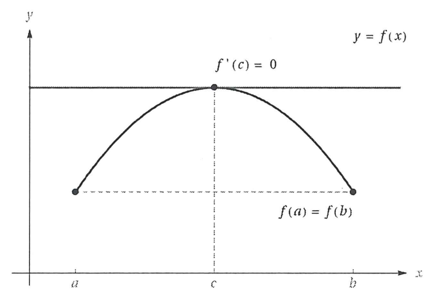
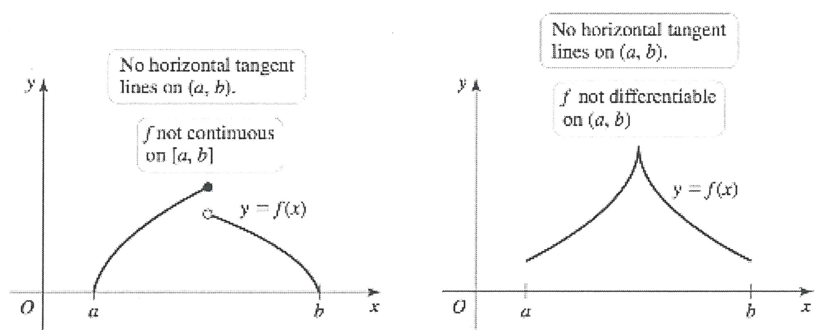
4.2 Mean Value Theorem

Before studying the *Mean Value Theorem*, we must first learn *Rolle's Theorem*:

> **Theorem 4.3: Rolle's Theorem**
> Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.



Note: Rolle's Theorem requires f be both *continuous* and *differentiable*:



(a) (b)

Example. Determine whether Rolle's Theorem applies for the following. If it applies, find the point(s) c such that $f'(c) = 0$.

$f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$

cont? ✓

diff? ✓

$f(0) = \sin(0) = 0$

$f(\pi/2) = \sin(2 \cdot \frac{\pi}{2}) = 0$

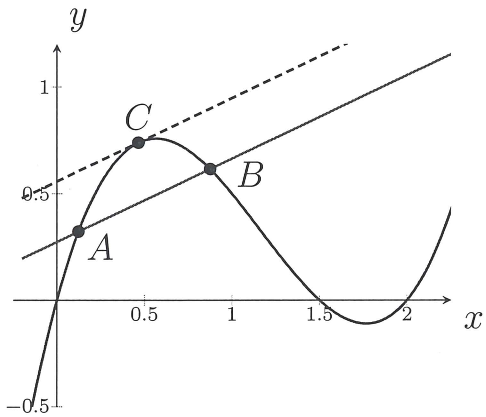
$f'(x) = 2\cos(2x)$

$2\cos(2x) = 0$

$2x = \frac{\pi}{2}$

$\boxed{x = \pi/4}$

Theorem 4.4: Mean Value Theorem
If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.



Example. For $f(x) = x^{\frac{2}{3}}$ on the interval $[0, 1]$, does the Mean Value Theorem apply? If so, find the point(s) c such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Cont on $[0,1]$? ✓

Diff on $(0,1)$? $f'(x) = \frac{2}{3\sqrt[3]{x}}$ $x \neq 0$

So diff on $(0,1)$

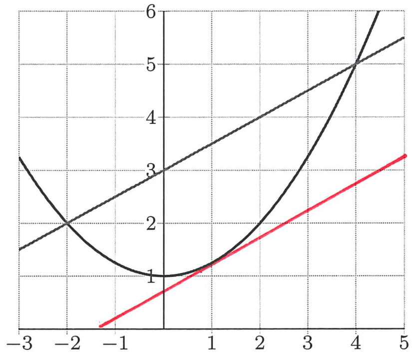
$\frac{f(1)-f(0)}{1-0} = 1$

$\frac{2}{3\sqrt[3]{x}} = 1 \iff \frac{2}{3} = \sqrt[3]{x}$

$\iff \boxed{\frac{8}{27} = x}$

Example. For each function, associated interval and graph determine if the conditions for the Mean Value Theorem are met and find the value(s) of c such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

1. $f(x) = \dfrac{x^2}{4} + 1$ on $[-2, 4]$



Cont? ✓

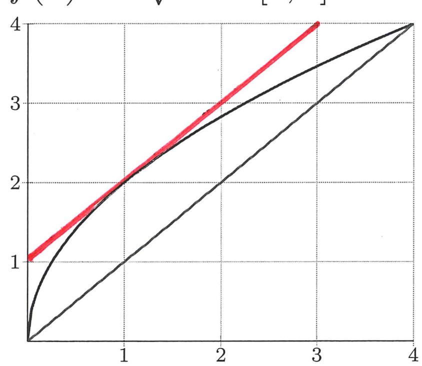
diff? ✓

$$\frac{f(4)-f(-2)}{4-(-2)} = \frac{1}{2}$$

$f'(x) = x/2$

$\frac{x}{2} = 1/2 \iff \boxed{x=1}$

2. $f(x) = 2\sqrt{x}$ on $[0, 4]$



cont? ✓

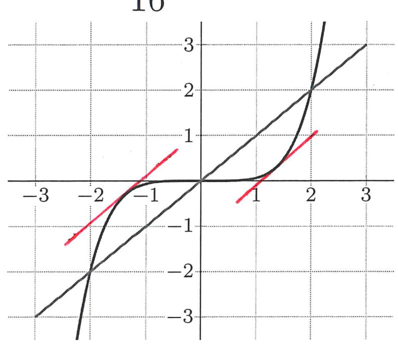
diff? ✓

$$\frac{f(4)-f(0)}{4-0} = 1$$

$f'(x) = \dfrac{1}{\sqrt{x}}$

$\dfrac{1}{\sqrt{x}} = 1 \iff \boxed{x=1}$

3. $f(x) = \dfrac{x^5}{16}$ on $[-2, 2]$



Cont? ✓

Diff? ✓

$$\frac{f(2)-f(-2)}{2-(-2)} = 1$$

$f'(x) = \dfrac{5}{16}x^4$

$\dfrac{5}{16}x^4 = 1$

$x^4 = \dfrac{16}{5}$

$\boxed{x = \pm\dfrac{2}{\sqrt[4]{5}}}$

$f(x) = x(x^2 - 2x + 1) = x^3 - 2x^2 + x$

Example. Determine whether Rolle's Theorem applies to the following functions and find the point(s) c if applicable.

