4.4: Graphing Functions

Graphing Guidelines for $y = f(x)$

1. Identify the domain or interval of interest.
2. Determine if the function is symmetric.
3. Use $f'(x)$ and $f''(x)$ to determine
   - intervals of increasing/decreasing ..... $f'(x) > 0$ or $f'(x) < 0$
   - relative max/mins .... 1st or 2nd derivative test
   - concave up/concave down .... $f''(x) > 0$ or $f''(x) < 0$
   - inflection points .... $f''(x)$ changes signs
4. Find asymptotes
   - vertical .... $\lim_{x\to a} f(x) = \pm\infty$
   - horizontal .... $\lim_{x\to\pm\infty} f(x) = c$
   - slant .... $y = ax + b$
5. Intercepts
   - y-intercept .... evaluate $f(0)$
   - x-intercept .... solve $f(x) = 0$
6. Sketch using the characteristics found above

Example. Sketch the following functions using derivatives:

1. $f(x) = x^4 - 6x^2$

① Domain $(-\infty, \infty)$

② $f(-x) = (-x)^4 - 6(-x)^2 = x^4 - 6x^2 = f(x)$ Symm

③ $f'(x) = 4x^3 - 12x \overset{\text{set}}{=} 0$

$4x(x^2-3) = 0$

$x = 0,\ x = \pm\sqrt{3}$

	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2

$4x$	$-$	$-$	$+$	$+$	
x^2-3	$+$	$-$	$-$	$+$	
$f'(x)$	$-$	$+$	$-$	$+$	
	↘	↗	↘	↗	

$$\boxed{\begin{array}{l}\text{Dec: } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \\ \text{Inc: } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)\end{array}}$$

$$\boxed{\begin{array}{l}\text{local min: } (-\sqrt{3}, -9) \\ \quad (\sqrt{3}, -9) \\ \text{local max } (0,0)\end{array}}$$

$f''(x) = 12x^2 - 12 \overset{\text{set}}{=} 0$

$x = \pm 1$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

$f''(x)$	$+$	$-$	$+$
	$\cup$	$\cap$	$\cup$

$$\boxed{\begin{array}{l}\text{Concave up: } (-\infty, -1) \cup (1, \infty) \\ \text{Concave down: } (-1, 1)\end{array}}$$

