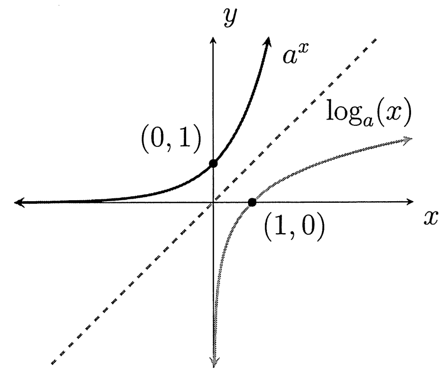
3.9: Derivatives of Logarithmic and Exponential Functions

Recall that $y = \log_a(x)$ and $y = a^x$ are inverse functions:

Inverse Properites of a^x and $\log_a(x)$

1. $a^{\log_a(x)} = x$, for $x > 0$, and $\log_a(a^x) = x$, for all x.
2. $y = \log_a(x)$ if and only if $x = a^y$.
3. For real numbers x and $b > 0$, $b^x = a^{\log_a(b^x)} = a^{x\log_a(b)}$.

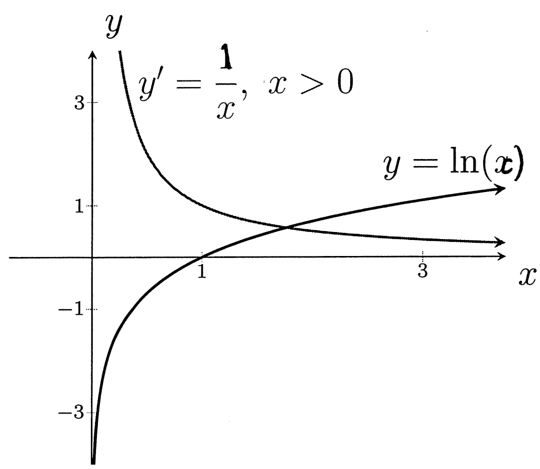


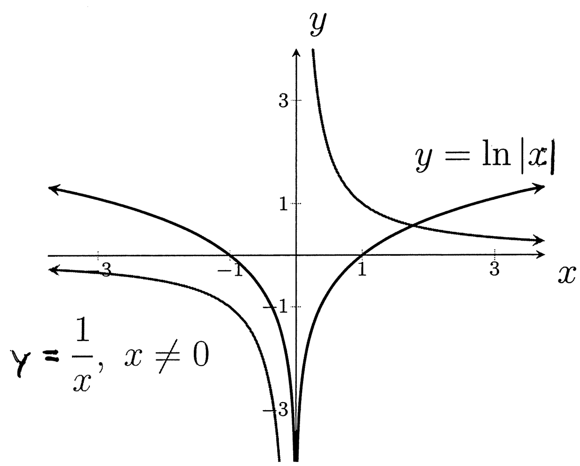
Theorem 3.15: Derivative of $\ln(x)$.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \text{ for } x > 0 \qquad \frac{d}{dx}(\ln|x|) = \frac{1}{x}, \text{ for } x \neq 0$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}$$





Example. Use implicit differentiation to prove $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Then, use the piecewise definition of $|x|$ to prove that $\frac{d}{dx}\ln|x| = \frac{1}{x}$.

$y = \ln(x)$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\boxed{\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}}$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\ln|x| = \begin{cases} \ln(x), & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

$\boxed{\frac{d}{dx}\left[\ln|x|\right] = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x}, & x < 0 \end{cases} = \frac{1}{x}.}$

Example. Find the derivatives of the following functions:

