

3.11 Related Rates

Related rates are problems that use a mathematical relationship between two or more objects under specific constraints. From this, we can differentiate this relationship and examine how each variable changes with respect to time.

Example. An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30m/hr , how fast is the area of the patch increasing when the patch has a radius of 100m ?

Solution:

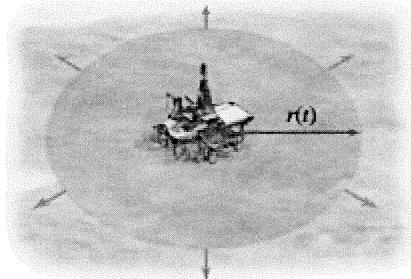
We are given the radius $r = 100\text{ m}$ which is increasing at a rate of $\frac{dr}{dt} = 30\text{ m/hr}$.

First, we note the formula for the area of a circle:

$$A = \pi r^2$$

Now, we differentiate *with respect to time t*:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$



Note that we want to solve for $\frac{dA}{dt}$ — the rate of change of the area of the oil patch:

$$\frac{dA}{dt} = 2\pi(100\text{ m})(30\text{ m/hr}) = 6000\pi\text{ m}^2/\text{hr}$$

Example. Two small planes approach an airport, one flying due west at 120 mi/hr and the other flying due north at 150 mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 mi from the airport and the northbound plane is 225 mi from the airport?

Solution:

From the problem statement, we know that

$$\begin{aligned}x &= 180 \text{ mi} & y &= 225 \text{ mi} \\ \frac{dx}{dt} &= -120 \text{ mi/hr} & \frac{dy}{dt} &= -150 \text{ mi/hr}\end{aligned}$$

Now we use the Pythagorean theorem to relate the three sides:

$$x^2 + y^2 = z^2$$

Next, we differentiate and get the resulting relationship between the related rates:

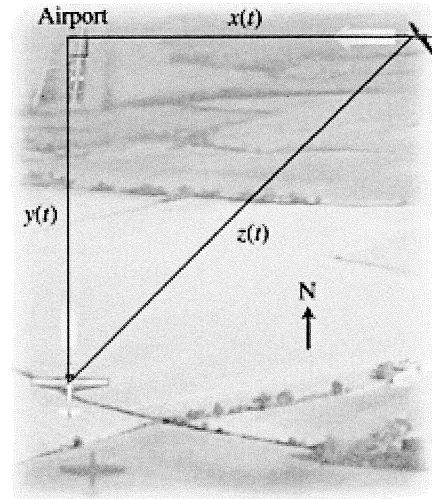
$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

Before we solve for $\frac{dz}{dt}$, we must find z :

$$z = \sqrt{180^2 + 225^2} = 45\sqrt{4^2 + 5^2} = 45\sqrt{41} \approx 288 \text{ mi}$$

This gives us

$$\begin{aligned}\frac{dz}{dt} &= \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z} = \frac{(180 \text{ mi})(-120 \text{ mi/hr}) + (225 \text{ mi})(-150 \text{ mi/hr})}{45\sqrt{41} \text{ mi}} \\ &= \frac{1230}{\sqrt{41}} \text{ mi/hr} \approx -192 \text{ mi/hr}\end{aligned}$$

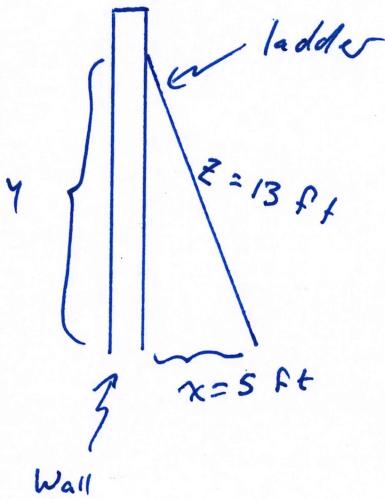


Steps for Related-Rate Problems

1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t .
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

Note: The *Just-In-Time* book has some examples in chapter 14 that are helpful in setting up the relationships outlined in these types of word problems.

