4.9: Antiderivatives

Definition. (Antiderivative)
A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$, for x in I.

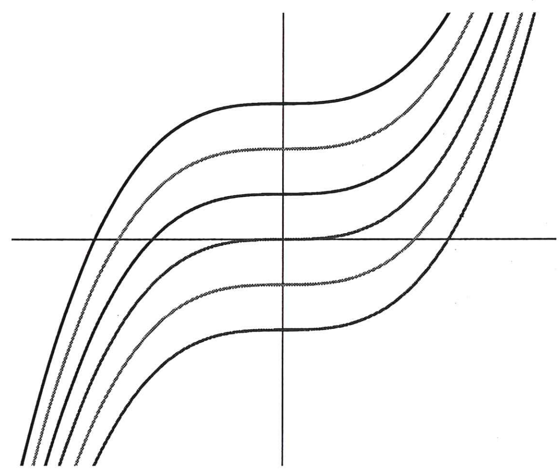
Note: we will denote this relationship in the following way:

Function	Anti-derivative
$f'(x)$	$f(x)$
$f(x)$	$F(x)$

Example. If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$. In this case, $\tan(x)$ is the *antiderivative* of $\sec^2(x)$.

Theorem 4.15: The Family of Antiderivatives
Let F be any antiderivative of f on an interval I. Then *all* antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

Example. If $f'(x) = x^2$, then $f(x) = \dfrac{x^3}{3} + C$ is the family of antiderivatives of $f'(x)$.



Example. Find the most general antiderivative of the following functions

$f(x) = \sin(x)$

$F(x) = -\cos(x) + C$

$k(x) = \dfrac{1}{1+x^2}$

$K(x) = \arctan(x) + C$

$g(x) = x^n,\ n \neq -1$

$G(x) = \dfrac{x^{n+1}}{n+1} + C$

$h(x) = {}^1\!/_x$

$H(x) = \ln|x| + C$

$j(x) = 3x^2$

$J(x) = x^3 + C$

$n(x) = 6x^5$

$N(x) = x^6 + C$

$\ell(x) = \dfrac{41}{x} + 4e^x$

$L(x) = 41\ln|x| + 4e^x + C$

$m(x) = \dfrac{1}{2x^3} = \frac{1}{2}x^{-3}$

$M(x) = \frac{1}{2}\cdot\dfrac{x^{-2}}{-2} + C$

$= \dfrac{-1}{4x^2} + C$

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2(x)$	$\tan(x)$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec(x)\tan(x)$	$\sec(x)$
$x^n \ (n \neq 1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\dfrac{1}{x}$	$\ln\|x\|$	$\dfrac{1}{1+x^2}$	$\tan^{-1}(x)$
$\cos(x)$	$\sin(x)$	$\sin(x)$	$-\cos(x)$
e^x	e^x		

Note: There are some more 'complicated' antiderivatives as well:

$$f(x) = e^{g(x)} \quad \Rightarrow \quad F(x) = \frac{e^{g(x)}}{g'(x)}$$
$$f(x) = k\sec^2(kx) \quad \Rightarrow \quad F(x) = \tan(kx)$$

Focus more on "What can I take the derivative of to get ..." rather than memorizing formulas.

Definition.
Recall that $\frac{d}{dx}[f(x)]$ represents taking the derivative of $f(x)$ with respect to x.

- Finding the antiderivative of f with respect to x is the **indefinite integral** $\int f(x)\,dx$.
- The **integrand** is the function $f(x)$ we are integrating.
- The **variable of integration**, dx, indicates which variable we are integrating with respect to.

Theorem 4.16: Power Rule for Indefinite Integrals

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

Theorem 4.17: Constant Multiple and Sum Rules

Constant Multiple Rule: $\int cf(x)\,dx = c\int f(x)\,dx$, for real numbers c.

