## 5.5: Substitution Rule

**Theorem 5.6: Substitution Rule for Indefinite Integrals**
Let $u = g(x)$, where $g$ is differentiable on an interval, and let $f$ be continuous on the corresponding range of $g$. On that interval,

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

**Example.** We know

$$\frac{d}{dx}\left[\frac{(2x+1)^4}{4}\right] = 2(2x+1)^3$$

Thus, if $f(x) = x^3$ and $g(x) = 2x+1$ then $g'(x) = 2$, so we let $u = 2x+1$, then

$$\begin{aligned}\int 2(2x+1)^3\,dx &= \int f\big(g(x)\big)g'(x)\,dx \\ &= \int u^3\,du \\ &= \frac{u^4}{4} + C \\ &= \frac{(2x+1)^4}{4} + C\end{aligned}$$

**Procedure: Substitution Rule (Change of Variables)**

1. Given an indefinite integral involving a composite function $f\big(g(x)\big)$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
3. Evaluate the new indefinite integral with respect to $u$.
4. Write the result in terms of $x$ using $u = g(x)$.

**Example.** Evaluate the following integrals:

a) $\int 2x(x^2+3)^4\,dx$

$u = x^2+3$

$du = 2x\,dx$

$= \int u^4\,du = \frac{u^5}{5} + C$

$= \boxed{\frac{1}{5}(x^2+3)^5 + C}$

b) $\int (2x+1)^3\,dx$

$u = 2x+1$

$du = 2\,dx \rightarrow \frac{1}{2}du = dx$

$= \frac{1}{2}\int u^3\,du = \frac{1}{8}u^4 + C$

$= \boxed{\frac{1}{8}(2x+1)^4 + C}$

c) $\int x^2\sqrt{x^3+1}\,dx$

$u = x^3+1$

$du = 3x^2\,dx \rightarrow \frac{1}{3}du = x^2\,dx$

$= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{9}(x^3+1)^{3/2} + C}$

d) $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

$u = 1-\theta^2$

$du = -2\theta\,d\theta \rightarrow -\frac{1}{2}du = \theta\,d\theta$

$= -\frac{1}{2}\int u^{1/4}\,du = -\frac{1}{2}\cdot\frac{4}{5}u^{5/4} + C$

$= \boxed{-\frac{2}{5}(1-\theta^2)^{5/4} + C}$

e) $\int \sqrt{4-t}\,dt$

$u = 4-t$

$du = -dt$

$= -\int u^{1/2}\,du = -\frac{2}{3}u^{3/2} + C$

$= \boxed{-\frac{2}{3}(4-t)^{3/2} + C}$

f) $\int (2-x)^6\,dx$

$u = 2-x$

$du = -dx$

$= -\int u^6\,du = -\frac{1}{7}u^7 + C$

$= \boxed{-\frac{1}{7}(2-x)^7 + C}$

**Example.** Evaluate the following integrals:

a) $\int \sec(2\theta)\tan(2\theta)\, d\theta$

$u = 2\theta$

$du = 2\, d\theta \to \frac{1}{2} du = d\theta$

$= \frac{1}{2}\int \sec(u)\tan(u)\, du = \frac{1}{2}\sec(u) + C$

$= \boxed{\frac{1}{2}\sec(2\theta) + C}$

b) $\int \csc^2\left(\frac{t}{3}\right) dt$

$u = \frac{t}{3}$

$du = \frac{1}{3} dt \to 3\, du = dt$

$= 3\int \csc^2(u)\, du = -3\cot(u) + C$

$= \boxed{-3\cot\left(\frac{t}{3}\right) + C}$

c) $\int \frac{\sin(x)}{1+\cos^2(x)}\, dx$

$u = \cos(x)$

$du = -\sin(x)\, dx \to -du = \sin(x)\, dx$

$= -\int \frac{1}{1+u^2}\, du = -\tan^{-1}(u) + C$

$= \boxed{-\tan^{-1}(\cos(x)) + C}$

d) $\int \frac{\tan^{-1}(x)}{1+x^2}\, dx$

$u = \tan^{-1}(x)$

$du = \frac{1}{1+x^2}\, dx$

$= \int u\, du = \frac{u^2}{2} + C = \boxed{\frac{(\tan^{-1}(x))^2}{2} + C}$

The acceleration of a particle moving back and forth on a line is $a(t) = \frac{d^2s}{dt^2} = \pi^2\cos(\pi t)\ m/s^2$ for all $t$. If $s = 0$ and $v = 8\ m/s$ when $t = 0$, find the value of $s$ when $t = 1$ sec.

