4.4 Graphing Functions

Graphing Guidelines for $y = f(x)$

1. Identify the domain or interval of interest.
2. Determine if the function is symmetric.
3. Use $f'(x)$ and $f''(x)$ to determine
   - intervals of increasing/decreasing ..... $f'(x) > 0$ or $f'(x) < 0$
   - relative max/mins ..... 1st or 2nd derivative test
   - concave up/concave down ..... $f''(x) > 0$ or $f''(x) < 0$
   - inflection points ..... $f''(x)$ changes signs
4. Find asymptotes
   - vertical ..... $\lim_{x \to a} f(x) = \pm\infty$
   - horizontal ..... $\lim_{x \to \pm\infty} f(x) = c$
   - slant ..... $y = ax + b$
5. Intercepts
   - y-intercept ..... evaluate $f(0)$
   - x-intercept ..... solve $f(x) = 0$
6. Sketch using the characteristics found above

Example. Sketch the following functions using derivatives:

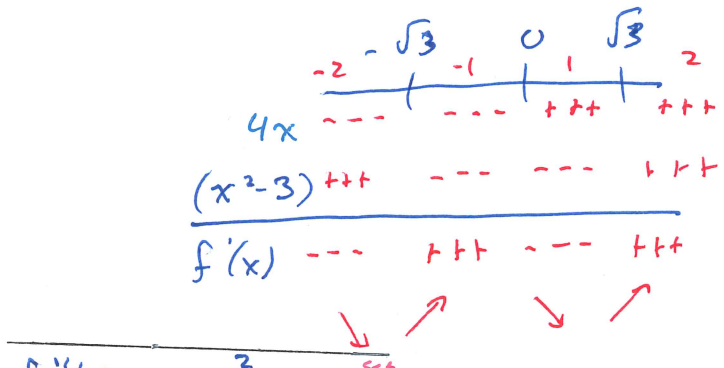
1. $f(x) = x^4 - 6x^2$

① Domain $(-\infty, \infty)$

② $f(-x) = (-x)^4 - 6(-x)^2 = x^4 - 6x^2 = f(x)$

③ $f'(x) = 4x^3 - 12x \overset{\text{set}}{=} 0$

$4x(x^2-3) = 0 \longrightarrow x=0,\ x=-\sqrt{3},\ x=\sqrt{3}$



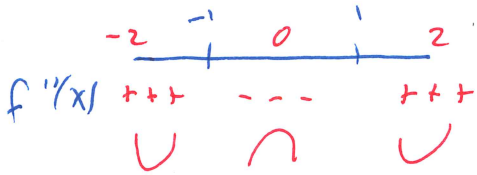
Inc $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Dec $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

local max $(0, 0)$

local min(s) $(-\sqrt{3}, -9)$, $(\sqrt{3}, -9)$

$f''(x) = 12x^2 - 12 \overset{\text{set}}{=} 0 \longrightarrow x = \pm 1$



Concave up $(-\infty, -1) \cup (1, \infty)$

Concave down $(-1, 1)$

Inflection point(s) $(-1, -5)$, $(1, -5)$

Note: (2nd Deriv test)

$f''(-\sqrt{3}) = 24 > 0 \rightarrow$ min

$f''(0) = -12 < 0 \rightarrow$ max

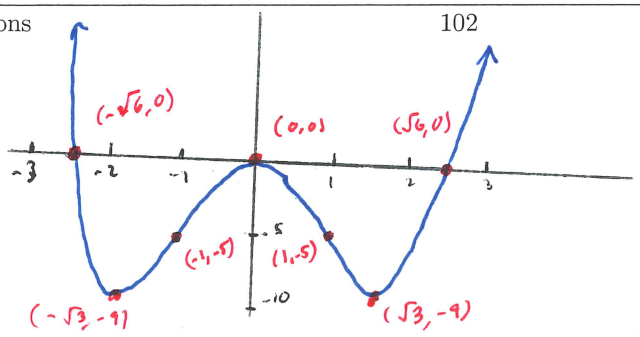
$f''(\sqrt{3}) = 24 > 0 \rightarrow$ min

④ No asymptotes

⑤ y-int $f(0) = 0$

x-int $f(x) = 0$

$x^2(x^2-6) = 0 \Rightarrow x = 0,\ x = -\sqrt{6},\ x = \sqrt{6}$



2. $g(x) = 200 + 8x^3 + x^4$

① Domain: $(-\infty, \infty)$

② $g(-x) = 200 + 8(-x)^3 + (x)^4$

$= 200 - 8x^3 + x^4 \neq g(x)$

Not symmetric

③ $g'(x) = 24x^2 + 4x^3$ set $= 0$

$4x^2(6+x) = 0$

$\Rightarrow x = 0, \quad x = -6$

	$(-\infty, -6)$	$(-6, 0)$	$(0, \infty)$
$4x^2$	+++	+++	+++
$6+x$	− − −	+++	+++
$f'(x)$	− − −	+++	+++
	↘	↗	

