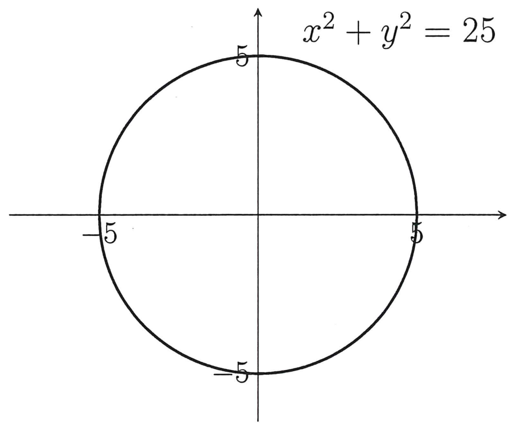
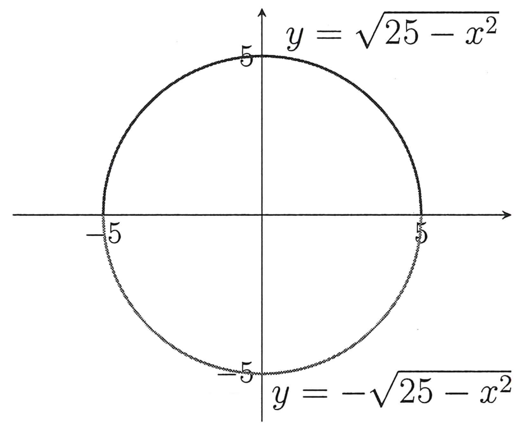
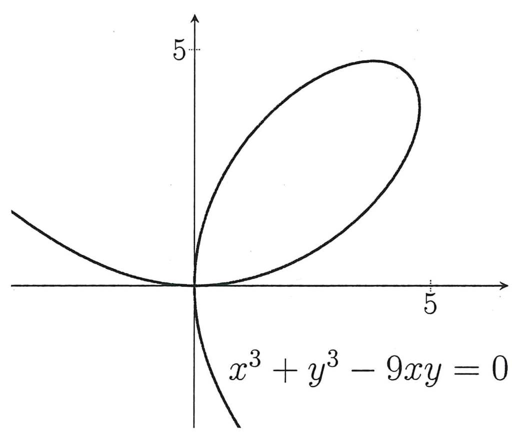
3.8: Implicit Differentiation

Up until now, we have only taken the derivatives of *explicitly* defined functions (functions defined in terms of only x).
An *implicitly* defined function will be written in terms of both x and y:

$$x^2 + y^2 = 25$$







Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example. Find the derivatives of the following by rewriting each function explicitly before taking the derivative, and by using implicit differentiation. Compare the results.

$y^2 = x$

① $y = \pm\sqrt{x} = \pm x^{1/2}$

$$\frac{dy}{dx} = \pm\frac{1}{2}x^{-1/2} = \boxed{\pm\frac{1}{2\sqrt{x}}}$$

chain rule

② $2y\frac{dy}{dx} = 1 \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2y}}$

Since $y = \pm\sqrt{x}$, these are equivalent

$\sqrt{x} + \sqrt{y} = 4$

① $y = (4-\sqrt{x})^2 \rightarrow \frac{dy}{dx} = 2(4-\sqrt{x})\left(-\frac{1}{2}x^{-1/2}\right)$

$$= \boxed{\frac{4-\sqrt{x}}{\sqrt{x}}}$$

chain rule

② $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}y^{-1/2}} = \boxed{\frac{\sqrt{y}}{\sqrt{x}}}$

Since $\sqrt{y} = 4-\sqrt{x}$, these are equivalent

Example. Find the derivatives of the following equations:

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \boxed{-\frac{x}{y}}$$

$$x^3 + y^3 - 9xy = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \boxed{\frac{3y - x^2}{y^2 - 3x}}$$

$$2y = x^2 + \sin y$$

$$2\frac{dy}{dx} = 2x + \cos(y)\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2 - \cos(y)\right) = 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{2 - \cos(y)}}$$

$$x^2y^2 + x\sin y = 4$$

$$2xy^2 + 2x^2y\frac{dy}{dx} + \sin(y) + x\cos(y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(2x^2y - x\cos(y)\right) = 2xy^2 + \sin(y)$$

$$\boxed{\frac{dy}{dx} = \frac{2xy^2 + \sin(y)}{2x^2y - x\cos(y)}}$$

$$y^5 + x^2y^3 = 1 + x^4y$$

$$5y^4\frac{dy}{dx} + 2xy^3 + 3x^2y^2\frac{dy}{dx} = 4x^3y + x^4\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(5y^4 + 3x^2y^2 - x^4\right) = -2xy^3 + 4x^3y$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}}$$

$$1 + x = \sin\left(xy^2\right)$$

Product rule

$$1 = \cos(xy^2)\left(y^2 + 2xy\frac{dy}{dx}\right)$$

chain rule

$$\sec(xy^2) = y^2 + 2xy\frac{dy}{dx}$$

$$\boxed{\frac{\sec(xy^2) - y^2}{2xy} = \frac{dy}{dx}}$$

Example. Find the derivatives of the following equations:

$$x^3 - xy + y^3 = 1$$

$$3x^2 - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(3y^2 - x\right) = y - 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}}$$

$$xe^y = x - y$$

$$e^y + xe^y\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(xe^y + 1\right) = 1 - e^y$$

$$\boxed{\frac{dy}{dx} = \frac{1-e^y}{xe^y+1}}$$

$$x^{-1} + y^{-1} = \frac{1}{x} + \frac{1}{y} = 1$$

$$-x^{-2} - y^{-2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-2}}{y^{-2}} = \boxed{-\left(\frac{y}{x}\right)^2}$$

$$x^2 - 2x^3y^4 + y^2 = 30y$$

$$2x - 2\left(3x^2y^4 + 4x^3y^3\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 30\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(-8x^3y^3 + 2y - 30\right) = -2x + 6x^2y^4$$

$$\boxed{\frac{dy}{dx} = \frac{x - 3x^2y^4}{4x^3y^3 - y + 15}}$$

$$\tan(xy) = x + y$$

$$\sec^2(xy)\left(y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x\sec^2(xy) - 1\right) = 1 - y\sec^2(xy)$$

$$\boxed{\frac{dy}{dx} = \frac{1-y\sec^2(xy)}{x\sec^2(xy)-1}}$$

$$x^2 = \frac{x-y}{x+y} \Rightarrow x^2(x+y) = x - y$$

$$2x(x+y) + x^2\left(1 + \frac{dy}{dx}\right) = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(x^2+1\right) = 1 - 2x(x+y) - x^2$$

$$\boxed{\frac{dy}{dx} = \frac{1-3x^2-2xy}{x^2+1}}$$

Example. Find the second derivative implicitly for the following equations:

$y^2 - 2x = 1 - 2y$

$$2y\frac{dy}{dx} - 2 = -2\frac{dy}{dx}$$

$$\frac{dy}{dx}(2y+2) = 2$$

$$\frac{dy}{dx} = \frac{2}{2y+2} = (y+1)^{-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(y+1)^{-2}\frac{dy}{dx}$$

$$= \boxed{-(y+1)^{-3}}$$

$xy = \cot(xy)$

$$y + x\frac{dy}{dx} = -\csc^2(xy)\left(y + x\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(x + x\csc^2(xy)\right) = -y - y\csc^2(xy)$$

$$\frac{dy}{dx} = \frac{-y(1+\csc^2(xy))}{x(1+\csc^2(xy))} = \frac{-y}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y}{x^2}$$

$$= \frac{-x(-\frac{y}{x}) + y}{x^2}$$

$$= \boxed{2y/x^2}$$

$x^3 + y^3 = 1$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-y^2(2x) - (-x^2)(2y)\frac{dy}{dx}}{y^4}$$

$$= \frac{-\frac{2xy^3}{y} + \frac{2x^3}{y}}{y^4} = \frac{-2x}{y^5}\underbrace{(y^3 + x^3)}_{1}$$

$$= \boxed{\frac{-2x}{y^5}}$$

$x = e^y$

$$1 = e^y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = e^{-y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -e^{-y}\frac{dy}{dx}$$

$$= -e^{-y}\cdot e^{-y}$$

$$= \boxed{-e^{-2y}}$$

$$y - f(a) = \frac{dy}{dx}\Big|_{(a,\,f(a))}(x-a)$$

Example. Find the equation of all lines tangent to the curve $x + y^3 - y = 1$ at $x = 1$.

