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Strategic information transmission despite conflict

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Abstract We analyze a standard Crawford and Sobel's (1982) cheap talk game in a two-dimensional framework, with uniform prior, quadratic preferences and a binary disclosure rule. Information might be credibly revealed by the Sender to the Receiver when players are able to strategically set aside their conflict. We exploit the few symmetries of the game parameters to derive multiple continua of equilibria, when varying the Sender's bias over the entire euclidean space. In particular, credible information might be revealed whatever the bias. Then we show that the equilibria exhibited characterize the game's full set of pure strategy equilibria.

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1 Introduction

Conflicts of interest prevail in most economic activities: for instance, employees and managers conflict over the employee's optimal effort, managers and shareholders conflict over the optimal allocation of capital, government and firms conflict over policy (hence the existence of lobbyists) and firm objectives (hence the existence of regulators), and so on. In this context, Barnard (1938)[1] distinguishes two ways of doing things:

> An organization can secure the efforts necessary to its existence, then, either by the objective inducements it provides or by changing states of mind. [...] We shall call the processes of offering objective incentives "the method of incentives"; and the processes of changing subjective attitudes "the method of persuasion".

Though monetary transfers are central to the design of incentives in organizations, there are a significant number of decisions that do not involve *direct* costs or benefits. In many situations, information is at the core of the decision-making process.

In this perspective the seminal work of Crawford and Sobel (1982) offers some guidelines related to the strategic aspects of information transmission between economic agents.[2] Crawford and Sobel (1982) study the effect of a conflict of interest on the influence that an informed agent (the Sender, she) might have on the action

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[1] Cited by Laffont and Martimort (2002, p. 12).
[2] See for instance Sobel (2013) for a review of the literature.

of an uninformed agent (the Receiver, he). The Sender possesses private information on a state of the world. She sends a message to the Receiver, who then takes an action. Players conflict over their preferred action. The difference is the Sender's *bias*. It provides reasons for the Receiver to be skeptical about the information disclosed. So the disclosed information might not influence his action. Crawford and Sobel (1982) show that multiple influential equilibria are associated with a limited bias. These authors characterize all equilibria. Equilibria are differentiated by their informativeness. An increased bias diminishes both the number of equilibria and their informativeness. A large bias precludes the possibility of influence.

In Crawford and Sobel's (1982) model, states and actions are one-dimensional. However many real-life interactions are multi-dimensional. In that case communication might not be restricted to communicating in each dimension separately. Spillover effects may arise across several dimensions, making analysis more complex. So in the multi-dimensional framework, results in the literature only refer to existence conditions. In the present paper we study an extension of Crawford and Sobel's (1982) model to include two dimensions under sharp assumptions which enable equilibria to be fully characterized. In particular, we consider two messages for the Sender's disclosure strategy,[3] and we assume quadratic preferences for the players, and a uniform distribution over $[0,1]^2$ for the states. These assumptions rule out the one-dimensional multiplicity of equilibria in each dimension, make the analysis tractable, and in our view provide non-trivial results.

To support the economic relevance of our study, consider for instance a manager who holds private information (θ_1, θ_2) concerning two tasks an employee has to carry out. Conditional on her private information, the manager makes a recommendation to the employee. The recommendation takes the form of a message m, chosen from a set of two alternatives m_1 and m_2. Upon observing m, the employee makes an inference concerning (θ_1, θ_2), and exerts efforts a_1 and a_2 on the respective tasks.[4] Let us assume that the employee prefers to adjust his efforts to be as close as possible to the state (θ_1, θ_2), whereas the manager prefers the employee to exert efforts as close as possible to $(\theta_1 + b_1, \theta_2 + b_2)$, where $(b_1, b_2) \in \mathbb{R}^2$ is the manager's bias.[5] We are interested in characterizing the associations of states (θ_1, θ_2) with messages m_1 or m_2 that influence the efforts. Strategically recommendation m_i results from the manager's anticipation of the efforts obtained, which in turn result from the information revealed.

For instance, if b_1 and b_2 are substantial, then the manager is not able credibly to differentiate between "low" (θ_1, θ_2)s and "high" (θ_1, θ_2)s. She always prefers the employee to perceive a high value of (θ_1, θ_2) in order to induce the greatest efforts. Hence, regardless of the state, it is always in her interest to deviate from the lower recommendation to the higher one. Yet, as shown by Chakraborty and Harbaugh (2007), comparative statements between θ_1 and θ_2 might be credible. If the manager recommends "$\theta_1 \geq \theta_2$" (m_1), then the employee simultaneously perceives a high value for θ_1 and low one for θ_2. This induces him to make a great effort $a_1|m_1 = \mathbb{E}[\theta_1|m_1]$ and a lower one $a_2|m_1 = \mathbb{E}[\theta_2|m_2]$. The symmetry of the state distribution implies that the alternative symmetric recommendation "$\theta_1 < \theta_2$" (m_2) induces symmetric efforts $a_1|m_2 = a_2|m_1$ and $a_2|m_2 = a_1|m_1$. The Sender prefers recommendation m_1 to m_2 whenever $(a_1, a_2)|m_1$ is closer than $(a_1, a_2)|m_2 = (a_2, a_1)|m_1$ to her most highly preferred action $(\theta_1 + b_1, \theta_2 + b_2)$. So she prefers m_1 whenever (θ_1, θ_2) satisfies $\theta_1 + b_1 \geq \theta_2 + b_2$.

[3] Many information disclosures are binary, for instance: endorsement, labeling, hiring, promotions, voting, court decisions. Moreover, the bit is the elementary element of information. So one could consider breaking down complex information disclosures into multiple binary ones.

[4] Alternatively the manager makes one of two *decisions*, which only rely on her private information and the different efforts they potentially induce.

[5] For instance, if $b_1 > b_2$, the manager has an exogenous asymmetric interest concerning the tasks, and *ceteris paribus* prefers a higher effort in the first task relative to the second. Such preferences could represent some favoritism from the manager concerning one of the tasks, in line with the literature on multi-tasks in organizations (Dewatripont et al., 2000).

In particular, if $b_1 = b_2$, then she has incentives to reveal m_1:"$\theta_1 \geq \theta_2$" or m_2:"$\theta_1 < \theta_2$" truthfully. This makes her recommendation to the employee credible.

We start by noting that the result obtained by Chakraborty and Harbaugh (2007) can be extended to cover multiple directions. For instance, if $b_1 = -b_2$, the manager might truthfully reveal whether $\theta_1 + \theta_2 \geq 1$ or not. More generally if Sender's preferences are symmetric relative to a symmetry of the distribution of states, then the symmetry of the two Receiver's posterior beliefs (given by $\mathbb{E}[(\theta_1, \theta_2)|m_i]$, $i \in \{1,2\}$) results from and conditions the symmetry of the Sender's recommendations. So when the state distribution has multiple symmetries, each one might support the possibility of credible recommendations, conditional on a corresponding alignment with the bias. Given the four axial symmetries of the uniform distribution over $[0,1]^2$, we obtain four *symmetric equilibria*, with each one associated with a family of symmetric bias.

In any symmetric equilibrium, information is revealed and processed strictly orthogonally to the dimension of the bias (*e.g.* when $b_1 = b_2$ the revealed sign of $\theta_1 - \theta_2$ says nothing about $\theta_1 + \theta_2$). So the bias has no effect in the informative dimension, and no information is revealed in the bias dimension. However, according to Crawford and Sobel (1982), the existence of a limited bias in a specific dimension allows information revelation in that dimension. If players plays a one-dimensional game with a limited bias, they have a strategy which aligns their choices. This mechanism does not simply extend to the multi-dimensional setup insofar as any adjustment of a player's strategy in one dimension prompts the other player's best response to vary in multiple dimensions. We show that in a symmetric equilibrium, this cross dimensional effect does not arise. The uniform distribution over $[0,1]^2$ renders a statistical independence relation between the informative and uninformative dimensions of a symmetric equilibrium. Accordingly, players might play a one-dimensional information-transmission game in the informative dimension of a symmetric equilibrium. We thus obtain four continua of *deviated symmetric equilibria*. They substantially extend the set of biases that allow players to reach agreement. The set of biases is extended in the orthogonal dimension of each of the symmetric biases associated with the four symmetric equilibria.

Crucial to the existence of these deviated symmetric equilibria is the possibility of being informative along a dimension, while remaining silent orthogonally to that dimension. The uniform distribution over $[0,1]^2$ lacks the independence required for this to occur for any random pair of orthogonal dimensions. However, we show next that given *any* dimension, players have the opportunity to strategically choose *some* information to be revealed along this dimension so that the recommendations are uninformative in its orthogonal dimension. Hence the uniform distribution presents a sufficiently weak dependence between the dimensions of a random pair of orthogonal dimensions. For instance, if the manager reveals asymmetrically and truthfully whether or not $\theta_1 \geq \theta_2/2$, then the Receiver infers $(\mathbb{E}[\theta_1|m_1], \mathbb{E}[\theta_2|m_1]) = (11/18, 4/9)$ and $(\mathbb{E}[\theta_1|m_2], \mathbb{E}[\theta_2|m_2]) = (1/6, 2/3)$, so that no information is revealed concerning $\theta_1/2 + \theta_2$ (we have $\mathbb{E}[\theta_1|m_i]/2 + \mathbb{E}[\theta_2|m_i] = 3/4$ for $i \in \{1,2\}$). This defines a profile of equilibrium strategies when the Sender's bias provides her with the incentives to reveal the corresponding asymmetric information. Then, we obtain a continuum of *asymmetric equilibria*, associated with a large family of specific non-zero biases.

The asymmetric equilibria also extend the set of biases that allow players to reach agreement. Then we show that the multiple continua of equilibria permit *any bias* to be associated with an influential equilibrium. In organizations, given our assumptions, any conflict might "sneak" through the dimensions, to become an influential factor in the decisions and actions of the economic agents.

We also observe that among the family of biases associated with an asymmetric equilibrium, there are four symmetric biases, associated with the four symmetries of the distribution of the states. So the corresponding equilibrium strategies are asymmetric despite the symmetry of all of the parameters of the corresponding game.

This shows the possibility of an *endogenously asymmetric* information transmission. In organizations, despite an *a priori* symmetric dependence of the agents' payoffs with respect to the tasks, the agents' conflict provides them with incentives to potentially treat the dimensions of their work asymmetrically.

Finally, we show that the equilibria exhibited cover the full set of the game's pure strategy equilibria, which results in the characterization of this set.

Extensions of Crawford and Sobel's (1982) model to include multiple dimensions have been investigated in the literature. However, results concern the possibility of influence, using either full revelation or binary disclosure rules, and do not seek to characterize the equilibria. When players do not conflict, a large number of messages and an equal number of actions might occur in equilibrium, up to full revelation.[6] Conflict introduces spillover effects between the multiple dimensions that make a general characterization beyond reach.

Battaglini (2002) notes that Sender and Receiver always have a dimension of agreement. Then when the state space is $\mathbb{R}^2$, the Receiver might extract full information from two senders, through an equilibrium alignment of her and each of the sender's interest in their dimension of agreement. The author also notes that with quadratic preferences, a possibility of communication occurs in the dimension which is strictly orthogonal to the Sender's bias. In contrast, Ambrus and Takahashi (2008) show that full extraction may be impossible if states are restricted to belong to a closed subset of $\mathbb{R}^d$. We focus on binary disclosure rules, which are necessarily not fully revealing when the state space is infinite. Our result establishes that communication is possible if the state space is restricted, whatever the bias. Moreover, we show that communication does not necessarily occur strictly orthogonally to the bias. Indeed, in an asymmetric equilibrium, the bias does impact communication.

Chakraborty and Harbaugh (2007) show the existence of the symmetric comparative equilibrium, conditional on a symmetric distribution of states and additively separable and super-modular utility functions of the players.[7] We note that Chakraborty and Harbaugh's (2007) result can be extended by symmetry to cover any direction such that the symmetric conditions are fulfilled with respect to that direction.

In a paper that complements Chakraborty and Harbaugh (2007), Levy and Razin (2007) investigate the large-conflict case, by assuming lexicographic preferences for the Sender. Note that when the Sender's preferences are quadratic, they tend to be lexicographic when the bias tends to infinity in a given direction. When the Sender has lexicographic preferences, Levy and Razin (2007) establish a necessary condition on the distribution of the states for communication to occur with k messages. We show that the uniform distribution over $[0,1]^2$ satisfies this condition in any direction when $k = 2$. Hence when states are uniformly distributed on $[0,1]^2$, Levy and Razin's (2007) result does not allow to establish that communication is impossible, whatever the direction of a Sender's arbitrary large bias. On the contrary, we find that for any extent and any direction of the bias, multiple possibilities of communication are possible. Moreover we find a continuum of equilibria such that when the bias tends to infinity, equilibrium conditions of the "finite" game (in which preferences are quadratic) do not necessarily converge to equilibrium conditions of the "infinite" game (in which preferences are lexicographic). This occurs although players' preferences in the former game do converge to players' preferences in the latter one. This result limits the scope of application of the large-conflict situation.

More generally, Levy and Razin (2007) show that multi-dimensionality might preclude the possibility of communication if game parameters lack sufficient symmetry relative to a specific direction. In contrast, Chakraborty and Harbaugh (2007) show that communication is possible whenever conflict is aligned with a symmetry of the

[6] In the multi-dimensional framework, these conflict-free equilibria occur as so called *centroidal Voronoi tessellations*, see Du et al. (1999) for a review. The work of Jäger et al. (2011) investigates the Voronoi tessellations with many cells, in a communication framework. In Jäger et al. (2011) players do not conflict.

[7] Chakraborty and Harbaugh (2010) derive the existence of an influential equilibrium whatever the distribution of states. However, they assume that the Sender's preferences do not depend on the state.

state distribution. We exploit the simultaneous existence of the few symmetries of the state distribution. This precludes strong asymmetries in any direction. Combined with the flexibility of the one-dimensional framework, we obtain a possibility of communication whatever the direction and extent of the conflict.

Asymmetric equilibria have not been investigated in the literature in the context of more general distributions of states or more general utility functions. Using a slightly different setup, Kamphorst and Swank (2016) exhibit an endogenous asymmetric equilibrium, assuming a uniform distribution, a binary disclosure rule,[8] quadratic preferences for the Receiver, and symmetric linear preferences for the Sender. The linear preferences assumption implies that Sender always prefers Receiver to take the highest possible action. We explore any fixed differences for the players' preferred action. Kamphorst and Swank (2016) interpret the asymmetric equilibrium as a possible discriminatory practice by the players in organizations. Our result suggests that the discrimination considered by Kamphorst and Swank (2016) is precisely the consequence of the multiple symmetries of the game parameters. It allows players to agree in multiple ways. It might occur without conflict of interest, with an exogenous asymmetric conflict, or, in line with Kamphorst and Swank (2016), be endogenously driven by a symmetric conflict.

Section 2 presents the formal model. Section 3 establishes the equilibrium conditions and provides examples and the vocabulary to state the results. Section 4 extends the examples to cover many directions, and shows the existence of influential equilibrium strategies for any bias. Section 5 characterizes the equilibria as the full set of pure strategy equilibria of the game. Section 6 concludes. The proofs are provided in the Appendix.