Dec $(-\infty, -6)$

Inc $(-6, 0) \cup (0, \infty)$

local min $(-6, -232)$

No local max

$g''(x) = 48x + 12x^2$ set $= 0$

$12x(4+x) = 0$

$\Rightarrow x = 0, \quad x = -4$

	$(-\infty, -4)$	$(-4, 0)$	$(0, \infty)$
$12x$	− − −	− − −	+++
$4+x$	− − −	+++	+++
$f''(x)$	+++	− − −	+++
	∪	∩	∪

Concave down $(-4, 0)$

Concave up $(-\infty, -4) \cup (0, \infty)$

Inflection pt. $(-4, -56)$

④ No asymptotes

⑤ y int $g(0) = 200$

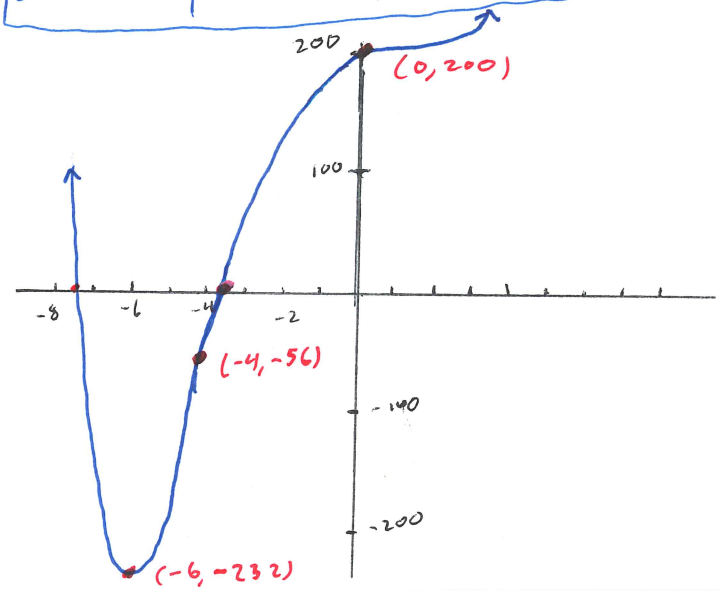
x int $g(x) = 0$

$200 + 8x^3 + x^4 = 0$

can't solve !!

$x \approx -7.532$

$x \approx -3.557$



① Domain $x \neq 1$

② $\dfrac{(-x)}{((-x)-1)^2} = \dfrac{-x}{(x+1)^2} \neq y$

3. $y = \dfrac{x}{(x-1)^2}$

③ $y' = \dfrac{(x-1)^2 - x\,2(x-1)}{(x-1)^4} = -\dfrac{x+1}{(x-1)^3}$ set $= 0$ $\Rightarrow$ $x = -1$, $x \neq 1$

Dec $(-\infty, -1) \cup (1, \infty)$

Inc $(-1, 1)$

local min $(-1, -1/4)$

No local max since DNE @ $x = 1$

	-2		0		2
		-1		1	
$-(x+1)$	+ + +		- - -		- - -
$(x-1)^3$	- - -		- - -		+ + +
$f'(x)$	- - -		+ + +		- - -
	↘		↗		↘

$y'' = -\dfrac{(x-1)^3 - (x+1)\,3(x-1)}{(x-1)^6} = \dfrac{2(x+2)}{(x-1)^4}$ set $= 0$ $\rightarrow$ $x = -2$, $x \neq 1$

	-3		-2		0		1		2
$2(x+2)$	- - -		0	+ + +		+ + +		+ + +	
$(x-1)^4$	+ + +			+ + +		+ + +		+ + +	
$f''(x)$	- - -		0	+ + +		+ + +		+ + +	
	∩			∪		∪		∪	

Concave up $(-2, 1) \cup (1, \infty)$

Concave down $(-\infty, -2)$

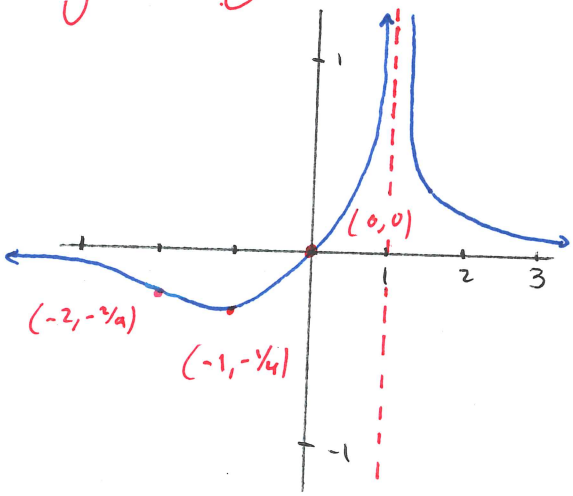
I. P. $(-2, -2/9)$

④ V. A.

