# Strategic information transmission with sender's approval: the single-crossing case

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May 2, 2022‡

**Abstract**

We consider games in which an informed sender first talks at no cost to a receiver; then, the latter proposes a decision and, finally, the sender accepts the proposal or "exits". We make the following assumptions: the sender has finitely many types, the receiver's decision is real-valued, utility functions over decisions are concave, single-peaked and single-crossing, exit is damaging to the receiver. In this setup, it may happen that babbling equilibria necessarily involve exit. We nevertheless propose a constructive algorithm that achieves a pure perfect Bayesian equilibrium without exit in every game of the class considered.

*Keywords:* discrete cheap talk, participation constraints, single-crossing

*JEL Classification:* C72, D82

## 1 Introduction

A standard sender-receiver game is played in two stages: first, the – informed – agent sends a message to the – uninformed – decision maker; then, the latter makes his decision. However, in practice, there is often a third stage: the agent can reject the decision in favor of an outside option.

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‡We benefited from the comments of Alexander Frug, Sidartha Gordon, Frédéric Koessler and the participants of various seminars (OSGAD, Paris, POLARIS, Grenoble, BETA, Strasbourg).

As an example (to be developed further below), consider the interaction between patients and a public hospital. Patients are asked to report personal information (on previous diseases, allergies, etc.), physicians decide on medical treatments, but these cannot be implemented without the patients' approval.

In this paper, we consider three-stage sender-receiver games in which, given the sender's message, the receiver proposes a decision. If the sender accepts the proposal, it is implemented. Otherwise, the sender "exits", and his outside option is implemented. The sender's utility (on decisions, but also on the outside option) depends on his type.

As in the most general version of Crawford and Sobel (1982), the receiver's decisions belong to a real interval and the type-dependent utility functions, over decisions, satisfy desirable properties, namely, they are concave, single-peaked and single-crossing; the sender is upwardly biased. As in Frug (2016), the sender has finitely many types, which are originally ordered to make sense of the previous properties.

By contrast with the previous papers, the sender's reservation utility matters in our model. We do not assume that it varies in any specific way with the sender's type. For instance, many relationships are conceivable between a patient's type and his utility for his outside option (to get no treatment, to go a private hospital, etc.).

We show that the sender-receiver game with sender's approval has a perfect Bayesian equilibrium (PBE) in which exit does not happen. Such equilibria are meaningful if the receiver's utility in case of rejection is sufficiently low and the receiver is conceived as an expected utility maximizing principal.

Without the "no exit" restriction, existence of a PBE is not an issue, because our game always has a babbling equilibrium, in which the sender possibly exits, for some of his types. A natural motivation for PBE without exit, under the assumption that the receiver's utility, when exit occurs, is sufficiently low, is that the maximal utility the receiver can expect, over all conceivable PBE, is achieved at a PBE without exit (provided there exists one, of course).[1] In other words, if, as Bester and Strausz (2001), we view the receiver as a principal who cannot commit to a mechanism, namely, must play a cheap talk game with the agent, but can choose among the equilibria of this game, a PBE without exit will emerge as the solution of the

[1]This quite intuitive result is established formally by Forges and Renault (2021) in a model that is more general than the current one.

principal's optimization problem.[2]

To illustrate further the possible merits of our solution concept, let us make our example more concrete by focusing on a specific disease with severe consequences for society. Assume that public hospitals can provide various levels of treatment against the disease, that every patient can only go to a given public hospital and that the effect of the treatment depends on aspects of the patient's health that cannot be checked on the spot (as suggested above, previous diseases, allergies, etc.). Our technical assumptions on the utility functions (in particular, the informed player's upward bias) make sense. More importantly, the exit of patients can be very costly to the hospital (due to sunk costs, but also to reputation effects). In other words, given the mission of public institutions, it seems reasonable to ask whether there exists an equilibrium in which treatments are proposed so as to keep all possible types of patients on board.[3]

As another example, suggested by the "France Telecom case", think of a company in need for management reorganization. After an interview with the employee (which can reveal his degree of psychological weakness), the company proposes new working conditions, an extreme case being employment termination with a severance pay. Exit, which can go as far as the employee's suicide, can have dramatic consequences for the company (e.g., trials with managers sentenced to jail). In such a context, the company would be wise to consider equilibria without exit.

Guided by the previous insights and building on Forges and Renault (2021), we study the limit game in which the receiver's utility level in case of exit is $-\infty$. In this setup, the receiver's expected utility, at any PBE in which exit occurs with positive probability, is $-\infty$, while it is finite in every PBE without exit.

As will be illustrated in Section 4.2, existence of a PBE without exit requires specific assumptions. We provide a counterexample in which utility functions are concave and single-peaked, but do not satisfy the single-crossing condition.

The conditions that we impose on utility functions, when exit does not occur, are standard in economic applications (see, e.g., Kreps and Sobel (1994), Sobel (2013) and the references below). They cover, but go much beyond, the popular case in which the utility functions are quadratic,

[2]The same applies if instead of maximizing the receiver's expected utility, we seek to maximize a weighted sum of the players' expected utilities with a positive weight to the receiver's utility.

