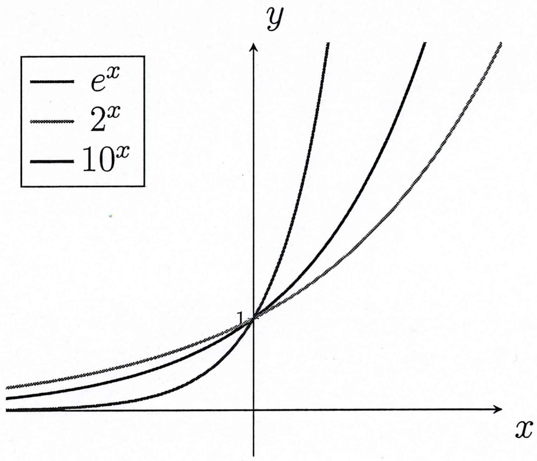
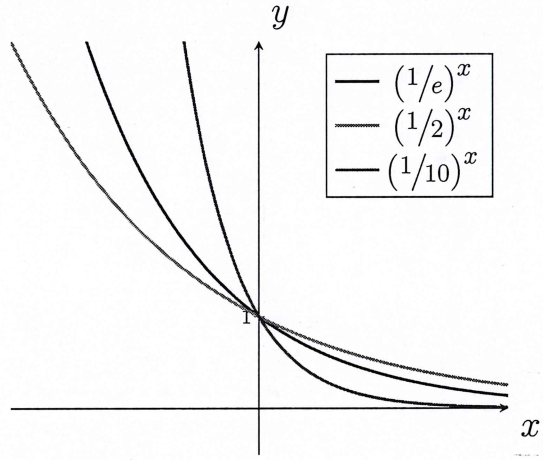
1.3 Inverse, Exponential and Logarithmic Functions

Definition.
An **exponential function** has the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. Exponential functions have a horizontal asymptote of $y =$ 0 and y-intercept of $(0,$ 1$)$. When b is such that $0 < b < 1$, then $f(x)$ is decreasing and when $b > 1$, then $f(x)$ is increasing. Exponential functions have domain $(-\infty,\infty)$ and range $(0,\infty)$.





Definition.
The **natural exponential function** is

$$f(x) = e^x.$$

where e is the irrational constant $e \approx 2.718281828459045\ldots$.

Laws of Exponents: For $a > 0$, we have the following laws:

a) $a^{x+y} = a^x a^y$

b) $a^{x-y} = \frac{a^x}{a^y}$

c) $(a^x)^y = a^{xy}$

d) $(ab)^x = a^x b^x$

Example. For the following expressions, use the Laws of Exponents to simplify:

a) $(x^2y^3)^5$

$= (x^2)^5 (y^3)^5$

$= \boxed{x^{10} y^{15}}$

b) $\left(\sqrt{3}\right)^{1/2} \cdot \left(\sqrt{12}\right)^{1/2}$

$= \left(3^{1/2} \cdot 12^{1/2}\right)^{1/2}$

$= (3 \cdot 12)^{1/2 \cdot 1/2}$

$= \left(\sqrt{36}\right)^{1/2}$

$= \boxed{\sqrt{6}}$

c) $\left(\dfrac{x^{-2}}{x^8}\right)^{-2}$

$= \dfrac{x^4}{x^{-16}}$

$= x^{4-(-16)}$

$= \boxed{x^{20}}$

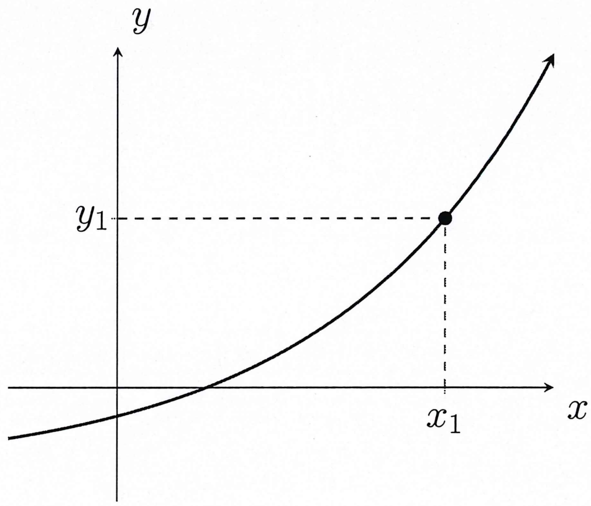
d) $\left(\dfrac{-1}{27}\right)^{4/3}$

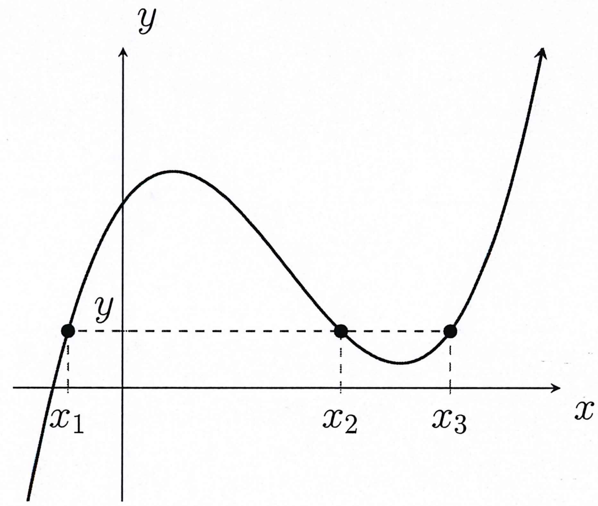
$= \left[\left(\dfrac{-1}{27}\right)^{1/3}\right]^4$

$= \left[\dfrac{-1}{3}\right]^4$

$= \boxed{\dfrac{1}{81}}$

Definition. (One-to-One Functions and the Horizontal Line Test)
A function f is **one-to-one** on a domain D if each value of $f(x)$ corresponds to exactly one value of x in D. More precisely, f is one-to-one on d if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, for x_1 and x_2 in D.
The **horizontal line test** says that every horizontal line intersects the graph of a one-to-one function at most once.





One-to-one

NOT one-to-one

Note: Both functions pass the vertical line test, but only one-to-one functions also pass the horizontal line test.

Finding an Inverse Function Suppose f is one-to-one on an interval I. To find f^{-1}, use the following steps:

1. Solve $y = f(x)$ for x. If necessary, choose the function that corresponds to I.
2. Interchange x and y and write $y = f^{-1}(x)$.

Example. Find $f^{-1}(x)$:

$f(x) = x^2 - 2x + 1, \ x \geq 1$

$y = (x-1)^2 \longrightarrow x = \sqrt{y} + 1 \longrightarrow f^{-1}(x) = \sqrt{x} + 1$

$g(x) = \dfrac{x}{2} - \dfrac{7}{2}$

$y = \dfrac{x-7}{2} \longrightarrow x = 2y + 7 \longrightarrow f^{-1}(x) = 2x + 7$

$h(x) = \sqrt[3]{5x+1}$

$y = (5x+1)^{1/3} \longrightarrow x = \dfrac{y^3 - 1}{5} \longrightarrow f^{-1}(x) = \dfrac{x^3 - 1}{5}$

$j(x) = \dfrac{2x}{1-x}$

$y = \dfrac{2x}{1-x} \longrightarrow y - xy = 2x \longrightarrow y = x(2+y) \longrightarrow x = \dfrac{y}{2+y}$

$\longrightarrow f^{-1}(x) = \dfrac{x}{2+x}$

$k(x) = e^x$

$y = e^x \longrightarrow x = \ln(y) \longrightarrow f^{-1}(x) = \ln(x)$

Existence of Inverse Functions
Let f be a one-to-one function on a domain D with a range R. Then f has a unique inverse f^{-1} with domain R and range D such that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f\big(f^{-1}(y)\big) = y$$

where x is in D and y is in R.