1. $f(x) = x(x-1)^2$ on $[0, 1]$.

Cont? ✓ Diff? ✓

$f(0) = 0$, $f(1) = 0$

$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1) \overset{\text{set}}{=} 0$

$\boxed{x = \frac{1}{3}, \ x = 1}$

2. $f(x) = x^3 - x^2 - 5x - 3$ on $[-1, 3]$.

Cont? ✓ Diff? ✓

$f(-1) = 0$, $f(3) = 0$

$f'(x) = 3x^2 - 2x - 5 = (3x-5)(x+1) \overset{\text{set}}{=} 0$

$\boxed{x = \frac{5}{3}, \ x = -1}$

Example. Determine whether the Mean Value Theorem applies to the following functions and find the point(s) c if applicable.

1. $f(x) = 3x^2 + 2x + 5$ on $[-1, 1]$.

Cont? diff? ✓✓

$\frac{f(1) - f(-1)}{1 - (-1)} = 2 \longrightarrow f'(x) = 6x + 2 \overset{\text{set}}{=} 0$

$\boxed{x = -\frac{1}{3}}$

2. $f(x) = x^{-\frac{1}{3}}$ on $\left[\frac{1}{8}, 8\right]$.

cont? ✓ Diff? ✓ $(x \neq 0, \ 0 \notin (1/8, 8))$

$\frac{f(8) - f(1/8)}{8 - 1/8} = -\frac{4}{21} \rightarrow f'(x) = -\frac{1}{3\sqrt[3]{x^4}} \overset{\text{set}}{=} -\frac{4}{21}$

$\Longrightarrow x = \left(\frac{7}{4}\right)^{3/4}$

3. $f(x) = \begin{cases} \frac{\sin(x)}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$.

Cont?

$f(0) = 0$

$\lim_{x \to 0} f(x) = 1$ $\Big\}$ Not cont

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4. $f(x) = |x - 1|$ on $[-1, 4]$.

Cont? ✓

Diff? X

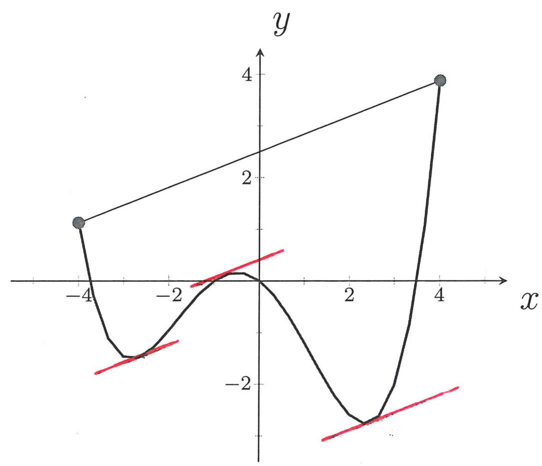
$f(x) = \begin{cases} x-1, & x \ge 1 \\ -(x-1), & x < 1 \end{cases}$

$f'(x) = \begin{cases} 1, & x \ge 1 \\ -1, & x < 1 \end{cases}$

$f'(x)$ DNE for $x = 1$

Example. Mean Value Theorem and graphs

Locate all points on the graph at which the slope of the tangent line equals the secant line on the interval $[-4, 4]$.



Example. Find the number that satisfies the hypotheses of the Mean Value Theorem for $f(x) = \sqrt{x}$ on $[0,4]$. Graph the function, the secant line through the endpoints, and the tangent line at c, visually verify that the secant and tangent are parallel.

Cont? ✓
diff? ✓

$$f'(x) = \frac{1}{2\sqrt{x}} \overset{\text{Set}}{=} \frac{1}{2}$$

$$\frac{f(4)-f(0)}{4-0} = \frac{1}{2}$$

$$\boxed{\Rightarrow x = 1}$$

Example. Drag racer acceleration

The fastest drag racers can reach a speed of $330\, mi/hr$ over a quarter-mile strip in 4.45 (from a standing start). Complete the following sentence about such a drag racer: At some point during the race, the maximum acceleration of the drag racer is at least 74.157 $mi/hr/s$.

$$\left.\begin{array}{l} f(0) = 0 \text{ mph} \\ f(4.45) = 330 \text{ mph} \end{array}\right\} \quad \frac{f(4.45)-f(0)}{4.45} = \frac{330}{4.45} \frac{\text{mph}}{s}$$

$$\approx 74.157$$

Example. A state patrol officer saw a car start from rest at a highway on-ramp. She radioed ahead to another officer 35 miles from the on-ramp. When the car reached the location of the second officer, 30 minutes later, it was clocked going $60\, mph$. The driver of the car was given a ticket for exceeding the $65-mph$ speed limit. Why can the officer conclude that the driver exceeded the speed limit?

$$\left.\begin{array}{l} f(0) = 0 \\ f(1/2) = 35 \end{array}\right\} \quad \frac{f(1/2)-f(0)}{1/2 - 0} = \frac{35 \text{ mi}}{1/2 \text{ hr}} = 70 \text{ mph}$$

At some point, the car had to be going at least 70 mph.