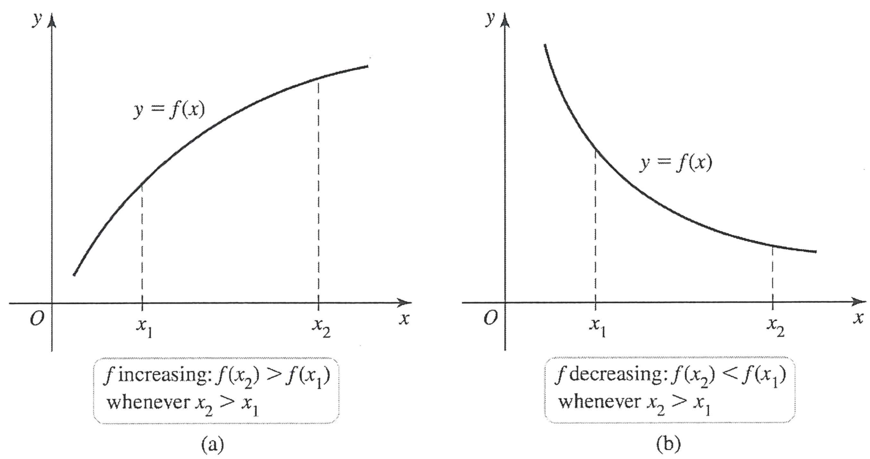
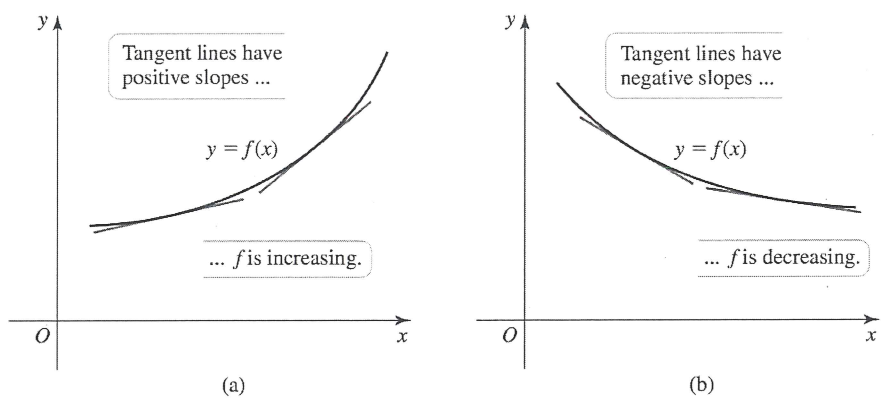
## 4.3 What Derivatives Tell Us

**Definition. (Increasing and Decreasing Functions)**

Suppose a function $f$ is defined on an interval $I$. We say that $f$ is increasing on $I$ if $f(x_2) > f(x_1)$ whenever $x_1$ and $x_2$ are in $I$ and $x_2 > x_1$. We say that $f$ is decreasing on $I$ if $f(x_2) < f(x_1)$ whenever $x_1$ and $x_2$ are in $I$ and $x_2 > x_1$.




**Theorem 4.7: Test for Intervals of Increase and Decrease**
Suppose $f$ is continuous on an interval $I$ and differentiable at every interior point of $I$. If $f'(x) > 0$ at all interior points of $I$, then $f$ is increasing on $I$. If $f'(x) < 0$ at all interior points of $I$, then $f$ is decreasing on $I$.



*Proof.* **(Theorem 4.7: Test for Intervals of Increase and Decrease) p258**
Let $a$ and $b$ be any two distinct points in the interval $I$ with $b > a$. By the Mean Value Theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

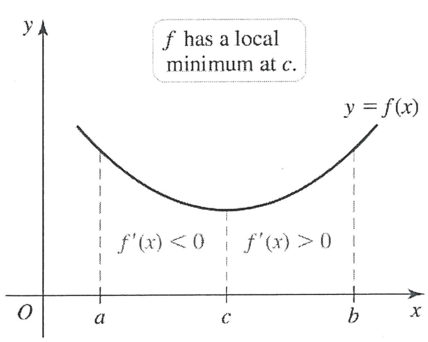
for some $c$ between $a$ and $b$. Equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