Example. For $f(x) = \sqrt[3]{4x-1} + 2$, so that $f^{-1}(f(x)) = f\big(f^{-1}(x)\big) = x$

$f^{-1}(x) = \dfrac{(x-2)^3+1}{4}$

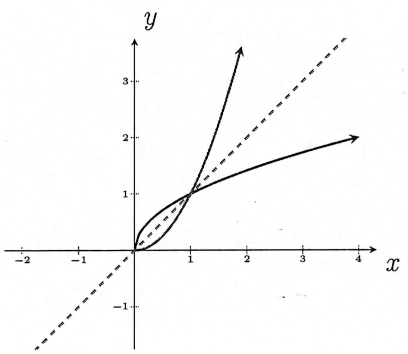
$f^{-1}(f(x)) = \dfrac{(\sqrt[3]{4x-1}+2-2)^3+1}{4}$

$= \dfrac{4x-1+1}{4} = \boxed{x}$

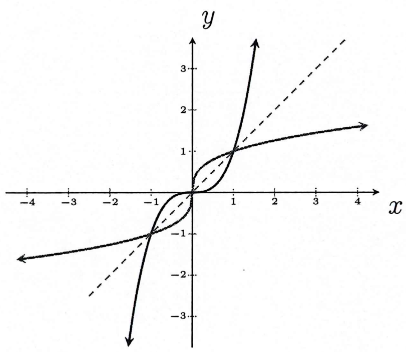
$f(f^{-1}(x))$ omitted

Note: A function is symmetric with it's inverse with respect to $y = x$.

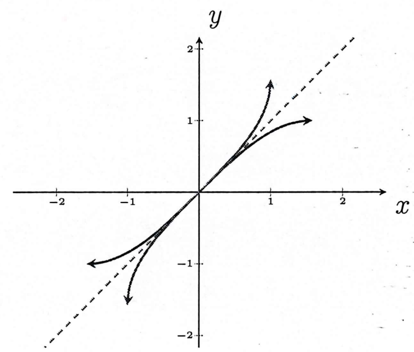
$f(x) = \sqrt{x}$
$f^{-1}(x) = x^2,\ x > 0$