Example. A ladder 13 feet long rests against a vertical wall and is sliding down the wall at a constant rate of 3 ft/s. How fast is the foot of the ladder moving away from the wall when the foot of the ladder is 5 ft from the base of the wall?



$$\text{Given } x = 5 \text{ ft}$$

$$z = 13 \text{ ft}$$

$$\frac{dy}{dt} = -3 \text{ ft/s}$$

Using Pythagorean theorem, we have

$$x^2 + y^2 = z^2 \rightarrow y = 12 \text{ ft}$$

Differentiating with respect to t, we have

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

We wish to solve for $\frac{dx}{dt}$. Note that $\frac{dz}{dt} = 0$

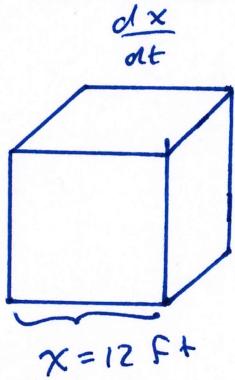
$$\Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt} = -\frac{12 \text{ ft}}{5 \text{ ft}} (-3 \text{ ft/s})$$

$$= \frac{36}{5} \text{ ft/s}$$

$$= 7.2 \text{ ft/s}$$

Example. The volume of a cube decreases at a rate of $0.5 \text{ ft}^3/\text{min}$. What is the rate of change of the side length when the side lengths are 12 ft ?



$$V = x^3$$



$$\frac{dV}{dt} = -0.5 \text{ ft}^3/\text{min}$$

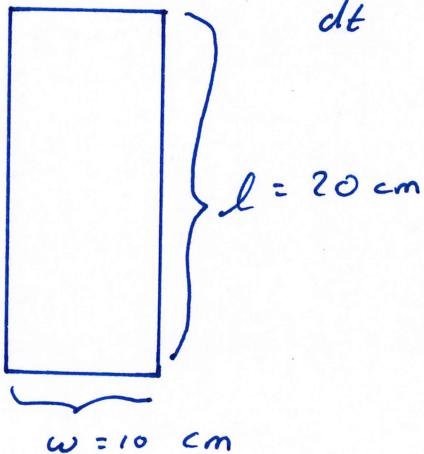
$$\frac{dV}{dt} = 3x^2 \boxed{\frac{dx}{dt}}$$

$$\frac{dx}{dt} = \frac{1}{3x^2} \frac{dV}{dt}$$

$$= \frac{1}{3(12 \text{ ft})^2} (-0.5 \text{ ft}^3/\text{min})$$

$$= \boxed{\frac{-1}{864} \text{ ft/min}}$$

Example. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?



$$\frac{dA}{dt}$$

$$\frac{dl}{dt} = 8 \text{ cm/s}$$

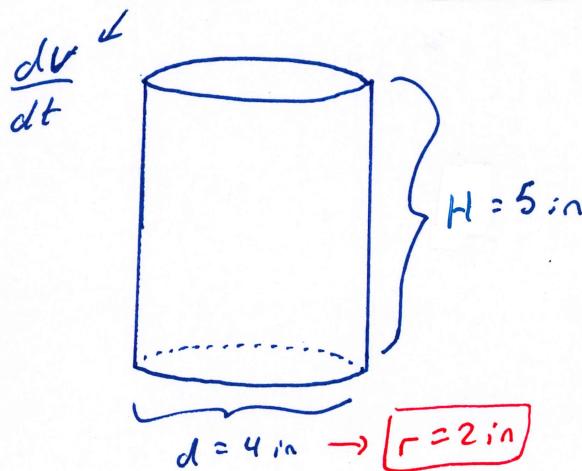
$$\frac{dw}{dt} = 3 \text{ cm/s}$$

$$A = lw$$

$$\downarrow$$

$$\begin{aligned}\frac{dA}{dt} &= \frac{dl}{dt}w + l\frac{dw}{dt} \\ &= (8 \text{ cm/s})(10 \text{ cm}) + (20 \text{ cm})(3 \text{ cm/s}) \\ &= \boxed{140 \text{ cm}^2/\text{s}}\end{aligned}$$

Example. A coffee mug has the shape of a right circular cylinder with inner diameter 4 inches and height 5 inches. If the mug is filled with hot chocolate at a constant rate of $2 \text{ in}^3/\text{sec}$, how fast is the level of the liquid rising?



$$\frac{dh}{dt}$$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2 \boxed{\frac{dh}{dt}}$$

