False Information from Near and Far*

Christophe Bravard[a], Jacques Durieu[a], Sudipta Sarangi[b], Stéphan Sémirat[a]

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Abstract

We study message credibility in social networks with biased and unbiased agents. Biased agents prefer a specific outcome while unbiased agents prefer the true state of the world. Each agent who receives a message knows the identity (but not type) of the message creator and only the identity and types of their immediate neighbors. We characterize the perfect Bayesian equilibria of this game and demonstrate filtering by the network: the posterior beliefs of agents depend on the distance a message travels. Unbiased agents, who receive a message from a biased agent, are more likely to assign a higher credibility and transmit it further when they are further away from the source. For a given network, we compute the probability that it will always support the communication of messages by unbiased agents. Finally, we establish that under certain parameters, this probability increases when agents are uncertain about their network location.

[a] *Univ. Grenoble Alpes, CNRS, INRA, Grenoble INP, GAEL, CREG, 38000 Grenoble, France Email: christophe.bravard@univ-grenoble-alpes.fr, jacques.durieu@univ-grenoble-alpes.fr* . [b] *DIW Berlin and Department of Economics, Virginia Tech Blacksburg VA 24061 - 0316, USA. Email: ssarangi@vt.edu.*

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1 Introduction

Suppose Alice sends you information (a message) about an event you have not observed. She sends you this message because you are friends. So *you know* (i) whether *she created* the mes-

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sage – because she is the one who observes the event – *or she transmits* a *message* obtained from someone else – because she is not the one who observed the event, *and* (ii) her opinions or *her bias*. Therefore, when you receive a message from her, you take into account her bias and whether she is the creator of the message or not in assessing the credibility of the message. What makes a network setting different is that Alice may not be the creator of the initial message, but simply passing along a message that her friend Bob sent her. For example, Alice forwards you an email she received from Bob. So you know that the message was created by an individual named Bob, but since Bob is not your friend, you do not know his bias.

How does your perception of the event change under these circumstances? Suppose you know that Alice is biased in favor of a specific event and wants to convince you as well. When she observes an event and sends you a message she has created, knowing her bias, you do not believe her if the a priori likelihood of such an event is low, i.e., Alice's message does not influence you. But, when Alice sends you a message that she received from Bob, you will reason that if the event was not consistent with her bias, then Alice would not forward to you that message. However, since Alice is not the creator of the message but is simply a transmitter, the credibility you put on this message depends on your beliefs about Bob's bias. Furthermore, it is possible that Bob is not the person who observed the event, and like Alice, is simply a transmitter. *You now have to determine whether the social network is acting as a filter for false information or not.* This is the question we address in this paper.

From this story, it is clear that the distance, between the source of the message – the person observing the event and sending the first message – and you, plays an important role. To illustrate this, suppose that there are two states of the world 0 and 1, and the latter occurs with probability 0.45. Let Bob be biased in favor of 1, and the one who observes the state of the world, i.e., he creates the message. Suppose that you and Alice are Bob's friends, i.e., directly linked to Bob in the network. Then your posterior, as an unbiased agent, is 0.45, and and you do not believe Bob's message – you believe messages when your posterior is at least 0.5. Since you do not believe the message, you do not transmit it. Now suppose that you are a friend of Alice who is a friend of Bob. Moreover, like 10% of the agents in the network, Alice is biased. Finally, Bob creates message 1, sends it to Alice who sends it to you. Then, your posterior beliefs that the state of the world is 1 is: $0.55/(0.1 \times 0.45 + 0.55) = 0.92$. Thus, you strongly believe that the message you received is true and you transmit the message. This example illustrates that in some situations an agent who is at distance 1 from a biased agent that observed the message thinks that his message is not very credible, whereas an agent who is at a distance 2 from the agent that observed the message may believe in the message he receives from a biased agent with a high probability. In the latter case, he transmits the message, in the former he does not. Thus distance traveled by information in a network can play an important role in exacerbating false information because of the role played by agents at each step in spreading or stopping a message.

In this paper, we are interested in identifying conditions under which a biased message will always be transmitted (and therefore in what affects its credibility) in social networks where the distance between the agent who creates the message and the agent receiving a message plays a crucial role in determining the receiver's posterior beliefs. Given the social network setting of our problem, we adopt some of the assumptions made in Bloch et al. (2018). In particular, we assume that *the social network is (i) connected, and (ii) does not contain cycles*. Due to (i) Bob can receive a message from any agent if it is not blocked by at least one agent. Because of (ii), if Bob sends a message to Alice about the event, then Bob cannot receive a message back from Alice. This assumption allows us to avoid feedback loops: we do not want the credibility of the message to increase for Bob because he transmitted a message. Formally, we focus our attention on social networks that are *trees*. Furthermore, to simplify the presentation, we assume that only the agent who observed the event can lie about it. Other agents can only transmit, or not, the message they have received.[1]

We present two frameworks in which agents only know their own type and the types of their immediate neighbors in the social network. In addition, *only agents who obtain a message know the identity of the creator of the message*. In a social network, when a message is forwarded, the identity of the creator of the message is generally known, but one cannot be certain of his type.[2] We start by analyzing a benchmark model where *each agent knows the architecture of the social network* in which the communication of messages takes place. A crucial role is devoted to *specific strategies* where (i) regardless of the true state, each agent biased in favor of state 1 always creates message 1, transmits it, and blocks the other message, and (ii) each unbiased agent creates message 1 only when it matches the true state of the world and transmits it when he thinks it is correct. In our first result, we show that these strategies constitute an equilibrium, called a *maximal communication equilibrium*. We also provide the

[1] In the discussion section, we relax this assumption. We thank an anonymous advisory editor for their suggestion on how to relax this assumption.

[2] In an email forward, for instance, it is not possible to know the creator's type. On a social media platform while it may be possible to learn more about the creator it is not always feasible to determine their bias. Hence, we assume that agents have localized knowledge and only know the types of their direct neighbors. We do not concern's ourselves with agents whose biases are well known. In that case, everyone is at a distance 1 from these people.

posterior beliefs of unbiased agents when these specific strategies are used. More precisely, the credibility of message 1, sent by a biased agent to an unbiased agent i, increases with the distance between i and the agent who observed the event, say i_0. By contrast, the credibility of message 1, sent by an unbiased agent to an unbiased agent i, decreases with the distance between i and i_0. Moreover, when the agent who observed the event is biased and transmits a message to an unbiased agent, the latter does not transmit this message. Thus, we provide conditions under which, in equilibrium, unbiased agents believe that the state of the world is 1 when they receive message 1 through the social network. We call a network where each unbiased agent believes message 1 in equilibrium a *full communication rooted-tree*. Since a full communication rooted-tree does not always exist, we provide the probability of its occurrence given a social network and the exogenous probability of an agent's bias. Moreover, we provide the class of networks that maximize the probability of obtaining a full communication rooted-tree when the probability that an agent is biased is less than 0.5. Next, we characterize the class of networks that minimize the probability of obtaining a full communication rooted-tree when the creator of the message sends false information.

Second, we take into account that in many social situations, agents do not know their distance from the creator of the message.[3] Hence, we modify the benchmark model by assuming that some of the agents who obtain a message *do not know their location in the social network*; they only know who sends them the message, his type, and the types and identities of agents in their neighborhood. This can be viewed as the model with incomplete information about the location of agents in the social network. To summarize, with the exception of the agent observing the event and his immediate neighbors, the remaining agents do not know how far they are from the agent that observed the event. More precisely, agents do not know the architecture of the social network. We assume that each tree has the same probability to be chosen as the social network. Hence, given their neighborhood and the agent who sends them the message, each agent can calculate probabilities about his distance from the creator of the message. In our first result, we establish that the above specific strategies, which satisfy (i) and (ii), constitute a maximal communication equilibrium for some parameters. Next, we establish that there are parameters for which the probability of obtaining a full communication rooted-tree is higher in the model with incomplete information about location than in the benchmark model. Finally, we provide the conditions under which there is a full commu-

[3]For example, the social network Twitter lets you know who is sending you the message, but it is not always easy to determine which other people in the network have acted as intermediaries. Agents only have information about their direct neighbors.

nication rooted-tree in the model with incomplete information about the locations of agents (respectively in the benchmark model) when there is a full communication rooted-tree in the benchmark model (respectively in the model with incomplete information).

The paper is organized as follows. In Section 2, we examine the related literature. In Section 3, we present the model setup. In Section 4, we study the benchmark model. In Section 5, we analyze the model with incomplete information regarding the location of agents in the social network. In Section 6, we successively relax several assumptions and discuss the consequences of these changes. We also explain how to compute the social benefit of fact-checking sites. Proofs of all results are reported in the Appendix (Section 7).

2 Related Literature

The seminal paper that examines a biased agent observing an event and sending a message to an unbiased agent is the Crawford and Sobel (1982) paper. The authors study the conditions under which the biased agent can influence the action of the unbiased agent in cheap talk situations. We approach this problem from a network perspective to examine how the depth of a network relates to the spread of false information.[4]

Our paper is most closely related to the recent paper by Bloch et al. (2018) on the diffusion of rumors in social networks. In this paper agents do not know who started the rumor, i.e., the creator of the message. Because the social network on which the rumor spreads is a tree, the agents know that the message was created by an agent located above them in the tree. Moreover, every agent knows the type of each agent but not the identity of the agent who created the message. The authors provide a necessary and sufficient condition for a network to be a full communication network, i.e., one where the message associated with the state for which some agents are biased, say message 1, is believed by every unbiased agent. This condition depends on the proportion of biased and unbiased agents in the population of potential message creators. More precisely, for each agent, the proportion of biased agents must be sufficiently low. They also extend this framework to the situation where each agent knows the identity and type of his neighbors and the proportion of biased and unbiased agents who are connected to him through any of his neighbors. They show that their result is preserved in this extension.

By contrast, in our benchmark model, the agents, who obtain a message, know the identity

[4]For a recent survey on the spread of false information on social media we refer the interested reader to Kumar and Shah (2018).

of the person creating the message and thus his location, but they do not know his type (except the immediate neighbors of this person). Next, in our model agents do not know the types of other agents – just the type of their immediate neighbors. In addition, agents do not know the proportion of unbiased and biased agents that are connected to them or their neighbors. They calculate the probability that the message they receive is true given the type of agent from whom they receive the message, given that they receive the message. These assumptions allow us to highlight the role played by the distance between the source of the message and agent i on the credibility that agent i assigns to the message he receives. We address the same question as Bloch et al., namely what are the conditions that allow us to obtain a network in which the message is never blocked? We establish that it is necessary and sufficient that *no unbiased agent close enough to the creator of the message obtains the message from a biased agent*.

Bloch et al. study full communication equilibria, i.e., equilibria where message 1 is always transmitted. More specifically, they build an algorithm for obtaining a maximal diffusion of message 1 within the network when the original network does not allow for the diffusion of the message. However, it may not always be possible to modify the original network. Hence we focus on the probability of obtaining a full communication network given that Nature draws both the type of each agent and the creator of the message. We also explore situations where unbiased agents do not transmit message 1 when they receive it.

In the second part of the paper, following Bloch et al., we also introduce the possibility that agents do not know the identity and the location of the creator of the message. However, there are several differences with Bloch et al. In our case, agents do not know the architecture of the social network. Each agent knows if one of his neighbors is the creator of the message. Thus, each agent knows when he is at distance 1 from the creator of the message. But, the agents, located at a distance 2 or more from the creator of the message, do not know who the creator of the message is and how far they are from this creator. Moreover, these agents do not know the proportion of biased and unbiased among the agents who have possibly received the signal from Nature. We assume that Nature draws a social network in the set of trees. Each tree is drawn with the same probability. This allows each agent to calculate probabilities associated with his location. For this case, we establish that the conditions associated with the maximal probability of obtaining a full communication rooted-tree are weaker than in the benchmark model. Again this clearly demonstrates that distance between the agent that must transmit or block the message and the agent that creates the message plays a crucial role in the spread of messages.

Lenoir (2020) extends the paper of Bloch et al. by allowing for the possibility of persuasion of biased agents and characterizes the equilibria in such a context. More precisely, the author studies a communication network where agents can publicly commit *ex ante* to fact-checking any message they send with a reliability of their choice. He shows that unbiased agents use fact-checking as a device to verify information while biased agents use fact-checking as a persuasion device to improve their credibility. In particular, he establishes a relationship between the cost of the persuasion device and the type of agents who will incur this cost in order to enable the diffusion of information in the network. In the discussion section of the paper we also examine the implications and the social benefits of having third party verification.

It is also worth mentioning that Acemoglu et al. (2020) examine the spread of false information in a network. However, their perspective is quite different – they examine the trade-off between aggregating information that agents share with each other in the presence of different types of agents and the spread of misinformation. Finally, there is also a recent literature that examines the incentives for individuals to share private information with others. Through this transmission, agents can influence and benefit from the actions of others. For example, agents have to be coordinated on the same action (see for instance Hagenbach and Koessler, 2010; Galeotti et al., 2013; Chatterjee and Dutta, 2016).

3 Model Setup

3.1 Agents and States of the World

Let $N = [\![1,n]\!]$, with $n \geq 3$, be the set of agents. The set of states of the world is denoted by $\Theta = \{0,1\}$, with θ a typical member of Θ. Agents have a common prior belief that $\theta = 1$ with probability π, where $\pi < 1/2$. Agents earn payoffs from a collective decision denoted by x. We assume that there are two types of agents according to their preferences.

Unbiased agents. They belong to the set $\mathcal{U}$, and prefer the outcome to match the state of the world. Their utility function, $v_{\mathcal{U}}$, satisfies: for every $\theta \in \Theta$,

$$v_{\mathcal{U}}(x,\theta) = \begin{cases} a & \text{if } x = \theta, \\ b' & \text{otherwise,} \end{cases}$$

with $a > b'$.

Biased agents. They belong to set $\mathcal{B}$, and prefer outcome $x = 1$ regardless of the state of the world. Hence their utility function, $v_{\mathcal{B}}$, satisfies: for every $\theta \in \Theta$,

$$v_{\mathcal{B}}(1,\theta) > v_{\mathcal{B}}(0,\theta).$$

The configuration of types, $\mathcal{C} = (X_1, \ldots, X_i, \ldots, X_n) \in \{\mathcal{B},\mathcal{U}\}^n$, summarizes the types of agents in the population; with a slight abuse of notation $X_i = \mathcal{B}$ means that agent i is of type $\mathcal{B}$, and $X_i = \mathcal{U}$ means that agent i is of type $\mathcal{U}$.

3.2 Network

Agents belong to a fixed social network that we model using an undirected graph $g = (N,E)$, with $E \subset N \times N$ as the set of links between agents. With a slight abuse of notation we denote the link between i and j in g by ij instead of $\{ij\} \in E$. The set of neighbors of agent i in g is denoted by $N_i = \{j \in N \setminus \{i\} : ij \in E\}$. A path between i and j in g, Pa$(i,j;g)$, is a sequence $i = i_0$, i_0i_1,i_1, …,i_{m-1}, $i_{m-1}i_m$, $i_m = j$. A network is connected if there exists a path between any pair of agents $(i,j) \in N \times N \setminus \{i\}$. A network is *acyclic* if there is at most one path between two agents. We assume that g is an undirected tree, i.e., an acyclic connected network. The set of undirected trees is denoted by G.