Sum Rule: $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

Table 4.9: Indefinite Integrals of Trigonometric Functions

$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\Longrightarrow$	$\int \cos(x)\,dx = \sin(x) + C$
$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\Longrightarrow$	$\int \sin(x)\,dx = -\cos(x) + C$
$\frac{d}{dx}[\tan(x)] = \sec^2(x)$	$\Longrightarrow$	$\int \sec^2(x)\,dx = \tan(x) + C$
$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$	$\Longrightarrow$	$\int \csc^2(x)\,dx = -\cot(x) + C$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\Longrightarrow$	$\int \sec(x)\tan(x)\,dx = \sec(x) + C$
$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	$\Longrightarrow$	$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

Table 4.10: Other Indefinite Integrals

$\frac{d}{dx}[e^x]$	$= e^x$	$\Longrightarrow$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}[\ln\lvert x\rvert]$	$= \frac{1}{x}$	$\Longrightarrow$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\left[\sin^{-1}(x)\right]$	$= \frac{1}{\sqrt{1-x^2}}$	$\Longrightarrow$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$
$\frac{d}{dx}\left[\tan^{-1}(x)\right]$	$= \frac{1}{1+x^2}$	$\Longrightarrow$	$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$
$\frac{d}{dx}\left[\sec^{-1}\lvert x\rvert\right]$	$= \frac{1}{x\sqrt{x^2-1}}$	$\Longrightarrow$	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}\lvert x\rvert + C$

Example. Verify the following integration formulas using differentiation.

$$\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$$

$$\frac{d}{dx}\left[\frac{x}{x+1} + C\right] = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$\int \sec^2(5x-1)\,dx = \frac{1}{5}\tan(5x-1) + C$$

$$\frac{d}{dx}\left[\frac{1}{5}\tan(5x-1) + C\right] = \frac{5}{5}\sec^2(5x-1) = \sec^2(5x-1)$$

$$\int \cos^3(x)\,dx = \sin(x) - \frac{1}{3}\sin^3(x) + C$$

$$\frac{d}{dx}\left[\sin(x) - \frac{1}{3}\sin^3(x) + C\right] = \cos(x) - \frac{3}{3}\sin^2(x)\cos(x)$$

$$= \cos(x)\,[\underbrace{1 - \sin^2(x)}_{\cos^2(x)}]$$

$$= \cos^3(x)$$

$$\int \frac{x}{\sqrt{x^2+1}} = \sqrt{x^2+1} + C$$

$$\frac{d}{dx}\left[(x^2+1)^{1/2} + C\right] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

