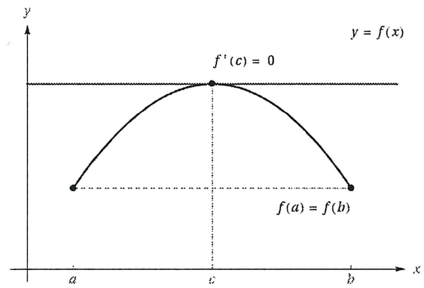
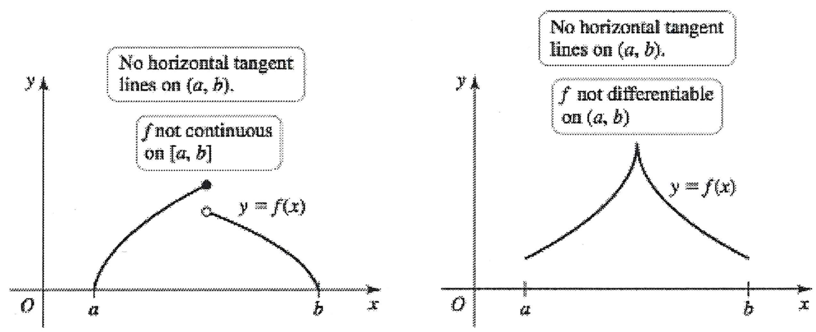
4.2: Mean Value Theorem

Before studying the *Mean Value Theorem*, we must first learn *Rolle's Theorem*:

> **Theorem 4.3: Rolle's Theorem**
> Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.



Note: Rolle's Theorem requires f be both *continuous* and *differentiable*:



(a) (b)

Example. Determine whether Rolle's Theorem applies for the following. If it applies, find the point(s) c such that $f'(c) = 0$.

$f(x) = \sin(2x)$ on $[0, \frac{\pi}{2}]$

① cont $[0, \pi/2]$ ✓

② diff $(0, \pi/2)$ ✓

$f(0) = \sin(0) = 0$

$f(\pi/2) = \sin(2 \cdot \frac{\pi}{2}) = 0$

$f'(x) = 2\cos(2x)$

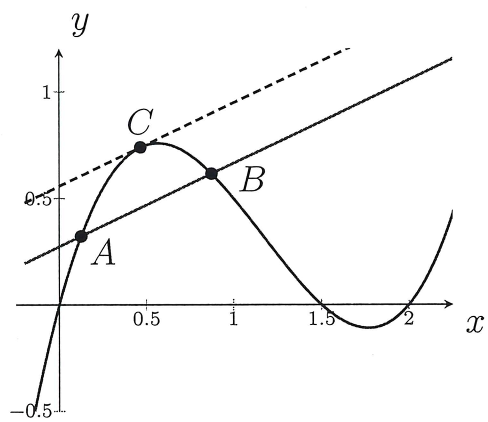
$2\cos(2x) = 0$

$\cos(2x) = 0$

$2x = \pi/2$

$\boxed{x = \pi/4}$

Theorem 4.4: Mean Value Theorem
If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.



Example. For $f(x) = x^{\frac{2}{3}}$ on the interval $[0, 1]$, does the Mean Value Theorem apply? If so, find the point(s) c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

① cont $[0,1]$ ✓

② Diff $(0,1)$ $\quad f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}, \quad x \neq 0$

$\Rightarrow$ Is diff on $(0,1)$. ✓

Solve $f'(x) = \frac{f(1)-f(0)}{1-0}$

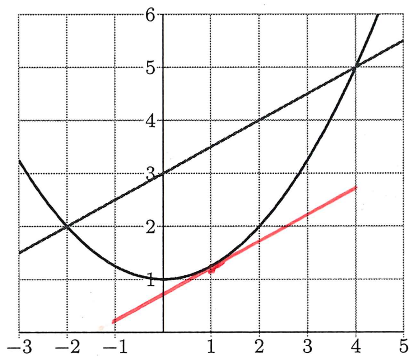
$$\frac{2}{3x^{1/3}} = \frac{1-0}{1-0} = 1$$

$$\frac{2}{3} = x^{1/3}$$

$$\boxed{8/27 = x}$$

Example. For each function, associated interval and graph determine if the conditions for the Mean Value Theorem are met and find the value(s) of c such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

