4.5 Optimization Problems

Note: Please note that these 'word problems' are different from the related rates problems that we did in section 3.11.

Guidelines for Optimization Problems

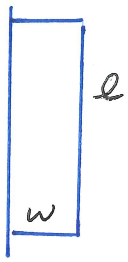
1. Read the problem and identify variables with a diagram. Only put numbers on the diagram if they are constant.
2. Express the function that will be optimized.
3. Identify the constraint(s). Use the constraint(s) to eliminate/rewrite all but the single independent variable of the objective function.
4. Use derivatives to find the absolute max/min of the objective function.

   Make sure you are finding the correct extrema!

   - First derivative test ($f'(x)$ changes signs at $x = c$)
   - Second derivative test ($f'(c) < 0$ (neg) means $f(c)$ is a max).
   - Evaluate at endpoints and critical points.
5. Summarize your result in a sentence.

1. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He decides not to fence along the river. What are the dimensions of the field that has the largest area?

Draw a diagram. Define your variables.



ℓ is length
w is width

Express your constraints and the variable that needs optimizing.

constraint: $2w + \ell = 2400 \Rightarrow \ell = 2400 - 2w$

objective: $A = w\ell \Rightarrow A = w(2400 - 2w)$
$= 2400w - 2w^2$

Find the local maximum.

$A' = 2400 - 4w \overset{\text{set}}{=} 0$

$2400 = 4w$

$\boxed{600 = w} \Rightarrow \ell = 2400 - 2(600)$

$\boxed{\ell = 1200}$

Verify the critical point is the absolute maximum.

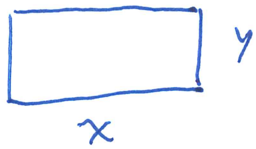
$A'' = -4 < 0$ $\frown$ $\Rightarrow$ global max

$\boxed{\text{2nd Derivative test}}$

State your answer ensuring that you are answering the question asked.

The field's area is maximized when $w = 600$ ft and $\ell = 1200$ ft.

2. Show among all rectangles with an 8 meter perimeter, the one with the largest area is a square.



$2x+2y=8 \Rightarrow y=4-x$

$A=xy \Rightarrow A=x(4-x)=4x-x^2$

$A'=4-2x \overset{\text{set}}{=} 0 \Rightarrow 2x=4$

$\boxed{x=2}$

Verification

$\Rightarrow y=4-2=\boxed{2}$

$A''=-2<0 \Rightarrow$ maximum

$\cap$ concave down

3. Find x and y such that $xy = 50$ but $x + y$ is minimal.

min $f = x+y$

$xy=50 \Rightarrow x=\frac{50}{y}$

$\Rightarrow f=\frac{50}{y}+y$

$f'=-\frac{50}{y^2}+1 \overset{\text{set}}{=} 0 \Rightarrow 1=\frac{50}{y^2}$

$y^2=50$

$y=\pm 5\sqrt{2} \rightarrow x=\frac{50}{\pm 5\sqrt{2}} = \pm 5\sqrt{2}$

2nd Deriv test

$f''=\frac{100}{y^3}$

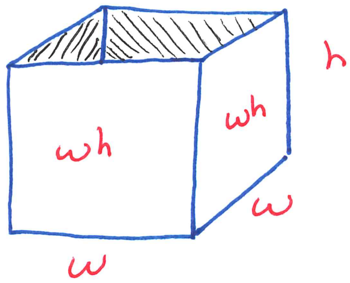
when $y=-5\sqrt{2}$, f'' is neg $\Rightarrow$ concave down $\rightarrow$ max

when $y=5\sqrt{2}$, f'' is pos $\Rightarrow$ concave up $\rightarrow$ min

$\boxed{\text{When } x=y=5\sqrt{2}, \text{ then } x+y \text{ is minimized.}}$

4. A box with a square base and an open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimizes the amount of material used.

S.A.



$$V = \underbrace{w^2 h = 32000}$$

$$h = \frac{32000}{w^2}$$

min $\quad m = w^2 + 4wh$

$$\Rightarrow m = w^2 + 4w\left(\frac{32000}{w^2}\right)$$

$$= w^2 + 128000\,w^{-1}$$

$$\Rightarrow m' = 2w - 128000\,w^{-2} \overset{\text{set}}{=} 0$$

$$2w = \frac{128000}{w^2}$$

$$w^3 = 64000$$

$$\boxed{w = 40} \Rightarrow h = \frac{32000}{(40)^2} = \boxed{20}$$

Verify

$$m'' = 2 + \frac{256000}{w^3}$$

$$w = 40 \Rightarrow 2 + \frac{256000}{40^3} > 0 \rightarrow \text{concave up } \cup$$

$$\Rightarrow \text{minimum}$$

Thus, the box with volume 32000 cm^3 is made with minimal material when $w = 40$ and $h = 20$.