Radius constant

$$\Rightarrow \frac{dr}{dt} = 0$$

Let h be the height of the fluid

$$\begin{aligned} \Rightarrow \frac{dh}{dt} &= \frac{1}{\pi r^2} \frac{dV}{dt} \\ &= \frac{1}{\pi (2\text{in})^2} (2\text{in}^3/\text{sec}) \\ &= \boxed{\frac{1}{2\pi} \text{ in/sec}} \end{aligned}$$

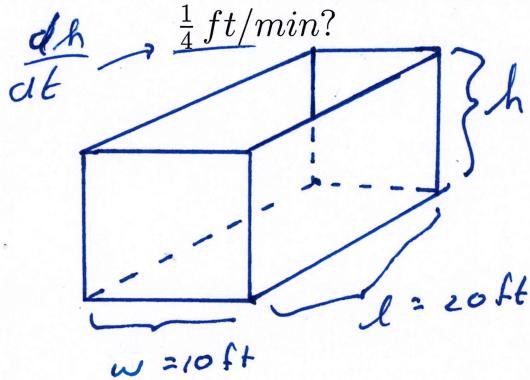
Note: chain rule can be used as well

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

but $\frac{dr}{dt} = 0$, so this is equivalent

Example. A rectangular swimming pool 10 ft wide by 20 ft long and of uniform depth is being filled with water.

a) At what rate is the volume of the water increasing if the water level is rising at



$$V = lwh$$

$$\frac{dV}{dt} = lwh \frac{dh}{dt}$$

$$= (20 \text{ ft})(10 \text{ ft}) \left(\frac{1}{4} \text{ ft/min}\right)$$

$$= \boxed{50 \text{ ft}^3/\text{min}}$$

Note: l and w are constant!

b) At what rate is the water level rising if the pool is filled at a rate of $10 \text{ ft}^3/\text{min}$?

$$\Rightarrow \frac{dh}{dt} = \frac{1}{lw} \frac{dV}{dt}$$

$$= \frac{1}{(20 \text{ ft})(10 \text{ ft})} (10 \text{ ft}^3/\text{min})$$

$$= \frac{1}{20} \text{ ft/min}$$

$$= \boxed{0.05 \text{ ft/min}}$$

Example. At all times, the length of a rectangle is twice the width w of the rectangle as the area of the rectangle changes with respect to time t .

- a) Find the equation that relates A to w .

$$l = 2w \quad A = lw$$

$$A = (2w)w = 2w^2$$

- b) Find the equation that relates dA/dt to dw/dt .

$$\frac{dA}{dt} = 4w \frac{dw}{dt}$$

Example. Assume x , y and z are functions of t with $z = x + y^3$. Find $\frac{dz}{dt}$ when $\underline{dx/dt = -1}$, $\underline{dy/dt = 5}$ and $\underline{y = 2}$.

$$Z = x + y^3$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{dx}{dt} + 3y^2 \frac{dy}{dt} \\ &= -1 + 3(z)^2 5\end{aligned}$$

$$\boxed{\frac{dz}{dt} = 59}$$

Example. Assume $w = x^2y^4$, where x and y are functions of t . Find dw/dt when $x = 3$, $dx/dt = 2$, $dy/dt = 4$, and $y = 1$.

$$w = x^2 y^4$$

$$\begin{aligned}\frac{dw}{dt} &= (2x) \frac{dx}{dt} y^4 + x^2 (4y^3) \frac{dy}{dt} \\ &= (6)(2)(1^4) + (3^2)(4)(4)\end{aligned}$$

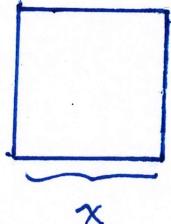
$$\boxed{\frac{dw}{dt} = 156}$$

$$\frac{dx}{dt}$$

Example. The sides of a square decrease in length at a rate of 1 m/s.

