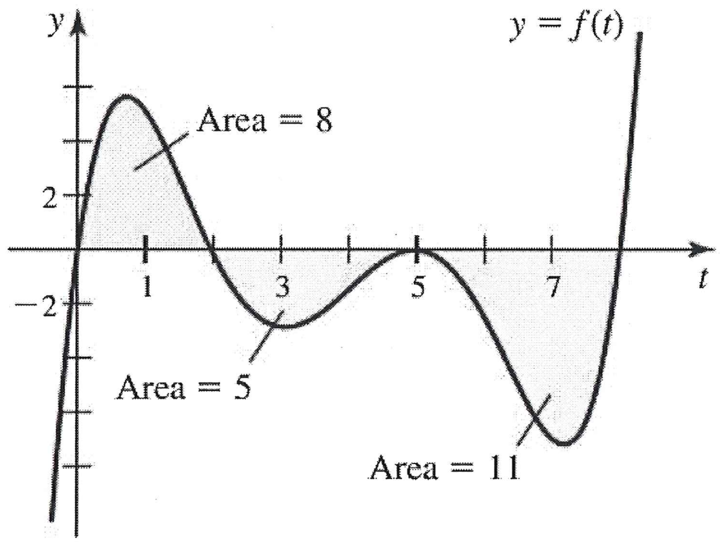
5.3: Fundamental Theorem of Calculus

Definition. (Area Function)
Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a is**

$$A(x) = \int_a^x f(t)\,dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Example. The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions:



a) $A(2)$ = 8

b) $F(5)$ = −5

c) $A(0)$ = 0

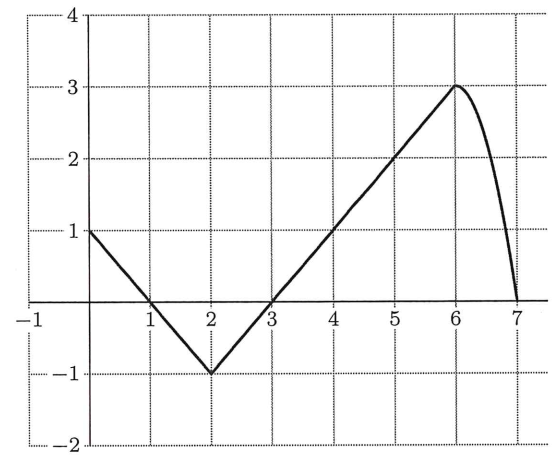
d) $F(8)$ = (−5) + (−11) = −16

e) $A(8)$ = 2 − 5 − 11 = −14

f) $A(5)$ = 8 − 5 = 3

g) $F(2)$ = 0

Example. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.



a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$ and 6.

x	0	1	2	3	4	5	6
$g(x)$	0	$1/2$	0	$-1/2$	0	$3/2$	4

b) Estimate $g(7)$.

$g(7) \approx 4 + 2.1 = 6.1$

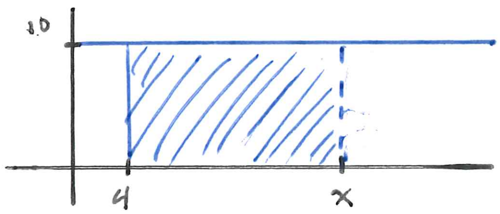
c) Where does g have a maximum value? Where does it have a minimum value?

g has max of 6.1 at $x = 7$

g has min of $-1/2$ at $x = 3$

Example. For the area function $A(x) = \int_a^x f(t)\,dt$, graph the area function and then verify that $A'(x) = f(x)$.