[3]Pushing the idea beyond the mere analysis of a three stage game, equilibria with exit would make public hospitals attractive to some types of patients only, which would give rise to an adverse selection phenomenon.

and the players' ideal decisions differ by a bias that is independent of the sender's type. We only restrict the sender's (type-dependent) reservation utility by assuming that, under complete information, the receiver would be able to make a decision inducing the sender's participation.

We establish that the limit game has a PBE in pure strategies, in which the sender accepts the receiver's proposal, whatever his type. Every such equilibrium corresponds to a partition of the sender's (finite) set of types satisfying two properties: individual rationality (IR) and incentive compatibility (IC). The idea is that the sender reveals the cell $\pi$ of the partition containing his type and that, for every cell $\pi$ of the partition, the receiver proposes a decision that maximizes his own utility, subject to the constraint that all types in $\pi$ approve the decision. In order to avoid exit, there must exist, for every cell $\pi$, a decision giving at least as much as the outside option to every sender's type in $\pi$. This is the (IR) property. If the partition is IR, let, for every cell $\pi$ of the partition, $y^{\pi}$ be the best choice of the decision maker when the sender's type is in $\pi$. The (IC) property expresses that every sender's type in $\pi$ prefers the decision $y^{\pi}$ associated with $\pi$ to the decision $y^{\pi'}$ associated with any other cell $\pi'$. Our existence proof takes the form of a step by step algorithm, which starts with the finest partition of the sender's types (namely, the set of singletons) and gradually modifies it – by moving types from one cell to another and possibly merging cells – so as to eventually satisfy (IR) and (IC).

Let us go back to the principal-agent interpretation suggested above. The receiver cannot commit to a mechanism, so that the revelation principle does not hold. Hence to maximize his expected utility, the receiver can just choose the PBE to be played. Given the previous characterization of pure PBE without exit, the receiver has to select an IR and IC partition of the sender's types, with the understanding that the sender will report the partition cell containing his true type.

The paper is organized as follows: we first complete the introduction by indicating further links between this paper and related ones. Then, in Section 2, we introduce the formal model and, in Section 3, we state our main result. Section 4 contains some elementary examples (Subsection 4.1) and, as already announced, a counterexample, in Subsection 4.2. The algorithm is described in Section 5. The section starts with basic lemmas which are proved in an appendix. The different steps of the algorithm are detailed and its convergence is established. A representative case is described in the body of the paper while the other cases are detailed in the appendix. Section 6 makes some suggestions for future research.

## Related papers

As pointed out above, we add an approval stage to the cheap talk game considered by Frug (2016), namely, Crawford and Sobel's (1982) model with finitely many types for the sender. In other words, Frug's (2016) model can be viewed as a particular case of ours, when the participation constraints are never binding. Frug (2016) convincingly argues that finitely many types appear in relevant applications. Among other results, in the uniform quadratic case, he proposes a procedure to determine a specific IR and IC partition of the types, which corresponds to an ex ante optimal equilibrium.

Matthews (1989) already considers a cheap talk game with an approval stage. He makes the same basic assumptions as Crawford and Sobel (1982). In particular, the sender has a continuum of unidimensional types. In Matthews' (1989) game, rejection of the receiver's proposal leaves the players at the status quo, which is not particularly harmful to the receiver. As a consequence, in typical equilibria of the game studied by Matthews (1989), some sender's types do veto the receiver's proposal. Actually, one of Matthews' (1989) insights is that incomplete information (together with cheap talk) can explain rational exit in a framework where complete information would preclude it.

By contrast, equilibria without exit are thoroughly analyzed in Shimizu (2017). The model is again a direct extension of Crawford and Sobel (1982), with a focus on quadratic utility functions, uniformly distributed types and type-independent reservation utility levels. Shimizu's (2017) main point is that the sender's outside options can make cheap talk informative in spite of a substantial conflict of interest between the players. To this aim, the author shows rigorously how a parameter representing the credibility of exit is as important as the sender's bias in the standard uniform quadratic model (see also Section 4.1). Shimizu (2013) illustrates the same phenomenon in the case where the receiver has a binary decision.

Finally, Forges and Renault (2021) can be viewed as a companion paper. The three-stage game and the solution concept are the same as in this paper. But Forges and Renault (2021) only assume that the receiver's decision set is compact and that the utility functions are continuous. In particular, the sender's types can be multidimensional, so that they cannot be ordered in a relevant way. Forges and Renault (2021) show that a PBE without exit may fail to exist when the receiver has finitely many decisions, over which he can randomize. Their analysis indicates that to establish the existence of a PBE without exit, one has to solve constrained optimization problems for the receiver (to account for the IR conditions) together with (IC) conditions for the sender, which can reliably be formulated for finitely many types (but looks intractable with a continuum of types). They prove that a pure PBE without exit exists in three specific cases. First, when the sender has two types only. Second, when decisions belong to the real line (as in the current paper) and, for every type, the sender's utility function is monotonic in the receiver's decision (in sharp contrast with the current paper). Third, when the receiver's utility is type-independent. In the latter case, the proof proceeds by merging cells of the finest partition of the sender's types, which is one of the basic ingredients of the algorithm below. Forges and Renault (2021) leave open the question of the existence of a PBE without exit in the popular case of single-crossing utility functions. The current paper provides an answer.

