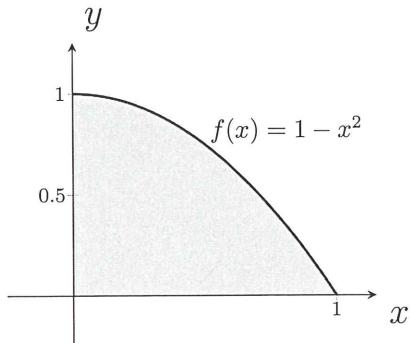
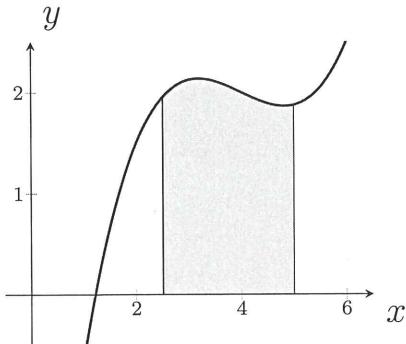
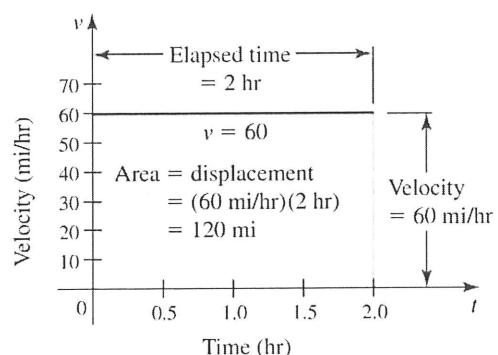
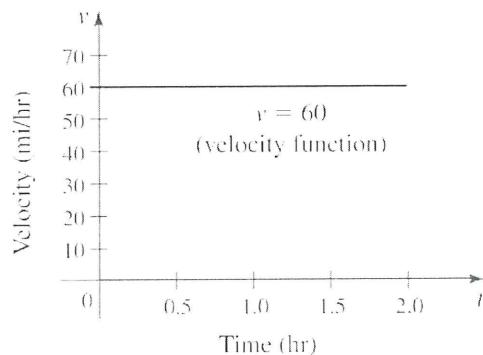
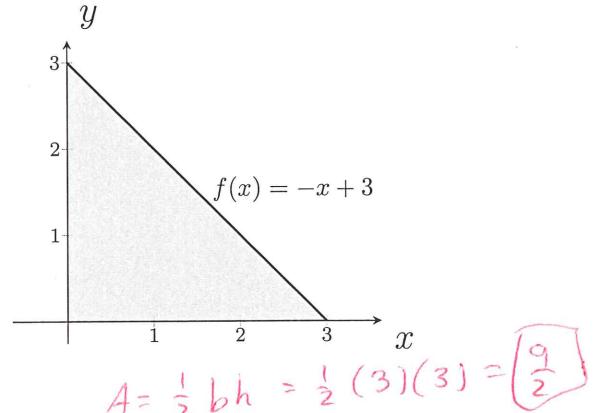
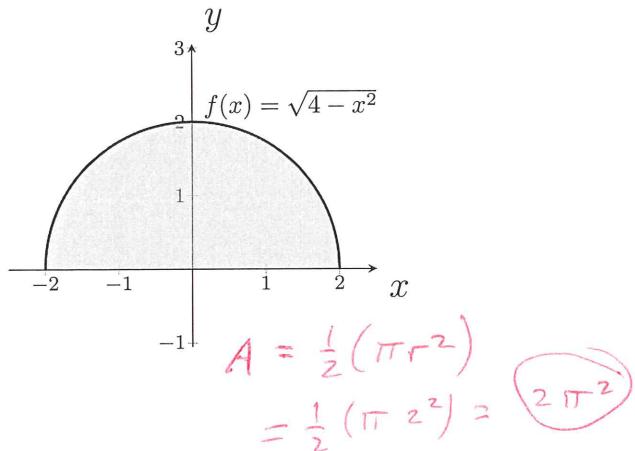


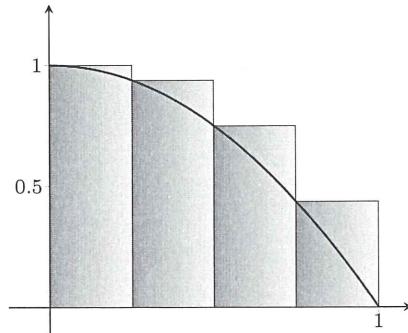
## 5.1: Approximating Areas under Curves