$v(t) = \int a(t)\, dt = \int \pi^2 \cos(\pi t)\, dt = \int \pi \cos(u)\, du = \pi \sin(\pi t) + C$

$u = \pi t$

$du = \pi\, dt$

$v(0) = 8 = \pi \sin(0) + C$

$\Rightarrow C = 8$

$s(t) = \int v(t)\, dt = \int (\pi \sin(\pi t) + 8)\, dt = \int \sin(u)\, du + \int 8\, dt$

$= -\cos(\pi t) + 8t + C \longrightarrow s(0) = 0 = -\cos(0) + 0 + C$

$\Rightarrow C = 1$



$\Rightarrow s(t) = -\cos(\pi t) + 8t + 1$

$\Rightarrow \boxed{s(1) = 10\ m}$

**Example.** Evaluate the following integrals:

a) $\int (6x^2+2)\sin(x^3+x+1)\,dx$

$u = x^3 + x + 1$

$du = 3x^2 + 1\, dx$

$= \int 2\sin(u)\,du = -2\cos(u) + c$

$= \boxed{-2\cos(x^3+x+1) + c}$

b) $\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$= -\int u^{-5}\,du = \frac{u^{-4}}{4} + c$

$= \boxed{\frac{1}{4}\sec^4(x) + c}$

c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int e^u\,du = 2e^u + c$

$= \boxed{2e^{\sqrt{x}} + c}$

d) $\int \frac{2^t}{2^t+3}\,dt$

$u = 2^t + 3$

$du = \ln(2)\, 2^t$

$= \frac{1}{\ln(2)}\int \frac{1}{u}\,du = \frac{\ln|u|}{\ln(2)} + c$

$= \boxed{\frac{\ln(2^t+3)}{\ln(2)} + c}$

e) $\int 6x^2 4^{x^3}\,dx$

$u = x^3$

$du = 3x^2\,dx$

$= 2\int 4^u\,du$

$v = 4^u$

$dv = \ln(4)\, 4^u\, du$

$= \frac{2}{\ln(4)}\int dv$

$= \frac{2}{\ln(4)} v + c = \frac{2}{\ln(4)} 4^u + c$

$= \boxed{\frac{2}{\ln(4)} 4^{x^3} + c}$

f) $\int \frac{dx}{\sqrt{36-4x^2}} = \int \frac{dx}{2\sqrt{9-\frac{4}{9}x^2}}$

$= \int \frac{dx}{6\sqrt{1-\left(\frac{x}{3}\right)^2}}$

$u = \frac{x}{3}$ $\quad du = \frac{1}{3}\,dx$

$= \int \frac{du}{2\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}(u) + c$

$= \boxed{\frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right) + c}$

g) $\int \sin(t)\sec^2(\cos(t))\,dt$

$u = \cos(t)$ $\quad du = -\sin(t)\,dt$

$= -\int \sec^2(u)\,du$

$= -\tan(u) + C$

$= \boxed{-\tan(\cos(t)) + C}$

h) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int \frac{1}{u^2}\,du$

$= -\frac{2}{u} + C$

$= \boxed{\frac{-2}{(1+\sqrt{x})^2} + C}$

i) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= 2\int \sin(u)\,du$

$= -2\cos(u) + C$

$= \boxed{-2\cos(\sqrt{x}) + C}$

j) $\int 5\cos(7x+5)\,dx$

$u = 7x+5$

$du = 7\,dx$

$= \frac{5}{7}\int \cos(u)\,du$

$= \frac{5}{7}\sin(u) + C$

$= \boxed{\frac{5}{7}\sin(7x+5) + C}$

k) $\int \frac{3}{\sqrt{1-25x^2}}\,dx$

$u = 5x$

$du = 5x$

$= \frac{3}{5}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{3}{5}\sin^{-1}(u) + C$

$= \boxed{\frac{3}{5}\sin^{-1}(5x) + C}$

l) $\int \frac{dx}{\sqrt{1-9x^2}}$

$u = 3x$

$du = 3\,dx$

$= \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{3}\sin^{-1}(u) + C$

$= \boxed{\frac{1}{3}\sin^{-1}(3x) + C}$

**Example.** Evaluate the following integrals using the recommended substitution:

a) $\int \sec^2(x)\tan(x)\,dx$

where $u = \tan(x)$.

$du = \sec^2(x)\,dx$

$= \int u\,du$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{1}{2}\tan^2(x) + C}$

b) $\int \sec^2(x)\tan(x)\,dx$

where $u = \sec(x)$.

$du = \sec(x)\tan(x)\,dx$

$= \int u\,du$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{1}{2}\sec^2(x) + C}$

