Monopoly's optimal pricing and quality choice on a Network of consumers

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Abstract

A two stage game where consumers are located in a network and the monopoly chooses price and quality are studied. We solved for the *Subgame Perfect Equilibrium* and presented a numerical experiment to illustrate the effect of networks on the equilibrium allocation.

1 INTRODUCTION

In the standard model of consumption, it is commonly assumed that people's demands for a good are independent of one another, which might be a very restrictive assumption. A person's demand may be affected by the number of other people who have purchased the good and are within the same social network.A positive network externality exists if the quantity of a good demanded by a consumer increases if the others' demand increases.

We followed Goyal et. al (2010), approach to represent this relationships, better known as strategic complements. [1] The goal of this study is to investigate the impact of the network effect and externalities on the quality-price game equilibrium. In particular, a monopolist first determines his optimal quality level and uniform price simultaneously and then customers choose their consumption levels that maximize their own utilities. The utility function of consumers is increasing in the quality level of the product and the consumption level of their peers. Such function captures the positive local network effect.

A main feature of our model is that the monopolist is choosing both quality level and a uniform price at the same time subject to the utility function of consumers which captures their purchasing behavior. Our work relates to the vast growing literature of games played on networks with externalities. In Galeotti et al. (2010), the paper has provided a framework analyzing the network interactions as the game is modeled in terms of an underlying network of interactions affects payoffs. The pricing game of the monopolist selling network goods and who could discriminate prices taking into consideration the value of network information has been well studied by Fainmesser, I. P., and Galeotti, A. (2013). In Candogan, Bimpikis and Ozdaglar (2012), the optimal pricing strategies of a monopolist in a market of embedded consumers in a social network with positive

[1] A payoff function exhibits *strategic complements* if it has increasing differences: $\forall k, x_i > x_i'$ and $x \geq x'$, $v_k(x_i,x) - v_k(x_i',x) \geq v_k(x_i,x') - v_k(x_i',x')$.
A payoff function exhibits *positive externalities* if for each k, and for all $x \geq x'$, $v_k(x_i,x) \geq v_k(x_i,x')$.

externalities has been studied. They consider a setting where the monopolist can offer a uniform price as a function of network effect. Our model is a modify version of the one in Candogan, Bimpikis and Ozdaglar (2012), where they derive a price discrimination schedule but there is no choice in quality for the firm.

2 THE MODEL

There are n consumers in this economy, the consumers set is $N = \{1, 2, ..., n\}$ The consumers are located in a network G. Given the structure of the network G, all consumers choose their level of consumption x_i.
Consumers care about the local networks effect, that is to say, they observe just their neighborhood $N(i) = \{j \in N | g_{ij} = 1\}$.[2] Consumers have a quadratic utility affected by the quality of the product that includes network externalities.

Definition 1. An allocation for this model is a vector $x = (x_i, x_{-i})$ which indicates the quantity chosen or demanded for each or the consumers at price p,and quality q.

Consumer i (or node i) has utility function,

$$U_i(x_i, x-i, p, q) = a_i x_i + tqx_i - b_i x_i^2 + \lambda \sum_{j \in N(i)} g_{ij} x_i x_j - px_i$$

where $\lambda > 0$ is the parameter of the network externalities given by $\sum_{j \in N(i)} x_i x_j$, as the one presented in Galeotti, Goyal et al.(2010).

Basically, the game is described by the following stages.

1. **Stage 1**(Pricing and quality choice) The Firm decides strategy (p, q) price and quality such that maximizes its profits.

2. **Stage 2** Each agent i chooses to buy/consume x_i units of the good by maximizing his utility; given the optimal strategy of the firm (p, q), and the choices of the other players x_{-1}.

3 EQUILIBRIUM

We look for subgame perfect Nash equilibrium which is the standard solution for a dynamic game with perfect information.

Definition 2. Fudenberg and Tirole (1991). A Nash equilibrium is said to be **subgame perfect** if an only if it is a Nash equilibrium in every subgame of the game.

[2]A network (g) is *directed* if it is possible that $g_{ij} \neq g_{ji}$, and is *undirected* if $g_{ij} = g_{ji}$ for all nodes i and j. Jackson (2008).

Intuitively speaking, this notion of equilibrium for finite horizon games it is found found by backward induction. The important feature of this procedure is that, when restricted to a subgame of the game,the equilibrium computed remains being equilibrium of the subgame.The existence of a solution is given by the next result.

Theorem 1. Fudenberg and Ti role (1991).Backwards induction gives the entire set of SPE.