④ No asymptotes

⑤ y-int $(x=0)$

$f(0) = 0$

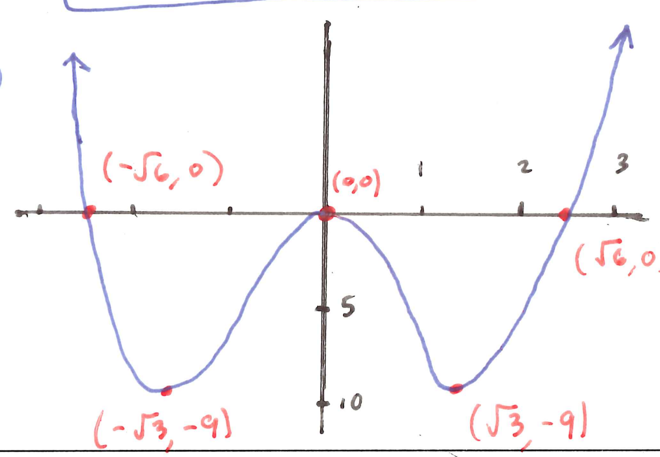
x-int $(f(x)=0)$

$x^4 - 6x^2$

$x^2(x^2-6) = 0$

$x = 0,\ x = \pm\sqrt{6}$

⑥



2. $g(x) = 200 + 8x^3 + x^4$

① Domain $(-\infty, \infty)$

② $g(-x) = 200 + 8(-x)^3 + (-x)^4$

$= 200 - 8x^3 + x^4$

$\neq g(x)$ Not symmetric

③ $g'(x) = 24x^2 + 4x^3 \overset{\text{set}}{=} 0$

$4x^2(6+x) = 0 \Rightarrow \boxed{x=0,\ x=-6}$

	-10	-6	-1	0	1
$4x^2$	+		+		+
$6+x$	-		+		+
$f'(x)$	-		+		+
	↘		↗		↗

dec: $(-\infty, -6)$

Inc $(-6,0)\cup(0,\infty)$

local min $(-6, -232)$

No local max

$g''(x) = 48x + 12x^2 \overset{\text{set}}{=} 0$

$12x(4+x) = 0 \Rightarrow \boxed{x=0,\ x=-4}$

	-6	-4	-2	0	2
$12x$	-		-		+
$4+x$	-		+		+
$f''(x)$	+		-		+
	∪		∩		∪

Concave down $(-4, 0)$

Concave up $(-\infty, -4)\cup(0,\infty)$

Inflection pt. $(-4, -56), (0, 200)$

④ No asymptotes

⑤ y-int $(x=0)$

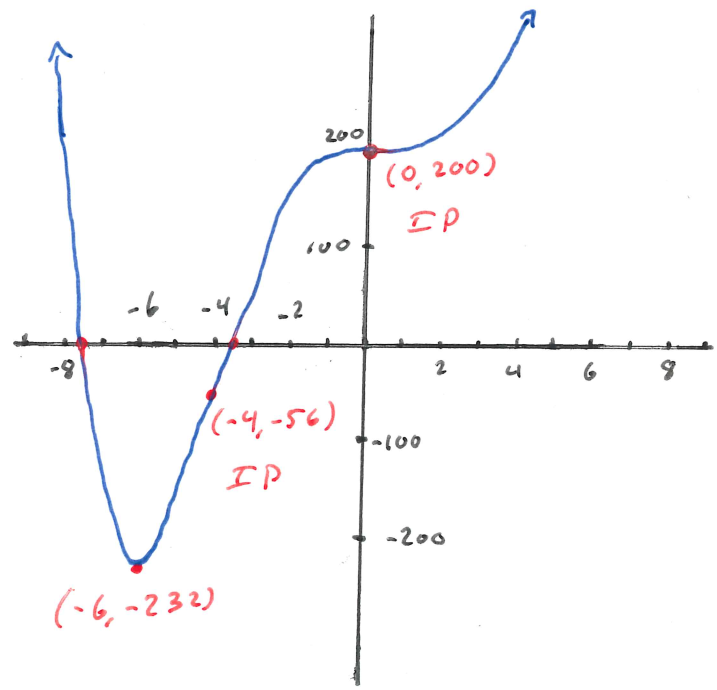
$\boxed{g(0) = 200}$

x-int $(g(x)=0)$

$200 + 8x^3 + x^4 = 0$

$\Rightarrow \boxed{\begin{array}{l} x \approx -7.532 \\ x \approx -3.557 \end{array}}$

Note: No closed form solution to find roots



3. $y = \dfrac{x}{(x-1)^2}$

① Domain: $x \neq 1$ $\quad (-\infty, 1) \cup (1, \infty)$

② $y(-x) = \dfrac{-x}{(-x-1)^2} = \dfrac{-x}{(x+1)^2} \neq y(x)$

Not symmetric

③ $y' = \dfrac{(x-1)^2 - 2x(x-1)}{(x-1)^4} = -\dfrac{x+1}{(x-1)^3}$

Crit pts

$\boxed{x = -1 \quad x \neq 1}$

	-2	-1	0	1	2
$-(x+1)$	+		−		−
$(x-1)^3$	−		−		+
y'	−		+		−
	↘		↗		↘

Dec: $(-\infty, -1) \cup (1, \infty)$

Inc: $(-1, 1)$

Local min $(-1, -1/4)$

No local max $\quad y(1)$ DNE

$y'' = \dfrac{2(x+2)}{(x-1)^4}$ $\quad$ Crit pts $\quad x = -2$, $\quad x \neq 1$

	-3	-2	0	1	2
$2(x+2)$	−		+		+
$(x-1)^4$	+		+		+
y''	−		+		+
	∩		∪		∪

c. down $(-\infty, -2)$

c. up $(0, 2) \cup (2, \infty)$

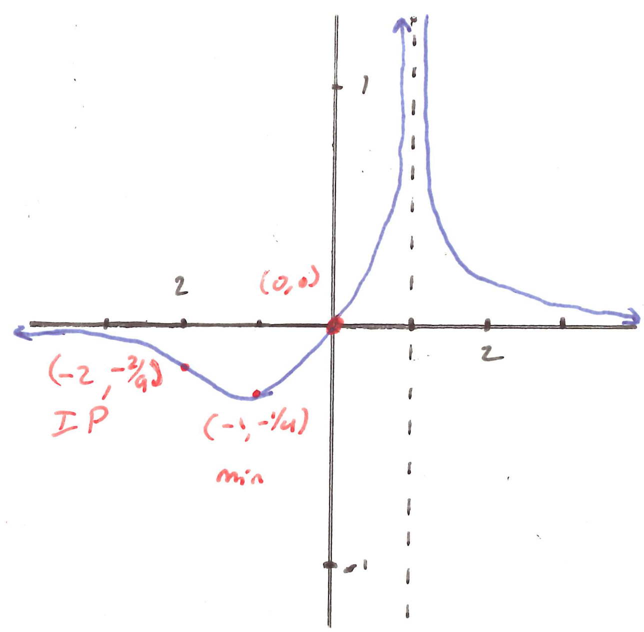
I.P. $(-2, -2/9)$

④ V.A. $\displaystyle\lim_{x \to 1} \frac{x}{(x-1)^2} = \infty$ $\quad \boxed{x \neq 1}$

