Chapter 1

Power Series

Definition 1. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} = c_n x^n = c_0 + c_1 x + c_2 x + ... \tag{1}$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x.

Example 1. For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution. We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty.$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus, the given series converges only when x = 0.

$\square$

Example 2. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

Solution. Let $a_n = \frac{(x-3)^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \text{ as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent when $|x-3|<1$, and divergent when $|x-3|>1$. Now

$$|x-3|<1 \iff -1<x-3<1 \iff 2<x<4$$

so the series converges when $2<x<4$, and diverges if $x<2$ or $x>4$. The Ratio Test gives no information when $|x-3|=1$ so we must consider $x=2$ and $x=4$ separately. If we put $x=4$ in the series, it becomes $\sum \frac{1}{n}$, the harmonic series, which is divergent. If $x=2$, the series is $\sum \frac{-1^n}{n}$, which converges by the Alternating Series Test. Thus, the given power series converges for $2 \leq x < 4$.

$\square$

Theorem 1. *For a given power series there are only three possibilities:*

1. The series converges only when $x=a$

2. The series converges for all x.

3. There is a positive number such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

The number R in case (3) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R=0$ in case (1) and $R=\infty$ in case (2). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (1) the interval consists of just a single point a. In case (2) the interval is $(-\infty,\infty)$. In case (3) note that the inequality $|x-a|<R$ can be rewritten as $a-R<x<a+R$. When x is an **endpoint** of the interval, that is, $x=a\pm R$, anything can happen; the series might converge at one or both endpoints or it might diverge at both endpoints. Thus, in case (3) there are four possibilities for the interval of convergence:

$$(a-R,a+R),\ (a-R,a+R],\ [a-R,a+R) \text{ or } [a-R,a+R]$$

Series	Radius Of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	$R=1$	$(-1,1)$
$\sum_{n=0}^{\infty} n!x^n$	$R=0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$	$R=1$	$[2,4)$