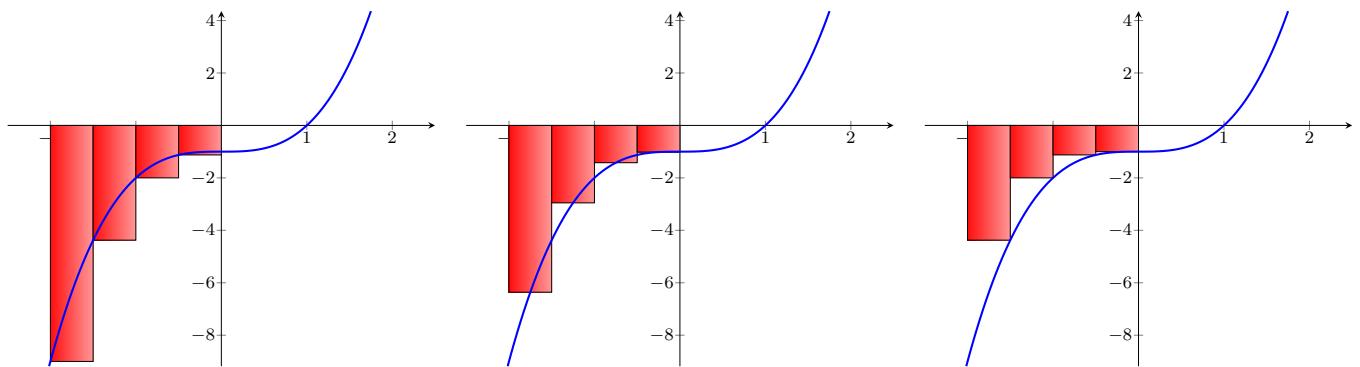


## 5.2: Definite Integrals

In section 5.1, we assumed  $f$  was nonnegative on the interval  $[a, b]$ .

When  $f(x) \leq 0$  on the interval  $[a, b]$ , then the area between  $f(x)$  and the  $x$ -axis is negative.

**Example.** Consider the function  $x^3 - 1$  on  $[-2, 0]$ . Using Riemann sums, we can approximate the area between the curve and the  $x$ -axis:



$$L_4 = f(-2)\frac{1}{2} + f(-1.5)\frac{1}{2} + f(-1)\frac{1}{2} + f(-0.5)\frac{1}{2} = -8.25$$

$$M_4 = f(-1.75)\frac{1}{2} + f(-1.25)\frac{1}{2} + f(-0.75)\frac{1}{2} + f(-0.25)\frac{1}{2} = -5.875$$

$$R_4 = f(-1.5)\frac{1}{2} + f(-1)\frac{1}{2} + f(-0.5)\frac{1}{2} + f(0)\frac{1}{2} = -4.25$$

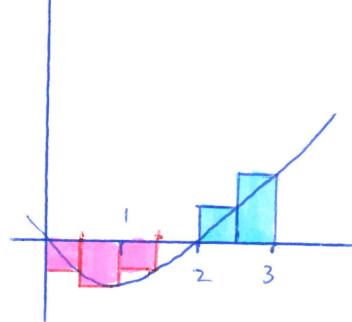
Actual area: -6

### Definition. (Net Area)

Consider the region  $R$  bounded by the graph of a continuous function  $f$  and the  $x$ -axis between  $x = a$  and  $x = b$ . The **net area** of  $R$  is the sum of the areas of the parts of  $R$  that lie above the  $x$ -axis *minus* the sum of the areas of the parts of  $R$  that lie below the  $x$ -axis on  $[a, b]$ .

**Example.** If  $f(x) = x^2 - 2x$ ,  $0 \leq x \leq 3$ , evaluate the Riemann sum with  $n = 6$ , taking the sample points to be right endpoints.