H.A. $\displaystyle\lim_{x \to \pm\infty} \frac{x}{(x-1)^2} = 0$ $\quad \boxed{y = 0}$

⑤ y-int $(x=0)$ $\quad y = \dfrac{0}{(0-1)^2} = 0$ $\quad \boxed{(0,0)}$

x-int $(y=0)$ $\quad \dfrac{x}{(x-1)^2} = 0 \Rightarrow x = 0$



4. $y = \dfrac{x^2-4}{x^2-2x} = \dfrac{x+2}{x}, \ x \neq 2$

① Domain: $\boxed{x \neq 0, \ x \neq 2}$

② $y(-x) = \dfrac{(-x)^2-4}{(-x^2)-2(-x)} = \dfrac{x^2-4}{x^2+2x} \neq y(x)$ Not symm.

③ $y' = -\dfrac{2}{x^2} \overset{\text{set}}{=} 0$, $\boxed{x \neq 0}$

	-1	0	1
-2	$-$		$-$
x^2	$+$		$+$
y'	$-$		$-$
	↘		↘

$\boxed{\text{Dec } (-\infty, 0) \cup (0, \infty)}$

$y'' = \dfrac{4}{x^3} \overset{\text{set}}{=} 0$

	-1	0	1
4	$+$		$+$
x^3	$-$		$+$
y''	$-$		$+$
	∩		∪

Concave down $(-\infty, 0)$

Concave up $(0, \infty)$

No inflection pt

④ V.A.

$$\lim_{x\to 0^-} \frac{x^2-4}{x^2-2x} = \lim_{x\to 0^-} \frac{x+2}{x} = \boxed{-\infty}$$

$$\lim_{x\to 0^+} \frac{x^2-4}{x^2-2x} = \lim_{x\to 0^+} \frac{x+2}{x} = \boxed{\infty}$$

$$\lim_{x\to 2} \frac{x^2-4}{x^2-2x} = \frac{x+2}{x} = \boxed{2}$$

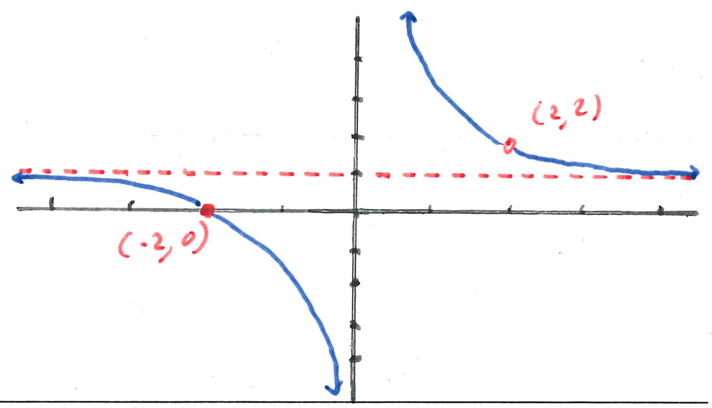
H.A.

$$\lim_{x\to \infty} \frac{x^2-4}{x^2-2x} = \boxed{1}$$

⑤ y-int $(x=0)$ DNE

x-int $(y=0)$

$\dfrac{x^2-4}{x^2-2x} = 0 \Rightarrow \boxed{x=-2}, \ x=2$ Not in domain



5. $f(x) = \dfrac{(x+1)^3}{(x-1)^2}$

① Domain: $\boxed{x \neq 1}$

② $f(-x) = \dfrac{(-x+1)^3}{(-x-1)^2} = \dfrac{-(x-1)^3}{(x+1)^2} \neq f(x)$

Not symmetric

③ $f'(x) = \dfrac{(x+1)^2(x-5)}{(x-1)^3} \overset{\text{set}}{=} 0$ $\boxed{\begin{matrix} x=-1, x=5 \\ x \neq 1 \end{matrix}}$

