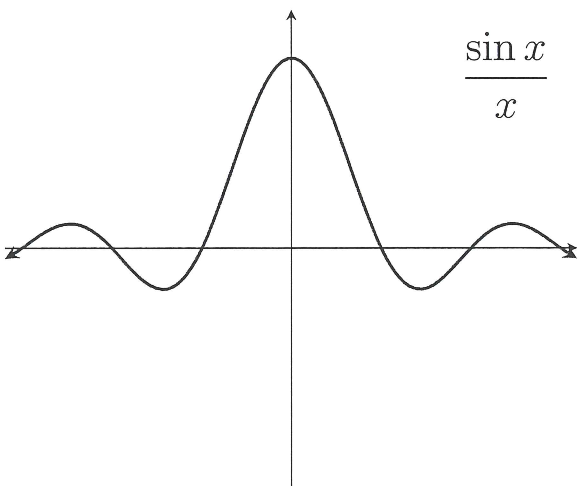
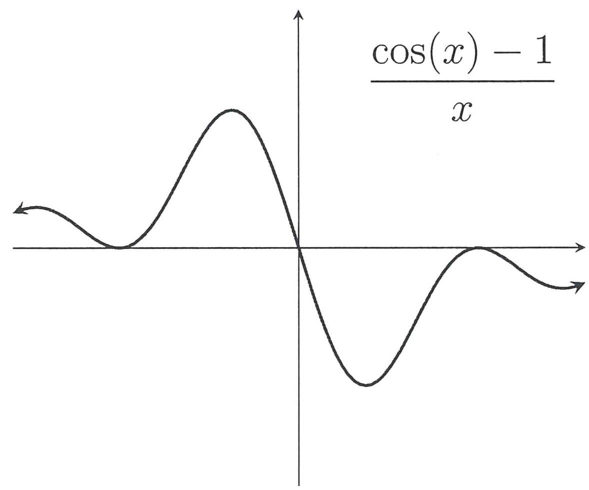
3.5 Derivatives of Trigonometric Functions

Theorem 3.10 Trigonometric Limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$





Example. Evaluate the following limits:

$$\lim_{x\to 0}\frac{\sin(3x)}{3x} = \boxed{1}$$

$$\lim_{x\to 0}\frac{\sin(4x)}{x} \left(\frac{4}{4}\right)$$

$$= \lim_{x\to 0} 4\,\frac{\sin(4x)}{4}$$

$$= \boxed{4}$$

$$\lim_{h\to 0}\frac{5h}{\sin(3h)} \left(\frac{3/5}{3/5}\right)$$

$$= \lim_{h\to 0} \frac{1}{3/5}\,\frac{3h}{\sin(3h)}$$

$$= \boxed{5/3}$$

$$\lim_{t\to\frac{\pi}{2}}\frac{\sin\left(t-\frac{\pi}{2}\right)}{t-\frac{\pi}{2}} = \boxed{1}$$

$$\lim_{x\to 0}\frac{\tan(2x)}{x}$$

$$= \lim_{h\to 0}\frac{\sin(2x)}{x\cos(x)}\left(\frac{2}{2}\right)$$

$$= \lim_{h\to 0}\frac{\sin(2x)}{2x}\cdot\frac{2}{\cos(x)}$$

$$= 1\cdot 2 = \boxed{2}$$

$$\lim_{x\to 0}\frac{\sin(7x)}{\sin(5x)}$$

$$= \lim_{x\to 0}\frac{\sin(7x)}{x}\cdot\left(\frac{7}{7}\right)\frac{x}{\sin(5x)}\left(\frac{5}{5}\right)$$

$$= \lim_{x\to 0} 7\cdot\frac{\sin(7x)}{7x}\cdot\frac{1}{5}\,\frac{5x}{\sin(5x)}$$