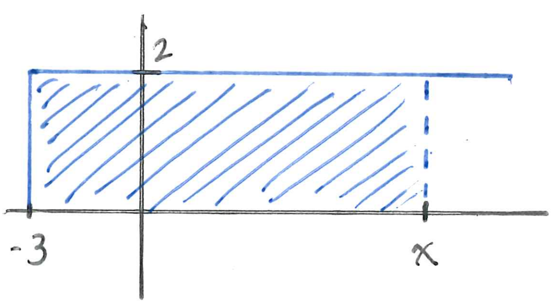
a) $f(t) = 10,\ a = 4$



$A(x) = 10(x-4) = 10x - 40$

$A'(x) = 10 = f(x)$

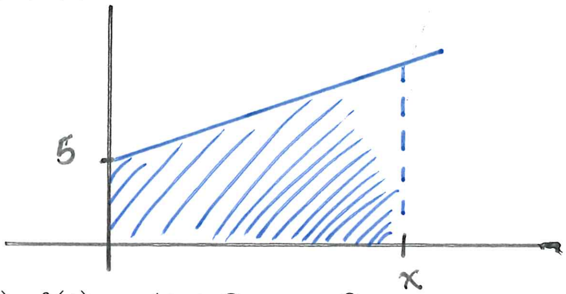
b) $f(t) = 2,\ a = -3$



$A(x) = 2(x+3) = 2x + 6$

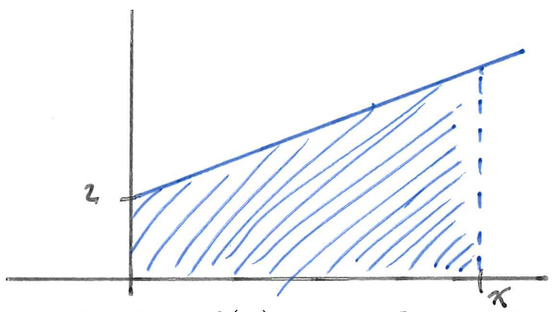
$A'(x) = 2 = f(x)$

c) $f(t) = 2t + 5,\ a = 0$



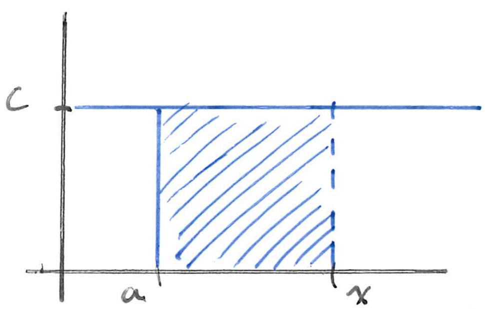
$$A(x) = \frac{1}{2}\left(5 + (2x+5)\right)(x-0)$$
$$= x^2 + 5x$$
$$A'(x) = 2x + 5 = f(x)$$

d) $f(t) = 4t + 2,\ a = 0$



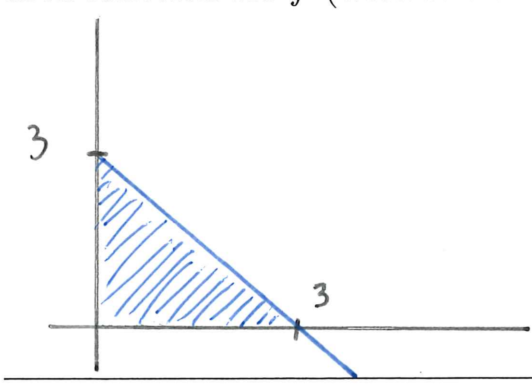
$$A(x) = \frac{1}{2}\left(2 + (4x+2)\right)(x-0)$$
$$= 2x^2 + 2x$$
$$A'(x) = 4x + 2 = f(x)$$

Example. Let $f(x) = c$, where c is a positive constant. Explain why an area function of f is an increasing function.



$A(x)$ represents the area of a rectangle with positive height and increasing width.

Example. The linear function $f(x) = 3 - x$ is decreasing on the interval $[0, 3]$. Is its area function for f (with left endpoint 0) increasing or decreasing on the interval $[0, 3]$?



$A(x)$ is increasing on $[0,3]$ since it is accumulating area on this interval.