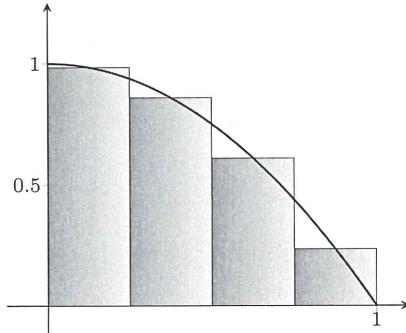
Finding the area under the curve is simple in some cases:



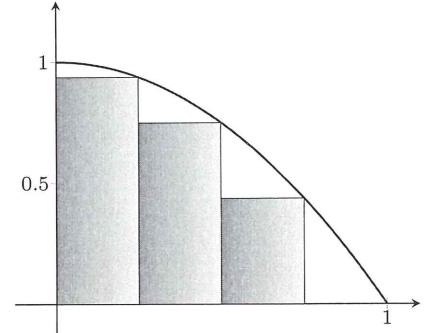
For functions whose curves are irregular shapes, we can approximate the area using rectangles:



Left rectangles

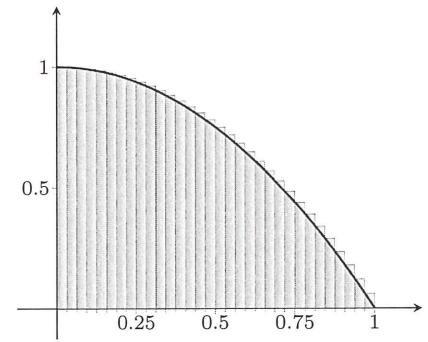
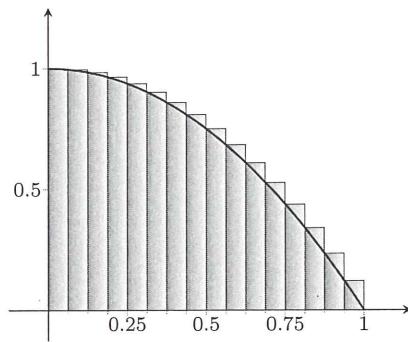
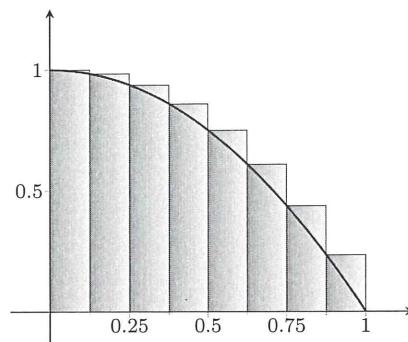


Midpoint rectangles



Right rectangles

These approximations are much more accurate when more rectangles are used:



### Definition. (Riemann Sum)

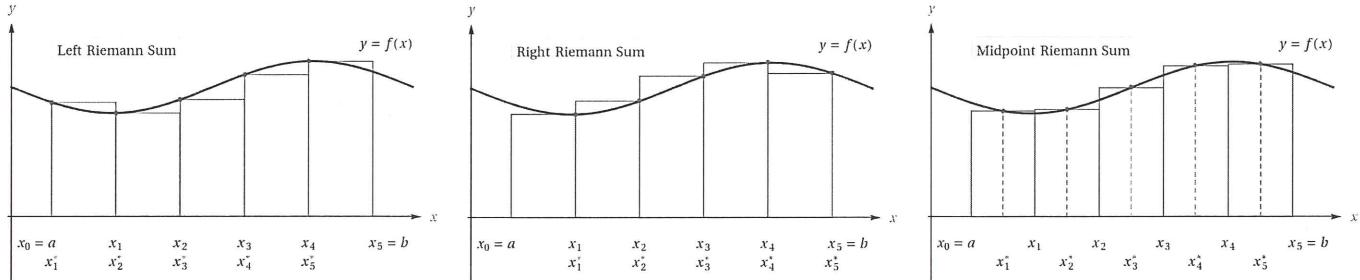
Suppose  $f$  is defined on a closed interval  $[a, b]$ , which is divided into  $n$  subintervals of equal length  $\Delta x$ . If  $x_k^*$  is any point in the  $k$ -th subinterval  $[x_{k-1}, x_k]$ , for  $k = 1, 2, \dots, n$ , then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for  $f$  on  $[a, b]$ . This sum is called

- a **left Riemann sum** if  $x_k^*$  is the left endpoint of  $[x_{k-1}, x_k]$ ,
- a **right Riemann sum** if  $x_k^*$  is the right endpoint of  $[x_{k-1}, x_k]$ , and
- a **midpoint Riemann sum** if  $x_k^*$  is the midpoint of  $[x_{k-1}, x_k]$ , for  $k = 1, 2, \dots, n$ .

In general, midpoint rectangles give better approximations than left or right rectangles.