Note:
Equivalent since $\tan^2\theta + 1 = \sec^2\theta$

**Example.** Solve the initial value problem: $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}$, $y(0) = 0$.

$y(x) = \int 4x(x^2+8)^{-1/3}\,dx = 2\int u^{-1/3}\,du = -3u^{2/3} + C$

$= -3(x^2+8)^{2/3} + C$

$u = x^2 + 8$

$du = 2x\,dx$

$y(0) = 0 = -3(0+8)^{2/3} + C \Rightarrow C = 12$

$\boxed{y(x) = -3(x^2+8)^{2/3} + 12}$

**Example.** Evaluate the following integrals:

a) $\int xe^{-x^2}\,dx$ $\quad u = -x^2$, $du = -2x\,dx$

$= -\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^u + c$

$= \boxed{-\frac{1}{2}e^{-x^2} + c}$

b) $\int \frac{e^{1/x}}{x^2}\,dx$ $\quad u = 1/x$, $du = -1/x^2\,dx$

$= -\int e^u\,du$

$= -e^u + c = \boxed{-e^{1/x} + c}$

c) $\int \frac{dt}{8-3t}$ $\quad u = 8-3t$, $du = -3\,dt$

$= -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|u| + c = \boxed{-\frac{1}{3}\ln|8-3t| + c}$

d) $\int 5^t \sin(5^t)\,dt$ $\quad u = 5^t$, $du = 5^t dt$

$= \frac{1}{\ln(5)}\int \sin(u)\,du$

$= \frac{-\cos(u)}{\ln(5)} + c$

$= \boxed{\frac{-\cos(5^t)}{\ln(5)} + c}$

e) $\int \frac{e^w}{36+e^{2w}}\,dw$ $\quad u = e^w$, $du = e^w\,dw$

$= \int \frac{du}{36 + \left(\frac{36}{36}\right)u^2} = \frac{1}{36}\int \frac{du}{1+\left(\frac{u}{6}\right)^2}$ $\quad v = u/6$, $dv = \frac{1}{6}\,du$

$= \frac{1}{6}\int \frac{dv}{1+v^2}$

$= \frac{1}{6}\tan^{-1}(v) + c = \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + c = \boxed{\frac{1}{6}\tan^{-1}\left(\frac{e^w}{6}\right) + c}$

**Theorem 5.7: Substitution Rule for Definite Integrals**
Let $u = g(x)$, where $g'$ is continuous on $[a, b]$, and let $f$ be continuous on the range of $g$. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

**Example.** Evaluate the integrals:

a) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$

$u = x^4 + 9$
$du = 4x^3\,dx$

$x=0,\ u=9$
$x=1,\ u=10$

$$= \frac{1}{4}\int_9^{10} u^{-1/2}\,du = \frac{1}{2}u^{1/2}\Big|_9^{10}$$

$$= \boxed{\frac{1}{2}(\sqrt{10}-3)}$$

b) $\displaystyle\int_1^3 \frac{dt}{(t-4)^2}$

$u = t-4$
$du = dt$

$t=1,\ u=-3$
$t=3,\ u=-1$

$$= \int_{-3}^{-1} u^{-2}\,du$$

$$= -u^{-1}\Big|_{-3}^{-1} = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

c) $\displaystyle\int_0^3 \frac{v^2+1}{\sqrt{v^3+3v+4}}\,dv$

$u = v^3 + 3v + 4$
$du = 3v^2 + 3\,dv$

$v=0,\ u=4$
$v=3,\ u=40$

$$= \frac{1}{3}\int_4^{40} u^{-1/2}\,du$$

$$= \frac{2}{3}u^{1/2}\Big|_4^{40} = \boxed{\frac{4}{3}(\sqrt{10}-2)}$$

d) $\displaystyle\int_0^1 2x(4-x^2)\,dx$

$u = 4 - x^2$
$du = -2x\,dx$

$x=0,\ u=4$
$x=1,\ u=3$

$$= \int_4^3 -u\,du = -\frac{u^2}{2}\Big|_4^3$$

$$= -(9-16)$$

$$= \boxed{7}$$

e) $\displaystyle\int_2^3 \frac{x}{\sqrt[3]{x^2-1}}\,dx$ $u = x^2-1$, $du = 2x\,dx$

$x=2,\ u=3$

$x=3,\ u=8$

$= \frac{1}{2}\int_3^8 u^{-1/3}\,du$

$= \frac{3}{4} u^{2/3}\Big|_3^8$

$= \boxed{\frac{3}{4}\left(4-\sqrt[3]{9}\right)}$

f) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$ $u = 1+\cos(x)$, $du = -\sin(x)\,dx$

$x=0,\ u=2$

$x=\pi/2,\ u=1$

$= -\int_2^1 \frac{1}{u}\,du$

$= -\ln(u)\Big|_2^1$

$= \boxed{\ln(2)}$

g) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$ $u=\cos(x)$, $du = -\sin(x)\,dx$

$x=0,\ u=1$

$x=\pi/4,\ u=\sqrt{2}/2$

$= -\int_1^{\sqrt{2}/2} u^{-2}\,du$

$= \frac{1}{u}\Big|_1^{\sqrt{2}/2} = \boxed{\sqrt{2}-1}$

h) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2(2y)\,dy$ $u=2y$, $du = 2\,dy$