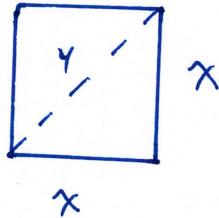
- a) At what rate is the area of the square changing when the sides are 5 m long?

$$x = 5 \text{ m}$$



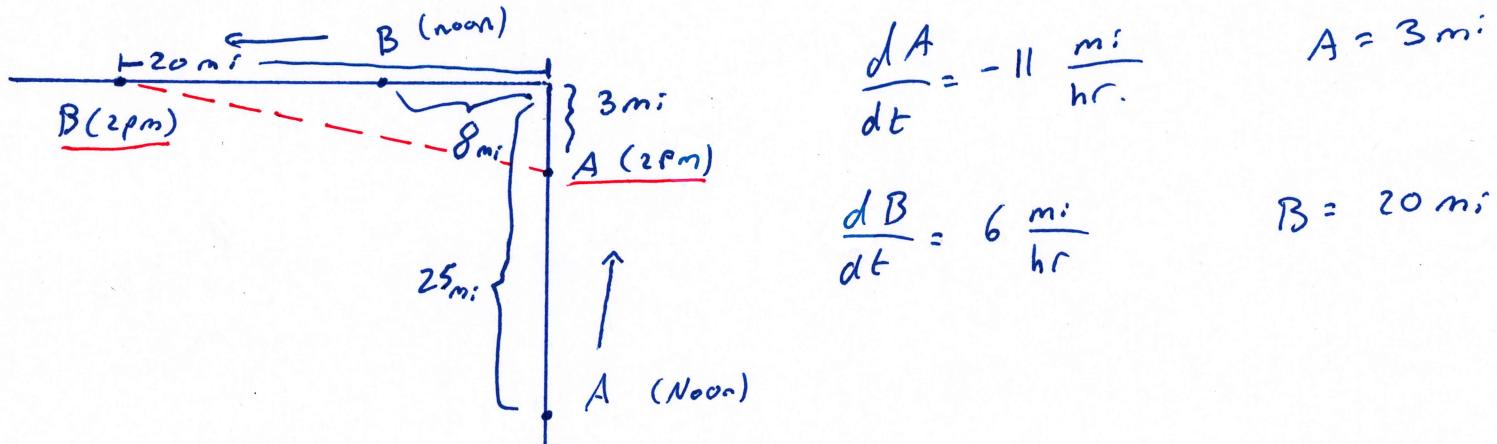
$$\begin{aligned} A &= x^2 \\ \frac{dA}{dt} &= 2x \frac{dx}{dt} \\ &= 2(5 \text{ m})(-1 \text{ m/s}) \\ &= \boxed{-10 \text{ m}^2/\text{s}} \end{aligned}$$

- b) At what rate are the lengths of the diagonals of the square changing?



$$\begin{aligned} y &= \sqrt{2x^2} = \sqrt{2} x \\ \frac{dy}{dt} &= \sqrt{2} \frac{dx}{dt} = \sqrt{2} (-1 \text{ m/s}) = \boxed{-\sqrt{2} \text{ m/s}} \end{aligned}$$

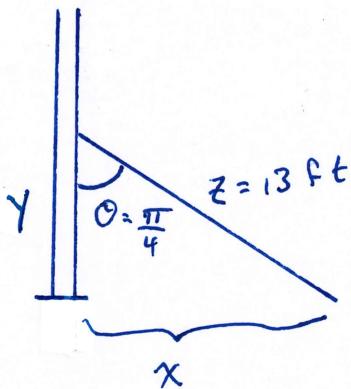
Example. At noon bicyclist A is 25 miles south of an intersection and bicyclist B is 8 miles west of the same intersection. Bicyclist A is traveling north at 11 miles per hour and bicyclist B is traveling 6 miles per hour west of the intersection. How is the distance between riders changing at 2pm?



Let $D = \sqrt{A^2 + B^2}$ be the distance between the bikers where A and B represent their distance from the intersection.

$$\begin{aligned}
 \Rightarrow \frac{dD}{dt} &= \frac{1}{2} (A^2 + B^2)^{-\frac{1}{2}} \left(2A \frac{dA}{dt} + 2B \frac{dB}{dt} \right) \\
 &= \frac{A \frac{dA}{dt} + B \frac{dB}{dt}}{\sqrt{A^2 + B^2}} \\
 &= \frac{(3 \text{ mi}) (-11 \text{ mi/hr}) + (20 \text{ mi})(6 \text{ mi/hr})}{\sqrt{(3 \text{ mi})^2 + (20 \text{ mi})^2}} = \frac{87}{\sqrt{409}} \text{ mi/hr} \\
 &\approx \boxed{4.302 \text{ mi/hr}}
 \end{aligned}$$