In our model, there are two proper subgames and the game itself, therefore there are three subgames. Which we solve by backwards induction in Section 3.1 and Section 3.2.

3.1 Optimal consumption choice

We begin solving the second stage of the game where the consumers choose a level of consumption.

$$\max_{x_i} U_i(x_i, x{-}i, p, q) = a_i x_i + tqx_i - b_i x_i^2 + \lambda \sum_{j \in N(i)} g_{ij} x_i x_j - px_i \tag{3.1.1}$$

Subject to

$$x_i > 0$$

So the solution will give us the best response of the consumer i given the choice of (p,q) from the firm, and the consumption choice of his neighbors x_{-i}.

$$\beta(x_i) = x_i^* = \frac{1}{2b_i}\left[a_i + tq - p + \lambda \sum_{j \in N(i)} g_{ij} x_j\right] \tag{3.1.2}$$

3.2 Optimal price and quality choice)

A firm produces a perfectly divisible good (or service) at cost $c(q)$.The production function is such that higher quality increases cost. For convenience, we assume a linear relationship so that the per unit profits are given by where the coefficient $c > 0$ characterizes the quality production technology.

$$\max_{(p,q)} \Pi(x_i, x{-}i, p, q) = \sum_{i=1}^{N} x_i [p - c(q)] \tag{3.2.1}$$

subject to

$$x_i \in arg \max_{x_i} U_i(x_i, x_{-i}, p, q)$$

First order conditions

$$\frac{\partial \Pi}{\partial p} = \frac{1}{2b_i}[a_i + tq - 2p + c(q) + \lambda \sum_j g_{ij} x_j] \tag{3.2.2}$$

$$\frac{\partial \Pi}{\partial q} = \frac{1}{2b_i}[pt - a_i c'(q) - t[qc'(q) + c(q)] + pc'(q) - \lambda \sum_j g_{ij} x_j c'(q)] \tag{3.2.3}$$

Assuming $c(q) = c(q-t)$, with $c > 0$, and $t > 0$. We get

$$p(q)* = \frac{(a_i - ct) + (t+c)q + \lambda \sum_j g_{ij}x_j}{2} \quad (1)(3.2.4)$$

$$q* = \frac{(t+c)p - ac - c\lambda \sum_j g_{ij}x_j}{2tc} \quad (3.2.5)$$

Solving for both equations we get p and q in terms of c, λ and $\sum_j g_{ij}x_j$.

$$p* = \frac{4ct(a_i - ct) - 2a_ic(t+c) + \lambda \sum_j g_{ij}x_j(4tc - (t+c)^2 + (t+c))}{4tc - (t+c)^2} \quad (3.2.6)$$

and quality

$$q* = \frac{(a_i - ct)(t+c) - 2a_ic + \lambda \sum_j g_{ij}x_j(t-c)}{4tc - (t+c)^2} \quad (3.2.7)$$

Therefore a **subgame perfect equilibrium** for this game is given by the strategy profile $\sigma = (x*, p*, q*)$ where $x = (x_1, x_2, ..., x_n)$.

4 NUMERICAL EXPERIMENTS

In this section, we describe computational experiments with the proposed game for a different network structures. The objective of this experiment is to investigate the effect of the network structure on the equilibrium of the underlying game model. The optimal quality and uniform price were determined by the monopolist and the consumers determine their optimal consumption level accordingly. All numerical computations were carried out on Intel Core i3 2.30 GHz PC running MATLAB R2015b on Windows 7 Home Premium with memory 4 GB RAM.