$\lim\limits_{x \to 1} \dfrac{x}{(x-1)^2} = \dfrac{1}{\text{Sml +}} = \infty$

H. A.

$\lim\limits_{x \to \pm\infty} \dfrac{x}{(x-1)^2} = 0$



⑤ y-int $(x = 0)$ $\dfrac{0}{(0-1)^2} = 0$ } $(0,0)$

x-int $(y = 0)$

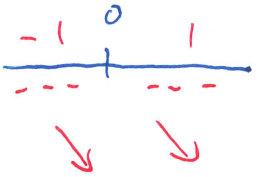
$\dfrac{x}{(x-1)^2} = 0 \Rightarrow x = 0$

4. $y = \dfrac{x^2-4}{x^2-2x} = \dfrac{x+2}{x}$, $x \neq 2$

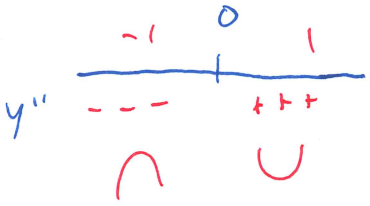
① Domain $x \neq 0$, $x \neq 2$

② $\dfrac{(-x)^2-4}{(-x)^2-2(-x)} = \dfrac{x^2-4}{x^2+2} \neq y$

③ $y' = \dfrac{x-(x+2)}{x^2} = -\dfrac{2}{x^2}$ set $= 0$ $\Rightarrow$ $-\dfrac{2}{x^2} \neq 0$, $\boxed{x \neq 0}$



$y'' = \dfrac{4}{x^3}$ set $= 0$ $\Rightarrow$ $\dfrac{4}{x^3} \neq 0$, $\boxed{x \neq 0}$



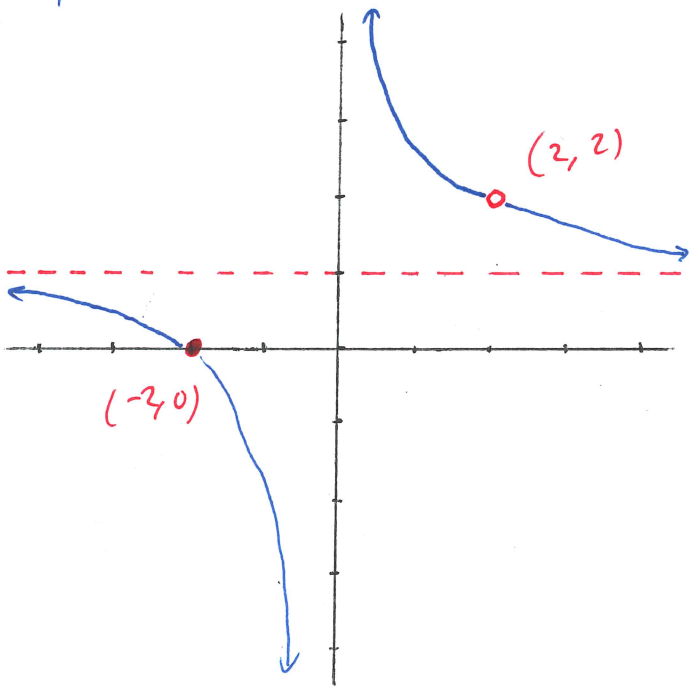
④ V. A.

$\lim\limits_{x \to 0^-} \dfrac{x^2-4}{x^2-2x} = -\infty$ $\quad$ $\lim\limits_{x \to 2} \dfrac{x^2-4}{x^2-2x} = 2$

$\lim\limits_{x \to 0^+} \dfrac{x^2-4}{x^2-2x} = \infty$ $\quad$ $\lim\limits_{x \to \pm\infty} \dfrac{x^2-4}{x^2-2x} = 1$

⑤ y-int $(x=0)$ DNE

x-int $\dfrac{x^2-4}{x^2-2x} = 0 \rightarrow \boxed{x = -2}$



5. $f(x) = \dfrac{(x+1)^3}{(x-1)^2}$

① Domain $x \neq 1$

② $f(-x) = \dfrac{(-x+1)^3}{(-x-1)^2} = \dfrac{-(x-1)^3}{(x-1)^2} \neq f(x)$

③ $f(x)' = \dfrac{(x+1)^2(x-5)}{(x-1)^3}$ set $= 0$ $\quad x=-1,\ x=5,\ x\neq 1$

	-1		1		5	
	-3		0		3	7
$(x+1)^2$	+++		+++		+++	+++
$(x-5)$	---		---		---	+++
$(x-1)^3$	---		---		+++	+++
$f(x)'$	+++	↗	+++	↗	---	+++ ↗

$f(x)'' = \dfrac{24(x+1)}{(x-1)^4}$ set $= 0 \Rightarrow x=-1$, $x\neq 1$

	-1		1	
	-2		0	2
$24(x+1)$	---		+++	+++
$(x-1)^4$	+++		+++	+++
$f(x)''$	---		+++	+++
	∩		∪	∪

Inc $(-\infty,-1) \cup (-1,1) \cup (5,\infty)$

Dec $(1,5]$

local min $(5, 27/2)$

No local max

Concave down $(-\infty,-1)$

Concave up $(-1,1) \cup (1,\infty)$

④ V.A.