$y=-\pi/12,\ u=-\pi/6$

$y=\pi/8,\ u=\pi/4$

$= \frac{1}{2}\int_{-\pi/6}^{\pi/4} \sec^2(u)\,du$

$= \frac{1}{2}\tan(u)\Big|_{-\pi/6}^{\pi/4}$

$= \boxed{\frac{1}{2}\left(1-\frac{\sqrt{3}}{3}\right)}$

i) $\int_0^1 (1-2x^9)\,dx$

$= x - \frac{x^{10}}{5}\Big|_0^1$

$= 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$

j) $\int_0^1 (1-2x)^9\,dx$

$u = 1-2x$

$du = -2\,dx$

$x=0,\ u=1$

$x=1,\ u=-1$

$= -\frac{1}{2}\int_1^{-1} u^9\,dx$

$= -\frac{1}{2}\,\frac{u^{10}}{10}\Big|_1^{-1}$

$= -\frac{1}{2}\left(\frac{1}{10}-\frac{1}{10}\right) = \boxed{0}$

k) $\int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$

$u = 2x$

$du = 2du$

$x=0,\ u=0$

$x=\frac{1}{2},\ u=1$

$= \frac{1}{2}\int_0^1 \frac{du}{1+u^2}$

$= \frac{1}{2}\tan^{-1}(u)\Big|_0^1$

$= \boxed{\frac{\pi}{8}}$

l) $\int_0^4 \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$x=0,\ u=1$

$x=4,\ u=17$

$= \frac{1}{2}\int_1^{17} \frac{1}{u}\,du = \frac{1}{2}\ln(u)\Big|_1^{17}$

$= \boxed{\frac{1}{2}\ln(17)}$

m) $\int_0^{\pi} 3\cos^2(x)\sin(x)\,dx$

$u = \cos(x)$
$du = -\sin(x)$

$x = 0,\ u = 1$
$x = \pi,\ u = -1$

$= -3\int_1^{-1} u^2\,du$

$= -u^3\Big|_1^{-1}$

$= \boxed{2}$

n) $\int_0^{\frac{\pi}{8}} \sec(2\theta)\tan(2\theta)\,d\theta$

$u = 2\theta$
$du = 2d\theta$

$\theta = 0,\ u = 0$
$\theta = \pi/8,\ u = \pi/4$

$= \frac{1}{2}\int_0^{\pi/4} \sec(u)\tan(u)\,du$

$= \frac{1}{2}\sec(u)\Big|_0^{\pi/4}$

$= \boxed{\frac{1}{2}(\sqrt{2}-1)}$

o) $\int_0^{1} (3t-1)^{50}\,d\,t$

$u = 3t-1$
$du = 3\,dt$

$t = 0,\ u = -1$
$t = 1,\ u = 2$

$= \frac{1}{3}\int_{-1}^{2} u\,du = \frac{1}{3}\frac{u^2}{2}\Big|_{-1}^{2}$

$= \frac{1}{3}\left(2 - \frac{1}{2}\right)$

$= \boxed{\frac{1}{2}}$

p) $\int_0^{3} \frac{1}{5x+1}\,dx$

$u = 5x+1$
$du = 5\,dx$

$x = 0,\ u = 1$
$x = 3,\ u = 16$

$= \frac{1}{5}\int_1^{16} \frac{1}{u}\,du$

$= \frac{1}{5}\ln(u)\Big|_1^{16}$

$= \boxed{\frac{1}{5}\ln(16)}$

$= \frac{4}{5}\ln(2)$

q) $\int_0^1 xe^{-x^2}\,dx$ $u = -x^2$, $du = -2x\,dx$

$x=0,\ u=0$

$x=1,\ u=-1$

$= -\frac{1}{2}\int_0^{-1} e^u\,du = -\frac{1}{2}e^u\Big|_0^{-1}$

$= -\frac{1}{2}(e^{-1} - e^0) = \boxed{\frac{1}{2}(1-e^{-1})}$

r) $\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}\,dx$ $u = \ln(x)$, $du = \frac{1}{x}\,dx$

$x=e,\ u=1$

$x=e^4,\ u=4$

$= \int_1^4 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_1^4$

$= 2(2-1) = \boxed{2}$

s) $\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}\,dx$ $u = \sin^{-1}(x)$, $du = \frac{dx}{\sqrt{1-x^2}}$

$x=0,\ u=0$

$x=\frac{1}{2},\ u=\frac{\pi}{6}$

$\int_0^{\pi/6} u\,du = \frac{u^2}{2}\Big|_0^{\pi/6}$

$= \boxed{\frac{\pi^2}{72}}$

t) $\int_0^1 \frac{e^z+1}{e^z+z}\,dz$ $u = e^z + z$, $du = e^z + 1\ dz$

$z=0,\ u=1$

$z=1,\ u=e+1$

$\int_1^{e+1} \frac{1}{u}\,du = \ln(u)\Big|_1^{e+1}$

$= \ln(e+1) - \ln(1)$

$= \boxed{\ln(e+1)}$

u) $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

$u = 1+\sqrt{y}$

$du = \frac{1}{2\sqrt{y}}\,dy$

$y=1,\ u=2$

$y=4,\ u=3$

$$= \int_2^3 u^{-2}\,du = -\frac{1}{u}\Big|_2^3$$

$$= -\frac{1}{3} - \left(-\frac{1}{2}\right)$$

$$= \boxed{\frac{1}{6}}$$

v) $\displaystyle\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln\left(\frac{\pi}{2}\right)} e^w \cos(e^w)\,dw$