For every pair $(g,i_0) \in G \times N$, we define a *directed* i_0*-rooted-tree*, $T(g;i_0) = T$, as follows. Agent i_0 is the root of the tree: there is a directed link, $\overrightarrow{i_0j}$, from i_0 to each of her neighbors j. We say that agents in N_{i_0} are located at Level L_1 of T. Similarly, for agent $j_1 \in N_{i_0}$, there is a directed link, $\overrightarrow{j_1k}$ from j_1 to other agents k in $N_{j_1} \setminus \{i_0\}$. We say that agents in $\cup_{j \in N_{i_0}} N_j \setminus \{i_0\}$ belong to Level L_2 of T. We build subsequent levels of the directed tree T accordingly and denote by L_λ the last level of T.

Any agent i has two potential types of neighbors in T: a direct predecessor who may send him a message about the state of the world, and direct successors to whom i may send a message. We denote by $N_i^+(T)$ the set of direct successors of i in T – clearly, any agent located at L_λ has no direct successors and i_0 has no direct predecessor. The successors of j in the i_0-rooted-tree T are agents $i \neq j$ such that Pa$(i_0,i;g)$ contains agent j. The set of successors of j in T is denoted by $N_j^s(T)$.

It is useful to describe the sequence of agent types between agent i_0 and agent j in T. Let $\mathcal{S}(k) = \left\{(X_{i_\ell})_{\ell \in [\![0,k-1]\!]} : \forall \ell \in [\![0, k-1]\!], X_{i_\ell} \in \{\mathcal{B},\mathcal{U}\}\right\}$ be the set of sequences with k agent types. For example, $\mathcal{S}(2) = \{(\mathcal{B};\mathcal{B}), (\mathcal{B};\mathcal{U}), (\mathcal{U};\mathcal{U}), (\mathcal{U};\mathcal{B})\}$. Moreover, let $S_X(k) \in \mathcal{S}(k)$ be a sequence with k elements equal to X, $X \in \{\mathcal{B},\mathcal{U}\}$. For example, $S_{\mathcal{B}}(2) = (\mathcal{B};\mathcal{B})$. Finally, $\mathcal{S}(k,N\mathcal{B})$ is the set of sequences of length k where there is no $\mathcal{B}$ that is a predecessor of $\mathcal{U}$. For example, $\mathcal{S}(2,N\mathcal{B}) = \{(\mathcal{B};\mathcal{B}), (\mathcal{U};\mathcal{B}), (\mathcal{U};\mathcal{U})\}$.

3.3 Collective vote

Following Bloch et al. (2018), we assume that payoffs are determined by a probabilistic voting model. Agents have the opportunity to vote either for alternative 0, or alternative 1. Let $f(z) = z/n$ be the probability that alternative 1 is implemented when z agents vote for it. Obviously, $1 - f(z)$ is the probability that alternative 0 is implemented. This assumption rules out strategic voting by the agents.

3.4 Timing of Moves in the Benchmark Model

Stage 1. Given a network g, Nature draws three types of events:

(a) Nature independently draws the type of each agent $i \in N$. More precisely, Nature assigns to each agent i type $\mathcal{B}$ with probability b, and type $\mathcal{U}$ with probability $1 - b$. Hence, Nature draws a vector $\mathcal{C} = (X_1, \ldots, X_j, \ldots, X_n)$ where $X_j \in \{\mathcal{B}, \mathcal{U}\}$ is the realized type of agent $j \in N$;

(b) Nature draws the state of the world given that $\theta = 1$ occurs with probability $\pi \in (0, 1/2)$;

(c) Nature draws one agent, say i_0, chosen randomly and called the rooted agent. Every agent has the same probability, $\frac{1}{n}$, and Nature sends a *perfect signal* about the state of the world, $\sigma \in \{0,1\}$.

At the end of Stage 1, there is a realization of a pair $(T(g; i_0); \mathcal{C})$ which indicates for each agent what level he is at and his specific type as well as the specific type of the agent who can send him a message.

Stage 2. After agent i_0 receives $\sigma \in \{0,1\}$ from Nature, he has the opportunity to create a message $M_{i_0}(\sigma) \in \{0,1,\emptyset\}$, and sends it to agents who belong to Level L_1 of $T(g; i_0)$. For the game to be interesting, we assume that agent i_0 who obtains Nature's signal *can send any message* – in particular $M_{i_0}(0) = 1$ is a possibility.

Stage 3. Agent i_1, located at Level L_1, who receives message $M_{i_0} \in \{0,1\}$ from agent i_0 either transmits this message, $m_{i_1} = M_{i_0}$, or he blocks it, $m_{i_1} = \emptyset$.
Similarly, every agent i_k, located at Level L_k, $k \geq 2$, who receives message $m_{i_{k-1}} \in \{0,1\}$ from agent i_{k-1} at Level L_{k-1} has two possibilities: either he transmits message $m_{i_k} = m_{i_{k-1}}$ to all his neighbors, or he does not transmit any message, $m_{i_k} = \emptyset$. In short, agents located at Level L_k, $k \geq 1$, *can only pass or block the message they received.*

Stage 4. Every agent $i \in N$ votes for one of the two states of the world given the information (message or signal) he has obtained and his preferences. The outcome of the collective choice is revealed after all agents have voted.

Since every agent i is biased with probability b and unbiased with probability $(1-b)$, we associate with any sequence $S \in \mathcal{S}(k)$ a probability $w(S) = b^q(1-b)^{k-q}$, where q is the number of biased agents in sequence S.

In Stages 2 and 3, we have assumed that *only the agent who creates the message has the possibility to lie*, i.e., to transmit a message different from the one he receives. Actually this assumption is not required. Indeed, if we assume that each agent has the possibility to lie by creating their own message, we obtain results that are qualitatively the same.[5]

3.5 Information Set of Agents

Let us now define the information available to each agent $j \in N$. Recall that every agent i has a prior $\pi < 1/2$ that $\theta = 1$. Similarly, every agent $i \in N$ knows that an agent is biased with probability b. We assume that every agent $j \in N$ knows his type and the type of every agent in $N_j(g)$. It is common knowledge that all agents have this information. Finally, we explore two frameworks.

1. In the *benchmark model*, every agent $j \in N$ *knows* the architecture of the social *network* g. Note that by construction if agent j belongs to Level L_k, $k \geq 1$, then the message comes from an agent in L_{k-1}. Hence, we simplify information obtained by agent j by using pair (m_{L_k}, X) where m_{L_k}, $k \in [\![0, \lambda-1]\!]$, indicates that the message has been sent by an agent located at Level L_k, and $X \in \{\mathcal{B}, \mathcal{U}\}$ indicates the type of the agent who sent the message to j. We denote by $\rho_j = \rho(m_{L_{k-1}}, X)$ the posterior belief on $\theta = 1$ of agent j who obtains a message from an agent located at Level L_{k-1}, of type X. Similarly, with a slight abuse of notation, the posterior belief of agent j when he receives no message is $\rho_j(\emptyset)$. To simplify the presentation, we say that agent j believes message 1 he receives when $\rho_j \geq 1/2$. Only agents who receive a message know the identity of the creator of the message and therefore their distance from him – agents who do not receive any message have no information about the identity of the creator of the message.

2. In the *model with incomplete information about locations of agents*, agents do not know

[5] We present the differences between the two frameworks in the discussion section.

the architecture of the network in which they are embedded. Formally, in addition to points (a) – (c) listed in Stage 1, Nature draws an undirected tree g from G. Although they do not know the network, the agents know that each tree has the same probability of being drawn by Nature. In addition, each agent knows the type and identity of all his neighbors, who sends him the message and whether the latter is the creator of the message. Except when the creator of the message is one of his neighbors, an agent does not know the identity of the creator of the message.[6] Due to all this information, each agent is able to compute probabilities concerning his location in the network.

Note that in both frameworks, for an unbiased agent who receives the signal from Nature his posterior belief is 1 when $\sigma = 1$ and 0 when $\sigma = 0$.

3.6 Equilibrium Strategies

The strategy of each agent i describes, for each rooted agent and configuration drawn by Nature, the decisions of agent i. More precisely, when Nature draws agent i_0 and configuration $\mathcal{C}$, strategy of agent i is:

1. if $i = i_0$, then there is a mapping $M_i : \{0{,}1\} \to \{0{,}1{,}\emptyset\}, \sigma \mapsto M_i(\sigma)$, that describes the action of i when he is drawn by Nature; and
2. if $i \neq i_0$, then there is a mapping $t_i : \{0{,}1\} \to \{0{,}1{,}\emptyset\}, m \mapsto t_i(m)$ such that $t_i(m) \in \{m{,}\emptyset\}$ that describes the action of i when he does not obtain Nature's signal.

An equilibrium consists of message creation strategies, transmission strategies, and beliefs for every choice of Nature, for every agent i, such that each agent's strategy is sequentially rational given the strategies of others, and beliefs that are formed using Bayes' rule whenever possible.

4 Analysis of the Benchmark Model

Given the collective vote defined in Section 3.3, in Lemma 1 (see Appendix) we establish that in equilibrium:[7]

1. each biased agent votes for alternative 1;
2. each unbiased agent j votes for alternative 1 if $\rho_j > 1/2$, 0 if $\rho_j < 1/2$, and votes 0 or 1 with equal probability if $\rho_j = 1/2$.

[6] This assumption does not qualitatively change our results – see the discussion section.

[7] This result follows from Bloch et al. (2018).

4.1 An Introductory Example

We begin with an example that explains how agents form their beliefs.

Example 1 Let g be as shown in Figure 1. We assume that, following Stage 1 given in Section 3.4, agents 1, 2, 4, 6, 8, 11, 12, 13 and 14 colored blue belong to $\mathcal{U}$, and agents 3, 5, 7, 9 and 10 colored red belong to $\mathcal{B}$. Moreover, agent 1 is the agent who obtains signal $\sigma = 1$ from Nature. In Figure 2, we draw 1-rooted-tree $T(g;1)$ that describes the flow of messages. Agent $2 \in N_1^+(T(g;1))$ obtains the following message: $m_1 = 1$. Since agent 2 knows that agent 1 is unbiased and obtains σ from Nature, agent 2 knows that $\theta = 1$. Consider the information that agent 4 has: he knows that he is located at Level L_2, $m_2 = 1$ and agent $2 \in \mathcal{U}$. For agent 4, there are two possibilities concerning agent 1 located at Level L_0. (i) Agent 1 is biased, so $M_1 = 1$, and the posterior of agent 2 is equal to his prior $\rho_2 = \pi < 1/2$. By Lemma 1, agent 2 has no incentive to transmit the message: $t_2(M_1) = \emptyset$. (ii) Agent 1 is unbiased, $M_1 = \sigma$. Then, agent 2 transmits message $t_2(M_1) = 1$. In other words, Agent 4 knows that if $m_2 = 1$, then $\theta = 1$.

Agent 11 knows that there is only one possibility for obtaining message 1 when $\theta = 0$: agent 1 is biased and $\sigma = 0$. There are two possibilities for obtaining message 1 when $\theta = 1$: $\sigma = 1$, and agent 1 is biased or agent 1 is unbiased. Consequently, by using Bayes' rule $\rho_{11}(m_{L_1} = 1,\mathcal{B}) = \frac{\pi}{(1-\pi)b+\pi}$. By Lemma 1, agent 11 transmits message 1 – when he has a successor – if and only if $\rho_{11}(m_{L_1} = 1,\mathcal{B}) \geq 1/2$. In this example, we assume π and b are such that $\rho_{11}(m_{L_1} = 1,\mathcal{B}) < 1/2$.

Finally, agents 8 and 12 have the same information set, so they will have the same posterior. Indeed, they receive the same message $m_5 = m_{10} = 1$, the message is transmitted by the same type of agents – agents 5 and 10 are biased – and agents 8 and 12 are located at the same level, L_3. What are the posterior of agents 8 and 12? Let us examine posterior of agent 8 since posterior of agent 12 is the same. We know that the only case where $\theta = 0$ and message 1 is received by agent 8 occurs when $\sigma = 0$ and agents $1,2 \in \mathcal{B}$. Similarly, cases where $\theta = 1$ and message 1 is sent to agent 8 occurs when $\sigma = 1$, and agents $1,2 \in \mathcal{B}$ or agents $1,2 \in \mathcal{U}$ or agents $1 \in \mathcal{U}$ and $2 \in \mathcal{B}$. Consequently, $\rho_8(m_{L_2} = 1,\mathcal{B}) = \frac{\pi\left(b^2+1-b\right)}{b^2+(1-b)\pi}$. If $\rho_8 \geq 1/2$, then agent 8 transmits message 1 to his successor – if he exists.

In Example 1, posterior of agent 8 can be written as follows:

$$\begin{aligned}\rho_8 &= \frac{\pi \sum_{S\in\mathcal{S}(2,N\mathcal{B})} w(S)}{(1-\pi)w(\mathcal{B};\mathcal{B})+\pi\sum_{S\in\mathcal{S}(2,N\mathcal{B})} w(S)}\\ &= \frac{\pi(w(\mathcal{B};\mathcal{B})+w(\mathcal{U};\mathcal{B})+w(\mathcal{U};\mathcal{U}))}{(1-\pi)w(\mathcal{B};\mathcal{B})+\pi(w(\mathcal{B};\mathcal{B})+w(\mathcal{U};\mathcal{B})+w(\mathcal{U};\mathcal{U}))}.\end{aligned}$$

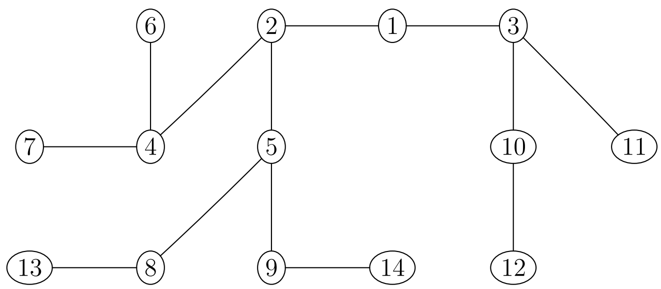


Figure 1: Network g of Example 1

How do we determine the posterior of an unbiased agent, say 14, located at Level L_4 who receives message 1 from a biased agent? There are two possibilities. Agent 14 receives message 1 when $\theta = 0$: if $\sigma = 0$ and sequence $S_{\mathcal{B}}(3) = (\mathcal{B}; \mathcal{B}; \mathcal{B})$ occurs since unbiased agents located at Levels L_1 and L_2 block message 1 that they receive from biased agents. Similarly, Agent 14 receives message 1 when $\theta = 1$ if $\sigma = 1$ and one of the following sequences occur: $(\mathcal{B}; \mathcal{B}; \mathcal{B})$ or $(\mathcal{U}; \mathcal{B}; \mathcal{B})$ or $(\mathcal{U}; \mathcal{U}; \mathcal{B})$ or $(\mathcal{U}; \mathcal{U}; \mathcal{U})$. These sequences together form the set