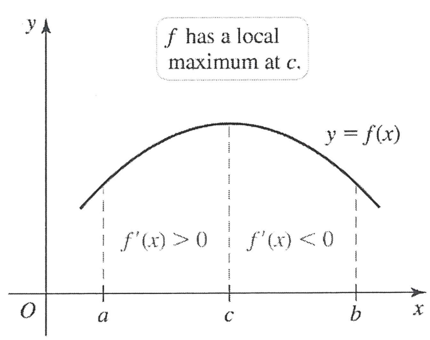
Notice that $b - a > 0$ by assumption. So if $f'(c) > 0$, then $f(b) - f(a) > 0$. Therefore, for all $a$ and $b$ in $I$ with $b > a$, we have $f(b) > f(a)$, which implies that $f$ is increasing on $I$. Similarly, if $f'(c) < 0$, then $f(b) - f(a) < 0$ or $f(b) < f(a)$. It follows that $f$ is decreasing on $I$. □

**Theorem 4.8: First Derivative Test**
Assume that $f$ is continuous on an interval that contains a critical point $c$ and assume $f$ is differentiable on an interval containing $c$, except perhaps at $c$ itself.

- If $f'$ changes sign from positive to negative as $x$ increases through $c$, then $f$ has a local maximum at $c$.
- If $f'$ changes sign from negative to positive as $x$ increases through $c$, then $f$ has a local minimum at $c$.
- If $f'$ does not change sign at $c$ (from positive to negative or vice versa), then $f$ has no local extreme value at $c$.



(a)



(b)

**Example.** Consider the function $f(x) = 6x - x^3$.

a) $f'(x) = 6 - 3x^2$

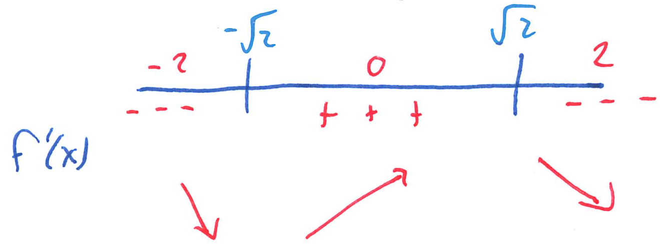
b) Find the intervals on which the function is increasing and decreasing.

Find $f'(x) = 0$

$6 - 3x^2 = 0$

$\frac{1}{2} = x^2$

$\pm\sqrt{2} = x$




dec $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

inc $(-\sqrt{2}, \sqrt{2})$

c) Identify the function's local extreme values, if any. (e.g. "local max of __ at $x =$ __")

$f(-\sqrt{2}) = -6\sqrt{2} + 2\sqrt{2}$

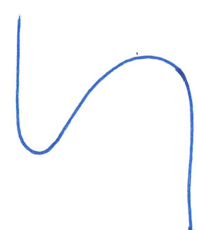
$f(\sqrt{2}) = 6\sqrt{2} - 2\sqrt{2}$

local min of $-4\sqrt{2}$ at $-\sqrt{2}$

local max of $4\sqrt{2}$ at $\sqrt{2}$

d) Which, if any, of the extreme values are absolute?

Neither, unbounded interval



**Example.** Consider the function $f(t) = 12t - t^3$ on $-3 \leq t < \infty$.

a) $f'(t) =$ $12 - 3t^2$

b) Find the intervals on which the function is increasing and decreasing.

$f'(t) = 0 \Rightarrow 12 - 3t^2 = 0$

$4 = t^2$

$\pm 2 = t$

-3, -2, 0, 2, 3

--- +++ ---

dec $[-3,-2) \cup (2,\infty)$

inc $(-2,2)$

c) Identify the function's local extreme values, if any. (e.g. "local max of __ at $x =$ __")

$f(-2) = -16$

$f(2) = 16$

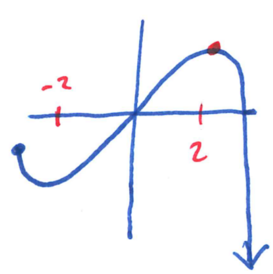
local min of -16 at $x = -2$

local max of 16 at $x = 2$

d) Which, if any, of the extreme values are absolute?