$\lim\limits_{x\to 1} \dfrac{(x+1)^3}{(x-1)^2} = \infty$

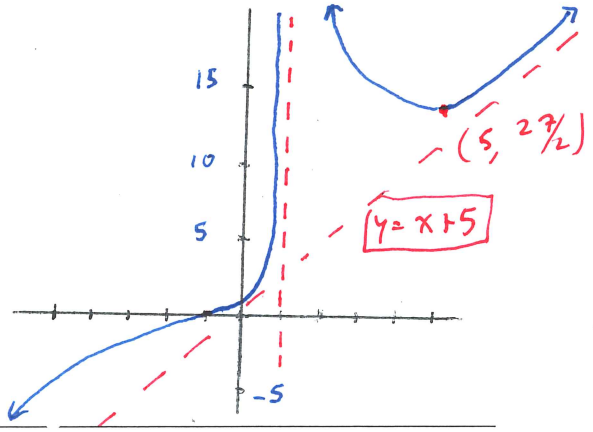
Slant asymptotes $y = x+5$

$$\begin{array}{r} x^2-2x+1 \;\big)\; x^3+3x^2+3x+1 \\ -(x^3-2x^2+x) \\ \hline 5x^2+2x+1 \\ -(5x^2-10x+5) \\ \hline 12x-4 \end{array} \qquad \text{quotient } x+5$$

⑤ y-int $(x=0)$ $\quad f(0)=1$

x-int solve $f(x)=0$

$\dfrac{(x+1)^3}{(x-1)^2} = 0 \Rightarrow x=-1$



Note:

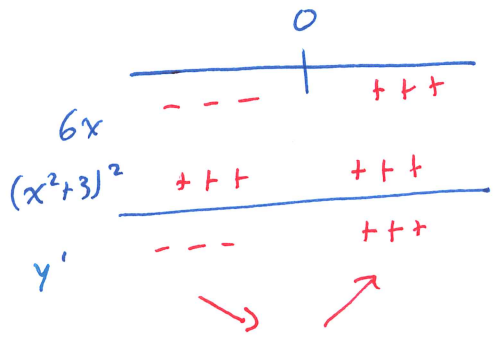
$\lim\limits_{x\to\pm\infty} \dfrac{(x+1)^3}{(x-1)^2} = \lim\limits_{x\to\pm\infty} \underbrace{x+5}_{\text{slant asymptote}} + \underbrace{\dfrac{12x-4}{(x-1)^2}}_{\to 0}$

① Domain $(-\infty, \infty)$

② $\dfrac{(-x)^2}{(-x)^2+3} = \dfrac{x^2}{x^2+3} = y \Rightarrow$ Symmetric

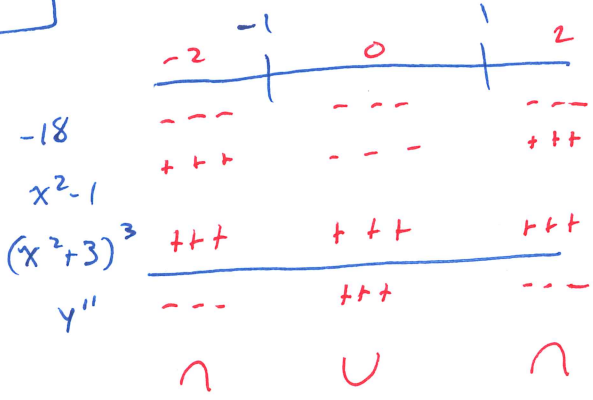
6. $y = \dfrac{x^2}{x^2+3}$

③ $y' = \dfrac{6x}{(x^2+3)^2}$ Set $= 0 \Rightarrow x = 0$



dec	$(-\infty, 0)$	local min	$(0,0)$
inc	$(0, \infty)$	No local max	

$y'' = \dfrac{-18(x^2-1)}{(x^2+3)^3}$ Set $= 0 \Rightarrow x = \pm 1$

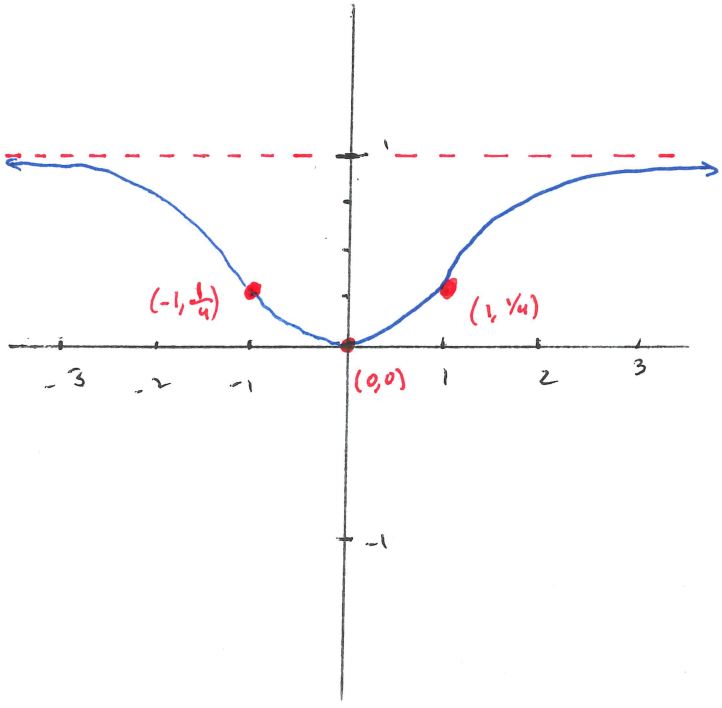


Concave down	$(-\infty, -1) \cup (1, \infty)$
concave up	$(-1, 1)$
I.P.	$(-1, 1/4)$, $(1, 1/4)$

④ No V.A.