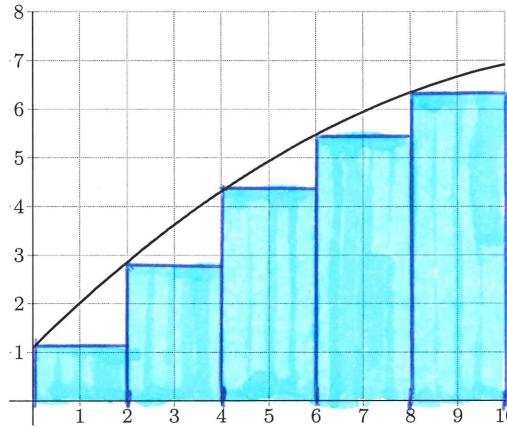
When estimating the area under the graph on the interval  $[a, b]$ , we define

$$\Delta x = \frac{b - a}{n}$$

to be the width of the rectangles. The height of the rectangles is given by  $f(x_i)$ , where the  $x_i$ 's are  $\Delta x$  apart.

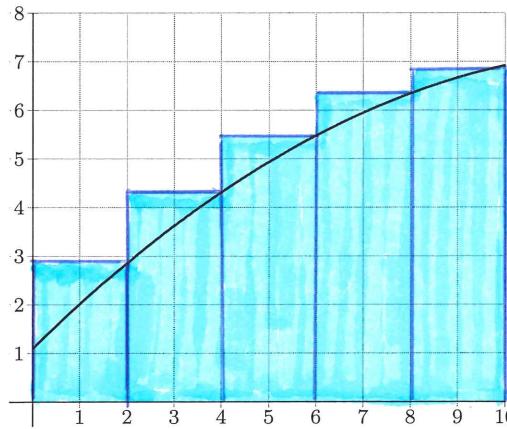
**Example.** Use the plots below to estimate the area under the given graph of  $f(x)$ :

Sketch five rectangles and use them to find a lower estimate for the area under the given graph of  $f(x)$  from  $x = 0$  to  $x = 10$ .



$$\begin{aligned}A &\approx 2(1.1) + 2(2.9) + 2(4.3) + 2(5.5) + 2(6.2) \\&= 2(20) = \underline{\underline{40}}\end{aligned}$$

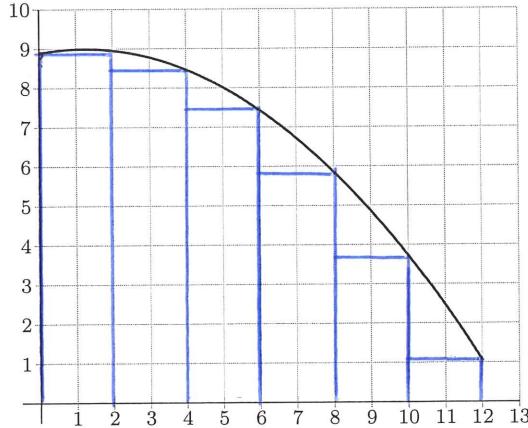
Sketch five rectangles and use them to find a upper estimate for the area under the given graph of  $f(x)$  from  $x = 0$  to  $x = 10$ .



$$\begin{aligned}A &\approx 2(3 + 4.3 + 5.5 + 6.2 + 6.8) \\&= 2(25.8) = \underline{\underline{51.6}}\end{aligned}$$

Use six right rectangles to estimate the area under the given graph of  $f(x)$  from  $x = 0$  to  $x = 12$ . Is the estimate an under-approximation or an over-approximation? Why?

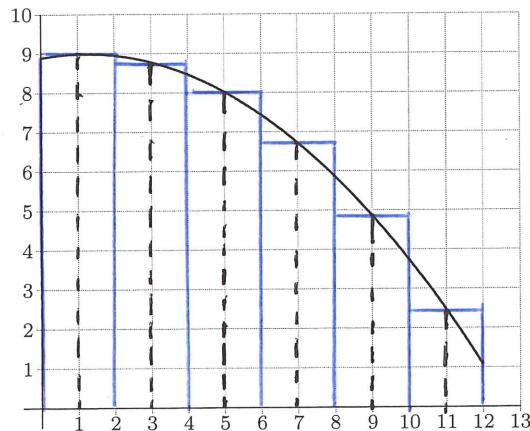
$$\Delta x = \frac{12-0}{6} = 2$$



$$A \approx 2(9) + 2(8.5) + 2(7.5) + 2(5.9) + 2(3.7) + 2(1.1) \\ = 2(35.7) = \boxed{71.4}$$

Use six midpoint rectangles to estimate the area under the given graph of  $f(x)$  from  $x = 0$  to  $x = 12$ . Talk about the quality of this estimate.