Example. Find the most general anti-derivative or indefinite integral

$$\int \left(\frac{t^2}{2} + 4t^3\right) dt$$

$$= \frac{1}{2}\frac{t^3}{3} + 4\frac{t^4}{4} + C$$

$$= \boxed{\frac{t^3}{6} + t^4 + C}$$

$$\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$$

$$= \boxed{\frac{1}{5}x + \frac{1}{x^2} + x^2 + C}$$

$$\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$$

$$= \frac{1}{2}\frac{x^{3/2}}{3/2} + 2\frac{x^{1/2}}{1/2} + C$$

$$= \boxed{\frac{x^{3/2}}{3} + 4x^{1/2} + C}$$

$$\int \frac{\csc\theta\cot\theta}{2}\, d\theta$$

$$= \boxed{-\frac{1}{2}\csc\theta + C}$$

$$\int \left(1+\frac{1}{4u^3}-(3u)^2\right) du$$

$$= \int 1 + \frac{1}{4} u^{-3} - 9u^2 \, du$$

$$= u - \frac{1}{4} \frac{u^{-2}}{-2} - 9\frac{u^3}{3} + c$$

$$= \boxed{u + \frac{1}{2u^2} - 3u^3 + c}$$

$$\int \left(\frac{7}{\sqrt{1-x^2}}-\frac{3}{\cos^2(x)}\right) dx$$

$$= \int 7 \frac{1}{\sqrt{1-x^2}} - 3\sec^2(x) \, dx$$

$$= \boxed{7 \tan^{-1}(x) - 3\tan(x) + C}$$

$$\int \left(\frac{1}{4e^x}-\frac{4}{x}+4^x\right) dx$$

$$= \int \frac{1}{4} e^{-x} - 4\frac{1}{x} + 4^x \, dx$$

$$= \boxed{-\frac{1}{4} e^{-x} - 4\ln|x| + \frac{4^x}{\ln 4} + C}$$

$$\int \frac{x^2-36}{x-6} dx = \int x+6 \, dx$$

$$= \boxed{\frac{x^2}{2} + 6x + C}$$

$$\int \frac{2+3\cos(y)}{\sin^2(y)}\,dy$$

$$= \int 2\csc^2(y) + 3\cot(y)\csc(y)\,dy$$

$$= \boxed{-2\cot(y) - 3\csc(y) + C}$$

$$\int (u+4)(2u+1)\,du$$

$$= \int 2u^2 + 9u + 4\,du$$

$$= \boxed{u^3 + \frac{9u}{2} + 4u + C}$$

$$\int e^{x+2}\,dx$$

$$= \frac{e^{x+2}}{\frac{d}{dx}[x+2]} + C$$

$$= \boxed{e^{x+2} + C}$$

$$\int \left(\sqrt[4]{x^3} + \sqrt{x^5}\right) dx$$

$$= \int x^{3/4} + x^{5/2}\,dx$$

$$= \frac{x^{7/4}}{7/4} + \frac{x^{7/2}}{7/2} + C$$

$$= \boxed{\frac{4x^{7/4}}{7} + \frac{2x^{7/2}}{7} + C}$$

$$\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx$$

$$= \boxed{\frac{x^3}{3} + x + \tan^{-1}(x) + C}$$

$$\int \left(\sin(4x) - \frac{3}{\sin^2(x)}\right) dx$$

$$= \int \sin(4x) - 3\csc^2(x)\, dx$$

$$= \boxed{\frac{-\cos(4x)}{4} + 3\cot(x) + C}$$

$$\int \left(\csc^2(2t) - 2e^t\right) dt$$

$$= \boxed{\frac{-\cot(2t)}{2} - 2e^t + C}$$

$$\int x\left(1 + 2x^4\right) dx$$

$$= \int x + 2x^5\, dx$$

$$= \boxed{\frac{x^2}{2} + \frac{x^6}{3} + C}$$

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot\theta \csc\theta \, d\theta$$

$$= \boxed{-\csc\theta + C}$$

$$\int \frac{\sin(2x)}{\sin(x)}\,dx$$

$$= \int \frac{2\sin(x)\cos(x)}{\sin(x)}\,dx$$

$$= 2\int \cos(x)\,dx$$

$$= \boxed{2\sin(x) + C}$$

$$\int \left(\pi + \frac{2}{yt}\right) dt$$

$$= \int \pi + \frac{2}{y}\cdot\frac{1}{t}\,dt$$

$$= \boxed{\pi t + \frac{2}{y}\ln|t| + C}$$

$$\int \frac{t^2 - e^{2t}}{t + e^t}\,dt$$

$$= \int \frac{(t-e^t)(t+e^t)}{t+e^t}\,dt$$

$$= \int t - e^t \, dt$$

$$= \boxed{\frac{t^2}{2} - e^t + C}$$

Definition.

- An equation involving an unknown function and its derivative is called a **differential equation**.
- An **initial condition** allows us to determine the arbitrary constant C.
- A differential equation coupled with an initial condition is called an **initial value problem**.

$f'(x) = g(x)$,	where g is given, and	Differential equation
$f(a) = b$,	where a and b are given.	Initial condition

Example. Solve the initial value problem:

$$\frac{dy}{dx} = 9x^2 - 4x + 5,\ y(-1) = 0$$

$y(x) = 3x^3 - 2x^2 + 5x + C$

$y(-1) = 3(-1)^3 - 2(-1)^2 + 5(-1) + C = 0$

$-3 - 2 - 5 + C = 0$

$C = 10$

$\Rightarrow \boxed{y(x) = 3x^3 - 2x^2 + 5x + 10}$

$$f'(x) = 8x^3 + 12x + 3,\ f(1) = 6$$

$f(x) = 2x^4 + 6x^2 + 3x + C$

$f(1) = 2 + 6 + 3 + C = 6$

$C = -5$

$\Rightarrow \boxed{f(x) = 2x^4 + 6x^2 + 3x - 5}$

$$f'(x) = 1 + 3\sqrt{x},\ f(4) = 25$$

$f(x) = x + 2x^{3/2} + C$

$f(4) = 4 + 2 \cdot 8 + C = 25$

$C = 5$

$\Rightarrow \boxed{f(x) = x + 2x^{3/2} + 5}$

$$\frac{dr}{d\theta} = \cos(\pi\theta),\ r(0) = 1$$

$r(\theta) = \frac{\sin(\pi\theta)}{\pi} + C$

$r(0) = 0 + C = 1$

$C = 1$

$\Rightarrow \boxed{r(\theta) = \frac{\sin(\pi\theta)}{\pi} + 1}$

$$f'(t) = 2\cos(t) + \sec^2(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}, \quad f\left(\frac{\pi}{3}\right) = 4$$