Example. A ladder 13 feet long rests against a vertical wall and is sliding down the wall at a constant rate of 3 ft/s. How fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ radians?



$$\boxed{\frac{d\theta}{dt}}$$

Note:

z is constant!

$$\frac{dy}{dt} = -3 \text{ ft/s}$$

$$\cos \theta = \frac{y}{z}$$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{z} \frac{dy}{dt}$$

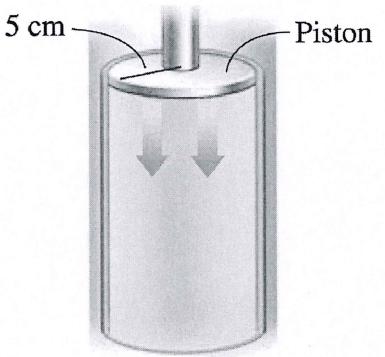
$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{-z \sin \theta} \frac{dy}{dt}$$

$$= \frac{1}{-(13 \text{ ft}) \frac{\sqrt{2}}{2}} (-3 \text{ ft/sec})$$

$$= \boxed{\frac{3\sqrt{2}}{13} \text{ radians/sec}}$$

Example. Piston compression A piston is seated at the top of a cylindrical chamber with radius 5 cm when it starts moving into the chamber at a constant speed of 3 cm/s. What is the rate of change of the volume of the cylinder when the piston is 2 cm from the base of the chamber?

$$r = 5 \text{ cm} \quad \frac{dr}{dt} = 0 \text{ cm/s. (constant)}$$



$$h = 2 \text{ cm} \quad \frac{dh}{dt} = -3 \text{ cm/s}$$

Find $\frac{dV}{dt}$:

$$V = \pi r^2 h = 25\pi h \quad (r \text{ is constant})$$

$$\begin{aligned} \frac{dV}{dt} &= 25\pi \frac{dh}{dt} \\ &= (25\pi \text{ cm}^2) (-3 \text{ cm/s}) \\ &= \boxed{-75\pi \text{ cm}^3/\text{s}} \end{aligned}$$

Example. Suppose we have a snowball that is a perfect sphere. If the snowball is melting at a rate of $5 \text{ in}^3/\text{min}$, how fast is the radius changing when the radius is 10 inches? How fast is the surface area changing?

$$\begin{aligned} V &= \frac{4}{3} \pi r^3 \\ SA &= 4 \pi r^2 \end{aligned}$$

$$\frac{dr}{dt}$$



$$V = ?$$

$$\frac{dV}{dt} = -5 \text{ in}^3/\text{min}$$

$$r = 10 \text{ in}$$

$$\frac{dr}{dt} \leftarrow \text{Find}$$

$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (10 \text{ in})^3 = \boxed{\frac{4000}{3} \pi \text{ in}^3 = V}$$

↓ Derivation

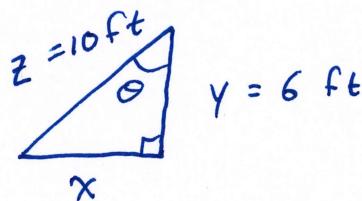
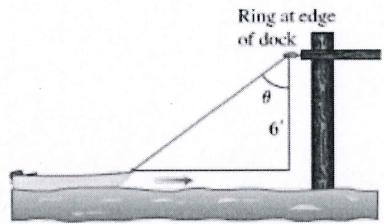
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

$$= \frac{1}{4\pi(10 \text{ in})^2} (-5 \text{ in}^3/\text{min})$$

$$\boxed{\frac{dr}{dt} = -\frac{1}{80\pi} \text{ in/min}}$$

$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt} = -\frac{8\pi (10 \text{ in})}{80\pi} \text{ in/min} = \boxed{-1 \text{ in}^2/\text{min}}$$

Example. A dingy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at a rate of 2 ft/sec. At what rate is the angle θ changing when 10 ft of rope is out?



$$\frac{d\theta}{dt}$$

$$\cos \theta = \frac{y}{z} = \frac{6 \text{ ft}}{10 \text{ ft}} = \frac{3}{5}$$

$\sin \theta = \frac{4}{5}$ (Pythagorean thm)

