

5.3: Fundamental Theorem of Calculus

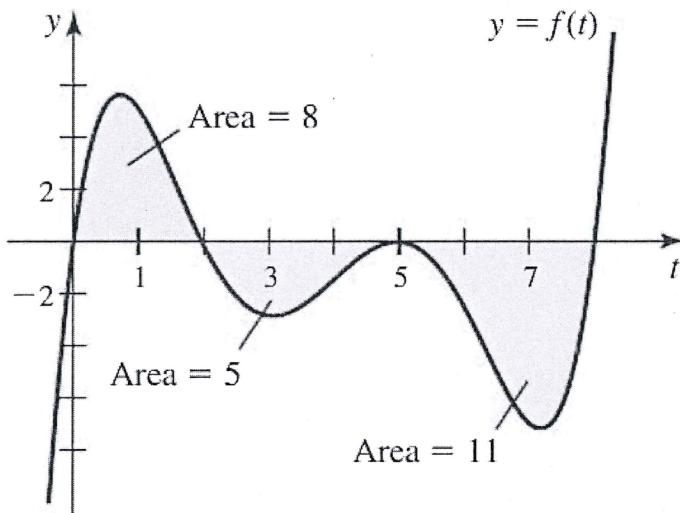
Definition. (Area Function)

Let f be a continuous function, for $t \geq a$. The **area function for f** with left endpoint a is

$$A(x) = \int_a^x f(t) dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t -axis on the interval $[a, x]$.

Example. The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t) dt$ and $F(x) = \int_2^x f(t) dt$ be two area functions for f . Evaluate the following area functions:



- | | | | |
|------------------------------|------------------------------|------------------------------|------------------------------|
| a) $A(2) = \int_0^2 f(t) dt$ | b) $F(5) = \int_2^5 f(t) dt$ | c) $A(0) = \int_0^0 f(t) dt$ | d) $F(8) = \int_2^8 f(t) dt$ |
| $= [8]$ | $= [-5]$ | $= [6]$ | $= -5 - 11$ |
| e) $A(8) = 8 - 5 - 11$ | f) $A(5) = 8 - 5$ | g) $F(2) = [0]$ | $= [-16]$ |
| $= [-8]$ | $= [3]$ | | |

Example. Let $g(x) = \int_0^x f(t) dt$, where f is the function whose graph is shown.

- a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$ and 6 .

x	0	1	2	3	4	5	6
$g(x)$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1.5	9

- b) Estimate $g(7)$.

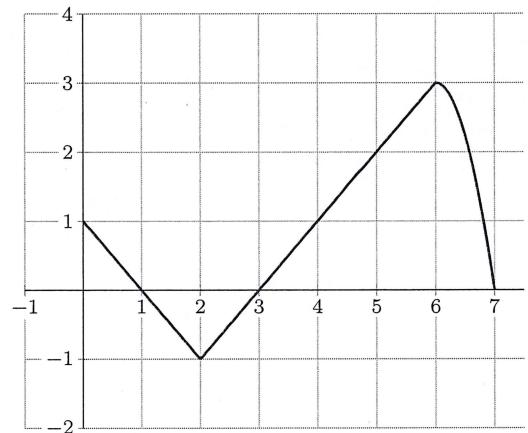
$$g(7) \approx 4 + 2.1 = 6.1$$

- c) Where does g have a maximum value?

Where does it have a minimum value?

