4.6: Linear Approximation and Differentials

Example. Find the equation of the line tangent to $y = \underbrace{1 + \sin(x)}_{f(x)}$ at $a = 0$. State the line in the slope–intercept form.

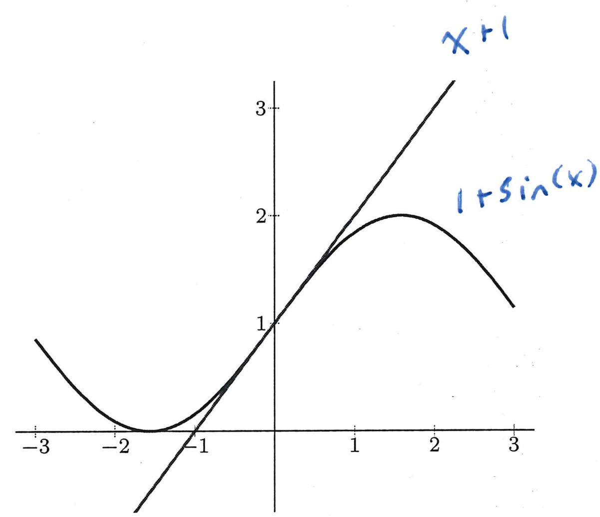
$f(0) = 1$

$y - f(a) = f'(a)(x - a)$

$f'(x) = \cos(x) \quad f'(0) = 1$

$y - 1 = 1(x - 0)$

$\boxed{y = x + 1}$



Definition.
Linear Approximation of f at a.
Suppose f if differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function.

$$L(x) = f(a) + f'(a)(x - a), \quad \text{for } x \text{ in } I.$$

Example. Consider the function $f(x) = \dfrac{x}{x+1}$. Find the linearization at $a = 1$. Use the linearization to estimate $f(1.1)$ and compare with the true value of $f(1.1)$.

$L(x) = f(a) + f'(a)(x-a)$

$f(x) = \dfrac{x}{x+1}$ $\qquad$ $f(1) = \dfrac{1}{2}$

$f'(x) = \dfrac{1}{(x+1)^2}$ $\qquad$ $f'(1) = \dfrac{1}{4}$

$\Rightarrow L(x) = \dfrac{1}{2} + \dfrac{1}{4}(x-1) = \dfrac{1}{4} + \dfrac{1}{4}x = \dfrac{1}{4}(1+x)$

$L(1.1) = \dfrac{2.1}{4} = \dfrac{21}{40} = 0,525$

$f(1.1) = \dfrac{1.1}{2.1} = \dfrac{11}{21} \approx 0.5238$

Example. Find the linearization $L(x)$ of $f(x) = e^{3x-6}$ at $a = 2$.

$f(2) = 1$

$f'(x) = 3e^{3x-6}$ $\qquad$ $f'(2) = 3$

$L(x) = 1 + 3(x-2)$

$= \boxed{3x - 5}$

Example. Find the linearization $L(x)$ of $f(x) = 9(4x + 11)^{\frac{2}{3}}$ at $a = 4$.

$f(4) = 81$

$f'(x) = 6(4x+11)^{-1/3}(4)$ $\qquad$ $f'(4) = 8$

$L(x) = 81 + 8(x-4)$

$= \boxed{8x + 49}$

Example. Find a linearization over an interval which will contain the given point x_0. Choose your center at a point *near* x_0 but not at x_0 so that the given function and its derivative are easy to evaluate. Lastly, use the linearization to approximate $f(x_0)$.

a) $f(x) = x^2 + 2x$, $x_0 = 0.1$

$f(0) = 0$ $\qquad$ $L(x) = 0 + 2(x-0) = \boxed{2x}$

$f'(x) = 2x + 2$ $\qquad$ $f'(0) = 2$ $\qquad$ $f(1.1) \approx L(1.1) = \boxed{2.2}$

b) $f(x) = \sqrt[3]{x}$, $x_0 = 8.5$.