$u = e^w$

$du = e^w\,dw$

$w = \ln(\pi/4),\ u = \pi/4$

$w = \ln(\pi/2),\ u = \pi/2$

$$= \int_{\pi/4}^{\pi/2} \cos(u)\,du$$

$$= \sin(u)\Big|_{\pi/4}^{\pi/2}$$

$$= \boxed{1 - \frac{\sqrt{2}}{2}}$$

w) $\displaystyle\int_0^{\frac{1}{8}} \frac{x}{\sqrt{1-16x^2}}\,dx$

$u = 1-16x^2$

$du = -32\,x\,dx$

$x=0,\ u=1$

$x=\frac{1}{8},\ u=\frac{3}{4}$

$$= -\frac{1}{32}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = -\frac{\sqrt{u}}{64}\Big|_1^{3/4}$$

$$= \boxed{\frac{1-\sqrt{3}/2}{64}}$$

x) $\displaystyle\int_1^{e^2} \frac{\ln(p)}{p}\,dp$

$u = \ln(p)$

$du = \frac{1}{p}\,dp$

$p=1,\ u=0$

$p=e^2,\ u=2$

$$= \int_0^2 u\,du = \frac{u^2}{2}\Big|_0^2$$

$$= \boxed{2}$$

y) $\displaystyle\int_0^{\frac{\pi}{4}} e^{\sin^2(x)} \sin(2x)\,dx$

$u = \sin^2(x)$

$du = 2\sin(x)\cos(x)\,dx$

$du = \sin(2x)\,dx$

$x=0,\ u=0$

$x=\pi/4,\ u=\frac{\sqrt{2}}{2}$

$$= \int_0^{\sqrt{2}/2} e^u\,du = e^u\Big|_0^{\sqrt{2}/2}$$

$$= \boxed{e^{\sqrt{2}/2} - 1}$$

z) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin(7x^3)\,dx$

$u = 7x^3$

$du = 21x^2\,dx$

$x=-\pi,\ u=-7\pi^3$

$x=\pi,\ u=7\pi^3$

$$= \frac{1}{21}\int_{-7\pi^3}^{7\pi^3} \sin(u)\,du$$

$$= \boxed{0} \quad \because \text{ odd function}$$

**Example. Average velocity:** An object moves in one dimension with a velocity in $m/s$ given by $v(t) = 8\sin(\pi t) + 2t$. Find its average velocity over the time interval from $t = 0$ to $t = 10$, where $t$ is measured in seconds.

$$\bar{v} = \frac{1}{10-0}\int_0^{10} 8\sin(\pi t) + 2t\,dt$$

$u = \pi t$

$du = \pi\,dt$

$t=0,\ u=0$

$t=10,\ u=10\pi$

$$= \frac{8}{10\pi}\int_0^{10\pi} \sin(u)\,du + \frac{2}{10}\int_0^{10} t\,dt$$

$$= \frac{-4}{5\pi}\cos(u)\Big|_0^{10\pi} + \frac{1}{10}t^2\Big|_0^{10}$$

$$= -\frac{4}{5\pi}(1-1) + \frac{1}{10}(100-0) = \boxed{10}$$

**Example.** Prove $\displaystyle\int \tan(x)\, dx = \ln|\sec(x)| + C$.

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

$$= \ln|\sec(x)| + C$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

**Example.** Evaluate the integrals:

a) $\displaystyle\int \frac{x}{(x-2)^3}\,dx$

$u = x-2$

$du = dx$

$$= \int \frac{u+2}{u^3}\,du$$

$$= \int u^{-2} + 2u^{-3}\,du$$

$$= -u^{-1} - u^{-2} + C$$

$$= -\frac{u-1}{u^2} + C$$

$$= \boxed{-\frac{x-3}{(x-2)^2} + C}$$

b) $\displaystyle\int x\sqrt{x-1}\,dx$

$u = x-1$

$du = dx$

$$= \int (u+1)\sqrt{u}\,du$$

$$= \int u^{3/2} + u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= 2u^{3/2}\left(\frac{3u+10}{15}\right) + C$$

$$= \boxed{2(x-1)^{3/2}\left(\frac{3x+7}{15}\right) + C}$$

c) $\int x^3(1+x^2)^{\frac{3}{2}}\,dx$ $\quad u = x^2$, $du = 2x\,dx$

$= \frac{1}{2}\int u(1+u)^{3/2}\,du$

$t = 1+u$

$dt = du$

$= \frac{1}{2}\int (t-1)\,t^{3/2}\,dt$

$= \frac{1}{2}\int t^{5/2} - t^{3/2}\,dt$

$= \frac{t^{7/2}}{7} - \frac{t^{5/2}}{5} + C = \boxed{\frac{(1+x^2)^{7/2}}{7} - \frac{(1+x^2)^{5/2}}{5} + C}$

d) $\int \frac{y^2}{(y+1)^4}\,dy$ $\quad u = y+1$, $du = dy$

$\int \frac{(u-1)^2}{u^4}\,du = \int \frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4}\,du$