## 2 Model

We consider a sender-receiver game $\Gamma$, in which the receiver's decision cannot be implemented unless the sender approves it. The sender's set of types $\Theta$ is a finite ordered set, $p$ denotes a probability distribution over $\Theta$, such that $p(\theta) > 0$ for every type $\theta$. In the same way as types, messages belong to a finite set, denoted as $M$, such that $|M| \geq |\Theta|$. Finally, $u_0^\theta$ is type $\theta$'s reservation utility.

The timing of the game $\Gamma$ is as follows:

- The sender's type $\theta$ is chosen in $\Theta$ according to $p$.
- The sender learns his type and sends a message in $M$ to the receiver.
- The receiver proposes a decision $x \in \mathbb{R}$ to the sender.
- If the sender accepts the proposal, the sender's utility is $U^\theta(x)$ and the receiver's utility is $V^\theta(x)$. Otherwise, the sender's utility is $u_0^\theta$ and the receiver's utility is $-\infty$.

At a perfect Bayesian equilibrium (PBE) of $\Gamma$, the sender accepts decision $x$ if $U^\theta(x) > u_0^\theta$ and exits if $U^\theta(x) < u_0^\theta$. We will focus on PBE without exit and such that, at equilibrium, $x$ is accepted if and only if $U^\theta(x) \geq u_0^\theta$. This means that the effective utility function of the sender is $\max\{U^\theta, u_0^\theta\}$. At a PBE without exit, the receiver's utility is finite.

As pointed out in the Introduction, Forges and Renault (2021) consider the more general case where the set of decisions is a compact subset of $\mathbb{R}^n$. They start by considering games $\Gamma(v_0)$, in which the receiver's utility, when the sender exits, is a finite number $v_0$. They show that the PBE without exit of the limit game $\Gamma$ are relevant to those of $\Gamma(v_0)$. Indeed, on the one hand, if $\Gamma$ does not have a PBE without exit, then, the same happens in every $\Gamma(v_0)$; on the other hand, if $\Gamma$ has a PBE without exit, then every $\Gamma(v_0)$ with a sufficiently low $v_0$ also has a PBE without exit.

In this paper, we directly concentrate on the game $\Gamma$, but we assume that the receiver's decision is a real number and that the functions $(\theta, x) \mapsto U^\theta(x)$ and $(\theta, x) \mapsto V^\theta(x)$ satisfy the following properties:

- *Strict concavity*:

  For every $\theta \in \Theta$, functions $x \mapsto U^\theta(x)$ and $x \mapsto V^\theta(x)$ are twice continuously differentiable and for every $x \in \mathbb{R}$, $\frac{\partial^2 U^\theta(x)}{\partial x^2} < 0$ and $\frac{\partial^2 V^\theta(x)}{\partial x^2} < 0$;

$$\tag{A0}$$

- *Single-crossing*:

$$\begin{aligned} &\text{For every } (\theta_1, \theta_2, x_1, x_2) \in \Theta^2 \times \mathbb{R}^2, \text{ with } \theta_2 > \theta_1 \text{ and } x_2 > x_1, \\ &\text{if } U^{\theta_1}(x_2) - U^{\theta_1}(x_1) \geq 0, \text{ then } U^{\theta_2}(x_2) - U^{\theta_2}(x_1) > 0, \text{ and} \\ &\text{if } V^{\theta_1}(x_2) - V^{\theta_1}(x_1) \geq 0, \text{ then } V^{\theta_2}(x_2) - V^{\theta_2}(x_1) > 0; \end{aligned} \tag{A1}$$

- *Existence of a unique maximizing argument*:

$$\begin{aligned} &\text{For every } \theta \in \Theta, \text{ there exists a unique } x^*(\theta) \in \mathbb{R} \text{ and a unique } y^*(\theta) \in \mathbb{R} \\ &\text{such that } \left.\frac{\partial U^\theta(x)}{\partial x}\right|_{x=x^*(\theta)} = 0 \text{ and } \left.\frac{\partial V^\theta(x)}{\partial x}\right|_{x=y^*(\theta)} = 0; \end{aligned} \tag{A2}$$

- *The sender is right biased*:

$$\text{For every } \theta \in \Theta,\ x^*(\theta) > y^*(\theta); \tag{A3}$$

- *For every type, the receiver has the opportunity to induce the sender's participation*:

$$\text{For every } \theta \in \Theta, \text{ there exists } x \in \mathbb{R} \text{ such that } U^\theta(x) > u_0^\theta; \tag{A4}$$

Note that if the functions $(x, \theta) \mapsto U^\theta(x)$ satisfy (A0)–(A3), then so do the functions $(x, \theta) \mapsto U^\theta(x) - u_0^\theta$. Therefore, we can assume w.l.o.g. that for each $\theta \in \Theta$, $u_0^\theta = 0$.[4] The effective utility function of the sender becomes $\max\{U^\theta, 0\}$, which, of course, does not necessarily satisfy properties (A0) - (A3).

[4]Replacing $U^\theta$ by $U^\theta - u_0^\theta$ makes reservation utilities type-independent but of course modifies the utility

# 3 Existence of an equilibrium without exit

In Section 5, we establish the following proposition.

