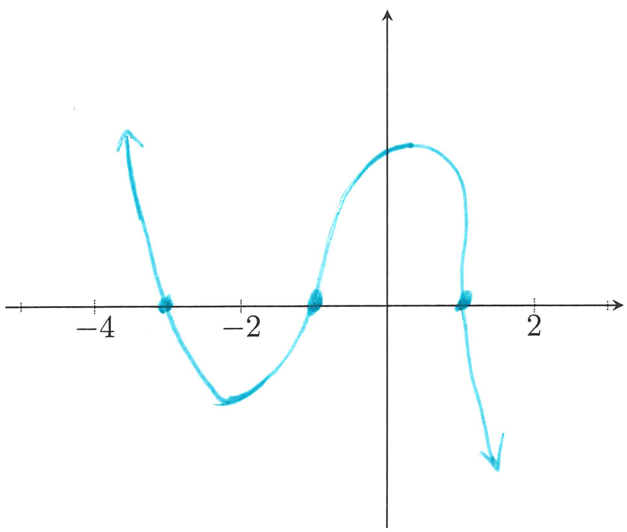
$$= \boxed{2ax + b}$$



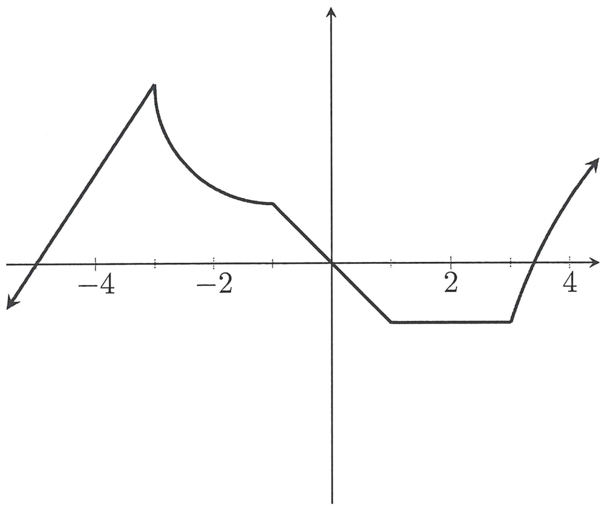
Function	Derivative
Increasing	Positive
Decreasing	Negative
Smooth Min/Max	zero
Constant	zero
Linear	constant
Quadratic	linear

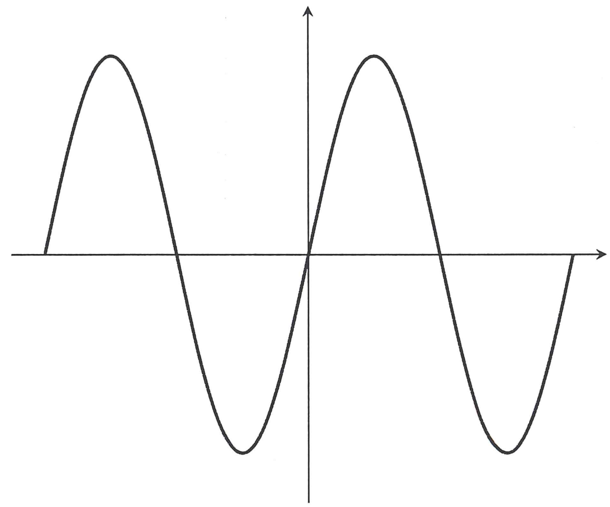
Example. Graph the slope graph of the following function