2 Model setup

We consider an agent, the Sender (S, she), who is informed about a state of the world $\boldsymbol{\theta}$, and an agent, the Receiver (R, he), who is not. The state of the world $\boldsymbol{\theta} = (\theta_1, \theta_2)$ is the realization of a random variable uniformly distributed over the compact convex set $\Theta = [0,1]^2$. The Sender sends a message m, one of two alternatives m_1 and m_2. The Receiver observes the Sender's message, and takes an action $\boldsymbol{a} \in \mathbb{R}^2$. [9] Both players' preferences rely on the state of the world $\boldsymbol{\theta}$ and on the Receiver's action $\boldsymbol{a}$, but not on the message m, which is purely informative. The conflict of interest between the players occurs as a constant difference in their preferred action $\boldsymbol{a}$. The Receiver prefers his action to be as close as possible to the state of the world. His utility U^R decreases with the (Euclidean) distance between $\boldsymbol{a}$ and $\boldsymbol{\theta}$. We set

$$U^R(\boldsymbol{a}, \boldsymbol{\theta}) = -\|\boldsymbol{a} - \boldsymbol{\theta}\|^2.$$

The Sender prefers the Receiver's action to be as close as possible to a shift in the state of the world. The Sender's utility U^S decreases with the distance between $\boldsymbol{a}$ and $\boldsymbol{\theta} + \boldsymbol{b}$, where $\boldsymbol{b} \in \mathbb{R}^2$ denotes the Sender's bias relative to the Receiver's preferred action, and represents the conflict of interest between the players. We set

$$U^S_{\boldsymbol{b}}(\boldsymbol{a}, \boldsymbol{\theta}) = -\|\boldsymbol{a} - (\boldsymbol{\theta} + \boldsymbol{b})\|^2.$$

The game is denoted $\Gamma_{\boldsymbol{b}}$. Its timing is as follows:

1. Nature draws the state of the world $\boldsymbol{\theta} = (\theta_1, \theta_2)$, and reveals it to the Sender, but not to the Receiver (he has a uniform prior over $[0,1]^2$, which is common knowledge);
2. the Sender sends a message $m \in \{m_1, m_2\}$ upon her observation of $\boldsymbol{\theta}$;
3. the Receiver observes the Sender's message m and updates his prior belief about $\boldsymbol{\theta}$;

[8] Crutzen et al. (2013) show that in this setup, the number of messages cannot exceed 3.

[9] The multidimensionality of the game also represents the situation in which a single Sender sends a public message to two Receivers who each takes a one-dimensional action.

4. the Receiver chooses his action $\boldsymbol{a} \in \mathbb{R}^2$ according to his posterior belief;
5. payoffs are realized.

We look for the perfect Bayesian equilibria of this game, that is: (i) the Receiver's action strategy is optimal, given his posterior belief about the state of the world; (ii) the Sender's disclosure strategy is optimal, given the Receiver's action strategy and belief updating; (iii) whenever possible, beliefs are updated according to Bayes's rule.

3 Analysis and examples

3.1 Strategies

Players' pure strategies are as follows. The Sender chooses $\mathfrak{m}(\boldsymbol{\theta}) \in \{m_1, m_2\}$ based on her observation of the state of the world $\boldsymbol{\theta}$. Let $\boldsymbol{a}(m_i) \in \mathbb{R}^2$ be the action played by the Receiver, given his posterior belief upon receiving m_i. For $i \in \{1,2\}$, let $\mathfrak{m}^{-1}(m_i)$ be the set of $\boldsymbol{\theta}$ for which message m_i is disclosed. If $\mathfrak{m}^{-1}(m_i) \neq \varnothing$, then based on his observation of m_i, the Receiver updates his information according to Bayes's rule to $\boldsymbol{\theta} \in \mathfrak{m}^{-1}(m_i)$ and subsequently takes the action that maximizes his expected utility, at

$$\boldsymbol{a}(m_i) = \arg\max_{\boldsymbol{a}\in\mathbb{R}^2} \int_{\boldsymbol{\theta}\in\mathfrak{m}^{-1}(m_i)} U^R(\boldsymbol{a},\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = \mathbb{E}[\boldsymbol{\theta}|\mathfrak{m}(\boldsymbol{\theta}) = m_i]. \tag{1}$$

Reciprocally, given the Receiver's belief updating and the corresponding action strategy $\boldsymbol{a} : m \mapsto \boldsymbol{a}(m)$, the Sender sends the message that maximizes her utility based on her observation of $\boldsymbol{\theta}$, and so

$$\mathfrak{m}(\boldsymbol{\theta}) = \arg\max_{m\in\{m_1,m_2\}} U^S_{\boldsymbol{b}}(\boldsymbol{a}(m),\boldsymbol{\theta}) = \{m_i,\, U^S_{\boldsymbol{b}}(\boldsymbol{a}(m_i),\boldsymbol{\theta}) \geq U^S_{\boldsymbol{b}}(\boldsymbol{a}(m_{-i}),\boldsymbol{\theta})\}, \tag{2}$$

where $-i$ represents the element of $\{1,2\} \setminus \{i\}$.

In equilibrium, strategies $\mathfrak{m} : \boldsymbol{\theta} \mapsto \mathfrak{m}(\boldsymbol{\theta})$ and $\boldsymbol{a} : m \mapsto \boldsymbol{a}(m)$ are each player's best response.

The followings points are worth noting.

First, there are always equilibria in which no information is revealed, *i.e. babbling equilibria*. The construction is as follows: the Sender sends the same message m_i, for some $i \in \{1,2\}$, regardless of $\boldsymbol{\theta} \in \Theta$; the Receiver always takes his action at $\boldsymbol{a}(m_i) = \mathbb{E}[\boldsymbol{\theta}|\mathfrak{m}(\boldsymbol{\theta}) = m_i] = \mathbb{E}[\Theta] = \left(\frac{1}{2}, \frac{1}{2}\right)$, regardless of the message received, and his off-path belief on receiving m_{-i} is the same as his belief on receiving m_i (in line with Farrell (1993, Section 3), m_1 and m_2 mean the same to the Receiver). We say that an equilibrium is *influential* if on the contrary, $\boldsymbol{a}(m_i) \neq \boldsymbol{a}(m_{-i})$.

Second, given a Receiver strategy $\boldsymbol{a}$, for any $\boldsymbol{\theta} \in \Theta$, we have $U^S_{\boldsymbol{b}}(\boldsymbol{a}(m_i),\boldsymbol{\theta}) \geq U^S_{\boldsymbol{b}}(\boldsymbol{a}(m_{-i}),\boldsymbol{\theta})$ *iff*

$$\begin{aligned} &-\|\boldsymbol{a}(m_i) - (\boldsymbol{\theta}+\boldsymbol{b})\|^2 \geq -\|\boldsymbol{a}(m_{-i}) - (\boldsymbol{\theta}+\boldsymbol{b})\|^2, \\ \textit{iff}\ &\|(\boldsymbol{a}(m_i) - \boldsymbol{b}) - \boldsymbol{\theta}\|^2 \leq \|(\boldsymbol{a}(m_{-i}) - \boldsymbol{b}) - \boldsymbol{\theta}\|^2, \end{aligned}$$

so that the equilibrium condition of the Sender's strategy might be stated geometrically as: the Sender necessarily discloses m_i or m_{-i} conditional on $\boldsymbol{\theta}$ belonging to one side of the *perpendicular bisector* of the line that supports $\boldsymbol{a}(m_1) - \boldsymbol{b}$ and $\boldsymbol{a}(m_2) - \boldsymbol{b}$ (see Figure 1 for an illustration of this point). Given $\boldsymbol{a}$ and $\boldsymbol{b}$, this bisector represents the set of states for which the Sender is indifferent between the two induced actions. In particular, a Sender's non-babbling strategy, up to the set of indifferent states, is fully characterized by a line that partitions Θ, and by the association of a message with either side of it. Now given a Sender' strategy characterized in this way, from (1), the Receiver's actions are represented by the mass centres of the partition elements.[10]

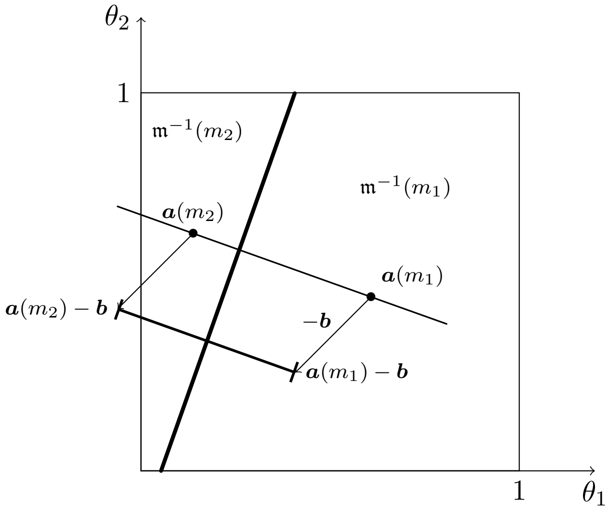

Fig. 1 Sender and Receiver's strategies

Third, given equilibrium strategies $\mathfrak{m}$ and $\boldsymbol{a}$, strategies $\mathfrak{m}'$ and $\boldsymbol{a}'$ respectively given by $\mathfrak{m}'(\boldsymbol{\theta}) = m_1$ if $\mathfrak{m}(\boldsymbol{\theta}) = m_2$ and $\mathfrak{m}'(\boldsymbol{\theta}) = m_2$ if $\mathfrak{m}(\boldsymbol{\theta}) = m_1$, and $\boldsymbol{a}'(m_i) = \boldsymbol{a}(m_{-i})$, $i \in \{1,2\}$, also are in equilibrium. Profiles of strategies $(\mathfrak{m}, \boldsymbol{a})$ and $(\mathfrak{m}', \boldsymbol{a}')$ are symmetric relative to the messages. We say that they are *message-symmetric*.

Fourth, let us consider potential *information symmetries*, i.e. relative to the way information is disclosed and processed. Let us consider orthogonal symmetries $s_{\boldsymbol{\tau}}$ with respect to an affine line of $\mathbb{R}^2$ with direction vector $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathbb{R}^2$. Given a set $X \subseteq \mathbb{R}^2$ (which could be a (set of) state(s), action(s), or bias(es)), we say that X is $\boldsymbol{\tau}$*-symmetric* if $s_{\boldsymbol{\tau}}(X) = X$.

Note that the uniform distribution of states over $\Theta = [0,1]^2$ is $\boldsymbol{\tau}$-symmetric for any $\boldsymbol{\tau} \in \{(-1,1),(0,1),(1,-1),(1,0)\}$. Hence all the parameters of the game $\Gamma_{\boldsymbol{b}}$ are $\boldsymbol{\tau}$-symmetric if the bias $\boldsymbol{b}$ is $\boldsymbol{\tau}$-symmetric, *i.e.* if it is collinear to $\boldsymbol{\tau}$ for some $\boldsymbol{\tau} \in \{(-1,1),(0,1),(1,-1),(1,0)\}$. In that case, we say that *the game is* $\boldsymbol{\tau}$*-symmetric*, and that it is symmetric if it is symmetric for some $\boldsymbol{\tau}$.

We extend the notion of symmetry to strategies, and say that Sender's strategy $\mathfrak{m} : \boldsymbol{\theta} \mapsto \mathfrak{m}(\boldsymbol{\theta})$ is $\boldsymbol{\tau}$-symmetric if the Sender discloses her information $\boldsymbol{\tau}$-symmetrically, *i.e.* if $s_{\boldsymbol{\tau}}(\mathfrak{m}^{-1}(m_1)) = \mathfrak{m}^{-1}(m_2)$. Similarly, Receiver's strategy $\boldsymbol{a}$ is $\boldsymbol{\tau}$-symmetric if the Receiver acts $\boldsymbol{\tau}$-symmetrically, *i.e.* if $s_{\boldsymbol{\tau}}(\boldsymbol{a}(m_1)) = \boldsymbol{a}(m_2)$. If players' strategies are $\boldsymbol{\tau}$-symmetric and in equilibrium, then so is the equilibrium. Given a $\boldsymbol{\tau}$-symmetric game, we say that *an equilibrium is symmetric* if it is $\boldsymbol{\tau}$-symmetric; otherwise it is *asymmetric*.

In the next two sub-sections we give examples of equilibrium strategies in which symmetry plays a crucial role. These examples are the basis for addressing the issue of the existence of an influential equilibrium in $\Gamma_{\boldsymbol{b}}$, for any $\boldsymbol{b} \in \mathbb{R}^2$, in Section 4.

[10] It is easy to show that given his belief, the Receiver has no interest in mixing his responses, so that the Sender discloses m_i or m_{-i} with probability 1 for the states that do not belong to the perpendicular bisector of $\boldsymbol{a}(m_1) - \boldsymbol{b}$ and $\boldsymbol{a}(m_2) - \boldsymbol{b}$. In particular, any mixed strategy equilibrium outcome is equivalent to a pure strategy equilibrium outcome. See Levy and Razin (2004, Proposition 1) for a generalization of this result.

3.2 Symmetric equilibria and their deviations

Chakraborty and Harbaugh (2007) show that regardless of the extent of the conflict of interest, if the conflict and prior are both $(1,1)$-symmetric, then comparison of the state components is influential. Basically the $(1,1)$-symmetry implies that the $(1,1)$-symmetric comparison induces $(1,1)$-symmetric posterior beliefs, and a consequently $(1,1)$-symmetric action strategy $\boldsymbol{a}$. Reciprocally, a $(1,1)$-symmetric conflict implies that the Sender's best response to a Receiver's $(1,1)$-symmetric strategy is a $(1,1)$-symmetric disclosure rule, which necessarily compares θ_1 and θ_2.

The conditions of existence of such an equilibrium might not be restricted to the $(1,1)$-direction, but could be extended to any $\boldsymbol{\tau}$ for which the symmetric conditions are fulfilled. Specifically, given the quadratic preferences of the Sender, it needs the bias $\boldsymbol{b}$ to be $\boldsymbol{\tau}$-symmetric for some $\boldsymbol{\tau} \in \{(1,1),(0,1),(1,0),(1,-1)\}$ of the $\boldsymbol{\tau}$-symmetric state space. We obtain the following four *symmetric equilibria.*

Example 1 Conditional on the bias $\boldsymbol{b} \in \mathbb{R}^2$, the following strategies are equilibrium strategies:

- if the bias $\boldsymbol{b} = (b,b)$, $b \in \mathbb{R}$ is $(1,1)$-symmetric, *comparative* $(1,1)$-symmetric strategies are
$$\mathfrak{m}_C(\boldsymbol{\theta}) = \begin{cases} m_1 \text{ if } \theta_1 \geq \theta_2, \\ m_2 \text{ if } \theta_1 < \theta_2, \end{cases} \quad \boldsymbol{a}_C(m) = \begin{cases} (\frac{2}{3}, \frac{1}{3}) \text{ if } m = m_1 \\ (\frac{2}{3}, \frac{1}{3}) \text{ if } m = m_2; \end{cases}$$
- if the bias $\boldsymbol{b} = (b,-b)$, $b \in \mathbb{R}$, is $(1,-1)$-symmetric, *aggregative* $(1,-1)$-symmetric strategies are
$$\mathfrak{m}_A(\boldsymbol{\theta}) = \begin{cases} m_1 \text{ if } \theta_1 + \theta_2 \geq 1, \\ m_2 \text{ if } \theta_1 + \theta_2 < 1, \end{cases} \quad \boldsymbol{a}_A(m) = \begin{cases} (\frac{2}{3}, \frac{2}{3}) \text{ if } m = m_1 \\ (\frac{1}{3}, \frac{1}{3}) \text{ if } m = m_2; \end{cases}$$
- if the bias $\boldsymbol{b} = (0,b)$ (resp. $\boldsymbol{b} = (b,0)$), $b \in \mathbb{R}$, is $(0,1)$-symmetric (resp. $(1,0)$-symmetric), *half-babbling* $(1,0)$-symmetric (resp. $(0,1)$-symmetric) strategies are
$$\mathfrak{m}_H(\boldsymbol{\theta}) = \begin{cases} m_1, \text{ if } \theta_1 \geq \frac{1}{2} \text{ (resp. } \theta_2 \leq \frac{1}{2}), \\ m_2, \text{ if } \theta_i < \frac{1}{2} \text{ (resp. } \theta_2 > \frac{1}{2}), \end{cases} \quad \boldsymbol{a}_H(m) = \begin{cases} (\frac{3}{4}, \frac{1}{2}) \text{ (resp. } (\frac{1}{2}, \frac{1}{4})) \text{ if } m = m_1 \\ (\frac{1}{4}, \frac{1}{2}) \text{ (resp. } (\frac{1}{2}, \frac{3}{4})) \text{ if } m = m_2. \end{cases}$$

Chakraborty and Harbaugh (2007) obtain the equilibrium strategies $(\mathfrak{m}_C, \boldsymbol{a}_C)$ from the symmetry of the state space, and the symmetry and super-modularity of the utility functions. By symmetry, all strategies exhibited in Example 1 could easily be derived under these more general conditions. However, Example 1 helps to highlight an important relationship between the different strategies. Specifically, if $\boldsymbol{b} = (0,0)$, then strategies $\mathfrak{m}_C$, $\mathfrak{m}_A$ and $\mathfrak{m}_H$ simultaneously occur as a potential Sender' equilibrium strategy. This highlights a new multiplicity of equilibria relative to the one-dimensional framework, which arises *across the dimensions.*[11] Strategies exhibited in Example 1 do not depend on the extent $\|\boldsymbol{b}\| = b$ of the bias $\boldsymbol{b}$, but rather on its direction. When the direction is well defined, *i.e.* when $b \neq 0$, Example 1 points to a unique equilibrium strategy of $\Gamma_{\boldsymbol{b}}$. In particular, conflict might rule out the above-mentioned conflict-free multiplicity. In the next section, we show that this is not the case. When b continuously increases from 0 to $+\infty$, multiple strategies move continuously and remain in equilibrium.