$$\Delta x = \frac{12-0}{6} = 2$$



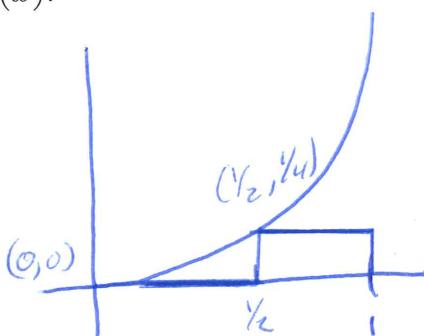
$$A \approx 2(9) + 2(8.7) + 2(8) + 2(7.7) + 2(6.7) + 2(5) \\ = 2(40.8) = \boxed{81.6}$$

The midpoint Riemann sum is a good approximation. It follows  $f(x)$  closely and some of the over/under estimates cancel.

Actual area is  $\frac{1}{3} \approx 0.3$

**Example.** Consider  $f(x) = x^2$  on the interval  $[0, 1]$ .

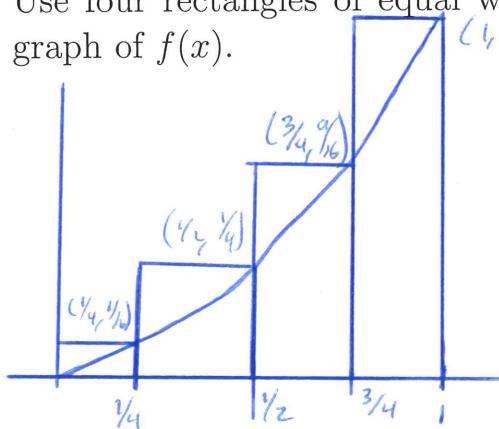
- a) Use two rectangles of equal width to find a lower sum for the area under the graph of  $f(x)$ .



$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

$$A \approx \frac{1}{2}(0) + \frac{1}{2}\left(\frac{1}{4}\right) = \boxed{\frac{1}{8}} = 0.125$$

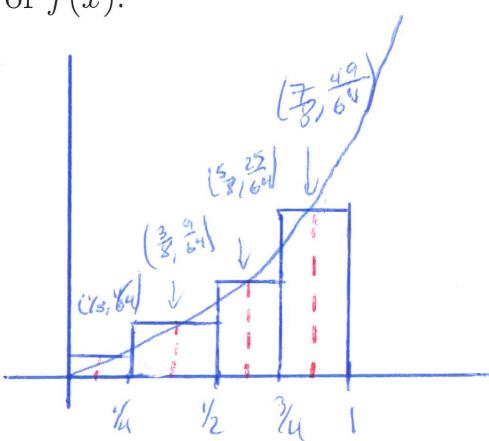
- b) Use four rectangles of equal width to find an upper sum for the area under the graph of  $f(x)$ .



$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$A \approx \frac{1}{4}(y_1) + \frac{1}{4}(y_2) + \frac{1}{4}(y_3) + \frac{1}{4}(y_4) \\ = \frac{1}{4}\left(\frac{15}{16}\right) = \boxed{\frac{15}{32}} = 0.46875$$

- c) Use four midpoint rectangles of equal width to estimate the area under the graph of  $f(x)$ .



$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$A \approx \frac{1}{4}\left(\frac{1}{64}\right) + \frac{1}{4}\left(\frac{9}{64}\right) + \frac{1}{4}\left(\frac{25}{64}\right) + \frac{1}{4}\left(\frac{49}{64}\right) \\ = \frac{1}{4}\left(\frac{84}{64}\right) = \boxed{\frac{21}{64}} = 0.328125$$

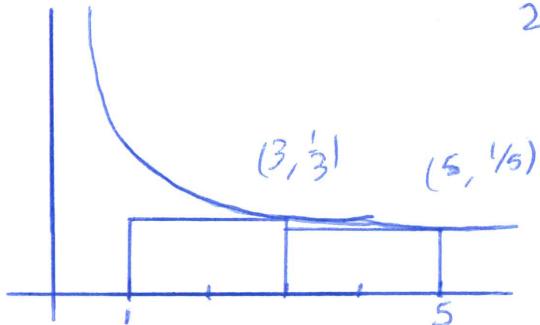
$$A := \ln(5) \approx 1.60944$$

**Example.** Consider  $f(x) = \frac{1}{x}$  on the interval  $[1, 5]$ .

- a) Use two rectangles of equal width to find a lower sum for the area under the graph of  $f(x)$ .