5. If $y = 2x - 89$, what is the minimum value of the product xy?

minimize $f = xy$ $\quad\quad y = 2x - 89$

$\Rightarrow f = x(2x-89)$

$= 2x^2 - 89x$

$f' = 4x - 89 \overset{\text{set}}{=} 0 \Rightarrow 4x = 89$

$\boxed{x = 89/4}$ $\quad y = 2(89/4) - 89$

$y = \frac{89}{2} - 89 = \boxed{\frac{-89}{2}}$

verify

$f'' = 4 > 0 \Rightarrow$ concave up

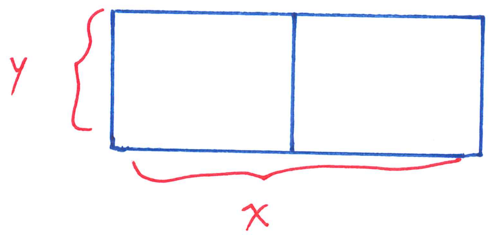
U

$\Rightarrow$ minimum

The product xy is minimized when $x = \frac{89}{4}$ and $y = -\frac{89}{2}$.

6. A farmer has 900 meters of fencing. The fencing is to be used to enclose a rectangular field and to divide it in half. Find the dimensions of the field that has maximum area.

$$2x + 3y = 900 \implies y = 300 - \frac{2}{3}x$$



minimize $A = xy$

$$\implies A = x\left(300 - \frac{2}{3}x\right)$$

$$\implies A = 300x - \frac{2}{3}x^2$$

$$A' = 300 - \frac{4}{3}x \overset{\text{set}}{=} 0$$

$$300 = \frac{4}{3}x$$

$$900 = 4x$$

$$\boxed{225 = x} \implies y = 300 - \frac{2}{3}(225)$$

$$= 300 - 150 = \boxed{150}$$

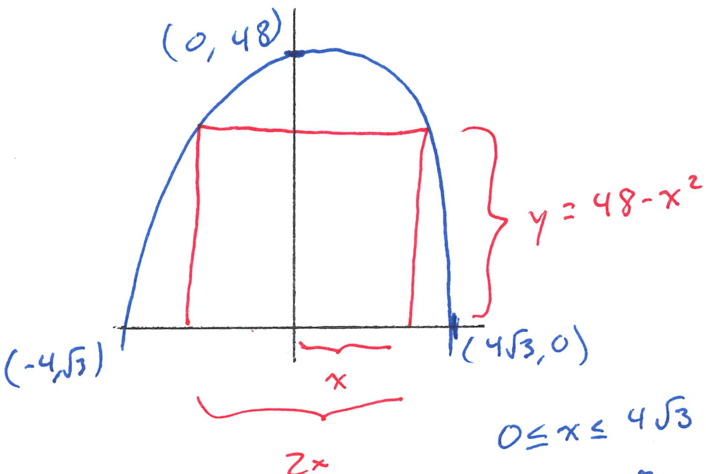
Verification

$$A'' = -\frac{4}{3} < 0 \implies \text{concave down}$$

$$\implies \text{maximum}$$

The area is maximized when $x = 225$ meters and $y = 150$ meters.

7. A rectangle is constructed with its base on the x-axis and two of its vertices on the parabola $y = 48 - x^2$. What are the dimensions of the rectangle with the maximum area? What is the area?



$0 \le x \le 4\sqrt{3}$

$0 \le y \le 48$

maximize $\quad A = 2x\,y$

$\Rightarrow A = 2x(48 - x^2) = 96x - 2x^3$

$A' = 96 - 6x^2 \overset{\text{Set}}{=} 0$

$96 = 6x^2$

$16 = x^2$

$\pm 4 = x$

$\boxed{\text{Since } 0 \le x \le 4\sqrt{3} \qquad x = 4}$

$\boxed{y = 48 - 4^2 = 32}$

Verify

$A'' = -12x$

when $x = 4$, $\quad A'' = -24 < 0$

$\Rightarrow$ concave down $\cap$

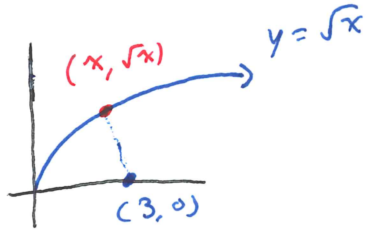
$\Rightarrow$ maximum

The rectangle's area is maximized when $x = 4$ and $y = 32$.

