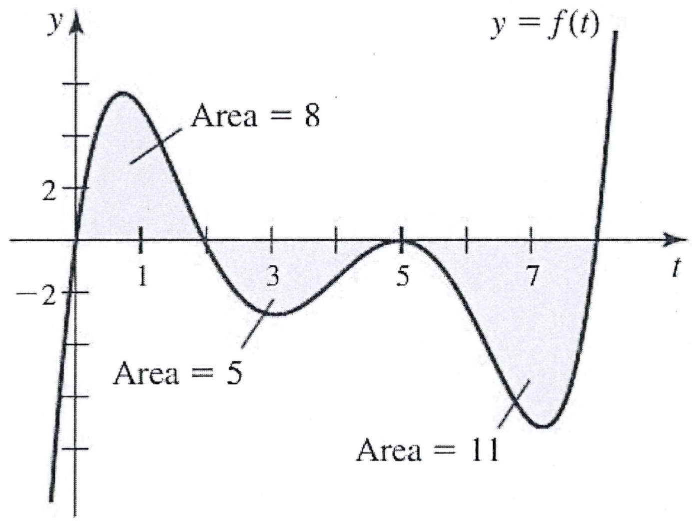
5.3: Fundamental Theorem of Calculus

Definition. (Area Function)
Let f be a continuous function, for $t \geq a$. The **area function for** f **with left endpoint** a **is**

$$A(x) = \int_a^x f(t)\,dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Example. The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions:



a) $A(2) = \int_0^2 f(t)\,dt = \boxed{8}$

b) $F(5) = \int_2^5 f(t)\,dt = \boxed{-5}$

c) $A(0) = \int_0^0 f(t)\,dt = \boxed{0}$

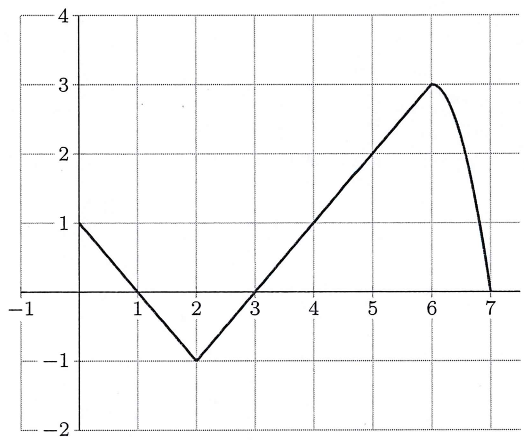
d) $F(8) = \int_2^8 f(t)\,dt = -5-11 = \boxed{-16}$

e) $A(8) = 8-5-11 = \boxed{-8}$

f) $A(5) = 8-5 = \boxed{3}$

g) $F(2) = \boxed{0}$

Example. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.



a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$ and 6.

x	0	1	2	3	4	5	6
$g(x)$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1.5	4

b) Estimate $g(7)$.

$g(7) \approx 4 + 2.1 = \boxed{6.1}$

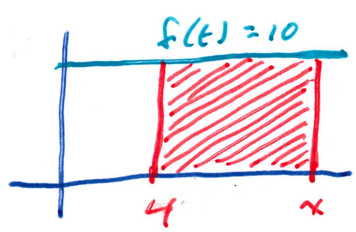
c) Where does g have a maximum value? Where does it have a minimum value?

max of 6.1 at $x = 7$

min of $-\frac{1}{2}$ at $x = 3$

Example. For the area function $A(x) = \int_a^x f(t)\,dt$, graph the area function and then verify that $A'(x) = f(x)$.