$y = \ln(x)$

$\boxed{y' = \frac{1}{x}}$

$y = \ln(4x)$

$\boxed{y' = \frac{1}{4x}(4) = \frac{1}{x}}$

-or-

$y = \ln(4) + \ln(x)$

$\boxed{y' = \frac{1}{x}}$

$y = \ln\left(4x^2 + 2\right)$

$\boxed{y' = \frac{1}{4x^2+2}(8x)}$

$f(x) = \sqrt{x}\ln(x^2)$

$f'(x) = \frac{1}{2}x^{-1/2}\ln(x^2) + \sqrt{x}\,\frac{1}{x^2}(2x)$

$= \frac{\ln(x^2)}{2\sqrt{x}} + \frac{2\sqrt{x}}{x}$

-or-

$\ln(x^2) = 2\ln(x)$

$f(x) = \ln\left(\frac{10}{x}\right)$

$\boxed{f'(x) = \frac{1}{10x^{-1}}(-10x^{-2}) = -\frac{1}{x}}$

-or-

$f(x) = \ln(10) - \ln(x)$

$\boxed{f'(x) = -\frac{1}{x}}$

$f(x) = \frac{\ln(x)}{1+\ln(x)}$

$f'(x) = \frac{[1+\ln(x)]\frac{1}{x} - [\ln(x)]\frac{1}{x}}{[1+\ln(x)]^2}$

$= \frac{1}{x[1+\ln(x)]^2}$

$\rightarrow \boxed{f'(x) = x^{-1/2}\ln(x^2) + 2\sqrt{x}\,\frac{1}{x}}$

$$f(x) = \sqrt[5]{\ln(3x^4)} = \left(\ln(3x^4)\right)^{1/5}$$

$$f'(x) = \frac{1}{5}\left(\ln(3x^4)\right)^{-4/5} \frac{1}{3x^4}(12x^3)$$

$$= \boxed{\frac{4}{5x\left(\ln(3x^4)\right)^{4/5}}}$$

$$f(x) = \ln(\ln(\ln(4x)))$$

$$f'(x) = \frac{1}{\ln(\ln(4x))} \cdot \frac{1}{\ln(4x)} \cdot \frac{1}{4x} \cdot (4)$$

$$= \frac{1}{x \ln(4x) \ln(\ln(4x))}$$

$$y = \ln\left(\sec^2\theta\right)$$

$$y' = \frac{1}{\sec^2\theta}(2\sec\theta)(\sec\theta\tan\theta)$$

$$= \boxed{2\tan\theta}$$

- or -

$$y = 2\ln(\sec\theta)$$

$$y' = \frac{2}{\sec\theta}\sec\theta\tan\theta = \boxed{2\tan\theta}$$

$$f(x) = \ln\sqrt[5]{3x} = \ln(3x)^{1/5} = \frac{1}{5}\ln(3x)$$

$$f'(x) = \frac{1}{5} \cdot \frac{1}{3x}(3) = \boxed{\frac{1}{5x}}$$

$$f(x) = \ln\left|x^2 - 1\right|$$

$$f'(x) = \frac{1}{x^2-1}(2x) = \frac{2x}{x^2-1}$$

$$y = (\ln(\sin(3x)))^2$$

$$y' = 2(\ln(\sin(3x)))\frac{1}{\sin(3x)} \cdot \cos(3x)(3)$$

$$= 6\cot(3x)\ln(\sin(3x))$$

Theorem 3.16: Derivative of b^x.
If $b > 0$ and $b \neq 1$, then for all x.

$$\frac{d}{dx}[b^x] = b^x \ln(b).$$

Example. Using the properties of exponents and logarithms, prove the above theorem. Extend this theorem by stating the derivative of $y = b^{f(x)}$.

$$y = b^x = e^{\ln(b^x)} = e^{x \ln(b)}$$

$$y' = \underbrace{e^{x\ln(b)}}_{b^x} \ln(b) = \boxed{b^x \ln(b)}$$

$$y = b^{f(x)}$$

$$\boxed{y' = b^{f(x)} \ln(b) \, f'(x)}$$

Example. Find the derivatives of the following functions:

$y = 5^{3x}$

$$y' = 5^{3x} \ln(5)\,(3)$$

$$\boxed{\text{Note: } 3 \cdot 5^{3x} \neq 15^{3x}}$$

$s(t) = \cos(2^t)$

$$s'(t) = -\sin(2^t) \cdot 2^t \ln(2)$$

$g(v) = 10^v(\ln(10^v) - 1)$

$$g'(v) = 10^v \ln(10)(\ln(10^v) - 1) + 10^v\left(\frac{1}{10^v} 10^v \ln(10)\right)$$

$$= 10^v \ln(10)(\ln(10^v) - 1 + 1)$$

$$= v\, 10^v (\ln(10))^2.$$

$y = 6^{x\ln(x)}$

$$y' = 6^{x\ln(x)} \cdot \ln 6\left((1)\ln(x) + x\frac{1}{x}\right)$$

$$= \boxed{6^{x\ln(x)} \ln(6)(\ln(x) + 1)}$$

Theorem 3.18: Derivative of $\log_b(x)$.
If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln b}, \text{ for } x > \text{ and } \frac{d}{dx}[\log_b |x|] = \frac{1}{x \ln b}, \text{ for } x \neq 0.$$

Example. Using the properties of exponents and logarithms, prove the above theorem. Extend this theorem by stating the derivative of $y = \log_b(g(x))$.