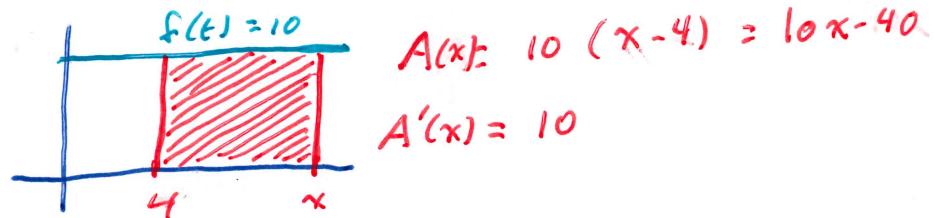
max of 6.1 at $x = 7$

min of $-\frac{1}{2}$ at $x = 3$

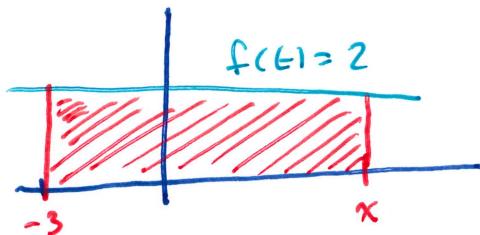


Example. For the area function $A(x) = \int_a^x f(t) dt$, graph the area function and then verify that $A'(x) = f(x)$.

- a) $f(t) = 10$, $a = 4$



- b) $f(t) = 2$, $a = -3$



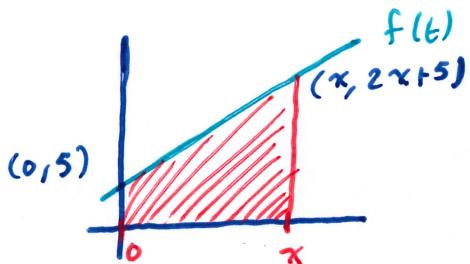
$$A(x) = 2(x + 3) = 2x + 6$$

$$A'(x) = 2$$

Area of trapezoid

$$\frac{1}{2}(b_1 + b_2)w$$

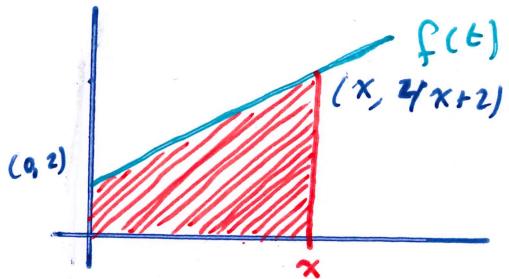
c) $f(t) = 2t + 5, a = 0$



$$A(x) = \frac{1}{2}(5 + 2x + 5)(x - 0)$$
$$= x^2 + 5x$$

$$A'(x) = 2x + 5$$

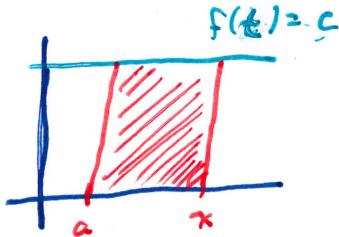
d) $f(t) = 4t + 2, a = 0$



$$A(x) = \frac{1}{2}(2 + 4x + 2)(x - 0)$$
$$= 2x^2 + 2x$$

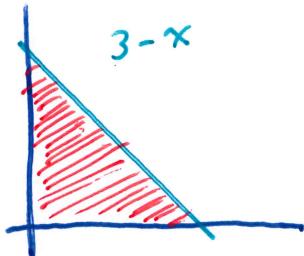
$$A'(x) = 4x + 2$$

Example. Let $f(x) = c$, where c is a positive constant. Explain why an area function of f is an increasing function.



$A(x)$ represents the area of a rectangle with positive height and increasing width.

Example. The linear function $f(x) = 3 - x$ is decreasing on the interval $[0, 3]$. Is its area function for f (with left endpoint 0) increasing or decreasing on the interval $[0, 3]$?



The area function is increasing because it is positive on $[0, 3]$.

Theorem 5.3 (Part I) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t) dt, \quad \text{for } a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b) . The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Example. For the following functions, find the derivatives

a) $g(x) = \int_0^x \sqrt{1-2t} dt$

$$g'(x) = \boxed{\sqrt{1-2x}}$$

b) $g(x) = \int_3^x e^{t^2-t} dt$

$$g'(x) = \boxed{e^{x^2-x}}$$

c) $g(y) = \int_2^y t^2 \sin(t) dt$

$$g'(y) = \boxed{y^2 \sin(y)}$$

d) $y = \int_x^2 \cos(t^2) dt = - \int_2^x \cos(t^2) dt$

$$y' = \boxed{-\cos(x^2)}$$

e) $y = \int_1^{\cos(x)} (t + \sin(t)) dt$

$$\begin{aligned} u &= \cos(x) & y &= \int_1^u (t + \sin(t)) dt \\ \frac{du}{dx} &= -\sin(x) & y' &= (u + \sin(u)) \frac{du}{dx} \\ &&&= \boxed{-(\cos(x) + \sin(\cos(x))) \sin(x)} \end{aligned}$$