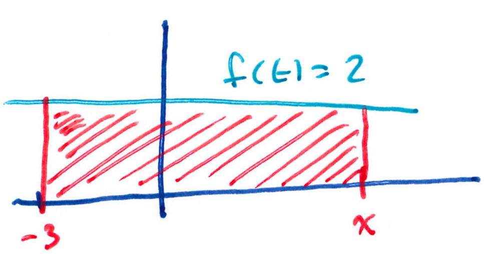
a) $f(t) = 10,\ a = 4$



$A(x) = 10(x-4) = 10x - 40$

$A'(x) = 10$

b) $f(t) = 2,\ a = -3$



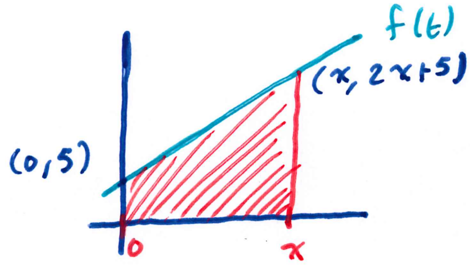
$A(x) = 2(x+3) = 2x + 6$

$A'(x) = 2$

Area of trapezoid

$\frac{1}{2}(b_1+b_2)w$

c) $f(t) = 2t + 5, \ a = 0$

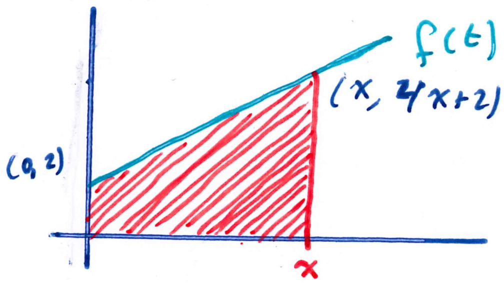


$A(x) = \frac{1}{2}(5 + 2x+5)(x-0)$

$= x^2 + 5x$

$A'(x) = 2x + 5$

d) $f(t) = 4t + 2, \ a = 0$

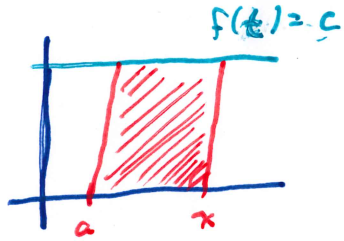


$A(x) = \frac{1}{2}(2 + 4x+2)(x-0)$

$= 2x^2 + 2x$

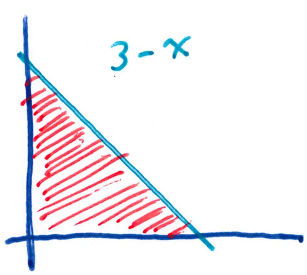
$A'(x) = 4x + 2$

Example. Let $f(x) = c$, where c is a positive constant. Explain why an area function of f is an increasing function.



A(x) represents the area of a rectangle with positive height and increasing width.

Example. The linear function $f(x) = 3 - x$ is decreasing on the interval $[0, 3]$. Is its area function for f (with left endpoint 0) increasing or decreasing on the interval $[0, 3]$?



The area function is increasing because it is positive on $[0, 3]$.

Theorem 5.3 (Part I) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \qquad \text{for } a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example. For the following functions, find the derivatives

a) $g(x) = \int_0^x \sqrt{1-2t}\,dt$

$g'(x) = \boxed{\sqrt{1-2x}}$

b) $g(x) = \int_3^x e^{t^2-t}\,dt$

$g'(x) = \boxed{e^{x^2-x}}$

c) $g(y) = \int_2^y t^2\sin(t)\,dt$

$g'(y) = \boxed{y^2 \sin(y)}$

d) $y = \int_x^2 \cos(t^2)\,dt = -\int_2^x \cos(t^2)\,dt$

$y' = \boxed{-\cos(x^2)}$

e) $y = \int_1^{\cos(x)} (t + \sin(t))\,dt$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$y = \int_1^u (t + \sin(t))dt$

$y' = (u + \sin(u))\frac{du}{dx}$

$= \boxed{-(\cos(x) + \sin(\cos(x)))\sin(x)}$

f) $y = \int_{-3}^{3x^4} \frac{t}{t^2-4t}\,dt = \int_{-3}^u \frac{t}{t^2-4t}\,dt$

$u = 3x^4$

$\frac{du}{dx} = 12x^3$

$y' = \left(\frac{u}{u^2-4u}\right)\frac{du}{dx}$

$= \frac{3x^4}{9x^8-12x^4}(12x^3)$

$= \boxed{\frac{12x^3}{3x^4-4}}$

g) $y = \int_1^{e^x} \ln(t)\, dt = \int_1^{u} \ln(t)dt$

$u = e^x$

$\frac{du}{dx} = e^x$

$y' = \ln(u) \frac{du}{dx}$

$= \ln(e^x)\, e^x$

$= \boxed{x e^x}$

h) $y = \int_0^{x^4} \cos^2(\theta)\, d\theta = \int_0^{u} \cos^2(\theta)\, d\theta$