H.A.

$\lim_{x \to \pm\infty} \dfrac{x^2}{x^2+3} = 1$



⑤ y-int $(x=0)$

$\dfrac{0}{0+3} = 0$ $\Big\}$ $(0,0)$

x-int

$\dfrac{x^2}{x^2+3} = 0$ $\Big\}$ $(0,0)$

7. $f(x) = \dfrac{x^3}{(x+1)^2}$

① Domain $\boxed{x \neq -1}$

② $f(-x) = \dfrac{(-x)^3}{(-x)+1)^2} = \dfrac{-x^3}{-(x-1)} \neq f(x)$

Not symm

③ $f'(x) = \dfrac{x^2(x+3)}{(x+1)^3} \overset{\text{set}}{=} 0 \Rightarrow \boxed{\begin{array}{l} x = 0 \\ x = -3 \\ x \neq -1 \end{array}}$

	-4		-2		-½		1
		-3		-1		0	
x^2	+++		+++		+++		+++
$x+3$	---		+++		+++		+++
$(x+1)^3$	---		---		+++		+++
$f'(x)$	+++		---		+++		+++

$f''(x) = \dfrac{6x}{(x+1)^4} \overset{\text{set}}{=} 0 \Rightarrow x = 0$, $x \neq -1$

		-1		0	
$6x$	---		---		+++
$(x+1)^4$	+++		+++		+++
$f''(x)$	---		---		+++
	∩		∩		∪

Dec $(-3, -1)$

Inc $(-\infty, -3) \cup (-1, 0) \cup (0, \infty)$

local max $(-3, -27/4)$

No local min

Concave down $(-\infty, -1) \cup (-1, 0)$

Concave up $(0, \infty)$

I. P. $(0, 0)$

④ V.A.

$\lim\limits_{x \to -1} \dfrac{x^3}{(x+1)^2} = -\infty$

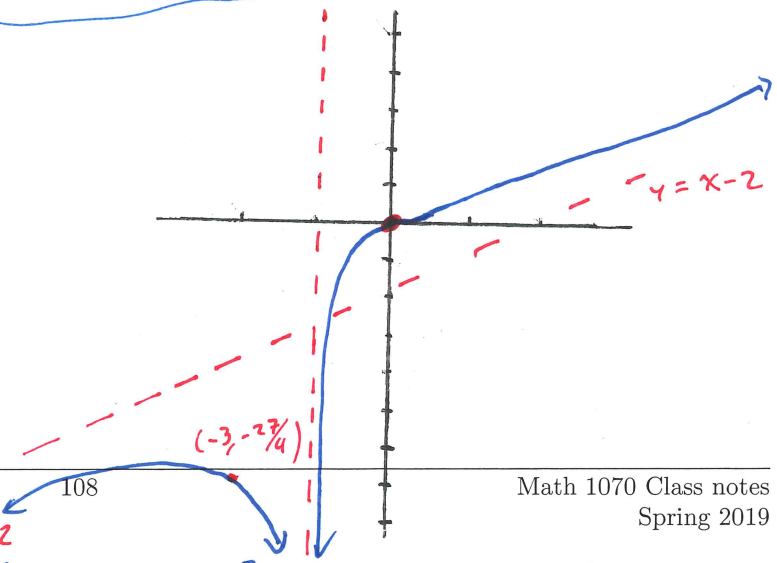
H.A. N/A

Slant. $y = x - 2$

⑤ y-int $f(0) = 0$

x-int $\dfrac{x^3}{(x+1)^2} = 0 \Rightarrow x = 0$

$(0, 0)$



Note:

$$\begin{array}{r} x - 2 \\ x^2 + 2x + 1 \overline{)\, x^3 + 0x^2 + 0x + 0} \\ -(x^3 + 2x^2 + x) \\ \hline -2x^2 - x + 0 \\ -(-2x^2 - 4x - 2) \\ \hline 3x + 2 \end{array}$$

$\Rightarrow \dfrac{x^3}{(x+1)^2} = (x-2) + \underbrace{\dfrac{3x+2}{(x+1)^2}}_{\to 0 \text{ as } x \to \infty}$

8. $y = x^{1/5}$

① Domain $(-\infty, \infty)$

② $(-x)^{1/5} = -x^{1/5} = -y$ odd (symmetric to origin)