$f(-3) = 9$

| $x$ | $f(x)$ |
|---|---|
| -3 | 9 |
| -2 | -16 |
| 2 | 16 |



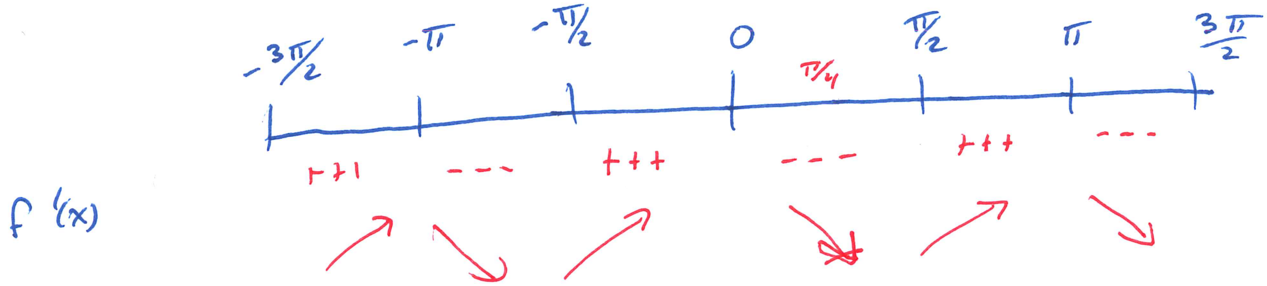

$x = 2$ is local max AND ABS max

No ABS Min

**Example.** Consider the function $f(x) = \cos^2(x)$ on $[-\pi, \pi]$. Find the intervals on which $f$ is increasing and the intervals on which it is decreasing.

$$f'(x) = -2\underbrace{\cos(x)}_{\text{solve}}\ \underbrace{\sin(x)}_{\text{solve}} \overset{\text{set}}{=} 0$$

$$\left.\begin{array}{lll} \cos(x) = 0 \iff & x = \frac{\pi}{2} + k\pi, & k \text{ integer} \\ \sin(x) = 0 \iff & x = k\pi, & k \text{ integer} \end{array}\right\} x = \frac{k\pi}{2}$$