1. $f(x) = \dfrac{x^2}{4} + 1$ on $[-2, 4]$



① cont $[-2, 4]$ ✓

② diff $(-2, 4)$ ✓

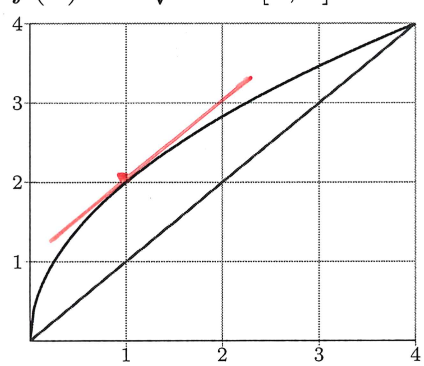
$f'(x) = \frac{1}{2}x$

Solve $f'(x) = \dfrac{f(4) - f(-2)}{4 - (-2)}$

$\frac{1}{2}x = \frac{1}{2}$

$\boxed{x = 1}$

2. $f(x) = 2\sqrt{x}$ on $[0, 4]$



① cont $[0, 4]$

② diff $(0, 4)$

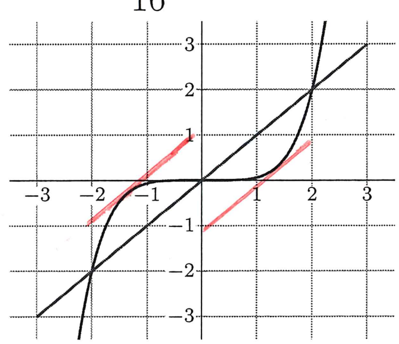
$f'(x) = \dfrac{1}{\sqrt{x}}$

Solve $f'(x) = \dfrac{f(4) - f(0)}{4 - 0}$

$\dfrac{1}{\sqrt{x}} = 1$

$\boxed{x = 1}$

3. $f(x) = \dfrac{x^5}{16}$ on $[-2, 2]$



① cont $[-2, 2]$

② cont $(-2, 2)$

$f'(x) = \dfrac{5x^4}{16}$

Solve $f'(x) = \dfrac{f(2) - f(-2)}{2 - (-2)}$

$\dfrac{5x^4}{16} = 1$

$x = \pm\sqrt[4]{\dfrac{16}{5}} = \pm\dfrac{2}{\sqrt[4]{5}}$

Example. Determine whether Rolle's Theorem applies to the following functions and find the point(s) c if applicable.

1. $f(x) = x(x-1)^2$ on $[0, 1]$.

① cont ✓ ② diff ✓

$f(0) = 0$, $f(1) = 0$ $\Rightarrow f(0) = f(1)$

$f'(x) = 3x^2 - 4x + 1 \overset{\text{set}}{=} 0$

$(3x-1)(x-1) = 0$

$\boxed{x = 1/3}$ $x = 1$ ← Endpoint

2. $f(x) = x^3 - x^2 - 5x - 3$ on $[-1, 3]$.

① cont ✓

② diff ✓

$f(-1) = 0$, $f(3) = 0$ $\Rightarrow f(-1) = f(3)$

$f'(x) = 3x^2 - 2x - 5 \overset{\text{set}}{=} 0$

$(3x-5)(x+1) = 0$

$\boxed{x = 5/3}$ $x = -1$ ← endpoint

Example. Determine whether the Mean Value Theorem applies to the following functions and find the point(s) c if applicable.

1. $f(x) = 3x^2 + 2x + 5$ on $[-1, 1]$.

① cont. ✓

② diff ✓

$f'(x) = 6x + 2$

Solve $f'(x) = \dfrac{f(1) - f(-1)}{1 - (-1)}$

$6x + 2 = 2$

$\boxed{x = 0}$

2. $f(x) = x^{-\frac{1}{3}}$ on $\left[\frac{1}{8}, 8\right]$.

① cont. ✓

② diff?