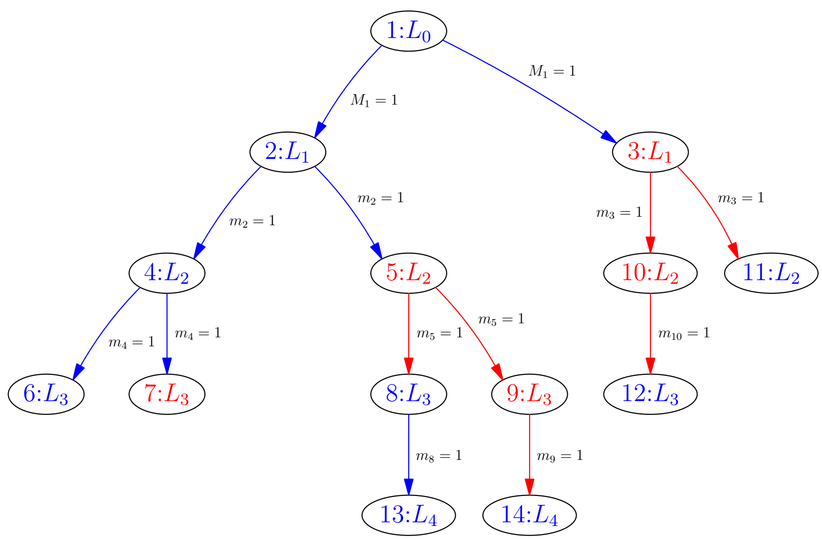


Figure 2: 1-rooted-tree, $T(g; 1)$, of Example 1

$\mathcal{S}(3,N\mathcal{B})$. Consequently, the posterior belief of agent 14 is:

$$\begin{aligned}
\rho_{14} &= \frac{\pi \sum_{S\in\mathcal{S}(3,N\mathcal{B})} w(S)}{(1-\pi)w(\mathcal{B};\mathcal{B};\mathcal{B})+\pi\sum_{S\in\mathcal{S}(3,N\mathcal{B})} w(S)} \\
&= \frac{\pi(w(\mathcal{U};\mathcal{U};\mathcal{U})+w(\mathcal{U};\mathcal{U},\mathcal{B})+w(\mathcal{U};\mathcal{B},\mathcal{B})+w(\mathcal{B};\mathcal{B};\mathcal{B}))}{(1-\pi)w(\mathcal{B};\mathcal{B};\mathcal{B})+\pi(w(\mathcal{U};\mathcal{U};\mathcal{U})+w(\mathcal{U};\mathcal{U},\mathcal{B})+w(\mathcal{U};\mathcal{B},\mathcal{B})+w(\mathcal{B};\mathcal{B};\mathcal{B}))} \\
&= \frac{\pi(w(\mathcal{B};\mathcal{B})+w(\mathcal{U};\mathcal{B})+w(\mathcal{U};\mathcal{U}))+\frac{\pi}{b}w(\mathcal{U};\mathcal{U};\mathcal{U})}{(1-\pi)w(\mathcal{B};\mathcal{B})+\pi(w(\mathcal{B};\mathcal{B})+w(\mathcal{U};\mathcal{B})+w(\mathcal{U};\mathcal{U}))+\frac{\pi}{b}w(\mathcal{U};\mathcal{U};\mathcal{U})} \\
&> \rho_8,
\end{aligned}$$

since $\frac{a+\xi}{b+a+\xi} > \frac{a}{b+a}$ for every $a,b,\xi > 0$. Let $\hat{\rho}(k)$ be the posterior of belief of agent $j \in \mathcal{U}$ located at Level L_k, $k \geq 2$, who obtains message $m_i = 1$ from $i \in \mathcal{B}$. We have:

$$\hat{\rho}(k) = \frac{\pi \sum_{S\in\mathcal{S}(k-1,N\mathcal{B})} w(S)}{(1-\pi)w(S_{\mathcal{B}}(k-1)) + \pi\sum_{S\in\mathcal{S}(k-1,N\mathcal{B})} w(S)}. \tag{1}$$

Let us start with a remark concerning function $\hat{\rho}(\cdot)$. This function will play a crucial role in the rest of the analysis.

Remark 1 *$\hat{\rho}(k)$ is increasing in k, $k \geq 2$.*

In other words, as long as the unbiased agents block message 1 received from a biased agent, the credibility of message 1 received by an unbiased agent from a biased agent increases with the distance between the former and the agent obtaining the Nature's signal. This result follows from the fact that the spread of false information through the network requires that the event "*only biased agents are involved in the path between the creator of the message and the agent*" occurs. The probability of this event decreases with the length of this path. We draw contours of $\hat{\rho}(\cdot)$ associated with pairs (b,π) for Levels L_k, $k \in [\![2,5]\!]$, in Figure 3. The posterior of message 1 is higher than $1/2$ for pairs above the curves.

4.2 Equilibrium Analysis

Our game allows for a multiplicity of equilibria – in particular babbling equilibria where messages are never transmitted will certainly exist. In the following, we are interested in a specific type of equilibrium: we examine an equilibrium in which the strategy of transmitting message 1 by unbiased agent j is consistent with the posterior beliefs of j.

Clearly, due to Remark 1, if $\hat{\rho}(\lambda-1) < 1/2$, then unbiased agent j_k located at Level L_k, $k \leq \lambda - 1$, has posterior beliefs lower than $1/2$ when he receives message 1 from a biased agent, and in equilibrium he will not transmit message 1, and will not vote 1.

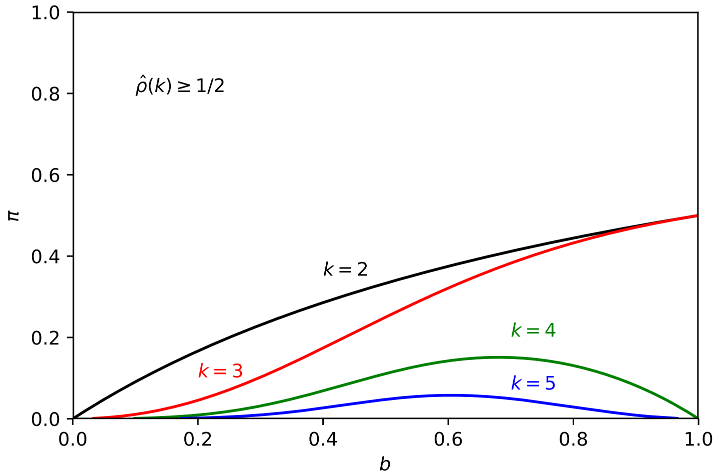


Figure 3: Contour curves of $\hat{\rho}(k) = 0.5$ associated with pairs (b,π) for $k \in [\![2,5]\!]$

Assume that $\pi \geq \bar{\pi}^{\lambda} = \frac{w(S_{\mathcal{B}}(\lambda-2))}{w(S_{\mathcal{B}}(\lambda-2))+\sum_{S\in\mathcal{S}(\lambda-2,N\mathcal{B})} w(S)}$, i.e., $\hat{\rho}(\lambda - 1) \geq 1/2$, and let $k^{\star} = \min\{k \in [\![2,\lambda - 1]\!] : \hat{\rho}(k) \geq 1/2\}$. By definition of $k^{\star}$, $L_{k^{\star}}$ is the level after which unbiased agents transmit message 1 they receive from a biased agent, while unbiased agents located at Level L_k, $k < k^{\star}$, block message 1 received from biased agents. In the following result, we focus on a specific Bayesian equilibrium that will play a crucial role in our analysis.

Theorem 1 *The following strategies constitute a – Bayesian – equilibrium.*

1. *For* $\sigma \in \{0,1\}$*:*
$$M_i(\sigma) = 1, \quad \textit{if } i \in \mathcal{B},$$
$$M_i(\sigma) = \sigma, \quad \textit{if } i \in \mathcal{U}.$$
2. *For* $i \in \mathcal{B}$*,* i *located at Level* L_k*,* $k \in [\![1, \lambda - 1]\!]$*,* $t_i(1) = 1$ *and* $t_i(0) = \emptyset$*.*
3. *For* $j \in \mathcal{U}$*,* j *located at Level* L_k*,* $k \in [\![1,\lambda - 1]\!]$*,* $t_j(0) = 0$*, and*
$$t_j(1) = \begin{cases} 1 & \textit{if } \rho_j \geq 1/2, \\ \emptyset & \textit{otherwise,} \end{cases} \tag{2}$$
with
$$\rho_j = \rho_j(m_{L_{k-1}} = 1,\mathcal{B}) = \begin{cases} \hat{\rho}(k^{\star}+1) & \textit{if } k > k^{\star}, \\ \hat{\rho}(k^{\star}) & \textit{if } k = k^{\star}, \\ \hat{\rho}(k) & \textit{if } k \in [\![2,k^{\star}-1]\!], \\ \pi & \textit{if } k = 1 \end{cases}$$

when agent who sent message 1 *to* j *is biased, and*

$$\rho_j = \rho_j(m_{L_{k-1}} = 1,\mathcal{U}) = \begin{cases} \hat{\rho}(k^\star + 1) & \text{if } k > k^\star, \\ 1 & \text{if } k \leq k^\star, \end{cases}$$

when agent who sent message 1 *to* j *is unbiased. Moreover,* $\rho_j(\emptyset) \leq \pi$*. We assume the following beliefs off the equilibrium path for agent* $j \in \mathcal{U}$*:* $\rho_j(M_{L_0} = 0,\mathcal{B}) = \rho_j(m_{L_k} = 0,\mathcal{B}) = \pi$ *for* $k \geq 1$*.*

In Theorem 1, we observe that the properties of the posteriors of the unbiased agents depend on the type of the agent that sends message 1. Specifically, the posterior of unbiased agents that receive message 1 from a biased agent, are non-decreasing with their distance from the root of the tree – and strictly increasing up to Level $L_{k^\star}$. On the other hand, the posteriors of unbiased agents that receive message 1 from an unbiased agent are non-increasing with their distance from the root.

We now illustrate that the strategies given in Theorem 1 are not the only equilibrium strategies. However, as is often the case in games of strategic communication some of these equilibria are not particularly interesting and we will ignore them in our detailed analysis. Consider for instance that the root of the rooted-tree, say i_0, is biased and all his neighbors are unbiased. Then, because of Bayes rule, message 1 sent by $i_0 \in \mathcal{B}$ will never be transmitted. Consequently, agent i_0 is indifferent between his different actions, and in equilibrium he may choose for example to not send a message. Now suppose that due to Nature's random draw, all neighbors of $i_0 \in \mathcal{B}$ are biased, and all of them do not transmit the message they obtain from i_0. In that case, i_0 plays a best response when he sends message 0, given the strategies of his neighbors, and his neighbors plays a best response given i_0 sends 0. Finally, suppose that i_0 is unbiased and that all his neighbors are biased. Then i_0 is indifferent between creating the message 0 and not creating a message when he receives $\sigma = 0$. Indeed, neighbors of agent i_0 do not transmit message 0.

This examples clearly highlight the potential for coordination failures. Note that in all of these examples, some agents are indifferent between some of their strategies because they do not affect the voting of unbiased agents and therefore, these equilibria are not interesting in terms of the behavior of agents. In the rest of the section, we focus on a class of equilibria, $\mathcal{E}$, where strategies of agents can affect the voting of unbiased agents.

Equilibria in $\mathcal{E}$ satisfy the two following properties:

(TR1) biased agents create message 1 when they obtain σ from Nature, and they transmit message 1 when they receive it;

(TR2) unbiased agents create message that matches with the signal given by Nature, and every unbiased agent j plays the following strategy: $t_j(0) = 0$, and $t_j(1) = 1$ if and only if $\rho_j \geq 1/2$.

Clearly, the specific equilibrium identified in Theorem 1 belongs to $\mathcal{E}$. Note that (TR1) and (TR2) can be interpreted as tie-breaking rules to get around coordination problems. Moreover, for agent j, whose successors are *all* biased agents, in $T(g; i_0)$ given $\mathcal{C}$, his strategy cannot modify the vote of other agents – all agents who can be influenced by the message sent by j always vote 1 regardless of what the strategy of j is. We assume as a tie-breaking rule according to which agent j chooses his strategy in line with his posterior.

In the next result, we highlight an equilibrium that plays a specific role in $\mathcal{E}$: it maximizes the transmission and the communication of message 1 between agents.

Proposition 1 *If there exists an equilibrium in $\mathcal{E}$ where an unbiased agent j believes message* 1*, then this agent j believes message* 1 *in the equilibrium described in Theorem 1.*

Our goal in this paper is to study how strategic considerations affect cheap talk, especially in the context of a network. Thus, following Bloch et al. (2018), we are interested in equilibria where the communication of message 1 is maximized. Following Proposition 1, we say that strategies and beliefs given in Theorem 1 constitute a *Maximal Communication Equilibrium* in $\mathcal{E}$.

Given g, $(T(g; i_0)$,$\mathcal{C})$ leads to a *Full Communication Rooted-Tree* if for strategies that constitute a maximal communication equilibrium – those given in Theorem 1 – *message* 1 *is believed by all unbiased agents* when they receive it, i.e., all unbiased agents vote 1, and consequently transmit messages when they can. Full communication rooted-trees are thus a very specific outcome where the spread of information, possibly false, does not stop and message 1 influences all unbiased agents. However, it is important from an agent's perspective because they do not know that they are part of a full communication rooted-tree as they are unaware of the complete draw of types or the entire vector $\mathcal{C}$.

In what follows, we examine the probability of obtaining a full communication rooted-tree given the social network g. For example, we consider social network g drawn in Figure 4.(i), called a *line network* since it is a tree where only two agents have a unique neighbor. Social network g allows for three rooted-trees drawn in Figure 4.(ii) – (iv) according to the agent drawn as the creator of the message by Nature. These rooted-trees can be combined with $2^3 = 8$ configurations of types, $\mathcal{C}$, that Nature can randomly draw. Consequently, there are $3 \times 8 = 24$ possible pairs $(T(g; i_0)$,$\mathcal{C})$. In some rooted-trees and configurations, message 1

(i) g (ii) $T(g;1)$ (iii) $T(g;2)$ (iv) $T(g;3)$

Figure 4: Tree g and its rooted trees

is not believed by all unbiased agents when they receive it while this message is believed in others. For example, in $T(g;1)$ agent 2 does not believe message 1 and does not transmit it when $1 \in \mathcal{B}$ and $2 \in \mathcal{U}$, while $T(g;1)$ is a full communication rooted-tree when $1,2,3 \in \mathcal{U}$. Our main result, Theorem 2, consists in providing the probability that Nature draws a pair $(T(g;i_0),\mathcal{C})$ such that $T(g;i_0)$ is a full communication rooted-tree. We begin by providing an additional property which ensures that $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree.

(P1) In $(T(g;i_0),\mathcal{C})$, for every agent $i_k \in N$, located at Level L_k, $k \in [\![1,k^\star-1]\!]$, the sequence of types of agents in the path between i_0 and i_k belongs to $\mathcal{S}(k+1,N\mathcal{B})$.