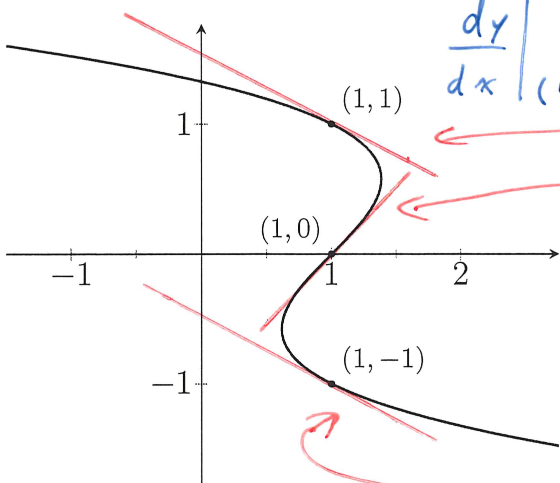
$$1 + 3y^2\frac{dy}{dx} - \frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = \frac{1}{1-3y^2}$$

$x = 1$

$\Rightarrow 1 + y^3 - y = 1$

$y(y^2-1) = 0$

$y = 0, \pm 1$



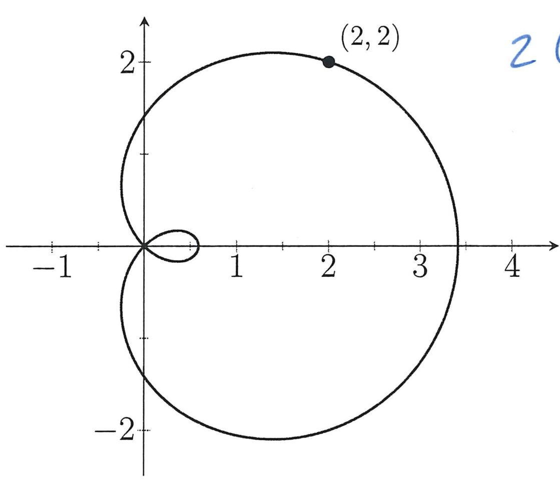
$$\frac{dy}{dx}\Big|_{(1,-1)} = \frac{1}{-2}$$

$$\boxed{y + 1 = -\tfrac{1}{2}(x-1)}$$

$$\frac{dy}{dx}\Big|_{(1,0)} = 1 \qquad \boxed{y - 0 = 1(x-1)}$$

$$\frac{dy}{dx}\Big|_{(1,1)} = \frac{1}{-2} \qquad \boxed{y - 1 = -\tfrac{1}{2}(x-1)}$$

Example. Find the equation of the tangent line and normal line for $\left(x^2 + y^2 - 2x\right)^2 = 2\left(x^2 + y^2\right)$ at $(x, y) = (2, 2)$.



$$2(x^2+y^2-2x)\left(2x + 2y\frac{dy}{dx} - 2\right) = 4x + 4y\frac{dy}{dx}$$

Evaluate at $(2,2)$

$$2(4+4-4)\left(4 + 4\frac{dy}{dx}\Big|_{(2,2)} - 2\right) = 8 + 8\frac{dy}{dx}\Big|_{(2,2)}$$

$$32 + 32\frac{dy}{dx}\Big|_{(2,2)} - 16 = 8 + 8\frac{dy}{dx}\Big|_{(2,2)}$$

$$24\frac{dy}{dx}\Big|_{(2,2)} = -8$$

$$\frac{dy}{dx}\Big|_{(2,2)} = -\frac{1}{3}$$

tangent line

$$\boxed{y - 2 = -\tfrac{1}{3}(x-2)}$$

normal line

$$\boxed{y - 2 = 3(x-2)}$$