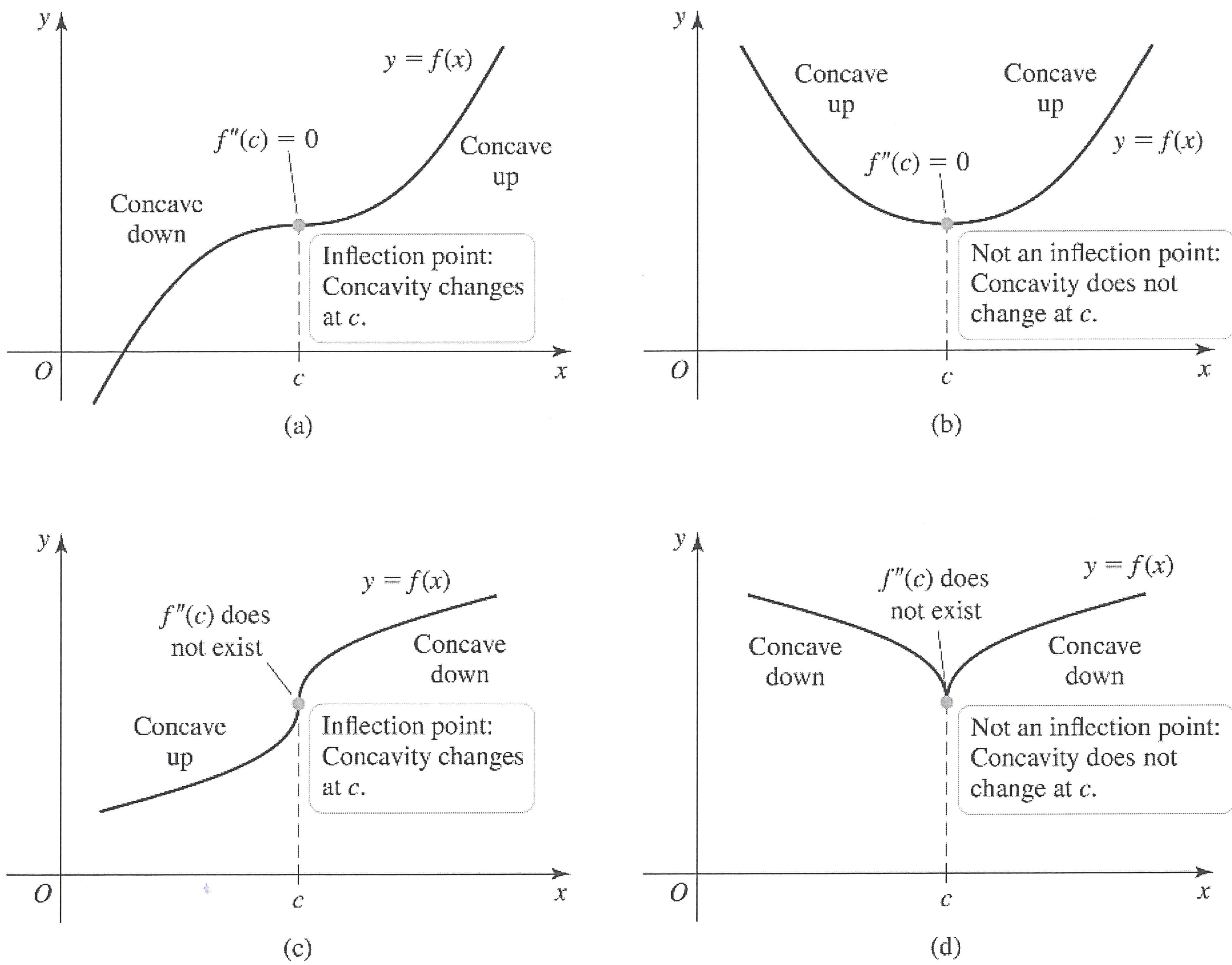

$\Rightarrow$ dec: $\ldots \cup (-\pi, -\pi/2) \cup (0, \pi/2) \cup (\pi, 3\pi/2) \cup \ldots$

inc: $\ldots \cup (-\frac{3\pi}{2}, -\pi) \cup (-\pi/2, 0) \cup (\pi/2, \pi) \cup \ldots$

**Definition. (Concavity and Inflection Point)**

Let $f$ be differentiable on an open interval $I$.

- If $f'$ is increasing on $I$, then $f$ is *concave up* on $I$.
- If $f'$ is decreasing on $I$, then $f$ is *concave down* on $I$.
- If $f$ is continuous at $c$ and $f$ changes concavity at $c$, then $f$ has an *inflection point* at $c$.



**Theorem 4.10: Test for Concavity**
Suppose $f''$ exists on an open interval $I$.

- If $f'' > 0$ on $I$, then $f$ is concave up on $I$.
- If $f'' < 0$ on $I$, then $f$ is concave down on $I$.
- If $c$ is a point of $I$ at which $f''$ changes sign at $c$, then $f$ has an inflection point at $c$.

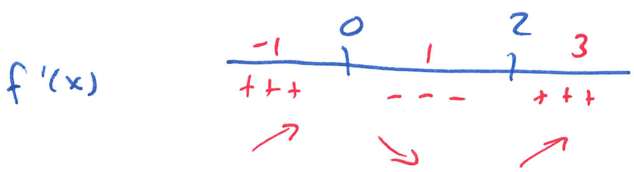
**Example.** Consider $f(x) = 5 - 3x^2 + x^3$

a) Find the intervals of increasing/decreasing and local max/min.

$f'(x) = -6x + 3x^2 \overset{set}{=} 0 \Rightarrow x = 0, x = 2$

local min $(2, 1)$

local max $(0, 5)$

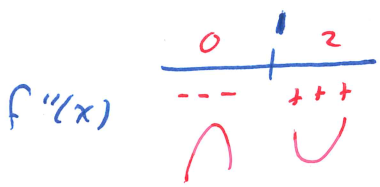



inc $(-\infty, 0) \cup (2, \infty)$

dec $(0, 2)$

b) Find intervals of concavity and inflection points.

$f''(x) = -6 + 6x \overset{set}{=} 0 \Rightarrow x = 1$



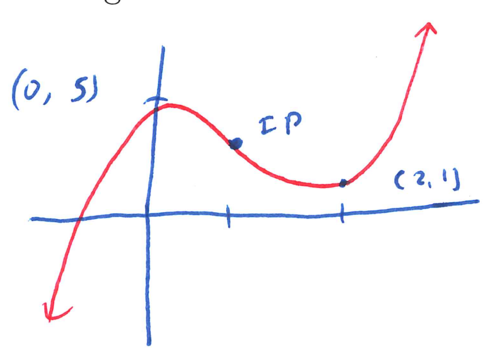

$f(1)$

Inflection point $(1, 3)$

Concave up $(1, \infty)$

Concave down $(-\infty, 1)$

c) Draw a rough sketch of the function.



**Example.** Consider $f(x) = xe^{-x^2/2}$

a) Find the intervals of increasing/decreasing and local max/min.

$$f'(x) = e^{-x^2/2} + x e^{-x^2/2}(-x) = e^{-x^2/2}(1-x^2) \overset{set}{=} 0 \implies x = \pm 1$$

| | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f'(x)$ | $---$ | | $+++$ | | $---$ |
| | ↘ | | ↗ | | ↘ |

Inc: $(-1, 1)$ dec $(-\infty, -1) \cup (1, \infty)$

local min $(-1, -e^{-1/2})$

local max $(1, e^{-1/2})$

b) Find intervals of concavity and inflection points.