	-2	-1	0	1	2	5	6
$(x+1)^2$	+		+		+		+
$(x-5)$	−		−		−		+
$(x-1)^3$	−		−		+		+
$f'(x)$	+		+		−		+
	↗		↗		↘		↗

Dec: $(1, 5)$

Inc: $(-\infty, -1) \cup (-1, 1) \cup (5, \infty)$

Local min $(5, {}^{27}\!/\!_2)$

No local max

$f''(x) = \dfrac{24(x+1)}{(x-1)^4} \overset{\text{set}}{=} 0$ $\boxed{\begin{matrix} x=-1 \\ x \neq 1 \end{matrix}}$

	-2	-1	0	1	2
24	+		+		+
$(x+1)$	−		+		+
$(x-1)^4$	+		+		+
$f''(x)$	−		+		+
	∩		∪		∪

Concave down: $(-\infty, -2)$

Concave up: $(-1, 1), (1, \infty)$

Inflection pt. $(-1, 0)$

④ VA

$\lim_{x \to 1} f(x) = \infty$

Slant

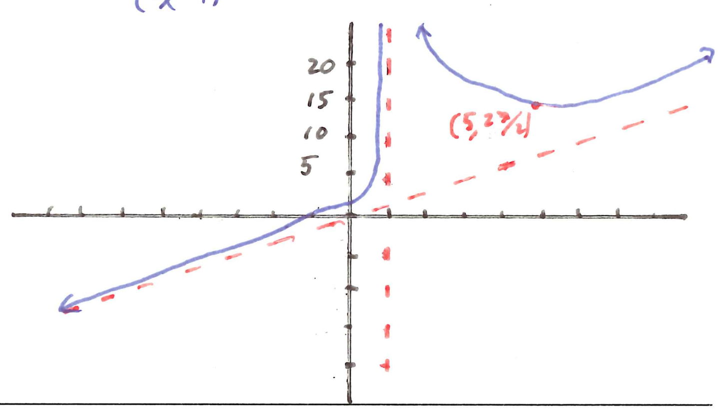
$$\begin{array}{r} x + 5 \\ x^2 - 2x + 1 \overline{\smash{)}\, x^3 + 3x^2 + 3x + 1} \\ -(x^3 - 2x^2 + x) \\ \hline 5x^2 + 2x + 1 \\ -(5x^2 - 10x + 5) \\ \hline 12x - 4 \end{array}$$

$\boxed{y = x + 5}$

⑤ y-int $(x=0)$ $\boxed{(0, 1)}$

x-int $(f(x)=0)$

$\dfrac{(x+1)^3}{(x-1)^2} = 0 \Rightarrow \boxed{(-1, 0)}$



6. $y = \dfrac{x^2}{x^2+3}$

① Domain : $(-\infty, \infty)$

② $y(-x) = \dfrac{(-x)^2}{(-x)^2+3} = \dfrac{x^2}{x^2+3} = y(x)$

symmetric

③ $y' = \dfrac{6x}{(x^2+3)^2} \overset{\text{set}}{=} 0 \Rightarrow \boxed{x=0}$

	-1	0	1
$6x$	$-$		$+$
$(x^2+3)^2$	$+$		$+$
y'	$-$		$+$
	↘		↗

Dec: $(-\infty, 0)$
Inc: $(0, \infty)$
L. min $(0,0)$
No L. max

$y'' = \dfrac{-18(x^2-1)}{(x^2+3)^3} \overset{\text{set}}{=} 0 \Rightarrow \boxed{x = \pm 1}$

	-2	-1	0	1	2
-18	$-$		$-$		$-$
x^2-1	$+$		$-$		$+$
$(x^2+3)^3$	$+$		$+$		$+$
y''	$-$		$+$		$-$
	$\cap$		$\cup$		$\cap$

C. down
$(-\infty,-1) \cup (1,\infty)$
C. up
$(-1,1)$
I. P.
$(-1, 1/4)$
$(1, 1/4)$

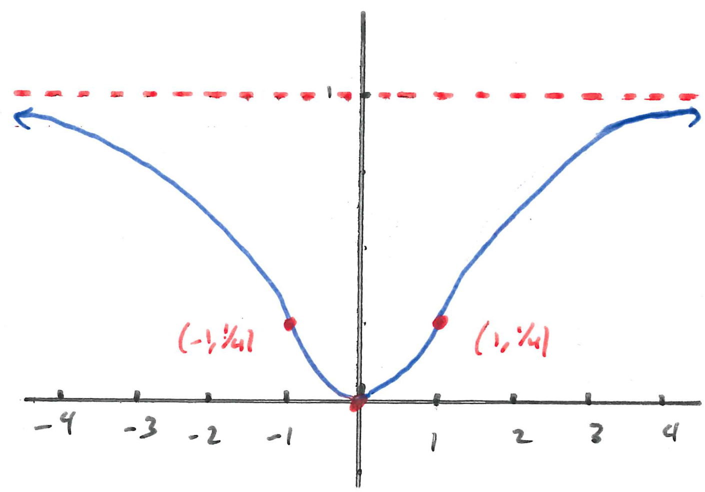
④ V.A. N/A

H.A.