$$= \boxed{\frac{7}{5}}$$

Theorem 3.11 Derivatives of Sine and Cosine

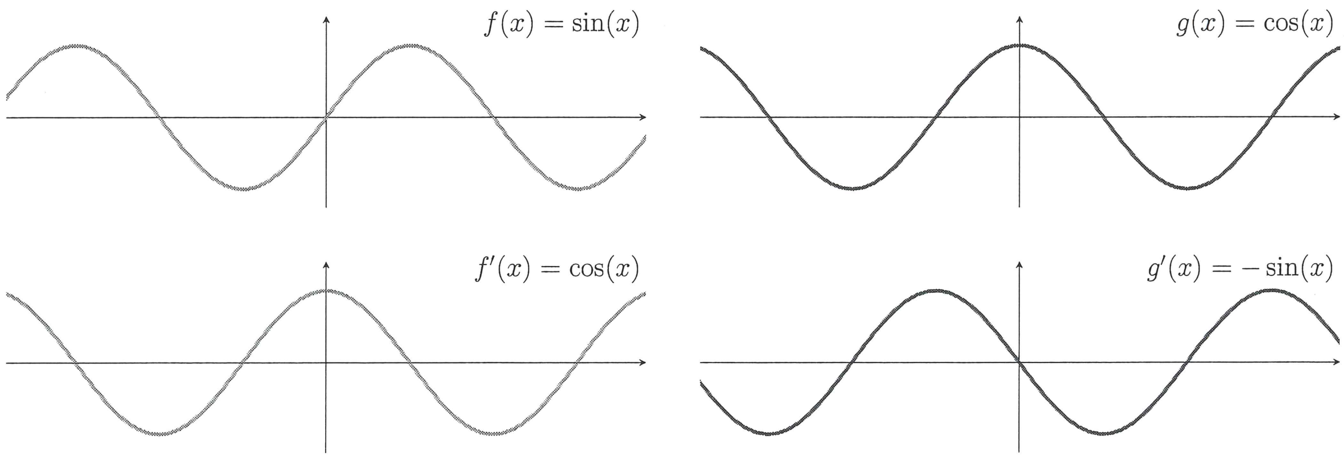
$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad\qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} = \cos(x)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} = -\sin(x)
\end{aligned}$$

□



Example. Find the derivative of the following functions:

$$y = 3\cos(x) - 2x^{\frac{3}{2}}$$

$$y' = -3\sin(x) - 3x^{1/2}$$

$$z = \frac{\sin(x)}{x}$$

$$z' = \frac{x[\cos(x)] - \sin(x)[1]}{x^2}$$

$$= \frac{x\cos(x) - \sin(x)}{x^2}$$

$$w = \frac{x}{\cos(x)}$$

$$w' = \frac{\cos(x)[1] - x[-\sin(x)]}{\cos(x)}$$

$$= \frac{\cos(x) + x\sin(x)}{\cos(x)}$$

$$\ell = e^x\cos(x)$$

$$\ell' = [e^x]\cos(x) + e^x[-\sin(x)]$$

$$= e^x(\cos(x) - \sin(x))$$

$$m = \frac{\cos(x)}{\sin(x)}$$

$$m' = \frac{\sin(x)[-\sin(x)] - \cos(x)[\cos(x)]}{\sin^2(x)}$$

$$= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)}$$

$$= -\frac{1}{\sin^2(x)} = -\csc^2(x)$$

$$n = \sin^2(x) + \cos^2(x) = 1$$

$$\Rightarrow n' = 0$$

$$n = \sin(x)\cdot\sin(x) + \cos(x)\cos(x)$$

$$n' = [\cos(x)]\sin(x) + \sin(x)[\cos(x)] + [-\sin(x)]\cos(x) + \sin(x)[-\cos(x)] = 0$$

Example. Find the equation of the line tangent to $y = \cos(x)$ at $x = \frac{\pi}{4}$.

$$y' = -\sin(x)$$

$$y(\pi/4) = \frac{\sqrt{2}}{2}, \quad y'(\pi/4) = -\sin(\pi/4) = -\frac{\sqrt{2}}{2}$$