Theorem 5.3 (Part I) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\, dt, \qquad \text{for } a \leq x \leq b,$$

$$A(g(x)) = \int_a^{g(x)} f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

$$\frac{d}{dx}\left[A(g(x))\right] = f(g(x)) \cdot g'(x)$$

Example. For the following functions, find the derivatives

a) $g(x) = \int_0^x \sqrt{1 - 2t}\, dt$

$g'(x) = \sqrt{1-2x}$

b) $g(x) = \int_3^x e^{t^2 - t}\, dt$

$g'(x) = e^{x^2 - x}$

c) $g(y) = \int_2^y t^2 \sin(t)\, dt$

$g'(y) = y^2 \sin(y)$

d) $y = \int_x^2 \cos(t^2)\, dt = -\int_2^x \cos(t^2)\, dt$

$y' = -\cos(x^2)$

e) $y = \int_1^{\cos(x)} (t + \sin(t))\, dt$

Let $u = \cos(x)$

$y = \int_1^u (t + \sin(t))\, dt$ chain rule

$\frac{d}{dx}[y] = (u + \sin(u)) \frac{du}{dx}$

$= \boxed{-(\cos(x) + \sin(\cos(x)))\sin(x)}$

f) $y = \int_{-3}^{3x^4} \frac{t}{t^2 - 4t}\, dt$

$y' = \frac{3x^4}{(3x^4)^2 - 4(3x^4)} \frac{d}{dx}[3x^4]$

$= \frac{3x^4 \cdot 12x^3}{(3x^4)^2 - 4(3x^4)} = \boxed{\frac{12x^3}{3x^4 - 4}}$

g) $y = \int_1^{e^x} \ln(t)\, dt$

$y' = \ln(e^x) \frac{d}{dx}[e^x]$

$= \boxed{x e^x}$

h) $y = \int_0^{x^4} \cos^2(\theta)\, d\theta$

$y' = \cos^2(x^4) \frac{d}{dx}[x^4]$

$= \boxed{4x^3 \cos^2(x^4)}$

i) $y = \int_{\tan(x)}^0 \frac{dt}{1 + t^2}$

$y' = -\frac{1}{1 + \tan^2(x)} \frac{d}{dx}[\tan(x)]$ Pythagorean Identity

$= -\frac{\sec^2(x)}{\sec^2(x)}$

$= \boxed{-1}$

j) $y = \int_{\sin(x)}^1 \sqrt{1 + t^2}\, dt$

$y' = -\sqrt{1 + \sin^2(x)} \frac{d}{dx}[\sin(x)]$

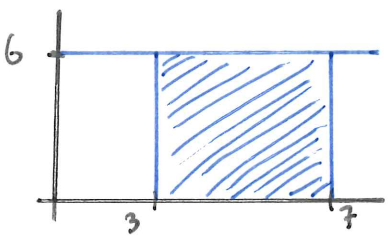
$= \boxed{-\cos(x)\sqrt{1 + \sin^2(x)}}$

Theorem 5.3 (Part II) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

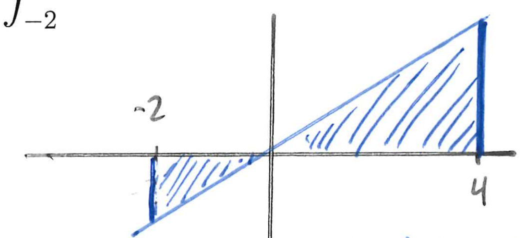
Example. Evaluate the following integrals using graphs and the Fundamental Theorem of Calculus.

a) $\displaystyle\int_3^7 6\,du = 6x\Big|_3^7 = 6(7) - 6(3) = \boxed{24}$



$A = 6(7-3) = \boxed{24}$

b) $\displaystyle\int_{-2}^4 x\,dx = \frac{x^2}{2}\Big|_{-2}^4 = \frac{4^2}{2} - \frac{(-2)^2}{2} = \boxed{6}$