$$y = \log_b(x)$$

$$b^y = x$$

$$b^y \ln(b) \frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{b^y \ln(b)} = \frac{1}{x \ln(b)}}$$

$$y = \log_b(x) = \frac{\ln(x)}{\ln(b)}$$

$$\boxed{y' = \frac{1}{x} \cdot \frac{1}{\ln(b)}}$$

$$y = \log_b(g(x))$$

$$\boxed{y' = \frac{g'(x)}{g(x) \ln(b)}}$$

Example. Find the derivatives of the following functions:

$$f(x) = \log_4(4x^2 + 3x)$$

$$\boxed{f'(x) = \frac{8x+3}{\ln(4)[4x^2+3x]}}$$

$$f(x) = \log_5(xe^x)$$

$$\boxed{f'(x) = \frac{e^x + xe^x}{\ln(5)\, xe^x}}$$

$$y = 2x \log_{10} \sqrt{x} = 2x \log_{10} x^{1/2}$$

$$y' = (2)\log_{10}\sqrt{x} + 2x \frac{1}{\ln(10)\sqrt{x}} \frac{1}{2} x^{-1/2}$$

$$= \boxed{2\log_{10}\sqrt{x} + \frac{1}{\ln(10)}}$$

-or-

$$y = x \log_{10} x \rightarrow \boxed{y' = \log_{10} x + \frac{x}{\ln(10)x}}$$

$$y = \frac{\log_3(\tan(e^2x))}{\pi \cdot e^{-4x}}$$

$$y' = \frac{\pi e^{-4x}\left(\frac{\sec^2(e^2x)\, e^2}{\ln(3)\tan(e^2x)}\right) - \left(\log_3(\tan(e^2x))\right)\left(\pi e^{-4x}(-4)\right)}{(\pi e^{-4x})^2}$$

$$= \frac{\frac{e^2 \sec(e^2x)\csc(e^2x)}{\ln(3)} + 4\log_3(\tan(e^2x))}{\pi e^{-4x}}$$

Derivative rules for exponential functions:

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}[b^x] = \ln(b) \cdot b^x \qquad \frac{d}{dx}\left[b^{g(x)}\right] = \ln(b) \cdot b^{g(x)} \cdot g'(x)$$

Derivative rules for logarithmic functions:

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad \frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{\ln(b)x} \qquad \frac{d}{dx}[\log_a(g(x))] = \frac{g'(x)}{\ln(b)g(x)}$$

Laws of Logarithms

For $x, y > 0$:

$$\log_a(xy) = \log_a(x) + \log_a(y) \qquad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(x)$$

$$\log_a(x^r) = r\log_a(x) \qquad \log_a(1) = 0$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Example. For the following functions, use the laws of logarithms to rewrite the function before taking the derivative:

$$F(t) = \ln\left(\frac{(2t+1)^3}{(3t+1)^4}\right) = \ln(2t+1)^3 + \ln(3t+1)^4 = 3\ln(2t+1) + 4\ln(3t+1)$$

$$\boxed{F'(t) = 3\,\frac{2}{2t+1} + 4\,\frac{3}{3t+1}}$$

$$y = \ln\sqrt[3]{\frac{1+x}{1-x}} = \frac{1}{3}\left(\ln(1+x) - \ln(1-x)\right)$$

$$y' = \frac{1}{3}\left(\frac{1}{1+x} - \frac{1}{1-x}(-1)\right) = \frac{1}{3}\left(\frac{(1-x)+(1+x)}{1-x^2}\right) = \boxed{\frac{2}{3(1-x^2)}}$$

$$y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \ln(x+1)^{5/2} - \ln(x+2)^{10}$$

$$\boxed{y' = \frac{5}{2}\left(\frac{1}{x+1}\right) - 10\left(\frac{1}{x+2}\right)}$$

Logarithmic Differentiation:

1. Take the natural logarithm of both sides of the equation.
2. Use logarithm laws to simplify.
3. Use implicit differentiation to take the derivative of both sides.
4. Solve for $\frac{dy}{dx}$.