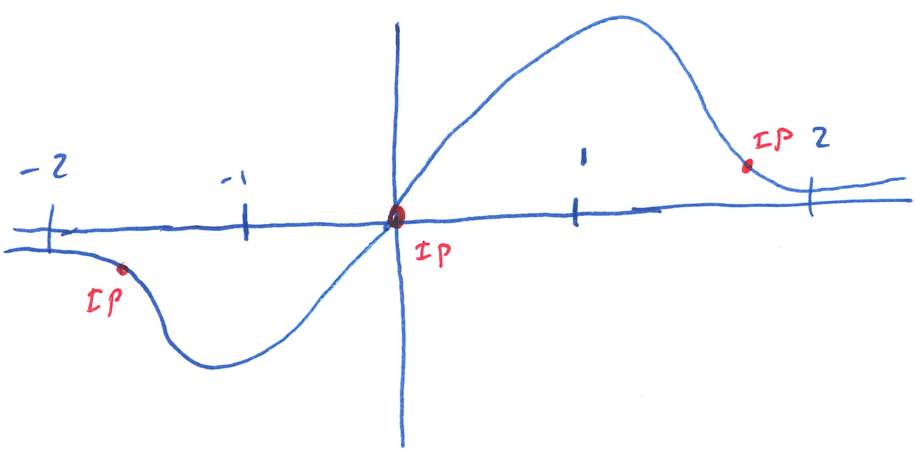
$$f''(x) = -x e^{-x^2/2}(1-x^2) - 2x e^{-x^2/2} = x e^{-x^2/2}(x^2-3) \overset{set}{=} 0$$

$$\implies x = 0, \quad x = \pm\sqrt{3}$$

IP @ $x = 0, \pm\sqrt{3}$

| | $-2$ | $-\sqrt{3}$ | $-1$ | $0$ | $1$ | $\sqrt{3}$ | $2$ |
|---|---|---|---|---|---|---|---|
| $f''(x)$ | $---$ | | $+++$ | | $---$ | | $+++$ |
| | $\cap$ | | $\cup$ | | $\cap$ | | $\cup$ |

c) Draw a rough sketch of the function.



**Example.** Consider $f(x) = x\left(\sqrt[3]{(x-3)^2}\right)$

a) Find the intervals of increasing/decreasing and local max/min.

$$f'(x) = (x-3)^{2/3} + \frac{2x}{3(x-3)^{1/3}} \overset{\text{set}}{=} 0 \Rightarrow (x-3) + \frac{2}{3}x = 0$$

$$\Rightarrow x = 9/5, \; x \neq 3$$

$f'(x)$: test points 0, 2, 4; marks $9/5$, $3$; signs $+++$ | $---$ | $+++$

inc $(-\infty, 9/5) \cup (3, \infty)$

dec $(9/5, 3)$

local max $\left(9/5, \; \frac{9}{5}\left(\frac{6}{5}\right)^{2/3}\right)$

local min $(3, 0)$

b) Find intervals of concavity and inflection points.

$$f''(x) = \frac{2}{3(x-3)^{1/3}} + \frac{2}{9}\,\frac{2x-9}{(x-3)^{4/3}} \overset{\text{set}}{=} 0 \Rightarrow x = 18/5$$

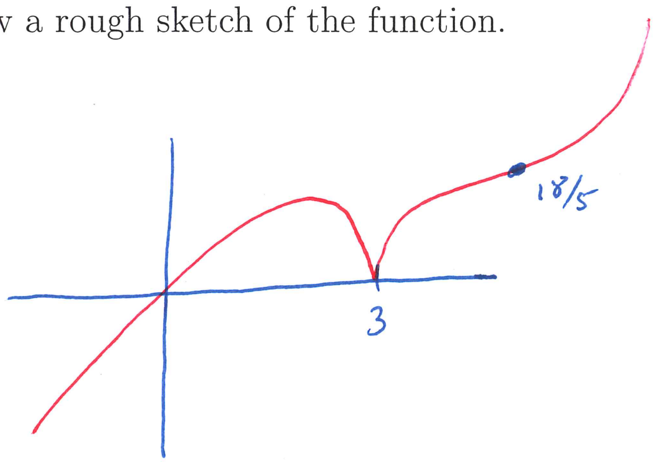
$$x \neq 3$$

$f''(x)$: test points 0, $\frac{16}{5}$, 4; marks $3$, $18/5$; signs $---$ | $---$ | $+++$

Concave down $(-\infty, 3) \cup (3, 18/5)$

concave up $(18/5, \infty)$

c) Draw a rough sketch of the function.