$f(8) = 2$ $\qquad$ $L(x) = 2 + \frac{1}{12}(x-8)$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad$ $f'(8) = \frac{1}{12}$

$= \boxed{\frac{x}{12} + \frac{4}{3}}$

$f(8.5) \approx L(8.5) = \boxed{\frac{49}{24}}$

Example. Use a linear approximation to estimate $\sqrt{146}$. ≈ 12.083046

$f(x) = \sqrt{x}$ $\quad f(144) = 12$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(144) = \frac{1}{24}$

$L(x) = 12 + \frac{1}{24}(x-144) = \frac{x}{24} + 6$

$f(146) \approx L(146) = \frac{146}{24} + 6 = \boxed{\frac{145}{12}} = 12.08\overline{3}$

Example. Use a linear approximation to estimate $(1.999)^4$. ≈ 15.968024

$f(x) = x^4$ $\quad f(2) = 16$

$f'(x) = 4x^3$ $\quad f'(2) = 32$

$L(x) = 16 + 32(x-2) = 32x - 48$

$f(1.999) \approx L(1.999) = \boxed{15.968}$

Example. Use a linear approximation to estimate $\sqrt{\dfrac{5}{29}}$. ≈ 0.415227

$f(x) = \sqrt{x}$ $\quad f(0.16) = \frac{2}{5}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(0.16) = \frac{5}{4}$

$L(x) = \frac{2}{5} + \frac{5}{4}\left(x - \frac{4}{25}\right) = \frac{5}{4}x + \frac{1}{5}$

$f\left(\frac{5}{29}\right) \approx L\left(\frac{5}{29}\right) = \boxed{\frac{241}{580}} \approx 0.415517$

Example. Find the linearization $L(x)$ of $f(x) = \sqrt{x^2+9}$ at $a = -4$. Use $L(x)$ to estimate $f(-4.1)$ and $\sqrt{23.44}$.

$f(x) = \sqrt{x^2+9}$ $\qquad$ $f(-4) = 5$

$f'(x) = \dfrac{x}{\sqrt{x^2+9}}$ $\qquad$ $f'(-4) = \dfrac{-4}{5}$

$\Rightarrow L(x) = 5 - \frac{4}{5}(x+4) = -\frac{4}{5}x + \frac{9}{5}$

$f(-4.1) \approx L(-4.1) = \dfrac{25.4}{5} = \boxed{\dfrac{127}{25}}$

$\sqrt{23.44} = \sqrt{14.44+9}$

$= \sqrt{(-3.8)^2+9}$

$= f(-3.8) \approx L(-3.8) = \dfrac{24.2}{5} = \boxed{\dfrac{121}{25}}$

Example. Use a linearization to show that 0.05 is a good approximation for ln(1.05).

$f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = \dfrac{1}{1+x}$ $\qquad f'(0) = 1$

$L(x) = 0 + 1(x-0) = x$

$\ln(1.05) \approx L(0.05) = 0.05$

Alternatively

$f(x) = \ln(x)$ $\qquad f(1) = 0$ $\qquad L(x) = x - 1$

$f'(x) = \dfrac{1}{x}$ $\qquad f'(1) = 1$ $\qquad \ln(1.05) \approx L(1.05) = 0.05$

Example. Find the linearization of the following functions at the given point and use concavity to identify if the linearization is an overestimate or an underestimate.

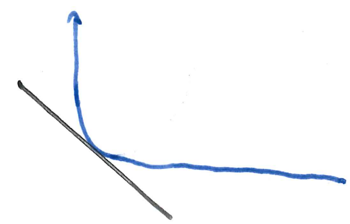
a) $f(x) = \dfrac{2}{x}$; $a = 1$

$f(1) = 2$

$f'(1) = -2$

$L(x) = \boxed{-2x + 4}$

$f'(x) = -\dfrac{2}{x^2}$



$f''(x) = \dfrac{4}{x^3}$ $\quad f''(1) = 4 > 0$ → concave up

→ under estimate

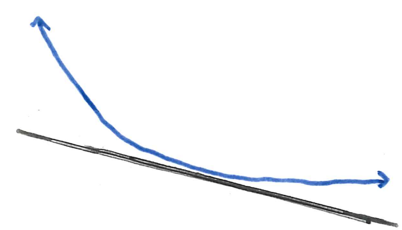
b) $f(x) = e^{-x}$; $a = \ln(2)$

$f(\ln(2)) = 1/2$

$f'(\ln(2)) = -1/2$

$L(x)$

$f'(x) = -e^{-x}$



$f''(x) = e^{-x}$ $\quad f''(\ln(2)) = 1/2 > 0$

Summary: Uses of Linear Approximation

1. To approximate f near $x = a$, use
$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$
2. To approximate the change Δy in the dependent variable when the independent variable x changes from a to $a + \Delta x$, use
$$\Delta y \approx f'(a)\Delta x.$$

