3.10 Derivatives of Inverse Trigonometric Functions

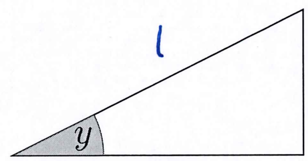
Example. Recall that $y = \sin^{-1}(x) \iff \sin(y) = x$. Use this fact and implicit differentiation to derive the derivative of $\sin^{-1}(x)$.

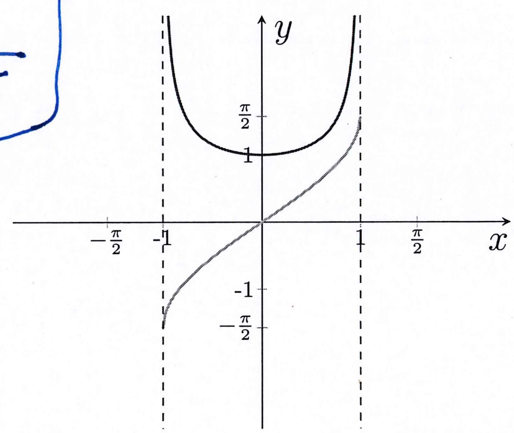
$y = \sin^{-1}(x)$

$\sin(y) = x$

$\cos(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}/1} = \boxed{\frac{1}{\sqrt{1-x^2}}}$





Note

$\sin^2(y) + \cos^2(y) = 1$

$\Rightarrow \cos(y) = \sqrt{1-\sin^2(y)}$

$= \sqrt{1-x^2}$

Next, extend this definition for the derivative of $\sin^{-1}(f(x))$.

$$\frac{d}{dx}\left[\sin^{-1}(f(x))\right] = \frac{1}{\sqrt{1-[f(x)]^2}} f'(x).$$

Example. Find the derivative of the following

$f(x) = \sqrt{1-x^2} \cdot \arcsin(x)$

$f(x) = (1-x^2)^{1/2} \cdot \sin^{-1}(x)$

$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} \sin^{-1}(x) + (1-x^2)^{1/2} \frac{-2x}{\sqrt{1-x^2}}$

$= \boxed{-\frac{x\sin^{-1}(x)}{\sqrt{1-x^2}} - 2x}$

$y = \sin^{-1}\left(\sqrt{2}t\right)$

$y' = \frac{\sqrt{2}}{\sqrt{1-(\sqrt{2}t)^2}} = \boxed{\frac{\sqrt{2}}{\sqrt{1-2t^2}}}$

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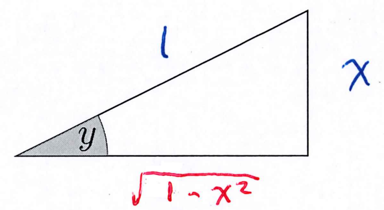

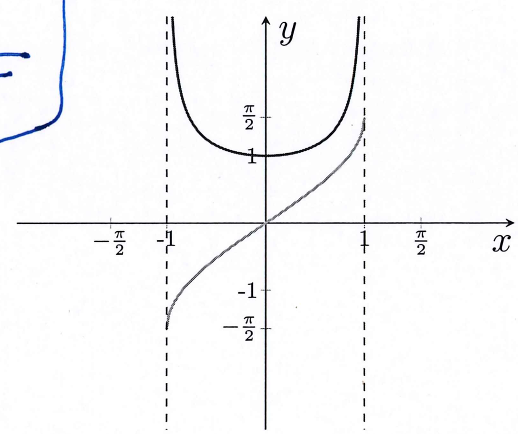

Note

$$\sin^2(y) + \cos^2(y) = 1$$

$$\Rightarrow \cos(y) = \sqrt{1-\sin^2(y)}$$

$$= \sqrt{1-x^2}$$

Next, extend this definition for the derivative of $\sin^{-1}(f(x))$.