Now, let us highlight that the strategies in Example 1 derive from a one-dimensional game of information transmission, relative to a specific dimension. For instance, let us decompose the Sender's utility in the $(1,-1)$-directed "dimension of comparison" and its orthogonal $(1,1)$-directed dimension:

$$-\|\boldsymbol{a}(m_i) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2 = -\left(((1,1)\cdot(\boldsymbol{a}(m_i) - (\boldsymbol{\theta}+\boldsymbol{b})))^2 + ((1,-1)\cdot(\boldsymbol{a}(m_i) - (\boldsymbol{\theta}+\boldsymbol{b})))^2\right), \tag{3}$$

[11] In the one-dimensional framework, different sets of messages are associated with different equilibria. This multiplicity of equilibria also occurs in the framework considered here. Example 2 specifies the one-dimensional aspect of the strategies presented in Example 1.

where $\cdot$ denotes the (Euclidean) scalar product. Note that for any $(1,1)$-symmetric action $\boldsymbol{a} = \boldsymbol{a}_C$, we have $(1,1) \cdot \boldsymbol{a}_C(m_1) = (1,1) \cdot \boldsymbol{a}_C(m_2)$, so that the first term of (3) is *independent* of $i \in \{1,2\}$. This shows that the $(1,-1)$-dimension fully determines the Sender's utility, and so it fully determines her strategy. Conversely, let $\mathfrak{m}$ be a Sender's strategy which is fully determined by the $(1,-1)$-dimension. Then we necessarily have $(1,1) \cdot \boldsymbol{a}(m_1) = (1,1) \cdot \boldsymbol{a}(m_2)$. Let $\boldsymbol{a}(m_i) = (\mathbb{E}[\theta_1|m_i], \mathbb{E}[\theta_2|m_i])$ denotes the Receiver's expectation given his observation of the message m_i associated with $\mathfrak{m}$. From $\theta_1 - \theta_2 = (1-\theta_2) - (1-\theta_1)$ and the $(1,-1)$-symmetry of $\Theta = [0,1]^2$, we have $\mathfrak{m}((\theta_1,\theta_2)) = m_i$ *iff* $\mathfrak{m}((1-\theta_2, 1-\theta_1)) = m_i$. This implies $\mathbb{E}[\theta_1|m_i] = \mathbb{E}[1-\theta_2|m_i]$, *i.e.* $\mathbb{E}[\theta_1+\theta_2|m_i] = 1$, regardless of i.

Therefore, the revealed information is restricted to the $(1,-1)$-dimension *iff* actions are aligned with the $(1,-1)$-dimension. This means that players might choose to play a one-dimensional game in the $(1,-1)$-dimension in order to reach agreement.[12] Then according to the one-dimensional framework of Crawford and Sobel (1982), players might reach agreement if the bias $b_1 - b_2 = \Delta b$ in this dimension is limited. In that case, an agreement is obtained regardless of the bias in the uninformative $(1,1)$-dimension, *i.e.* regardless of $b_1 + b_2$.

Example 2 Consider the following Sender's strategy, denoted $\mathfrak{m}_{C(c)}$, with $c \in (-1,1)$:[13]

$$\mathfrak{m}_{C(c)}(\boldsymbol{\theta}) = \begin{cases} m_1, \text{ if } \theta_1 \geq \theta_2 + c \\ m_2, \text{ if } \theta_1 < \theta_2 + c. \end{cases}$$

Let the bias be given by $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$, and set $\Delta b = |b_2 - b_1|$. Then there exists $c \in (-1,1)$ such that $\mathfrak{m}_{C(c)}$ is an equilibrium strategy of $\Gamma_{\boldsymbol{b}}$ *iff* $\Delta b < \frac{1}{2}$.

Figure 2 illustrates the Sender's corresponding strategies $\mathfrak{m}_{C(c)}$.

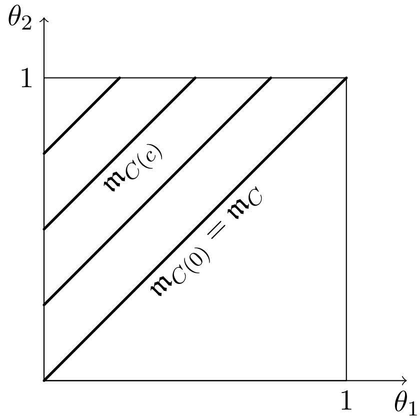

Fig. 2 Deviations of the comparative $(1,1)$-symmetric disclosure rule $\mathfrak{m}_C$, due to bias $\boldsymbol{b} = (b_1, b_1 + \Delta b)$

Note that each symmetric equilibrium of Example 1 results from a single symmetry of the distribution of states. In contrast, the deviated equilibria of Example 2 are the result of the two $(1,1)$ and $(1,-1)$-symmetries of $[0,1]^2$, and of their orthogonality. Similar deviated equilibrium strategies hold with regard to the aggregative $\mathfrak{m}_A$ and half-babbling $\mathfrak{m}_H$ strategies. Let $\mathfrak{m}_{A(c)}$ and $\mathfrak{m}_{H(c)}$ denote the corresponding strategies.

[12] A by-product is that the binary nature of our setting in the present case is very restrictive. According to the one dimensional framework, if the bias is sufficiently limited in the dimension of agreement, more than two messages might be used in equilibrium. However, investigating the corresponding model is beyond the scope of the present study.

[13] Notice that when $|c| \to 1$, strategy $\mathfrak{m}_{C(c)}$ tends to babble: either m_1 or m_2 would always be revealed, whatever $\boldsymbol{\theta}$.

3.3 Asymmetric equilibria

In previous examples, $\boldsymbol{\tau}$-asymmetric equilibria of a $\boldsymbol{\tau}$-asymmetric game are interpreted as deviations of a $\boldsymbol{\tau}$-symmetric equilibria of a $\boldsymbol{\tau}$-symmetric game. They are obtained from a limited $\boldsymbol{\tau}$-asymmetry of the corresponding $\boldsymbol{\tau}$-symmetric bias. The four symmetries of $\Theta = [0,1]^2$ condition the existence of these equilibria. The next example exhibits equilibrium strategies not based on these symmetries. It is depicted in Figure 3.

Example 3 Players' strategies

$$\mathfrak{m}(\boldsymbol{\theta}) = \begin{cases} m_1 \text{ if } \theta_1 \geq \frac{1}{2}\theta_2, \\ m_2 \text{ if } \theta_1 < \frac{1}{2}\theta_2, \end{cases} \quad \boldsymbol{a}(m) = \begin{cases} \left(\frac{11}{18}, \frac{4}{9}\right) \text{ if } m = m_1, \\ \left(\frac{1}{6}, \frac{2}{3}\right) \text{ if } m = m_2, \end{cases}$$

are equilibrium strategies of $\Gamma_{\boldsymbol{b}(b)}$, where $\boldsymbol{b}(b) = \left(\frac{4}{45} + \frac{1}{2}b, \frac{-2}{45} + b\right)$, $b \in \mathbb{R}$.

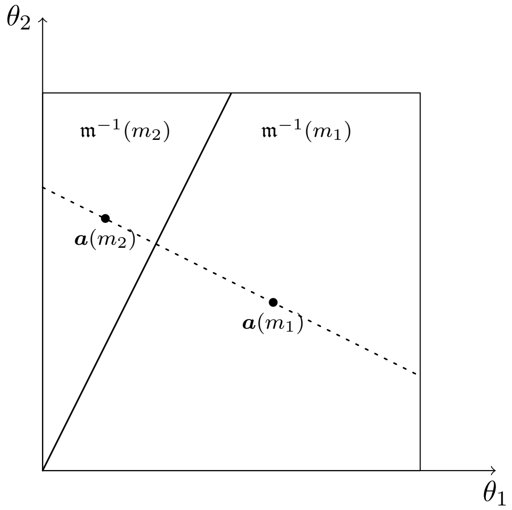

Fig. 3 Asymmetric equilibrium strategies

In Example 3 the revealed information concerns $\theta_1 - \frac{1}{2}\theta_2$ and thus it is revealed in the $\left(1, -\frac{1}{2}\right)$-dimension. The corresponding induced actions align with that dimension with $\left(\frac{1}{2}, 1\right) \cdot \boldsymbol{a}(m_1) = \left(\frac{1}{2}, 1\right) \cdot \boldsymbol{a}(m_2)$, *i.e.* $\frac{1}{2}\mathbb{E}[\theta_1|m_1] + \mathbb{E}[\theta_2|m_1] = \frac{1}{2}\mathbb{E}[\theta_1|m_2] + \mathbb{E}[\theta_2 m_2]$. Hence the *specific* information revealed about $\theta_1 - \frac{1}{2}\theta_2$ (whether it is positive or not) is not informative about $\frac{1}{2}\theta_1 + \theta_2$, *i.e.* in the dimension which is orthogonal to the informative dimension. Players exploit a *sufficiently weak dependence*[14] between two orthogonal dimensions of the distribution of states in order to reach agreement. In particular, the Sender is uninformative in the $\left(\frac{1}{2}, 1\right)$-dimension, so that her bias is irrelevant in that dimension. Then any $\left(\frac{1}{2}b, b\right)$, $b \in \mathbb{R}$, could be included added to the Sender's bias, without it changing the equilibrium conditions. In contrast, in the informative $\left(1, -\frac{1}{2}\right)$-dimension, information is revealed asymmetrically. This imposes a non-zero (and limited) bias. In that dimension the bias must match the asymmetry of the disclosed information. We find $\left(\frac{4}{45}, \frac{-2}{45}\right)$ for the adequate $\left(1, -\frac{1}{2}\right)$-dimensional component of $\boldsymbol{b}$.

Note that if $b = \frac{4}{45}$, then $\boldsymbol{b} = \left(\frac{2}{9}, \frac{2}{9}\right)$ is $(1,1)$-symmetric. So the game is $(1,1)$-symmetric. Therefore, strategies in Example 3 are *endogenously asymmetric*. Similarly if $b = \frac{-8}{45}$ (resp. $b = \frac{2}{45}$, $b = \frac{-4}{135}$), then $\boldsymbol{b} = \left(0, -\frac{2}{9}\right)$ (resp.

[14] Note that $\frac{1}{2}\theta_1 + \theta_2$ is not, in general, independent of $\theta_1 - \frac{1}{2}\theta_2$. For instance, if the Sender reveals $\theta_1 - \frac{1}{2}\theta_2 = 1$, then the Receiver infers $\theta_1 = 1$ and $\theta_2 = 0$ so that $\frac{1}{2}\theta_1 + \theta_2 = \frac{1}{2}$. If the Sender reveals $\theta_1 - \frac{1}{2}\theta_2 = \frac{-1}{2}$, then the Receiver infers $\theta_1 = 0$ and $\theta_2 = 1$ so that $\frac{1}{2}\theta_1 + \theta_2 = 1 \neq \frac{1}{2}$. This contrasts with the independence between the informative and uninformative *dimensions* of a symmetric equilibrium. Here, the independence is derived from the equilibrium *issues* in these dimensions.

$\boldsymbol{b} = \left(\frac{1}{9}, 0\right)$, $\boldsymbol{b} = \left(\frac{2}{27}, \frac{-2}{27}\right)$) and the strategies are asymmetric strategies of a $(0,1)$-symmetric (resp. $(1,0)$-symmetric, $(1,-1)$-symmetric) game. In the context of organizations, this result means that despite *a priori* symmetric dependencies of the agents' payoffs with respect to a specific dimension, asymmetric treatments relative to that dimension are possible in equilibrium.

Strategies in Example 3 are no exception. Figure 4 illustrates a continuum of $(1,1)$-asymmetric Sender's strategies $\mathfrak{m}$ of the $(1,1)$-symmetric game $\Gamma_{\boldsymbol{b}}$, with $\boldsymbol{b} = (b, b)$. In particular conflict does not rule out the multiplicity of conflict-free equilibria referred to in the previous section. As $\boldsymbol{b} = (b, b)$ increases to infinity, strategies $(\mathfrak{m}_{A(c)}, \boldsymbol{a}_{A(c)})$ and $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$ are ruled out but the asymmetric strategies depicted in Figure 4 are not, and neither are the symmetric strategies $(\mathfrak{m}_C, \boldsymbol{a}_C)$, so that multiple equilibria remain.

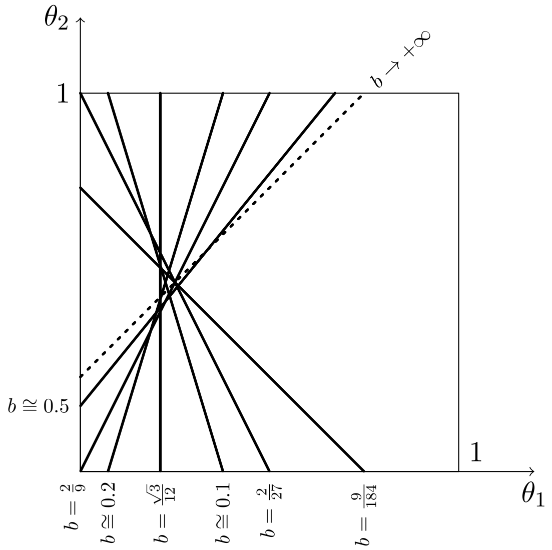

Fig. 4 Endogeneous $(1,1)$-asymmetric disclosure rules associated to $(1,1)$-symmetric bias $\boldsymbol{b} = (b, b)$

Finally, let us discuss the impact of the extent of conflict on the players' interaction. In a symmetric equilibrium, no information is revealed in the (symmetric) bias dimension. Since players' conflict is restricted to that dimension, the extent of the conflict has no impact on the outcomes. In contrast, as Figure 4 shows, the extent of the underlying symmetric conflict of an asymmetric equilibrium impacts the outcomes. In Figure 4, the dimension of agreement $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$ is aligned with the dimension of the symmetric bias $= (b, b)$ when b is close to 0, and tends to be orthogonal to it as b tends to infinity.

Let us provide some details on the convergence of the equilibrium conditions when the bias tends to infinity, in relation with Levy and Razin's (2004; 2007) important contributions. As argued by Levy and Razin (2007), when $b \to \infty$, quadratic preferences of the Sender converge to lexicographic preferences. To see this, consider for instance $\boldsymbol{b} = (b, b) \in \mathbb{R}^2$. Then given $\boldsymbol{\theta} = (\theta_1, \theta_2)$, the Sender prefers $\boldsymbol{a} = (a_1, a_2)$ to $\boldsymbol{a}' = (a'_1, a'_2)$ *iff* $-(a_1 - (\theta_1 + b))^2 - (a_2 - (\theta_2 + b))^2 \geq -(a'_1 - (\theta_1 + b))^2 - (a'_2 - (\theta_2 + b))^2$, *i.e. iff*

$$(a_1 - a'_1)\left(\theta_1 - \frac{a_1 + a'_1}{2}\right) + b(a_1 + a_2) \geq (a'_2 - a_2)\left(\theta_2 - \frac{a_2 + a'_2}{2}\right) + b(a'_1 + a'_2). \tag{4}$$

If b is sufficiently large relative to $a_1 + a_2 - (a'_1 + a'_2)$, so that in (4) the difference between the terms $b(a_1 + a_2)$ and $b(a'_1 + a'_2)$ dominates the difference of the other terms, then inequality (4) holds *iff* $a_1 + a_2 \geq a'_1 + a'_2$. In that case the Sender's preferred action is based on the comparison of $a_1 + a_2$ and $a'_1 + a'_2$. This is a one-dimensional comparison, in the $(1,1)$-dimension of $\mathbb{R}^2$. The Sender has lexicographic preferences because in that dimension, she always prefers the element with the highest value, and if values are equal, (4) might be written $\theta_1 - \frac{a_1 - a_2}{2} \geq \theta_2 - \frac{a'_1 - a'_2}{2}$, in which only the $(1,-1)$-dimension is involved.