Distance between 2 points

$$d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

8. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.



minimize $d = \sqrt{(3-x)^2 + (0-\sqrt{x})^2}$

$$d = \left((3-x)^2 + x\right)^{1/2}$$

$$= \left(x^2 - 5x + 9\right)^{1/2}$$

$$d' = \frac{1}{2}\left(x^2-5x+9\right)^{-1/2}(2x-5) = \frac{2x-5}{2\sqrt{x^2-5x+9}} \overset{\text{set}}{=} 0$$

$$\Rightarrow 2x - 5 = 0$$

$$\boxed{x = \frac{5}{2}}$$

Verify (Note: d'' is messy)

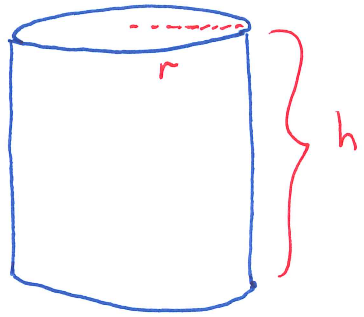
1st Deriv test

$$\begin{array}{c|c|c} & 1 \quad \frac{5}{2} & 10 \\ \hline d' & - - - & + + + \\ & \searrow & \nearrow \end{array}$$

$\Rightarrow x = 5/2$ is where the minimum occurs

The distance is minimized when $x = \frac{5}{2}$ and $y = \sqrt{5/2}$.

9. A cylindrical can is to be made to hold $1\,L$ $(1000\,cm^3)$ of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.

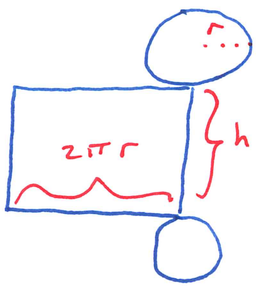


$$V = \pi r^2 h = 1000 \quad \rightarrow \quad h = \frac{1000}{\pi r^2}$$

minimize $\quad m = 2\pi r^2 + 2\pi r h$

$$m = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + 2000 r^{-1}$$



$$m' = 4\pi r - 2000 r^{-2} \;\underline{\underline{\text{set}}}\; 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt{\frac{500}{\pi}}$$

$$h = \frac{1000}{\pi\left(\sqrt{\frac{500}{\pi}}\right)^2} = 2$$

Verify

$$m'' = 4\pi + \frac{4000}{r^3}$$

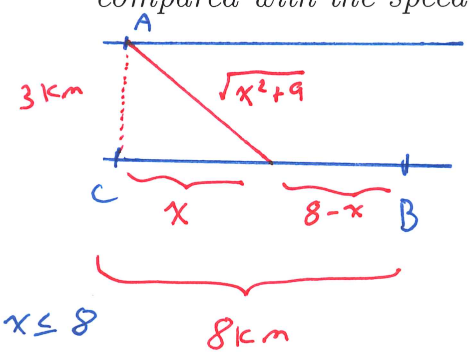
Since $r > 0$, then $m'' > 0$

$\Rightarrow$ Concave down

$\Rightarrow$ minimum occurs where $r = \sqrt{\frac{500}{\pi}}$.

The can with Volume $1000\,cm^3$ is made with minimal material when $r = \sqrt{\frac{500}{\pi}}$ and $h = 2$.