Definition. (Differentials)
Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)\,dx.$$

Example. Find the differential dy.

$y = \cos(x^2)$

$dy = -2x\sin(x^2)\,dx$

$y = \sqrt{1 - x^2}$

$dy = \dfrac{-x}{\sqrt{1-x^2}}\,dx$

$y = 4x^2 - 3x + 2$

$dy = (8x-3)\,dx$

$y = x\tan(x^3)$

$dy = \left(\tan(x^3) + 3x^3\sec^2(x^3)\right)dx$

$y = \cos^5(x)$

$dy = -5\cos^4(x)\sin(x)\,dx$

$f(x) = \sin^{-1}(x)$

$dy = \dfrac{1}{\sqrt{1-x^2}}\,dx$

Example. Let $y = x^2$

a) Find dy

$$dy = 2x\ dx$$

b) If $x = 1$ and $dx = 0.01$, find dy.

$$dy = 2(1)(0.01) = \boxed{0.02}$$

c) Compare dy and Δy at this point.

$$\Delta y = f(1.01) - f(1) = 1.0201 - 1 = \boxed{0.0201}$$

Example. Let $y = \sqrt{3 + x^2}$

a) Find dy

$$dy = \frac{x}{\sqrt{3 + x^2}}\ dx$$

b) If $x = 1$ and $dx = -0.1$, find dy.

$$dy = \frac{1}{2}(-0.1) = -\frac{1}{20} = \boxed{-0.05}$$

c) Compare dy and Δy at this point.

$$\Delta y = y(0.9) - y(1) \approx 1.9519 - 2 = \boxed{-0.0481}$$

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.01) - f(5) = \underbrace{0.25}_{\Delta y}$. Use linear approximation to estimate the value of $f'(5)$.

$$\boxed{\Delta y \approx dy = f'(x)\,dx \Rightarrow f'(x) \approx \frac{\Delta y}{dx}}$$

$$\left.\begin{array}{l} \Delta y = 0.25 \\ dx = 5.01 - 5 = 0.01 \end{array}\right\} \Rightarrow f'(x) \approx \frac{0.25}{0.01} = \boxed{25}$$

Example. Suppose f is differentiable on $(-\infty, \infty)$ and $f(5.99) = 7$ and $f(6) = 7.002$. Use linear approximation to estimate the value of $f'(6)$.

$$\Delta y = f(6) - f(5.99) = 0.002$$

$$dx = 6 - 5.99 = 0.01$$

$$f'(6) \approx \frac{0.002}{0.01} = \boxed{0.2}$$

Example. Compute dy and Δy for $y = e^x$ when $x = 0$ and $\Delta x = 0.5$.

$$dy = f'(0)\, 0.5 = e^0 (0.5) = \boxed{0.5}$$

$$\Delta y = e^{0.5} - e^0 \approx \boxed{0.648721}$$

Example. Approximate the change in the area of a circle when its radius increases from 2.00 to 2.02 m.

$$A = \pi r^2$$

$$r = 2$$
$$dr = 0.02$$

$$\Delta A \approx dA = 2\pi r \cdot dr$$
$$= 2\pi(2)\cdot(0.02)$$
$$= 0.08\pi \ m^2 \approx \boxed{0.251327}$$

Example. Approximate the change in the magnitude of the electrostatic force between two charges when the distance between them increases from $r = 20\,m$ to $r = 21\,m$ ($F(r) = 0.01/r^2$).

$$r = 20m$$
$$dr = 1m$$

$$\Delta y \approx dy = F'(r)\, dr$$
$$= \frac{-0.02}{(20)^3}(1)$$
$$= \boxed{-\frac{1}{400{,}000}}$$

Example. Approximate the change in the volume of a right circular cylinder of fixed radius $r = 20\,cm$ when its height decreases from $h = 12\,cm$ to $h = 11.9\,cm$ ($V(h) = \pi r^2 h$).