Tan line:

$$y - f(a) = f'(a)(x-a)$$

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}(x - \pi/4)$$

$$\boxed{y = -\frac{\sqrt{2}}{2}x + \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2}}$$

Example. Find the derivative of $y = \dfrac{x\cos(x)}{1+\sin(x)}$ and simplify.

$$y' = \frac{(1+\sin(x))\left[[1]\cos(x) + x[-\sin(x)]\right] - x\cos(x)\left[\cos(x)\right]}{(1+\sin(x))^2}$$

$$= \frac{\cos(x)(1+\sin(x)) - x\sin(x) \underbrace{- x\sin^2(x) - x\cos^2(x)}_{-x}}{(1+\sin(x))^2}$$

$$= \frac{\cos(x)(1+\sin(x)) - x\sin(x) - x}{(1+\sin(x))^2}$$

$$= \frac{\cos(x)(1+\sin(x)) - x(1+\sin(x))}{(1+\sin(x))^2} = \boxed{\frac{\cos(x) - x}{1+\sin(x)}}$$

Theorem 3.12 Derivatives of the Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Example. Find the derivatives of the following:

$y = \frac{4}{x} - \frac{9}{13}\tan(x) = 4x^{-1} - \frac{9}{13}\tan(x)$

$y' = -4x^{-2} - \frac{9}{13}\sec^2(x)$

$f(x) = -4x^3\cot(x)$

$f'(x) = [-12x^2]\cot(x) - 4x^3[-\csc^2(x)]$

$= -12x^2\cot(x) + 4x^3\csc^2(x)$

$g(\theta) = \dfrac{\sec(\theta)}{1+\sec(\theta)}$

$g'(\theta) = \dfrac{(1+\sec\theta)[\sec\theta\tan\theta] - \sec\theta[\sec\theta\tan\theta]}{(1+\sec\theta)^2}$

$= \dfrac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$

$h(w) = e^w\csc(w)$

$h'(w) = [e^w]\csc(w) + e^w[-\csc(w)\cot(w)]$

$= (1-\cot(w))e^w\csc(w)$

Example. Evaluate

$$\frac{d}{dx}[\tan(x)]\bigg|_{x=\frac{\pi}{4}}$$

$$= \sec^2(x)\Big|_{x=\pi/4}$$

$$= (\sqrt{2})^2$$

$$= \boxed{2}$$

$$\frac{d}{dx}[(\sin(x)+\cos(x))\csc(x)]$$

$$= [\cos(x) - \sin(x)]\csc(x)$$
$$+ (\sin(x) + \cos(x))[-\csc(x)\cot(x)]$$

$$= \csc(x)\Big(\underbrace{\cos(x)} - \sin(x) - \underbrace{\cot(x)(\sin(x)+\cos(x))}_{-\cos(x)}\Big)$$

$$= \csc(x)(-\sin(x) - \cot(x)\cos(x))$$

$$= -(1 + \cot^2(x))$$

$$= -\csc^2(x)$$

$$\frac{d}{d\theta}[\theta^2\sin(\theta)\tan(\theta)]$$

$$= \frac{d}{d\theta}[\theta^2]\sin\theta\tan\theta + \theta^2\frac{d}{d\theta}[\sin\theta]\tan\theta + \theta^2\sin\theta\frac{d}{d\theta}[\tan\theta]$$

$$= 2\theta\sin\theta\tan\theta + \theta^2\underbrace{\cos(\theta)\tan\theta}_{\sin\theta} + \theta^2\underbrace{\sin\theta\sec^2\theta}_{\tan\theta\sec\theta}$$

Example. Find the following higher order derivatives:

y'' when $y = \cos(x)$

$y' = -\sin(x)$

$\boxed{y'' = -\cos(x)}$

$f''(x)$ when $f(x) = \sin(x)$

$f'(x) = \cos(x)$

$\boxed{f''(x) = -\sin(x)}$

$y^{(42)}$ when $y = \cos(x)$

$$4\overline{)42}\;\; 10,\quad -40,\quad 2 \;\Rightarrow\; y^{(42)} = y^{(2)} = \boxed{-\cos(x)}$$