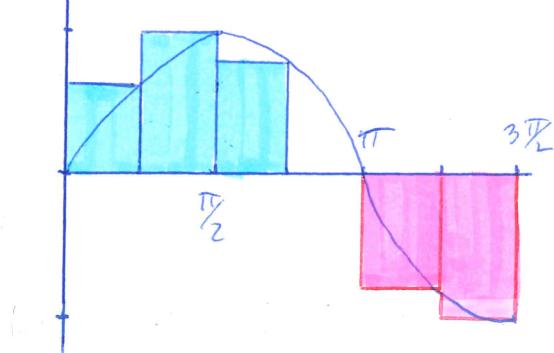
$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$



$$\begin{aligned} \sum_{k=1}^6 f(x_k) \Delta x &= f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x \\ &\quad + f(x_4) \Delta x + f(x_5) \Delta x + f(x_6) \Delta x \\ &= \frac{1}{2} (-0.75 - 1 - 0.75 + 0 + 1.25 + 3, 0) \\ &= \frac{1}{2} (1.75) = 0.875 = \boxed{\frac{7}{8}} \end{aligned}$$

**Example.** Find the Riemann sum for  $f(x) = \sin(x)$ ,  $0 \leq x \leq \frac{3\pi}{2}$ , with six terms, taking the sample points to be right endpoints.

$$\Delta x = \frac{\frac{3\pi}{2} - 0}{6} = \frac{\pi}{4}$$

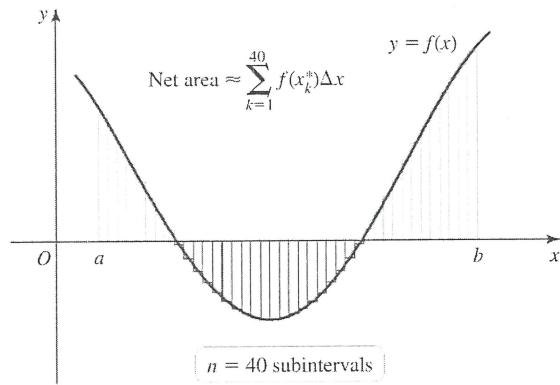
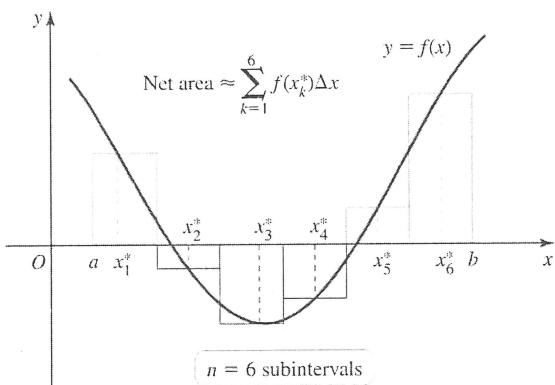


$$\begin{aligned} \sum_{k=1}^6 f(x_k) \Delta x &= \frac{\pi}{4} \left( \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1 \right) \\ &= \boxed{\frac{(\sqrt{2}+2)\pi}{8}} \end{aligned}$$

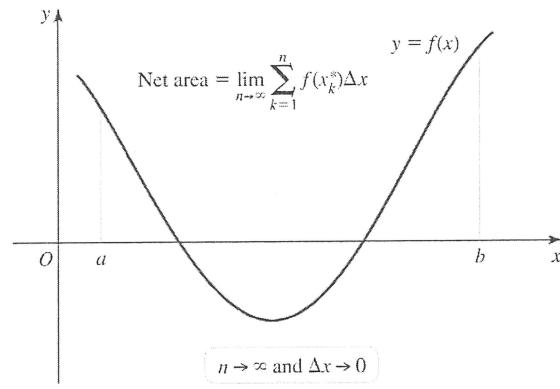
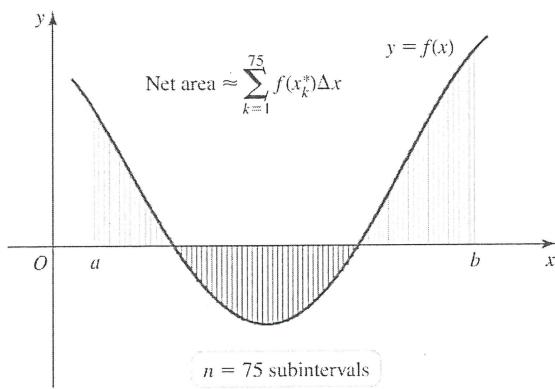
## Definition. (Definite Integral)