Property (P1) means that there is no sequence of agents in $T(g;i_0)$ such that a biased agent is a predecessor of an unbiased agent up to Level $L_{k^\star}$. Note that false information can be transmitted when (P1) is true. Consider for example the rooted-tree T given in Figure 5 and suppose $\pi \in [b^2/(2b^2-b+1),b/(1+b))$, i.e., $k^\star = 3$. We assume that agents 1, 2 and 3 are biased and Nature transmits signal $\sigma = 0$ to agent 1. Clearly, unbiased agents 4, 5 and 6 believe message 1 created and transmitted by biased agents. Note that T together with the previous specific configurations of the type of the agents lead to a full communication rooted-tree.

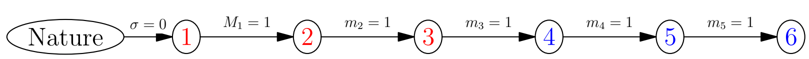


Figure 5: Rooted-tree T

Proposition 2 *Suppose $i_0 \in N$ is the agent who obtains the signal σ from Nature. $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree if and only if (P1) holds.*

Note that in Proposition 2 we provide a property under which all unbiased agents believe message 1 when they receive it. Because of the specific role played by the strategies and beliefs given in Theorem 1, we are interested in the probability of obtaining a full communication rooted-tree depending on the architecture of the social network g when such strategies and beliefs are used.

For any rooted-tree $T = T(g; i_0)$, $N(T,k)$ is the set of agents who are located at Level $L_{k'}$, $k' < k$. Moreover, let $N_i^s(T,k) = N_i^s(T) \cap N(T,k)$ be the set of agents located at $L_{k'}$, with $k' < k$, who are successors of i in T. Finally, we define a recurrence equation on the neighbors of agent $j \in N$, given rooted-tree T, that allows us to provide the probability of obtaining a full communication rooted-tree.

For each agent j located at Level L_k, we define $\Phi_j(T,k)$ as follows. For $j \in N(T, k^\star)$,

$$\Phi_j(T,k) = b^{|N_j^s(T,k^\star)|+1} + (1-b) \prod_{\ell \in N_j^+(T)} \Phi_\ell(T, k+1), \tag{3}$$

and for $j \notin N(T, k^\star)$, $\Phi_j(T,k) = 1$. We observe that $\Phi_{i_0}(T,0)$ is the recurrence equation associated with the agent who obtains signal σ from Nature, i_0 – every agent in N is a successor of i_0 in $T(g; i_0)$. The following result relies on the fact that in a full communication rooted-tree, all successors of a biased agent located at Levels L_k with $k < k^\star$, must be biased.

Theorem 2 *Given a social network g, the probability of a full communication rooted-tree occurring is:*

$$\frac{1}{n} \sum_{i_0 \in N} \Phi_{i_0}(T(g; i_0),0).$$

Moreover, the probability of a full communication rooted-tree occurring is at most:

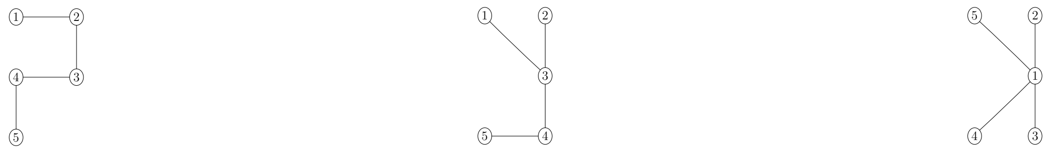
$$1 - \frac{1}{n} \sum_{i_0 \in N} b\big(1 - b^{|N_{i_0}|}\big).$$

Clearly, the architecture of the social network plays a crucial role in Equation (3), and thus in the probability of obtaining a full communication rooted-tree.

We now turn to the following question: What is the best architecture for social networks for the communication of messages? In other words, what is the architecture that allows message 1 to be believed with the highest probability by unbiased agents?

To answer this question, we define star networks as trees in which one agent, called the center agent, has a link with all the other agents and other agents, called peripheral agents, have a unique link with the center. In the first result, we restrict our attention to $\pi \geq b/(1+b)$, i.e., $k^\star = 2$. This expression also suggests that a high prior of the state being 1 implies that a false message 1 will be transmited further. Note that while $k^\star$ is endogenous, below we sometimes use $k^\star$ for ease of exposition.

Proposition 3 *Let $\pi \geq b/(1+b)$. Then, star networks are the trees that generate full communication rooted-trees with the highest probability.*



(i) g_1 (ii) g_2 (iii) g_3

Figure 6: Trees with 5 agents

It is difficult to obtain general results concerning architectures of social networks that maximize the probability of obtaining full communication rooted-trees. Indeed, such a maximizer is parameter dependent. For instance, let us examine the situation with 5 agents. We draw the three possible trees with 5 agents in Figure 6. Suppose $\pi \in [b^2/(2b^2-b+1), b/(1+b))$, i.e., $k^\star = 3$. Network g_3 maximizes the probability for obtaining a full communication rooted-tree when $b = 0.25$, while network g_1 maximizes this probability when $b = 0.5$. In the following result, we show that when agents are more likely to be unbiased than biased, there are (weak) conditions under which star networks are the trees that generate full communication rooted-trees with the highest probability.

Proposition 4 *Suppose $n > 5$, $\pi \in [\bar{\pi}^\lambda, b^3/(b^3+2b^2-2b+1))$, and $b \leq 0.5$. Then, star networks are the trees that generate full communication rooted-trees with the highest probability.*

Another important question concerns the most efficient social network architecture for reducing the spread of false information. Intuitively, star networks seem to be the social networks with the highest potential for filtering false information. In other words, they should be social networks that minimize the probability of obtaining full communication rooted-tree when the creator of the message sends a false message. This intuition is correct when $\pi \in [\bar{\pi}^\lambda, b/(1+b))$, i.e., $k^\star > 2$. However, the following example established that this intuition is not correct when $\pi \geq b/(1+b)$, i.e., $k^\star = 2$.

Example 2 Let $N = [\![1, 4]\!]$ and $\pi \geq b/(1+b)$, i.e., $k^\star = 2$. There are two possible social networks when $|N| = 4$: the line network called network g_1 (drawn in Figure 7.(i)) and the star network called network g_2 (drawn in Figure 7.(ii)). In the figure, we indicate inside the square for each agent i the probability of obtaining a full communication rooted-tree, i.e., a configuration, where no agent votes 0 when $\theta = 0$. For instance, in the star network, each rooted-tree associated with peripheral agents has a probability b^2 to be a full communication rooted-tree where all agents vote 1 when $\sigma = 0$. Indeed, since $k^\star = 2$, agent who obtains the signal from Nature and all his direct neighbors – and only them – have to be biased. Clearly, the probability of obtaining a full communication rooted-tree under these conditions is $\frac{1}{4}(2b^2 + 2b^3)$ for g_1 and $\frac{1}{4}(3b^2+b^4)$ for g_2. Clearly, $3b^2+b^4-(2b^2+2b^3) = (b(1-b))^2 > 0$. Therefore, the probability of obtaining a full communication rooted-tree is higher in the star network than in the line network.

(i) g_1

(ii) g_2

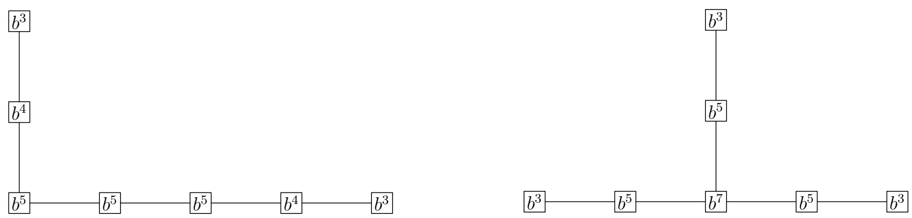
Figure 7: Diffusion of false information when $k^\star = 2$

We are able to obtain the following general result for social networks that minimize the diffusion of false information.

Proposition 5 *Trees that minimize the probability of obtaining full communication rooted-trees when message 0 is sent by Nature to a biased agent are line networks when $\pi \geq b/(1+b)$, and star networks when $\pi \in [\bar{\pi}^\lambda, b/(1+b))$.*

We now deal with social networks that maximize the diffusion of false information. We know that line networks are not the social networks that maximize the diffusion of false information when $k^\star = 2$. Actually, by using arguments similar to those given in the proof of Proposition 5, it is possible to show that *star networks maximize the diffusion of false information when $\pi \geq b/(1+b)$, i.e., $k^\star = 2$.* Intuitively, line networks seem to be the most efficient networks when $\pi \in [\bar{\pi}^\lambda, b/(1+b))$, i.e., $k^\star > 2$ for maximizing the diffusion of false information. We show through an example that this intuition is not true: the architecture of social networks that maximizes the diffusion of false information is parameter dependent.

Example 3 Suppose $N = [\![1,7]\!]$ and $\pi \in [b^2/(2b^2-b+1), b/(1+b))$, i.e., $k^\star = 3$. We consider two social networks g_1 and g_2 drawn in Figure 8. Again, we report for each agent i the probability of obtaining a full communication network, where no agent votes 0 and $\theta = 0$, when agent i obtains the Nature signal. Since $k^\star = 3$ the agent who receives the signal from Nature and all agents at distance 2 or less of the latter – and only them – have to be biased. Probability of obtaining a full communication rooted-tree in line network g_1 is $\frac{1}{7}(2(b^3+b^4)+3b^5)$ and this probability is $\frac{1}{7}(3(b^3+b^5)+b^7)$ in g_2. The sign of the difference between these probabilities depends on the sign of $2b^4 - b^3 - b^7 = (1-b)b^3(b^3+b^2+b-1)$. For $b = 1/2$, $b^3+b^2+b-1 = -1/8$, and for $b = 3/5$, $b^3+b^2+b-1 = 22/125$. It follows that the line network is not always the network that maximize the spread of false information.



(i) g_1 (ii) g_2

Figure 8: Diffusion of false information when $k^\star = 3$

In the next section, we examine how important it is for an agent to know the distance at which it receives the signal, i_0, by looking at cases where agents have only a probability distribution on the distance between themselves and agent i_0.

5 Uncertainty about Network Distance

In the previous section we assumed that the agents know the network structure. However, in practice this may not be the case: agents may not be aware of the entire network and may only have some idea about the network. We model this by assuming that the agents do not know the network structure which is tantamount to not knowing distance from the source. This allows us to study the importance of the knowledge of the network itself in the spread of false information. Formally, in this section, we assume that social network g is not given. Nature draws a social network from G, say g; each tree has the same probability of being drawn. Then, Nature reveals to each agent i:

1. the identity of the agent who sends him the message and whether this agent has received the signal from Nature; and
2. his set of neighbors $N_i(g)$.

Only agents located at Levels L_0 or L_1 know their distance from the root i_0 since agents at Level L_1 know that i_0 obtains the signal from Nature. Other agents who obtain a message do not know the identity of i_0.[8] In this section we provide conditions under which *agents who do not know where they are located in the network, believe message* 1 *when they receive it*.

5.1 Posterior Beliefs of Agents: an Example

Recall that message 1 is never blocked by an agent located at level L_k, $k \geq 2$. We begin with the following question: Given that agents located at Level L_k, $k \geq 2$, do not know their location in the network, what is the probability that agent j receives true message 1 from a biased agent, given his location? Denote by $\mathfrak{p}_2(j)$ the probability agent j assigns to the event he is located at Level L_2.

Example 4 We assume that agent $j \in \mathcal{U}$ receives message 1 from agent $i \in \mathcal{B}$, and his neighbors have not received the signal from Nature.

1. agent j thinks he is located at level L_2 with probability $\mathfrak{p}_2(j)$. In that case, the probability he receives a true message is equal to $\frac{\pi}{(1-\pi)b+\pi}$.
2. With probability $1 - \mathfrak{p}_2(j)$, agent j is located at Level L_k, with $k > 2$. Given that all agents transmit message 1, we have: $\frac{\pi\left(b^2+1-b\right)}{b^2+(1-b)\pi}$, for every $k > 2$.

Consequently, posterior beliefs of agent j when he receives message 1 from biased agent i is $\tilde{\rho}_j(\mathcal{B}) = \frac{\pi}{(1-\pi)b+\pi}\mathfrak{p}_2(j) + \frac{\pi\left(b^2-b+1\right)}{b^2+(1-b)\pi}(1 - \mathfrak{p}_2(j))$. To ensure that agent j located at Level L_k, $k \geq 2$, has an incentive to transmit message 1 sent by a biased agent, we must have $\tilde{\rho}_j(\mathcal{B}) \geq 1/2$.

By inspecting the previous example, we define the posteriors beliefs for agent $j \in \mathcal{U}$ located at Level L_k, $k \geq 2$, when he receives message 1 from a biased agent, $\tilde{\rho}_j(\mathcal{B})$, and from an unbiased agent, $\tilde{\rho}_j(\mathcal{U})$. We have:

$$\tilde{\rho}_j(\mathcal{B}) = \hat{\rho}(2)\mathfrak{p}_2(j) + \hat{\rho}(3)(1 - \mathfrak{p}_2(j)), \tag{4}$$

$$\tilde{\rho}_j(\mathcal{U}) = \mathfrak{p}_2(j) + \hat{\rho}(3)(1 - \mathfrak{p}_2(j)). \tag{5}$$

[8] In the discussion section, we present some alternative possibilities regarding the information provided to agents.

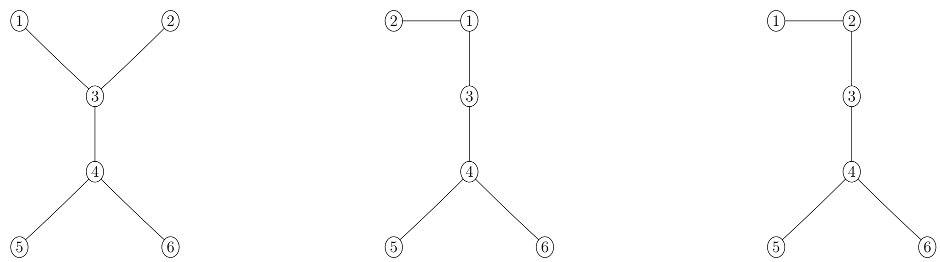
5.2 Agents' Beliefs about Their Location

In this section, we provide results that allow us to study the beliefs of agent $j \in N$ about his location in a tree, given that he knows when he is at level L_0 or L_1, who his neighbors are, and who sent him the message. As we established in the previous section, we are only interested in the beliefs of j about his probability of being located at Level L_2, $\mathfrak{p}_2(j)$. We denote the set of successors of agent j given that his neighbor i sends him the message in rooted-tree T, $N_j^s(T|i)$. Let us begin with an example that illustrates the calculation of $\mathfrak{p}_2(j)$ by agent j.