$\displaystyle\lim_{x\to\infty} \frac{x^2}{x^2+3} = 1$

⑤ y-int $(x=0)$ $(0,0)$

x-int $\dfrac{x^2}{x^2+3} = 0 \Rightarrow (0,0)$



7. $f(x) = \dfrac{x^3}{(x+1)^2}$

① Domain $x \neq -1$

② $f(-x) = \dfrac{(-x)^3}{(-x+1)^2} = \dfrac{-x^3}{(x-1)^2} \neq f(x)$

not symm

③ $f'(x) = \dfrac{x^2(x+3)}{(x+1)^3} \overset{\text{set}}{=} 0$ $\boxed{\begin{array}{l} x=0 \\ x=-3 \\ x\neq -1 \end{array}}$

	-4	-3	-2	-1	-½	0	1
x^2	+		+		+		+
$x+3$	-		+		+		+
$(x+1)^3$	-		-		+		+
$f'(x)$	+		-		+		+
	↗		↘		↗		↗

Dec $(-3,-1)$

Inc $(-\infty,-3) \cup (-1,0) \cup (0,\infty)$

L. Min DNE

L. max $(-3, -27/4)$

$f''(x) = \dfrac{6x}{(x+1)^4} \overset{\text{set}}{=} 0$

$x=0,\ x \neq -1$

	-2	-1	-½	0	1
$6x$	-		-		+
$(x+1)^4$	+		+		+
$f''(x)$	-		-		+
	∩		∩		∪

Concave down $(-\infty,-1) \cup (-1,0)$

Concave up $(0,\infty)$

Inflection Pt. $(0,0)$

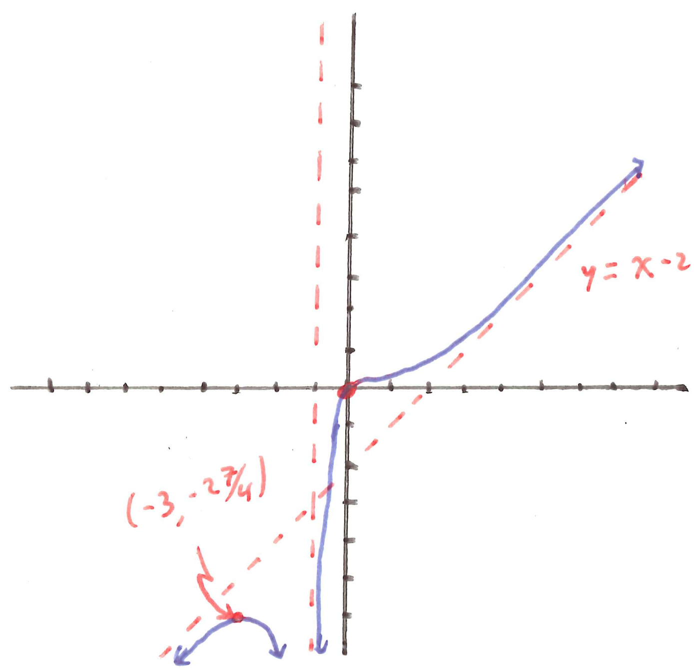
④ VA

$\lim_{x\to -1} \dfrac{x^3}{(x+1)^2} = -\infty$

Slant

$$\begin{array}{r} x - 2 \\ x^2+2x+1 \,\overline{\big)\, x^3 + 0x^2 + 0x + 0} \\ -(x^3 + 2x^2 + x) \\ \hline -2x^2 - x + 0 \\ -(-2x^2 - 4x - 2) \\ \hline 3x - 2 \end{array}$$