A function  $f$  defined on  $[a, b]$  is **integrable** on  $[a, b]$  if  $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x_k$  exists and is unique over all partitions of  $[a, b]$  and all choices of  $x_k^*$  on a partition. This limit is the **definite integral of  $f$  from  $a$  to  $b$** , which we write

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x_k$$



As the number of subintervals  $n$  increases, the Riemann sums approach the net area of the region between the curve  $y = f(x)$  and the  $x$ -axis on  $[a, b]$ .



**Example.**

- a) For the function  $f(x) = 3x + 2x^2$ , find a formula for the upper sum obtained by dividing the interval  $[0, 1]$  into  $n$  equal subintervals.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_k = 0 + k\Delta x = \frac{k}{n}$$

$$f(x_k) = \frac{3k}{n} + \frac{2k^2}{n^2}$$

$$\begin{aligned} \sum_{k=1}^n f(x_k) \Delta x &= \sum_{k=1}^n \left( \frac{3k}{n} + \frac{2k^2}{n^2} \right) \frac{1}{n} \\ &= \frac{3}{n^2} \sum_{k=1}^n k + \frac{2}{n^3} \sum_{k=1}^n k^2 \\ &= \frac{3}{n^2} \frac{n(n+1)}{2} + \frac{2}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= \frac{3(n+1)}{2n} + \frac{2n^2+3n+1}{3n^2} \end{aligned}$$

- b) Take the limit of the sum as  $n \rightarrow \infty$  to calculate the area under  $f(x) = 3x + 2x^2$  over  $[0, 1]$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x &= \lim_{n \rightarrow \infty} \frac{3(n+1)}{2n} + \frac{2n^2+3n+1}{3n^2} \\ &= \frac{3}{2} + \frac{2}{3} = \boxed{\frac{13}{6}} \end{aligned}$$

**Example.** Use the limit of the Riemann sum notation to evaluate  $\int_0^2 (2 - x^2) dx$ .

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = 0 + k\Delta x = \frac{2k}{n}$$

$$f(x_k) = 2 - \left(\frac{2k}{n}\right)^2 = 2 - \frac{4k^2}{n^2}$$

$$\sum_{k=1}^n \left(2 - \frac{4k^2}{n^2}\right) \frac{2}{n} = \frac{1}{n} \sum_{k=1}^n 4 - \frac{8}{n^3} \sum_{k=1}^n k^2 = 4 - \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= 4 - \frac{4(2n^2+3n+1)}{3n^2}$$

$$\int_0^2 (2-x^2) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x = \lim_{n \rightarrow \infty} 4 - \frac{4(2n^2+3n+1)}{3n^2} = 4 - \frac{8}{3} = \boxed{\frac{4}{3}}$$

**Example.** Use the definition of the definite integral to evaluate  $\int_1^4 (x^2 - 1) dx$ .

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_k = 1 + k\Delta x = 1 + \frac{3k}{n}$$

$$f(x_k) = \left(1 + \frac{3k}{n}\right)^2 - 1 = \frac{6k}{n} + \frac{9k^2}{n^2}$$

$$\sum_{k=1}^n \left(\frac{6k}{n} + \frac{9k^2}{n^2}\right) \frac{3}{n} = \sum_{k=1}^n \frac{18k}{n^2} + \frac{27k^2}{n^3} = \frac{18}{n^2} \sum_{k=1}^n k + \frac{27}{n^3} \sum_{k=1}^n k^2$$