③ $y' = \frac{1}{5x^{4/5}}$ set $= 0$

$y' \neq 0$

$x \neq 0$

	0	
	-1	1
y'	+++	+++
	↗	↗

Increasing $(-\infty, 0) \cup (0, \infty)$

Not decreasing

No local max/min

$y'' = \frac{-4}{25x^{9/5}}$ set $= 0$

$y'' \neq 0$

$x \neq 0$

	0	
	-1	1
y''	+++	− − −
	∪	∩

Concave up $(-\infty, 0)$

Concave down $(0, \infty)$

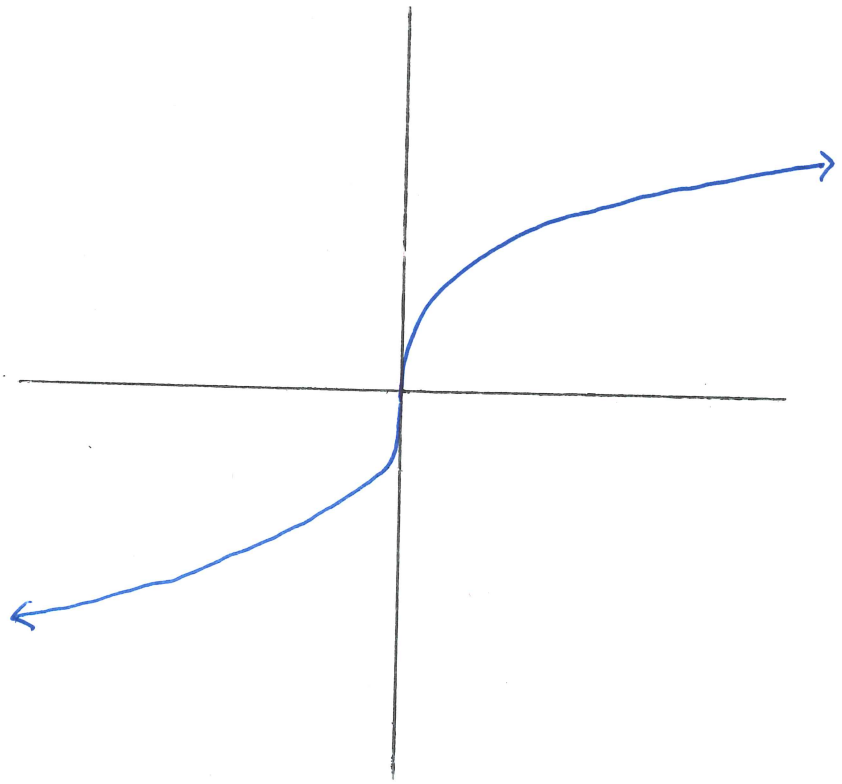
I.P. $(0, 0)$

④ No V.A.

No H.A.

No slant Asymptotes

⑤ y-int, x-int $\}$ $(0, 0)$



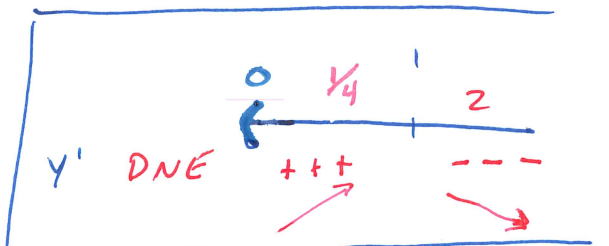
① Domain $[0, \infty)$

② $2\sqrt{-x} - (-x) \neq 2\sqrt{x} - x$ } No symm.
$\neq -(2\sqrt{x} - x)$ }

9. $y = 2\sqrt{x} - x$

③ $y' = \frac{1}{\sqrt{x}} - 1 \overset{\text{set}}{=} 0 \Rightarrow x = 1$

$x \neq 0$



Inc $[0, 1)$	rel max $(1, 1)$
Dec $(0, \infty)$	No rel min.

$y'' = -\frac{1}{2x^{3/2}} \overset{\text{set}}{=} 0 \Rightarrow y'' \neq 0$ anywhere

$x \neq 0$

Concave down	Not concave up
$[0, \infty)$	No I.P.

④ No V.A.
No H.A. / slant

⑤ y-int ($x = 0$) $\Rightarrow$ $(0, 0)$

x-int

$2\sqrt{x} - x = 0$

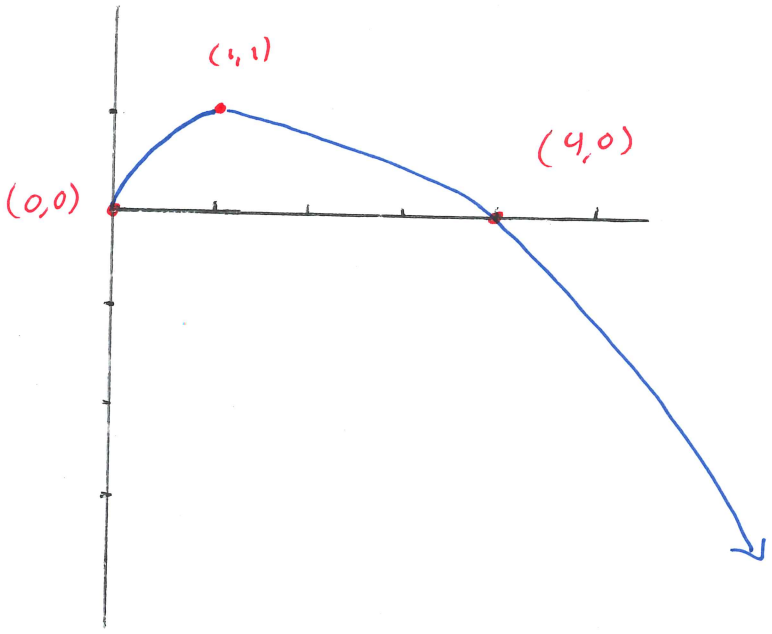
$2\sqrt{x} = x$

$4x = x^2$

$0 = (4 - x)x$

$\Rightarrow x = 0 \quad (0, 0)$

$x = 4 \quad (4, 0)$



① Domain: $1+e^{-x} \neq 0 \Rightarrow e^{-x} \neq -1$ $(-\infty, \infty)$

② $\dfrac{1}{1+e^{-(-x)}} = \dfrac{1}{1+e^{x}}$ ← No symm

10. $y = \frac{1}{1+e^{-x}}$

③ $y' = \dfrac{e^{-x}}{(1+e^{-x})^2} \overset{\text{set}}{=} 0$ $\quad y' \neq 0$

y': $+ + +$ (over 0)