f) $y = \int_{-3}^{3x^4} \frac{t}{t^2 - 4t} dt = \int_{-3}^u \frac{t}{t^2 - 4t} dt$

$$\begin{aligned} u &= 3x^4 & y &= \left(\frac{u}{u^2 - 4u} \right) \frac{du}{dx} \\ \frac{du}{dx} &= 12x^3 & &= \frac{3x^4}{9x^8 - 12x^4} (12x^3) \end{aligned}$$

$$= \boxed{\frac{12x^3}{3x^4 - 4}}$$

$$g) y = \int_1^{e^x} \ln(t) dt = \int_1^u \ln(u) du$$

$$\begin{aligned} u &= e^x \\ \frac{du}{dx} &= e^x \quad y' = \ln(u) \frac{du}{dx} \\ &= \ln(e^x) e^x \\ &= \boxed{x e^x} \end{aligned}$$

$$h) y = \int_0^{x^4} \cos^2(\theta) d\theta = \int_0^u \cos^2(u) du$$

$$\begin{aligned} u &= x^4 \\ \frac{du}{dx} &= 4x^3 \quad y' = \cos^2(u) \frac{du}{dx} \\ &= \boxed{\cos^2(x^4) (4x^3)} \end{aligned}$$

$$i) y = \int_{\tan(x)}^0 \frac{dt}{1+t^2} = - \int_0^u \frac{dt}{1+t^2}$$

$$\begin{aligned} u &= \tan(x) \\ \frac{du}{dx} &= \sec^2(x) \quad y' = - \frac{1}{1+u^2} \frac{du}{dx} \\ &= - \frac{\sec^2(x)}{1+\tan^2(x)} \\ &= - \frac{\sec^2(x)}{\sec^2(x)} \\ &= \boxed{-1} \end{aligned}$$

$$j) y = \int_{\sin(x)}^1 \sqrt{1+t^2} dt = - \int_1^u \sqrt{1+t^2} dt$$

$$\begin{aligned} u &= \sin(x) \\ \frac{du}{dx} &= \cos(x) \quad y' = - \sqrt{1+u^2} \frac{du}{dt} \\ &= \boxed{-\sqrt{1+\sin^2(x)} \cos(x)} \end{aligned}$$

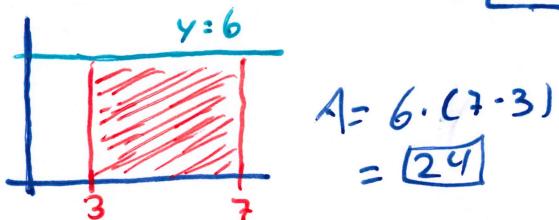
Theorem 5.3 (Part II) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

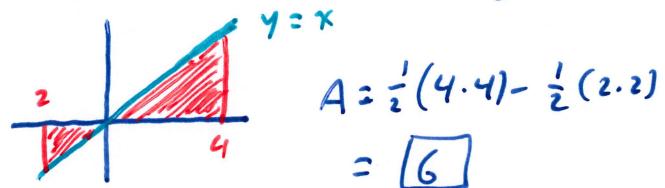
$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b$$

Example. Evaluate the following integrals using graphs and the Fundamental Theorem of Calculus.

$$a) \int_3^7 6 du = 6u \Big|_3^7 = 6(7) - 6(3) = [24]$$



$$b) \int_{-2}^4 x dx = \frac{x^2}{2} \Big|_{-2}^4 = \frac{(4)^2}{2} - \frac{(-2)^2}{2} = [6]$$