$$= \frac{9(n+1)}{n} + \frac{9(2n^2+3n+1)}{2n^2}$$

$$\int_1^4 x^2 - 1 dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x = \lim_{n \rightarrow \infty} \frac{9(n+1)}{n} + \frac{9(2n^2+3n+1)}{2n^2}$$

$$= 9 + 9 = \boxed{18}$$

**Example.** Use the definition of the definite integral to evaluate  $\int_1^4 (x^2 - 4x + 2) dx$ .

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_k = 1 + k\Delta x = 1 + \frac{3k}{n}$$

$$f(x_k) = (1 + \frac{3k}{n})^2 - 4(1 + \frac{3k}{n}) + 2 = \frac{9k^2}{n^2} - \frac{6k}{n} - 1$$

$$\begin{aligned}\sum_{k=1}^n f(x_k) \Delta x &= \sum_{k=1}^n \left( \frac{9k^2}{n^2} - \frac{6k}{n} - 1 \right) \frac{3}{n} = \frac{27}{n^3} \sum_{k=1}^n k^2 - \frac{18}{n^2} \sum_{k=1}^n k - \frac{3}{n}(n) \\ &= \frac{9(2n^2+3n+1)}{2n^2} - \frac{18(n+1)}{n} - 3\end{aligned}$$

$$\int_1^4 (x^2 - 4x + 2) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x = \lim_{n \rightarrow \infty} \frac{9(2n^2+3n+1)}{2n^2} - \frac{9(n+1)}{n} - 3 = 9 - 9 - 3 = \boxed{-3}$$

**Example.** Use the definition of the definite integral to evaluate  $\int_0^2 4x^3 dx$ .

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = 0 + k\Delta x = \frac{2k}{n}$$

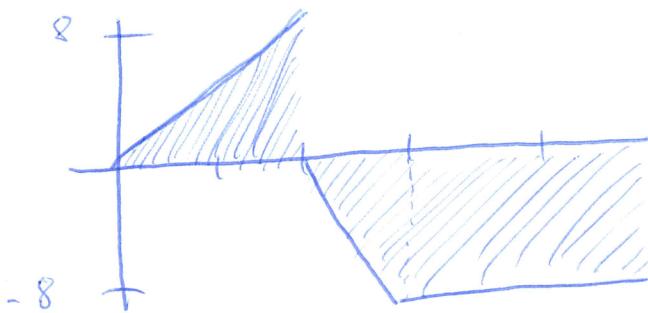
$$f(x_k) = 4 \left( \frac{2k}{n} \right)^3 = \frac{32k^3}{n^3}$$

$$\sum_{k=1}^n f(x_k) \Delta x = \sum_{k=1}^n \left( \frac{32k^3}{n^3} \right) \left( \frac{2}{n} \right) = \frac{64}{n^4} \sum_{k=1}^n k^3 = \frac{64}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{16(n+1)^2}{n^2}$$

$$\int_0^2 4x^3 dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x = \lim_{n \rightarrow \infty} \frac{16(n+1)^2}{n^2} = \boxed{16}$$

**Example.** Use a sketch and geometry to evaluate the following integral

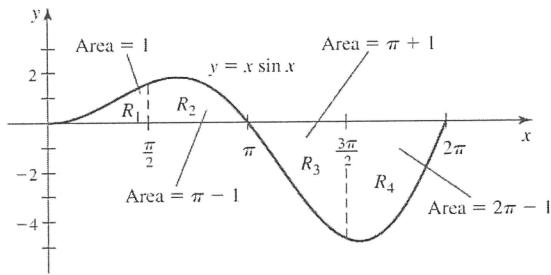
$$\int_1^{10} g(x) dx, \text{ where } g(x) = \begin{cases} 4x, & \text{if } 0 \leq x \leq 2 \\ -8x + 16, & \text{if } 2 < x \leq 3 \\ -8, & \text{if } x > 3 \end{cases}$$



