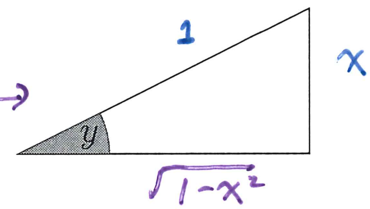
3.10: Derivatives of Inverse Trigonometric Functions

Example. Recall that $y = \sin^{-1}(x) \iff \sin(y) = x$. Use this fact and implicit differentiation to derive the derivative of $\sin^{-1}(x)$.

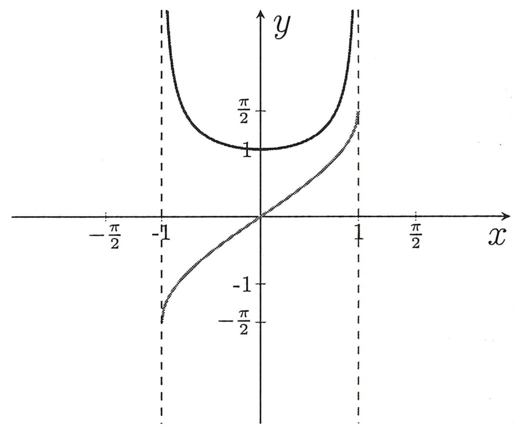
$y = \sin^{-1}(x)$

$\sin(y) = x$

$\cos(y)\dfrac{dy}{dx} = 1$



$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}/1} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$



Note:

$\sin^2(y) + \cos^2(y) = 1$

$\Rightarrow \cos(y) = \sqrt{1-\sin^2(y)}$

$= \sqrt{1-x^2}$

Next, extend this definition for the derivative of $\sin^{-1}(f(x))$.

$$\frac{d}{dx}\left[\sin^{-1}(f(x))\right] = \frac{1}{\sqrt{1-(f(x))^2}}\, f'(x)$$

Example. Find the derivative of the following

$$f(x) = \sqrt{1-x^2}\,\arcsin(x) \qquad\qquad y = \sin^{-1}\left(\sqrt{2}t\right)$$

$f'(x) = \frac{1}{2}(1-x^2)^{-1/2}(-2x)\sin^{-1}(x) + \sqrt{1-x^2}\,\frac{1}{\sqrt{1-x^2}}$

$= \boxed{\dfrac{-x\sin^{-1}(x)}{\sqrt{1-x^2}} + 1}$

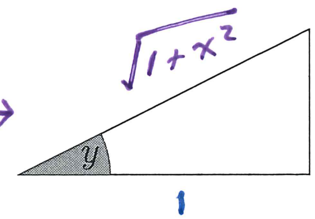
$y' = \dfrac{1}{\sqrt{1-(\sqrt{2}t)^2}}\sqrt{2}$

$= \boxed{\dfrac{\sqrt{2}}{\sqrt{1-2t^2}}}$

Example. Recall that $y = \tan^{-1}(x) \iff \tan(y) = x$. Use this fact and implicit differentiation to derive the derivative of $\tan^{-1}(x)$.

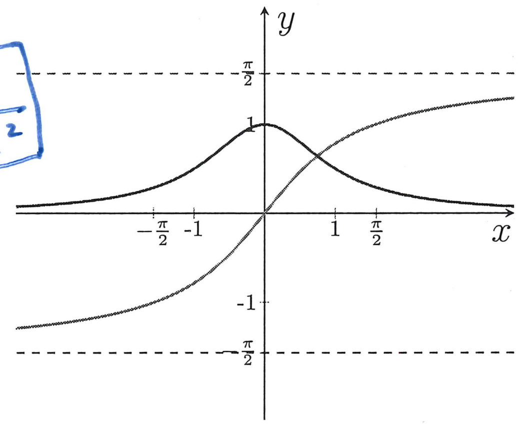
$y = \tan^{-1}(x)$

$\tan(y) = x \longrightarrow$



$\sec^2(y)\dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \boxed{\frac{1}{1+x^2}}$$



Note

$\frac{1}{\cos^2(x)}$

$\sin^2(y) + \cos^2(y) = 1$

$\underbrace{\tan^2(y)}_{x^2} + 1 = \sec^2(y)$

$\frac{1}{\cos^2(x)}$

Next, extend this definition for the derivative of $\tan^{-1}(f(x))$.

$$\frac{d}{dx}\left[\tan^{-1}(f(x))\right] = \frac{1}{1+(f(x))^2} f'(x)$$

Example. Find the derivative of the following

$y = \sqrt{\tan^{-1}(x)}$

$$y' = \frac{1}{2}\left(\tan^{-1}(x)\right)^{-1/2} \frac{1}{1+x^2}$$

$$= \boxed{\frac{1}{2(1+x^2)\sqrt{\tan^{-1}(x)}}}$$

$y = \tan^{-1}\left(\sqrt{x}\right)$

$$y' = \frac{1}{1+(\sqrt{x})^2} \frac{1}{2}x^{-1/2}$$

$$= \boxed{\frac{1}{2\sqrt{x}(1+x)}}$$

Derivatives of Inverse Trigonometric Functions

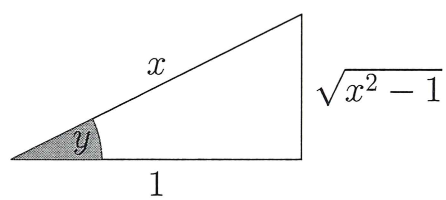
$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\csc^{-1}(x)\right] = -\frac{1}{|x|\sqrt{x^2-1}}$$