**Proposition 1.** *The game* $\Gamma$ *has a pure perfect Bayesian equilibrium without exit.*

Set for every $\theta \in \Theta$,

$$X^\theta = \{x \in \mathbb{R},\, U^\theta(x) \geq 0\}.$$

The set $X^\theta$ contains the decisions that are approved by type $\theta$. By assumption (A4), $X^\theta \neq \varnothing$. We show below that $X^\theta$ is a closed interval of $\mathbb{R}$. Since $X^\theta \neq \varnothing$,

$$y^\theta = \arg\max_{x \in X^\theta} V^\theta(x)$$

is well-defined.

More generally, let, for every $L \subseteq \Theta$ and every $x \in \mathbb{R}$,

$$X^L = \bigcap_{\theta \in L} X^\theta,$$

$$V^L(x) = \sum_{\theta \in L} p(\theta) V^\theta(x).$$

The set $X^L$ contains the decisions that are approved by all types in $L$. The function $V^L$ is proportional to the receiver's conditional expected utility, given that the sender's type belongs to $L$; it is concave and single-peaked on $\mathbb{R}$. If $X^L \neq \varnothing$,

$$y^L = \arg\max_{x \in X^L} V^L(x)$$

is well-defined. Note that $y^{\{\theta\}} = y^\theta$.

To prove Proposition 1, we construct a partition $\Pi$ of $\Theta$ such that $\Pi$ induces an equilibrium without exit of $\Gamma$.

Consider the following conditions:

(IR) *Individual Rationality*:

$$\text{For every cell } \pi \in \Pi, X^\pi \neq \varnothing;$$

(IC) *Incentive Compatibility*:

$$\text{If } |\Pi| > 1, \text{ for every } \pi, \pi' \in \Pi \text{ and every } \theta \in \pi, U^\theta(y^\pi) \geq U^\theta(y^{\pi'}).$$

functions when exit does not occur. Hence, assuming a specific functional form, e.g., quadratic, for $U^\theta$ together with type-independent reservation utilities, as in Shimizu (2017), may entail a loss of generality (see Subsection 4.1).

Let $\Pi$ be a partition of $\Theta$ satisfying (IR) and (IC).[5] To construct strategies from $\Pi$, let us first associate a message $m_\pi$ to every cell $\pi$ of $\Pi$. Let the sender's strategy $\sigma$ associated to $\Pi$ be:

$$\text{for every } \theta \in \Theta, \sigma(\theta) = m_{\pi(\theta)},$$

where $\pi(\theta)$ is the cell that contains $\theta$. Let the receiver's strategy $\tau$ associated to $\Pi$ be:

$$\begin{aligned} \text{for every } m \in M, \quad & \text{if } m = m_\pi \text{ for some } \pi \in \Pi, \quad && \tau(m_\pi) = y^\pi, \\ & \text{otherwise,} && \tau(m) = y^{\pi_0}, \end{aligned}$$

where $\pi_0$ is an arbitrary cell of $\Pi$. It is straightforward to check that $(\sigma, \tau)$ defines a PBE without exit of $\Gamma$.[6]

As a first particular case, the coarsest partition $\Pi = \{\Theta\}$ corresponds to a nonrevealing (aka pooling) strategy of the sender, for which (IC) is automatically satisfied. However, in our model, $X^\Theta$ may be empty. In this case, (IR) cannot hold for $\Pi = \{\Theta\}$ and a nonrevealing PBE without exit does not exist. At the other extreme, the finest partition $\Pi = \{\{\theta\} : \theta \in \Theta\}$ corresponds to a fully revealing (aka separating) strategy for the sender, for which (IR) is satisfied as a consequence of assumption (A4). But in this case, (IC) is quite demanding.

# 4 Examples

## 4.1 Elementary examples

Let us consider quadratic utility functions, with type-independent bias $b > 0$ and type-independent maximal utility $c^2$ for the sender[7]:

$$U^\theta(x) = c^2 - (\theta + b - x)^2$$

$$V^\theta(x) = -(\theta - x)^2$$

[5]Note that single-crossing (i.e., (A1)) implies that a partition $\Pi$ satisfying (IR) and (IC) is necessarily an *interval* partition, in which every cell consists of consecutive types, i.e., if $\theta_1 < \theta_2$ belong to the same cell $\pi$ of $\Pi$, then every $\theta$ such that $\theta_1 < \theta < \theta_2$ also belongs to $\pi$.

[6]Note that conversely, every pure PBE without exit $(\sigma, \tau)$ of $\Gamma$ induces a partition $\Pi$ satisfying (IR) and (IC). It is the partition whose cells contain the types that lead to the same action, for every action on the equilibrium path.

[7]We have pointed out above that w.l.o.g., we can assume that the sender's reservation utility $u_0^\theta$ equals 0. However, imposing at the same time that $U^\theta$ has a specific quadratic form is of course restrictive.