Example. Evaluate the following integrals

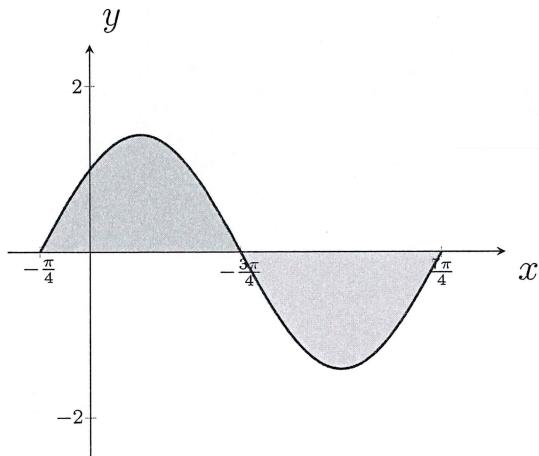
$$a) \int_{-1}^3 x^2 dx = \frac{x^3}{3} \Big|_{-1}^3 = \frac{(3)^3}{3} - \frac{(-1)^3}{3} = [28/3]$$

$$c) \int_{-5}^5 e dx = e x \Big|_{-5}^5 = [5e] - [-5e] = [10e]$$

$$b) \int_0^\pi (1 + \cos(x)) dx = x + \sin(x) \Big|_0^\pi = [\pi + \sin(\pi)] - [0 + \sin(0)] = [\pi]$$

$$d) \int_0^{\pi/4} \sec \theta \tan \theta d\theta = \sec \theta \Big|_0^{\pi/4} = [\sqrt{2} - 1]$$

Example. Evaluate the following integrals using the Fundamental Theorem of Calculus



$$\begin{aligned}
 & \int_{-\pi/4}^{7\pi/4} (\sin(x) + \cos(x)) dx \\
 &= \left[-\cos(x) + \sin(x) \right] \Big|_{-\pi/4}^{7\pi/4} \\
 &= \left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) \right] - \left[-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) \right] \\
 &= 0
 \end{aligned}$$

Example. Evaluate $\int_3^8 f'(t) dt$, where f' is continuous on $[3, 8]$, $f(3) = 4$, and $f(8) = 20$.

$$\int_3^8 f'(t) dt = f(t) \Big|_3^8 = f(8) - f(3) = 20 - 4 = \boxed{16}$$

Example. Find $\frac{d}{dt} \int_0^{t^4} \sqrt{u} du$ by evaluating the integral directly and then differentiating the result.

$$\frac{d}{dt} \left(\int_0^{t^4} u^{1/2} du \right) = \frac{d}{dt} \left(\frac{2}{3} u^{3/2} \Big|_0^{t^4} \right) = \frac{d}{dt} \left(\frac{2}{3} t^6 \right) = \boxed{4t^5}$$

Example. Find $\frac{d}{dt} \int_0^{t^4} \sqrt{u} du$ by differentiating the integral directly.

$$\begin{aligned}
 \frac{d}{dt} \int_0^v \sqrt{u} du &= \sqrt{v} \frac{dv}{dt} = \sqrt{t^4} \cdot 4t^3 = \boxed{4t^5} \\
 v &= t^4 \quad \frac{dv}{dt} = 4t^3
 \end{aligned}$$

Example. Find $\frac{d}{d\theta} \int_0^{\tan(\theta)} \sec^2(y) dy$ by evaluating the integral directly and then differentiating the result.

$$\begin{aligned} \frac{d}{d\theta} \left[\int_0^{\tan(\theta)} \sec^2(y) dy \right] &= \frac{d}{d\theta} \left[\tan(y) \Big|_0^{\tan(\theta)} \right] = \frac{d}{d\theta} [\tan(\tan(\theta)) - \tan(0)] \\ &= \frac{d}{d\theta} [\tan(\tan(\theta))] = \boxed{\sec^2(\tan(\theta)) \sec^2(\theta)} \end{aligned}$$

chain
rule

Example. Find $\frac{d}{d\theta} \int_0^{\tan(\theta)} \sec^2(y) dy$ by differentiating the integral directly.