$$\Delta x = \frac{5-1}{2} = 2$$

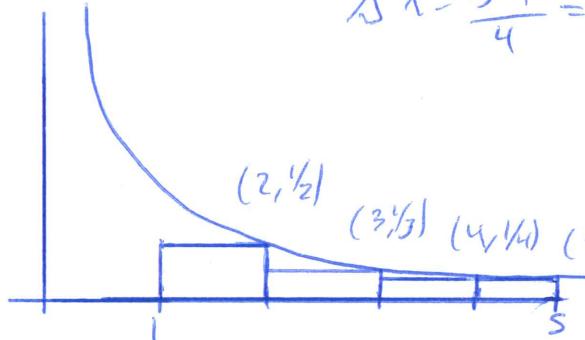
$$A = 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{5}\right) = \left(\frac{16}{15}\right) \approx 0.53$$



- b) Use four rectangles of equal width to find a lower sum for the area under the graph of  $f(x)$ .

$$\Delta x = \frac{5-1}{4} = 1$$

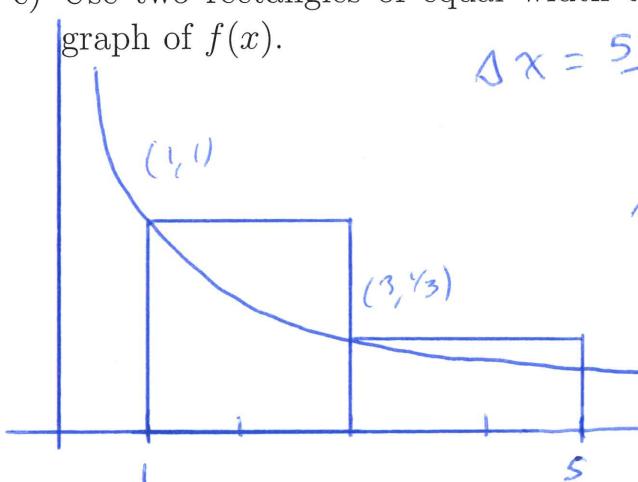
$$A \approx 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right) + 1\left(\frac{1}{5}\right) = \left(\frac{77}{60}\right) = 1.283$$



- c) Use two rectangles of equal width to find an upper sum for the area under the graph of  $f(x)$ .

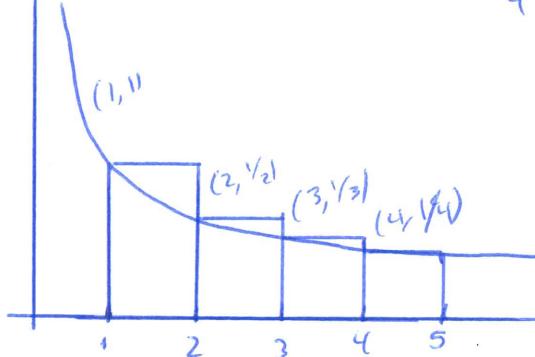
$$\Delta x = \frac{5-1}{2} = 2$$

$$A \approx 2(1) + 2\left(\frac{1}{3}\right) = \left(\frac{8}{3}\right) = 2.66$$



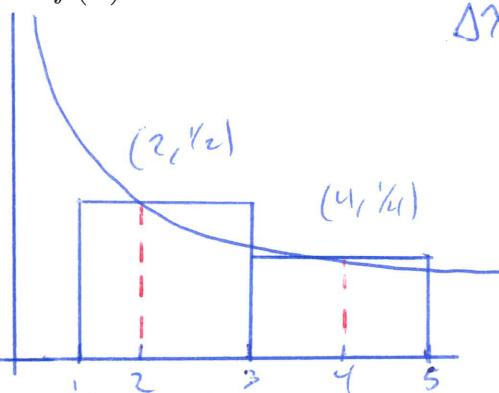
- d) Use four rectangles of equal width to find an upper sum for the area under the graph of  $f(x)$ .

$$\Delta x = \frac{5-1}{4} = 1$$



$$A \approx 1(1) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right) = \frac{25}{12} = 2.08\bar{3}$$

- e) Use two midpoint rectangles of equal width to estimate the area under the graph of  $f(x)$ .