Increasing $(-\infty, \infty)$
No local max/min

$y'' = \dfrac{-e^{-x}(1-e^{-x})}{(1+e^{-x})^3} \overset{\text{set}}{=} 0 \Rightarrow -e^{-x}\underbrace{(1-e^{-x})}_{=0} = 0$

$1 = e^{-x} \Rightarrow x = 0$

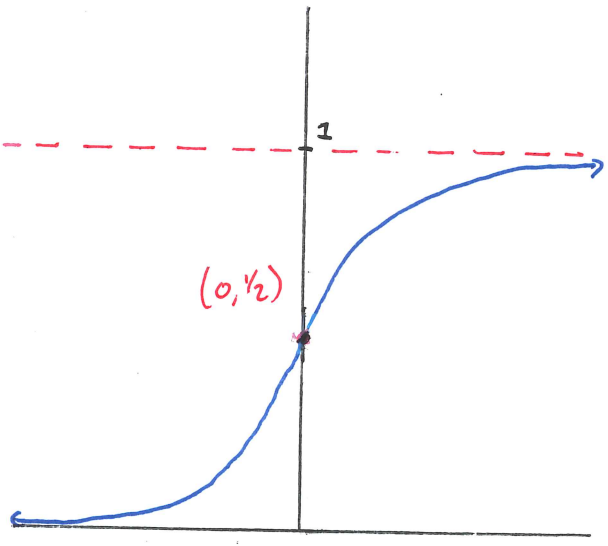
	-1		1	
$-e^{-x}$	$---$		$---$	
$1-e^{-x}$	$---$		$+++$	
$(1+e^{-x})^3$	$+++$		$+++$	
y''	$+++$		$---$	

Concave up $(-\infty, 0)$
Concave down $(0, \infty)$
I.P. $(0, 1/2)$

④ No V.A.

$\lim_{x \to -\infty} \dfrac{1}{1+e^{-x}} = \dfrac{\lim_{x \to -\infty} 1}{\lim_{x \to -\infty} (1+e^{-x})} = \boxed{0}$ ($\lim_{x \to -\infty}(1+e^{-x}) \to \infty$)

$\lim_{x \to \infty} \dfrac{1}{1+e^{-x}} = \dfrac{1}{\lim_{x \to \infty}(1+e^{-x})} = \dfrac{1}{1} = \boxed{1}$



⑤ y-int $\dfrac{1}{1+e^{0}} = 1 \to (0,1)$

No x-int

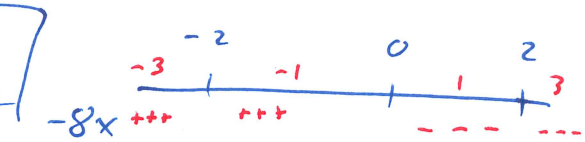
"Logistic function"

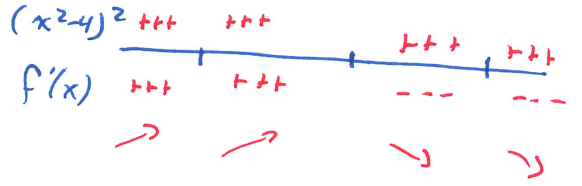
① Domain $x \neq \pm 2$

② $f(-x) = \dfrac{(-x)^2}{(-x)^2+4} = \dfrac{x^2}{x^2+4} = f(x)$ ← Even, Symm about y-axis

11. $f(x) = \dfrac{x^2}{x^2-4}$

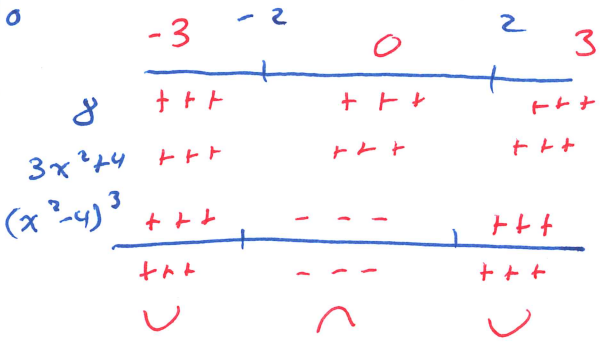
③ $f'(x) = -\dfrac{8x}{(x^2-4)^2} \overset{\text{set}}{=} 0 \Rightarrow x = 0,\ x \neq \pm 2$





Inc	$(-\infty, -2) \cup (-2, 0)$
Dec	$(0, 2) \cup (2, \infty)$
local max	$(0, 0)$

$f''(x) = 8\dfrac{(3x^2+4)}{(x^2-4)^3} \overset{\text{set}}{=} 0$ $\quad f''(x) \neq 0$ $\quad x \neq \pm 2$



Concave up	$(-\infty, -2) \cup (2, \infty)$
Concave down	$(-2, 2)$
No I.P.	(Dis cont.)