$= -\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3} + C$

$= \boxed{-\frac{1}{y+1} + \frac{1}{(y+1)^2} - \frac{1}{3(y+1)^3} + C}$

e) $\int (z+1)\sqrt{3z+2}\,dz$ $\quad u = 3z+2$, $du = 3\,dz$

$= \frac{1}{3}\int \left(\frac{u}{3} + \frac{1}{3}\right)\sqrt{u}\,du$

$= \frac{1}{9}\int u^{3/2} + u^{1/2}\,du$

$= \frac{1}{9}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$

$= \frac{2u^{3/2}}{9}\left(\frac{3u+10}{15}\right) + C$

$= \boxed{\frac{2}{135}(3z+2)^{3/2}(9z+16) + C}$

f) $\int_0^1 \frac{x}{(x+2)^3}\,dx$ $\quad u = x+2$, $du = dx$

$x = 0,\ u = 2$

$x = 1,\ u = 3$

$= \int_2^3 \frac{u-2}{u^3}\,du$

$= \int_2^3 u^{-2} - 2u^{-3}\,du$

$= -u^{-1} + u^{-2}\Big|_2^3$

$= \left(-\frac{1}{3} + \frac{1}{9}\right) - \left(-\frac{1}{2} + \frac{1}{4}\right)$

$= \boxed{1/36}$

**Half-Angle Formulas**

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

**Example.** Evaluate the integrals:

a) $\int \cos^2(x)\,dx$

$$= \int \frac{1+\cos(2x)}{2}\,dx$$

$u = 2x$
$du = 2dx$

$$= \frac{1}{2}\int dx + \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{x}{2} + \frac{\sin(u)}{4} + C$$

$$= \boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

b) $\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx$

$u = 2x$
$du = 2\,dx$

$x = 0,\ u = 0$
$x = \pi/2,\ u = \pi$

$$= \frac{1}{2}\int_0^{\pi/2} dx + \frac{1}{4}\int_0^{\pi} \cos(u)\,du$$

$$= \frac{\pi}{4} + \frac{1}{4}(-1-1)$$

$$= \boxed{\frac{\pi - 2}{4}}$$

c) $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$ $\quad u = 1/x$, $\quad du = -1/x^2\,dx$

$= -\int \cos^2(u)\,du$

$= -\frac{1}{2}\int du - \frac{1}{2}\int \cos\left(\frac{u}{2}\right)du$ $\quad t = u/2$, $\quad dt = 1/2\,du$

$= -\frac{u}{2} - \int \cos(t)\,dt$

$= \boxed{-\frac{1}{2x} - \sin\left(\frac{1}{2x}\right) + C}$

d) $\displaystyle\int x\sin^2(x^2)\,dx$ $\quad u = x^2$, $\quad du = 2x\,dx$

$= \frac{1}{2}\int \frac{1-\cos(u/2)}{2}\,du$ $\quad t = u/2$, $\quad dt = \frac{1}{2}du$

$= \frac{1}{4}\int du - \int \cos(t)\,dt$

$= \frac{u}{4} - \sin(t) + c$

$= \boxed{\frac{x^2}{4} - \sin\left(\frac{x^2}{2}\right) + c}$

e) $\displaystyle\int \sin^2\left(\theta + \frac{\pi}{6}\right)d\theta$ $\quad u = \theta + \pi/6$, $\quad du = d\theta$

$= \int \frac{1-\cos(\frac{u}{2})\,du}{2}$ $\quad t = u/2$, $\quad dt = 1/2\,du$

$= \frac{1}{2}\int du - \int \cos(t)\,dt$

$= \frac{u}{2} - \sin(t) + C$

$= \boxed{\frac{\theta}{2} + \frac{\pi}{12} - \sin\left(\frac{\theta}{2} + \frac{\pi}{12}\right) + c}$

f) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(8\theta)\,d\theta$ $\quad u = 8\theta$, $\quad du = 8\,d\theta$

$= \frac{1}{8}\int_0^{2\pi} \cos^2(u)\,du$ $\quad \theta = 0, u = 0$; $\quad \theta = \pi/4, u = 2\pi$

$= \frac{1}{8}\int_0^{2\pi} \frac{1}{2}\,du + \frac{1}{8}\int_0^{2\pi} \frac{\cos(u/2)}{2}\,du$ $\quad t = u/2$, $\quad dt = \frac{1}{2}du$; $\quad u = 0, t = 0$; $\quad u = 2\pi, t = \pi$

$= \frac{u}{16}\Big|_0^{2\pi} + \frac{1}{8}\int_0^{\pi} \cos(t)\,dt$

$= \frac{\pi}{8} + \frac{1}{8}\sin(t)\Big|_0^{\pi} = \boxed{\frac{\pi}{8}}$

**Example.** If $f$ is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

$u = 2x$

$du = 2\,dx$

$x = 0,\ u = 0$

$x = 2,\ u = 4$

$$\int_0^2 f(2x)\,dx = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = \boxed{5}$$

**Example.** If $f$ is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$.

$u = x^2$

$du = 2x\,dx$

$x = 0,\ u = 0$

$x = 3,\ u = 9$

$$\int_0^3 x f(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = \boxed{2}$$

**Example.** Suppose $f$ is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the following:

a) $\int_{-1}^{1} x f(x^2)\,dx$.