Example 5 Suppose that $N = [\![1,6]\!]$ and agent 4 is not located at Level L_0 or L_1. We assume that $N_4(g) = \{3,5,6\}$ and agent 4 receives message 1 from agent 3. Because of our assumptions, agent 4 knows that agent 3 did not obtain σ from Nature. Thus, for agent 4 there are 3 possibilities concerning the social network g that has been drawn by Nature given that $N_4(g) = \{3,5,6\}$ and it generates a rooted-tree T where agent 4 receives a message from agent 3: $N_4^s(T|3) = \{5,6\}$ or $N_4^s(T|3) = \{1,5,6\}$ or $N_4^s(T|3) = \{2,5,6\}$. Note that $N_4^s(T|3) = \{1,2,5,6\}$ is impossible because of our assumptions. In Figure 9, we draw all social networks, g_1, g_2 and g_3, that generate a rooted-tree T such that $N_4^s(T) = \{5,6\}$ and agent 4 is linked with agent 3 – by Cayley's theorem[9] (see Harary and Palmer, p. 20, 1973) there are 3 possible trees. Clearly, there is one tree, g_1 drawn in Figure 9.(i), where agent 3 has two direct neighbors. In that case, agent 4 is located at Level L_2 with probability 1. Similarly, there are two trees, g_2 and g_3 drawn in Figures 9.(ii) and 9.(iii), where agent 3 has only one direct neighbor. In these cases, agent 4 is located at level L_2 with probability 0.5 since agents 1 and 2 have the same probability to obtain the signal from Nature. Moreover, there are 2 trees, g_4, g_5, drawn in Figure 10.(i) and (ii), that generate a rooted-tree T such that $N_4^s(T|3) = \{2,5,6\}$ and $N_4(g_4) = N_4(g_5) = \{3,5,6\}$. In each of these trees, agent 1 obtained the signal with probability 1 and agent 4 is surely located at Level L_2. Finally, there are 2 trees, g_6, g_7, drawn in Figure10.(iii) and (iv), that generate a rooted-tree T such that $N_4^s(T|3) = \{1,5,6\}$ and $N_4(g_6) = N_4(g_7) = \{3,5,6\}$. Again in each of these trees, agent 4 is surely located at Level L_2. By assumption, each tree has the same probability of occurring, i.e., $1/7$. It follows that agent 4 is located at Level L_2 with probability $5 \times 1/7 \times 1 + 2 \times 1/7 \times 1/2 = 6/7$.

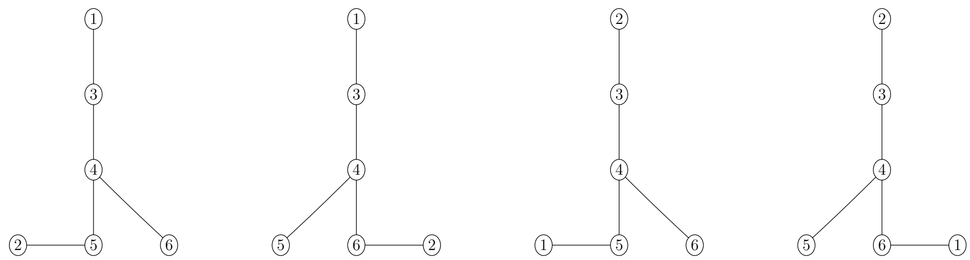
We need to present an additional graph theoretic result for obtaining $\mathfrak{p}_2(j)$ in any situation. The idea behind the calculation of $\mathfrak{p}_2(j)$ follows what we developed in the previous example, i.e., find trees compatible with the information held by agent j, then compute the probability

[9] For every positive integer n, the number of trees on n labeled vertices is n^{n-2}.



(i) g_1 (ii) g_2 (iii) g_3

Figure 9: Social networks that allow for $N_4^s(T;3) = \{5,6\}$



(i) g_4 (ii) g_5 (iii) g_6 (iv) g_7

Figure 10: Social networks that allow for $|N_4^s(T;3)| = 3$

that j is located at level L_2 in each of these trees. The following theorem allows us to determine the total number of trees where a specific agent, say i, has n_i neighbors in a tree with m agents.

Theorem 3 (Clarke, 1958) *The number of trees g with m agents in which a given agent, say i, has $n_i = |N_i(g)|$ neighbors, $n_i \in [\![1,m]\!]$, is*

$$C\ell(n_i,m) = \binom{m-2}{n_i-1}(m-1)^{m-n_i-1}.$$

Corollary 1 *Let $W \subset N \setminus \{i\}$ be a subset of agents, with $|W| = n_i$, $n_i \in [\![1,n-1]\!]$. The number of trees g in which a given node, say i, satisfies $N_i(g) = W$, is*

$$C\ell(W,m) = \frac{C\ell(n_i,m)}{\binom{n-1}{n_i}}.$$

From Corollary 1, we obtain two lemmas (2 and 3), given in the appendix, that allow us to state the main result of this section. Lemma 2 provides the number of social networks, $\Gamma_{N_j} = \sum_{v=n_j-1}^{n-3} \binom{n-1-n_j}{v-n_j+1}(n-v-1)^{n-v-3} C\ell(N_j(g)\setminus\{i\},v+1)$ with $n_j = |N_j(g)|$, that are compatible with the information obtained by agent j from Nature. In particular, (1) j knows that he receives the message from a specific agent, say i, and (2) j knows the identity of his neighbors – they belong to $N_j(g)$. In Lemma 3, we provide the number of social networks, $\Gamma_{N_j}(n_i,v) = \binom{n-1-n_j}{v-n_j+1} C\ell(n_i-1,n-v-1) C\ell(N_j(g)\setminus\{i\},v+1)$, that satisfy (1) and (2), and where agent j has v successors and $|N_i(g)| = n_i$. Lemmas 2 and 3 allow us to determine $\mathfrak{p}_2(j)$. Indeed, $\Gamma_{N_j}(n_i,v)/\Gamma_{n_j}$ provides the proportion of trees compatible with the information owned by agent j where agent i has n_i neighbors and generate a rooted-tree where the number predecessors of i is $n-v-2$. Because agent j is located at Level L_2 when the root is one of the $n_i - 1$ neighbors of i, and the number of agents candidate for being the root is $n-v-2$, the probability that agent j is located at Level L_2 is $(n_i-1)/(n-v-2)$. Finally, we have to take into account all the possible values for n_i and v. Using these arguments, in the following proposition, we state the value of $\mathfrak{p}_2(j)$.

Proposition 6 *Suppose that agent j receives message from agent i. Then,*

$$\mathfrak{p}_2(j) = \sum_{v=n_j-1}^{n-3} \sum_{n_i=1}^{n-v-1} \frac{(n_i-1)\Gamma_{N_j}(n_i,v)}{(n-v-2)\Gamma_{N_j}}.$$

5.3 Full Communication Rooted-Trees

In the following, we are interested in situations where all unbiased agents located at Level L_k, $k > 1$, believe message 1 when they receive it. Let $j' \in \arg\min_{j\in\mathcal{U}}\{\tilde{\rho}_j(\mathcal{B}) = \hat{\rho}(2)\mathfrak{p}_2(j) + \hat{\rho}(3)(1-\mathfrak{p}_2(j))\}$ be the unbiased agent who has the lowest posterior when he receives message 1 from a biased agent in the rooted-tree. Since $\hat{\rho}(\cdot)$ is increasing, $j' \in \arg\max_{j\in\mathcal{U}}\{\mathfrak{p}_2(j)\}$. Let $\tilde{\rho}_{j'}(\mathcal{B}) = \tilde{\rho}(\mathcal{B})$ and $\mathfrak{p}_2(j') = \mathfrak{p}_2$. We define the two following properties:

(P1') $\tilde{\rho}(\mathcal{B}) \geq 1/2$;

(P2') if $i_0 \in \mathcal{B}$, then every $j \in N_{i_0}(g)$ belongs to $\mathcal{B}$.

Property (P1') ensures that message 1 sent by a biased agent to an unbiased agent located at Level L_k, $k > 1$, is not blocked. Property (P2') ensures that the agent, say i_0, who obtains the signal from Nature has no neighbors who are unbiased when i_0 is biased.

Proposition 7 *If properties (P1') and (P2') hold, then $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree.*

In the rest of the section, we assume $\rho_j(\emptyset) \leq \pi$, and for agent $j \in \mathcal{U}$: $\rho_j(M_{L_0} = 0,\mathcal{B}) = \rho_j(m_{L_k} = 0,\mathcal{B}) = \pi$, for $k \geq 1$ as in Section 4. By using the same argument as in the benchmark model, we obtain the following result.

Theorem 4 *Suppose agents use strategies given in Theorem 1. Moreover, let g be the social network drawn by Nature. If $\tilde{\rho}(\mathcal{B}) \geq 1/2$, then the probability of a full communication rooted-tree occurring is:*

$$1 - \frac{1}{n} \sum_{i_0 \in N} b(1 - b^{|N_{i_0}(g)|}). \tag{6}$$

Recall that in the benchmark model, full communication rooted-trees occur with the probability given in Equation (6) only when $\hat{\rho}(2) \geq 1/2$. This is the maximum probability for obtaining a full communication rooted-tree in the benchmark model. As $\tilde{\rho}(\mathcal{B}) > \hat{\rho}(2)$ in networks with at least 3 levels, under the incomplete information model there is a larger class of parameters than in the benchmark model in which the maximum probability of obtaining a full communication rooted-tree is achieved. In the following result, we provide the conditions under which $(T(g; i_0), \mathcal{C})$ leads to a full communication rooted-tree in the model with incomplete information when it is a full communication rooted-tree in the benchmark model. Similarly, we provide conditions under which $(T(g; i_0), \mathcal{C})$ leads to a full communication rooted-tree in the benchmark model when it is a full communication rooted-tree in the model with incomplete information. We consider agents to be embedded in the same network, g, in both settings.

Proposition 8 *Suppose that agents use strategies given in Theorem 1. Moreover, Nature sent the signal to i_0, and draws configuration $\mathcal{C}$.*

1. *Suppose that $(T(g; i_0), \mathcal{C})$ leads to a full communication rooted-tree in the model with incomplete information on locations. If there is no agent $j \in \mathcal{U}$ who obtains message from agent $i \in \mathcal{B}$ located at Level L_2 in $T(g; i_0)$, then $(T(g; i_0), \mathcal{C})$ leads to a full communication rooted-tree in the benchmark model.*
2. *Suppose that $(T(g; i_0), \mathcal{C})$ leads to a full communication rooted-tree in the benchmark model. If there exists an agent $j \in \mathcal{U}$ located at level L_2 who obtains message from agent $i \in \mathcal{B}$, then $(T(g; i_0), \mathcal{C})$ leads to a full communication rooted-tree in the model with incomplete information on locations.*

The latter proposition highlights the following point. Whether or not players have information about the architecture of the social network does not determine whether it is easier to get an full communication rooted-tree since this will depend on the drawn configuration $\mathcal{C}$.

6 Discussion

Till now, we have assumed that only the agent receiving the message from Nature has the ability to transmit a false message, all other agents can only transmit or block the messages they have received. Let us examine what happens in a context where we relax this assumption and allow each agent who receives a message to modify it. First, note that the agents' strategies are modified since they must take into account the strategy that will be used by the agent to whom they send a message. For example, an unbiased agent who gets message 0 and believes it to be a true message will not send this message to a biased agent since the latter will transmit message 1. Similarly, a biased agent does not send message 1 to an unbiased agent if the latter will not believe this message since he will transmit message 0. It follows that, as in our model, message 0 cannot pass through a path that contains biased agents even in this situation. The difference is that in our model, message 0 is blocked by biased agents, while it is blocked by unbiased agents that are the predecessors of biased agents in the current context. Likewise, message 1 cannot pass through a path that contains unbiased agents who have a biased predecessor located at Level L_k, $k < k^{\star}$. Consequently, strategies we describe in Theorem 1 constitute an equilibrium and our results are preserved.[10] However, there is a slight difference. Indeed, in equilibrium, unbiased agents, located at Level L_k with $k < k^{\star}$, cannot obtain message 1 from a biased agent. Therefore, beliefs about this event must be assumptions because they are off the equilibrium path.

We have also assumed that neighbors of agent i_0, who creates the message, are aware that i_0 is the creator of the message. What happens if agent i_0's neighbors do not know that he received the signal?[11] We now address this question in our benchmark model when the social network is the line network g drawn in Figure 11 where agent 2, who is a biased agent, gets the signal.

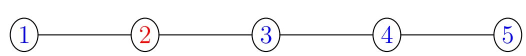


Figure 11: Line network g

[10]It is worth noting that in our model, each agent must know the type of the agent who sends him the message, but the type of his direct successors has no consequence on his decision. If each agent can lie, then each agent must know and take into account not only the type of his predecessor but also his direct successors.

[11]Under this assumption the setting is the same as the setting explored by Bloch et al. (2018) in Theorem 4, but here we deal with situations where message 1 is not transmitted.

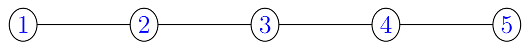


Figure 12: Line network g'

Then agent 3, who observes g, does not know whether he is at distance 1 or distance 2 from the agent who obtained σ. Since each agent has the same probability of being drawn by Nature and agent 3 does not know the type of agent 1, his posterior is equal to $\varpi = \frac{\pi}{2} + \frac{\pi}{2((1-\pi)b+\pi)}$. In other words, agent 3 assigns a credibility to the message sent by 2 that is higher than the credibility (π) he assigns when he knows that agent 2 obtains σ. Note that when agent 2 is an unbiased agent as in network g' drawn in Figure 12, agent 3 knows that the message is true. Indeed, either agent 2 obtains the signal and the message is true, or agent 1 obtains the signal and agent 2 transmits the message to agent 3 only if agent 1 is unbiased. This type of arguments can be used for agent 4 in network g' when $\varpi < 1/2$.

We now discuss the assumptions of the model with incomplete information about the location of agents. In the current paper, we have chosen to examine situations where agents, who are not neighbors, do not know the identity of the agent, say i_0, who receives the signal from Nature. By inspecting Lemmas 9 and 10, it is clear that if we assume that each agent knows the identity of i_0, the results given in Section 5 will be qualitatively the same. Similarly, it is possible to assume that agents do not know the identity of their neighbors, but only their numbers, without qualitatively changing the results. This assumption is relevant to the analysis of certain social media such as Twitter where some agents have several million followers.

Next, we have assumed that any agent can observe the event, and thus that Nature can send its signal to any of them. In some contexts, only one agent is likely to observe the event – for example, because of his spatial position. So, we examine only one rooted-tree, the one where the agent observing the event is the root. In this case, it can be established (using arguments similar to those given in the Appendix) that the i_0-rooted-trees that maximize the probability of being a full communication rooted-tree is either the star network where i_0 is the center, or the i_0-rooted-trees where there is only one agent at each Level L_k with $k \in [\![0,k^\star-1]\!]$. Moreover, the i_0-rooted-tree that minimizes the probability of being a full communication rooted-tree when i_0 sends a false message is the star network with i_0 as the center.

Finally, the models we have presented allow us to see the role played by fact-checking sites (for example sites like FactCheck in USA, CaptainFact in France and Full Fact in UK). These sites have been created by journalists who check whether a piece of information is true or not. Every – unbiased – agent can use these sites to check whether the message they receive is true.

In our model, if a biased agent obtains the signal and has unbiased neighbors with successors, then there is no possibility of obtaining a full communication rooted tree. When it is possible for unbiased agents to verify information through an exogenous mechanism, it is possible to achieve a maximal communication equilibrium in the previous situation if the reliability of the fact-checking site, i.e., the probability that its verification is correct, is higher than $1 - \pi$. In other words, sufficiently reliable fact checking is a device that can ensure that there is always a full communication rooted-tree. Finally, our paper suggests that there exists a threshold $k^\star$ based on the parameters of the model after which message 1 will be believed by all agents in the social network. A simple device to counter this might be to require that all social media platforms automatically include a link to a fact-checking website in a message that is getting forwarded many times to facilitate verification by receivers.