$u = x^4$

$\frac{du}{dx} = 4x^3$

$y' = \cos^2(u) \frac{du}{dx}$

$= \boxed{\cos^2(x^4)\,(4x^3)}$

i) $y = \int_{\tan(x)}^{0} \frac{dt}{1+t^2} = -\int_0^{u} \frac{dt}{1+t^2}$

$u = \tan(x)$

$\frac{du}{dx} = \sec^2(x)$

$y' = -\frac{1}{1+u^2} \frac{du}{dx}$

$= -\frac{\sec^2(x)}{1+\tan^2(x)}$

$= -\frac{\sec^2(x)}{\sec^2(x)}$

$= \boxed{-1}$

j) $y = \int_{\sin(x)}^{1} \sqrt{1+t^2}\, dt = -\int_1^{u} \sqrt{1+t^2}\, dt$

$u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$y' = -\sqrt{1+u^2} \frac{du}{dt}$

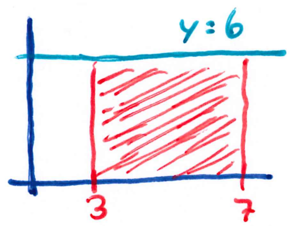
$= \boxed{-\sqrt{1+\sin^2(x)}\ \cos(x)}$

Theorem 5.3 (Part II) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Example. Evaluate the following integrals using graphs and the Fundamental Theorem of Calculus.

a) $\displaystyle\int_3^7 6\,du = 6u\Big|_3^7 = 6(7) - 6(3) = \boxed{24}$



$A = 6\cdot(7-3) = \boxed{24}$

b) $\displaystyle\int_{-2}^4 x\,dx = \frac{x^2}{2}\Big|_{-2}^4 = \frac{(4)^2}{2} - \frac{(-2)^2}{2} = \boxed{6}$

$A = \frac{1}{2}(4\cdot 4) - \frac{1}{2}(2\cdot 2) = \boxed{6}$

Example. Evaluate the following integrals

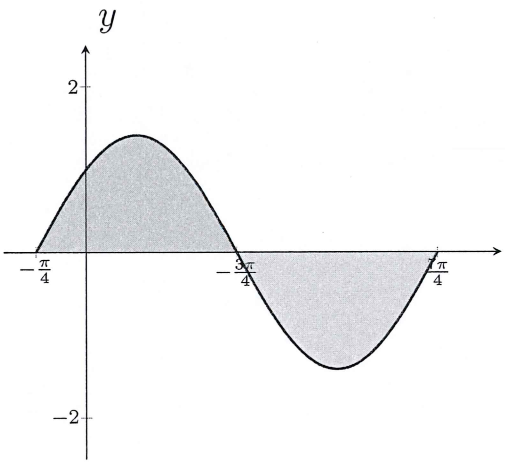
a) $\displaystyle\int_{-1}^3 x^2\,dx = \frac{x^3}{3}\Big|_{-1}^3 = \frac{(3)^3}{3} - \frac{(-1)^3}{3} = \boxed{28/3}$

b) $\displaystyle\int_0^\pi (1 + \cos(x))\,dx = x + \sin(x)\Big|_0^\pi = [\pi + \sin(\pi)] - [0 + \sin(0)] = \boxed{\pi}$

c) $\displaystyle\int_{-5}^5 e\,dx = ex\Big|_{-5}^5 = [5e] - [-5e] = \boxed{10e}$

d) $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta = \sec\theta\Big|_0^{\pi/4} = \boxed{\sqrt{2} - 1}$

Example. Evaluate the following integrals using the Fundamental Theorem of Calculus



$$\int_{-\pi/4}^{7\pi/4} (\sin(x) + \cos(x))\, dx$$

$$= -\cos(x) + \sin(x) \Big|_{-\pi/4}^{7\pi/4}$$

$$= \left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\right] - \left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= 0$$

Example. Evaluate $\int_3^8 f'(t)\, dt$, where f' is continuous on $[3, 8]$, $f(3) = 4$, and $f(8) = 20$.

$$\int_3^8 f'(t)\, dt = f(t)\Big|_3^8 = f(8) - f(3) = 20 - 4 = \boxed{16}$$

Example. Find $\frac{d}{dt} \int_0^{t^4} \sqrt{u}\, du$ by evaluating the integral directly and then differentiating the result.