4.1 Experiment of Different consumer Network Structures.

Figure 1: Complete Network

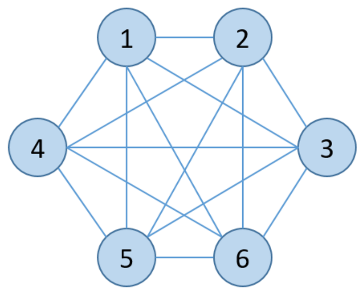


Figure 2: Line Network

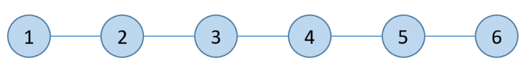


Figure 3: Ring Network

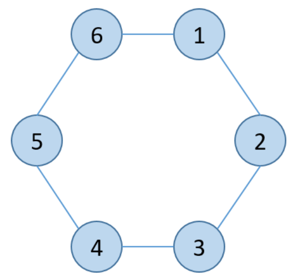


Figure 4: Star Network

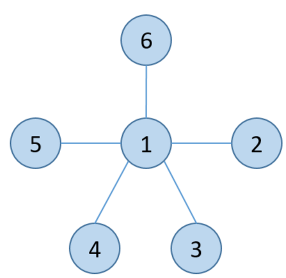


Figure 5: Tree Network

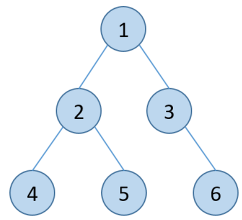
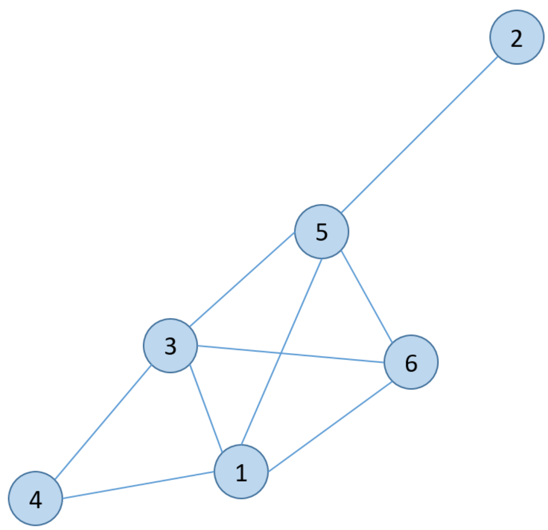


Figure 6: Random Network



4.2 Results and Discussion

In particular, the consumer network consists of 6 consumers. To ensure that consumption level is always strictly positive, in the experiments carried out, we assume $2bi > \sum_{j \in N(i)} g_{ij}$ to ensure the feasibility of the problem. Also, we assume the marginal production cost $c > 1$ to ensure that the model is well-behaved. As in the computation of the optimal p and q there is a term of $(c-1)$, this assumption would help avoiding the case of having zero solution. Moreover, we assume the parameter $t \geq 2$ to ensure that both the profit and the consumption level to be have positive values. In addition, we set the column vector $\mathbf{a} = [7; 4; 8; 5; 1; 3]$ for each consumer according to their position in the corresponding network type.For the sake of simplicity we set $\lambda = 1$.

As seen in the figure (Consumption), the star network structure affect the consumption level of consumers especially consumer number 1, the center node, who has the highest consumption level among his peer due to two factors, his unique and advantageous position in the network and his relatively high valuation factor of the product a. Also, consumer number 5 has the lowest consumption level.

Figure 7: Network Effect on Consumption Level

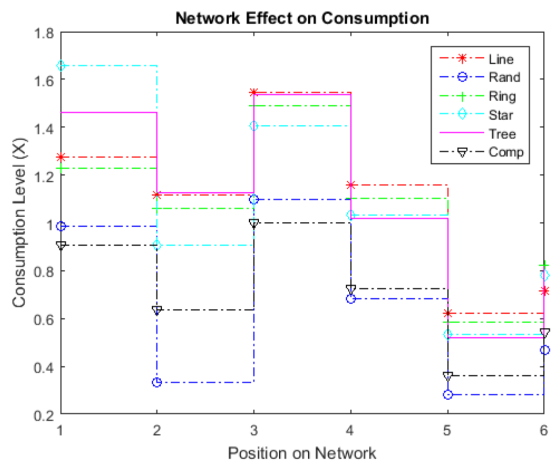
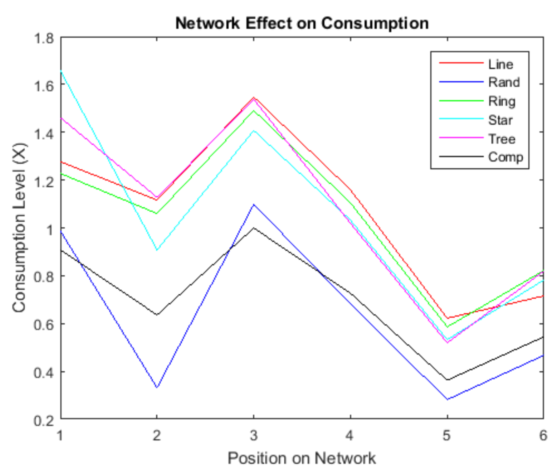


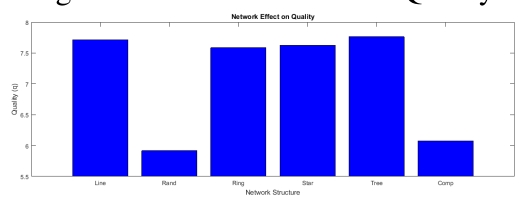
Figure 8: Network Effect on Consumption Level



Consumer number 6 is the closest to number 5 because she is linked to only one other node. The degree of an individual alone is not as significant as her relative degree compared to her peers in the same network. In other words, in the complete network, since all consumers have the same degree, the network effect becomes almost insignificant and the consumption level is affected mainly by the corresponding value of a for each customer. That's why the consumption level of the complete network has been dominated by most of the other network structures. The network effect becomes very obvious when looking and comparing between the ring and the line networks. As both networks almost have the same structure, they produce very close results in terms of consumption levels.