Note: $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = -2 \text{ ft/s.}$

$$-\sin \theta \frac{d\theta}{dt} = -\frac{y}{z^2} \frac{dz}{dt}$$

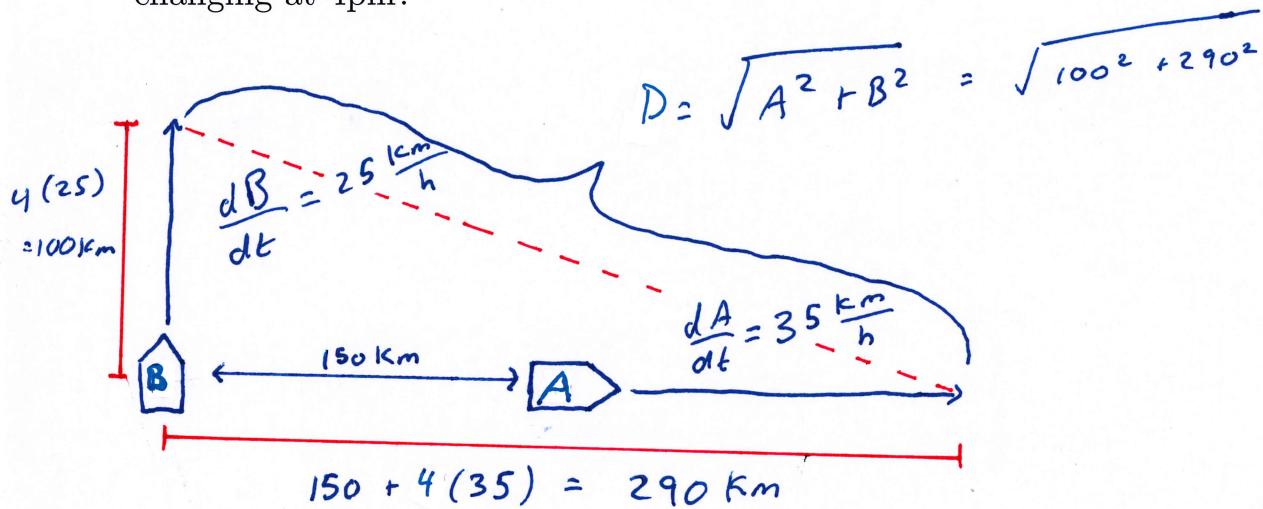
$$\Rightarrow \frac{d\theta}{dt} = \frac{y}{z^2 \sin \theta} \frac{dz}{dt}$$

$$= \frac{6 \text{ ft}}{(10 \text{ ft})^2 (4/5)} (-2 \text{ ft/sec})$$

$$= -3/2 \text{ rad/sec}$$

$\frac{d\theta}{dt} = -1.5 \text{ rad/sec}$

Example. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4pm?



$$D = (A^2 + B^2)^{1/2}$$

$$\frac{dD}{dt} = \frac{1}{2} (A^2 + B^2)^{-1/2} \left(2A \frac{dA}{dt} + 2B \frac{dB}{dt} \right)$$

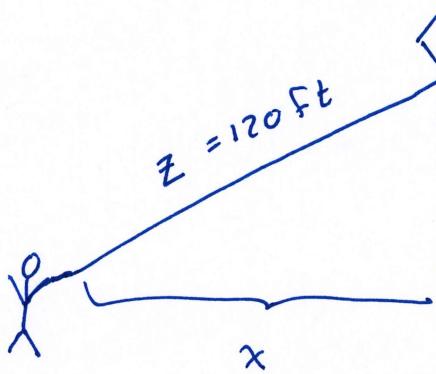
$$= \frac{A \frac{dA}{dt} + B \frac{dB}{dt}}{\sqrt{A^2 + B^2}}$$

$$= \frac{(290 \text{ km}) (35 \frac{\text{km}}{\text{h}}) + (100 \text{ km}) (25 \frac{\text{km}}{\text{hr}})}{\sqrt{(290 \text{ km})^2 + (100 \text{ km})^2}}$$

$$= \frac{1265}{\sqrt{941}} \text{ km/h} \approx \boxed{41.238 \text{ km/h}}$$

$$\frac{dy}{dt} = 0$$

Example. Once Kate's kite reaches a height of 50 ft (above her hands), it rises no higher but drifts due east in a wind blowing 5 ft/s. How fast is the string running through Kate's hands at the moment that she has released 120 ft of string?