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7 Appendix

7.1 Maximal Communication Equilibrium

Lemma 1 [Adapted from Lemma 1, Bloch et al., 2018] *A Nash equilibrium of the voting game consists of the following strategies: Each biased agent votes for alternative* $x = 1$. *Each unbiased agent* i *votes for alternative* x, *with*

$$x = \begin{cases} 1 & \text{if } \rho_i > 1/2, \\ 0 & \text{if } \rho_i < 1/2, \end{cases}$$

and i *votes* 0 *and* 1 *with equal probability if* $\rho_i = 1/2$.

Proof of Lemma 1 This proof is a straightforward adaptation of the proof of Bloch et al. (2018, Lemma 1). At the end of the transmission stage, we consider unbiased agent i and $\mathcal{I}$ his information set. The set $\mathcal{I}$ includes other's strategies as well as the message or signal i may have received and who sent him the message. Other's vote depend on their own information but not on i's vote. Let us denote the number of agents who vote for alternative 1 by z and by $\mathrm{P}(\theta, z \mid \mathcal{I})$ their joint probability with the state as is perceived by i. Thus i's expected utility from voting for $\ell \in \{0,1\}$, $\mathbb{E}(v_{\mathcal{U}}(\tilde{x},\tilde{\theta} \mid \ell,\mathcal{I}))$, is equal to:

$$\sum_{\substack{z\in[\![0,n-1]\!]\\ \theta\in[\![0,1]\!]}} \left(\frac{z+\ell}{n} v_{\mathcal{U}}(1,\theta) + \left(1 - \frac{z+\ell}{n}\right) v_{\mathcal{U}}(0,\theta)\right) \mathrm{P}(\theta, z \mid \mathcal{I}).$$

Clearly, agent $i \in \mathcal{U}$ votes 1 if and only if $A = \sum_{z,\theta} \frac{1}{n}(v_{\mathcal{U}}(1,\theta) - v_{\mathcal{U}}(0,\theta))\mathrm{P}(\theta, z \mid \mathcal{I}) \geq 0$. We have $v_{\mathcal{U}}(1,1) - v_{\mathcal{U}}(0,1) = a - b'$ and $v_{\mathcal{U}}(1,0) - v_{\mathcal{U}}(0,0) = b' - a$. Moreover, note that $\sum_z[\mathrm{P}(1,z|I)] = \mathrm{P}(1|I)$ and $\sum_z[\mathrm{P}(0,z|I)] = \mathrm{P}(0|I)$. Consequently, we obtain that the incentives to vote 1 or 0 only depend on the sign of $2\sum_z[\mathrm{P}(1,z|I)] - 1 = 2\rho_i - 1$. The result follows. $\square$

Proof of Remark 1 We have for $k \geq 1$:

$$\begin{aligned}
\hat{\rho}(k+1) &= 1 - \frac{(1-\pi)w(S_{\mathcal{B}}(k))}{(1-\pi)w(S_{\mathcal{B}}(k))+\pi\sum_{S\in\mathcal{S}(k,N\mathcal{B})}w(S)} \\
&= 1 - \frac{(1-\pi)bw(S_{\mathcal{B}}(k-1))}{(1-\pi)bw(S_{\mathcal{B}}(k-1))+\pi(w(S_{\mathcal{U}}(k))+b\sum_{S\in\mathcal{S}(k-1,N\mathcal{B})}w(S))} \\
&> 1 - \frac{(1-\pi)bw(S_{\mathcal{B}}(k-1))}{(1-\pi)bw(S_{\mathcal{B}}(k-1))+b\pi\sum_{S\in\mathcal{S}(k-1,N\mathcal{B})}w(S))} \\
&= 1 - \frac{(1-\pi)w(S_{\mathcal{B}}(k-1))}{(1-\pi)w(S_{\mathcal{B}}(k-1))+\pi\sum_{S\in\mathcal{S}(k-1,N\mathcal{B})}w(S)} \\
&= \hat{\rho}(k).
\end{aligned}$$

Finally, $\hat{\rho}(1) = \pi < \frac{\pi}{(1-\pi)b+\pi} = \hat{\rho}(2)$ since $(1-\pi)b + \pi < 1$. $\square$

Proof of Theorem 1 We establish that strategy of every agent is a best response given strategies of others and his beliefs by using Bayes rule, and show that these beliefs are consistent with strategies. We assume that Nature sends a message to agent i_0. First, we provide beliefs obtained from strategies by using Bayes' rule.

1. For $j \in \mathcal{U}$ located at Level L_1, $\rho_j(m_{L_0} = 1{,}\mathcal{U}) = 1$, $\rho_j(m_{L_0} = 0{,}\mathcal{U}) = 0$, and $\rho_j(m_{L_0} = 1{,}\mathcal{B}) = \pi$. This result derives from strategy $M_{j_0}(\sigma) = \sigma$, $M_{i_0}(\sigma) = 1$ used by $j_0 \in \mathcal{U}$ and $i_0 \in \mathcal{B}$ located at Level L_0, and Bayes' rule.

2. For $j \in \mathcal{U}$ located at Level L_k, $k < k^\star$, we have $\rho_j(m_{L_{k-1}} = 1{,}\mathcal{B}) = \hat{\rho}(k)$. This result follows Bayes' rule and the fact that unbiased agents do not transmit message 1 obtained from a biased agent before Level $L_{k^\star}$. Indeed, by construction, for $k < k^\star$, $j \in \mathcal{U}$, $\rho_j(m_{L_{k-1}} = 1{,}\mathcal{B}) = \hat{\rho}(k) < 1/2$, and $t_j(1) = 1$ if and only if $\rho_j \geq 1/2$.

3. For $j \in \mathcal{U}$ located at Level L_k, $k \in [\![2{,}k^\star]\!]$, $\rho_j(m_{L_{k-1}} = 1{,}\mathcal{U}) = 1$. We know that unbiased agents do not transmit message obtained from biased agents before Level $L_{k^\star}$. Hence, agent $j \in \mathcal{U}$ located at Level L_k, $k \leq k^\star$, obtains message 1 from an unbiased agent only when $\theta = 1$ and sequence $S_{\mathcal{U}}(k)$ occurs.

4. For $j_{k^\star} \in \mathcal{U}$ located at Level $L_{k^\star}$, $\rho_{j_{k^\star}}(m_{L_{k^\star-1}} = 1{,}\mathcal{B}) = \hat{\rho}(k^\star)$. Indeed, we know that there is only one possibility for $j_{k^\star} \in \mathcal{U}$ to obtain $m_i = 1$, $i \in \mathcal{B}$, and $\theta = 0$: when sequence $S_{\mathcal{B}}(k^\star - 1)$ occurs – we do not take into account the biased agent who sends message 1 to agent $j_{k^\star}$. Moreover, agent $j_{k^\star}$ obtains message $m_i = 1$, $i \in \mathcal{B}$ when $\theta = 1$, in cases where one of the sequences in $\mathcal{S}(k^\star - 1{,}N\mathcal{B})$ occur and again we do not take into account the biased agent who sends message 1 to agent $j_{k^\star}$. Consequently, given strategies played by agents and by Bayes' rule, the posterior of agent $j_{k^\star}$ is $\hat{\rho}(k^\star)$.

5. For $j_k \in \mathcal{U}$ located at Level L_k, with $k \geq k^\star + 1$, we have: $\rho_{j_k}(m_{L_{k-1}} = 1, \mathcal{B}) = \rho_{j_k}(m_{L_{k-1}} = 1, \mathcal{U}) = \hat{\rho}(k^\star + 1)$. We only establish the result for $\rho_{j_k}(m_{L_{k-1}} = 1, \mathcal{B})$ since the arguments are similar for $\rho_{j_k}(m_{L_{k-1}} = 1, \mathcal{U})$. We know that no agent $j \in \mathcal{U}$ located at Level L_ℓ, $\ell < k^\star$, transmits message 1 received from a biased agent. It follows that message 1, obtained by j_k, is false when $\theta = 0$ and $k^\star$ first elements of the sequence between the root i_0 and j_k together form the sequence $S_{\mathcal{B}}(k^\star)$. Moreover, message 1, obtained by agent $j_{k^\star+1}$, is true when $\theta = 1$ and $k^\star$ first elements of the sequence between i_0 and j_k belong to $\mathcal{S}(k^\star, N\mathcal{B})$. Consequently, posterior of agent j_k is given by $\hat{\rho}(k^\star + 1)$.

We now establish that beliefs given in Theorem 1 are consistent with strategies, given Bayes' rule and strategies of agents.

1. Posterior $\rho_j(m_{L_0} = 0, \mathcal{B}) = \rho_j(m_{L_k} = 0, \mathcal{B}) = \pi$, for $k \geq 1$, provides a probability to the event: a biased agent sends message 0 to agent $j \in \mathcal{U}$. This event occurs off the equilibrium path.

2. Posterior $\rho_j(\emptyset) \leq \pi$. First, agents may not receive any message because a biased agent has blocked the transmission. Then, θ is surely equal to 0. Second, agents may not receive a message because it has been blocked by an unbiased agent. In that case, posterior of the unbiased agent who has blocked the message is strictly lower than $1/2$. Finally, the receipt of no message would occur off the equilibrium path if a biased agent fails to transmit message 1 or an unbiased agent fails to transmit a message. Posterior $\rho_j(\emptyset) \leq \pi$ is consistent with all these possibilities.

Finally, we show that strategies given in Theorem 1 constitute an equilibrium together with the previous beliefs. For $j \in \mathcal{U}$, j located at Level L_k, $k \geq 1$,

$$t_j(m_i = m, \mathcal{B}) = t_j(m_i = m, \mathcal{U}) = \begin{cases} m & \text{if } \rho_i \geq 1/2, \\ \emptyset & \text{otherwise.} \end{cases} \tag{7}$$

For every $j \in \mathcal{U}$, we have $\rho_j(m, \mathcal{X}) \geq 1/2$, for every $\mathcal{X} \in \{\mathcal{B}, \mathcal{U}\}$ j believes that $\theta = m$. By Lemma 1, j votes m. Moreover, his expected payoff increases in the number of agents who vote m. Consequently, j has an incentive to transmit m given other strategies and beliefs. We use the same arguments for strategies described in Points 1 and 2.

□

Proof of Proposition 1 We call the equilibrium obtained from beliefs and strategies given in Theorem 1 the original equilibrium. Due to (TR1) and (TR2) we know that message sent by the agent who obtains the signal from Nature uses the same strategy in an alternative equilibrium in $\mathcal{E}$ as in the original one. Moreover, strategies used by biased agents are the same in the original equilibrium and the alternative one. Recall that agent j knows the distance between him and the agent who receives the signal and the type of the neighbor who sent him the message. In any equilibrium, the posterior ρ_j is obtained by using Bayes' rule given the type of agent who sends message 1 to agent j and the location of j. Clearly, the type of agent who sends a message to j and location of agent j do not depend on the strategies of others. Moreover, due to (TR2), in any equilibrium, every unbiased agent believes and transmits message 1 *only if* his posterior is at least $1/2$. Consequently, agent j knows that unbiased agents located at Level L_k, $k < k^\star$, do not believe and transmit message 1 sent by biased agents. Hence, Bayes' rule leads to the same posterior for agent j in the original equilibrium and in any alternative one when he receives message 1. Since agent j believes and transmits a message in the alternative equilibrium, then, by Lemma 1, j increases the probability that an agent votes 1. It follows that in the alternative equilibrium the posterior of agent j, associated with his location and the type of agent who sends him message 1, ρ_j is at least $1/2$. By construction of strategies given in Theorem 1, in the original equilibrium agent j believes and transmits message 1 since $\rho_j \geq 1/2$. □

Proof of Proposition 2 Recall that we use strategies and beliefs given in Theorem 1.

Sufficiency. By (P1), every agent $j \in \mathcal{U}$ located at Level L_k, $k < k^\star$, receives message 1 from an unbiased agent. Hence, j believes and transmits message 1. Note that $\rho_j(m_{L_k-1} = 1,\mathcal{U}) = 1$. By inspecting the proof of Theorem 1, for agent $j' \in \mathcal{U}$ located at Level L_k, $k \geq k^\star$, we have $\rho_{j'}(m_{L_k-1} = 1,\mathcal{U}) \geq \rho_{j'}(m_{L_k-1} = 1,\mathcal{B}) \geq 1/2$. Hence, agent $j' \in \mathcal{U}$ always believes and transmits message 1 when (P1) is satisfied.

Necessity. Suppose that (P1) does not hold. Then, there exists an unbiased agent j, located at Level $L_{k'}$, $k' < k^\star$, who receives message m from a biased agent i. Suppose that $m = 1$. Then, $\rho_j(m_{L_{k'-1}} = 1,\mathcal{B}) = \hat{\rho}(k') < 1/2$. Hence by strategies given in Theorem 1, unbiased agent j does not believe and transmit message 1. □

Proof of Theorem 2 We prove successively the two parts of the Theorem.

1. By Proposition 2, when agents use strategies given in Theorem 1, $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree if and only if (P1) is satisfied. Every biased agent, and every unbiased agent located at Level L_k, $k \geq k^\star$, believes message 1 when he receives it. Hence, for obtaining the probability that $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree, it is sufficient to obtain the probability that every unbiased agent i, located at

Level L_k, $k < k^\star$, believes message 1 when he receives it. Due to the fact that strategies of Theorem 1 are used, we know that unbiased agent i, located at Level L_k, $k < k^\star$, believes message 1 only when he receives it from an unbiased agent. It follows that every agent i located at Level L_k, $k < k^\star$, is either a biased agent, or an unbiased agent who receives message from an unbiased agent in $T(g; i_0)$.

In Equation (3), the first term of the sum, $b^{|N_j^s(T,k^\star)|+1}$, takes into account the fact that if $j \in N(T(g; i_0), k^\star)$ is biased, then all his successors who belong to $N(T(g; i_0), k^\star)$ have to be biased in a full communication rooted-tree. Note that the location of j at Level $k^\star - 1$ imposes no restriction on the type of agents in $N_j^+(T(g; i_0))$ due to the recurrence equation. The second term of the sum, $(1 - b) \prod_{\ell \in N_j^+(T)} \Phi_\ell(T, k + 1)$, takes into account the fact that if j is not biased, then there is no restriction on the type of his successors. Product operator follows the fact that the event "agent $j' \in N$ is biased" is independent on the event "agent $j'' \in N \setminus \{j'\}$ is biased". The result follows.

2. If $(T(g; i_0)$,$\mathcal{C})$ leads to a full communication rooted-tree when $\hat{\rho}(k) \geq 1/2$, for $k > 2$, then $(T(g; i_0)$,$\mathcal{C})$ leads to a full communication rooted-tree when $\hat{\rho}(2) \geq 1/2$. Therefore, the maximal probability that $(T(g; i_0)$,$\mathcal{C})$ leads to a full communication rooted-tree occurs when $\hat{\rho}(2) \geq 1/2$.