$x > 0$

**Example.** Consider $f(x) = x^2 - x - \ln(x)$

a) Find the intervals of increasing/decreasing and local max/min.

$f'(x) = 2x - 1 - \frac{1}{x} \overset{\text{set}}{=} 0 \Rightarrow 2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{2}, 1$

Not in domain

$f'(0)$ DNE $\quad x \neq 0$

| | 1/2 | 2 |
|---|---|---|
| $f'(x)$ | $- - -$ | $+ + +$ |
| | ↘ | ↗ |

Dec $(0, 1)$

Inc $(1, \infty)$

local min $(1, 0)$

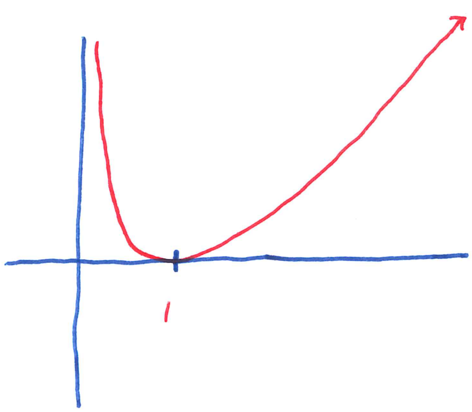
b) Find intervals of concavity and inflection points.

$f''(x) = 2 + \frac{1}{x^2} \overset{\text{set}}{=} 0 \Rightarrow$ This function is never zero and is positive everywhere

$\Rightarrow$ No inflection point

$\Rightarrow$ Concave up on $(-\infty, \infty)$

c) Draw a rough sketch of the function.



**Example.** Given the first derivative $y' = (x-2)^{-1/3}$

a) Find the intervals of increasing/decreasing and local max/min.

Solve $y' = 0 \Rightarrow \dfrac{1}{(x-2)^{1/3}} = 0 \leftarrow$ Never zero.

$x \neq 2$

$y'$: $- - -$ | $+ + +$ (test points 0, 3; critical point 2)

local min at $x = 2$

dec $(-\infty, 2)$

inc $(2, \infty)$

b) Find intervals of concavity and inflection points.

$y'' = -\frac{1}{3}(x-2)^{-4/3} \leftarrow$ Never zero.

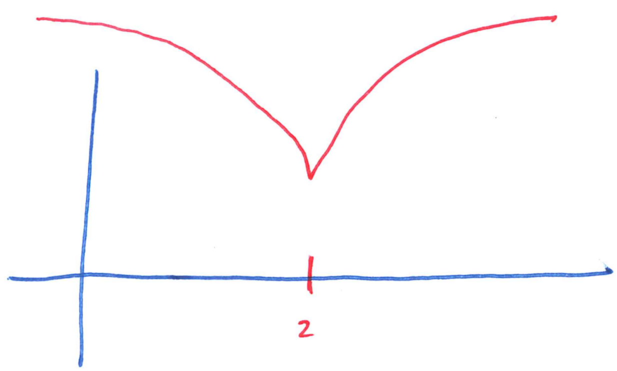
$x \neq 2$

$y''$: $- - -$ | $- - -$ (test points 0, 3; point 2)