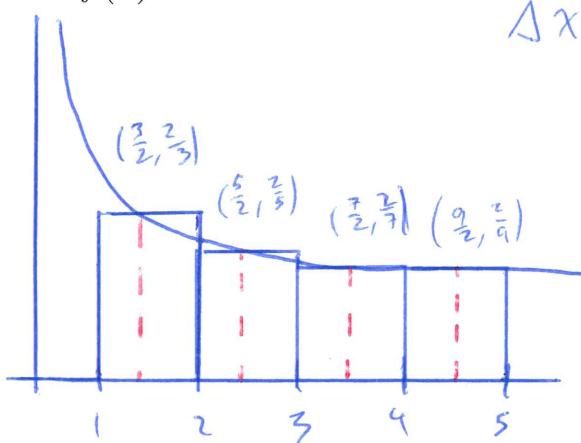


$$\Delta x = \frac{5-1}{2} = 2$$

$$A \approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = \frac{3}{2} = 1.5$$

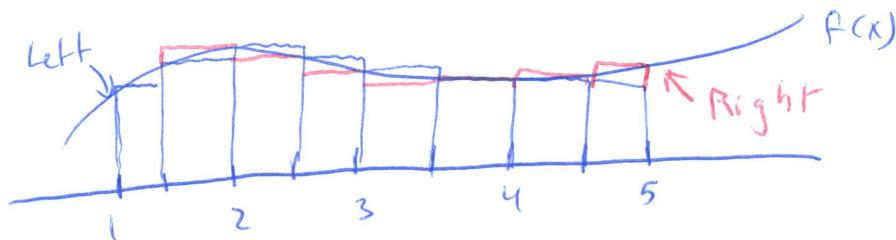
- f) Use four midpoint rectangles of equal width to estimate the area under the graph of  $f(x)$ .

$$\Delta x = \frac{5-1}{4} = 1$$



$$A \approx 1\left(\frac{2}{3}\right) + 1\left(\frac{2}{5}\right) + 1\left(\frac{2}{7}\right) + 1\left(\frac{2}{9}\right)$$

$$= \frac{496}{315} \approx 1.5746$$



**Example.** Use the tabulated values of  $f$  to evaluate both the left and right Riemann sums. ( $n = 8$ ,  $[1, 5]$ )

$x$	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	0	2	3	2	2	1	0	2	3

$$\Delta x = \frac{5-1}{8} = \frac{1}{2}$$

$$A_L = \frac{1}{2}(0) + \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(2) + \frac{1}{2}(2) + \frac{1}{2}(1) + \frac{1}{2}(0) + \frac{1}{2}(2) \\ = 6$$

$$A_R = \frac{1}{2}(2) + \frac{1}{2}(3) + \frac{1}{2}(2) + \frac{1}{2}(2) + \frac{1}{2}(1) + \frac{1}{2}(0) + \frac{1}{2}(2) + \frac{1}{2}(3) = \left(\frac{11}{2}\right)$$

When velocity is a continuous function, the area between the curve and the  $x$ -axis gives the displacement.

**Example.** The velocities (in  $m/s$ ) of an automobile moving along a straight freeway over a four-second period are given in the following table. Find the midpoint Riemann sum approximation to the displacement on  $[0, 4]$  with  $n = 2$  and  $n = 4$  subintervals.

$t$	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	20	25	30	35	30	30	35	40	40

$$n=2 \quad \Delta t = \frac{4-0}{2} = 2 \quad \begin{array}{ccccccc} & | & | & | & | & | & | \\ 0 & 1 & 2 & 3 & 4 \end{array} \quad S \approx 2(30) + 2(35) = 130 \text{ m}$$

$$n=4 \quad \Delta t = \frac{4-0}{4} = 1 \quad \begin{array}{ccccccc} & | & | & | & | & | & | \\ 0 & 0.5 & 1 & 1.5 & 2 & 3 & 4 \\ & 2.5 & 3.5 \end{array} \quad S \approx 1(25) + 1(35) + 1(30) + 1(40) = 130 \text{ m}$$

 **Example.** Use the following table of recorded velocities answer the following questions:

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1.0	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

- a) Estimate the total displacement using 12 subintervals of length 5 with left-endpoint values.

$$s \approx 300(1.0 + 1.2 + 1.7 + 2.0 + 1.8 + 1.6 + 1.4 + 1.2 + 1.0 + 1.8 + 1.5 + 1.2) \\ = 300(17.4) = \boxed{5220 \text{ m}}$$

- b) Estimate the total displacement using 12 subintervals of length 5 with right-endpoint values.

$$s \approx 300(1.2 + 1.7 + 2.0 + 1.8 + 1.6 + 1.4 + 1.2 + 1.0 + 1.8 + 1.5 + 1.2 + 0) \\ = 300(16.4) = \boxed{4920 \text{ m}}$$

**Definition.**

If  $a_m, a_{m+1}, \dots, a_n$  are real numbers and  $m$  and  $n$  are integers such that  $m \leq n$ , then