In this case, assuming $c > 0$,

$$\begin{aligned} X^\theta &= [\theta + b - c, \theta + b + c] \\ x^*(\theta) &= \theta + b \\ y^*(\theta) &= \theta \end{aligned}$$

Recalling $y^\theta = \arg\max_{x \in X^\theta} V^\theta(x)$, we have here

$$y^\theta = \begin{cases} \theta + b - c & \text{if } c < b \\ \theta & \text{if } c \geq b. \end{cases}$$

This expression indicates that if $c < b$, the sender's participation constraint $x \in X^\theta$ is binding, for *every* $\theta$ while for $c \geq b$, it is not binding, for *every* $\theta$. Furthermore, $y^\theta$ increases with $\theta$. As next examples will show, these properties are not always satisfied in our model.

Let us show how we can construct a PBE without exit when $c < b$. Let us set $\Theta = \{\theta_1, \ldots, \theta_n\}$, with $\theta_1 < \ldots < \theta_n$. For every subset $L = \{\theta_{j+1}, \ldots, \theta_{j+\ell}\}$ of $\ell$ consecutive types such that $X^L \neq \varnothing$, the participation constraint, namely, $x \in X^L$, is binding:

$$\sum_{\theta \in L} p(\theta)\theta \leq \theta_{j+\ell} \leq \theta_{j+\ell} + b - c = \min X^L$$

so that $y^L = \theta_{j+\ell} + b - c$.

The partition $\Pi_0 = \{\{\theta_i\}, i = 1, \ldots, n\}$ satisfies (IR), but every type just receives his reservation value. Regarding (IC), if $\theta_i < \theta_j$, $y^{\theta_i} \notin X^{\theta_j}$ so that type $\theta_j$ cannot envy $\theta_i$. However, $\theta_j$ envies $\theta_k > \theta_j$ as soon as $\theta_k < \theta_j + 2c$. This suggests that the effect of the parameter $c$, which reflects the credibility of exit (since it determines the size of $X^\theta$) is as important as the effect of the bias $b$, which reflects the players' conflict of interest. This is a key insight in Shimizu (2017).

A partition satisfying (IR) and (IC) can be gradually constructed from $\Pi_0$. To see that, observe that the highest type $\theta_n$ does not envy any type. Let us keep the cells as in $\Pi_0$ until we find $\theta_k$ envying $\theta_{k+1}$; if this happens, we move $\theta_k$ to the next cell to form $\{\theta_k, \theta_{k+1}\}$. The receiver's constrained optimal decision over the new cell is $y^{\{\theta_k, \theta_{k+1}\}} = \theta_{k+1} + b - c = y^{\theta_{k+1}}$. Hence the move is helpful to fulfill type $\theta_k$'s (IC) condition but has no effect on the other types. We can go on by identifying the next type, among $\theta_1 < \ldots < \theta_{k-1}$, starting with $\theta_{k-1}$, which envies the cell on its right, and so on to end up with a partition satisfying both (IR) and (IC). These steps are consistent with the algorithm that we propose below and all correspond to case C. of Subsection 5.7.

In this particular example, the algorithm to go from partition $\Pi_0$ to an IR and IC partition is quite simple, because moving a type to a next cell does not generate a new decision for the receiver. As a result, checking (IC) from step to step is straightforward. Even with the utility functions above, the construction gets more intricate when $c \geq b$, because the participation constraint $x \in X^L$, which is not binding for $L = \{\theta\}$, can become binding when $L$ gets larger. Note that when $c$ is so large that the participation constraint is never binding, whatever the subset of types, we recover the standard case, in which there is no need to ask for the sender's approval.

Other examples are proposed in the sequel to illustrate the difficulties of a fully general algorithm. These examples are still generated by quadratic utility functions but both the bias and the maximum payoff of the sender depend on his type, namely

$$U^\theta(x) = c(\theta)^2 - (\theta + b(\theta) - x)^2$$

with $b(\theta) > 0$ and $\theta + b(\theta)$ increasing with $\theta$.

## 4.2 Counterexample

Existence of a partition $\Pi$ that satisfies (IR) and (IC) is not guaranteed in general.[8] Indeed, conditions (IR) and (IC) tend to exert opposite effects on the players' strategies:

- Condition (IC) directly reports on the sender's Nash equilibrium conditions. When condition (IR) is irrelevant (e.g. when the sender prefers the receiver's preferred action to his outside option), condition (IC) is obtained by *pooling* adjacent types, up to the situation in which players agree about the resulting pool-contingent actions.[9]

- Condition (IR) requires *separating* strategies for types whose participation constraints cannot be simultaneously satisfied.

Let us illustrate, in a setting close to ours, that conditions (IC) and (IR) may not be simultaneously satisfied. In the following example, we only relax the single-crossing assumption

[8]For instance, Forges and Renault (2021, Section 4.2 and 4.3) exhibit settings in which there is no pure strategy equilibrium without exit (Section 4.2), or in which there is no mixed strategy equilibrium without exit (Section 4.3).

[9]The resulting set of pools possibly consists of the single set $\Theta$. For instance, it is so in the leading example of Crawford and Sobel (1982), when the informed player's bias is sufficiently large.