10. A man launches his boat from point A on a bank of a straight river, $3\,km$ wide, and wants to reach point B, $8\,km$ downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point between point C and point B and then run to B. If he can row $6\,km/h$ and run $8\,km/h$, where should he land to reach B as soon as possible? (*Assume that the speed of the water is negligible compared with the speed at which the man rows.*)



$0 \le x \le 8$

Since we wish to minimize the time spent traveling, we divide each distance by the respective speed.

$$T = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8} = \frac{1}{6}(x^2+9)^{1/2} + 1 - \frac{x}{8}$$

$$T' = \frac{2x}{12\sqrt{x^2+9}} - \frac{1}{8} \overset{\text{set}}{=} 0 \implies \frac{x}{6\sqrt{x^2+9}} = \frac{1}{8}$$

$$\implies 8x = 6\sqrt{x^2+9}$$

$$\implies 64x^2 = 36x^2 + 36\cdot 9$$

$$\implies 28x^2 = 36\cdot 9$$

$$\implies x^2 = 81/7$$

$$x = \pm 9/\sqrt{7}$$

$$\boxed{x = 9/\sqrt{7}}$$

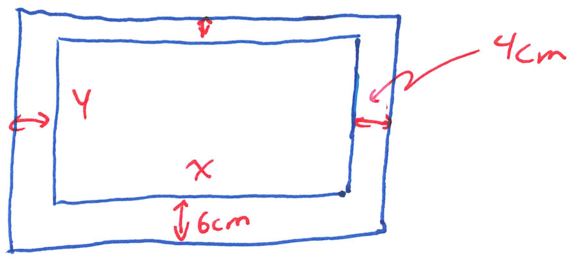
Verify (1st deriv test)

	0	$\sqrt{7}$	3
T'	$- - -$		$+ + +$
	↘		↗

$\implies \sqrt{7}$ is where the local minimum occurs.

When the man rows to the point that is $\sqrt{7}$ km away from C, the time spent traveling is minimized.

11. The top and bottom margins of a poster are each $6\,cm$ and the side margins are each $4\,cm$. If the area of printed material on the poster is fixed at $384\,cm^2$, find the dimensions of the poster with the smallest area.



$x \geq 0,\ y \geq 0$

$xy = 384 \rightarrow y = \frac{384}{x}$

$\min\ A = (x + 2(4))(y + 2(6))$

$A = (x+8)\left(\frac{384}{x} + 12\right)$

$= 384 + 12x + \frac{8(384)}{x} + 96$

$= 12x + 3072x^{-1} + 480$

$A' = 12 - 3072x^{-2} \overset{\text{set}}{=} 0$

$12 = \frac{3072}{x^2}$

$x^2 = 256$

$x = \pm 16$

$\boxed{x = 16}$

$\boxed{y = \frac{384}{16} = 24}$

Verify

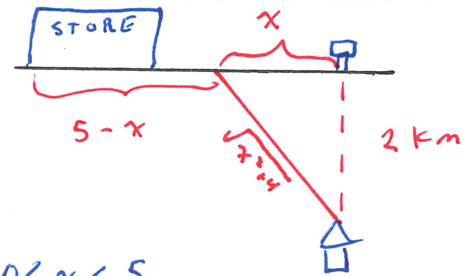
$A'' = \frac{6144}{x^3} > 0$ whenever $x > 0$

$\Rightarrow$ concave up $\cup$

$\Rightarrow$ minimum

The area of the poster and margins is minimized when $x = 16$ cm and $y = 24$ cm

12. A cabin is located 2 km directly into the woods from a mailbox on a straight road. A store is located on the road 5 km from the mailbox. A woman wished to walk from the cabin to the store. She can walk 3 km/h through the woods and 4 km/h along the road. Find the point on the road toward which she would walk in order to minimize the total time for her walk.



$0 \le x \le 5$

$$T = \frac{\sqrt{x^2+4}}{3} + \frac{5-x}{4}$$

$$T' = \frac{x}{3\sqrt{x^2+4}} - \frac{1}{4} = 0$$

$$\Rightarrow \quad 4x = 3\sqrt{x^2+4}$$

$$16x^2 = 9x^2 + 36$$

$$x^2 = \frac{36}{7}$$

$$x = \pm\frac{6}{\sqrt{7}}$$

$$x \ge 0 \Rightarrow \boxed{x = 6/\sqrt{7}}$$

Verify

$6/\sqrt{7}$

	0		3
T'	$- - -$		$+ + +$
	$\searrow$		$\nearrow$

A local minimum occurs at $x = 6/\sqrt{7}$

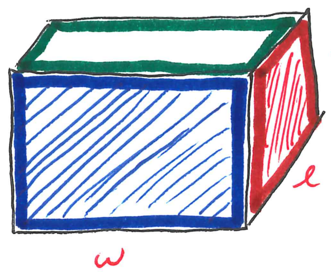
The total time is minimized if she walks to the point that is $6/\sqrt{7}$ km away from the mailbox.