   Clearly, when $\hat{\rho}(2) \geq 1/2$, $(T(g; i_0)$,$\mathcal{C})$ does not lead to a full communication rooted-tree if (i) $i_0 \in \mathcal{B}$, and (ii) at least one agent, say $j \in N_{i_0}(g)$, belongs to $\mathcal{U}$. This event occurs with probability $b\big(1 - b^{|N_{i_0}|}\big)$. Consequently, $(T(g; i_0)$,$\mathcal{C})$ leads to a full communication rooted-tree with probability $1 - b\big(1 - b^{|N_{i_0}|}\big)$. Since every agent $i \in N$ gets Nature's signal with the same probability, obtaining a full communication rooted-tree occurs with probability $\frac{1}{n} \sum_{i_0 \in N} \big(1 - b\big(1 - b^{|N_{i_0}|}\big)\big)$.

$\square$

Proof of Proposition 3 Note that that $\pi \geq b/(1 + b)$ implies $k^\star = 2$. Since $k^\star = 2$, agents at Level L_k, $k \geq 2$, believe message 1 they receive from any agent. Consequently, a i_0-rooted network $T(g; i_0)$ is not a full communication network if only if $i_0 \in \mathcal{B}$ and one of his neighbors is unbiased. Hence, the probability that a social network g leads to a full communication network is equal to $P(g) = \frac{1}{n} \sum_{\ell \in N} 1 - b(1 - b^{|N_\ell(g)|}) = \frac{1}{n} \big(n - \sum_{\ell \in N} b(1 - b^{|N_\ell(g)|})\big)$. Let $\hat{g}$ be a social network that is a maximizer of $P(\cdot)$, i.e., $P(\hat{g}) \geq P(g)$ for all $g \in G$. Consequently,

$$\hat{g} \in \arg\min_{g \in G} \left\{ \bar{P}(g) = \sum_{\ell \in N} b(1 - b^{|N_\ell(g)|}) \right\}.$$

To introduce a contradiction, suppose that $\hat{g}$ is not a star. Let agent i be such that $|N_i(\hat{g})| = \max_{\ell \in N}\{|N_\ell(\hat{g})|\}$. Since $\hat{g}$ is a tree that is not a star, there exist an agent j and a peripheral agent j' for whom $|N_{j'}| = 1$, such that $j' \notin N_i(\hat{g})$ and $j' \in N_j(\hat{g})$. By construction $|N_i(\hat{g})| \geq |N_j(\hat{g})|$. We build network g' that is identical to $\hat{g}$ except that we replace the link $j'j$ by $j'i$. We have

$$\begin{aligned}
\bar{P}(g') - \bar{P}(\hat{g}) &= b((1-b^{|N_i(\hat{g})|+1}) + (1-b^{|N_j(\hat{g})|-1})) \\
&\quad -b((1-b^{|N_i(\hat{g})|}) - (1-b^{|N_j(\hat{g})|})) \\
&= b(b^{|N_i(\hat{g})|} - b^{|N_i(\hat{g})|+1} + b^{|N_j(\hat{g})|} - b^{|N_j(\hat{g})|-1}) \\
&= b(b^{|N_i(\hat{g})|}(1-b) + b^{|N_j(\hat{g})|-1}(b-1)) \\
&= b(1-b)(b^{|N_i(\hat{g})|} - b^{|N_j(\hat{g})|-1}) \\
&< 0.
\end{aligned}$$

The inequality follows the fact that $|N_i(\hat{g})| \geq |N_j(\hat{g})| - 1$ and $b \in [0;1]$. Consequently, $\hat{g}$ is not a minimizer of $\bar{P}(\cdot)$, a contradiction. $\square$

Proof of Proposition 4 Note that $\pi \in [\bar{\pi}^\lambda, b^3/(b^3+2b^2-2b+1))$ implies $k^\star > 4$. Let $s_j = |N_j^s(T)|$ and $\nu(T)$ be the probability that rooted-tree T is a full communication rooted-tree (FCR). For a star network g^s, and a peripheral agent i_p, we have $\nu_p = \nu(T(g^s, i_p)) = (1-b)^2 + (1-b)b^{n-1} + b^n$, and for the central agent i_c, we have $\nu_c = \nu(T(g^s, i_c)) = (1-b) + b^n > \nu_p$. To introduce a contradiction suppose that social network g is a tree that is not a star network that maximizes the probability of a FCR. We define three types of agents in g. Peripheral agents belong to $A_1 = A_1(g)$, agents who are directly linked to at least one peripheral agent belong to $A_2 = A_2(g)$, and all other agents belong to $A_3 = A_3(g)$, A_3 is possibly empty. Since g is a tree $|A_1| \geq 2$, and since g is not a star network $|A_2| \geq 2$. The proof is divided into four main steps. First, we establish that a rooted-tree $T = T(g;i)$ that contains an agent at Level L_4 satisfies $\nu(T) < \nu_p$. Second, we show that for any rooted-tree $T = T(g;i)$ with $i \in A_3$, we have $\nu(T) < \nu_p$. Third, we establish that $\sum_{i \in A_2} \nu(T(g;i)) < \nu_c + (|A_2|-1)\nu_p$. Fourth, we show that for any rooted-tree $T = T(g;i)$ with $i \in A_1$, we have $\nu(T) < \nu_p$. Consequently, we conclude that $\sum_{i \in N} \nu(T(g;i)) < \nu_c + (n-1)\nu_p$, i.e., star networks maximize the probability of obtaining a FCR.

Consider the i-rooted tree, $T = T(g;i)$, such that there is an agent at Level L_4. Then, T is a FCR for configurations where types of agent i and his successors at Levels L_1, L_2, L_3 and L_4 belong to $\mathcal{S}(5, N\mathcal{B})$. The probability that one sequence in $\mathcal{S}(4, N\mathcal{B})$ occurs is $\sum_{\ell=0}^4 (1-b)^\ell b^{4-\ell} = (1-b)^4 + (1-b)^2 b^2 + b^4 \leq (1-b)^3 + b^4$ since $b \leq 0.5$. We have $\nu_p - \nu(T) > (1-b)^2 - ((1-b)^3 + b^4) = b(1-b)^2 - b^4 = b((1-b)^2 - b^3) > 0$ since

$b \leq 1/2$.

We establish that i-rooted trees, $T = T(g; i)$, associated with agents $i \in A_3$ are such that $\nu(T) < \nu_p$. We determine a maximum for $\nu(T)$. To introduce a contradiction suppose that there is a path i, j, j_2, j_1 in T with $j_\ell \in A_\ell$, $\ell \in \{1, 2\}$, and $j \in A_2 \cup A_3$. Note that there is no other agent in this path otherwise there exists an agent located at Level L_4 in T and $\nu(T) < \nu_p$. Similarly, j_2 is linked only with peripheral agents in the i-rooted-tree. Finally, $A_2 \cap (N \setminus N_j^s(T)) \neq \emptyset$ since in any $T(g; i_1)$, $i_1 \in A_1$, i has successors otherwise $i \notin A_3$. We build network g' that is identical to g except that the link jj_2 is replaced by the link ij_2 and j and all his successors become direct successors of j_2. Note that when $i \in \mathcal{B}$ only the configuration where all agents are biased allows $T(g; i)$ and $T' = T(g'; i)$ to be FCR since there is no agent located at Level L_4. Let $\nu_{\mathcal{U}}(T_j) \geq 0$ be the probability that T is a FCR when agent j and his successors in T are removed from T and $i \in \mathcal{U}$. Probability that T is a FCR is at most $(1-b)^2 \nu_{\mathcal{U}}(T_j)$ when j and j_2 are unbiased, $\left(b^{s_{j_2}+1}(1-b)\right) \nu_{\mathcal{U}}(T_j)$ when j is unbiased and j_2 is biased, and $b^{s_j+1} \nu_{\mathcal{U}}(T_j)$ when j is biased. In g', the probability that T' is a FCR when j_2 is unbiased is at most $(1-b)\nu_{\mathcal{U}}(T_j)$ and $b^{s_j+1}\nu_{\mathcal{U}}(T_j)$ when j_2 is biased. We have $1 - b + b^{s_j+1} > (1-b)^2 + b^{s_{j_2}+1}(1-b) + b^{s_j+1}$ since $s_{j_2} \geq 1$ and $b \leq 1/2$. Consequently, for obtaining the highest value for $\nu(T)$ we assume that there is no agents between $i \in A_3$ and any agent in A_2.

We now show that we obtain a higher value for $\nu(T)$ if we concentrate a maximal number of peripheral agents on the same agent in A_2. Suppose that $j, j' \in A_2$ and $s_j \geq s_{j'} > 1$. Let $\nu_{\mathcal{U}}(T_{j,j'})$ be the probability that T is a FCR when agents j and j' and their successors in T are removed, and $i \in \mathcal{U}$. If we increase the number of peripheral agents linked to j by one and decrease the number of peripheral agents linked to j' by one, the variation for the probability that T is a FCR is $\nu_{\mathcal{U}}(T_{j,j'})(1-b)^2(b^{s_{j'}-1} - b^{s_j}) > 0$. The last inequality follows the fact that $s_j \geq s_{j'} > 1$. We conclude that every agent in A_2 except one, say j, has one successor in T. We now establish that $\nu(T)$ increases when agent $j' \in A_2 \setminus \{j\}$ and his successor become successors of j in T. Indeed, the variation induces by this change is $\nu_{\mathcal{U}}(T_{j,j'})((1-b-(1-b)^2+(1-b)b^2+b^{s_j+1}(1-b))) = \nu_{\mathcal{U}}(T_{j,j'})(1-b)(b-b^2-b^{s_j+1}) \geq (1-b)b(1-2b) \geq 0$ since $s_j \geq 1$ and $b \leq 1/2$. Consequently, the highest value for $\nu(T)$ is obtained when $|A_2| = 2$, one agent in A_2, say j', satisfies $|N_{j'}^+(T)| = 1$ and the other agent in A_2, say j, satisfies $|N_j^+(T)| = n - 4$. Hence, we have $\nu(T) \leq (1-b)((1-b)^2 + (1-b)b^2 + (1-b)b^{n-3} + b^{n-1}) + b^n$. We have $\nu_p - \nu(T) = (1-b)^2(b - b^2 - b^{n-3}) > 0$ since $b \leq 1/2$ and $n > 5$.

Second, we deal with $\nu(T)$ for $T = T(g; i)$ with $i \in A_2$. We divide the analysis into two steps.

1. Suppose that $|A_2| \geq 3$. We start by obtaining a maximum for $B = \frac{1}{|A_2|}\sum_{\ell \in A_2} \nu(T(g;\ell))$ given that $|A_2| \geq 3$ and establish that this maximum is lower than $\frac{1}{|A_2|}(\nu_c + (|A_2| - 1)\nu_p)$. Since $|A_2| \geq 3$, there is at most one agent, say agent 1, such that all agents in A_2 are his direct neighbors. For all other agents in A_2, there exists an agent in A_2 that is not his direct neighbors, i.e., for every agent $i \in A_2 \setminus \{1\}$, there is at least one agent in A_2 located at Level L_2 in $T(g;i)$. We begin by assuming that there exists an agent in A_2, agent 1, who is linked with all other agents in A_2. First, we establish that the value associated with $\nu(T(g;i))$, $i \in A_2 \backslash \{1\}$, increases when the number of agents at Level L_1 in $T = T(g;i)$ decreases. Suppose that there are two agents $1, j \in A_2$ located at Level L_1 in T. Then $\nu_{\mathcal{U}}(T_{1,j})((1-b)^2+(1-b)b^{s_1+1}+(1-b)b^{s_j+1}+b^{s_1+s_j+2})$ is the probability that T is a FCR when $i \in \mathcal{U}$. Let g' be the network identical to g except that agents in $N^s_j(T) \cup \{j\}$ belong to $N^s_1(T(g';i))$. Probability that $T' = T(g';i)$ is a FCR when $i \in \mathcal{U}$ is $\nu_{\mathcal{U}}(T_{1,j})((1-b) + b^{s_1+s_j+2}) \geq \nu_{\mathcal{U}}(T_{1,j})((1-b)^2 + (1-b)b^{s_1+1} + (1-b)b^{s_j+1} + b^{s_1+s_j+2})$ since $s_1, s_j \geq 1$ and $b \leq 1/2$. By using the same argument, we establish that the value associated with $\nu(T(g;i))$, $i \in A_2 \setminus \{1\}$, increases when the number of agents at Level L_2 decreases – hence, the maximal value of $\nu(T(g;i))$ is obtained when there is at most one agent at Level L_2, i.e., when $|A_2| = 3$. Then, we use the same arguments as for $i \in A_3$ to establish that we obtain the highest value of $\nu(T(g;1))$ when agent 1 is directly linked to any agent in A_2 and when $|A_2| = 3$. Consequently, for maximizing B, we assume that g satisfies $A_2(g) = \{1,2,3\}$, $A_3(g) = \emptyset$, and $N_1(g) \supset \{2,3\}$. Suppose now that $s_2 \geq s_3$ and let g' be identical to g except that we replace the link between one agent, say j, in $N_3(g) \setminus \{1\}$ and agent 3 by a link between j and agent 2. By using the same arguments as for $i \in A_3$, $\nu(T(g',1)) - \nu(T(g,1)) \geq 0$. Moreover, we have $\nu(T(g';2)) + \nu(T(g';3)) - (\nu(T(g;2)) + \nu(T(g;3))) = (1-b)^3(b^{s_3} - b^{s_2+1}) + (1-b)^2\left(b^{n-s_2-1} - b^{n-s_3}\right) \geq 0$ since $s_2 \geq s_3$. Consequently, for maximizing B, we assume that g satisfies $A_2(g) = \{1,2,3\}$, $A_3(g) = \emptyset$, $N_1(g) \supset \{2,3\}$ and $|N_3(g)| = 2$. Clearly, $\nu(T(g;1)) + \nu(T(g;2)) + \nu(T(g;3))$ is equal to

$$\begin{aligned} f(s_1,s_2) &= 3((1-b)^3 + b^n) + 2(1-b)^2(b^2 + b^{s_2+1}) \\ &\quad +(1-b)\left(b^{n-s_1-1} + b^{n-s_2-1} + b^{n-1}\right), \end{aligned}$$

Suppose that $s_2 \geq s_1$. Then $f(s_2,s_1) - f(s_1,s_2) = 2(1-b)^2(b^{s_1+1} - b^{s_2+1}) \geq 0$. Consequently, to obtain a maximum for B, we assume that $s_2 < s_1$. Moreover, let us move one agent from $N_2(g)$ to $N_1(g)$. We have $f(s_1+1, s_2-1) - f(s_1,s_2) > 0$. It follows that a maximum for B is obtained in a network g where $A_2(g) = \{1,2,3\}$,

$A_3(g) = \emptyset$, $N_1(g) \supseteq \{2, 3\}$ and $|N_1(g) \cap A_1| = n - 5$, $|N_2(g)| = |N_3(g)| = 2$. Let us now establish that B associated with g is lower than $1/3(\nu_c + 2\nu_p)$. We have $\nu_c - \nu(T(g; 1)) = (1 - b) - ((1 - b)^3 + 2(1 - b)^2 b^2 + (1 - b)b^4) \geq (1 - b)(1 - (1 - b)^2 - (1 - b)b - b^4) = (1 - b)b(1 - b^3) > 0$ since $b \leq 1/2$. Moreover, we have $\nu_p - \nu(T(g; 2)) = \nu_p - \nu(T(g; 3)) \geq (1 - b)^2 - (1 - b)^3 - (1 - b)^2 b^2 - (1 - b)b^{n-2} \geq (1-b)^2(b-b^2-b^{n-3}) > (1-b)^2 b(1-2b^2) > 0$ since $b \leq 1/2$ and $n > 5$. Let us increase the number of agents in $A_2(g)$. By reiterating the same arguments, we establish that $\nu(T(g; 1))$ decreases, i.e., $\nu(T(g; 1)) < \nu_c$, and for any $i \in A_2 \setminus \{1\}$, $\nu(T(g; i)) < \nu_p$. It follows that $\sum_{i \in A_2} \nu(T(g; i)) < \nu_c + (|A_2| - 1)\nu_p$.