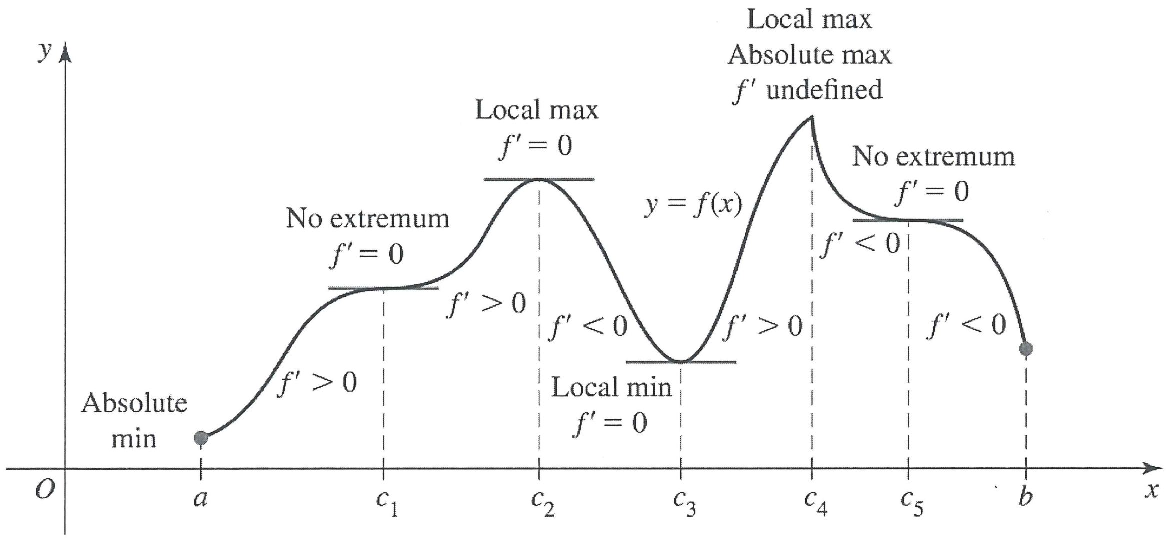
No change in sign

Concave down on $(-\infty, 2) \cup (2, \infty)$

c) Draw a rough sketch of the function.



As a visual summary, here is figure 4.27 from Briggs:

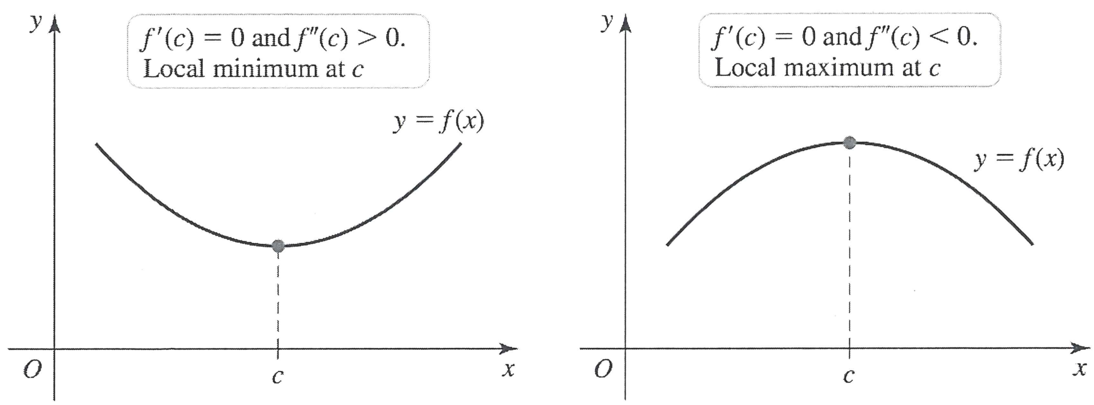



| $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|
| increasing | positive | — |
| decreasing | negative | — |
| max | zero (pos to neg) | — |
| min | zero (neg to pos) | — |
| concave up | increasing | positive |
| concave down | decreasing | negative |
| Inflection point | max/min | changes sign |

**Theorem 4.11: Second Derivative Test for Local Extrema**

Suppose f" is continuous on an open interval containing c with f'(c)=0.

- If $f''(c) > 0$, then $f$ has a local minimum at $c$ (Figure 4.40a).
- If $f''(c) < 0$, then $f$ has a local maximum at $c$ (Figure 4.40b).
- If $f''(c) = 0$, then the test is inconclusive; $f$ may have a local maximum, local minimum, or neither at $c$.




(a) (b)

**Example.** For each of the following, use the Second Derivative Test for Extrema to determine local max/mins:

a) $f(x) = x^5 - 5x + 3$

$f'(x) = 5x^4 - 5 \overset{set}{=} 0$

$\Rightarrow x = \pm 1$

$f''(x) = 20x^3$

$f''(-1) = -20 < 0$

local max $(-1, 7)$

$f''(1) = 20 > 0$

local min $(1, -1)$

b) $f(x) = 2x^3 - 3x^2 + 12$

$f'(x) = 6x^2 - 6x \overset{set}{=} 0$

$\Rightarrow x = -1, 0, 1$

$f''(x) = 12x - 6$

| $x$ | $f''(x)$ | |
|---|---|---|
| $-1$ | $-18$ | local max $(-1, 7)$ |
| $0$ | $-6$ | local max $(0, 12)$ |
| $1$ | $6$ | local min $(1, 11)$ |