$$\int_1^{10} g(x) dx = \frac{1}{2}(2+8) + \frac{1}{2}(-8+7)(-8)$$

$$= [-55]$$

**Example.** For the following figure to evaluate the integrals below:



$$a) \int_0^\pi x \sin(x) dx = R_1 + R_2 = 1 + (\pi - 1) = \textcircled{7}$$

$$b) \int_0^{3\pi/2} x \sin(x) dx = R_1 + R_2 - R_3$$

$$= 1 + (\pi - 1) - (\pi + 1)$$

$$= \textcircled{-1}$$

$$c) \int_0^{2\pi} x \sin(x) dx$$

$$= R_1 + R_2 - R_3 - R_4$$

$$= 1 + (\pi - 1) - (\pi + 1) - (2\pi - 1)$$

$$= \textcircled{-2\pi}$$

$$d) \int_{\pi/2}^{2\pi} x \sin(x) dx$$

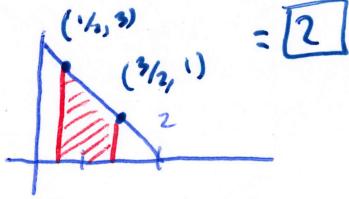
$$= R_2 - R_3 - R_4$$

$$= (\pi - 1) - (\pi + 1) - (2\pi - 1)$$

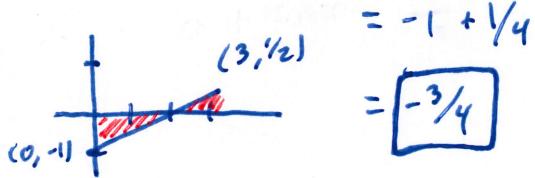
$$= \textcircled{-1 - 2\pi}$$

**Example.** Graph the following integrands and compute the areas to evaluate the integrals.

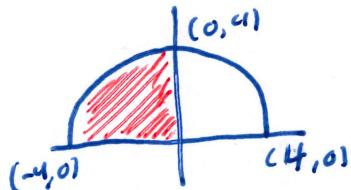
$$a) \int_{1/2}^{3/2} (-2x + 4) dx = \frac{1}{2}(3+1) \cdot 1$$



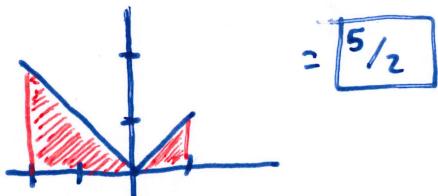
$$c) \int_0^3 \left(\frac{1}{2}x - 1\right) dx = -\frac{1}{2}(1)(2) + \frac{1}{2}\left(\frac{1}{2}\right)(1)$$



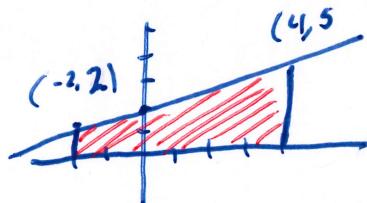
$$e) \int_{-4}^0 \sqrt{16 - x^2} dx = \frac{1}{4} \pi (4)^2 = 4\pi$$



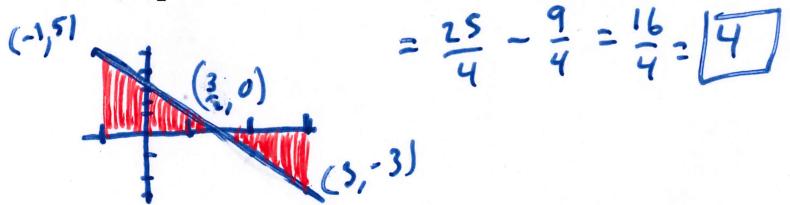
$$g) \int_{-2}^1 |x| dx = \frac{1}{2}(2)(2) + \frac{1}{2}(1)(1)$$