④ V.A.

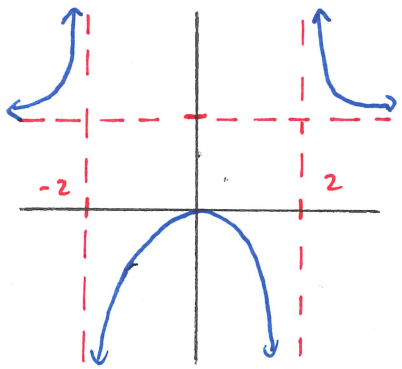
$\lim_{x \to -2^-} \dfrac{x^2}{x^2-4} = \dfrac{4}{\text{Sm+}} = \infty$

$\lim_{x \to -2^+} \dfrac{x^2}{x^2-4} = \dfrac{4}{\text{Sm-}} = -\infty$

$\lim_{x \to 2^\pm} f(x) = \pm\infty$ (by symm)

H.A.

$\lim_{x \to \pm\infty} f(x) = 1$



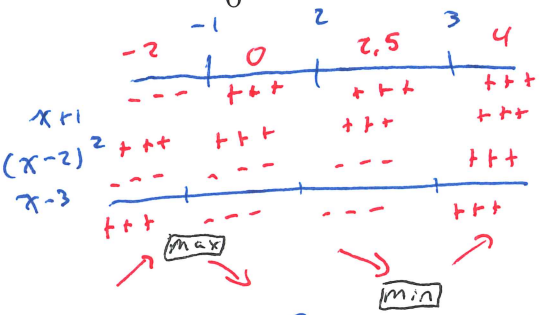
⑤ y-int

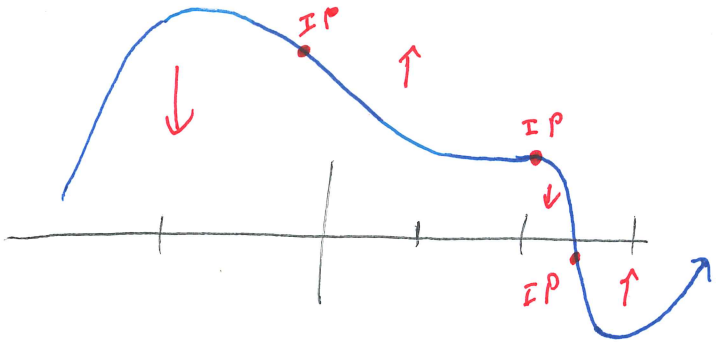
$f(0) = 0$

x-int $\quad \dfrac{x^2}{x^2+4} = 0 \Rightarrow x = 0$

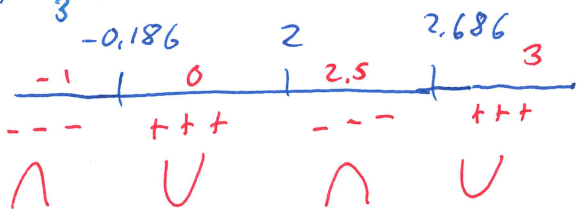
Example. Using the following derivatives, sketch a possible graph of the original function.

1. $f'(x) = \frac{1}{6}(x+1)(x-2)^2(x-3)$

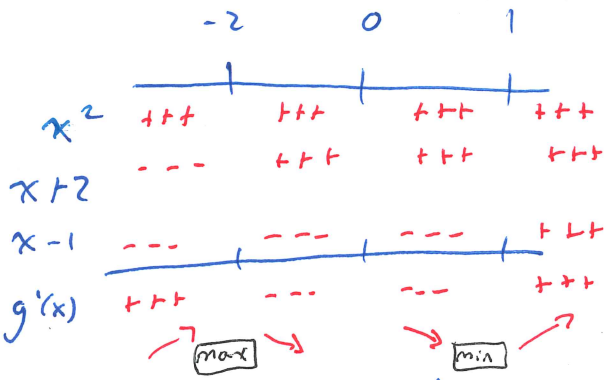


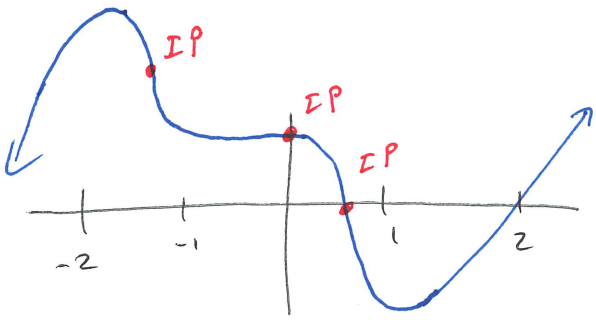


$f''(x) = \frac{1}{3}(2x^3 - 9x^2 + 9x + 2)$



2. $g'(x) = x^2(x+2)(x-1)$





$g''(x) = x(4x^2 + 3x - 4)$