$$\begin{aligned} \frac{d}{d\theta} \int_0^u \sec^2(y) dy &= \sec^2(u) \frac{du}{d\theta} \\ &= \boxed{\sec^2(\tan(\theta)) \sec^2(\theta)} \end{aligned}$$

$$u = \tan \theta$$

$$\frac{du}{d\theta} = \sec^2 \theta$$

Example. Evaluate the following integrals:

$$\text{a) } \int_1^8 \sqrt[3]{x} dx = \int_1^8 x^{1/3} dx$$

$$= \frac{3}{4} x^{4/3} \Big|_1^8$$

$$= \frac{3}{4} (8)^{4/3} - \frac{3}{4} (1)^{4/3}$$

$$= \frac{3}{4} \cdot 16 - \frac{3}{4} \cdot 1$$

$$= \boxed{\frac{45}{4}}$$

$$\text{c) } \int_0^2 x(2 + x^5) dx$$

$$= \int_0^2 2x + x^6 dx$$

$$= x^2 + \frac{x^7}{7} \Big|_0^2$$

$$= \left[2^2 + \frac{2^7}{7} \right] - \left[0^2 + \frac{0^7}{7} \right]$$

$$= 4 + \frac{128}{7}$$

$$= \boxed{\frac{156}{7}}$$

$$\text{b) } \int_{-2}^{-1} \frac{2}{x^2} dx = \int_{-2}^{-1} 2x^{-2} dx$$

$$= 2x^{-1} \Big|_{-2}^{-1}$$

$$= \left[-\frac{2}{-1} \right] - \left[-\frac{2}{-2} \right]$$

$$= 2 - 1 = \boxed{1}$$

$$\text{d) } \int_9^4 \frac{1 - \sqrt{u}}{\sqrt{u}} du$$

$$= \int_9^4 u^{-1/2} - 1 du$$

$$= 2u^{1/2} - u \Big|_9^4$$

$$= [2\sqrt{4} - 4] - [2\sqrt{9} - 9]$$

$$= 0 - [2\sqrt{9} - 9]$$

$$= \boxed{3}$$

$$e) \int_0^2 (y-1)(2y+1) dy$$

$$= \int_0^2 2y^3 - y^2 - y dy$$

$$= \left[\frac{y^4}{2} - \frac{y^3}{3} - \frac{y^2}{2} \right]_0^2$$

$$= \left[\frac{2^4}{2} - \frac{2^3}{3} - 2 \right] - \left[\frac{0^4}{2} - \frac{0^3}{3} - 0 \right]$$

$$= 8 - 2 - 2$$

$$= \boxed{4}$$

$$g) \int_0^1 (x^e + e^x) dx$$

$$= \left[\frac{x^{e+1}}{e+1} + e^x \right]_0^1$$

$$= \left[\frac{1^{e+1}}{e+1} + e^1 \right] - \left[\frac{0^{e+1}}{e+1} + e^0 \right]$$

$$= \frac{1}{e+1} + e - 1$$

$$= \boxed{\frac{e}{e+1}}$$

$$f) \int_0^4 \left(1 + 3y - y^2 - \frac{y^3}{4} \right) dy$$

$$= \left[y + \frac{3}{2}y^2 - \frac{1}{3}y^3 - \frac{y^4}{16} \right]_0^4$$

$$= \left[4 + \frac{3}{2}(4)^2 - \frac{1}{3}(4)^3 - \frac{4^4}{16} \right] - [0]$$

$$= 4 + 6 \cdot \frac{64}{3} - 16$$

$$= \boxed{\frac{46}{3}}$$

$$h) \int_0^3 (2\sin(x) - e^x) dx$$

$$= -2\cos(x) - e^x \Big|_0^3$$

$$= [-2\cos(3) - e^3] - [-2\cos(0)]$$

$$= -2\cos(3) - e^3 + 2 + 1$$

$$= \boxed{3 - 2\cos(3) - e^3}$$

$$\begin{aligned} \text{i) } & \int_{\frac{\pi}{2}}^0 \frac{1 - \cos(2t)}{2} dt \\ &= \int_{\frac{\pi}{2}}^0 \frac{1}{2} - \frac{\cos(2t)}{2} dt \\ &= t - \frac{\sin(2t)}{4} \Big|_{\frac{\pi}{2}}^0 \\ &= \left[0 - \frac{\sin(0)}{4} \right] - \left[\frac{\pi}{2} - \frac{\sin(\pi)}{4} \right] \\ &= \boxed{-\frac{\pi}{2}} \end{aligned}$$