$$\frac{d}{dt}\left(\int_0^{t^4} u^{1/2} du\right) = \frac{d}{dt}\left(\frac{2}{3} u^{3/2}\Big|_0^{t^4}\right) = \frac{d}{dt}\left(\frac{2}{3} t^6\right) = \boxed{4t^5}$$

Example. Find $\frac{d}{dt} \int_0^{t^4} \sqrt{u}\, du$ by differentiating the integral directly.

$$\frac{d}{dt}\int_0^{v} \sqrt{u}\, du = \sqrt{v}\, \frac{dv}{dt} = \sqrt{t^4} \cdot 4t^3 = \boxed{4t^5}$$

$v = t^4$ $\qquad \frac{dv}{dt} = 4t^3$

Example. Find $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan(\theta)} \sec^2(y)\,dy$ by evaluating the integral directly and then differentiating the result.

$$\frac{d}{d\theta}\left[\int_0^{\tan(\theta)} \sec^2(y)\,dy\right] = \frac{d}{d\theta}\left[\tan(y)\Big|_0^{\tan\theta}\right] = \frac{d}{d\theta}\left[\tan(\tan\theta) - \tan(0)\right]$$

$$= \frac{d}{d\theta}\left[\tan(\tan(\theta))\right] = \boxed{\sec^2(\tan(\theta))\sec^2(\theta)}$$

chain rule

Example. Find $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan(\theta)} \sec^2(y)\,dy$ by differentiating the integral directly.

$$\frac{d}{d\theta}\int_0^{u} \sec^2(y)\,dy = \sec^2(u)\frac{du}{d\theta}$$

$$= \boxed{\sec^2(\tan(\theta))\sec^2(\theta)}$$

$u = \tan\theta$

$\dfrac{du}{d\theta} = \sec^2\theta$

Example. Evaluate the following integrals:

a) $\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{1/3}\,dx$

$= \frac{3}{4} x^{4/3} \Big|_1^8$

$= \frac{3}{4}(8)^{4/3} - \frac{3}{4}(1)^{4/3}$

$= \frac{3}{4}\cdot 16 - \frac{3}{4}\cdot 1$

$= \boxed{\frac{45}{4}}$

b) $\int_{-2}^{-1} \frac{2}{x^2}\,dx = \int_{-2}^{-1} 2x^{-2}\,dx$

$= 2x^{-1}\Big|_{-2}^{-1}$

$= \left[-\frac{2}{-1}\right] - \left[-\frac{2}{-2}\right]$

$= 2 - 1 = \boxed{1}$

c) $\int_0^2 x(2+x^5)\,dx$

$= \int_0^2 2x + x^6\,dx$

$= x^2 + \frac{x^7}{7}\Big|_0^2$

$= \left[2^2 + \frac{2^7}{7}\right] - \left[0^2 + \frac{0^7}{7}\right]$

$= 4 + \frac{128}{7}$

$= \boxed{\frac{156}{7}}$

d) $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}\,du$

$= \int_9^4 u^{-1/2} - 1\,du$

$= 2u^{1/2} - u\Big|_9^4$

$= [2\sqrt{4} - 4] - [2\sqrt{9} - 9]$

$= 0 - [2\sqrt{9} - 9]$

$= \boxed{3}$

e) $\displaystyle\int_0^2 (y-1)(2y+1)\,dy$

$= \displaystyle\int_0^2 2y^3 - y - 1\,dy$

$= \dfrac{y^4}{2} - \dfrac{y^2}{2} - y \Big|_0^2$

$= \left[\dfrac{2^4}{2} - \dfrac{2^2}{2} - 2\right] - \left[\dfrac{0^4}{2} - \dfrac{0^2}{2} - 0\right]$

$= 8 - 2 - 2$

$= \boxed{4}$

f) $\displaystyle\int_0^4 \left(1 + 3y - y^2 - \frac{y^3}{4}\right) dy$

$= y + \frac{3}{2}y^2 - \frac{1}{3}y^3 - \dfrac{y^4}{16}\Big|_0^4$

$= \left[4 + \frac{3}{2}(4)^2 - \frac{1}{3}(4)^3 - \dfrac{4^4}{16}\right] - [0]$

$= 4 + 24 - \dfrac{64}{3} - 16$

$= \boxed{\dfrac{-28}{3}}$

g) $\displaystyle\int_0^1 (x^e + e^x)\,dx$

$= \dfrac{x^{e+1}}{e+1} + e^x \Big|_0^1$

$= \left[\dfrac{1^{e+1}}{e+1} + e^1\right] - \left[\dfrac{0^{e+1}}{e+1} + e^0\right]$