$f(x) = x^3$
$f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$

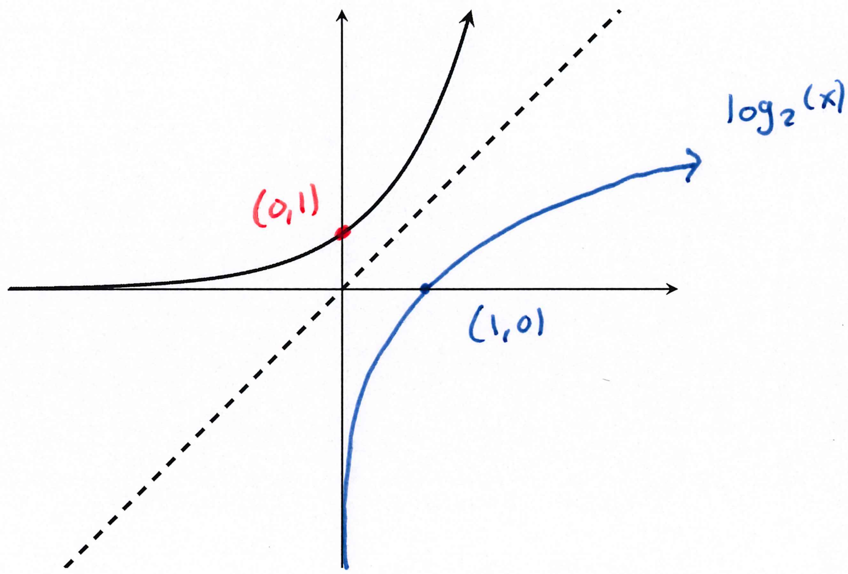


$f(x) = \sin x$ on $[-\pi/2, \pi/2]$
$f^{-1}(x) = \sin^{-1} x$

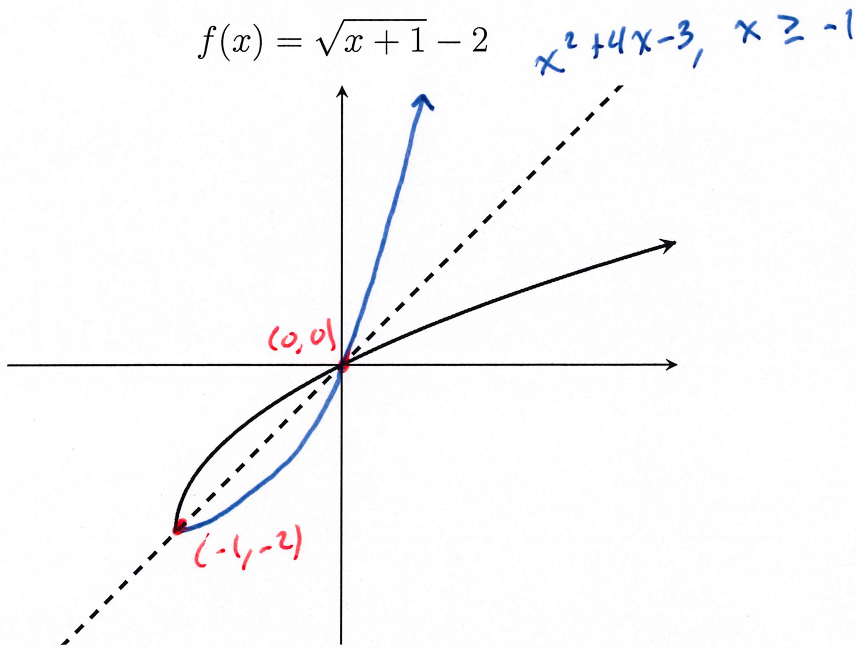


Example. Draw the function inverses:

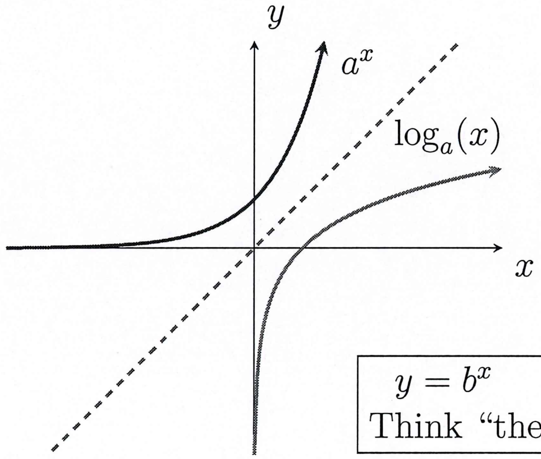
$$f(x) = 2^x$$



$$f(x) = \sqrt{x+1} - 2$$



> **Definition. (Logarithmic Function Base b)**
> For any base $b > 0$, with $b \neq 1$, the **logarithmic function base** b, denoted $y = \log_b(x)$, is the inverse of the exponential function $y = b^x$. The inverse of the natural exponential function with base $b = e$ is the **natural logarithm function**, denoted $y = \ln(x)$.



Note:

	a^x	$\log_a(x)$
Domain:	$(-\infty, \infty)$	$(0, \infty)$
Range:	$(0, \infty)$	$(-\infty, \infty)$

$$y = b^x \iff \log_b(y) = x$$
Think "the base stays the base"

Example. Evaluate:

$\log_9(81) = 2$

$9^x = 81$

$\log_3(\sqrt{3}) = 1/2$

$3^x = 3^{1/2}$

$\log_{\frac{1}{2}}(8) = -3$

$\frac{1}{2}^x = 8$

$(\log_5(5^{-3}))^2 = (-3)^2 = 9$

$5^x = 5^{-3}$

Note: In this course, the **common logarithm** is $\log_{10}(x)$ and is denoted by $\log(x)$.
– Sometimes other disciplines use $\log(x)$ to represent other bases.