$u = x^2$

$du = 2x\,dx$

$x = -1,\ u = 1$

$x = 1,\ u = 1$

$= \frac{1}{2}\int_1^1 f(u)\,du$

$= \boxed{0}$

b) $\int_{-2}^{2} x^2 f(x^3)\,dx$.

$u = x^3$

$du = 3x^2\,dx$

$x = -2,\ u = -8$

$x = 2,\ u = 8$

$= \frac{1}{3}\int_{-8}^{8} f(u)\,du = \frac{2}{3}\int_0^8 f(u)\,du$

$= \frac{2}{3}(9) = \boxed{6}$

**Example.** Evaluate the integrals:

a) $\int \sec^2(10x)\,dx$

$u = 10x \qquad du = 10\,dx$

$= \frac{1}{10}\int \sec^2(u)\,du$

$= \frac{1}{10}\tan(u) + C$

$= \boxed{\frac{1}{10}\tan(10x) + C}$

b) $\int \tan^{10}(4x)\sec^2(4x)\,dx$

$u = 4x \qquad du = 4\,dx$

$= \frac{1}{4}\int \tan^{10}(u)\sec^2(u)\,du$

$t = \tan(u) \qquad dt = \sec^2(u)\,du$

$= \frac{1}{4}\int t^{10}\,dt = \frac{1}{44}t^{11} + C$

$= \boxed{\frac{1}{44}\tan^{11}(4x) + C}$

c) $\int \left(x^{\frac{3}{2}} + 8\right)^5 \sqrt{x}\,dx$

$u = x^{3/2}$

$du = \frac{3}{2}x^{1/2}dx$

$= \frac{2}{3}\int (u+8)^5\,du$

$t = u+8$

$dt = du$

$= \frac{2}{3}\int t^5\,dt$

$= \frac{1}{9}t^6 + C = \boxed{\frac{1}{9}\left(x^{3/2}+8\right)^6 + C}$

d) $\int \frac{2x}{\sqrt{3x+2}}\,dx$

$u = 3x+2$

$du = 3\,dx$

$= \frac{1}{3}\int \frac{2(u-2)}{3\sqrt{u}}\,du$

$= \frac{2}{9}\int \sqrt{u} - \frac{2}{\sqrt{u}}\,du$

$= \frac{2}{9}\left(\frac{2}{3}u^{3/2} - 4\sqrt{u}\right) + C$

$= \boxed{\frac{4}{27}\sqrt{3x+2}\,(3x-4) + C}$

e) $\displaystyle\int \frac{7x^2+2x}{x}\,dx$

$= \displaystyle\int 7x+2\,dx$

$= \boxed{\frac{7}{2}x^2+2x+c}$

f) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

$u = e^x + e^{-x}$

$du = e^x - e^{-x}\,dx$

$\displaystyle\int \frac{1}{u}\,du = \ln|u| + c$

$= \boxed{\ln|e^x+e^{-x}| + c}$

g) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9+x^2}\,dx$

$u = \frac{x}{3}$

$du = \frac{1}{3}\,dx$

$= \displaystyle\int_0^{\sqrt{3}} \frac{3}{9+\frac{9}{9}x^2}\,dx$

$x=0,\ u=0$

$x=\sqrt{3},\ u=\sqrt{3}/3$

$= \displaystyle\int_0^{\sqrt{3}/3} \frac{1}{1+u^2}\,du$

$= \tan^{-1}(u)\Big|_0^{\sqrt{3}/3}$

$= \boxed{\pi/6}$

h) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin(2y)}{\sin^2(y)+2}\,dy$

$u = \sin^2(y)+2$

$du = 2\sin(y)\cos(y)\,dy$

$= \sin(2y)\,dy$

$y=0,\ u=2$

$y=\pi/6,\ u=9/4$

$= \displaystyle\int_2^{9/4} \frac{1}{u}\,du$

$= \ln(u)\Big|_2^{9/4} = \ln(9/4) - \ln(2)$

$= \boxed{\ln(9/8)}$

i) $\displaystyle\int \frac{\sec(z)\tan(z)}{\sqrt{\sec(z)}}\,dz$ $u = \sec(z)$, $du = \sec(z)\tan(z)\,dz$