$= \dfrac{1}{e+1} + e - 1$

$= \boxed{\dfrac{e^2}{e+1}}$

h) $\displaystyle\int_0^3 (2\sin(x) - e^x)\,dx$

$= -2\cos(x) - e^x \Big|_0^3$

$= \left[-2\cos(3) - e^3\right] - \left[-2\cos(0)\right.$

$= -2\cos(3) - e^3 + 2 + 1$

$= \boxed{3 - 2\cos(3) - e^3}$

i) $\displaystyle\int_{\frac{\pi}{2}}^{0} \frac{1-\cos(2t)}{2}\,dt$

$$= \int_{\pi/2}^{0} \frac{1}{2} - \frac{\cos(2t)}{2}\,dt$$

$$= t - \frac{\sin(2t)}{4}\Big|_{\pi/2}^{0}$$

$$= \left[0 - \frac{\sin(0)}{4}\right] - \left[\pi/4 - \frac{\sin(\pi)}{4}\right]$$

$$= \boxed{-\pi/4}$$

j) $\displaystyle\int_{\frac{1}{2}}^{2} \left(1-\frac{1}{x^2}\right) dx$

$$= x + \frac{1}{x}\Big|_{1/2}^{2}$$

$$= \left[2 + \tfrac{1}{2}\right] - \left[\tfrac{1}{2} + 2\right]$$

$$= \boxed{0}$$

k) $\displaystyle\int_{1}^{2} \left(\frac{2}{s^2}-\frac{4}{s^3}\right) ds$

$$= -\frac{2}{s} + \frac{2}{s^2}\Big|_{1}^{2}$$

$$= \left[-\frac{2}{2} + \frac{2}{2^2}\right] - \left[-\frac{2}{1} + \frac{2}{1^2}\right]$$

$$= \boxed{-\frac{1}{2}}$$

l) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

$$= 8\tan^{-1}(x)\Big|_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \left[8\tan^{-1}(\sqrt{3})\right] - \left[8\tan^{-1}(1/\sqrt{3})\right]$$

$$= 8(\pi/3) - 8(\pi/6)$$

$$= \boxed{\frac{4\pi}{3}}$$

m) $\int_0^5 (x^2 - 9)\, dx$

$= \frac{x^3}{3} - 9x \Big|_0^5$

$= \left[\frac{5^3}{3} - 9(5)\right] - [0]$

$= \frac{125}{3} - 45$

$= \boxed{-\frac{10}{3}}$

n) $\int_{1/2}^2 \left(1 - \frac{1}{x^2}\right) dx$

$= x + \frac{1}{x} \Big|_{1/2}^2$

$= [2 + \frac{1}{2}] - [\frac{1}{2} + 2]$

$= \boxed{0}$

o) $\int_{-1}^2 x^3\, dx$

$= \frac{x^4}{4} \Big|_{-1}^2$

$= \left[\frac{2^4}{4}\right] - \left[\frac{(-1)^4}{4}\right]$

$= \boxed{\frac{15}{4}}$

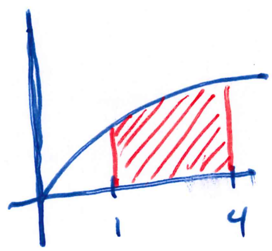
p) $\int_{\pi/6}^{2\pi} \cos(x)\, dx$

$= \sin(x) \Big|_{\pi/6}^{2\pi}$

$= \sin(2\pi) - \sin(\pi/6)$

$= \boxed{-\frac{1}{2}}$

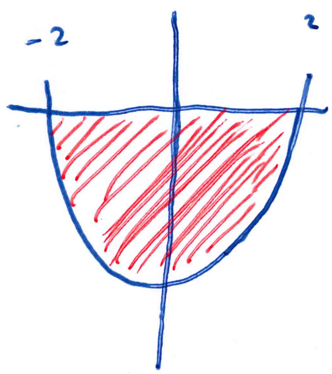
Example. Find the area of the region bounded by $y = \sqrt{x}$ between $x = 1$ and $x = 4$.



$$\int_1^4 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_1^4$$

$$= \left[\frac{2}{3}(4)^{3/2}\right] - \left[\frac{2}{3}(1)^{3/2}\right]$$

Example. Find the area of the region below the x-axis bounded by $y = x^4 - 16$.