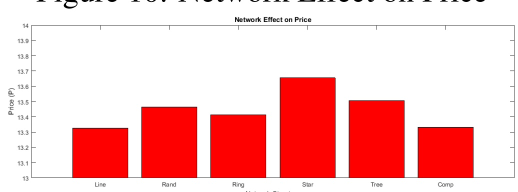
In general, most network structures resulted in the same quality levels except the complete and random networks which lead to major reduction in the quality level. This certainly plays a major role in increasing consumers utilities as the higher the quality of the product, the higher the consumption level. Thats why the consumption levels of both the complete and random networks are the lowest among the other structures. As investing in enhancing the quality under these two networks would not result in better outcomes because they have low demand. Thus, it sounds logical to focus on decreasing the marginal cost or the quality levels.

Figure 9: Network Effect on Quality



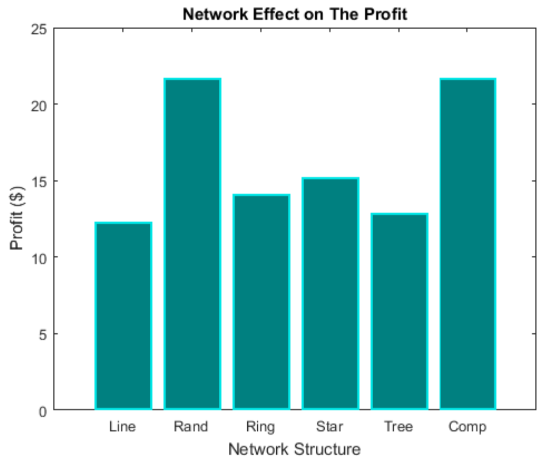
In terms of price, as seen in the figure below, it is obvious that the star network has the most favorable outcome. Such result is very reasonable as the star network result in the highest consumption level. Thus, raising the price is the best action to maximize the profit.

Figure 10: Network Effect on Price



In terms of profit, surprisingly enough the random network is the winner. This should come at no surprise as it has a very low optimal quality level the marginal production cost is diminished significantly. Such huge reduction gives the random network a competitive advantage over the other network structures. Similarly, the same goes for the complete network.

Figure 11: Network Effect on Profit



5 EXTENSIONS

A Natural extension to this model is a Bertrand competition between firms. That is to say, a duopoly competition in prices. Considering same technology, and an homogeneous and indivisible good. This setup could represent real economic interactions more accurate. One could imagine two big firms such as Samsung and Apple that even though they share very similar characteristics and information, could have different advantages depending on the consumer's network.
Summing up, we could consider a consumer i that buys product from firm k has utility

$$U_i(x-i, p_1, p_2) = a_i - p_j + \lambda \sum_{j \in N(i)} (x_{jk} + \gamma x_{jl}). \tag{5.1}$$

The consumer chooses to buy from firm k if

$$p_k < p_l - \lambda \sum_{j \in N(i)} (1-\gamma)(x_{jk} - x_{jl}). \tag{5.2}$$

Where,a_i is the willingness to pay for the product, γ is the degree of compatibility with the other product,p_k is the price for the good in firm k, and x_{jk} , x_{jl} are the measure of consumers at the at the network who consume form firm k and l, respectively.

Firms compete in prices. Firm k has profits given by

$$\Pi(x, p_k, p_l) = \sum_{i=1}^{N} x_{ik}[p_k - c].. \tag{5.3}$$

To solve this , we first can consider the case of a symmetric equilibrium, i.e. both firms use the same strategy $p_k = p_l$. Afterwards, we consider (without lost of generalization) the case when $p_k > p_l$.

Other variations could work, for example Cournot competition, where firms compete in quantity, or Stackelberg competition, where one of the firms has the advantage of making the first move.