$$\begin{aligned} \text{j) } & \int_{\frac{1}{2}}^2 \left(1 - \frac{1}{x^2} \right) dx \\ &= x + \frac{1}{x} \Big|_{\frac{1}{2}}^2 \\ &= \left[2 + \frac{1}{2} \right] - \left[\frac{1}{2} + 2 \right] \\ &= \boxed{0} \end{aligned}$$

$$\begin{aligned} \text{k) } & \int_1^2 \left(\frac{2}{s^2} - \frac{4}{s^3} \right) ds \\ &= -\frac{2}{s} + \frac{2}{s^2} \Big|_1^2 \\ &= \left[-\frac{2}{2} + \frac{2}{2^2} \right] - \left[-\frac{2}{1} + \frac{2}{1^2} \right] \\ &= \boxed{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \text{l) } & \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2} dx \\ &= 8 \tan^{-1}(x) \Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= \left[8 \tan^{-1}(\sqrt{3}) \right] - \left[8 \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \right] \\ &= 8(\pi/3) - 8(\pi/6) \\ &= \boxed{\frac{4\pi}{3}} \end{aligned}$$

$$m) \int_0^5 (x^2 - 9) dx$$

$$= \frac{x^3}{3} - 9x \Big|_0^5$$

$$= \left[\frac{5^3}{3} - 9(5) \right] - [0]$$

$$= \frac{125}{3} - 45$$

$$= \boxed{-\frac{10}{3}}$$

$$o) \int_{-1}^2 x^3 dx$$

$$= \frac{x^4}{4} \Big|_{-1}^2$$

$$= \left[\frac{2^4}{4} \right] - \left[\frac{(-1)^4}{4} \right]$$

$$= \boxed{\frac{15}{4}}$$

$$n) \int_{1/2}^2 \left(1 - \frac{1}{x^2} \right) dx$$

$$= x + \frac{1}{x} \Big|_{1/2}^2$$

$$= \left[2 + \frac{1}{2} \right] - \left[\frac{1}{2} + 2 \right]$$

$$= \boxed{0}$$

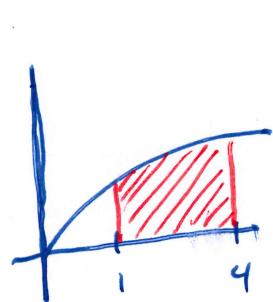
$$p) \int_{\pi/6}^{2\pi} \cos(x) dx$$

$$= \sin(x) \Big|_{\pi/6}^{2\pi}$$

$$= \sin(2\pi) - \sin(\pi/6)$$

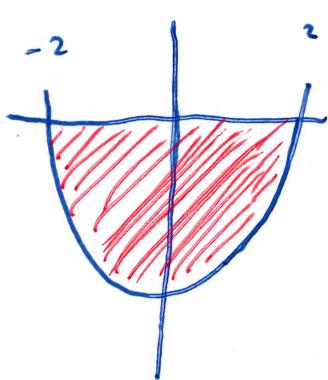
$$= \boxed{-\frac{1}{2}}$$

Example. Find the area of the region bounded by $y = \sqrt{x}$ between $x = 1$ and $x = 4$.



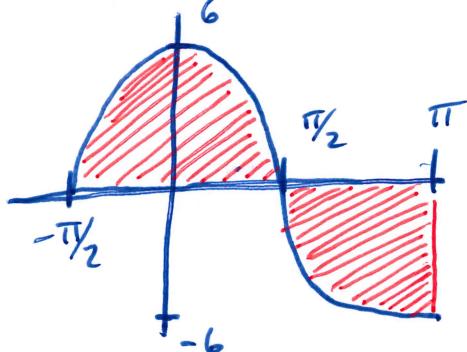
$$\begin{aligned} \int_1^4 \sqrt{x} \, dx &= \frac{2}{3} x^{3/2} \Big|_1^4 \\ &= \left[\frac{2}{3} (4)^{3/2} \right] - \left[\frac{2}{3} (1)^{3/2} \right] \end{aligned}$$

Example. Find the area of the region below the x -axis bounded by $y = x^4 - 16$.