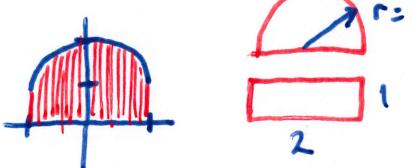
$$b) \int_{-2}^4 \left(\frac{x}{2} + 3\right) dx = \frac{1}{2}(2+5)6 = 21$$



$$d) \int_{-1}^3 (3 - 2x) dx = \frac{1}{2}\left(\frac{5}{2}\right)(5) - \frac{1}{2}\left(\frac{3}{2}\right)(3)$$



$$f) \int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right) dx = (1)(2) + \frac{1}{2}\pi(1)^2$$



$$h) \int_{-1}^1 (2 - |x|) dx = \frac{1}{2}(1+2)(1) + \frac{1}{2}(2+1)(1) = 3$$



## Properties of definite integrals

Let  $f$  and  $g$  be integrable functions on an interval that contains  $a$ ,  $b$ , and  $p$ .

$$1. \int_a^a f(x) dx = 0$$

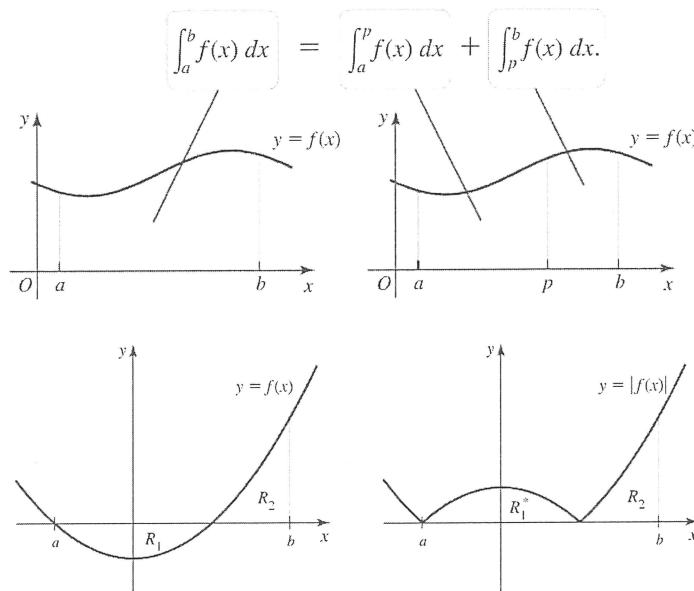
$$2. \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$3. \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$4. \int_a^b cf(x) dx = c \int_a^b f(x) dx, \text{ for any constant } c$$

$$5. \int_a^b f(x) dx = \int_a^p f(x) dx + \int_p^b f(x) dx$$

6. The function  $|f|$  is integrable on  $[a, b]$  and  $\int_a^b |f(x)| dx$  is the sum of the areas of the regions bounded by the graph of  $f$  and the  $x$ -axis on  $[a, b]$ .



**Example.** Suppose that  $\int_{-3}^0 g(t) dt = \sqrt{2}$ . Evaluate the following:

$$\text{a)} \int_{-3}^0 g(u) du = \boxed{\sqrt{2}}$$

$$\text{b)} \int_0^{-3} g(t) dt = - \int_{-3}^0 g(t) dt = \boxed{-\sqrt{2}}$$

$$\begin{aligned} \text{c)} \int_0^{-3} [-g(x)] dx &= - \int_0^{-3} g(x) dx \\ &= \int_{-3}^0 g(x) dx = \boxed{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{d)} \int_{-3}^0 \frac{g(r)}{\sqrt{2}} dr &= \frac{1}{\sqrt{2}} \int_{-3}^0 g(r) dr \\ &= \frac{1}{\sqrt{2}} \cdot \sqrt{2} = \boxed{1} \end{aligned}$$

**Example.** Suppose that  $\int_0^3 f(z) dz = 3$  and  $\int_0^4 f(z) dz = 7$ . Evaluate the following:

$$\begin{aligned} \text{a)} \int_3^4 f(z) dz &= \int_0^4 f(z) dz - \int_0^3 f(z) dz \\ &= 7 - 3 = \boxed{4} \end{aligned}$$

$$\begin{aligned} \text{b)} \int_4^3 f(z) dz &= - \int_3^4 f(z) dz \\ &= \boxed{-4} \end{aligned}$$