$f'(x) = -\frac{1}{3} x^{-4/3} = -\dfrac{1}{3\sqrt[3]{x^4}}, \; x \neq 0$

$\Rightarrow$ diff. ✓

Solve $f'(x) = \dfrac{f(8) - f(1/8)}{8 - 1/8}$

$-\dfrac{1}{3\sqrt[3]{x^4}} = -4/21$

$\dfrac{7}{4} = \sqrt[3]{x^4}$

$\boxed{x = \left(\frac{7}{4}\right)^{3/4}}$

3. $f(x) = \begin{cases} \frac{\sin(x)}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}.$

Cont?

$$\left.\begin{array}{l} f(0) = 0 \\ \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1 \end{array}\right\} \text{Not cont!}$$

4. $f(x) = |x - 1|$ on $[-1, 4]$.

cont ✓

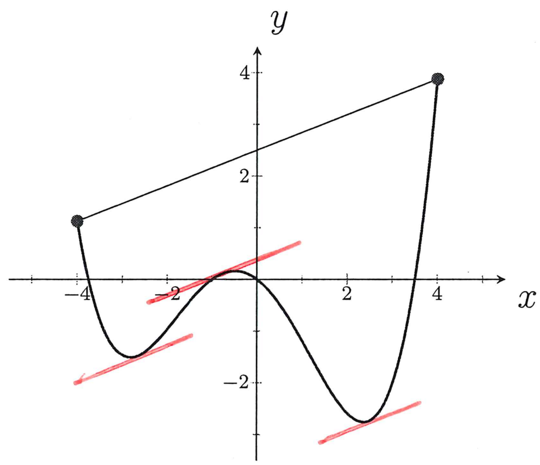
diff?

$$f(x) = \begin{cases} x - 1, & x \ge 1 \\ -(x-1), & x < 1 \end{cases}$$

$$f'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases} \Rightarrow \text{Not diff @ } x = 1$$

Example. Mean Value Theorem and graphs

Locate all points on the graph at which the slope of the tangent line equals the secant line on the interval $[-4, 4]$.



Example. Find the number that satisfies the hypotheses of the Mean Value Theorem for $f(x) = \sqrt{x}$ on $[0, 4]$. Graph the function, the secant line through the endpoints, and the tangent line at c, visually verify that the secant and tangent are parallel.

① cont ✓

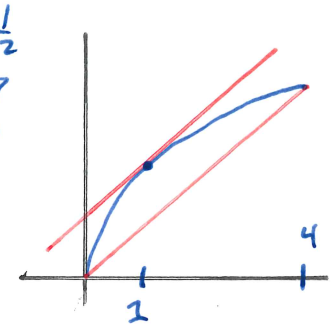
② diff $\Rightarrow f'(x) = \frac{1}{2\sqrt{x}},\ x \neq 0$

$\Rightarrow$ diff on $(0,4)$ ✓

Solve $f'(x) = \frac{f(4)-f(0)}{4-0} = \frac{1}{2}$

$\frac{1}{2\sqrt{x}} = \frac{1}{2}$

$\boxed{x = 1}$



Example. Drag racer acceleration

The fastest drag racers can reach a speed of 330 mi/hr over a quarter-mile strip in 5 secs (from a standing start). Complete the following sentence about such a drag racer: At some point during the race, the maximum acceleration of the drag racer is at least _______ $mi/hr/s$.

$\left.\begin{array}{l} f(0) = 0 \text{ mph} \\ f(5) = 330 \text{ mph} \end{array}\right\}$

$\frac{f(5)-f(0)}{5-0} = \frac{330}{5} \frac{\text{mph}}{\text{s}}$

$= \boxed{66 \text{ mph/s}}$

Example. A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to another officer 35 miles from the on-ramp. When the car reached the location of the second officer, 30 minutes later, it was clocked going 60 mph. The driver of the car was given a ticket for exceeding the $65 - mph$ speed limit. Why can the officer conclude that the driver exceeded the speed limit?

$\left.\begin{array}{l} f(0) = 0 \text{ mi} \\ f(1/2) = 35 \text{ mi} \end{array}\right\}$

$\frac{f(1/2)-f(0)}{1/2-0} = 70 \text{ mph}$

$\Rightarrow$ At some point, the car had to be going at least 70 mph.