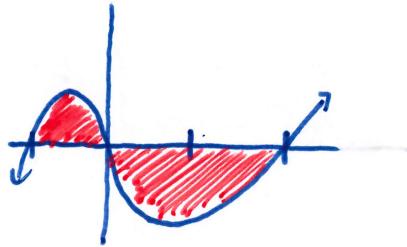
$$\begin{aligned} \int_{-2}^2 x^4 - 16 \, dx &= \frac{x^5}{5} - 16x \Big|_{-2}^2 \\ &= \left[\frac{2^5}{5} - 16(2) \right] - \left[\frac{(-2)^5}{5} - 16(-2) \right] \\ &= \boxed{-\frac{256}{5}} \end{aligned}$$

Example. Find the area of the region bounded by $y = 6 \cos(x)$ between $x = -\pi/2$ and $x = \pi$.



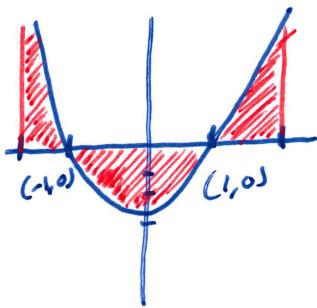
$$\begin{aligned} \int_{-\pi/2}^{\pi} 6 \cos(x) \, dx &= 6 \sin(x) \Big|_{-\pi/2}^{\pi} \\ &= [6 \sin(\pi)] - [6 \sin(-\pi/2)] \\ &= 0 - (-6) \\ &= \boxed{6} \end{aligned}$$

Example. Find the area of the region bounded by $f(x) = x(x+1)(x-2)$ and the x -axis on the interval $[-1, 2]$.



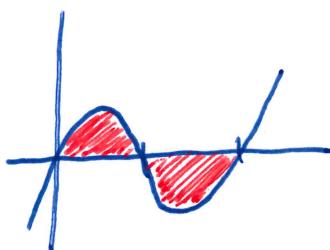
$$\begin{aligned} \int_{-1}^2 x(x+1)(x-2) dx &= \int_{-1}^2 x^3 - x^2 - 2x dx \\ &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^2 \\ &= \left[\frac{2^4}{4} - \frac{2^3}{3} - 2^2 \right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2 \right] = \boxed{-\frac{9}{4}} \end{aligned}$$

Example. Find the total area between $y = 3x^2 - 3$ and the x -axis on $-2 \leq x \leq 2$.



$$\begin{aligned} \int_{-2}^2 |3x^2 - 3| dx &= \left| \int_{-2}^{-1} 3x^2 - 3 dx \right| + \left| \int_{-1}^1 3x^2 - 3 dx \right| + \left| \int_1^2 3x^2 - 3 dx \right| \\ &= \left| (x^3 - 3x) \Big|_{-2}^{-1} \right| + \left| (x^3 - 3x) \Big|_{-1}^1 + \left| (x^3 - 3x) \Big|_1^2 \right| \\ &= |4| + |-4| + |4| = \boxed{12} \end{aligned}$$

Example. Find the total area between $y = x^3 - 3x^2 + 2x$ and the x axis on the interval $0 \leq x \leq 2$.



$$\begin{aligned} \int_0^2 |x^3 - 3x^2 + 2x| dx &= \left| \left(\frac{x^4}{4} - x^3 + x^2 \right) \Big|_0^1 \right| + \left| \left(\frac{x^4}{4} - x^3 + x^2 \right) \Big|_1^2 \right| \\ &= \left| \frac{1}{4} \right| + \left| -\frac{1}{4} \right| \\ &= \boxed{\frac{1}{2}} \end{aligned}$$