However the Sender's lexicographic preferences only concern actions such that b is sufficiently large relative to $(a_1+a_2)-(a_1'+a_2')$. When b tends to $+\infty$, this holds for almost all actions. Thus when b tends to $+\infty$, Sender's preferences tend to be lexicographic over the full action space. Levy and Razin (2007) specifically examine the game in which Sender has lexicographic preferences, interpreting it as a game with quadratic preference and an "infinite bias".

Now, let us show that the outcomes of such an "infinite game" do not necessarily correspond to the limit, when b tends to $+\infty$, of the outcomes of game $\Gamma_{\boldsymbol{b}}$, $\boldsymbol{b}=(b,b)$. To do this, let us examine the convergence of the equilibrium conditions of the finite games in which preferences converge to the $(1,1)$-oriented lexicographic preferences. When the Sender has $(1,1)$-oriented lexicographic preferences, then any information revelation in equilibrium necessarily induces $a_1(m_1)+a_2(m_1)=a_1(m_2)+a_2(m_2)$, since otherwise the message that induces the highest sum is always sent. Now if $a_1(m_1)+a_2(m_1)=a_1(m_2)+a_2(m_2)$, then inequality (4) is an equality at $\boldsymbol{\theta}=\frac{\boldsymbol{a}(m_1)+\boldsymbol{a}(m_2)}{2}$. In particular, in equilibrium, the Sender is necessarily indifferent between the induced actions when the state equals their mid-point. However, for instance the family of equilibria depicted in Figure 4 is such that when $b\to\infty$ the limit of the set of the Sender's indifferent states does not contain the mid-point of the limit of the induced actions.[15] Hence when the conflict tends to infinity, the limit of an equilibrium in the game with finite conflict (in which Sender's preferences are quadratic) does not necessarily converge to an equilibrium in the game with infinite conflict (in which Sender's preferences are lexicographic).

For a formal explanation, it should be noted that according to (4), $\boldsymbol{\theta}=\frac{\boldsymbol{a}(m_1)+\boldsymbol{a}(m_2)}{2}$ is a Sender's indifferent state *iff* $b(a_1(m_1)+a_2(m_1))=b(a_1(m_2)+a_2(m_2))$. The induced actions of the family of equilibria depicted in Figure 4 is such that $\lim_{b\to+\infty}(a_1(m_1)+a_2(m_1))=\lim_{b\to+\infty}(a_1(m_2)+a_2(m_2))$, so that the dimension of the induced actions tends to align orthogonally to the conflict. This does tend to be similar to the condition associated with the lexicographic preferences case, in which in equilibrium $a_1(m_1)+a_2(m_1)=a_1(m_2)+a_2(m_2)$ is necessary. However $\lim_{b\to+\infty}(a_1(m_1)+a_2(m_1))=\lim_{b\to+\infty}(a_1(m_2)+a_2(m_2))$ does not necessarily imply $\lim_{b\to+\infty}(b(a_1(m_1)+a_2(m_1)))=\lim_{b\to+\infty}(b(a_1(m_2)+a_2(m_2)))$.[16] If $\lim_{b\to+\infty}(b(a_1(m_1)+a_2(m_1)))\neq\lim_{b\to+\infty}(b(a_1(m_2)+a_2(m_2)))$, then the mid-point of the limit of the induced actions does not correspond to an indifferent state. Therefore, the mid-point of the limit of the induced actions does not necessarily correspond to an indifferent state of the limit strategies. [17]

4 Influence in the context of any conflict

Examples 1, 2 and 3 describe equilibrium strategies associated with some particular types of biases. In this section, we extend and exploit these strategies to derive the existence of equilibrium strategies for any bias.

Consider the correspondence

$$\mathcal{E}:\boldsymbol{b}\rightrightarrows(\mathfrak{m},\boldsymbol{a}),$$

which associates bias a $\boldsymbol{b}$ with the set of strategies $(\mathfrak{m},\boldsymbol{a})$ that occur as an influential equilibrium strategy in the game $\Gamma_{\boldsymbol{b}}$. We seek to show that for any $\boldsymbol{b}\in\mathbb{R}^2$, $\mathcal{E}(\boldsymbol{b})$ is not empty. To do this, we consider the inverse correspondence, denoted $\mathcal{E}^{-1}$, which associates strategies $(\boldsymbol{a},\mathfrak{m})$ with the set of biases $\boldsymbol{b}\in\mathbb{R}^2$ for which $(\mathfrak{m},\boldsymbol{a})\in\mathcal{E}(\boldsymbol{b})$. Clearly we have $\mathcal{E}(\boldsymbol{b})\neq\varnothing$ *iff* $\boldsymbol{b}\in\mathcal{E}^{-1}((\mathfrak{m},\boldsymbol{a}))$ for some $(\mathfrak{m},\boldsymbol{a})$. Hence it is sufficient to provide many

[15] This is easily derived by equation $\theta_2=\theta_1+1/4$ which characterize the set of indifferent states when $b\to+\infty$. A proof of this characterization is provided in the appendix, in proving Lemma 3.

[16] Stated in words, two vectors might tend to be orthogonal, but their scalar product does not necessarily converge to 0. For instance, when $b\to\infty$, vectors $(0,b)$ and $(1,1/b)$ tend to be orthogonal, in the $(0,1)$ and $(1,0)$ respective directions, but for any $b\in\mathbb{R}$, we have $(0,b)\cdot(1,1/b)=1$.

[17] In particular, Levy and Razin (2004, Corollary 1(iii), and the statement of Problem A) are not correct when a family of asymmetric equilibria such as the one depicted in Figure 4 is considered.

strategies $(\mathfrak{m}, \boldsymbol{a})$ such that the sets of biases $\mathcal{E}^{-1}((\mathfrak{m}, \boldsymbol{a}))$ cover $\mathbb{R}^2$. First, we characterize the set $\mathcal{E}^{-1}((\mathfrak{m}, \boldsymbol{a}))$, given a profile $(\mathfrak{m}, \boldsymbol{a})$ (Lemma 1). Then we extend Example 3 continuously to cover a large number of directions (Lemmas 2 and 3), and then to cover all directions (Lemma 4). We then show that the strategies exhibited yield the result (Proposition 1).

Lemma 1 *Let $(\mathfrak{m}, \boldsymbol{a})$ be such that $(\mathfrak{m}, \boldsymbol{a}) \in \mathcal{E}(\boldsymbol{b})$ for some $\boldsymbol{b} \in \mathbb{R}^2$. Then for any $\boldsymbol{b}' \in \mathbb{R}^2$, $(\mathfrak{m}, \boldsymbol{a}) \in \mathcal{E}(\boldsymbol{b}')$ iff $\boldsymbol{b}' \cdot (\boldsymbol{a}(m_1) - \boldsymbol{a}(m_2)) = \boldsymbol{b} \cdot (\boldsymbol{a}(m_1) - \boldsymbol{a}(m_2))$.*

From Lemma 1, the set of biases $\boldsymbol{b}'$ for which strategies $(\mathfrak{m}, \boldsymbol{a}) \in \mathcal{E}(\boldsymbol{b})$ are equilibrium strategies of $\Gamma_{\boldsymbol{b}'}$ is given by those $\boldsymbol{b}'$ that project orthogonally as does $\boldsymbol{b}$ onto the line $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$. Geometrically speaking (see Figure 5), a Sender's strategy $\mathfrak{m}$ is characterized by the association of a message m_i with each side of the perpendicular bisector of $\boldsymbol{a}(m_1) - \boldsymbol{b}$ and $\boldsymbol{a}(m_2) - \boldsymbol{b}$. Therefore $\mathcal{E}^{-1}(\mathfrak{m}, \boldsymbol{a})$ corresponds to bias $\boldsymbol{b}'$ such that $\boldsymbol{a}(m_1) - \boldsymbol{b}'$ and $\boldsymbol{a}(m_2) - \boldsymbol{b}'$ induce the same perpendicular bisector.

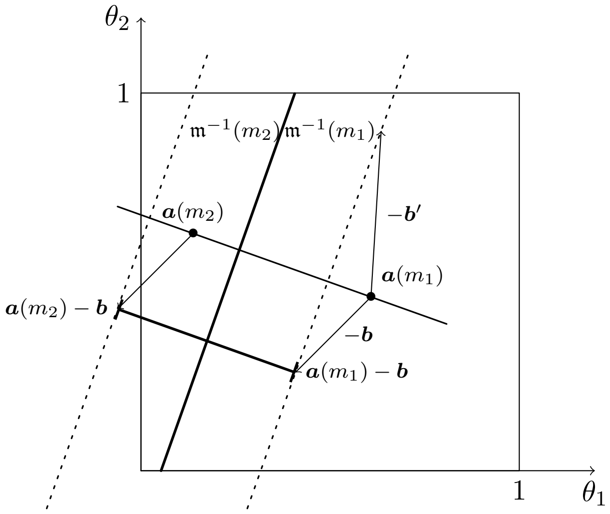

Fig. 5 $\boldsymbol{b}' \in \mathcal{E}^{-1}((\mathfrak{m}, \boldsymbol{a}))$

The following lemma extends Example 3 to cover a large number of directions. It is depicted in Figure 6.

Lemma 2 *For any line (OT), with $O = \mathbb{E}[\Theta] = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $T \in \{0\} \times \left(\frac{1}{2}, 1\right)$, there exists a strategy equilibrium profile $(\mathfrak{m}_T, \boldsymbol{a}_T)$ of the game $\Gamma_{\boldsymbol{b}_T}$, with $\boldsymbol{a}_T(m_1) \in (OT)$, $\boldsymbol{a}_T(m_2) \in (OT)$, and*

$$\boldsymbol{b}_T = \frac{\boldsymbol{a}_T(m_1) + \boldsymbol{a}_T(m_1)}{2} - \boldsymbol{\theta}_T, \tag{5}$$

for some $\boldsymbol{\theta}_T \in (OT)$.

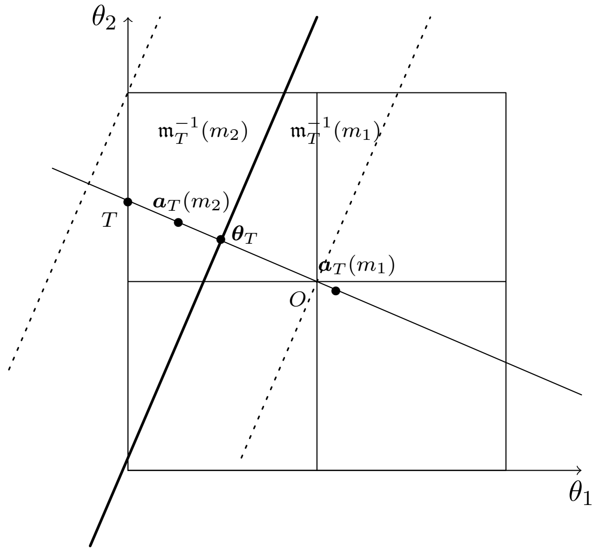

Fig. 6 Existence of an equilibrium strategy given T

We derive Lemma 2 by showing that the equilibrium conditions of Example 3 extend in all directions (OT), where $T \in \{0\} \times \left(\frac{1}{2}, 1\right)$. In particular, for all these directions, $\Theta = [0,1]^2$ always allows specific information to be disclosed in the (OT)-dimension so that no information is revealed in the corresponding orthogonal dimension.[18]

Lemma 3 provides us with the limit of the equilibrium strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ exhibited in Lemma 2, when T moves continuously to points $(0, \frac{1}{2})$ or $(0,1)$. It allows us to extend the strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ *continuously*[19] as influential strategies in all directions of the plane. Strategies $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$ are defined in Example 2 as deviated comparative symmetric strategies. Strategies $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$ are defined similarly with regard to symmetric half-babbling strategies.

Lemma 3 *Strategies* $(\mathfrak{m}_T, \boldsymbol{a}_T)$*,* $T \in \{0\} \times (\frac{1}{2}, 1)$ *extend continuously at* $T = (0, \frac{1}{2})$ *and* $T' = (0,1)$ *to asymmetric strategies* $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$ *and* $(\mathfrak{m}_{C(c')}, \boldsymbol{a}_{C(c')})$ *respectively, for some* $c, c' \in (0,1)$*.*

Lemma 4 allows us to further extend the set of equilibrium strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ to cover all directions in the plane by considering symmetric transformations of $(\mathfrak{m}_T, \boldsymbol{a}_T)$ with respect to the symmetries of Θ. Let ρ be one of the four axial symmetries of Θ, or a composition of them. From $\rho(\Theta) = \Theta$, strategy

$$\rho(\mathfrak{m}) : \boldsymbol{\theta} \mapsto \rho(\mathfrak{m})(\boldsymbol{\theta}) = m_i \text{ iff } \boldsymbol{\theta} \in \rho^{-1}(\mathfrak{m}^{-1}(m_i)),$$

associated with a given Sender's strategy $\mathfrak{m}$, is well defined. Similarly, given a Receiver's strategy $\boldsymbol{a}$, we define $\rho(\boldsymbol{a})$ by $\rho(\boldsymbol{a})(m_i) = \rho(\boldsymbol{a}(m_i))$, $i \in \{1,2\}$. Since ρ does not change measure and orthogonality, and since players' preferences are characterized as Euclidean distances, equilibrium conditions are invariant if all the parameters are transformed through ρ. Equilibrium strategies of $\Gamma_{\boldsymbol{b}}$ are transformed via ρ to equilibrium strategies of the transformed game $\Gamma_{\rho(\boldsymbol{b})}$. Since ρ^{-1} is also a symmetry of Θ, we obtain the equivalence stated in Lemma 4.

Lemma 4 *Let* ρ *be one of the axial symmetries of* Θ*, or a composition of them. Let* $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$*, and let* $\mathfrak{m}$ *(resp.* $\rho(\mathfrak{m})$*) and* $\boldsymbol{a}$ *(resp.* $\rho(\boldsymbol{a})$*) be any strategies of the game* $\Gamma_{\boldsymbol{b}}$ *(resp.* $\Gamma_{\rho(\boldsymbol{b})}$*). Then strategies* $(\mathfrak{m}, \boldsymbol{a})$ *are equilibrium strategies of* $\Gamma_{\boldsymbol{b}}$ *iff* $(\rho(\mathfrak{m}), \rho(\boldsymbol{a}))$ *are equilibrium strategies of* $\Gamma_{\rho(\boldsymbol{b})}$*.*

[18] In what follows, we show that this extends in all directions. Therefore the uniform distribution over $\Theta = [0,1]^2$ renders an *overall weak dependence* of the states relative to orthogonal dimension pairs. Reciprocally, the construction might provide a way of measuring the independence of the θs over several dimensions, *i.e.* to provide an overall measure of the symmetry of Θ. There is a substantial literature on this question, including Toth (2015).

[19] The continuity of the strategies is with regard to actions $\boldsymbol{a}_T(m_i)$ and set of states $\mathfrak{m}_T^{-1}(m_i)$, $i \in \{1,2\}$.

For example, Figure 7 represents equilibrium strategies $\rho(\mathfrak{m})$ and $\rho(\boldsymbol{a})$ associated with bias $\rho(\boldsymbol{b}) = (-b, b)$ derived from equilibrium strategies $\mathfrak{m}$ and $\boldsymbol{a}$ depicted in Figure 3, associated with bias $\boldsymbol{b} = (b, b)$. The strategies are derived from the composition of the $(0, 1)$ and $(1, 1)$-symmetries of Θ, *i.e.* the rotation $\rho_{\frac{\pi}{2}}$ of angle $\frac{\pi}{2}$ (in the trigonometric sense) and centre $\mathbb{E}[\Theta] = (\frac{1}{2}, \frac{1}{2})$. Strategies $(\mathfrak{m}_{A(c)}, \boldsymbol{a}_{A(c)})$ might be similarly derived from $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$, and one of the two half-babbling strategy profile $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$ might be similarly derived from the other.