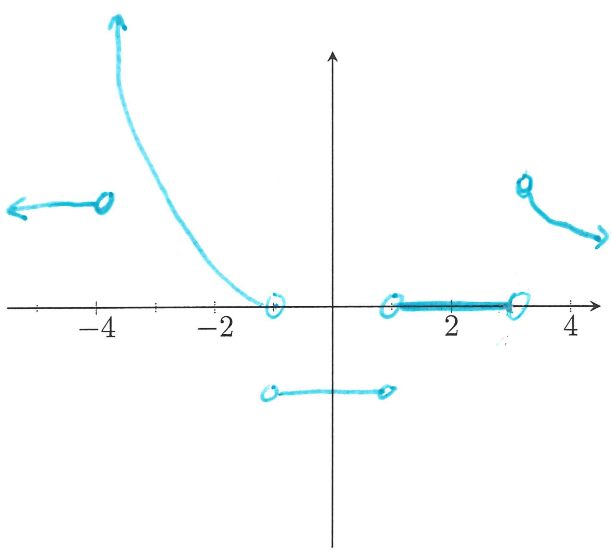


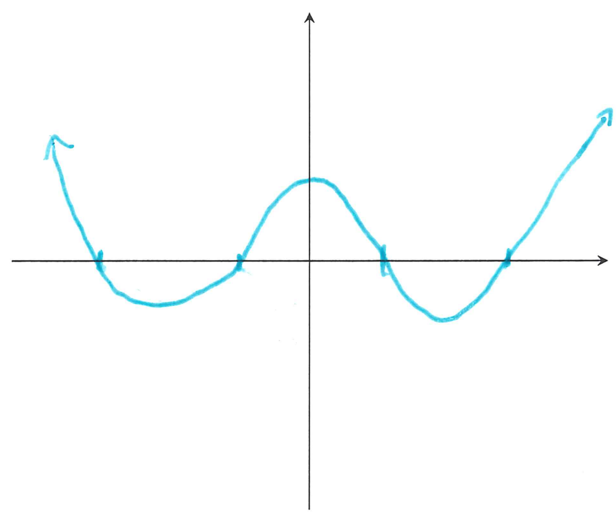


Example. Graph the slope graph of the following functions





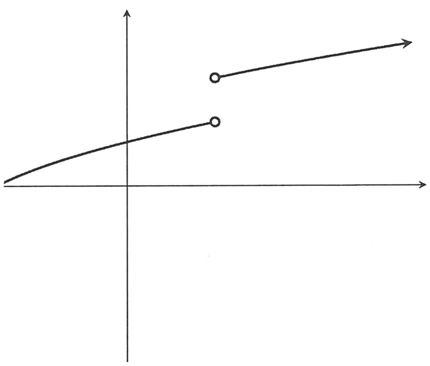


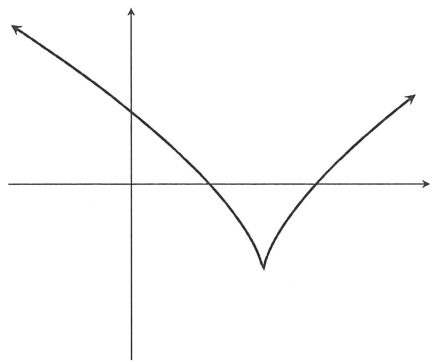


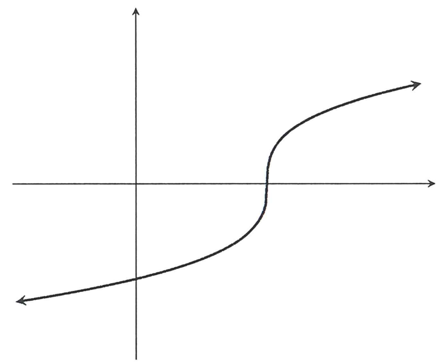
When is a Function Not Differentiable at a Point?

A function f is *not* differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a
2. f has a corner at a
3. f has a vertical tangent at a