$$l \frac{dz}{dt}$$

$$\frac{dx}{dt} = 5 \text{ ft/s}$$

$$z^2 = x^2 + y^2 \Rightarrow x = 10\sqrt{119} \approx 10.909$$

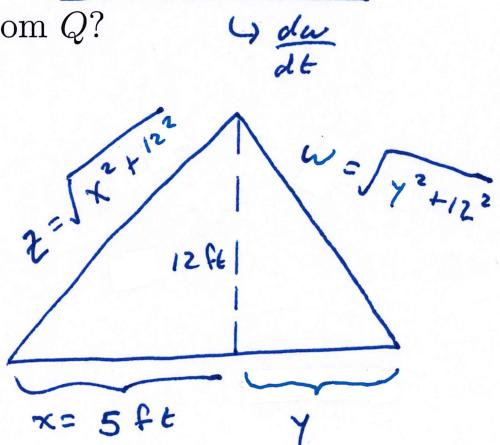
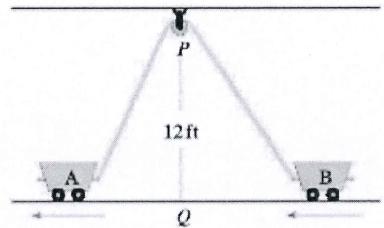
$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{x}{z} \frac{dx}{dt}$$

$$= \frac{10\sqrt{119} \text{ ft}}{120 \text{ ft}} (5 \text{ ft/s})$$

$$\approx 4.5453 \text{ ft/s}$$

Example. Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P . The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s . How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q ?



$$\frac{dx}{dt} = 2 \text{ ft/s}$$

$$39 \text{ ft} = \sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2}$$

$$\text{Since } x = 5 \text{ ft}$$

$$39 \text{ ft} = \underbrace{\sqrt{25 + 144}}_{13 \text{ ft}} + \sqrt{y^2 + 144}$$

$$\Rightarrow (26 \text{ ft})^2 = y^2 + 144$$

$$\Rightarrow y = \sqrt{676 - 144} = \sqrt{532} \text{ ft}$$

$$= 2\sqrt{133} \text{ ft}$$

Derivative:

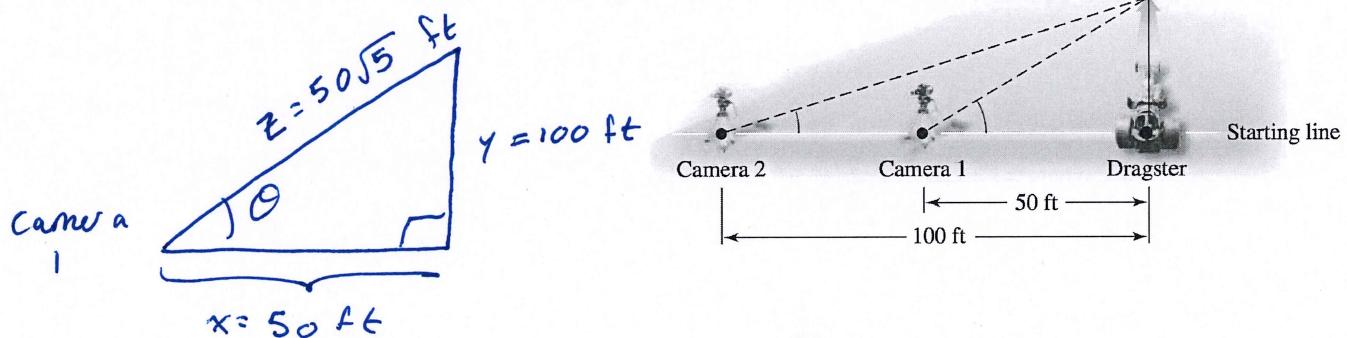
$$39 \text{ ft} = \sqrt{x^2 + 144} + \sqrt{y^2 + 144}$$

$$0 = \frac{x}{\sqrt{x^2 + 144}} \frac{dx}{dt} + \frac{y}{\sqrt{y^2 + 144}} \frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = - \frac{x \sqrt{y^2 + 144}}{y \sqrt{x^2 + 144}} \frac{dx}{dt} = - \frac{(5 \text{ ft})(26 \text{ ft})}{(2\sqrt{133} \text{ ft})(13 \text{ ft})} (2 \text{ ft/s}) = \boxed{\frac{10}{\sqrt{133}} \text{ ft/s}}$$

Example. A camera is set up at the starting line of a drag race 50ft from a dragster at the starting line (camera 1 in the figure). Two seconds after the start of the race, the dragster has traveled 100ft and the camera is turning at 0.75 rad/s while filming the dragster.