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$$= \boxed{-\frac{x\sin^{-1}(x)}{\sqrt{1-x^2}} - 2x}$$

$$y = \sin^{-1}\left(\sqrt{2}t\right)$$

$$y' = \frac{\sqrt{2}}{\sqrt{1-(\sqrt{2}t)^2}} = \boxed{\frac{\sqrt{2}}{\sqrt{1-2t^2}}}$$

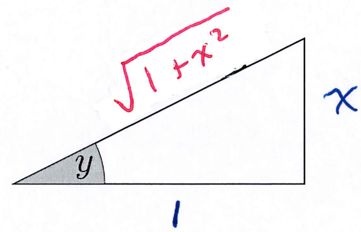
Example. Recall that $y = \tan^{-1}(x) \iff \tan(y) = x$. Use this fact and implicit differentiation to derive the derivative of $\tan^{-1}(x)$.

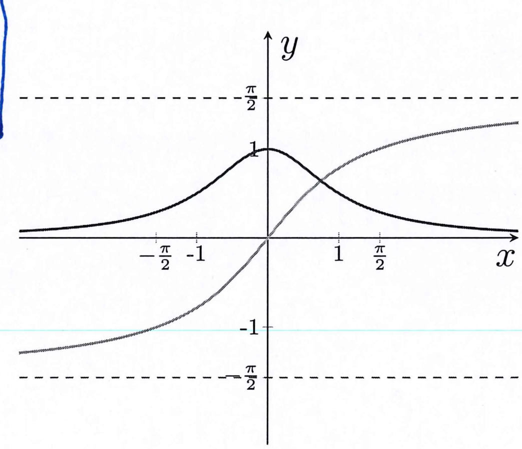
$y = \tan^{-1}(x)$

$\tan(y) = x$

$\sec^2(y) \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y) = \boxed{\frac{1}{1+x^2}}$





Note:

$\sin^2(y) + \cos^2(y) = 1$

$\underbrace{\tan^2(y)}_{x^2} + 1 = \sec^2(x)$

Next, extend this definition for the derivative of $\tan^{-1}(f(x))$.

$\frac{d}{dx}\left[\tan^{-1}(f(x))\right] = \frac{1}{1+[f(x)]^2} f'(x)$

Example. Find the derivative of the following

$y = \sqrt{\tan^{-1}(x)} = (\tan^{-1}(x))^{1/2}$

$\boxed{y' = \frac{1}{2\sqrt{\tan^{-1}(x)}} \cdot \frac{1}{1+x^2}}$

$y = \tan^{-1}(\sqrt{x})$

$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}$

$= \boxed{\frac{1}{2\sqrt{x}(1+x)}}$

Derivatives of Inverse Trigonometric Functions

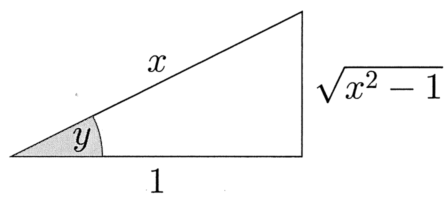
$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\csc^{-1}(x)\right] = -\frac{1}{|x|\sqrt{x^2-1}}$$

Derivative of $y = \sec^{-1}(x)$

$$\begin{aligned} y &= \sec^{-1}(x) \\ \sec(y) &= x \\ \sec(y)\tan(y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec(y)\tan(y)} \end{aligned}$$



$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \tan^2(x) + 1 &= \sec^2(x) \\ 1 + \cot^2(x) &= \csc^2(x) \end{aligned}$$