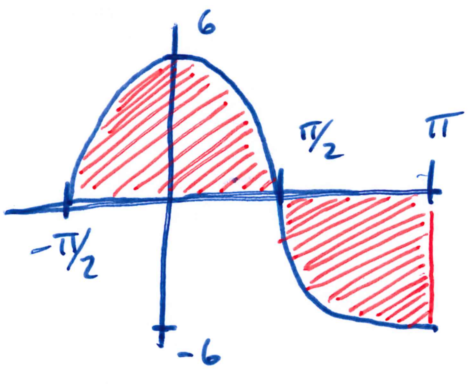


$$\int_{-2}^{2} x^4 - 16\, dx = \frac{x^5}{5} - 16x \Big|_{-2}^{2}$$

$$= \left[\frac{2^5}{5} - 16(2)\right] - \left[\frac{(-2)^5}{5} - 16(-2)\right]$$

$$= \boxed{-\frac{256}{5}}$$

Example. Find the area of the region bounded by $y = 6\cos(x)$ between $x = -\pi/2$ and $x = \pi$.



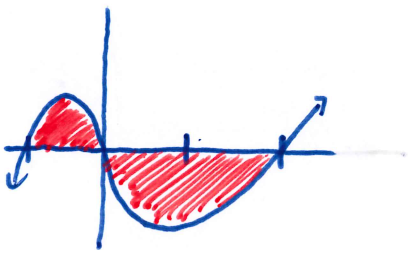
$$\int_{-\pi/2}^{\pi} 6\cos(x)\, dx = 6\sin(x) \Big|_{-\pi/2}^{\pi}$$

$$= \left[6\sin(\pi)\right] - \left[6\sin(-\pi/2)\right]$$

$$= 0 - (-6)$$

$$= \boxed{6}$$

Example. Find the area of the region bounded by $f(x) = x(x+1)(x-2)$ and the x-axis on the interval $[-1, 2]$.

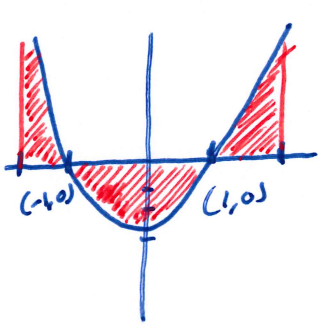


$$\int_{-1}^{2} x(x+1)(x-2)\,dx = \int_{-1}^{2} x^3 - x^2 - 2x\,dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3} - x^2 \Big|_{-1}^{2}$$

$$= \left[\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right] - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \boxed{-\frac{9}{4}}$$

Example. Find the total area between $y = 3x^2 - 3$ and the x-axis on $-2 \leq x \leq 2$.

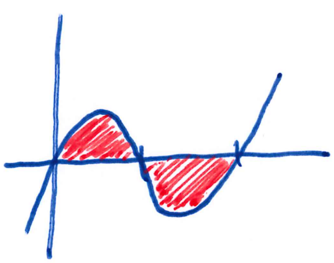


$$\int_{-2}^{2} |3x^2-3|\,dx = \left|\int_{-2}^{-1} 3x^2-3\,dx\right| + \left|\int_{-1}^{1} 3x^2-3\,dx\right| + \left|\int_{1}^{2} 3x^2-3\,dx\right|$$

$$= \left|(x^3-3x)\Big|_{-2}^{-1}\right| + \left|(x^3-3x)\Big|_{-1}^{1}\right| + \left|(x^3-3x)\Big|_{1}^{2}\right|$$

$$= |4| + |-4| + |4| = \boxed{12}$$

Example. Find the total area between $y = x^3 - 3x^2 + 2x$ and the x axis on the interval $0 \leq x \leq 2$.



$$\int_{0}^{2} |x^3 - 3x^2 + 2x|\,dx = \left|\left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_{0}^{1}\right| + \left|\left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_{1}^{2}\right|$$

$$= \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right|$$

$$= \boxed{\frac{1}{2}}$$