$A = \frac{1}{2}(2 \cdot -2) + \frac{1}{2}(4 \cdot 4) = \boxed{6}$

Example. Evaluate the following integrals

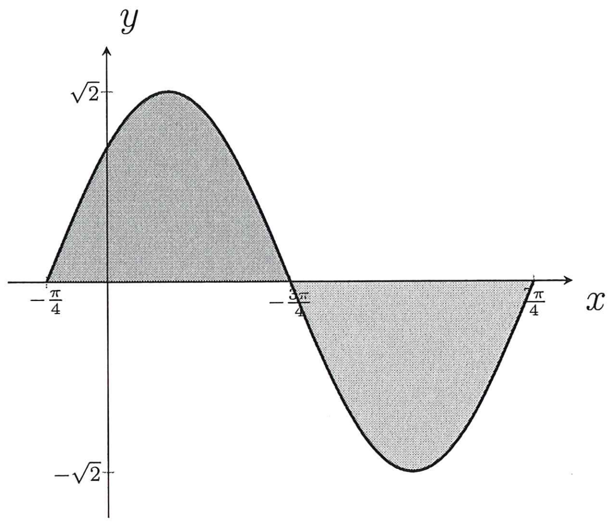
a) $\displaystyle\int_{-1}^3 x^2\,dx = \frac{x^3}{3}\Big|_{-1}^3 = \frac{3^3}{3} - \frac{(-1)^3}{3} = \boxed{\frac{28}{3}}$

b) $\displaystyle\int_0^\pi (1 + \cos(x))\,dx = x + \sin(x)\Big|_0^\pi = [\pi + \sin(\pi)] - [0 + \sin(0)] = \boxed{\pi}$

c) $\displaystyle\int_{-5}^5 e\,dx = ex\Big|_{-5}^5 = e(5) - e(-5) = \boxed{10e}$

d) $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta = \sec\theta\Big|_0^{\pi/4} = \boxed{\sqrt{2} - 1}$

Example. Evaluate the following integrals using the Fundamental Theorem of Calculus



$$\int_{-\pi/4}^{7\pi/4} (\sin(x) + \cos(x))\, dx$$

$$= -\cos(x) + \sin(x) \Big|_{-\pi/4}^{7\pi/4}$$

$$= \left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\right] - \left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= \boxed{0}$$

Example. Evaluate $\int_3^8 f'(t)\, dt$, where f' is continuous on $[3, 8]$, $f(3) = 4$, and $f(8) = 20$.

$$\int_3^8 f'(t)\, dt = f(t)\Big|_3^8 = f(8) - f(3) = 20 - 4 = \boxed{16}$$

Example. Find $\frac{d}{dt} \int_0^{t^4} \sqrt{u}\, du$ by evaluating the integral directly and then differentiating the result.

$$\frac{d}{dt}\int_0^{t^4} u^{1/2}\, du = \frac{d}{dt}\left[\frac{2}{3}u^{3/2}\Big|_0^{t^4}\right] = \frac{d}{dt}\left[\frac{2}{3}(t^4)^{3/2}\right] = \frac{d}{dt}\left[\frac{2}{3}t^6\right] = \boxed{4t^5}$$

Example. Find $\dfrac{d}{dt}\displaystyle\int_0^{t^4} \sqrt{u}\,du$ by differentiating the integral directly.

$$\frac{d}{dt}\int_0^{t^4}\sqrt{u}\,du = \sqrt{t^4}\,\frac{d}{dt}\left[t^4\right] = t^2(4t^3) = \boxed{4t^5}$$

Example. Find $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan(\theta)} \sec^2(y)\,dy$ by evaluating the integral directly and then differentiating the result.

$$\frac{d}{d\theta}\int_0^{\tan\theta}\sec^2(y)\,dy = \frac{d}{d\theta}\left[\tan(y)\Big|_0^{\tan\theta}\right] = \frac{d}{d\theta}\left[\tan(\tan(\theta))\right]$$

$$= \boxed{\sec^2(\tan(\theta))\sec^2(\theta)}$$

Example. Find $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan(\theta)} \sec^2(y)\,dy$ by differentiating the integral directly.