Now we rewrite $\sec(y)\tan(y)$ in terms of x. Note that the domain of $\sec(y)$ is restricted to $[0, \pi/2) \cup (\pi/2, \pi]$. If we look at these quadrants of the unit circle, we see that $\sec(y)\tan(y)$ is always positive, so the resulting derivative is always positive:

$$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$$

Example. Find the derivatives of the following functions:

$$h(t) = e^{\sec^{-1}(t)}$$

$$\boxed{h'(t) = e^{\sec^{-1}(t)} \frac{1}{|x|\sqrt{x^2-1}}}$$

$$y = \sin^{-1}(2x+1)$$

$$y' = \frac{2}{\sqrt{1-(2x+1)^2}}$$

$$= \frac{2}{\sqrt{-4x^2-4x}} = \boxed{\frac{1}{\sqrt{-x^2-x}}}$$

$$f(x) = \csc^{-1}\left(\tan(e^x)\right)$$

$$f'(x) = \boxed{-\frac{\sec^2(e^x)\,e^x}{|\tan(e^x)|\sqrt{\tan^2(e^x)-1}}}$$

$$y = \arccos\left(e^{2x}\right) = \cos^{-1}(e^{2x})$$

$$y' = -\frac{1}{\sqrt{1-(e^{2x})^2}}\, e^{2x}(2)$$

$$= -\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

$$y = \sec^{-1}(5r)$$

$$y' = \frac{5}{|5r|\sqrt{(5r)^2-1}}$$

$$= \boxed{\frac{1}{|r|\sqrt{25r^2-1}}}$$

$$f(x) = \tan^{-1}(10x)$$

$$f'(x) = \frac{1}{1+(10x)^2}(10)$$

$$= \boxed{\frac{10}{1+100x^2}}$$

$$y = x\sin^{-1}(x) + \sqrt{1-x^2}$$

$$y' = \sin^{-1}(x) + x\,\frac{1}{\sqrt{1-x^2}} + \frac{2x}{2\sqrt{1-x^2}}$$

$$= \boxed{\sin^{-1}(x)}$$

$$h(t) = \cot^{-1}(t) + \cot^{-1}\left(\tfrac{1}{t}\right)$$

$$h'(t) = -\frac{1}{1+t^2} - \frac{-t^{-2}}{1+(t^{-1})^2}\left(\frac{t^2}{t^2}\right)$$

$$= -\frac{1}{1+t^2} + \frac{1}{t^2+1} = \boxed{0}$$

$$f(x) = 2x\tan^{-1}(x) - \ln\left(1+x^2\right)$$

$$f'(x) = 2\tan^{-1}(x) + \frac{2x}{1+x^2} - \frac{2x}{1+x^2}$$

$$= \boxed{2\tan^{-1}(x)}$$

$$f(t) = \ln\left(\tan^{-1}(t)\right)$$

$$f'(t) = \boxed{\frac{1}{\tan^{-1}(t)}\,\frac{1}{1+t^2}}$$

Example. Find the equation of the tangent line to $f(x) = \cos^{-1}(x^2)$ at $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right)$.

$$f'(x) = -\frac{1}{\sqrt{1-(x^2)^2}} = \frac{-1}{\sqrt{1-x^4}}$$

$$f'\left(\tfrac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^4}} = \boxed{\frac{-2}{\sqrt{3}}}$$

$$y - y_1 = m(x - x_1)$$

$$y - \pi/3 = \frac{-2}{\sqrt{3}}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\boxed{y = \frac{-2\sqrt{3}}{3}x + \frac{\sqrt{6}+\pi}{3}}$$

Example. Find the equation of the tangent line to $f(x) = \sec^{-1}(e^x)$ at $\left(\ln(2), \frac{\pi}{3}\right)$.

$$f'(x) = \frac{1}{|e^x|\sqrt{e^{2x}-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}}$$

$$f'(\ln(2)) = \frac{1}{\sqrt{e^{2\ln(2)}-1}} = \frac{1}{\sqrt{3}}$$

$$y - y_1 = m(x - x_1)$$

$$y - \pi/3 = \frac{\sqrt{3}}{3}(x - \ln(2))$$

$$\boxed{y = \frac{\sqrt{3}}{3}x + \frac{\pi - \sqrt{3}\ln(2)}{3}}$$