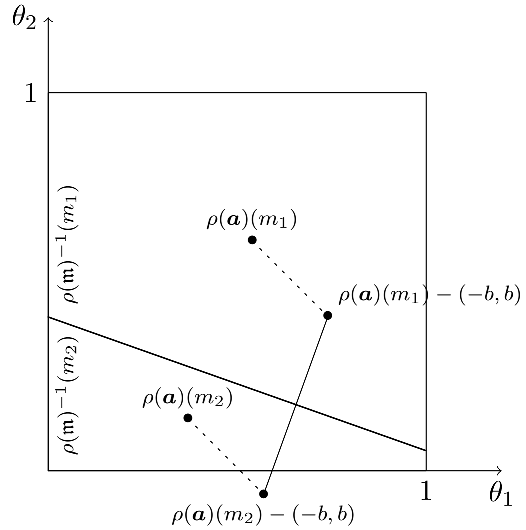

Fig. 7 Equilibrium strategies associated with the bias $\boldsymbol{b} = (-b, b)$

We can now derive the existence of an influential equilibrium for any bias as follows. First, if T spans the full border of Θ, strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ exhibited in Lemma 2 extend continuously in all directions as influential asymmetric strategies (Lemma 3 and 4). This provides us with an equilibrium strategy for all biases $\boldsymbol{b}_T$ given by (5). Since strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ are asymmetric, the extent of these biases is necessarily bounded from below by some strictly positive fixed number. Second, Lemma 1 allows the existence of equilibrium strategies to be extended to all biases $\boldsymbol{b}$ which are orthogonally projected as $\boldsymbol{b}_T$ on the (OT)s. Then we obtain equilibrium strategies for biases the extent of which is bounded from below in one of the (OT)s-dimensions and may be arbitrary large in the orthogonal dimension. Since the (OT)s cover all directions, we obtain equilibrium strategies for all biases which are bounded from below in at least one of their components. Finally, for the remaining biases $\boldsymbol{b}$ whose extent is small in both of their components, the symmetric strategies and their deviations yield the result.

Proposition 1 *For any $\boldsymbol{b} \in \mathbb{R}^2$, there exists a profile of strategies $(\mathfrak{m}, \boldsymbol{a})$ that defines an influential equilibrium of the game $\Gamma_{\boldsymbol{b}}$.*

Therefore, whatever the conflict of interest between the Sender and the Receiver, there are grounds for an agreement. Given our assumptions, any conflict of interest in organizations could influence decisions and actions.

5 Characterization of the equilibrium strategies

In Section 4, we associated a set of biases to specific equilibrium strategies, and showed that the set of strategies reached the entire bias space. Although we did not determine the set of pure strategies equilibria for a given game $\Gamma_{\boldsymbol{b}}$, we did exhibit at least one. In this section, we derive the necessary part of the equilibrium conditions by showing that the strategies exhibited are the only ones that are in equilibrium.

Let $\boldsymbol{b} \in \mathbb{R}^2$, and let $(\mathfrak{m}, \boldsymbol{a})$ be equilibrium strategies of $\Gamma_{\boldsymbol{b}}$. The law of iterated expectations gives $\mathbb{E}[\Theta] = |\mathfrak{m}^{-1}(m_1)|\boldsymbol{a}(m_1) + |\mathfrak{m}^{-1}(m_2)|\boldsymbol{a}(m_2)$, so that the line $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$ necessarily passes through $O = \mathbb{E}[\Theta]$. In particular, we necessarily have

$$(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2)) = (OT) \text{ for some } T \text{ on the border of } \Theta. \tag{6}$$

Given such a T, the set of the Sender's indifferent state $\mathcal{L}$ is the perpendicular bisector of $\boldsymbol{a}(m_1) - \boldsymbol{b}$ and $\boldsymbol{a}(m_2) - \boldsymbol{b}$, and thus, it is necessarily one of the lines perpendicular to (OT). The following lemma characterizes lines $\mathcal{L}_T$, derived from strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$, as the only perpendicular with this property.

Lemma 5 *For any $T \in \{0\} \times (0, \frac{1}{2})$, the perpendicular bisector $\mathcal{L}_T$ of $\boldsymbol{a}_T(m_1) - \boldsymbol{b}_T$ and $\boldsymbol{a}_T(m_2) - \boldsymbol{b}_T$ is the only line $\mathcal{L}$ that partitions Θ into two sets Θ_1 and Θ_2, with $|\Theta_i| > 0$, such that the line going through $\mathbb{E}[\Theta_1]$ and $\mathbb{E}[\Theta_2]$ is perpendicular to $\mathcal{L}$.*

By symmetry and according to the equivalence stated in Lemma 4, Lemma 5 extends to all Ts such that (OT) is not the horizontal, vertical nor any diagonal axes of Θ. Therefore, according to (6), out of these cases, strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ cover all possible equilibrium strategies $(\mathfrak{m}, \boldsymbol{a})$ of a given game $\Gamma_{\boldsymbol{b}}$. In the case in which $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$ is the horizontal, vertical or one of the diagonal axes of Θ, strategies $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$ and $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$, $c \in (-1, 1)$ and their transformations through one of the four symmetries of Θ cover all lines perpendicular to $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$. Hence, it necessarily includes any line that defines a Sender's indifferent set of states.

Proposition 2 *Given $\boldsymbol{b} \in \mathbb{R}^2$, a profile of equilibrium strategies of $\Gamma_{\boldsymbol{b}}$ is either*

- *an asymmetric profile of strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$, for some $T \in \{0\} \times (0, \frac{1}{2})\}$ (as defined in Lemma 2), or*
- *a symmetric profile of strategies $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$, or $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$, for some $c \in (-1, 1)\}$ (as defined in Example 2 and hereafter), or*
- *a transformation of one of these profiles of strategies by a composition of the four axial symmetries of $\Theta = [0, 1]^2$.*

Next we provide a geometric representation of influential strategies $(\mathfrak{m}, \boldsymbol{a})$ of a given game $\Gamma_{\boldsymbol{b}}$, $\boldsymbol{b} \in \mathbb{R}^2$. According to Lemma 1, $(\mathfrak{m}, \boldsymbol{a}) \in \mathcal{E}(\boldsymbol{b})$ is conditioned by the projection $\boldsymbol{b}^\pi$ of $\boldsymbol{b}$ onto the support $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$ of the induced actions (see Figure 5). For instance, $(\mathfrak{m}_T, \boldsymbol{a}_T)$ is an equilibrium profile of strategies of $\Gamma_{\boldsymbol{b}}$ *iff* $-\boldsymbol{b}$ projects as $-\boldsymbol{b}_T$ on $(\boldsymbol{a}_T(m_1)\boldsymbol{a}_T(m_2))$. Figures 8, 9 and 10 depict the inverse map of the corresponding projections associated with specific continua of equilibrium strategies $(\mathfrak{m}, \boldsymbol{a})$.

The construction is as follows. Given strategies $(\mathfrak{m}, \boldsymbol{a})$ which are equilibrium strategies of some game, we first represent the projected bias $-\boldsymbol{b}^\pi \in \mathcal{E}^{-1}((\mathfrak{m}, \boldsymbol{a}))$, which is aligned with $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$. Then for any $\boldsymbol{b} \in \mathbb{R}^2$, we have $\boldsymbol{b} \in \mathcal{E}^1((\mathfrak{m}, \boldsymbol{a}))$ *iff* $-\boldsymbol{b}$ projects as $-\boldsymbol{b}^\pi$ on $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$, *i.e.* on the $-\boldsymbol{b}^\pi$-directed line.

For instance, in Figure 8, the bold $(1, -1)$-directed segment corresponds to projected bias $-\boldsymbol{b}^\pi$ for $(\mathfrak{m}, \boldsymbol{a}) = (\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$, for all $c \in (-1, 1)$. Then bias $\boldsymbol{b} = (b_1, b_2)$ for which $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)}) \in \mathcal{E}(\boldsymbol{b})$ for some $c \in (-1, 1)$ corresponds to the points $(-b_1, -b_2)$ that project orthogonally at $-\boldsymbol{b}^\pi$, onto the line starting from the origin, *i.e.* in the corresponding $(1, -1)$-directed segment. A bias $\boldsymbol{b}$ projects orthogonally onto any segment *iff* the corresponding strategies are equilibrium strategies of $\Gamma_{\boldsymbol{b}}$.

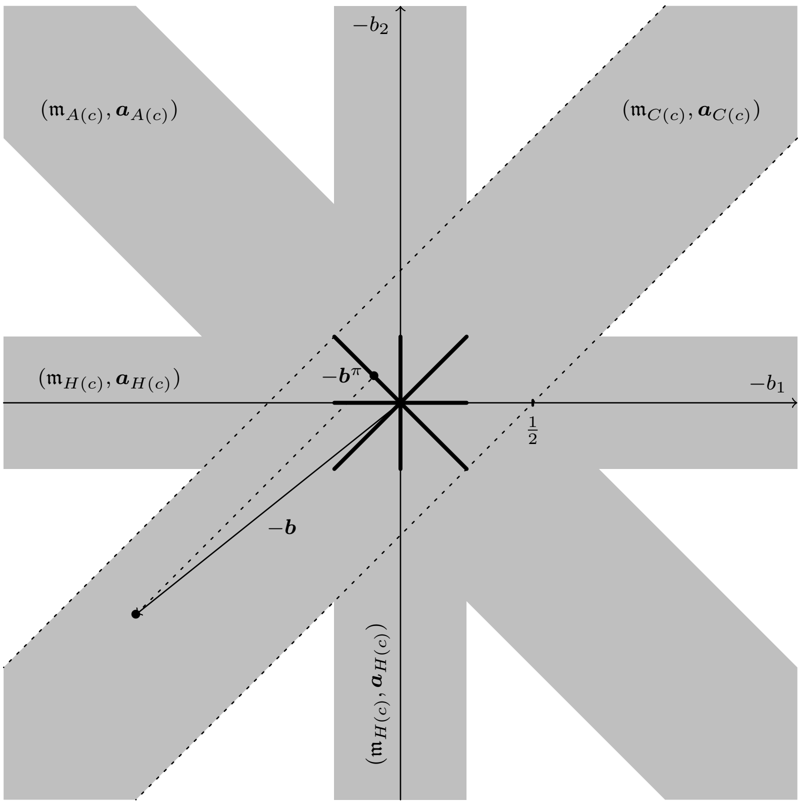

Fig. 8 Deviated symmetric equilibrium strategies $(\mathfrak{m}, \boldsymbol{a})$ associated with bias $\boldsymbol{b}$

In Figure 9, the same construction is used to derive bias $\boldsymbol{b}$ for which $(\mathfrak{m}, \boldsymbol{a}) = (\mathfrak{m}_T, \boldsymbol{a}_T) \in \mathcal{E}(\boldsymbol{b})$, for some $T \in \{0\} \times [\frac{1}{2}, 1]$. Given $-\boldsymbol{b}^{\pi}$ (corresponding to a bold point in the figure) a bias $-\boldsymbol{b}$ projects orthogonally at $-\boldsymbol{b}^{\pi}$ onto the line starting from the origin *iff* the corresponding strategies are equilibrium strategies of $\Gamma_{\boldsymbol{b}}$ (then the line from the origin characterizes the direction $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$ of the dimension of the players' agreement). The representation is extended by symmetry for T spanning the full border of Θ in Figure 10.

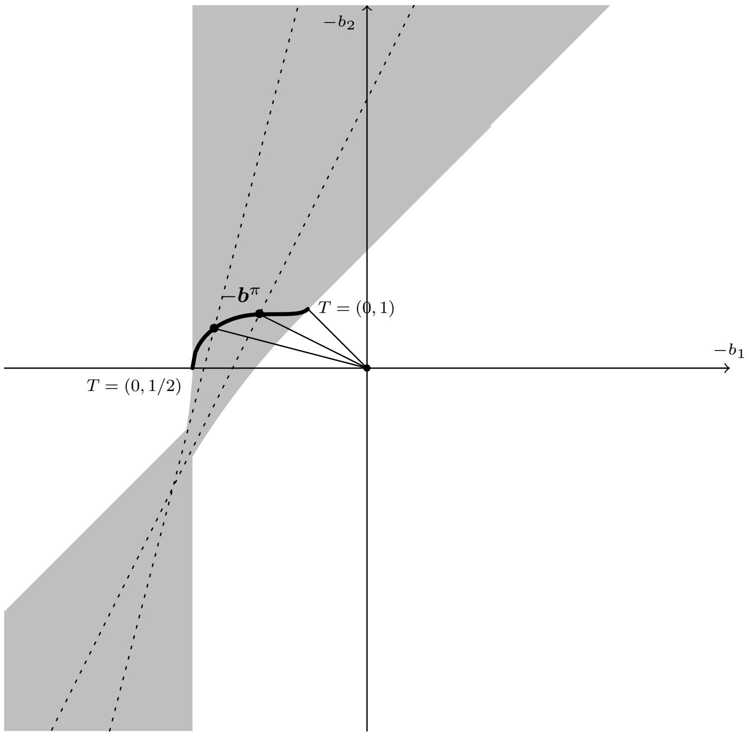

Fig. 9 Asymmetric equilibrium strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ associated with bias $\boldsymbol{b}$, for $T \in \{0\} \times [\frac{1}{2}, 1]$

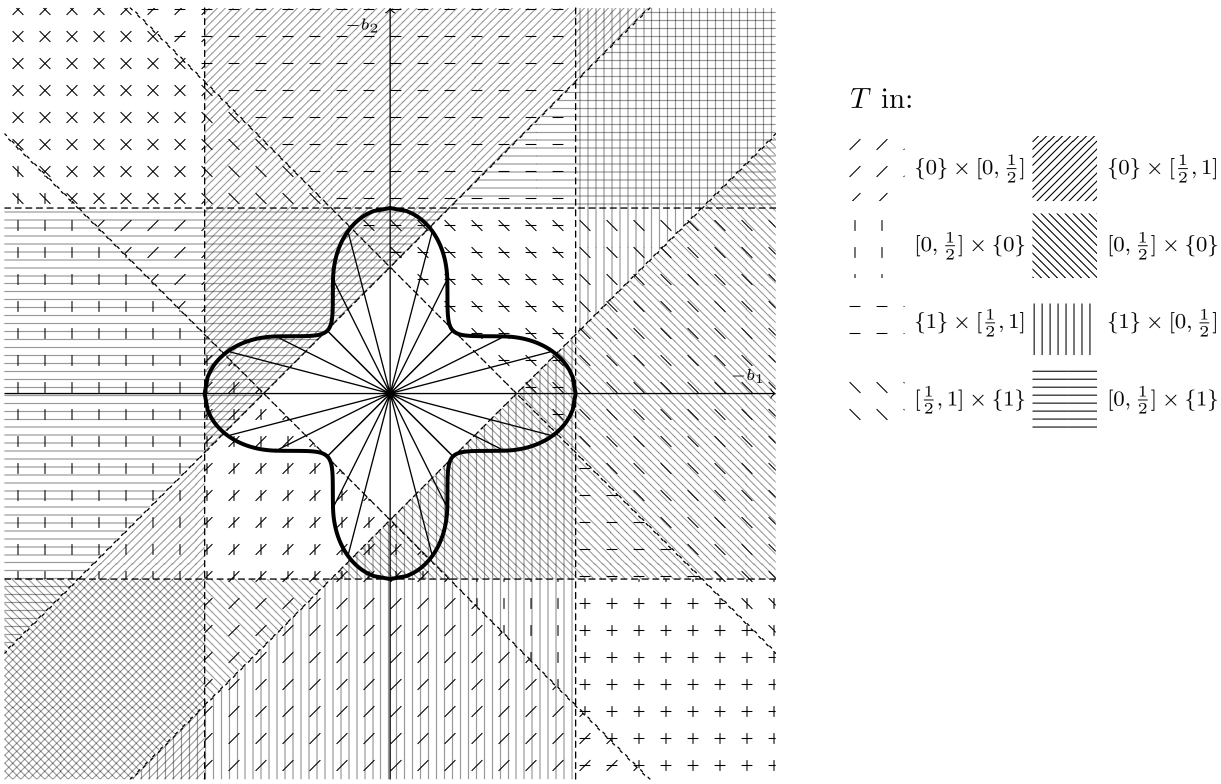

Fig. 10 Asymmetric equilibrium strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ associated with bias $\boldsymbol{b}$, for T on the border of Θ

6 Conclusion

In this paper, we characterize the set of equilibria of an extension of the Crawford and Sobel's (1982) model of information transmission. We consider two dimensions, and we assume a uniform prior, quadratic preferences, and a binary disclosure rule. To do so we show the existence of continua of equilibria in the various games $\Gamma_{\boldsymbol{b}}$, $\boldsymbol{b} \in \mathbb{R}^2$, enabling us to associate equilibria with biases. In particular, we show that at least one influential equilibrium might be associated with any bias.