Derivative of $y = \sec^{-1}(x)$

$$\begin{aligned} y &= \sec^{-1}(x) \\ \sec(y) &= x \\ \sec(y)\tan(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec(y)\tan(y)} \end{aligned}$$



$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \tan^2(x) + 1 &= \sec^2(x) \\ 1 + \cot^2(x) &= \csc^2(x) \end{aligned}$$

Now we rewrite $\sec(y)\tan(y)$ in terms of x. Note that the domain of $\sec(y)$ is restricted to $[0, \pi/2) \cup (\pi/2, \pi]$. If we look at these quadrants of the unit circle, we see that $\sec(y)\tan(y)$ is always positive, so the resulting derivative is always positive:

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

Example. Find the derivatives of the following functions:

$$h(t) = e^{\sec^{-1}(t)}$$

$$h'(t) = \boxed{e^{\sec^{-1}(t)} \frac{1}{|t|\sqrt{t^2-1}}}$$

$$y = \arccos\left(e^{2x}\right)$$

$$y' = -\frac{1}{\sqrt{1-(e^{2x})^2}} \quad e^{2x}(2)$$

$$= \boxed{-\frac{2e^{2x}}{\sqrt{1-e^{4x}}}}$$

$$y = \sin^{-1}(2x+1)$$

$$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \quad 2$$

$$= \frac{2}{\sqrt{-4x^2-4x}}$$

$$= \boxed{\frac{1}{\sqrt{-x^2-x}}}$$

$$y = \sec^{-1}(5r)$$

$$y' = \frac{1}{|5r|\sqrt{(5r)^2-1}} \quad 5$$

$$= \boxed{\frac{1}{|r|\sqrt{25r^2-1}}}$$

$$f(x) = \csc^{-1}(\tan(e^x))$$

$$f'(x) = \boxed{-\frac{\sec^2(e^x)\cdot e^x}{|\tan(e^x)|\sqrt{(\tan(e^x))^2-1}}}$$

$$f(x) = \tan^{-1}(10x)$$

$$f'(x) = \frac{1}{1+(10x)^2} \quad 10$$

$$= \boxed{\frac{10}{1+100x}}$$

$y = x\sin^{-1}(x) + \sqrt{1-x^2}$

$$y' = \sin^{-1}(x) + x\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \boxed{\sin^{-1}(x)}$$

$h(t) = \cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right)$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+\left(\frac{1}{t}\right)^2}(-t^{-2})$$

$$= -\frac{1}{1+t^2} + \frac{1}{t^2\left(1+\frac{1}{t^2}\right)} = \boxed{0}$$

$f(x) = 2x\tan^{-1}(x) - \ln\left(1+x^2\right)$

$$f'(x) = 2\tan^{-1}(x) + \frac{2x}{1+x^2} - \frac{2x}{1+x^2}$$

$$= \boxed{2\tan^{-1}(x)}$$

$f(t) = \ln\left(\tan^{-1}(t)\right)$

$$f'(t) = \boxed{\frac{1}{\tan^{-1}(t)}\frac{1}{1+t^2}}$$

Example. Find the equation of the tangent line to $f(x) = \cos^{-1}(x^2)$ at $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$.

$$f'(x) = -\frac{1}{\sqrt{1-(x^2)^2}}2x = \frac{-2x}{\sqrt{1-x^4}}$$

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{3} \quad\Rightarrow\quad \boxed{y - \pi/3 = -\frac{2\sqrt{6}}{3}\left(x - \frac{1}{\sqrt{2}}\right)}$$

$$f'\left(\frac{1}{\sqrt{2}}\right) = \frac{-\frac{2}{\sqrt{2}}}{\sqrt{1-(1/\sqrt{2})^4}} = \frac{-\sqrt{2}}{\sqrt{3/4}} = -\frac{2\sqrt{6}}{3}$$

Example. Find the equation of the tangent line to $f(x) = \sec^{-1}(e^x)$ at $\left(\ln(2), \frac{\pi}{3}\right)$.

$$f'(x) = \frac{1}{|e^x|\sqrt{(e^x)^2-1}}$$

$$f(\ln(2)) = \pi/3 \quad\Rightarrow\quad \boxed{y - \pi/3 = \frac{\sqrt{3}}{6}\left(x - \ln(2)\right)}$$

$$f'(\ln(2)) = \frac{1}{|e^{\ln(2)}|\sqrt{(e^{\ln(2)})^2-1}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$