3.3 Rules of Differentiation

Theorem 3.2 Constant Rule
If c is a real number, then $\frac{d}{dx}(c) = 0$.

Example. Find the derivatives of

$f(x) = 3$ $\qquad$ $g(x) = \pi$ $\qquad$ $h(x) = e^{\pi}$

$f'(x) = 0$ $\qquad$ $\frac{d}{dx}[\pi] = 0$ $\qquad$ $\frac{dh}{dx} = 0$

Note: All equivalent notation

Theorem 3.3 Power Rule
If n is a nonnegative integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

Example. Find the derivative of

$j(x) = x^3$ $\qquad$ $\ell(x) = x^{\pi}$ $\qquad$ $m(x) = \pi^{42\cos(e)}$

$j'(x) = 3x^2$ $\qquad$ $\ell'(x) = \pi x^{\pi-1}$ $\qquad$ $m'(x) = 0$

Note: No variable means $m(x)$ is constant!

Proof. (Briggs, p153)

Let $f(x) = x^n$ and use the definition of the derivative in the form

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

With $n = 1$ and $f(x) = x$, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1$$

as given by the Power Rule.

With $n \geq 2$ and $f(x) = x^n$, note that $f(x) - f(a) = x^n - a^n$. A factoring formula gives

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Therefore,

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= \underbrace{a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1}}_{n \text{ terms}} = na^{n-1}
\end{aligned}$$

$\square$

Theorem 3.4 Constant Multiple Rule
If f is differentiable at x and c is a constant, then

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

Example.

$$\frac{d}{dx}(-4x^9)$$

$= -4(9x^8)$

$= -36x^8$

$$\frac{d}{dx}\left(-\frac{7x^{11}}{8}\right)$$

$= -\frac{7}{8}(11x^{10})$

$= -\frac{77}{8}x^{10}$

$$\frac{d}{dx}\left(\tfrac{1}{3}x^3\right)$$

$= \frac{1}{3}(3x^2)$

$= x^2$

Theorem 3.5 Sum Rule
If f and g are differentiable at x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Example. Find the derivative of the following:

$$p(x) = 3x^{100} + 4x^e - 17x + 24 - \pi^{\cos(e)} \qquad t(w) = 2w^3 + 9w^2 - 6w + 4$$

$p'(x) = 300x^{99} + 4ex^{e-1} - 17 + 0 - 0$

$t'(w) = 6w^2 + 18w - 6 + 0$

Definition.
(The Number e)
The number $e = 2.718281828459\ldots$ satisfies

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

It is the base of the natural exponential function $f(x) = e^x$

Note: One way to show the above result is to recall that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Theorem 3.6 The Derivative of e^x
The function $f(x) = e^x$ is differentiable for all real numbers x, and

$$\frac{d}{dx}(e^x) = e^x$$

Proof.

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h} = e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x$$

□

Example. Find the derivatives of the following

e^x

$\frac{d}{dx}[e^x] = e^x$

$42e^x$

$\frac{d}{dx}[42e^x] = 42e^x$

$\frac{d}{dx}[7e^x - 14x^e] = 7e^x - 14ex^{e-1}$

Example. *Note:* Simplify the expression before taking the derivative

a) $\dfrac{d}{ds}\left(\dfrac{12s^3 - 8s^2 + 12s}{4s}\right)$

$= \dfrac{d}{ds}(3s^2 - 2s + 3)$

$= 6s - 2 + 0$

b) $h(x) = \dfrac{x^3 - 6x^2 + 8x}{x^2 - 2x} = \dfrac{x(x-2)(x-4)}{x\ (x-2)}$

$= x - 4, \quad x \neq 0, \ x \neq 2$

$h'(x) = 1 - 0 = 1$

c) $\dfrac{d}{dx}\left(\dfrac{x-a}{\sqrt{x} - \sqrt{a}}\right)$

$= \dfrac{d}{dx}\left(\dfrac{x-a}{\sqrt{x}-\sqrt{a}}\left(\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right)\right)$

$= \dfrac{d}{dx}\left(\dfrac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}\right)$

$= \dfrac{d}{dx}\left(x^{1/2} + a^{1/2}\right)$

$= \frac{1}{2}x^{-1/2} + 0 = \dfrac{1}{2\sqrt{x}}$

d) $g(w) = \begin{cases} w + 5e^w, & \text{if } w \leq 1 \\ 2w^3 + 4w + 5, & \text{if } w > 1 \end{cases}$

$g'(w) = \begin{cases} 1 + 5e^w, & \text{if } w \leq 1 \\ 6w^2 + 4, & \text{if } w > 1 \end{cases}$

Example. Use the table to find the following derivatives:

x	1	2	3	4	5
$f'(x)$	3	4	2	1	4
$g'(x)$	2	4	3	1	5

a) $\left.\frac{d}{dx}[f(x)+g(x)]\right|_{x=1}$

$= f'(1) + g'(1)$

$= 3 + 2 = \boxed{5}$

b) $\left.\frac{d}{dx}[1.5f(x)]\right|_{x=2}$

$= 1.5 f'(2)$

$= 1.5(4)$

$= \boxed{6}$

c) $\left.\frac{d}{dx}[2x - 3g(x)]\right|_{x=4}$

$= 2 - 3g'(4)$

$= 2 - 3 \cdot 1$

$= \boxed{-1}$

Example. Find the equation of the tangent line to $y = x^3 - 4x^2 + 2x - 1$ at $a = 2$

$y' = 3x^2 - 8x + 2 - 0$

$y(2) = 11 \quad y'(2) = -2$

Tan line

$y - f(a) = f'(a)(x - a)$

Tan line:

$y - 11 = -2(x-2)$

$\boxed{y = -2x + 15}$

Example. Find the equation of the tangent line to $y = \frac{e^x}{4} - x$ at $a = 0$.