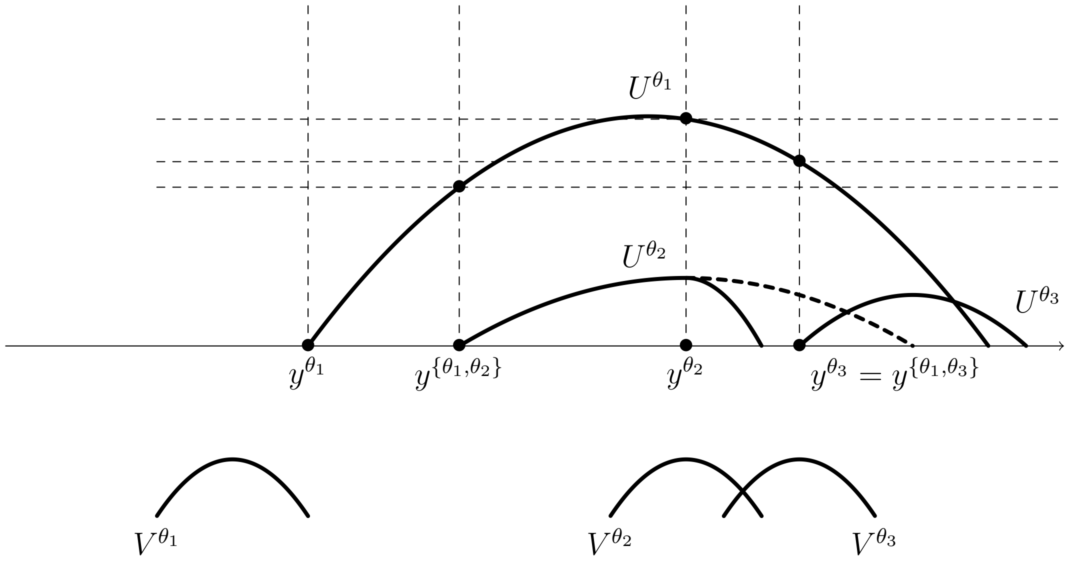


Figure 1: A situation in which (A1) does not hold, and in which (IR) and (IC) do not simultaneously hold.

(A1), and show that it is not possible to pool and separate the types in such a way that both conditions (IR) and (IC) hold.

**Example 1.** There are three types $\theta_1$, $\theta_2$, $\theta_3$, with $\theta_1 < \theta_2 < \theta_3$. The utility functions and cell-contingent optimal actions are as depicted in Figure 1. In particular, assumptions (A0), (A2), (A3) and (A4) hold. Moreover, we have $x^*(\theta_1) < x^*(\theta_2) < x^*(\theta_3)$ and $y^*(\theta_1) < y^*(\theta_2) < y^*(\theta_3)$. However, the single-crossing condition (A1) does not hold, because $\theta_2 > \theta_1$ and $y^{\theta_3} > y^{\{\theta_1,\theta_2\}}$, but

$$U^{\theta_1}(y^{\theta_3}) - U^{\theta_1}(y^{\{\theta_1,\theta_2\}}) > 0 \text{ and } U^{\theta_2}(y^{\theta_3}) - U^{\theta_2}(y^{\{\theta_1,\theta_2\}}) < 0.$$

In that context, there is no partition $\Pi$ of $\Theta = \{\theta_1, \theta_2, \theta_3\}$ such that both (IR) and (IC) are satisfied:

- If $\Pi = \{\{\theta_1, \theta_2, \theta_3\}\}$ (uninformative), or if $\Pi = \{\{\theta_1\}, \{\theta_2, \theta_3\}\}$ (partially informative), from $X^{\{\theta_1,\theta_2,\theta_3\}} \subset X^{\{\theta_2,\theta_3\}} = X^{\theta_2} \cap X^{\theta_3} = \varnothing$, (IR) does not hold.

- If $\Pi = \{\{\theta_1, \theta_2\}, \{\theta_3\}\}$ (partially informative), then (IR) holds, but from $U^{\theta_1}(y^{\{\theta_1,\theta_2\}}) < U^{\theta_1}(y^{\theta_3})$, (IC) does not hold.

- If $\Pi = \{\{\theta_2\}, \{\theta_1, \theta_3\}\}$ (partially informative), then (IR) holds, but from $U^{\theta_1}(y^{\{\theta_1,\theta_3\}}) < U^{\theta_1}(y^{\theta_2})$, (IC) does not hold.

- If $\Pi = \{\{\theta_1\},\{\theta_2\},\{\theta_3\}\}$ (fully informative), then (IR) holds, but from $U^{\theta_1}(y^{\theta_1}) < U^{\theta_1}(y^{\theta_2})$ or $U^{\theta_1}(y^{\theta_1}) < U^{\theta_1}(y^{\theta_3})$, (IC) does not hold.

Finally, note that assumption (A1) would hold if, in Figure 1, the alternative graph for utility function $U^{\theta_2}$ (the dashed one) was considered. In that case, $X^{\{\theta_1,\theta_2,\theta_3\}} = X^{\theta_2} \cap X^{\theta_3} \neq \varnothing$, $y^{\{\theta_1,\theta_2,\theta_3\}} = y^{\theta_3}$, and the uninformative partition $\Pi = \{\{\theta_1,\theta_2,\theta_3\}\}$ does satisfy (IR) and (IC).

In the next section, we provide an algorithm showing that under assumptions (A0)–(A4), it is always possible to pool and separate the types in such a way that conditions (IR) and (IC) simultaneously hold.