$f(t) = 2\sin(t) + \tan(t) + c$

$f\left(\frac{\pi}{3}\right) = \sqrt{3} + \sqrt{3} + c = 4$

$\Rightarrow c = 4 - 2\sqrt{3}$

$\Rightarrow \boxed{f(t) = 2\sin(t) + \tan(t) + (4 - 2\sqrt{3})}$

$$g'(x) = 7x\left(x^6 - \tfrac{1}{7}\right); \quad g(1) = 2$$

$= 7x^7 - x$

$g(x) = \frac{7}{8}x^8 - \frac{x^2}{2} + c$

$g(1) = \frac{7}{8} - \frac{1}{2} + c = 2$

$\Rightarrow c = \frac{13}{8}$

$\Rightarrow \boxed{g(x) = \frac{7}{8}x^8 - \frac{x^2}{2} + \frac{13}{8}}$

$$f'''(x) = \sin(x), \quad f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1$$

$f''(x) = -\cos(x) + c$

$f''(0) = -1 + c = 1$

$\Rightarrow c = 2$

$\Rightarrow f''(x) = -\cos(x) + 2$

$f'(x) = -\sin(x) + 2x + c$

$f'(0) = 0 + 0 + c = 1$

$\Rightarrow f'(x) = -\sin(x) + 2x + 1$

$f(x) = \cos(x) + x^2 + x + c$

$f(0) = 1 + 0 + 0 + c = 1$

$c = 0$

$\Rightarrow \boxed{f(x) = \cos(x) + x^2 + x}$

$$f'(x) = \frac{4}{\sqrt{1-x^2}}, \quad f\left(\frac{1}{2}\right) = 1$$

$f(x) = 4\sin^{-1}(x) + C$

$f(\tfrac{1}{2}) = 4(\pi/6) + C = 1$

$C = 1 - \frac{2\pi}{3}$

$\Rightarrow \boxed{f(x) = 4\sin^{-1}(x) + 1 - \frac{2\pi}{3}}$

$$\frac{d^2r}{dt^2} = \frac{2}{t^3}; \quad \left.\frac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$$

$r'(t) = -\frac{1}{t^2} + C$

$r'(1) = -1 + C = 1$

$C = 2$

$r'(t) = 2 - \frac{1}{t^2}$

$r(t) = 2t + \frac{1}{t} + C$

$r(1) = 2 + 1 + C = 1$

$C = -2$

$\Rightarrow \boxed{r(t) = 2t + \frac{1}{t} - 2}$

$$f''(x) = 4 + 6x + 4x^2, \quad f(0) = 3, \quad f(1) = 10$$

$f'(x) = 4x + 3x^2 + \frac{4}{3}x^3 + C$

$f(x) = 2x^2 + x^3 + \frac{1}{3}x^4 + Cx + D$

$$\begin{cases} f(0) = 0 + 0 + 0 + 0 + D = 3 \Rightarrow D = 3 \\ f(1) = 2 + 1 + \frac{1}{3} + C + 3 = 10 \Rightarrow C = 11/3 \end{cases}$$

$\Rightarrow \boxed{f(x) = 2x^2 + x^3 + \frac{1}{3}x^4 + 11/3\,x + 3}$

Initial Value Problems for Velocity and Position
Suppose an object moves along a line with a (known) velocity $v(t)$, for $t \geq 0$. Then its position is found by solving the initial value problem.

$$s'(t) = v(t),\ s(0) = s_0,\ \text{where } s_0 \text{ is the (known) initial position.}$$

If the (known) acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t),\ v(0) = v_0,\ \text{where } v_0 \text{ is the (known) initial velocity.}$$

Recall:

Position	$s(t)$
Velocity	$v(t) = s'(t)$
Acceleration	$a(t) = v'(t) = s''(t)$

Example. Solve the following velocity and position initial value problems

$$v(t) = \sin(t) + 3\cos(t); \quad s(0) = 4$$

$s(t) = -\cos(t) + 3\sin(t) + c$

$s(0) = -1 + 0 + c = 4$

$c = 5$

$\Rightarrow \boxed{s(t) = -\cos(t) + 3\sin(t) + 5}$

$$a(t) = 2e^t - 12, \quad v(0) = 1, \quad s(0) = 0$$

$v(t) = 2e^t - 12t + c$

$v(0) = 2 - 0 + c = 1$

$c = -1$

$\Rightarrow v(t) = 2e^t - 12t - 1$

$s(t) = 2e^t - 6t^2 - t + D$

$s(0) = 2 - 0 - 0 + D = 0$

$D = -2$

$\Rightarrow \boxed{s(t) = 2e^t - 6t^2 - t - 2}$

Example. The acceleration of gravity near the surface of Mars is $3.72 m/s^2$. A rock is thrown straight up from the surface with an initial velocity of 23 m/s. How high does the rock go?

a) Write the initial value problem

$a(t) = -3.72 \ m/s^2$

$v(0) = 23 \ m/s$

$s(0) = 0 \ m$

b) Find the velocity and position functions.

$v(t) = -3.72t + C$

$\left.\begin{array}{l} v(0) = 0 + C = 23 \\ C = 23 \end{array}\right\} \Rightarrow \boxed{v(t) = -3.72t + 23}$

$s(t) = -1.86t^2 + 23t + D$

$\left.\begin{array}{l} s(0) = 0 + 0 + D = 0 \\ D = 0 \end{array}\right\} \Rightarrow \boxed{s(t) = -1.86t^2 + 23t}$

c) Maximum height is reached when velocity is 0. Find the time when this happens and the maximum height.

Solve $v(t) = 0$

$-3.72t + 23 = 0$

$\boxed{t = \frac{23}{3.72} \text{ sec}}$

evaluate

$s\left(\frac{23}{3.72}\right) = -1.86\left(\frac{23}{3.72}\right)^2 + 23\left(\frac{23}{3.72}\right) \approx \boxed{71.102 \ m}$

Example. A ball is thrown vertically upward from a height of 48 feet above ground at a speed of $32 ft/s$. Assume the acceleration due to gravity is 32 ft/s^2.

a) How high above the ground will it get?

$a(t) = -32 \ m/s^2$

$v(0) = 32 \ ft/s$

$s(0) = 48 \ ft$

$v(t) = -32t + C$

$v(0) = 0 + C = 32$

$C = 32$

$\rightarrow v(t) = -32t + 32$

$s(t) = -16t^2 + 32t + D$

$s(0) = 0 + 0 + D = 48$

$D = 48$

$s(t) = -16t^2 + 32t + 48$

Solve $v(t) = 0$ $\quad -32t + 32 = 0$

$t = 1$

Eval

$s(1) = -16 + 32 + 48 = \boxed{64 \ ft}$

b) How long after it is thrown will it hit the ground?

Solve $s(t) = 0$

$-16t^2 + 32t + 48 = 0$

$-16(t^2 - 2t - 3) = 0$

$-16(t + 1)(t - 3) = 0 \quad \Rightarrow \quad t = -1, \ \boxed{t = 3}$

Example. A stone was dropped off a cliff and hits the ground with a speed of $120 ft/s$. What is the height of the cliff (assuming $a(t) = -32 ft/s^2$).

$a(t) = -32$
$v(0) = 0$

$v(t) = -32t + c$

$v(0) = 0 + c = 0$

$c = 0$

$v(T_f) = -120$

T_f is final time

$\Rightarrow v(t) = -32t$

Solve for t:

$-32t = -120$

$t = \frac{120}{32} = \frac{15}{4}$ sec

$\Rightarrow s(15/4) = 0$

$s(t) = -16t^2 + D$

$s(15/4) = -16(15/4)^2 + D = 0$

$D = \boxed{225 \text{ ft.}}$

$s(t) = -16t^2 + 225$

Example. Find the antiderivative of $f(x) = \dfrac{2 + x^2}{1 + x^2}$

$$\int \frac{2+x^2}{1+x^2}\,dx = \int \frac{1}{1+x^2} + \underbrace{\frac{1+x^2}{1+x^2}}_{1}\,dx$$

$$= \tan^{-1}(x) + x + c$$