$y^{(0)} = \cos(x)$

$y^{(1)} = -\sin(x)$

$y^{(2)} = -\cos(x)$

$y^{(3)} = \sin(x)$

$\frac{d^2}{d\theta^2}[\sin(\theta)\cos(\theta)]$

$\frac{d}{d\theta}[\sin\theta\cos\theta] = \cos^2\theta - \sin^2\theta$

$\frac{d}{d\theta}[\sin\theta\cos\theta] = [-\sin\theta]\cos\theta + \cos\theta[-\sin\theta]$

$- [\cos\theta]\sin\theta - \sin\theta[\cos\theta]$

$= \boxed{-4\sin\theta\cos\theta}$

$\frac{d^2}{dx^2}\left[\frac{1}{2}e^x\cos(x)\right]$

$\frac{d}{dx}\left[\frac{1}{2}e^x\cos(x)\right] = \frac{1}{2}e^x\cos(x) - \frac{1}{2}e^x\sin(x)$

$= \frac{1}{2}e^x(\cos(x) - \sin(x))$

$\frac{d^2}{dx^2}\left[\frac{1}{2}e^x\cos(x)\right] = \frac{1}{2}e^x(\cos(x) - \sin(x)) + \frac{1}{2}e^x(-\sin(x) - \cos(x))$

$= \boxed{-e^x\sin(x)}$

$\frac{d^2}{dx^2}[\cot x]$

$= \frac{d}{dx}[-\csc^2(x)]$

$= \frac{d}{dx}[-\csc(x)\cdot\csc(x)]$

$= \csc(x)\tan(x)\cdot\csc(x)$

$+ \csc(x)\cdot\csc(x)\tan(x)$

$= \boxed{2\csc^2(x)\tan(x)}$

Example. For

$$f = \begin{cases} \frac{3\sin(x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

Find a such that f is continuous.

Want $\lim_{x \to 0} f(x) = f(0)$

$f(0) = a$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{3\sin(x)}{x} = 3 \cdot 1 = 3 \Rightarrow$ Let $\boxed{a = 3}$

Example. Find the equation of the line tangent to $y = \dfrac{\cos(x)}{1 - \cos(x)}$ at $x = \frac{\pi}{3}$.

$$y' = \frac{(1 - \cos(x))[-\sin(x)] - \cos(x)[\sin(x)]}{(1 - \cos(x))^2}$$

$$= \frac{-\sin(x)}{(1 - \cos(x))^2}$$

$y(\pi/3) = \frac{1/2}{1/2} = 1$

$y'(\pi/3) = \frac{-\sqrt{3}/2}{(1/2)^2} = -2\sqrt{3}$

$y - 1 = -2\sqrt{3}(x - \pi/3)$

$$\boxed{y = -2\sqrt{3}\,x + \frac{2\pi\sqrt{3} + 3}{3}}$$

Example. For what values of x does $x - \sin(x)$ have a horizontal tangent line?

Solve $\frac{d}{dx}[x - \sin(x)] = 0$

$1 - \cos(x) = 0$

$\cos(x) = 1$

$\boxed{x = 2k\pi}$

Example. Evaluate the following limits

$$\lim_{x\to\pi/4} \frac{\tan(x)-1}{x-\pi/4} = \lim_{x\to\pi/4} \frac{\tan(x)-\tan(\pi/4)}{x-\pi/4} = \sec^2(\pi/4)$$

$$= \boxed{2}$$

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}}{h} = \cos(\pi/6) = \boxed{\frac{\sqrt{3}}{2}}$$

$$\lim_{x\to\pi/4} \frac{\cot x - 1}{x-\frac{\pi}{4}} = -\csc^2(\pi/4) = \boxed{-2}$$