Let us now assume that there does not exist an agent in A_2 that is linked with all other agents in A_2. Then, for every agent $i \in A_2$, there exists at least an agent located at Level L_3 in $T(g; i)$. Note that for agent $i \in A_2$ for which there exists at least one agent located at Level L_4 in $T(g; i)$, we know that $\nu(T(g; i) < \nu_p$. By restricting our analysis to agents i in A_2 such that there is no agent located at Level L_4 in $T(g; i)$, and by using the same arguments as in the case where there exists a unique agent in A_2 who is linked with all other agents in A_2, we obtain the result.

2. Suppose that $|A_2| = 2$, $A_2 = \{i, j\}$. First, by using the same arguments as in the previous points, $\nu(T(g; i)) + \nu(T(g; j))$ increases if both agents in A_2 are directly linked. Similarly, $\nu(T(g; i)) + \nu(T(g; j))$ increases when we concentrate peripheral agents on i or j. Indeed, let $s_i \geq s_j > 1$, if we replace a link between j and one of its successor j' by a link ij' in g', we obtain $\nu(T(g'; i)) + \nu(T(g'; j)) - \nu(T(g; i)) + \nu(T(g; j)) = (1 - b)^2(b^{s_j} - b^{s_i+1}) > 0$. Hence, $\nu(T(g; i)) + \nu(T(g; j))$ is bounded above by $2(1 - b)^2 + (1 - b)(b^2 + b^{n-2}) + 2b^n$. We obtain $\nu_c + \nu_p - (\nu(T(g; i)) + \nu(T(g; j))) \geq (1 - b) + (1 - b)b^{n-1} - ((1 - b)^2 + (1 - b)(b^2 + b^{n-2})) = b(1 - b)^2(1 - b^{n-3}) > 0$ since $b \leq 1/2$ and $n > 5$.

Finally, we establish that for $i \in A_1$, $\nu(T(g; i)) < \nu_p$. Given that there are at least two agents in A_2, say j and j', $\nu_p - \nu(T(g; i)) = (1-b)^2 - ((1-b)^3 + (1-b)^2 b^2) \geq b(1-b)^2(1-b) > 0$. The result follows. □

Proof of Proposition 5 Note that $\pi \geq b/(1 + b)$ implies $k^\star = 2$, and $\pi \in [\bar{\pi}^\lambda, b/(1 + b))$ implies $k^\star > 2$.

Suppose $k^\star = 2$. Assume that Nature sends message 0 to $i_0 \in \mathcal{B}$. Then, in $T(g; i_0)$, message 1 is believed by all unbiased agents if and only if all agents in $N_{i_0}(g)$ are biased. Hence, the probability that all unbiased agents believe message 1 in $T(g; i_0)$, when $\theta = 0$, is

$(1-\pi)b^{|N_{i_0}(g)|+1}$. Therefore, the probability that a social network g leads to a full communication network when Nature sends 0 to a biased agent is $P'(g) = (1-\pi)\sum_{\ell \in N} b^{|N_\ell(g)|+1}$. Let $\hat{g} = \arg\min_{g \in G}\{P'(g)\}$. We will establish that $\hat{g}$ is a line network. To introduce a contradiction, suppose that $\hat{g}$ is a tree that is not a line network. Let agent i be such that $|N_i(\hat{g})| = \max_{\ell \in N}\{|N_\ell(\hat{g})|\}$ since $\hat{g}$ is not a line network, $|N_i(\hat{g})| \geq 3$. There exist agents j and j' such that $j' \in N_i(\hat{g})$, $j' \notin N_j(\hat{g})$ and $N_j(\hat{g}) = 1$ since $|N_i(\hat{g})| = \max_{\ell \in N}\{|N_\ell(\hat{g})|\}$. To summarize $|N_i(\hat{g})| \geq 3$ and $|N_j(\hat{g})| = 1$. We build network g' that is identical to $\hat{g}$ except that we replace the link $j'i$ by $j'j$. We obtain $P'(g') - P'(\hat{g}) = (1-\pi)(1-b)(b^{|N_i(\hat{g})|} - b^2) < 0$ since $|N_i(\hat{g})| \geq 3$, a contradiction.

Suppose $k^\star > 2$. When the central agent in the star, i_c, is biased and receives 0 from Nature, no unbiased agent who receives message 1 from i_c believes it. Therefore, the only configuration where no (unbiased) agent votes 0 is the one where all agents are biased. When a peripheral agent in the star, i_p, is biased and receives 0 from Nature, no unbiased agent located at Levels L_1 or L_2 who receives message 1 from a biased agent believes it. Therefore, the only configuration, where no unbiased agent votes 0, is the one where all agents are biased. Clearly, in any rooted-tree, the configuration where all agents are biased leads to a full communication rooted-tree. Hence, star networks are the trees that minimize the probability of obtaining full communication rooted-trees when signal 0 is sent to a biased agent. □

7.2 Incomplete information on location of agents

Proof of Corollary 1 By Theorem 3, we know that there are $C\ell(n_i,m)$ trees where the number of neighbors of agent i is n_i. These neighbors have to be chosen from W, with $|W| = n_i$, whereas they are chosen in $N \setminus \{i\}$ in Clarke's theorem. The result follows. □

Lemma 2 *The number of trees g that generate at least one rooted-tree T where (1) agent j obtains message from i, and (2) agent j is located at level L_k, $k \geq 2$ in T, is*

$$\Gamma_{N_j} = \sum_{\ell=n_j-1}^{n-3} \binom{n-1-n_j}{v-n_j+1}(n-v-1)^{n-v-3} C\ell(N_j(g) \setminus \{i\}, v+1),$$

with $v \leq n-3$.

Proof Let $V \subseteq N \setminus \{i,j\}$, $V \supseteq N_j \setminus \{i\}$, be a subset of agents with $v = |V| \in [\![n_j - 1, n-3]\!]$. This set is identified to the set of successors of j in a rooted-tree.

1. We compute the number of trees g that generate at least one rooted-tree T such that $N_j^s(T|i) = V$, and agent j obtains message from i.

   (a) We are interested in trees that restrict the population of agents to j and his successors. By Corollary 1, there are $C\ell(N_j(g) \setminus \{i\}, v+1)$ trees g with the following properties (I) whose set of agents is $V \cup \{j\} \subseteq N \setminus \{i\}$, and ($II$) where agent i is a predecessor of j.

   (b) We are interested in trees that restrict the population of agents to $N \setminus (V \cup \{j\})$. By Cayley's theorem, there are $(n - v - 1)^{n-v-3}$ trees whose set of agents is $N \setminus (V \cup \{j\})$.

   (c) It follows from (I) and (II) that the number of trees g, built with the entire population of agents, that generate at least one rooted-tree T where $N_j^s(T|i) = V$, and there is a link between i and j is

$$(n - v - 1)^{n-v-3} C\ell(N_j(g) \setminus \{i\}, v+1).$$

2. Given that the set $V \setminus (N_j(g) \setminus \{i\})$ of agents is chosen among $n - 1 - n_j$ agents, there are

$$\binom{n-1-n_j}{v-n_j+1} (n - v - 1)^{n-v-3} C\ell(N_j(g) \setminus \{i\}, v+1)$$

   trees g that generate a rooted-tree T where (I) $|N_j^s(T|i)| = v$, and (II) j obtains message from i. Indeed, there are no trees g_1 and g_2 that satisfy the properties given in point 1 (a) – (b) that are isomorphic when the rooted-tree associated with g_1 and g_2 respectively $T_1 = T(g_1; i)$ and $T_2 = T(g_2; i)$ satisfy $N_j^s(T_1|i) \neq N_j^s(T_2|i)$. Indeed by construction, there exists an agent, say ℓ, such that $\ell \in N_j^s(T_1|i)$ and $\ell \notin N_j^s(T_2|i)$. Clearly, the path between i and ℓ goes through j in g_1, while this path does not go through j in g_2, hence g_1 and g_2 are not isomorphic.

We now observe that $v \leq n - 3$. Indeed, if $v \geq n - 2$, then agents i or j obtains signal σ and agent j knows that he is located at Level L_0 or L_1. Consequently, $v \in [\![n_j - 1, n - 3]\!]$ and the result follows.

□

Lemma 3 *The number of trees g that generate a rooted-tree T where (a) agent j obtains message from i, (b) $|N_j^s(T|i)| = v$, $v \in [\![n_j - 1, n - 3]\!]$, and (c) agent i has $n_i \in [\![2, n - v - 1]\!]$ neighbors, is*

$$\Gamma_{N_j}(n_i,v) = \binom{n-1-n_j}{v-n_j+1} C\ell(n_i-1,n-v-1) C\ell(N_j(g)\setminus\{i\},v+1).$$

Proof The proof is similar to the proof of Lemma 2 except for part 1.(b). Indeed, we have to compute the number of trees whose set of agents is $N \setminus (V \cup \{j\})$, given that agent $i \in N \setminus (V \cup \{j\})$, and i has $n_i - 1$ neighbors since agent $j \in N_i(g)$. By Clarke's theorem, the number of these trees is $C\ell(n_i - 1,n - v - 1)$. The result follows. $\square$

Proof of Proposition 6 The probability that Nature draws a tree that satisfies properties (a) – (c) given in Lemma 3 among trees that satisfy properties (a) – (b) given in Lemma 2 is $\Gamma_{n_j}(n_i,v)/\Gamma_{n_j}$. Moreover, in each of these trees g, agent j knows that he is located at Level L_2 if the signal σ is obtained by an agent in $N_i(g)\setminus\{j\}$. Moreover, j knows that the number of agents who can obtain signal σ is $n-v-2$ since agent i cannot obtain the signal. Indeed, when i gets the signal, agent j is located at Level L_1 and knows it. It follows that agent j knows that there is a probability $(n_i - 1)/(n - v - 2)$ that he is located at Level L_2 in rooted-trees T where the number of predecessors of i is $n - v - 2$. The result follows. $\square$

Proof of Proposition 7 We consider strategies and beliefs given in Theorem 1, $\rho_j(\emptyset) \leq \pi$, and for agent $j \in \mathcal{U}$: $\rho_j(M_{L_0} = 0,\mathcal{B}) = \rho_j(m_{L_k} = 0,\mathcal{B}) = \pi$, for $k \geq 1$. Due to (P2') the posterior beliefs of unbiased agents, located at Level L_1 and who can transmit the message to at least one other agent in $T(g; i_0)$, are equal to 1. By (P1') the posterior beliefs of every unbiased agent j, located at Level L_k, $k \geq 2$, who receives message 1 from a biased agent, is $\tilde{\rho}_j(\mathcal{B}) \geq \tilde{\rho}(\mathcal{B}) \geq 1/2$. Similarly, the posterior beliefs of every unbiased agent j, located at Level L_k, $k \geq 2$, who receives message 1 from an unbiased agent, is $\tilde{\rho}_j(\mathcal{U}) \geq \tilde{\rho}_j(\mathcal{B}) \geq \tilde{\rho}(\mathcal{B}) \geq 1/2$. It follows that the strategies given in Theorem 1 lead every unbiased agent located at Level L_k, $k \geq 2$, to transmit message 1 when he receives it. The result follows. $\square$

Proof of Theorem 4 The posterior of unbiased agent j, located at Level L_k, $k \geq 2$, who receives message 1 from a biased agent i, is given by Equation (4). Similarly, the posterior of unbiased agent j, located at Level L_k, $k \geq 2$, who receives message 1 from an unbiased agent i, is given by Equation (5). We have $\tilde{\rho}_j(\mathcal{U}) \geq \tilde{\rho}_j(\mathcal{B}) \geq \tilde{\rho}(\mathcal{B}) \geq 1/2$. Due to strategies given in Theorem 1, unbiased agents located at Level L_k, $k \geq 2$, who receive message 1 believe it. It follows that $(T(g; i_0),\mathcal{C})$ leads to a full communication rooted-tree when the following event does not occur: the agent who obtains the signal from Nature, i_0, is biased, and there is at least one unbiased agent in $N_{i_0}(g)$. The result follows.

$\square$

Proof of Proposition 8 We successively prove the two parts of the proposition.

1. Since $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree in the model with incomplete information on locations, message 1 is believed by unbiased agents located at Level L_1. Further, for any unbiased agent j located at Level L_k, $k \geq 2$, who receives message 1 from a biased agent i, we have $1/2 \leq \tilde{\rho}(\mathcal{B})$. Therefore, for every unbiased agent located at Level L_k, $k > 2$, we have $1/2 \leq \tilde{\rho}(\mathcal{B}) = \mathfrak{p}_2\hat{\rho}(2) + (1-\mathfrak{p}_2)\hat{\rho}(3) \leq \hat{\rho}(k)$. The last inequality follows from Remark 1: $\hat{\rho}(\cdot)$ is increasing. By Theorem 2, we have for every agent j located at Level L_k, $k > 2$, $\hat{\rho}(k) = \rho_j(m_{L_k} = 1,\mathcal{B}) \leq \rho_j(m_{L_k} = 1,\mathcal{U})$ in the benchmark model. It follows that posterior of every unbiased agent j is at least equal to $1/2$ when he receives message 1 in the benchmark model. By using strategies and beliefs given in Theorem 1, the result follows.

2. Since $(T(g;i_0),\mathcal{C})$ leads to a full communication rooted-tree in the benchmark model, no agent located at Level L_1 blocks message 1. Moreover, since there exists agent j_2, located at Level L_2, who obtains message from a biased agent, we have $\rho_{j_2} = \hat{\rho}(2) \geq 1/2$. Further, we have for every $j \in \mathcal{U}$, located at Level L_k, $k \geq 2$, who receives message 1 from biased agent, $\tilde{\rho}(\mathcal{B}) = \mathfrak{p}_2\hat{\rho}(2) + (1-\mathfrak{p}_2)\hat{\rho}(3) \geq \hat{\rho}(2)$ when every unbiased agent transmits message 1. Moreover, for every $j \in \mathcal{U}$, located at Level L_k, $k \geq 2$, who obtains message 1 from agent v_j, $\tilde{\rho}(\mathcal{U}) \geq \tilde{\rho}(\mathcal{B})$. Hence, the posterior of every agent $j \in \mathcal{U}$ is at least equal to $1/2$ in the model with incomplete information on locations. By using strategies and beliefs given in Theorem 1, the result follows

$\square$