# 5 Algorithm

We establish Proposition 1 by recursively defining a sequence of partitions $(\Pi_r)_{r\geq 0}$ that ends on a partition $\Pi$ of $\Theta$ satisfying (IR) and (IC).

Let us first state some lemmas about the receiver's constrained optimum (proofs are given in the appendix).

## 5.1 Preliminary lemmas

The first lemma states that $X^\theta$ is a closed interval.[10]

**Lemma 1.** *For every $\theta \in \Theta$, $X^\theta = [\min X^\theta, \max X^\theta]$.*

The next two lemmas deal with the receiver's constrained optimal action in the complete information case, namely, $y^\theta = \arg\max_{x\in X^\theta} V^\theta(x)$.

**Lemma 2.** *For every $\theta \in \Theta$, $y^\theta < x^*(\theta) < \max X^\theta$. In particular, $U^\theta$ is increasing at $x = y^\theta$.*

**Lemma 3.** *For every $\theta \in \Theta$, $y^\theta \geq y^*(\theta)$, with equality iff $y^*(\theta) \geq \min X^\theta$. In particular, $V^\theta$ is not increasing at $x = y^\theta$.*

Summing up, $y^\theta = \begin{cases} y^*(\theta) & \text{if } (x \in X^\theta) \text{ is not binding,} \\ \min X^\theta & \text{if } (x \in X^\theta) \text{ is binding.} \end{cases}$

[10] The strict inequality in (A4) guarantees that, for every type $\theta$, the set $X^\theta$ has a nonempty interior. This is to avoid extra case by case analysis that, we believe, would not add much to the result.

Observe that, as expected from the single-crossing condition (A1), $y^*(\theta)$ increases when $\theta$ increases according to the original order over $\Theta$, but that this monotonicity does not necessarily hold for $y^\theta$ (see Figure 2 below).

Finally, the next three lemmas identify relationships between the receiver's constrained optimal action in the partially informative case, namely, $y^L$, $L \subset \Theta$ and $y^\theta$, $\theta \in L$. Note that when $L$ is not a singleton, $y^L$ is possibly achieved at both ends of $X^L$:

$$y^L = \arg\max_{x \in X^L} \sum_{\theta \in L} p(\theta) V^\theta(x) = \begin{cases} y^*(L) & \text{if } (x \in X^L) \text{ is not binding} \\ \min X^L \text{ or } \max X^L & \text{if } (x \in X^L) \text{ is binding} \end{cases}$$

where $y^*(L) = \arg\max_{x \in \mathbb{R}} \sum_{\theta \in L} p(\theta) V^\theta(x)$.

**Lemma 4.** *Let $L \subseteq \Theta$ be such that $X^L \neq \varnothing$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \leq y^{\theta'}$. Then $y^\theta \leq y^L$.*

**Lemma 5.** *Let $L \subseteq \Theta$ be such that $X^L \neq \varnothing$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \geq y^{\theta'}$. Then $y^\theta \geq y^L$.*

**Lemma 6.** *Let $L_1, L_2 \subseteq \Theta$ be such that $X^{L_1 \cup L_2} \neq \varnothing$. If $y^{L_1} \geq y^{L_2}$ then $y^{L_1} \geq y^{L_1 \cup L_2} \geq y^{L_2}$.*

## 5.2 Initialization

Let us order the types in $\Theta$ as $\theta_{i_1}, \ldots, \theta_{i_n}$, so that

$$y^{\theta_{i_1}} < \ldots < y^{\theta_{i_n}}.$$

We start with the following partition $\Pi_0$:

$$\{\pi_0^1 = \{\theta_{i_1}\}, \ldots, \pi_0^n = \{\theta_{i_n}\}\},$$

whose cells are ordered in the same way as the types.[11]

Partition $\Pi_0$ clearly satisfies (IR). Moreover, since for each $\theta \in \Theta$, $U^\theta$ is increasing at $y^\theta$ (see Lemma 2), partition $\Pi_0$ is such that no type in a cell strictly prefers the action induced in a *preceding* cell. Hence partition $\Pi = \Pi_0$ also satisfies the following condition, which makes sense in any partition $\Pi$ with ordered cells:

[11] If there exist types $\theta$ and $\theta'$ such that $y^\theta = y^{\theta'}$, then it is straightforward to see that $y^{\{\theta,\theta'\}} = y^\theta = y^{\theta'}$, so that such types can be merged into a unique cell, up to obtaining a partition $\tilde{\Pi}_0$ that does satisfy $y^{\tilde{\pi}_0^1} < y^{\tilde{\pi}_0^2} < \ldots < y^{\tilde{\pi}_0^{|\tilde{\Pi}_0|}}$. In that case, the algorithm has to start from partition $\tilde{\Pi}_0$ instead of $\Pi_0$.