$\boxed{y = x - 2}$



8. $y = x^{1/5}$

① Domain: $(-\infty, \infty)$

② $y(-x) = (-x)^{1/5} = -x^{1/5} = -y(x)$

symm to origin

③ $y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}} \overset{set}{=} 0$

No x exists

$\boxed{x \neq 0}$

	-1	0	1
y'	+		+
	↗		↗

Dec: N/A

Inc: $(-\infty, 0) \cup (0, \infty)$

No min

No max

$y'' = -\frac{4}{25}x^{-9/5} = \frac{-4}{25x^{9/5}} \overset{set}{=} 0$

No x exists

$\boxed{x \neq 0}$

	-1	0	1
$-4/25$	−		−
$x^{-9/5}$	−		+
y''	+		−
	∪		∩

Concave down $(0, \infty)$

Concave up $(-\infty, 0)$

Inflection pt. $(0, 0)$

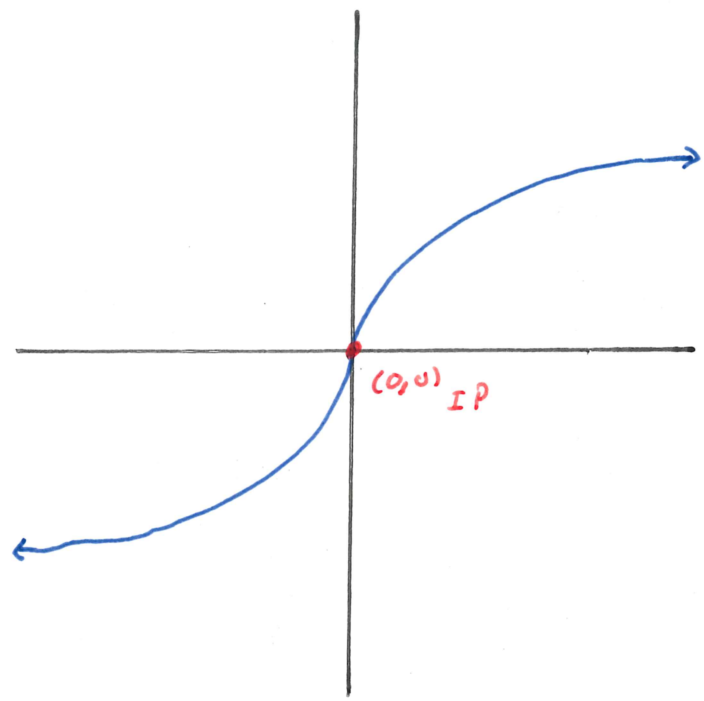
④ No V.A.

No H.A.

No slant

⑤ y-int $(0, 0)$

x-int $0, 0)$



9. $y = 2\sqrt{x} - x$

① Domain: $[0, \infty)$

② $y(-x) = 2\sqrt{-x} - (-x) \neq y(x)$

No symm

③ $y' = \frac{1}{\sqrt{x}} - 1 \overset{\text{set}}{=} 0 \Rightarrow x = 1$

$x \neq 0$

0 1/4 1 4

y' + −

$\boxed{\begin{array}{l} \text{Dec: } (1, \infty) \\ \text{Inc: } (0, 1) \\ \text{Local max } (1, 1) \end{array}}$

$y'' = \frac{-1}{2x^{3/2}} \overset{\text{set}}{=} 0$ No x exists

$x \neq 0$

0

$-1/2$ −

$x^{-3/2}$ +

y'' −

$\boxed{\begin{array}{l} \text{Concave down} \\ (0, \infty) \\ \text{No inflection pt} \end{array}}$

④ No VA

No HA/slant

⑤ $\underline{y\text{-int}}$ $(x = 0)$ $(0, 0)$

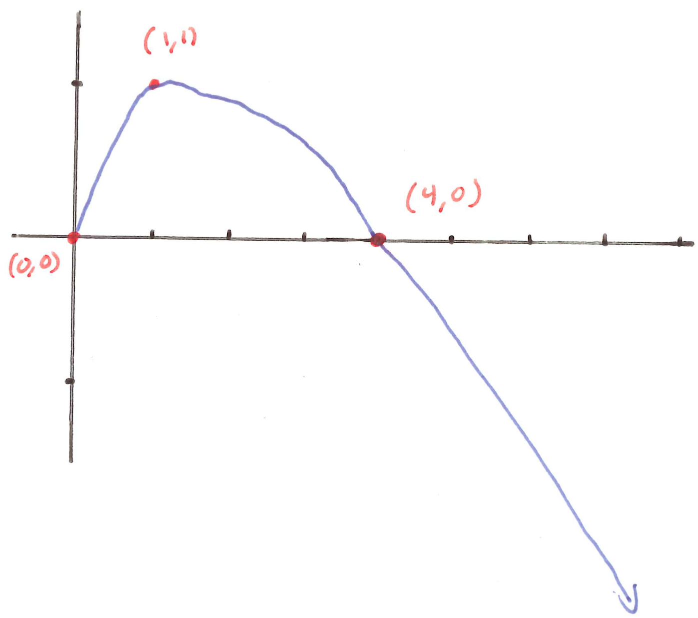
$\underline{x\text{-int}}$ $2\sqrt{x} - x = 0$