The symmetries of the game parameters underpin our results. In particular, the utility functions we consider make it possible to derive equilibrium conditions in simple geometric terms. We relate these conditions to the symmetries of the uniform distribution over $[0,1]^2$. The equilibria result from the multiplicity of dimensions along which information may be disclosed, combined with the flexibility of communication in any of these dimensions.

While the existing literature suggested the possibility of communication based on strict symmetries and any conflict, and the impossibility of communication based on large asymmetries and infinite conflict, our result show that adding dimensions to the communication environment introduces the possibility of other bases for agreement between players with arbitrary large finite conflict.

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Appendix: proofs

Example 1

Given the Sender's strategy $\mathfrak{m}_C$, we derive from (1) the induced actions

$$\boldsymbol{a}_C(m_1) = \left(\frac{2}{3}, \frac{1}{3}\right), \text{ and } \boldsymbol{a}_C(m_2) = \left(\frac{1}{3}, \frac{2}{3}\right).$$

For $\mathfrak{m}_A$ (resp. the two $\mathfrak{m}_H$), we obtain $\boldsymbol{a}_A(m_1) = \left(\frac{2}{3}, \frac{2}{3}\right)$ and $\boldsymbol{a}_A(m_2) = \left(\frac{1}{3}, \frac{1}{3}\right)$ (resp. $\boldsymbol{a}_H(m_1) = \left(\frac{3}{4}, \frac{1}{2}\right)$ and $\boldsymbol{a}_H(m_2) = \left(\frac{1}{4}, \frac{1}{2}\right)$, or $\boldsymbol{a}_H(m_1) = \left(\frac{1}{2}, \frac{3}{4}\right)$ and $\boldsymbol{a}_H(m_2) = \left(\frac{1}{2}, \frac{1}{4}\right)$).

Reciprocally, given the Receiver's strategy $\boldsymbol{a} : m \mapsto \boldsymbol{a}(m)$, for instance $\boldsymbol{a}(m_1) = \boldsymbol{a}_C(m_1) = \left(\frac{2}{3}, \frac{1}{3}\right)$, and $\boldsymbol{a}(m_2) = \boldsymbol{a}_C(m_2) = \left(\frac{1}{3}, \frac{2}{3}\right)$ for the game associated to the bias $\boldsymbol{b} = (b, b)$, $b \in \mathbb{R}$, from (2), we have $\mathfrak{m}(\boldsymbol{\theta}) = m_1$ if and only if (up to a null measure set)

$$\begin{aligned}
& - \|\boldsymbol{a}_C(m_1) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2 \geq -\|\boldsymbol{a}_C(m_2) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2 \\
\iff & - \left(\frac{2}{3} - (\theta_1 + b)\right)^2 - \left(\frac{1}{3} - (\theta_2 + b)\right)^2 \geq - \left(\frac{1}{3} - (\theta_1 + b)\right)^2 - \left(\frac{2}{3} - (\theta_2 + b)\right)^2 \\
\iff & \theta_1 \geq \theta_2
\end{aligned}$$

and similarly $\mathfrak{m}(\boldsymbol{\theta}) = m_2$ if and only if $\theta_1 < \theta_2$. Therefore, strategies $\mathfrak{m}_C$ and $\boldsymbol{a}_C$ are best response in equilibrium. Proofs for the profiles of strategies $(\mathfrak{m}_A, \boldsymbol{a}_A)$ and $(\mathfrak{m}_H, \boldsymbol{a}_H)$ are similar.

Example 2

Consider the Sender's strategy

$$\mathfrak{m}_{C(c)}(\boldsymbol{\theta}) = \begin{cases} m_1, \text{ if } \theta_1 \geq \theta_2 + c, \\ m_2, \text{ if } \theta_1 < \theta_2 + c, \end{cases}$$

for some $c \leq 0$. Then from (1), we obtain, for $c > -1$,

$$\boldsymbol{a}(m_1) = \mathbb{E}[\boldsymbol{\theta} | \theta_1 \geq \theta_2 + c] = \left(\frac{1}{3}\frac{c^3 + 3c^2 + 3c - 2}{c^2 + 2c - 1}, \frac{1}{3}\frac{-c^3 + 3c - 1}{c^2 + 2c - 1}\right),$$

and

$$\boldsymbol{a}(m_2) = \mathbb{E}[\boldsymbol{\theta} | \theta_1 < \theta_2 + c] = \left(\frac{c+1}{3}, \frac{2-c}{3}\right).$$

Reciprocally, given the Receiver's strategy $\boldsymbol{a}_{C(c)}$ defined by the above actions, from (2), message m_1 is disclosed if and only if $-\|\boldsymbol{a}_{C(c)}(m_1) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2 \geq -\|\boldsymbol{a}_{C(c)}(m_2) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2$, where $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$. Set $\boldsymbol{b} = (b_1, b_1 + \Delta b)$ with $\Delta b = b_2 - b_1 \in \mathbb{R}$. Then in equilibrium, we must have

$$\theta_1 \geq \theta_2 + \frac{1}{3}\frac{c(c+2)(2c-1)}{c^2 + 2c - 1} + \Delta b,$$

so that c must solve $\Delta b = \frac{1}{3}\frac{c^3 + 3c^2 - c}{c^2 + 2c - 1}$. This defines a strictly increasing function $\Delta b \mapsto c(\Delta b)$ from $\Delta b \in (-\frac{1}{2}, 0]$ to $(-1, 0]$. A similar bijection is obtained in case $c \geq 0$ and $\Delta b \in [0, \frac{1}{2})$.

Example 3

It is straightforward to verify that each strategy of the profile of strategies $(\mathfrak{m}, \boldsymbol{a})$ stated in the example is the best response of the other, *i.e.* satisfies (1) and (2).

Lemma 1

Let $(\mathfrak{m}, \boldsymbol{a})$ be such that $(\mathfrak{m}, \boldsymbol{a}) \in \mathcal{E}(\boldsymbol{b})$ for some $\boldsymbol{b} \in \mathbb{R}^2$. Let $\boldsymbol{b}' \in \mathbb{R}^2$. From $\|\boldsymbol{x}\|^2 = \boldsymbol{x} \cdot \boldsymbol{x}$, given $\boldsymbol{a}$, we obtain for instance $\mathfrak{m}'(\boldsymbol{\theta}) = m_1$ *iff*

$$\begin{aligned}
& - \|\boldsymbol{a}(m_1) - (\boldsymbol{\theta} + \boldsymbol{b}')\|^2 \geq - \|\boldsymbol{a}(m_2) - (\boldsymbol{\theta} + \boldsymbol{b}')\|^2 \\
\iff & - \|(\boldsymbol{a}(m_1) - \boldsymbol{b} - \boldsymbol{\theta}) - (\boldsymbol{b}' - \boldsymbol{b})\|^2 \geq - \|(\boldsymbol{a}(m_2) - \boldsymbol{b} - \boldsymbol{\theta}) - (\boldsymbol{b}' - \boldsymbol{b})\|^2 \\
\iff & - \|\boldsymbol{a}(m_1) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2 \geq - \|\boldsymbol{a}(m_2) - (\boldsymbol{\theta} + \boldsymbol{b})\|^2 - 2(\boldsymbol{b}' - \boldsymbol{b}) \cdot (\boldsymbol{a}(m_1) - \boldsymbol{a}(m_2)).
\end{aligned}$$

Therefore given $\boldsymbol{a}$, the Sender of $\Gamma_{\boldsymbol{b}'}$ uses the same disclosure strategy $\mathfrak{m}'$ as the Sender of $\Gamma_{\boldsymbol{b}}$ who uses $\mathfrak{m}$ for any observed $\boldsymbol{\theta} \in \Theta$ *iff* $(\boldsymbol{b}' - \boldsymbol{b}) \cdot (\boldsymbol{a}(m_1) - \boldsymbol{a}(m_2)) = 0$. Reciprocally, strategies $\mathfrak{m}'$ and $\mathfrak{m}$ induce the same actions.

Lemma 2

Consider a line (OT) with $O = \mathbb{E}[\Theta] = (\frac{1}{2}, \frac{1}{2})$ and $T \in \{0\} \times (\frac{1}{2}, 1)$ (see Figure 6 in the main text). We seek to show that for any such line, there exists a game $\Gamma_{\boldsymbol{b}_T}$ such that (OT) supports the two induced actions of the Receiver of $\Gamma_{\boldsymbol{b}_T}$, for some profile $(\mathfrak{m}_T, \boldsymbol{a}_T)$ of equilibrium strategies.

Since for any profile $(\mathfrak{m}, \boldsymbol{a})$ of equilibrium strategies, line $(\boldsymbol{a}(m_1)\boldsymbol{a}(m_2))$ is perpendicular to the Sender's set of indifferent states $\mathcal{L} = \{\boldsymbol{\theta},\, U_S(\boldsymbol{\theta}, \boldsymbol{a}(m_1)) = U_S(\boldsymbol{\theta}, \boldsymbol{a}(m_2))\}$, a necessary equilibrium condition is the orthogonality of $\mathcal{L}$ and (OT). Given $\mathcal{L}_T$ perpendicular to (OT), let us denote $\Theta_1(\mathcal{L}_T)$ and $\Theta_2(\mathcal{L}_T)$ the two regions situated on the sides of $\mathcal{L}_T$, with for instance $O \in \Theta_1(\mathcal{L}_T)$.[20] Then we have to show that the players' strategies $\mathfrak{m}_T$ and $\boldsymbol{a}_T$ given by

$$\mathfrak{m}_T(\theta) = \begin{cases} m_1 \text{ if } \boldsymbol{\theta} \in \Theta_1(\mathcal{L}_T), \\ m_2 \text{ if } \boldsymbol{\theta} \in \Theta_2(\mathcal{L}_T), \end{cases} \quad \text{and } \boldsymbol{a}_T(m) = \begin{cases} \mathbb{E}[\Theta_1(\mathcal{L}_T)] \text{ if } m = m_1, \\ \mathbb{E}[\Theta_2(\mathcal{L}_T)] \text{ if } m = m_2, \end{cases}$$

define a profile of equilibrium strategies which satisfies $\boldsymbol{a}_T(m_i) \in (OT)$ for $i \in \{1, 2\}$. Note that it is sufficient to show that $\boldsymbol{a}_T(m_2) \in (OT)$, since we necessarily have $\boldsymbol{a}_T(m_1) \in (O\boldsymbol{a}_T(m_2)) = (OT)$.

To do this, let us consider the orthogonal coordinate system $\mathcal{R}_T(O, x, y)$, with $(y^- O y^+)$ supported by (OT) (see Figure 11).

[20] By setting $O \in \Theta_2(\mathcal{L}_T)$, we can derive the message symmetric profile of strategies $(\mathfrak{m}'_T, \boldsymbol{a}'_T)$ such that for any $\boldsymbol{\theta} \in \Theta$, $\mathfrak{m}'_T(\boldsymbol{\theta}) = m_1$ *iff* $\mathfrak{m}_T(\boldsymbol{\theta}) = m_2$, and $\boldsymbol{a}'_T(m_1) = \boldsymbol{a}_T(m_2)$, $\boldsymbol{a}'_T(m_2) = \boldsymbol{a}_T(m_1)$.

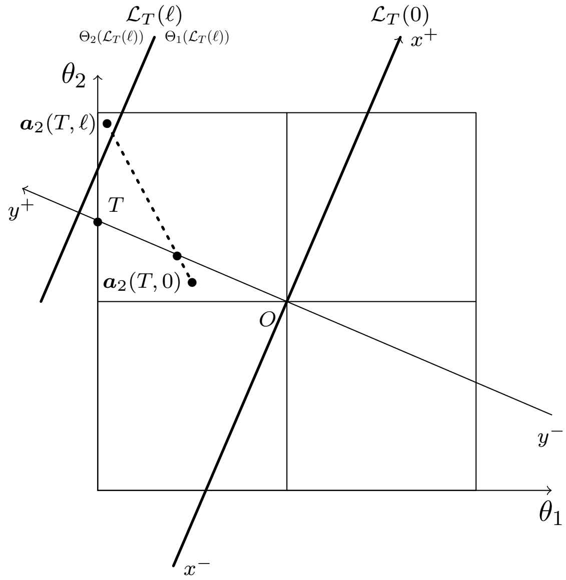

Fig. 11 $\mathcal{L}_T(\ell)$ and subsequent $\boldsymbol{a}_2(T,\ell)$

Given $\ell \in \mathbb{R}$, let $\mathcal{L}_T(\ell)$ be the line perpendicular to (OT) and situated at a distance ℓ of O, with $O \notin \Theta_2(\mathcal{L}_T(\ell))$, and let

$$\boldsymbol{a}_2(T,\ell) = \mathbb{E}[\boldsymbol{\theta}|\boldsymbol{\theta} \in \Theta_2(\mathcal{L}_T(\ell))]$$

denote the corresponding expectation. Let $\overline{\ell}$ be such that $(0,1) \in \mathcal{L}_T(\overline{\ell})$. For any $T \in \{0\} \times (\frac{1}{2},1)$, if ℓ is sufficiently close to $\overline{\ell}$, then $\Theta_2(\mathcal{L}_T(\ell)) \subset \{(x,y),\, x>0\}$, and thus $x(\boldsymbol{a}_2(T,\ell)) > 0$ for any such ℓ. Therefore, by continuity, if there exists $\underline{\ell}$ such that

$$x(\boldsymbol{a}_2(T,\underline{\ell})) < 0, \tag{7}$$

then there is some $\ell^* \in (\underline{\ell},\overline{\ell})$ such that $x(\boldsymbol{a}_2(A,\ell^*)) = 0$. Let us show that for any $T \in \{0\} \times (\frac{1}{2},1)$,

$$x(\boldsymbol{a}_2(T,0)) < 0, \tag{8}$$

i.e. $\underline{\ell} = 0$ satisfies Condition (7).

Let us set $\Theta^+ = \{\boldsymbol{\theta} \in \Theta,\, x(\boldsymbol{\theta}) > 0, y(\boldsymbol{\theta}) > 0\}$ and $\Theta^- = \{\boldsymbol{\theta} \in \Theta,\, x(\boldsymbol{\theta}) < 0, y(\boldsymbol{\theta}) > 0\}$, and $\boldsymbol{a}_2^+ = \mathbb{E}[\Theta^+]$, $\boldsymbol{a}_2^- = \mathbb{E}[\Theta^-]$ (see Figure 12).