c) $f(x) = x^3 - 6$

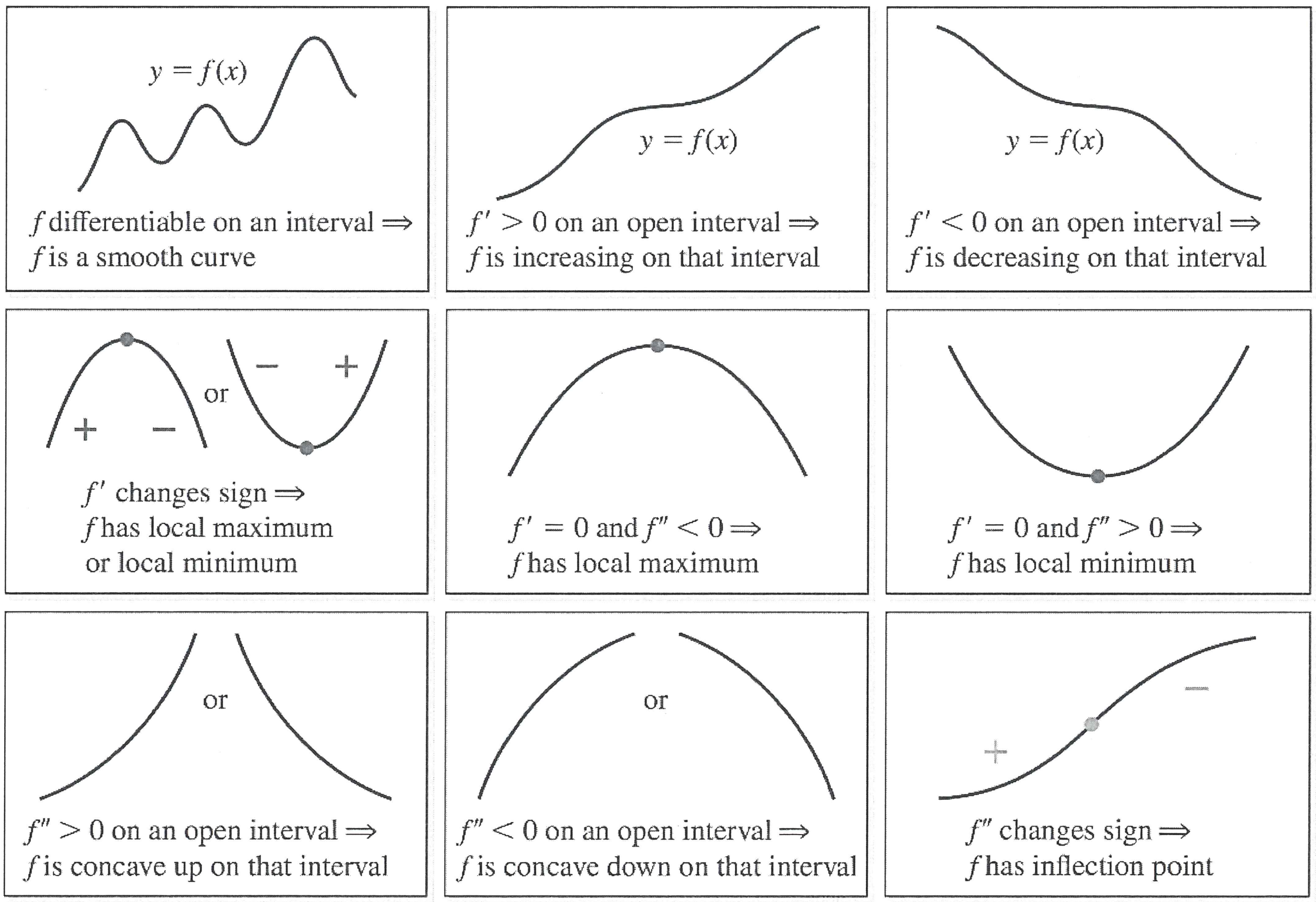
$f'(x) = 3x^2 \overset{set}{=} 0$

$\Rightarrow x = 0$

$f''(x) = 6x$

$f''(0) = 0$

$\Rightarrow$ Inconclusive.


$y = f(x)$
$f$ differentiable on an interval $\Rightarrow$ $f$ is a smooth curve
$y = f(x)$
$f' > 0$ on an open interval $\Rightarrow$ $f$ is increasing on that interval
$y = f(x)$
$f' < 0$ on an open interval $\Rightarrow$ $f$ is decreasing on that interval
$+$ $-$ or $-$ $+$
$f'$ changes sign $\Rightarrow$ $f$ has local maximum or local minimum
$f' = 0$ and $f'' < 0 \Rightarrow$ $f$ has local maximum
$f' = 0$ and $f'' > 0 \Rightarrow$ $f$ has local minimum
or
$f'' > 0$ on an open interval $\Rightarrow$ $f$ is concave up on that interval
or
$f'' < 0$ on an open interval $\Rightarrow$ $f$ is concave down on that interval
$+$ $-$
$f''$ changes sign $\Rightarrow$ $f$ has inflection point