Example. Evaluate:

$\log 100000 = 5$

$10^x = 100000$

$\log \frac{1}{1000} = -3$

$10^x = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$

Recall that for a function f and its inverse g:

- $f(g(x)) = x$
- $g(f(x)) = x$
- Domain of f=Range of g
- Domain of g=Range of f

Inverse Relations for Exponential and Logarithmic Functions
For any base $b > 0$, with $b \neq 1$, the following inverse relations hold:

$$b^{\log_b x} = x \qquad \log_b(b^x) = x, \text{ for all real values of } x$$

Example. Evaluate:

$2^{\log_2 8} = 8$ $\qquad$ $\log_b b^{\pi} = \pi$ $\qquad$ $\log 10^3 = 3$

Example. Write each expression in terms of one logarithm:

$\log_2 6 - \log_2 15 + \log_2 20$

$= \log_2\left(\frac{6 \cdot 20}{15}\right) = \log_2(8) = \boxed{3}$

$\log_3 100 - (\log_3 18 + \log_3 50)$

$= \log_3\left(\frac{100}{18 \cdot 50}\right) = \log_3\left(\frac{1}{9}\right) = \boxed{-2}$

Laws of Logarithms
For $x, y > 0$:

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(x)$
3. $\log_a(x^r) = r\log_a(x)$
4. $\log_a(1) = 0$
5. $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Example. Solve each equation checking for extraneous solutions:

$\log_{64} x^2 = \frac{1}{3}$ $\longrightarrow$ $64^{1/3} = x^2$

$4 = x^2$

$\boxed{\pm 2 = x}$

Check

$x^2 \geq 0$

$2^2 \geq 0$ ✓

$(-2)^2 \geq 0$ ✓

$\log(3x + 2) + \log(x - 1) = 2$

$10^2 = (3x+2)(x-1)$

$0 = 3x^2 - x - 102$

$0 = (x-6)(3x+17)$

$\rightarrow \boxed{x = 6}$

$x = -17/6$ $\longleftarrow$ No

Check

$x - 1 \geq 0$

$\boxed{x \geq 1}$

$3x + 2 \geq 0$

$x \geq -2/3$

$\log_2 x^2 - \log_2(3x - 8) = 2$

$2^2 = \dfrac{x^2}{3x-8}$ $\longrightarrow$ $x^2 - 12x + 32 = 0$

$(x-4)(x-8) = 0$

$\boxed{x = 4,\ x = 8}$

Check

$x^2 \geq 0$

$3x - 8 \geq 0 \rightarrow \boxed{x \geq \frac{8}{3}}$ ✓

$\log_4 x - \log_4(x - 1) = \frac{1}{2}$

$4^{1/2} = \dfrac{x}{x-1}$ $\longrightarrow$ $2(x-1) = x$

$x - 2 = 0$

$\boxed{x = 2}$

Check

$x \geq 0$

$x - 1 \geq 0 \rightarrow \boxed{x \geq 1}$ ✓

$\log_3(x + 6) - \log_3(x - 6) = 2$

$3^2 = \dfrac{x+6}{x-6}$ $\longrightarrow$ $9x - 54 = x + 6$

$8x = 60$

$x = 15/2$ $\longleftarrow$ No!

check

$x + 6 \geq 0 \rightarrow x \geq -6$

$x - 6 \geq 0 \rightarrow \boxed{x \geq 6}$

$\log_3(x^2 - 5) = 2$

$3^2 = x^2 - 5$

$0 = x^2 - 14$

$\boxed{x = \pm\sqrt{14}}$

Check

$x^2 - 5 \geq 0$

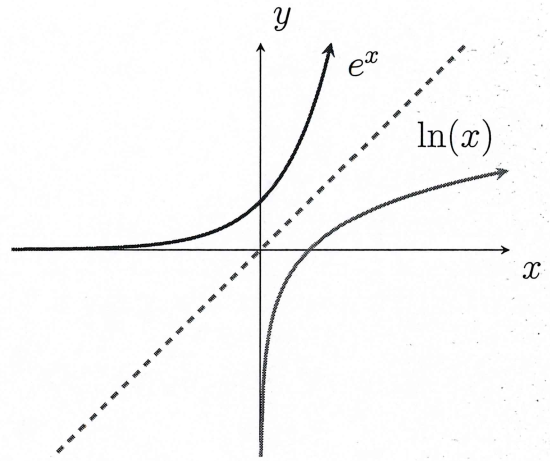
$x^2 \geq 5$

$\boxed{x \geq \sqrt{5}}$ $\quad$ $\boxed{x \leq -\sqrt{5}}$ $\rightarrow$ $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$