- $\frac{dy}{dt}$
- a) What is the speed of the dragster at this point? $\frac{d\theta}{dt}$
- b) A second camera (camera 2 in the figure) filming the dragster is located on the starting line 100 ft away from the dragster at the start of the race. How fast is this camera turning 2 s after the start of the race?



$$a) \tan \theta = \frac{y}{x} \implies \sec^2 \theta = \frac{\text{hyp}}{\text{adj}} = \frac{50\sqrt{5} \text{ ft}}{50 \text{ ft}} = \sqrt{5}$$



$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{x} \frac{dy}{dt} \implies \frac{dy}{dt} = x \sec^2 \theta \frac{d\theta}{dt}$$

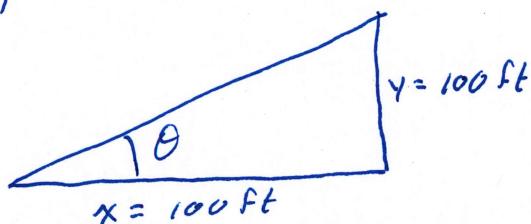
$$= (50 \text{ ft}) (\sqrt{5})^2 (0.75 \text{ rad/s})$$

$$= \frac{750}{4} \text{ ft/sec} = \boxed{187.5 \text{ ft/sec}}$$

Note:

radians are 'unitless'

b)



$$\tan \theta = \frac{y}{x} \implies \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{x} \frac{dy}{dt}$$

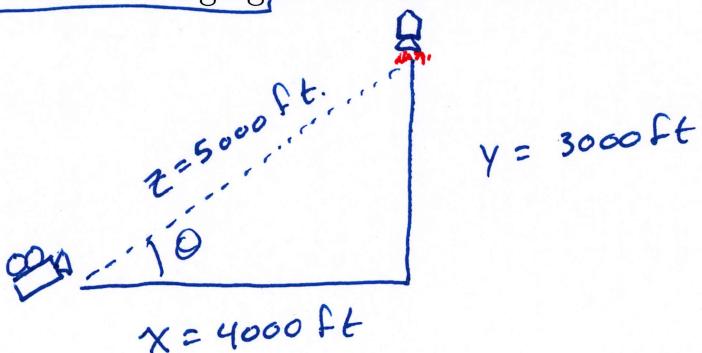
$$\implies \frac{d\theta}{dt} = \frac{\cos^2 \theta}{x} \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{750 \text{ ft}}{\text{sec}}$$

$$= \frac{\sqrt{2}}{2(100 \text{ ft})} 750 \frac{\text{ft}}{\text{sec}}$$

$$= \frac{5\sqrt{2}}{4} \text{ rad/sec} \approx \boxed{1.7678 \frac{\text{rad}}{\text{sec}}}$$

Example. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft. How fast is the distance from the television camera to the rocket changing at that moment? Also, if the television camera is always kept aimed at the rocket, how fast is the cameras angle of elevation changing at that same moment?



$$\frac{dx}{dt} = 0 \text{ ft/sec (constant)}$$

$$\frac{dy}{dt} = 600 \text{ ft/sec}$$

$$\frac{dz}{dt} \quad (\text{Find})$$

$$z^2 = x^2 + y^2$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{y}{z} \frac{dy}{dt} = \frac{3000 \text{ ft}}{5000 \text{ ft}} 600 \text{ ft/sec} = 360 \text{ ft/sec}$$

$$\tan \theta = \frac{y}{x}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{x} \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2 \theta}{x} \frac{dy}{dt} = \frac{(4/5)^2}{4000 \text{ ft}} 600 \text{ ft/sec} = \frac{12}{125} \text{ rad/sec} = 0.096 \text{ rad/sec}$$