$2\sqrt{x} = x$

$4x = x^2$

$0 = x(4 - x)$

$\boxed{x = 0, \ x = 4}$



10. $y = \dfrac{1}{1+e^{-x}} = (1+e^{-x})^{-1}$

① Domain:

$\left.\begin{array}{l} 1+e^{-x} \neq 0 \\ e^{-x} \neq 1 \end{array}\right\} \Rightarrow (-\infty, \infty)$

② $y(-x) = \dfrac{1}{1+e^{-(-x)}} = \dfrac{1}{1+e^{x}} \neq y(x)$

No symm

③ $y' = -(1+e^{-x})^{-2}(-e^{-x})$

$= \dfrac{e^{-x}}{(1+e^{-x})^2} = 0$ No x exists

	$-\infty$	0	∞
e^{-x}		+	
$(1+e^{-x})^2$		+	
y'		+ ↗	

$\boxed{\begin{array}{l} \text{Inc} \\ (-\infty, \infty) \\ \text{No local max/min} \end{array}}$

$y'' = \dfrac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} \overset{set}{=} 0$

$e^{-x}(e^{-x}-1) = 0$

$e^{-x} = 1$

$x = 0$

	-10	0	10
e^{-x}	+		+
$e^{-x}-1$	+		−
$(1+e^{-x})^3$	+		+
y''	+		−
	∪		∩

$\boxed{\begin{array}{ll} \text{Concave down} & (0, \infty) \\ \text{Concave up} & (-\infty, 0) \\ \text{Inflection pt} & (0, 1/2) \end{array}}$

④ No VA

H.A.

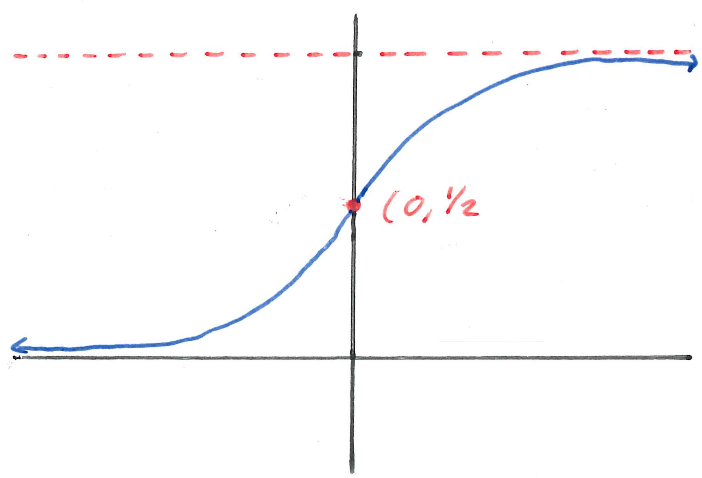
$\lim_{x \to -\infty} \dfrac{1}{\underbrace{1+e^{-x}}_{\text{large denom}}} = 0$

$\lim_{x \to \infty} \dfrac{1}{1+e^{-x}} = \dfrac{1}{1+0} = 1$

⑤ y-int $(x=0)$ $(0, 1/2)$

x-int

solve $\dfrac{1}{1+e^{-x}} = 0$ No such x exists



11. $f(x) = \dfrac{x^2}{x^2-4}$

① Domain: $x^2-4 \neq 0$

$x \neq \pm 2$

② $f(-x) = \dfrac{(-x)^2}{(-x)^2-4} = \dfrac{x^2}{x^2-4} = f(x)$ symm

③ $f'(x) = -\dfrac{8x}{(x^2-4)^2} \overset{\text{set}}{=} 0 \Rightarrow x = 0$

$x \neq \pm 2$

	-3	-1	1	3
$-8x$	+	+	−	−
$(x^2-4)^2$	+	+	+	+
$f'(x)$	+	+	−	−
	↗	↗	↘	↘

Dec $(0,2) \cup (2,\infty)$

Inc $(-\infty,-2) \cup (-2,0)$

Local max $(0,0)$

$f''(x) = \dfrac{8(3x^2+4)}{(x^2-4)^3} \overset{\text{set}}{=} 0$ No such x exists

$x \neq \pm 2$

	-3	0	3
8	+	+	+
$3x^2+4$	+	+	+
$(x^2-4)^3$	+	−	+
$f''(x)$	+	−	+
	∪	∩	∪

Concave down $(-2,2)$

concave up $(-\infty,-2) \cup (2,\infty)$

No inflection pts

④ V.A.