> **Definition.**
> The **Natural Logarithmic Function** uses base e and is denoted $\ln(x) = \log_e x$.

Note: The natural log is the inverse of e^x:

$$\ln(x) = y \iff e^y = x$$



> **Inverse Properties for a^x and $\log_a x$**
>
> Base a: $\quad a^{\log_a x} = x, \quad \log_a a^x = x, \quad a > 0,\ a \neq 1,\ x > 0$
>
> Base e: $\quad e^{\ln x} = x, \quad \ln e^x = x, \quad x > 0$

Example. Simplify

$e^{-\ln 0.3} = \boxed{.0.3}$

$e^{\ln \pi x - \ln 2} = e^{\ln\left(\frac{\pi x}{2}\right)} = \boxed{\frac{\pi x}{2}}$

$\ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = \boxed{-1}$

$e^{4\ln x} = e^{\ln(x^4)} = \boxed{x^4}$

Example. Write each expression in terms of one logarithm:

$$\ln(a+b) + \ln(a-b) - 2\ln c$$

$$= \ln\left(\frac{(a+b)(a-b)}{c^2}\right)$$

$$= \ln\left(\frac{a^2-b^2}{c^2}\right)$$

Laws of the Natural Logarithm

For $x, y > 0$:

1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(x)$
3. $\ln(x^r) = r\ln(x)$
4. $\ln(1) = 0$
5. $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$

$$\tfrac{1}{3}\ln(x+2)^3 + \tfrac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right]$$

$$= \ln(x+2) + \ln(x^{1/2}) - \ln(x^2+3x+2)$$

$$= \ln\left(\frac{(x+2)\,x^{1/2}}{(x+2)(x+1)}\right)$$

$$= \ln\left(\frac{x^{1/2}}{x+1}\right)$$

Note: Many common mistakes come from using the logarithm rules incorrectly:

$$\ln A - \ln B \neq \frac{\ln A}{\ln B} \qquad \ln(A+B) \neq \ln(A)\ln(B)$$

Example. Solve:

$2^x = 55$

$x = \log_2(55)$

$5^{3x} = 29$ $\rightarrow$ $125^x = 29$

$x = \log_{125}(29)$

Let $u = e^x$ $e^{2x} - 5e^x - 14 = 0$

$u^2 - 5u - 14 = 0$

$(u-7)(u+2) = 0$ $\rightarrow$ $e^x = 7$ $\boxed{x = \ln(7)}$

$e^x = -2$ No

$4e^{2x} - 7e^x = 15$

$4u^2 - 7u - 15 = 0$

$(4u+5)(u-3) = 0$

$e^x = -\frac{5}{4}$ No

$e^x = 3 \rightarrow \boxed{x = \ln(3)}$

Example. Solve for y in terms of x:

$\ln(y - 40) = 5x$

$e^{5x} = y - 40$

$\boxed{y = e^{5x} + 40}$

$\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

$\frac{(y+1)(y-1)}{y+1} = \sin(x)$

$\boxed{y = \sin(x) + 1}$

Example. Solve:

$\ln(t) + \ln(t^2) = 6$

$t^3 = e^6$

$\boxed{t = e^2}$

$e^{x^2+2x-3} = 1$ $e^0 = 1$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\boxed{x = -3,\ x = 1}$

$\ln x = -1$

$\boxed{x = e^{-1}}$

$e^{-0.3t} = 27$

$-0.3t = \ln(27)$

$\boxed{t = \frac{3\ln(3)}{-0.3} = -10\ln(3)}$