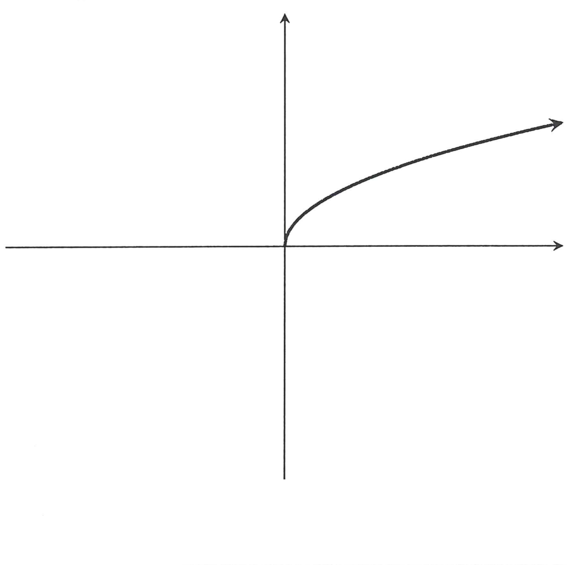


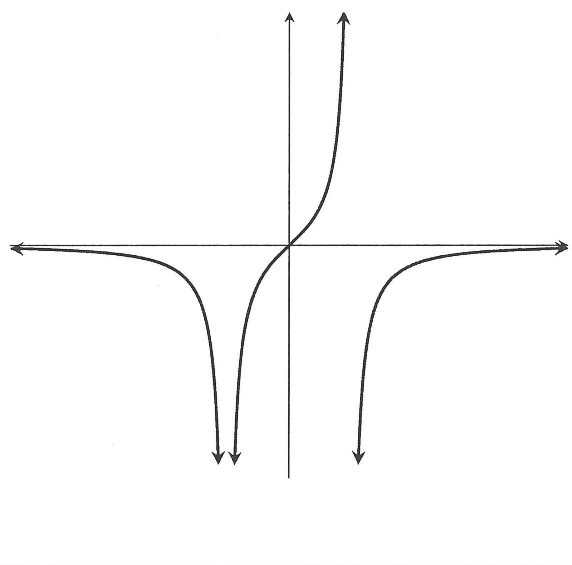




Theorem: Differentiable Implies Continuous
If f is differentiable at a, then f is continuous at a.

Theorem: Not Continuous Implies Not Differentiable
If f is not continuous at a, then f is not differentiable at a.





> **Definition.**
> The **normal** line at $(a, f(a))$ is the line perpendicular to the tangent line that crosses the point $(a, f(a))$.

Example. Find the derivative of $g(x) = \sqrt{x-2}$. Use your result to find the tangent line and the normal line at $x = 11$.

$$g'(x) = \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \left(\frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}} \right)$$

$$= \lim_{h \to 0} \frac{x+h-2-x+2}{h\left(\sqrt{x+h-2} + \sqrt{x-2}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h-2} + \sqrt{x-2}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \frac{1}{2\sqrt{x-2}}$$

$$y - g(11) = -\frac{1}{g'(11)}(x-11)$$

$$y = -6(x-11) + 3 \longrightarrow \boxed{y = -6x + 69}$$

Example. Find the tangent line and normal line of $h(x) = \dfrac{2}{\sqrt{x^2+x-2}}$ at $x = 4$.

$$h'(x) = \lim_{h\to 0} \frac{\left[\frac{2}{\sqrt{(x+h)^2+(x+h)-2}}\right] - \left[\frac{2}{\sqrt{x^2+x-2}}\right]}{h}$$

$$= \lim_{h\to 0} \frac{2\left(\sqrt{x^2+x-2} - \sqrt{(x+h)^2+(x+h)-2}\right)}{h\sqrt{(x+h)^2+(x+h)-2}\,\sqrt{x^2+x-2}} \left(\frac{\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}}{\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}}\right)$$

$$= \lim_{h\to 0} \frac{2\left[(x^2+x-2) - (x^2+2xh+h^2+x+h-2)\right]}{h\sqrt{(x+h)^2+(x+h)-2}\,\sqrt{x^2+x-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \lim_{h\to 0} \frac{-2(2xh+h^2+h)}{h\sqrt{(x+h)^2+(x+h)-2}\,\sqrt{x^2+x-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \lim_{h\to 0} \frac{-4x-2h-2}{\sqrt{(x+h)^2+(x+h)-2}\,\sqrt{x^2+x-2}\left(\sqrt{x^2+x-2} + \sqrt{(x+h)^2+(x+h)-2}\right)}$$

$$= \frac{-4x-2}{(x^2+x-2)\left(2\sqrt{x^2+x-2}\right)}$$

$$= \frac{-2x-1}{(x^2+x-2)^{3/2}}$$

tangent: $y - h(4) = h'(4)(x-4)$

$$\boxed{y = \frac{-9}{18^{3/2}}(x-4) + \frac{2}{3\sqrt{2}}}$$

normal: $y - h(4) = \frac{-1}{h'(4)}(x-4)$

$$\boxed{y = \frac{18^{3/2}}{9}(x-4) + \frac{2}{3\sqrt{2}}}$$