**Example.** Use the fact that  $\int_0^{\pi/2} (\cos(\theta) - 2\sin(\theta)) d\theta = -1$  to evaluate the following

$$\text{a)} \int_0^{\pi/2} (2\sin(\theta) - \cos(\theta)) d\theta$$

$$= \int_0^{\pi/2} (\cos(\theta) - 2\sin(\theta)) d\theta$$

$$= -(-1) = \boxed{1}$$

$$\text{b)} \int_{\pi/2}^0 (4\cos(\theta) - 8\sin(\theta)) d\theta$$

$$= - \int_0^{\pi/2} 4(\cos(\theta) - 2\sin(\theta)) d\theta$$

$$= -4 \int_0^{\pi/2} (\cos(\theta) - 2\sin(\theta)) d\theta = -4(-1) = \boxed{4}$$

**Example.** Suppose that  $\int_1^9 f(x) dx = -1$ ,  $\int_7^9 f(x) dx = 5$  and  $\int_7^9 h(x) dx = 4$ . Evaluate the following:

$$\text{a)} \int_1^9 -2f(x) dx = -2 \int_1^9 f(x) dx$$

$$= -2(-1) = \boxed{2}$$

$$\text{b)} \int_7^9 [f(x) + h(x)] dx$$

$$= \int_7^9 f(x) dx + \int_7^9 h(x) dx = 4 + 5 = \boxed{9}$$

$$\text{c)} \int_7^9 [2f(x) - 3h(x)] dx$$

$$= 2 \int_7^9 f(x) dx - 3 \int_7^9 h(x) dx$$

$$= 2(5) - 3(4) = \boxed{-2}$$

$$\text{d)} \int_9^1 f(x) dx = - \int_1^9 f(x) dx$$

$$= -(-1) = \boxed{1}$$

$$\text{e)} \int_1^7 f(x) dx$$

$$= \int_1^9 f(x) dx - \int_7^9 f(x) dx$$

$$= -1 - 5 = \boxed{-6}$$

$$\text{f)} \int_7^9 [h(x) - f(x)] dx$$

$$= \int_7^9 h(x) dx - \int_7^9 f(x) dx$$

$$= 4 - 5 = \boxed{-1}$$

**Example.** Given  $\int_1^3 e^x dx = e^3 - e$ , find  $\int_1^3 (2e^x - 1) dx$

$$\begin{aligned} \int_1^3 2e^x - 1 dx &= 2 \int_1^3 e^x dx - \int_1^3 1 dx \\ &= 2(e^3 - e) - (3 - 1) = \boxed{2e^3 - 2e - 2} \end{aligned}$$

**Example.** Suppose that  $f(x) \geq 0$  on  $[0, 2]$  and  $f(x) \leq 0$  on  $[2, 5]$  where  $\int_0^2 f(x) dx = 6$  and  $\int_2^5 f(x) dx = -8$ . Evaluate the following:

a)  $\int_0^5 f(x) dx$

$$= \int_0^2 f(x) dx + \int_2^5 f(x) dx$$

$$= 6 + (-8) = \textcircled{-2}$$

b)  $\int_0^5 |f(x)| dx$

$$= \int_0^2 |f(x)| dx + \int_2^5 |f(x)| dx$$

$$= |6| + |-8| = \boxed{14}$$

c)  $\int_2^5 4|f(x)| dx$

$$= 4 \int_2^5 |f(x)| dx$$

$$= 4|-8| = \boxed{32}$$

d)  $\int_0^5 (f(x) + |f(x)|) dx$

$$= \int_0^2 f(x) dx + \int_2^5 |f(x)| dx$$

$$= -2 + 14$$

$$= \boxed{12}$$