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + \dots + a_{n-1} + a_n$$

**Example.** Rewrite the sums without sigma notation and evaluate:

a)  $\sum_{k=1}^3 \frac{k-1}{k}$

$$= \frac{0}{1} + \frac{1}{2} + \frac{2}{3} = \left( \frac{7}{6} \right)$$

b)  $\sum_{k=1}^4 (-1)^k \cos(k\pi)$

$$= -\cos(\pi) + \cos(2\pi) - \cos(3\pi) + \cos(4\pi) \\ = -(-1) + 1 - (-1) + 1 = (4)$$

**Example.** Rewrite the following sum without sigma notation. Note that  $n$  denotes the number of rectangles and the letters  $i$ ,  $j$ , and  $k$  are typically used for indexing.

$$\begin{aligned} & \sum_{k=1}^n \frac{1}{n} (k^2 + 1) \\ &= \frac{1}{n} \left( (1+1) + (4+1) + (9+1) + \dots + (n^2+1) \right) \\ &= \frac{1}{n} (1+4+9+\dots+n^2) + \frac{1}{n} (\underbrace{1+1+\dots+1}_{n-\text{times}}) \\ &= \frac{1}{n} (1+4+9+\dots+n^2) + 1 \end{aligned}$$

**Example.** Which of the following summations represent the sum  $1 + 2 + 4 + 8 + 16 + 32$ ?

$$a) \sum_{k=1}^6 2^{k-1}$$

$$\begin{aligned} &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 1 + 2 + 4 + 8 + 16 + 32 \end{aligned}$$

Yes

$$b) \sum_{k=0}^5 2^k$$

$$\begin{aligned} &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 1 + 2 + 4 + 8 + 16 + 32 \end{aligned}$$

Yes

$$c) \sum_{k=-1}^4 2^{k+1}$$

$$\begin{aligned} &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 1 + 2 + 4 + 8 + 16 + 32 \end{aligned}$$

Yes

**Example.** Which of the following summations represent the sum  $1 - 2 + 4 - 8 + 16 - 32$ ?

$$a) \sum_{k=1}^6 (-2)^{k-1}$$

$$\begin{aligned} &= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 + (-2)^5 \\ &= 1 - 2 + 4 - 8 + 16 - 32 \end{aligned}$$

Yes

$$b) \sum_{k=0}^5 (-1)^k 2^k$$

$$\begin{aligned} &= (-1)^0 2^0 + (-1)^1 2^1 + (-1)^2 2^2 \\ &\quad + (-1)^3 2^3 + (-1)^4 2^4 + (-1)^5 2^5 \\ &= 1 - 2 + 4 - 8 + 16 - 32 \end{aligned}$$

Yes

$$c) \sum_{k=-2}^3 (-1)^{k+1} 2^{k+1}$$

$$\begin{aligned} &= (-1)^{-1} 2^0 + (-1)^0 2^1 + (-1)^1 2^2 \\ &\quad + (-1)^2 2^3 + (-1)^3 2^4 + (-1)^4 2^5 \\ &= -1 + 2 - 4 + 8 - 16 + 32 \end{aligned}$$

No

**Example.** Express the following sums in sigma notation:

a)  $-1 + 4 - 9 + 16 - 25$

$$= \sum_{k=1}^5 (-1)^k k^2$$

b)  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

$$= \sum_{k=1}^4 \left(\frac{1}{2}\right)^k = \sum_{k=1}^4 2^{-k}$$

c)  $\frac{5}{2} + \frac{10}{3} + \frac{15}{4} + \frac{20}{5} + \frac{25}{6} + \frac{30}{7}$

$$= \sum_{k=1}^6 \frac{5k}{k+1}$$

d)  $4 + 9 + 14 + \dots + 44$

$$= \sum_{k=0}^8 4 + 5k$$

**Example.** Suppose that  $\sum_{k=1}^n a_k = 0$  and  $\sum_{k=1}^n b_k = 1$ , evaluate the following

a)  $\sum_{k=1}^n 8a_k = 8 \underbrace{\sum_{k=1}^n a_k}_{0} = 8(0) = 0$

b)  $\sum_{k=1}^n 250b_k = 250 \underbrace{\sum_{k=1}^n b_k}_1 = 250(1) = 250$

c)  $\sum_{k=1}^n (a_k + 1)$

$$= \sum_{k=1}^n a_k + \sum_{k=1}^n 1$$

$$= 0 + n = n$$

d)  $\sum_{k=1}^n (b_k - 1) = \sum_{k=1}^n b_k - \sum_{k=1}^n 1$

$$= 1 - n$$

### Theorem 5.1: Sums of Powers of Integers

Let  $n$  be a positive integer and  $c$  a real number.