$V(h) = \pi r^2 h$ $\qquad h = 12\,cm$

$V'(h) = \pi r^2$ $\qquad dh = -0.1\,cm$

$$\Delta V \approx dV = V'(h)\,dh$$
$$= \pi (20cm)^2(-0.1\,cm)$$
$$= \boxed{-40\pi\,cm^3}$$
$$\approx -125.664\,cm^3$$

Example. Approximate the change in the volume of a right circular cylinder of a right circular cone of fixed height $h = 4\,m$ when its radius increases from $r = 3\,m$ to $r = 3.05\,m$ ($V(r) = \frac{1}{3}\pi r^2 h$).

$r = 3\,m$

$V'(r) = \frac{2}{3}\pi r h$ $\qquad dr = 0.05\,m$

$$\Delta V \approx dV = V'(r)\,dr$$
$$= \frac{2}{3}\pi (3m)(4m)(0.05\,m)$$
$$= \boxed{\frac{2}{5}\pi\,m^3}$$
$$\approx 1.256637\,m^3$$

Example. The radius of a sphere is measured and found to be 0.7 inches with a possible error in measurement of at most 0.01 inches.

a) What is the maximum error in using the value of the radius to compute the volume of the sphere?

$V(r) = \frac{4}{3}\pi r^3$ $\quad r = 0.7$ in

$V'(r) = 4\pi r^2$ $\quad dr = 0.01$ in

$\Delta V \approx dV = 4\pi (0.7 \text{ in})^2 (0.01 \text{ in})$

$= \boxed{\frac{49}{2500}\pi \text{ in}^3}$

$\approx 0.061575 \text{ in}^3$

b) Find the relative error of the volume: *relative error* $\frac{dV}{V}$

What is the percentage error?

rel. error: $\frac{4\pi r^2\, dr}{\frac{4}{3}\pi r^3} = \frac{3}{r} dr = \boxed{\frac{3}{70}} \approx 0.042857$

percentage error = rel. error × 100%

$\approx \boxed{4.286\ \%}$

Example. The radius of a circular disk is given as $24\,cm$ with a maximum error in measurement of $0.2\,cm$.

a) Use differentials to estimate the maximum error in the calculated area of the disk.

$$A = \pi r^2 \qquad r = 24\,cm \qquad dr = 0.2\,cm$$

$$\Delta A \approx dA = 2\pi r \cdot dr$$

$$= 2\pi (24\,cm)(0.2\,cm)$$

$$= \boxed{9.6\pi\ cm^2}$$

$$\approx 30.1593\ cm^2$$

b) What is the relative error? What is the percentage error?

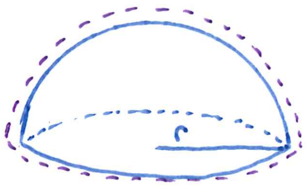
$$\text{rel. error: } \frac{dA}{A} = \frac{2\pi r \cdot dr}{\pi r^2} = \frac{2}{r} dr$$

$$= \frac{2}{24} 0.2 = \boxed{\frac{1}{60}} = 0.01\overline{6}$$

percentage error = rel. error $\times 100\%$

$$= \boxed{1.\overline{6}\,\%}$$

Example. Use differentials to estimate the amount of paint needed to apply a coat of paint $0.05\,cm$ thick to a hemispherical dome with diameter $50\,m$.



$r = 25\,m$

$dr = 0.05\,cm = 0.0005\,m$

$$S = \frac{1}{2}(4\pi r^2) = 2\pi r^2$$

$$\Delta S \approx dS = 4\pi r \cdot dr$$

$$= 4\pi(25\,m)(0.0005\,m)$$

$$= \boxed{\frac{\pi}{20}\,m^2}$$

$$\approx 0.157080\,m^2$$