green

constraint

13. We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $\$10/ft^2$ and the material used to build the sides costs $\$6/ft^2$. If the box must have a volume of 50 ft^3 determine the dimensions that will minimize the cost to build the box.

red & blue



$w \geq 0$

$h \geq 0$

$l \geq 0$

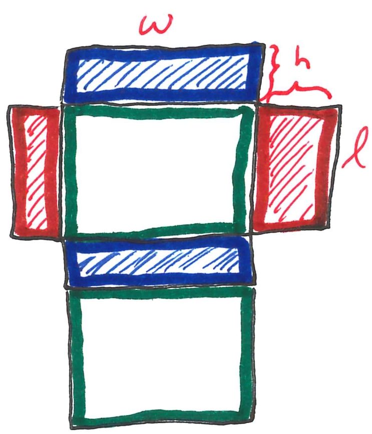
$$\left.\begin{array}{l} V = wlh = 50 \\ l = 3w \end{array}\right\} \quad h = \frac{50}{3w^2}$$

minimize cost

$$C = 10(2wl) + 6(2hl + 2hw)$$

$$= 20w^2 + 48hw$$

$$= 20w^2 + 800w^{-1}$$



$$C' = 40w - 800w^{-2} \overset{\text{set}}{=} 0$$

$$40w = \frac{800}{w^2}$$

$$w^3 = 20$$

$$\boxed{w = \sqrt[3]{20}}$$

$$\boxed{l = 3\sqrt[3]{20}}$$

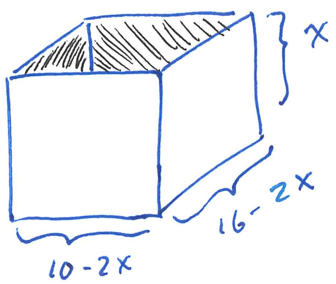
$$\boxed{h = \frac{50}{3(\sqrt[3]{20})^2}}$$

Verify

$$C'' = 40 + \frac{1600}{w^3} > 0 \quad \text{since } w > 0$$

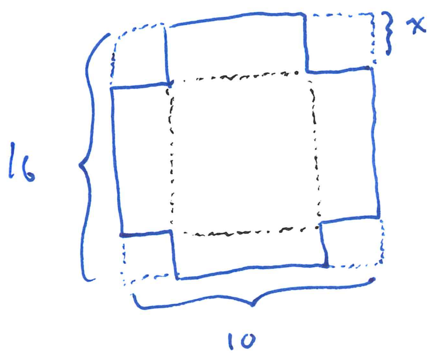
The box of volume 50 ft^3 has minimal cost when $w = \sqrt[3]{20}$ ft, $l = 3\sqrt[3]{20}$ ft and $h = \frac{50}{3(\sqrt[3]{20})^2}$ ft.

14. An open-top rectangular box is constructed from a 10 *in.* by 16 *in.* piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find the dimensions of the box of largest volume and the maximum volume.



$$V = (10-2x)(16-2x)\,x$$

$$= 4x^3 - 52x^2 + 160x$$



$$V' = 12x^2 - 104x + 160 \overset{\text{set}}{=} 0$$

$$\Rightarrow 3x^2 - 26x + 40 = 0$$

$$(3x-20)(x-2) = 0$$

$$\Rightarrow \boxed{x = 2}$$

$$x = \frac{20}{3} = 6.\overline{6}$$

$$\boxed{\begin{aligned} 10-2(2) &= 6 \\ 16-2(2) &= 14 \\ x &= 2 \end{aligned}}$$

$0 \le x \le 5$

- If $x=5$, then the side of width 10 is gone

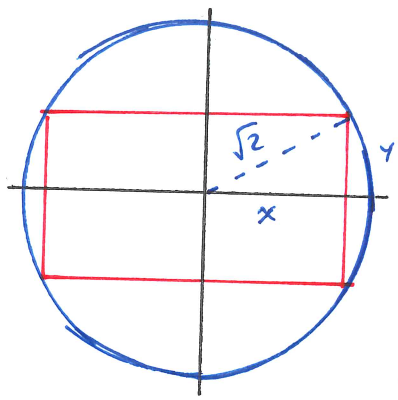
Verify

$V'' = 24x - 104$

$24(2) - 104 < 0 \rightarrow$ concave up $\rightarrow$ maximum

The volume of the box is maximized when the dimensions are $6 \times 14 \times 2$ in.

15. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius $\sqrt{2}\,cm$.



$$x^2 + y^2 = (\sqrt{2})^2 \longrightarrow y = \sqrt{2 - x^2}$$

maximize $A = xy$

$$\rightarrow A = x\sqrt{2 - x^2}$$

$$A' = \sqrt{2 - x^2} - \frac{x^2}{\sqrt{2 - x^2}} \stackrel{\text{set}}{=} 0$$

$0 \le x \le \sqrt{2}$

$0 \le y \le \sqrt{2}$

$$2 - x^2 = x^2$$

$$2 = 2x^2$$

$$\pm 1 = x$$

$$x \ge 0 \Rightarrow \boxed{x = 1}$$

$$\boxed{y = 1}$$

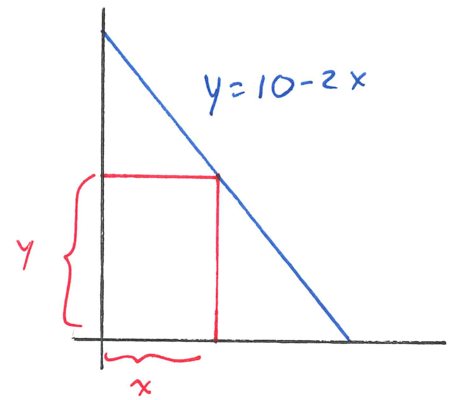
Verify

	0	$\frac{1}{2}$	$x = 1$	$\sqrt{3/2}$	$\sqrt{2}$
A'		$+++$		$---$	
		$\nearrow$		$\searrow$	

$\Rightarrow$ maximum at $x = 1$

The rectangle's area is maximized when $x = y = 1$.

16. A rectangle is constructed with one side on the positive x-axis, one side on the positive y-axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?



$y = 10 - 2x$

maximize $A = xy$

$\longrightarrow A = x(10-2x)$

$= 10x - 2x^2$

$A' = 10 - 4x \overset{\text{set}}{=} 0 \longrightarrow 10 = 4x$

$\boxed{\frac{5}{2} = x}$ $\boxed{y = 5}$

Verify

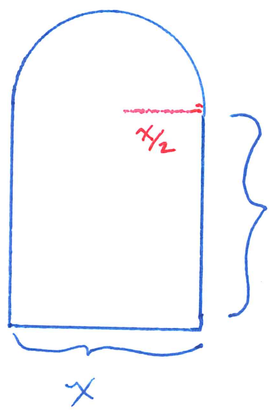
$A'' = -4 < 0$

$\Rightarrow$ concave down $\cap$

$\Rightarrow$ maximum

The rectangle's area is maximized when $x = 5/2$ and $y = 5$. This gives an area of $25/2$.

17. A Norman window has the shape of a rectangle surmounted by a semi-circle. (Thus, the diameter of the semi-circle is equal to the width of the rectangle.) If the perimeter of the window is 20 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.



$$20 = \pi r + 2y + x$$

$$= \left(\frac{\pi}{2} + 1\right)x + 2y \longrightarrow y = \frac{20 - \left(\frac{\pi}{2}+1\right)x}{2}$$

maximize $A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$$\rightarrow A = x\left(10 - \left(\frac{\pi}{4} + \frac{1}{2}\right)x\right) + \frac{\pi}{8}x^2$$

$$= 10x - \left(\frac{\pi}{4} + \frac{1}{2}\right)x^2 + \frac{\pi}{8}x^2$$

$$= 10x - \left(\frac{\pi}{8} + \frac{1}{2}\right)x^2$$

$$A' = 10 - \left(\frac{\pi}{4} + 1\right)x \overset{\text{set}}{=} 0$$

$$10 = \left(\frac{\pi}{4} + 1\right)x$$

$$\boxed{x = \frac{10}{\frac{\pi}{4} + 1} = \frac{40}{\pi + 4}} \qquad \boxed{y = 10 - \left(\frac{\pi}{4} + \frac{1}{2}\right)\left(\frac{40}{\pi + 4}\right)}$$

$$\boxed{A'' = -\left(\frac{\pi}{4} + 1\right) < 0}$$
$\rightarrow$ concave down
$\rightarrow$ local max

The window has a width $x = \frac{40}{\pi + 4}$ ft and height $y = 10 - \left(\frac{\pi}{4} + \frac{1}{2}\right)\left(\frac{40}{\pi + 4}\right)$