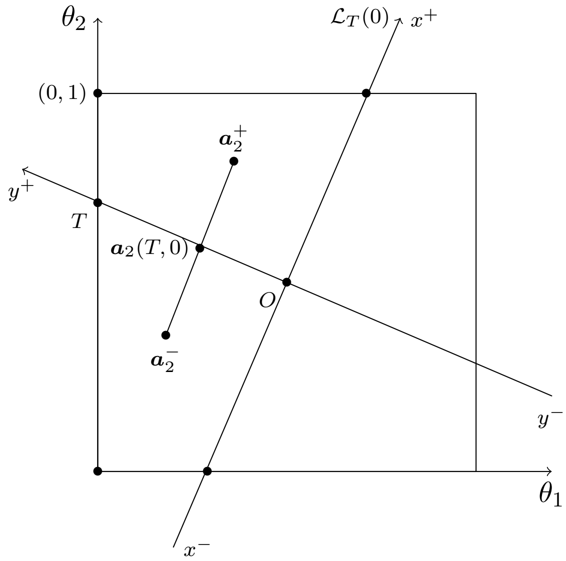

Fig. 12 $\boldsymbol{a}_2(T,0) = \frac{\boldsymbol{a}_2^+ + \boldsymbol{a}_2^-}{2}$

Then from the law of iterated expectations, we have

$$(|\Theta^+| + |\Theta^-|)\boldsymbol{a}_2(T,0) = |\Theta^+|\boldsymbol{a}_2^+ + |\Theta^+|\boldsymbol{a}_2^-,$$

where $|\cdot|$ denotes the Lebesgue measure. Now note that rotation $\rho_{\frac{\pi}{2}}$ of angle $\frac{\pi}{2}$ and centre O transforms Θ^+ to Θ^- point wise, so that for any $\boldsymbol{\theta} \in \Theta$, we have $(x(\boldsymbol{\theta}), y(\boldsymbol{\theta})) \in \Theta^+$ *iff* $(-y(\boldsymbol{\theta}), x(\boldsymbol{\theta})) \in \Theta^-$. Then on the one hand, we have $|\Theta^+| = |\Theta^-|$, which gives

$$\boldsymbol{a}_2(T,0) = \frac{\boldsymbol{a}_2^+ + \boldsymbol{a}_2^-}{2},$$

and on the other hand, we have $\rho_{\frac{\pi}{2}}(\boldsymbol{a}_2^+) = \boldsymbol{a}_2^-$, which gives

$$\begin{cases} x(\boldsymbol{a}_2^+) = y(\boldsymbol{a}_2^-), \\ y(\boldsymbol{a}_2^+) = -x(\boldsymbol{a}_2^-). \end{cases}$$

Consequently, Condition (8) is satisfied *iff* $\frac{x(\boldsymbol{a}_2^-)+x(\boldsymbol{a}_2^+)}{2} < 0$, *i.e.*

$$x(\boldsymbol{a}_2^+) < y(\boldsymbol{a}_2^+). \tag{9}$$

Now let T' be the point such that $\rho_{\frac{\pi}{2}}(T') = T$ (see Figure 13), and let us partition the set Θ^+ according to triangles $\Delta = (OTT')$ and $\Delta' = \Theta^+ \setminus (OTT')$. Again, from the law of iterated expectations, $\boldsymbol{a}_2^+$ is situated on line (GG'), where $G = \mathbb{E}[\boldsymbol{\theta}|\boldsymbol{\theta} \in \Delta]$ and $G' = \mathbb{E}[\boldsymbol{\theta}|\boldsymbol{\theta} \in \Delta']$ are the respective mass centres of triangles Δ and Δ'.[21]

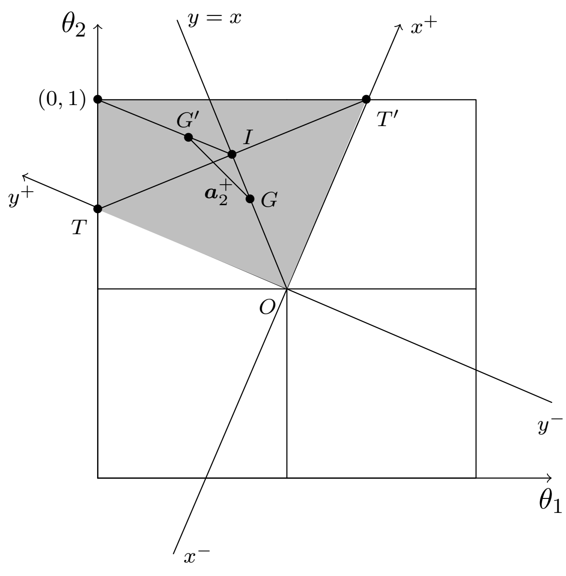

Fig. 13 $y(\boldsymbol{a}_2^+) > x(\boldsymbol{a}_2^+)$

Note that from $OT = OT'$, median (OI) of (OTT') has equation $x = y$ in $\mathcal{R}(0, x, y)$. Therefore $x(G) = y(G)$ and Condition (9) holds *iff*

$$x(G') < y(G'). \tag{10}$$

Finally, note that G' is situated on the median line of Δ' which passes through I and the point $(0,1)$, between I and $(0,1)$ which satisfy $y(I) = x(I)$ and $x((0,1)) < y((0,1))$ respectively. Therefore $G' \in \{(x,y),\, y > x\}$ and Condition (10) holds.

[21] Bearing in mind that the mass centre of a triangle is situated at the intersection of the median lines.

Since (10) hold, so do Conditions (9), (8) and (7). Therefore we have $x(\boldsymbol{a}_2(T,0)) < 0$, and we obtain the existence of $\ell^* \in (0, \bar{\ell})$ such that $\boldsymbol{a}_2(T, \ell^*) \in (OT)$ as wanted.

Next, we define the game $\Gamma_{\boldsymbol{b}_T}$. We have to find some $\boldsymbol{b}_T \in \mathbb{R}^2$ such that strategies $\mathfrak{m}_T$ and $\boldsymbol{a}_T$ given by

$$\mathfrak{m}_T(\boldsymbol{\theta}) = \begin{cases} m_1 \text{ if } \boldsymbol{\theta} \in \Theta_1(\mathcal{L}_T(\ell^*)), \\ m_2 \text{ if } \boldsymbol{\theta} \in \Theta_2(\mathcal{L}_T(\ell^*)), \end{cases} \quad \text{and } \boldsymbol{a}_T(m) = \begin{cases} \boldsymbol{a}_1(T, \ell^*) \text{ if } m = m_1, \\ \boldsymbol{a}_2(T, \ell^*) \text{ if } m = m_2, \end{cases}$$

are equilibrium strategies of $\Gamma_{\boldsymbol{b}_T}$. A sufficient condition is that $\mathcal{L}_T(\ell^*)$ is the perpendicular bisector of $\boldsymbol{a}_T(m_1) - \boldsymbol{b}_T$ and $\boldsymbol{a}_T(m_2) - \boldsymbol{b}_T$. Let us set $\boldsymbol{b}_T = \frac{\boldsymbol{a}_T(m_1)+\boldsymbol{a}_T(m_2)}{2} - \boldsymbol{\theta}_T$, where $\boldsymbol{\theta}_T = (OT) \cap \mathcal{L}_T(\ell^*)$, so that $\boldsymbol{\theta}_T$ is the mid-point of $\boldsymbol{a}_T(m_1) - \boldsymbol{b}_T$ and $\boldsymbol{a}_T(m_2) - \boldsymbol{b}_T$. Since $\mathcal{L}_T(\ell^*) \perp (OT)$, $\mathcal{L}_T(\ell^*)$ has the desired property.

Lemma 3

We show that strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ extend continuously to one of the symmetric deviated strategies. By construction, when T spans continuously the open segment $\{0\} \times (\frac{1}{2}, 1)$, line $\mathcal{L}_T$, and therefore[22] sets $\mathfrak{m}_T^{-1}(m_i)$, $i \in \{1, 2\}$, actions $\boldsymbol{a}_T(m_1)$ and $\boldsymbol{a}_T(m_2)$, and bias

$$T \mapsto \boldsymbol{b}_T = \frac{\boldsymbol{a}_T(m_1) + \boldsymbol{a}_T(m_2)}{2} + \boldsymbol{\theta}_T \in (OT) \tag{11}$$

can all be chosen to move continuously.

When $T \to T_1 = (0, 1)$, we have: (a) actions tend to be onto the diagonal line (OT_1), and (b) Sender's set of indifferent states $\mathcal{L}_T$ tends to intersect the diagonal line orthogonally at some $\boldsymbol{\theta}_{T_1} = (\theta_{T_1}, 1 - \theta_{T_1})$, for some $0 \leq \theta_{T_1} \leq \frac{1}{2}$.

If $\theta_{T_1} > 0$, this necessarily identifies a $(1, 1)$-symmetric (deviated if $\theta_{T_1} < \frac{1}{2}$) comparative profile of strategies $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$, $c \in (0, 1)$, since, as shown in Example 2, when c spans $(0, 1)$, the support of the induced actions and the sets of indifferent states associated with profiles $(\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$ span all such orthogonal lines.

If $\theta_{T_1} = 0$, this identifies a babbling equilibrium. Let us show that this case does not occur.

Let us set $T(0, \frac{1}{2} + t)$, where $t > 0$ is sufficiently close to T_1, and let us denote $(z(t), 1)$, with $0 \leq z(t) \leq \frac{1}{2}$, the coordinates of the intersection of $\mathcal{L}_T$ and the upper border of Θ (see Figure 14).

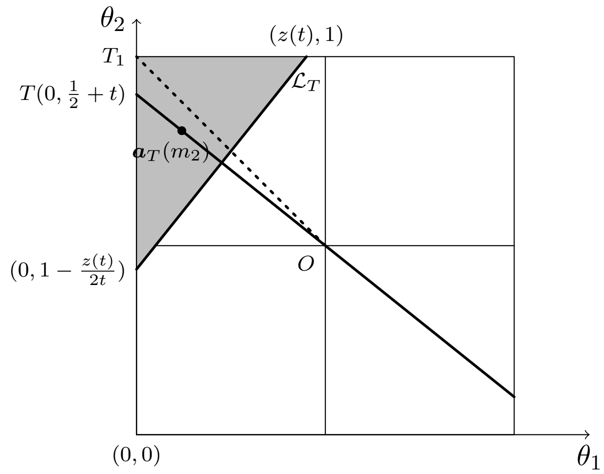

Fig. 14 $\Theta_2(\mathcal{L}_T)$ as $T \to T_1$

[22] The rule applied for choosing which region on the side of $\mathcal{L}_T(\ell^*)$ is associated with m_1 and which one is associated with m_2 must be consistent for all Ts. For instance, following the choice made in the proof of Lemma 2, we have to choose $O \in \Theta_1(\mathcal{L}_T)$ for every T.

Suppose $\theta_{T_1} = 0$. This implies that if T is sufficiently close to T_1, then line $\mathcal{L}_T$ intersects the diagonal line (OT_1) at some $\boldsymbol{\theta}_T = (\theta_T, 1-\theta_T)$ with $\lim\limits_{T\to T_1} \theta_T = \theta_{T_1} = 0$, so that $\lim\limits_{t\to\frac{1}{2}} z(t) = 0$.

Let us show that $\lim\limits_{t\to\frac{1}{2}} z(t) > 0$, so that $\theta_{T_1} = 0$ does not hold.

From $\mathcal{L}_T \perp (OT)$, it is easy to derive that the vertex of $\Theta_2(\mathcal{L}_T) = \mathfrak{m}_T^{-1}(m_2)$ are given by $(0,1)$, $(z(t),1)$ and $(0, 1-\frac{z(t)}{2t})$, as depicted in Figure 14. The coordinates of the mass centre $\boldsymbol{a}(m_2)$ of $\Theta_2(\mathcal{L}_T)$ are given by $\boldsymbol{a}(m_2) = \left(\frac{1}{3}z(t), 1-\frac{1}{3}\frac{z(t)}{2t}\right)$. From the equilibrium condition $\boldsymbol{a}(m_2) \in (OT)$, with $(OT) = \{(\theta_1,\theta_2) \in \Theta,\, \theta_2 = -2t\theta_1 + \frac{1}{2} + t\}$, we obtain, for any t sufficiently close to $\frac{1}{2}$,

$$1 - \frac{1}{3}\frac{z(t)}{2t} = -2t\frac{1}{3}z(t) + \frac{1}{2} + t.$$

This gives $z(t) = \frac{t-\frac{1}{2}}{\frac{-1}{6t}+\frac{2}{3}t} = \frac{6t}{2(2t+1)}$, and in particular, $\lim\limits_{t\to\frac{1}{2}} z(t) = \frac{3}{4} > 0$, as expected.

Let us now look at the other limit of $(\mathfrak{m}_T, \boldsymbol{a}_T)$, *i.e.* when $T \to T_0 = (0, \frac{1}{2})$. The arguments are similar, but $\Theta_2(\mathcal{L}_T)$ has a more complex shape.

We have: (a) actions $\boldsymbol{a}_T(m_1)$ and $\boldsymbol{a}_T(m_2)$ tend to be supported on the *horizontal line* (OT_0), and (b) line $\mathcal{L}_T$ of the Sender's indifferent states tends to intersect the horizontal line orthogonally at some $\boldsymbol{\theta}_{T_0} = (\theta_{T_0}, \frac{1}{2})$, for some $0 \le \theta_{T_0} \le \frac{1}{2}$.

Again, if $\theta_{T_0} > 0$, this necessarily identifies a $(0,1)$-symmetric (deviated if $\theta_{T_0} < \frac{1}{2}$) half-babbling profile of strategies $(\mathfrak{m}_{H(c)}, \boldsymbol{a}_{H(c)})$, $c \in (0,1)$, whereas if $\theta_{T_0} = 0$, this identifies a babbling equilibrium. We show that this cannot occur.

Suppose $\theta_{T_0} = 0$. This implies that if T is sufficiently close to T_0, the set of indifferent states $\mathcal{L}_T$ associated with strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ intersects the horizontal line (OT_0) at $\boldsymbol{\theta}_T = (\theta_T, \frac{1}{2})$, with $\lim\limits_{T\to T_1} \theta_T = \theta_{T_1} = 0$. Since $\mathcal{L}_T$ is orthogonal to (OT), and since (OT) tends to the horizontal line (OT_0), we have that $\mathcal{L}_T$ tends to the vertical line supported by $(0,0) - (0,1)$.

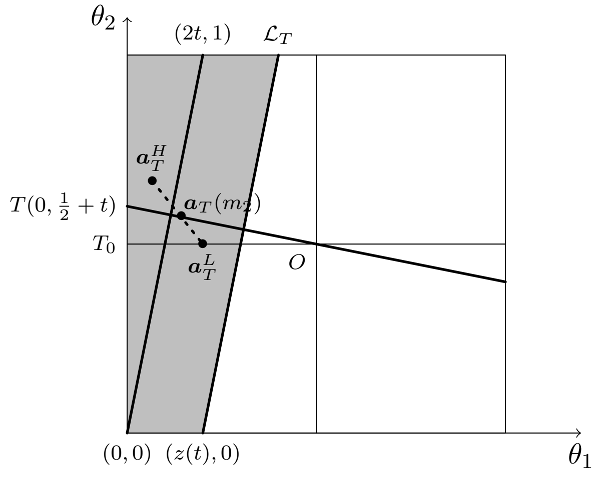

Fig. 15 Partition of $\Theta_2(\mathcal{L}_T)$ and subsequent expectations

Let us consider $T(0, \frac{1}{2}+t)$, with $t > 0$, sufficiently close to T_0, and let us parameterize the set $\Theta_2(\mathcal{L}_T) = \mathfrak{m}_T^{-1}(m_2)$ through a partition consisting of a triangle with $(0,0)$ as vertex and hypotenuse parallel to $\mathcal{L}_T$, and a parallelogram of width $z(t)$ for some $z(t) \in [0, \frac{1}{2})$, as depicted in Figure 15. Equality $\theta_{T_0} = 0$ implies $\lim\limits_{t\to 0} z(t) = 0$ and thus we seek to show $\lim\limits_{t\to 0} z(t) > 0$ in order to obtain a contradiction.

From the law of iterated expectations, $\boldsymbol{a}(m_2)$ might be written $\boldsymbol{a}_T(m_2) = (1-\lambda(z(t),t))\boldsymbol{a}_T^H + \lambda(z(t),t)\boldsymbol{a}_T^L$, with $\lambda(z(t),t) = \frac{V(z(t))}{V(z(t))+t} \in [0,1]$, where $V(z(t))$ and t are the respective Lebesgue measure of the parallelogram and the triangle. The coordinates of their respective mass centre are given by $\boldsymbol{a}_T^L = \left(\frac{2t+z(t)}{2}, \frac{1}{2}\right)$ and $\boldsymbol{a}_T^H = \left(\frac{2}{3}t, \frac{2}{3}\right)$ respectively. From $\boldsymbol{a}_T(m_2) \in (OT)$, with $(OT) = \{(\theta_1, \theta_2) \in \Theta, \theta_2 = -2t\theta_1 + \frac{1}{2} + t\}$, we obtain, for any t sufficiently close to 0,

$$(1-\lambda(z(t),t))\frac{2}{3} + \lambda(z(t),t)\frac{1}{2} = -2t\left((1-\lambda(z(t),t))\frac{2}{3}t + \lambda(z(t),t)\frac{2t+z(t)}{2}\right) + \frac{1}{2} + t. \tag{12}$$

Let us set $\lambda = \lim_{t\to 0} \lambda(z(t),t) \in [0,1]$. Then when $t \to 0$, Equation (12) gives $(1-\lambda)\frac{2}{3} + \lambda\frac{1}{2} = \frac{1}{2}$, so that

$$\lambda = \lim_{t\to 0} \lambda(z(t),t) = 1. \tag{13}$$

Now for $t > 0$, Equation (12) is

$$z(t) = \frac{-1}{6\lambda(z(t),t)} \frac{1-\lambda(z(t),t)}{t} + \frac{1}{\lambda(z(t),t)} - \frac{4}{3}t\frac{(1-\lambda(z(t),t))}{\lambda(z(t),t)} - 2t, \tag{14}$$

and note that

$$\frac{1-\lambda(z(t),t)}{t} = \frac{1-\frac{V(z(t))}{V(z(t))+t}}{t} = \frac{t}{t(V(z(t))+t)} = \frac{1}{V(z(t))+t}.$$

If $\lim_{t\to 0} z(t) = 0$, then we have $\lim_{t\to 0} V(z(t)) = 0$, and thus

$$\lim_{t\to 0} \frac{1-\lambda(z(t),t)}{t} = \infty.$$

Now (13) and (14) give the impossible $0 = \infty$. Therefore $\lim_{t\to 0} z(t) > 0$ as expected.