$\lim_{x \to -2^-} \dfrac{x^2}{x^2-4} = \infty$

$\lim_{x \to -2^+} \dfrac{x^2}{x^2-4} = -\infty$

$\lim_{x \to 2^-} \dfrac{x^2}{x^2-4} = -\infty$

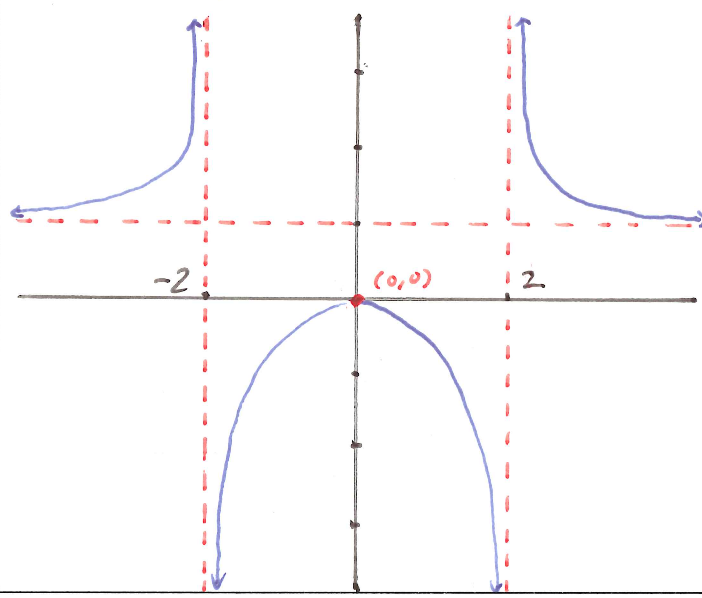
$\lim_{x \to 2^+} \dfrac{x^2}{x^2-4} = \infty$

H.A.

$\lim_{x \to -\infty} \dfrac{x^2}{x^2-4} = 1$ $\quad \lim_{x \to \infty} \dfrac{x^2}{x^2-4} = 1$

⑤ y-int $(x=0)$ $(0,0)$

x-int $\dfrac{x^2}{x^2-4} = 0 \Rightarrow (0,0)$



Example. Using the following derivatives, sketch a possible graph of the original function.

1. $f'(x) = \frac{1}{6}(x+1)(x-2)^2(x-3)$

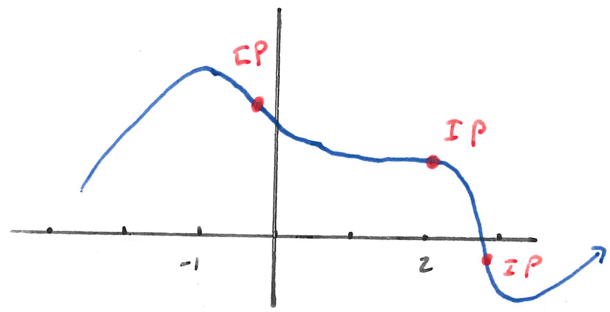
	-2	0	$5/2$	4
$1/6$	+	+	+	+
$x+1$	−	+	+	+
$(x-2)^2$	+	+	+	+
$x-3$	−	−	−	+
$f'(x)$	+	−	−	+
	↗	↘	↘	↗

Critical points: -1, 2, 3 — max at -1, min at 3.

$f''(x) = \frac{1}{3}(2x^3 - 9x^2 + 9x + 2)$

	-1	0	2.5	3
$f''(x)$	−	+	−	+
	∩	∪	∩	∪

Critical points: -0.186, 2, 2.686



2. $g'(x) = x^2(x+2)(x-1)$

	-3	-1	$1/2$	2
x^2	+	+	+	+
$x+2$	−	+	+	+
$x-1$	−	−	−	+
$g'(x)$	+	−	−	+
	↗	↘	↘	↗

Critical points: -2, 0, 1

$g''(x) = x(4x^2 + 3x - 4)$

	-2	-1	0.5	1
$g''(x)$	−	+	−	+
	∩	∪	∩	∪

Critical points: -1.443, 0, 0.693