$y' = \frac{1}{4}e^x - 1$

$y(0) = 1/4$

$y'(0) = -3/4$

Tan line

$y - 1/4 = -3/4(x - 0)$

$\boxed{y = -\frac{3}{4}x + 1/4}$

Example. Find the equation of the normal line to $f(x) = 1 - x^2$ at $x = 2$.

$f'(x) = -2x$

$f(2) = -3$

$f'(2) = -4$

Tan line

$y - f(2) = f'(2)(x-2)$

$y + 3 = -4(x-2)$

$\boxed{y = -4x + 5}$

Example. Find the equations of the tangent line and normal line to $y = \dfrac{1}{2}x^4$ at $a = 2$.

$y' = \frac{4}{2}x^3 = 2x^3$

$y(2) = 8$

$y'(2) = 16$

Tan line

$y - 8 = 16(x-2)$

$\boxed{y = 16x - 24}$

Example. At what x-values does $f(x) = x - 2x^2$ have horizontal tangents?

Where is $f'(x) = 0$?

$f'(x) = 1 - 4x$

$1 - 4x = 0$

$1 = 4x$

$\boxed{\frac{1}{4} = x}$

Example. Find an equation of the line having slope $\frac{1}{4}$ that is tangent to the curve $y = \sqrt{x}$. $= x^{1/2}$

$y' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

Solve $y' = \frac{1}{4}$

$\frac{1}{2\sqrt{x}} = \frac{1}{4}$

$2 = \sqrt{x}$

$4 = x$ $\quad \because x > 0$

$y(4) = 1/2$

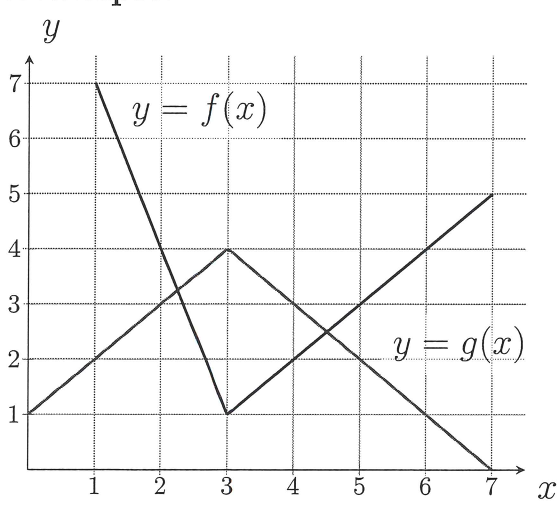
$y'(4) = 1/4$

Tan line

$y - \frac{1}{2} = \frac{1}{4}(x-4)$

$\boxed{y = \frac{1}{4}x - \frac{1}{2}}$

Example.



a) $f'(2)$ $= -3$

b) $g'(2)$ $= 1$

c) $f'(5)$ $= 1$

d) $g'(5)$ $= -1$

Example. The line tangent to the graph of f at $x = 5$ is $y = \frac{1}{10}x - 2$. Find $\left.\frac{d}{dx}(4f(x))\right|_{x=5}$

Since y is the line tangent to f at $x=5$,

then $f(5) = y(5)$ and $f'(5) = \frac{1}{10}$

$$\left.\frac{d}{dx}(4f(x))\right|_{x=5} = 4\left.\frac{d}{dx}(f(x))\right|_{x=5} = 4f'(5) = 4\left(\frac{1}{10}\right) = \boxed{\frac{2}{5}}$$

Example. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$.

$y' = 2e^x - 3$

$y = 3x - 5$

parallel means equal slopes. Solve $y' = 3$

$2e^x - 3 = 3$

$2e^x = 6$

$e^x = 3$

$\boxed{x = \ln(3)}$

Example. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and parallel to the line $12x - y = 1$.

$y(2) = 9$

$y(-2) = -7$

$y' = 3x^2$

$y = 12x - 1$

Solve $y' = 12$

$3x^2 = 12$

$x^2 = 4$

$\boxed{x = \pm 2}$

1st

$y - 9 = 12(x - 2)$

$\boxed{y = 12x - 15}$

2nd

$y + 7 = 12(x + 2)$

$\boxed{y = 12x + 17}$

> **Definition.**
> Higher-Order Derivatives
> Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is
> $$f''(x) = \frac{d}{dx}(f'(x))$$
> For integers $n \geq 1$, the ***n*th derivative** of f is
> $$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

Example. Find all the derivatives of $y = \dfrac{x^5}{120}$

$y' = \frac{5x^4}{120} = \frac{x^4}{24}$

$y'' = \frac{4x^3}{24} = \frac{x^3}{6}$

$y''' = \frac{3x^2}{6} = \frac{x^2}{2}$

$y^{(4)} = \frac{2x}{2} = x$

$y^{(5)} = 1$

$y^{(k)} = 0, \quad k = 6, 7, 8, \ldots$

Example. Find the first, second and third derivatives of $f(x) = 5x^4 + 10x^3 + 3x + 6$

$f'(x) = 20x^3 + 30x^2 + 3$

$f''(x) = 60x^2 + 60x$

$f'''(x) = 120x + 60$

Example. Find the first, second and third derivatives of $f(x) = x^2(2 + x^{-3})$. $= 2x^2 + x^{-1}$

$f'(x) = 4x - x^{-2}$

$f''(x) = 4 + 2x^{-3}$

$f'''(x) = -6x^{-4}$