$$\frac{d}{d\theta}\int_0^{\tan\theta}\sec^2(y)\,dy = \sec^2(\tan\theta)\,\frac{d}{d\theta}\left[\tan\theta\right]$$

$$= \boxed{\sec^2(\tan\theta)\sec^2(\theta)}$$

Example. Evaluate the following integrals:

a) $\displaystyle\int_1^8 \sqrt[3]{x}\, dx$

$= \displaystyle\int_1^8 x^{1/3}\, dx$

$= \dfrac{3}{4} x^{4/3} \Big|_1^8$

$= \dfrac{3}{4}(8)^{4/3} - \dfrac{3}{4}(1)^{4/3}$

$= \dfrac{3}{4}(16) - \dfrac{3}{4} = \boxed{\dfrac{45}{4}}$

b) $\displaystyle\int_{-2}^{-1} \frac{2}{x^2}\, dx = \int_{-2}^{-1} 2x^{-2}\, dx$

$= -2x^{-1} \Big|_{-2}^{-1}$

$= \dfrac{-2}{-1} - \dfrac{-2}{-2}$

$= 2 - 1 = \boxed{1}$

c) $\displaystyle\int_0^2 x(2+x^5)\, dx = \int_0^2 2x + x^6\, dx$

$= x^2 + \dfrac{x^7}{7} \Big|_0^2$

$= \left[2^2 + \dfrac{2^7}{7}\right] - \left[0^2 + \dfrac{0^7}{7}\right]$

$= 4 + \dfrac{128}{7}$

$= \boxed{\dfrac{156}{7}}$

d) $\displaystyle\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}\, du = \int_9^4 u^{-1/2} - 1\, du$

$= 2u^{1/2} - u \Big|_9^4$

$= \left[2(4)^{1/2} - 4\right] - \left[2(9)^{1/2} - 9\right]$

$= [0] - [-3]$

$= \boxed{3}$

e) $\displaystyle\int_0^2 (y-1)(2y+1)\,dy$

$= \displaystyle\int_0^2 2y^2 - y - 1 \, dy$

$= \frac{2}{3}y^3 - \frac{1}{2}y^2 - y \Big|_0^2$

$= \left[\frac{2}{3}2^3 - \frac{1}{2}2^2 - 2\right] - \left[\frac{2}{3}0^3 - \frac{1}{2}0^2 - 0\right]$

$= \frac{16}{3} - 2 - 2 = \boxed{\frac{4}{3}}$

f) $\displaystyle\int_0^4 \left(1 + 3y - y^2 - \frac{y^3}{4}\right) dy$

$= y + \frac{3}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{16}y^4 \Big|_0^4$

$= \left[4 + \frac{3}{2}(4)^2 - \frac{1}{3}(4)^3 - \frac{1}{16}(4)^4\right] - [0]$

$= 4 + 24 - \frac{64}{3} - 16$

$= \boxed{-\frac{28}{3}}$

g) $\displaystyle\int_0^1 (x^e + e^x)\,dx$

$= \frac{x^{e+1}}{e+1} + e^x \Big|_0^1$

$= \left[\frac{1^{e+1}}{e+1} + e^1\right] - \left[\frac{0^{e+1}}{e+1} + e^0\right]$

$= \frac{1}{e+1} + e - 1$

$= \boxed{\frac{e^2}{e+1}}$

h) $\displaystyle\int_0^3 (2\sin(x) - e^x)\,dx$

$= -2\cos(x) - e^x \Big|_0^3$

$= \left[-2\cos(3) - e^3\right] - \left[-2\cos(0) - e^0\right]$

$= \boxed{3 - 2\cos(3) - e^3}$

i) $\displaystyle\int_{\frac{\pi}{2}}^{0} \frac{1-\cos(2t)}{2}\, dt$

$= \displaystyle\int_{\pi/2}^{0} \frac{1}{2} - \frac{\cos(2t)}{2}\, dt$

$= \frac{1}{2}t - \frac{\sin(2t)}{4} \Big|_{\pi/2}^{0}$

$= [0-0] - \left[\frac{1}{2}(\pi/2) - \frac{\sin(\pi)}{4}\right]$

$= \boxed{-\pi/4}$

j) $\displaystyle\int_{\frac{1}{2}}^{2} \left(1-\frac{1}{x^2}\right) dx = x + \frac{1}{x}\Big|_{1/2}^{2}$