Example. Find the derivatives of the following functions:

$$y = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}$$

$$\ln y = 2\ln(\sin(x)) + 4\ln(\tan(x)) - 2\ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{\cos(x)}{\sin(x)} + 4\frac{\sec^2(x)}{\tan(x)} - 2\frac{2x}{x^2+1}$$

$$\frac{dy}{dx} = y\left(2\cot(x) + 4\sec(x)\csc(x) - \frac{4x}{x^2+1}\right)$$

$$= \boxed{\frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}\left(2\cot(x) - 4\sec(x)\csc(x) - \frac{4x}{x^2+1}\right)}$$

$$h(\theta) = \frac{\theta\sin(\theta)}{\sqrt{\sec(\theta)}}$$

$$\ln(h(\theta)) = \ln(\theta) + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta)$$

$$\frac{h'(\theta)}{h(\theta)} = \frac{1}{\theta} + \cot\theta - \frac{1}{2}\frac{\sec\theta\tan\theta}{\sec\theta}$$

$$h'(\theta) = h(\theta)\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]$$

$$\boxed{= \frac{\theta\sin(\theta)}{\sqrt{\sec(\theta)}}\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]}$$

$$g(x) = \sqrt[10]{\frac{3x+4}{2x-4}}$$

$$\ln(g(x)) = \frac{1}{10}\ln(3x+4) - \frac{1}{10}\ln(2x-4)$$

$$\frac{g'(x)}{g(x)} = \frac{1}{10}\frac{3}{3x+4} - \frac{1}{10}\frac{2}{2x-4}$$

$$\boxed{g'(x) = \sqrt[10]{\frac{3x+4}{2x-4}}\left(\frac{3}{10(3x+4)} - \frac{1}{5(2x-4)}\right)}$$

$$y = \frac{e^{-x}\cos^2(x)}{x^2+x+1}$$

$$\ln(y) = -1 + 2\ln(\cos(x)) - \ln(x^2+x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{-\sin(x)}{\cos(x)} - \frac{2x+1}{x^2+x+1}$$

$$\frac{dy}{dx} = \frac{e^{-x}\cos^2(x)}{x^2+x+1}\left[-2\cot(x) - \frac{2x+1}{x^2+x+1}\right]$$

Note: Whenever the function is of the form $f(x)^{g(x)}$, then *Logarithmic Differentiation* is the only option!

$$y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = (1)\ln(x) + x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left(\ln(x) + 1\right)$$

$$= \boxed{x^x(\ln(x)+1)}$$

$$y = (\ln(x))^x$$

$$\ln y = x \ln(\ln(x))$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln(x)) + x\frac{1}{\ln(x)}\cdot\frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = (\ln(x))^x\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right)}$$

$$y = (\tan x)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln(\tan x)$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\ln(\tan(x)) + \frac{1}{x}\frac{\sec^2(x)}{\tan(x)}$$

$$\boxed{\frac{dy}{dx} = (\tan x)^{1/x}\left[\frac{\sec(x)\csc(x)}{x} - \frac{\ln(\tan(x))}{x^2}\right]}$$

$$y = (2x)^{3x}$$

$$\ln(y) = 3x\ln(2x)$$

$$\frac{1}{y}\frac{dy}{dx} = 3\ln(2x) + 3x\frac{2}{2x}$$

$$\boxed{\frac{dy}{dx} = (2x)^{3x}\left[3\ln(2x) + 3\right]}$$

Example. Use the definition of the derivative to evaluate the following limits:

$$\lim_{x\to e} \frac{\ln(x) - 1}{x - e}$$

$\ln(e) = 1$

$\Rightarrow \frac{d}{dx}\left[\ln(x)\right]_{x=e}$

$= \frac{1}{x}\Big|_{x=e}$

$= \boxed{\frac{1}{e}}$

$$\lim_{h\to 0} \frac{\ln(e^8 + h) - 8}{h}$$

$\ln(e^8) = 8$

$\Rightarrow \frac{d}{dx}\left[\ln(x)\right]_{x=e^8}$

$= \frac{1}{x}\Big|_{x=e^8}$

$= \boxed{\frac{1}{e^8}}$

$$\lim_{x\to 2} \frac{5^x - 25}{x - 2}$$

$5^2 = 25$

$\Rightarrow \frac{d}{dx}\left[5^x\right]_{x=2}$

$= \ln(5)\, 5^x\Big|_{x=2}$

$= \boxed{\ln(5)\, 25}$

$$\lim_{h\to 0} \frac{(3 + h)^{3+h} - 27}{h}$$

$f(x) = x^x \qquad f(3) = 27$

$\Rightarrow \quad y = x^x \rightarrow \ln(y) = x\ln(x)$

$\frac{1}{y}\frac{dy}{dx} = x\ln(x)$

$\frac{dy}{dx} = x^x\left(\ln(x) + x\frac{1}{x}\right)$

$\boxed{\frac{dy}{dx}\Big|_{x=3} = 3^3\left(\ln(3) + 1\right) = 27\left(\ln(3) + 1\right)}$