6 CONCLUSIONS

We analyzed the effect of the network externality, in a local manner, since we only consider the neighborhood of each consumer. We could observe that the externality affects the choice of the firm and gives the firm a window to spend more on quality. Network effect plays a vital role in quality-price games. Incorporating it in the model in order to make use of it to maximize the profit is fairly difficult for the monopolist. Take full advantage the network structure, the consumer purchasing behavior could be characterized and utilized to find the optimal quality level and the optimal uniform price.

In this project, we consider a two-stage model where consumers are located in a network and a monopoly has to decide its allocations for price and quantity. In the model we use, which is a modified version of Candogan,Bimpikis and Ozdaglar (2012); we try to capture the consumer attraction to high product quality. The model is solved for different network structures that affect the consumers local network effect to analyze such effect on the choice of quality and price. Essential assumptions has been proposed and used to ensure the well-behavior of the model.

A sensitivity analysis is used to derive some insights related to proper selection of the optimal quality level and price under each network structure type. Moreover, an analysis of the effect of the network structure on the consumption level of consumers. Potentially, one could extend this work to incorporate uncertainty in the quality level and maximize over the expectation. Also, one could consider the competition of two firms over the same network of consumers.

APPENDIX

In this section we derive the matrix representation of the model that we used to compute the numerical example.

Consider the maximization problem of consumer i,

$$\max_{x_i>0} U_i = a_i x_i + tqx_i - b_i x_i^2 + \sum_{j\in N(i)} g_{ij} x_j x_i - P_0 x_i \tag{1}$$

$$\mathbf{X} = (\mathbf{B}-\mathbf{G})^{-1} * \Big(\mathbf{a} + (tq - P_0) * \mathbf{1}\Big) \tag{2}$$

Let $\mathbf{B}, \mathbf{G}$ be 6×6 symmetric matrices and $\mathbf{a}$ be a 6×1 vector. t, q, b_i, and P_0 are scalars.

$$\mathbf{B} = \begin{pmatrix} 2b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2b_6 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & 0 & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & 0 & g_{34} & g_{35} & g_{36} \\ g_{41} & g_{42} & g_{43} & 0 & g_{45} & g_{46} \\ g_{51} & g_{52} & g_{53} & g_{54} & 0 & g_{56} \\ g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & 0 \end{pmatrix},$$

$$\begin{aligned} \max_{p,q} \Pi &= \Big[P_0 - c(q)\Big] * X \\ &= \Big[P_0 - c(q)\Big] * \mathbf{1} * (\mathbf{B}-\mathbf{G})^{-1} * \Big(\mathbf{a} + (t*q - P_0) * \mathbf{1}\Big) \end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial \Pi}{\partial P_0} =& \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{a} + (c+1) * q * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1} \\ & - 2 * P_0 * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1} - c * t * \mathbf{1}^T * (\mathbf{B}-\mathbf{G}) * \mathbf{1} = 0\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\partial \Pi}{\partial q} =& - c * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{a} - 2 * c * q * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1} \\ & + (c+1) * P_0 * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1} + c * t * \mathbf{1}^T * (\mathbf{B}-\mathbf{G}) * \mathbf{1} = 0\end{aligned} \tag{5}$$

$$P_0^* = \frac{\left(\frac{c-1}{2}\right) * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{a} + \left(\frac{c-1}{2}\right) * t * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1}}{\left[\frac{(c+1)^2-4*c}{2*c}\right] \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1}} \tag{6}$$

$$\begin{aligned}q^* =& \frac{-\mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{a} + c * t * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1}}{(c+1) * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1}} \\ & + \frac{2 * c * (c-1) * \left[\mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{a} + t * \mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1}\right]}{\left[(c+1)^2 - 4 * c\right] * (c+1) * \left[\mathbf{1}^T * (\mathbf{B}-\mathbf{G})^{-1} * \mathbf{1}\right]^2}\end{aligned} \tag{7}$$

References

[1] Candogan, O., Bimpikis, K., and Ozdaglar, A. (2012). Optimal pricing in networks with externalities. Operations Research, 60(4), 883-905.

[2] Bolton, P., and Harris, C. (1999). Strategic experimentation. Econometrica, 67(2), 349-374.

[3] Fainmesser, I. P., and Galeotti, A. (2013). The value of network information. Available at SSRN 2366077.

[4] Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2010). Network games. The review of economic studies, 77(1), 218-244.

[5] Fudenberg, D.,and Tirole, J. (1991). Game theory.

[6] Jackson, M. O. (2008). Social and economic networks (Vol. 3). Princeton: Princeton university press.

[7] Vega-Redondo, F. (2007). Complex social networks (No. 44). Cambridge University Press.