$= \left[2+\frac{1}{2}\right] - \left[\frac{1}{2}+2\right]$

$= \boxed{0}$

k) $\displaystyle\int_{1}^{2} \left(\frac{2}{s^2}-\frac{4}{s^3}\right) ds$

$= -\frac{2}{s} + \frac{2}{s^2}\Big|_{1}^{2}$

$= \left[\frac{-2}{2}+\frac{2}{4}\right] - \left[\frac{-2}{1}+\frac{2}{1}\right]$

$= \boxed{-1/2}$

l) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx = 8\tan^{-1}(x)\Big|_{1/\sqrt{3}}^{\sqrt{3}}$

$= \left[8\tan^{-1}(\sqrt{3})\right] - \left[8\tan^{-1}(1/\sqrt{3})\right]$

$= 8(\pi/3) - 8(\pi/6)$

$= \boxed{\frac{4\pi}{3}}$

m) $\displaystyle\int_0^5 (x^2 - 9)\, dx$

$= \frac{x^3}{3} - 9x \Big|_0^5$

$= \left[\frac{5^3}{3} - 9(5)\right] - \left[\frac{0^3}{3} - 9(0)\right]$

$= \frac{125}{3} - 45$

$= \boxed{-\frac{10}{3}}$

n) $\displaystyle\int_{1/2}^2 \left(1 - \frac{1}{x^2}\right) dx$

$= x + \frac{1}{x} \Big|_{1/2}^2$

$= \left[2 + \frac{1}{2}\right] - \left[\frac{1}{2} + 2\right]$

$= \boxed{0}$

o) $\displaystyle\int_{-1}^2 x^3\, dx = \frac{x^4}{4} \Big|_{-1}^2$

$= \left[\frac{2^4}{4}\right] - \left[\frac{(-1)^4}{4}\right]$

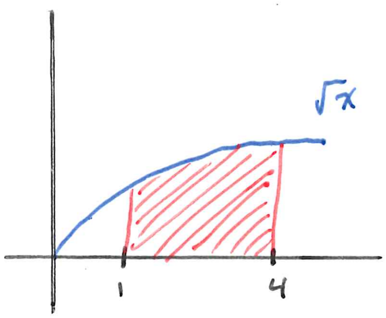
$= \boxed{\frac{15}{4}}$

p) $\displaystyle\int_{\pi/6}^{2\pi} \cos(x)\, dx = \sin(x) \Big|_{\pi/6}^{2\pi}$

$= \sin(2\pi) - \sin(\pi/6)$

$= \boxed{-1/2}$

Example. Find the area of the region bounded by $y = \sqrt{x}$ between $x = 1$ and $x = 4$.

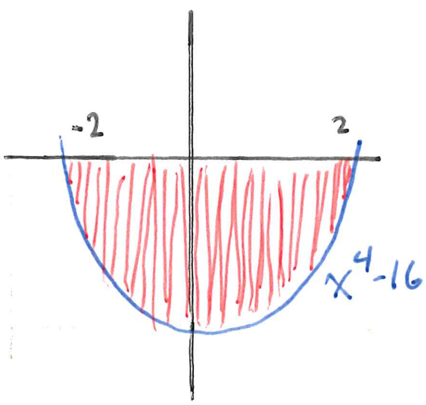


$$\int_1^4 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_1^4$$

$$= \left[\frac{2}{3}(4)^{3/2}\right] - \left[\frac{2}{3}(1)^{3/2}\right]$$

$$= \boxed{14/3}$$

Example. Find the area of the region below the x-axis bounded by $y = x^4 - 16$.