$$\sum_{k=1}^n c = cn$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

**Example.** Prove that  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

$$\sum_{k=1}^n k = 1 + 2 + \dots + (n-1) + n$$

$$+ \sum_{k=1}^n k = n + (n-1) + \dots + 2 + 1$$

$$2 \sum_{k=1}^n k = \underbrace{(n+1) + (n+1) + \dots + (n+1) + (n+1)}_{n\text{-times}} = n(n+1)$$

**Example.** Evaluate the following sums

a)  $\sum_{k=1}^{10} k = \frac{10(11)}{2} = 55$

b)  $\sum_{k=1}^{10} (1+k^2)$

c)  $\sum_{k=1}^{10} k^3 = \frac{10^2(11)^2}{4}$

$$= \sum_{k=1}^{10} 1 + \sum_{k=1}^{10} k^2$$

$$= \frac{100(121)}{4}$$

$$= 25(121) = 3025$$

d)  $\sum_{k=1}^7 (-2k - 4)$

$$= -2 \sum_{k=1}^7 k - \sum_{k=1}^7 4$$

$$= (-2) \frac{7(8)}{2} - 4(7)$$

e)  $\sum_{k=1}^6 (3 - k^2)$

$$= \sum_{k=1}^6 3 - \sum_{k=1}^6 k^2$$

$$= 3(6) - \frac{6(7)(13)}{6}$$

f)  $\sum_{m=1}^3 \frac{2m+2}{3}$

$$= \frac{2}{3} \sum_{m=1}^3 m + \sum_{m=1}^3 \frac{2}{3}$$

$$= \frac{2}{3} \frac{3(4)}{2} + \frac{2}{3}(3)$$

$$= -56 - 28$$

$$= 18 - 91$$

$$= 4 + 2 = \boxed{6}$$

$$= \boxed{-84}$$

$$= \boxed{-73}$$

**Definition. (Left, Right and Midpoint Riemann Sums in Sigma Notation)**

Suppose  $f$  is defined on a closed interval  $[a, b]$ , which is divided into  $n$  subintervals of equal length  $\Delta x$ . If  $x_k^*$  is a point in the  $k$ th subinterval  $[x_{k-1}, x_k]$ , for  $k = 1, 2, \dots, n$ , then the **Riemann sum** for  $f$  on  $[a, b]$  is  $\sum_{k=1}^n f(x_k^*)\Delta x$ . Three cases arise in practice.

- $\sum_{k=1}^n f(x_k^*)\Delta x$  is a **left Riemann sum** if  $x_k^* = a + (k - 1)\Delta x$ .
- $\sum_{k=1}^n f(x_k^*)\Delta x$  is a **right Riemann sum** if  $x_k^* = a + k\Delta x$ .
- $\sum_{k=1}^n f(x_k^*)\Delta x$  is a **midpoint Riemann sum** if  $x_k^* = a + (k - 1/2)\Delta x$ .

**Example.** For the function  $f(x) = 3x^2$ , find a formula for the upper sum obtained by dividing the interval  $[0, 1]$  into  $n$  equal subintervals.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n} \quad x_i = 0 + i \left(\frac{1}{n}\right) = \frac{i}{n} \quad f(x_i) = 3 \left(\frac{i}{n}\right)^2 = \frac{3i^2}{n^2}$$

$$\Rightarrow \sum_{i=1}^n \frac{3i^2}{n^2} \cdot \frac{1}{n} = \sum_{i=1}^n \frac{3i^2}{n^3} = \frac{3}{n^3} \sum_{i=1}^n i^2 = \frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} = \boxed{\frac{2n^2+3n+1}{2n^2}}$$

Now take the limit of the sum as  $n \rightarrow \infty$  to calculate the area under  $f(x) = 3x^2$  over  $[0, 1]$ .

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3i^2}{n^3} = \lim_{n \rightarrow \infty} \frac{2n^2+3n+1}{2n^2} = \boxed{1}$$

**Example.** For the function  $f(x) = 2x$ , find a formula for the upper sum obtained by dividing the interval  $[0, 3]$  into  $n$  equal subintervals.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n} \quad x_i = 0 + i \Delta x = \frac{3i}{n} \quad f(x_i) = \frac{6i}{n}$$

$$\Rightarrow \sum_{i=1}^n \frac{6i}{n} \cdot \frac{3}{n} = \sum_{i=1}^n \frac{18i}{n^2} = \frac{18}{n^2} \sum_{i=1}^n i = \frac{18}{n^2} \frac{n(n+1)}{2} = \boxed{\frac{9n+9}{n}}$$

Now take the limit of the sum as  $n \rightarrow \infty$  to calculate the area under  $f(x) = 2x$  over  $[0, 3]$ .

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{18i}{n^2} = \lim_{n \rightarrow \infty} \frac{9n+9}{n} = \boxed{9}$$