$= \displaystyle\int \frac{1}{\sqrt{u}}\,du$

$= 2\sqrt{u} + C$

$= \boxed{2\sqrt{\sec(z)} + C}$

j) $\displaystyle\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}}\,dx$ $u = \sin^{-1}(x)$, $du = \frac{1}{\sqrt{1-x^2}}\,dx$

$= \displaystyle\int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \boxed{\ln|\sin^{-1}(x)| + C}$

k) $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$ $u = 4-9x^2$, $du = -18x\,dx$

$= -\dfrac{1}{18}\displaystyle\int \frac{1}{\sqrt{u}}\,du$

$= -\dfrac{1}{9}\sqrt{u} + C$

$= \boxed{-\dfrac{1}{9}\sqrt{4-9x^2} + C}$

l) $\displaystyle\int \frac{x}{1+x^4}\,dx$ $u = x^2$, $du = 2x\,dx$

$= \dfrac{1}{2}\displaystyle\int \frac{1}{1+u^2}\,du$

$= \dfrac{1}{2}\tan^{-1}(u) + C$

$= \boxed{\dfrac{1}{2}\tan^{-1}(x^2) + C}$

m) $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$ $= 2\int \frac{1}{u^2}\,du = -2\frac{1}{u} + c$

$= \boxed{-2\csc(\sqrt{\theta}) + c}$

$u = \sin(\sqrt{\theta})$

$du = \frac{\cos(\sqrt{\theta})}{2\sqrt{\theta}}\,d\theta$

n) $\displaystyle\int x^2\sqrt{2+x}\,dx$ $= \int (u-2)^2\sqrt{u}\,du = \int u^{5/2} - 4u^{3/2} + 4u^{1/2}\,du$

$u = 2+x$

$du = dx$

$= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + c$

$= \boxed{\frac{2}{105}(x+2)^{3/2}(15x^2 - 24x + 32) + c}$

o) $\displaystyle\int \left(\sin^5(x) + 3\sin^3(x) - \sin(x)\right)\cos(x)\,dx$

$u = \sin(x)$

$du = \cos(x)dx$

$= \int (u^5 + 3u^3 - u)\,du = \frac{u^6}{6} + \frac{3}{4}u^4 - \frac{u^2}{2} + c$

$= \boxed{\frac{\sin^2(x)}{2}\left(\frac{\sin^4(x)}{3} + \frac{3}{2}\sin^2(x) - 1\right) + c}$

p) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(x^3 + x^4 \tan(x)\right) dx = \int_{-\pi/4}^{\pi/4} x^3 \, dx + \int_{-\pi/4}^{\pi/4} x^4 \tan(x) \, dx$

odd

$= 0 + 0 = \boxed{0}$

q) $\int_0^{\frac{\pi}{2}} \cos(x) \sin\left(\sin(x)\right) dx = \int_0^1 \sin(u) \, du = -\cos(u) \Big|_0^1$

$u = \sin(x)$

$du = \cos(x) \, dx$

$= \boxed{1 - \cos(1)}$

$x = 0, \; u = 0$

$x = \pi/2, \; u = 1$

r) $\int \frac{1+x}{1+x^2} \, dx = \int \frac{1}{1+x^2} \, dx + \int \frac{x}{1+x^2} \, dx$

$u = 1 + x^2$

$du = 2x \, dx$

$= \tan^{-1}(x) + \frac{1}{2} \int \frac{1}{u} \, du$

$= \tan^{-1}(x) + \frac{1}{2} \ln|u| + C$

$= \boxed{\tan^{-1}(x) + \frac{1}{2} \ln|1+x^2| + C}$

**Example.** Evaluate the more challenging integrals:

a) $\displaystyle\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$  $u = 1+x$, $du = dx$

$= \displaystyle\int \frac{du}{\sqrt{1+\sqrt{u}}}$  $t = 1+\sqrt{u}$, $dt = \frac{1}{2\sqrt{u}}\,du \rightarrow 2\sqrt{u}\,dt = du$

$\rightarrow 2(t-1)\,dt = du$

$= 2\displaystyle\int \frac{t-1}{\sqrt{t}}\,dt$

$= 2\displaystyle\int \sqrt{t} - \frac{1}{\sqrt{t}}\,dt$

$= 2\left(\frac{2}{3}t^{3/2} - 2\sqrt{t}\right) + C$

$= \dfrac{4\sqrt{1+\sqrt{u}}}{3}\left(\sqrt{u} - 2\right) + C$

$= \boxed{\dfrac{4\sqrt{1+\sqrt{1+x}}}{3}\left(\sqrt{1+x} - 2\right) + C}$

b) $\int x \sin^4(x^2)\cos(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\int \sin^4(u)\cos(u)\,du$

$t = \sin(u)$

$dt = \cos(u)\,du$

$= \frac{1}{2}\int t^4\,dt$

$= \frac{1}{10}t^5 + C$

$= \frac{1}{10}\sin^5(u) + C$

$= \boxed{\frac{1}{10}\sin^5(x^2) + C}$