Proposition 1

We show that the strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ exhibited in Lemma 2 allows any bias $\boldsymbol{b} \in \mathbb{R}^2$ to be associated with an influential equilibrium.

According to Lemma 3, maps

$$T \mapsto \boldsymbol{b}_T = \frac{\boldsymbol{a}_T(m_1) + \boldsymbol{a}_T(m_2)}{2} + \boldsymbol{\theta}_T,$$

and $T \mapsto \boldsymbol{a}_T(m_i)$, $i \in \{1,2\}$, can be extended continuously to the *closed* segment $\{0\} \times [\frac{1}{2}, 1]$, with *non babbling* strategies $(\mathfrak{m}_T, \boldsymbol{a}_T)$ for each T. According to the four symmetries of Θ, and according to Lemma 4, these maps extend continuously to the full border of Θ. Note that this needs a suitable choice of *message assignment* across the Ts. W.l.o.g., we assume $O = \mathbb{E}[\Theta] \in \mathfrak{m}_T^{-1}(m_1)$ for all T.

Now let us consider the continuous maps

$$b_\Theta : T \mapsto \|\boldsymbol{b}_T\|,$$

and, given $\boldsymbol{b} \in \mathbb{R}^2$,

$$\pi_{\boldsymbol{b}} : T \mapsto \left\| \boldsymbol{b} \cdot \frac{\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)}{\|\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)\|} \right\|.$$

Each map is defined on the full border of Θ. Map $\pi_{\boldsymbol{b}}$ is the normalized norm of the projection of $\boldsymbol{b}$ onto (OT). Note that

$$b_\Theta(T) = \pi_{\boldsymbol{b}}(T) \text{ iff } \|\boldsymbol{b}_T\|\|\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)\| = \|\boldsymbol{b} \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2))\|,$$

and that $\|\boldsymbol{b}_T\|\|\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)\| = \boldsymbol{b}_T \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2))$ according to our choice $O \in \mathfrak{m}_T^{-1}(m_1)$ (the alternative choice would have given $\|\boldsymbol{b}_T\|\|\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)\| = -\boldsymbol{b}_T \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2))$). Then we obtain:

$$b_\Theta(T) = \pi_{\boldsymbol{b}}(T) \text{ iff } \begin{cases} \boldsymbol{b}_T \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)) = \boldsymbol{b} \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)), \text{ or} \\ \boldsymbol{b}_T \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)) = -\boldsymbol{b} \cdot (\boldsymbol{a}_T(m_1) - \boldsymbol{a}_T(m_2)). \end{cases}$$

Now consider the central symmetry ρ of Θ. Strategies $\rho(\mathfrak{m}_T)$ and $\rho(\boldsymbol{a}_T)$ are given by strategies $\mathfrak{m}_{\rho(T)}$ and $\boldsymbol{a}_{\rho(T)}$, associated with bias $\boldsymbol{b}_{\rho(T)} = \rho(\boldsymbol{b}_T) = -\boldsymbol{b}_T$, and we have $\boldsymbol{a}_T(m_1) = \boldsymbol{a}_{\rho(T)}(m_2)$, $\boldsymbol{a}_T(m_2) = \boldsymbol{a}_{\rho(T)}(m_1)$. Then the second equation above might be written

$$\boldsymbol{b}_{\rho(T)} \cdot (\boldsymbol{a}_{\rho(T)}(m_1) - \boldsymbol{a}_{\rho(T)}(m_2)) = \boldsymbol{b} \cdot (\boldsymbol{a}_{\rho(T)}(m_1) - \boldsymbol{a}_{\rho(T)}(m_2)).$$

From Lemma 1, we obtain:

$$b_\Theta(T) = \pi_{\boldsymbol{b}}(T) \textit{ iff } [(\mathfrak{m}_T, \boldsymbol{a}_T) \in \mathcal{E}(\boldsymbol{b}), \text{ or } (\mathfrak{m}_{\rho(T)}, \boldsymbol{a}_{\rho(T)}) \in \mathcal{E}(\boldsymbol{b})].$$

This means: if $\boldsymbol{b}$ projects as $\boldsymbol{b}_T$ or $-\boldsymbol{b}_T$ onto (OT), then respectively $(\mathfrak{m}_T, \boldsymbol{a}_T)$ or $(\mathfrak{m}_{\rho(T)}, \boldsymbol{a}_{\rho(T)})$ are profiles of equilibrium strategies of $\Gamma_{\boldsymbol{b}}$. In particular, we have:

$$\text{if } b_\Theta(T) = \pi_{\boldsymbol{b}}(T) \text{ for some } T, \text{ then } \mathcal{E}(\boldsymbol{b}) \neq \varnothing. \tag{15}$$

Next, we apply Bolzano's Theorem to show that there is always such a T, unless $\boldsymbol{b}$ is small. Let $[\underline{b}_\Theta, \bar{b}_\Theta] \subset \mathbb{R}^+$ denotes the set of values spanned by b_Θ (continuous) when T spans the border of Θ (compact). We have $\underline{b}_\Theta \geq 0$, and according to Example 3, $\bar{b}_\Theta > 0$. Concerning $\pi_{\boldsymbol{b}}$, since (OT) spans every directions when T spans the border of Θ, $\pi_{\boldsymbol{b}}(T)$ spans $[0, \|\boldsymbol{b}\|]$ ($\pi_{\boldsymbol{b}}$ has been normalized).

Note that if T is such that $(OT) \perp \boldsymbol{b}$, then $\pi_{\boldsymbol{b}}(T) = 0$, and thus we have

$$b_\Theta(T) - \pi_{\boldsymbol{b}}(T) = b_\Theta(T) - 0 \geq 0$$

for any such T. Therefore, by continuity and Bolzano's Theorem, if there exists T such that

$$b_\Theta(T) - \pi_{\boldsymbol{b}}(T) \leq 0 \tag{16}$$

then there exists T onto the border of Θ such that $b_\Theta(T) - \pi_{\boldsymbol{b}}(T) = 0$, which implies $\mathcal{E}(\boldsymbol{b}) \neq \varnothing$ according to (15).

First, let us consider the case of a large bias. If $\|\boldsymbol{b}\| \geq \bar{b}_\Theta$, choose T such that (OT) and $\boldsymbol{b}$ have the same direction, so that $\pi_{\boldsymbol{b}}(T) = \|\boldsymbol{b}\|$. Then we have

$$b_\Theta(T) - \pi_{\boldsymbol{b}}(T) \leq \bar{b}_\Theta - \|\boldsymbol{b}\| \leq 0,$$

and (16) holds for such a T. Thus $\mathcal{E}(\boldsymbol{b}) \neq \varnothing$ for all biases $\boldsymbol{b}$ with $\|\boldsymbol{b}\| \geq \bar{b}_\Theta$.

Next, consider the case of a small bias. Suppose that (16) does not hold when T spans the full border of Θ. In particular, it does not hold if $T_1 = (0,1)$ or $T_2 = (0,0)$, so that for $i \in \{1,2\}$, $b_\Theta(T_i) - \pi_{\boldsymbol{b}}(T_i) > 0$, *i.e.*

$$\pi_{\boldsymbol{b}}(T_i) < b_\Theta(T_i). \tag{17}$$

Note that T_1 and T_2 are such that $(\mathfrak{m}_{T_1}, \boldsymbol{a}_{T_1}) = (\mathfrak{m}_{C(c)}, \boldsymbol{a}_{C(c)})$ for some $c \in (-1,0)$, and $(\mathfrak{m}_{T_2}, \boldsymbol{a}_{T_2}) = (\mathfrak{m}_{A(c'}, \boldsymbol{a}_{A(c')}) = \rho_{\frac{\pi}{2}}(\mathfrak{m}_{C(c')}, \boldsymbol{a}_{C(c')})$ for some $c' \in (-1,0)$.[23] In particular, according to Example 2, we have $\boldsymbol{b}_{T_i} = (b_1, b_2)$ for some $(b_1, b_2) \in \mathbb{R}^2$ such that $|b_2 - b_1| < \frac{1}{2}$. Furthermore, since $\boldsymbol{b}_{T_1}$ is supported on (OT_1), we have $b_2 = -b_1$. Hence $|b_2 - b_1| = 2|b_1|$ so that $|b_1| < \frac{1}{4}$. Then we obtain:

$$b_\Theta(T_1) = \|\boldsymbol{b}_{T_1}\| = \sqrt{b_1^2 + b_2^2} = |b_1|\sqrt{2} < \frac{\sqrt{2}}{4}.$$

Similarly, we have

$$b_\Theta(T_2) = \|\boldsymbol{b}_{T_2}\| < \frac{\sqrt{2}}{4}.$$

[23] By construction we derive $c' = c$ from the symmetries, but this is not useful for the proof.

Consequently, from (17), $\boldsymbol{b}$ projects onto (OT_1) and (OT_2) as vectors the norm of them is less than $\frac{\sqrt{2}}{4}$. Since (OT_1) and (OT_2) are orthogonal, we can deduce

$$\|\boldsymbol{b}\| = \sqrt{\pi_{\boldsymbol{b}}(T_1)^2 + \pi_{\boldsymbol{b}}(T_2)^2} < \sqrt{b_\Theta(T_1)^2 + b_\Theta(T_2)^2} < \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Then, setting $\boldsymbol{b} = (b'_1, b'_2) \in \mathbb{R}^2$, it is necessary that $|b'_1| < \frac{1}{2}$ and $|b'_2| < \frac{1}{2}$, so that $|b'_2 - b'_1| \le |b'_2| - |b'_1| < \frac{1}{2}$. Again, according to Example 2, there is some c'' such that $(\mathfrak{m}_{C(c'')}, \boldsymbol{a}_{C(c'')})$ is an equilibrium strategy of $\Gamma_{\boldsymbol{b}}$, and thus $\mathcal{E}(\boldsymbol{b}) \neq \varnothing$.

Lemma 5

We show that given $T(0, \frac{1}{2} + t) \in \{0\} \times (\frac{1}{2}, 1)$, for some $t \in (0, \frac{1}{2})$, there is a unique line $\mathcal{L}_T$ such that (i) $\mathcal{L}_T$ is orthogonal to (OT), (ii) $\mathcal{L}_T$ partitions Θ through $\Theta = \Theta_1 \cup \Theta_2$, with $|\Theta_i| > 0$, $i \in \{1, 2\}$ and w.l.o.g. $O \in \Theta_1$, and (iii) $\mathbb{E}[\Theta_i] \in (OT)$, $i \in \{1, 2\}$. Notice that the existence of $\mathcal{L}_T$ has already been established and it is sufficient to show that if conditions (i), (ii) and (iii) are satisfied for some $\mathcal{L}$, then $\mathcal{L}$ is uniquely determined.

Let us parameterize any line $\mathcal{L}$ satisfying these conditions through

$$\mathcal{L} = \{\boldsymbol{\theta} \in \Theta,\, \theta_1 = 2e\theta_2 + c - e\},$$

so that it passes through the points $(c, \frac{1}{2})$, with $c \le \frac{1}{2}$, and through $(c + e, 1)$ and $(c - e, 0)$ for some $e > 0$ (see Figure 16).

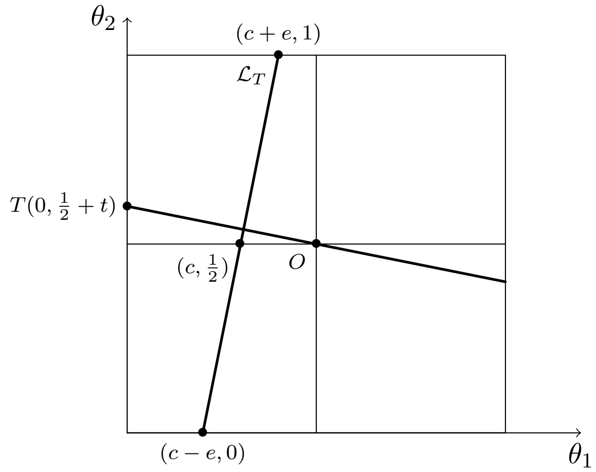

Fig. 16 Parametrization of $\mathcal{L}_T$

Notice that Condition (i) gives $((c + e, 1) - (c, \frac{1}{2})) \cdot ((\frac{1}{2}, \frac{1}{2}) - (0, \frac{1}{2} + t)) = 0$, *i.e.* $e = t$, and in particular,

$$e \in (0, \frac{1}{2}). \tag{18}$$

Condition (ii) implies

$$c + e < 1, \text{ and } c - e < \frac{1}{2}, \tag{19}$$

since otherwise, we would have $O \notin \Theta_1$. To derive Condition (iii) with respect to c and e, let us compute the expectations with respect to each side of $\mathcal{L}_T$ and with respect to parameters c and e. We compute

$$E_{11}(c, e) = \frac{\iint_{\theta_1 > 2e\theta_2 + c - e} \theta_1 \, \mathrm{d}\boldsymbol{\theta}}{\iint_{\theta_1 > 2e\theta_2 + c - e} \mathrm{d}\boldsymbol{\theta}} = \frac{3c^2 + e^2 - 3}{6(c - 1)}, \qquad E_{12}(c, e) = \frac{\iint_{\theta_1 < 2e\theta_2 + c - e} \theta_1 \, \mathrm{d}\boldsymbol{\theta}}{\iint_{\theta_1 < 2e\theta_2 + c - e} \mathrm{d}\boldsymbol{\theta}} = \frac{3c^2 + e^2}{6c},$$

$$E_{21}(c, e) = \frac{\iint_{\theta_1 > 2e\theta_2 + c - e} \theta_2 \, \mathrm{d}\boldsymbol{\theta}}{\iint_{\theta_1 > 2e\theta_2 + c - e} \mathrm{d}\boldsymbol{\theta}} = \frac{3c + e - 3}{6(c - 1)}, \qquad E_{22}(c, e) = \frac{\iint_{\theta_1 < 2e\theta_2 + c - e} \theta_2 \, \mathrm{d}\boldsymbol{\theta}}{\iint_{\theta_1 < 2e\theta_2 + c - e} \mathrm{d}\boldsymbol{\theta}} = \frac{3c + e}{6c}.$$

Now we have that $\mathcal{L}$ is orthogonal to $(\boldsymbol{E}_1\boldsymbol{E}_2)$, with $\boldsymbol{E}_1(E_{11}(c,e), E_{21}(c,e))$ and $\boldsymbol{E}_2(E_{12}(c,e), E_{22}(c,e))$, *iff* $(E_{11} - E_{12}, E_{21} - E_{22}) \cdot (e, \frac{1}{2}) = 0$, which gives

$$e^2 = 3c - 3c^2 - \frac{1}{2}. \tag{20}$$

According to (18), e^2 spans $(0, \frac{1}{4})$ and therefore (20) has two solutions $c_1 = \frac{3-\sqrt{3}\sqrt{1-4e^2}}{6}$ and $c_2 = \frac{3+\sqrt{3}\sqrt{1-4e^2}}{6}$. Now in particular, (19) gives $c_2 + e < 1$, which implies $e < \frac{1}{4}$. It also gives $c_2 - e < \frac{1}{2}$, which implies $e > \frac{1}{4}$. Hence, for any $e \in (0, \frac{1}{2})$, c_2 does not satisfy (19), and c_1 is the only solution to (20). Hence, there is a unique $\mathcal{L}$ that satisfies conditions (i), (ii) and (iii).