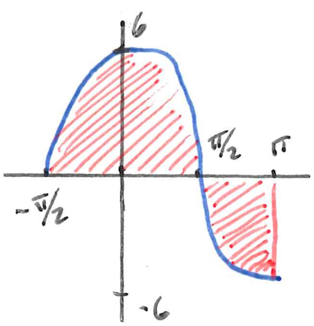


$$\int_{-2}^{2} x^4 - 16\, dx = \frac{x^5}{5} - 16x \Big|_{-2}^{2}$$

$$= \left[\frac{2^5}{5} - 16(2)\right] - \left[\frac{(-2)^5}{5} - 16(-2)\right]$$

$$= \boxed{-\frac{256}{5}}$$

Example. Find the area of the region bounded by $y = 6\cos(x)$ between $x = -\pi/2$ and $x = \pi$.



$$\int_{-\pi/2}^{\pi} 6\cos(x)\, dx = 6\sin(x) \Big|_{-\pi/2}^{\pi}$$

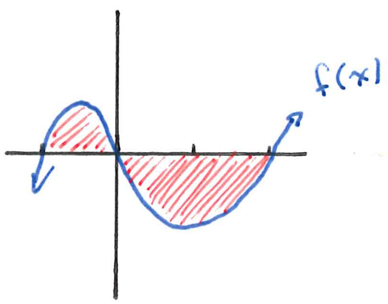
$$= 6\sin(\pi) - 6\sin(-\pi/2)$$

$$= 0 - (-6)$$

$$= \boxed{6}$$

Example. Find the area of the region bounded by $f(x) = x(x+1)(x-2)$ and the x-axis on the interval $[-1, 2]$.

$$\int_{-1}^{2} x(x+1)(x-2)\,dx = \int_{-1}^{2} x^3 - x^2 - 2x\,dx$$

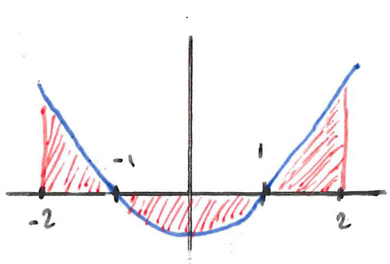


$$= \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{-1}^{2}$$

$$= \left[\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right] - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \boxed{-\frac{9}{4}}$$

Example. Find the total area between $y = 3x^2 - 3$ and the x-axis on $-2 \le x \le 2$.

$$\int_{-2}^{2} |3x^2 - 3|\,dx = \left|\int_{-2}^{-1} 3x^2 - 3\,dx\right| + \left|\int_{-1}^{1} 3x^2 - 3\,dx\right| + \left|\int_{1}^{2} 3x^2 - 3\,dx\right|$$



$$= \left|(x^3 - 3x)\Big|_{-2}^{-1}\right| + \left|(x^3 - 3x)\Big|_{-1}^{1}\right| + \left|(x^3 - 3x)\Big|_{1}^{2}\right|$$

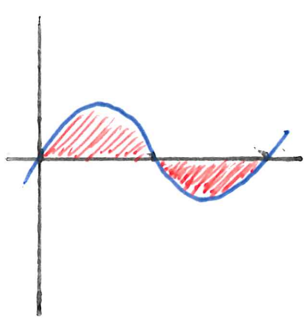
$$= |4| + |-4| + |4|$$

$$= \boxed{12}$$

Example. Find the total area between $y = x^3 - 3x^2 + 2x$ and the x axis on the interval $0 \le x \le 2$.

$$\int_{0}^{2} |x^3 - 3x^2 + 2x|\,dx = \left|\int_{0}^{1} x^3 - 3x^2 + 2x\,dx\right| + \left|\int_{1}^{2} x^3 - 3x^2 + 2x\,dx\right|$$

$$= \left|\left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_{0}^{1}\right| + \left|\left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_{1}^{2}\right|$$



$$= \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right| = \boxed{\frac{1}{2}}$$