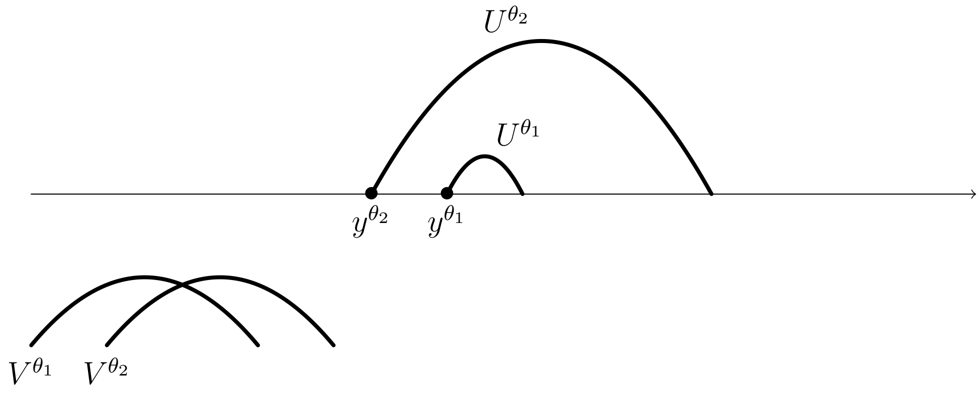


Figure 2: A situation in which $\theta_1 < \theta_2$, but $y^{\theta_1} > y^{\theta_2}$.

(L-IC) *Left-sided incentive compatibility*: If $|\Pi| > 1$, for every $n_2 \in \{2, \ldots, |\Pi|\}$, for every $\theta \in \pi^{n_2}$, for every $n_1 \in \{1, \ldots, n_2 - 1\}$, $U^\theta(y^{\pi^{n_1}}) < U^\theta(y^{\pi^{n_2}})$.

Condition (L-IC) can be seen as a half-way path to the more demanding condition (IC), which says that no type in a given cell strictly prefers the action induced in neither a preceding nor a succeeding cell. The recursive procedure detailed below will generate a sequence $(\Pi_r)_{r\geq 0}$ in which every $\Pi_r$ is an ordered partition satisfying conditions (IR) and (L-IC), and such that, as $r$ gets larger, $\Pi_r$ gradually fulfills the "right-sided incentive compatibility" (R-IC) constraint too. To achieve this goal, if $\Pi_r$ does not satisfy (R-IC), the idea is to construct $\Pi_{r+1}$ from $\Pi_r$ by merging cells of $\Pi_r$ or by moving a type $\theta \in \pi^n$, with $\pi^n \in \Pi_r$, to some succeeding cell $\pi^{n'}$, $n' > n$, of $\Pi_r$. But to keep (IR) and (L-IC) all along the sequence, some care is needed in the way cells are ordered in every partition $\Pi_r$. It turns out to be useful to identify further properties (namely, (P1) and (P2) below) to be satisfied by the partitions in $(\Pi_r)_{r\geq 0}$.

## 5.3 Orders on cells, types and induced actions

The type ordering used to define partition $\Pi_0$, for which (L-IC) holds, might not be the original one, for which single-crossing holds. Indeed a lower type may have a greater outside option than a higher one. Then, as illustrated in Figure 2, in case of a large bias of the sender, the receiver's optimal type-contingent actions may be *non monotonic* in the type.[12]

However, from the single-crossing condition, a partition that satisfies (IC) requires the receiver's optimization to associate types with actions *increasingly*. For instance, if, as in

[12]This is in sharp contrast with the standard setting (e.g. as in Crawford and Sobel (1982)), for which the receiver's (unconstrained) optimal actions are ordered in the same way as the original order of types, whatever the sender's bias.

Figure 2, $\theta_1 < \theta_2$ and $y^{\theta_2} < y^{\theta_1}$, so that the ordered partition $\Pi_0 = \{\{\theta_2\}, \{\theta_1\}\}$ associates types with actions decreasingly, then $\theta_1$ necessarily prefers $y^{\theta_1}$ to $y^{\theta_2}$, and from (A4), so does $\theta_2$. Thus partition $\Pi_0$ does not satisfy (IC).

As we shall see, an increasing association between types and actions from $\Pi_0$ can be achieved through merging adequately the types into cells. However, given that the action associated with a cell is a convex combination of the actions associated with the types that it contains (see Lemmas 4 and 5), the non monotonic association of type and type-contingent actions makes the task difficult. But we do have a monotonic association between types and type-contingent actions for at least a subset of the types. Indeed, as a consequence of the single-crossing condition on the receiver's utility function, according to which $\theta \mapsto y^*(\theta)$ increases, type and type-contingent actions are identically ordered at least for those types $\theta$s such that $y^\theta = y^*(\theta)$, *i.e.* for those types such that the receiver's constraint is not binding (such that $y^\theta > \min X^\theta$).

When considered in $\Pi_0$, since $y^{\pi_0^k} = y^{\theta_{i_k}}$, $k \in \{1, \ldots, n\}$, the previous property can also be stated as a monotonic association between types and *cell*-contingent actions. In this way, the property can be stated for any partition $\Pi = \{\pi^1, \ldots, \pi^{|\Pi|}\}$ through the following two conditions:

(P1) *Increasing cell-contingent actions*: $y^{\pi^1} < y^{\pi^2} < \ldots < y^{\pi^{|\Pi|}}$.

(P2) *Increasing unconstrained type-contingent actions*: If $|\Pi| > 1$, then for every $n \in \{1, ..., |\Pi|-1\}$, for every $\theta \in \pi^n$, if $y^\theta > \min X^\theta$, then for every $n' \in \{n+1, \ldots, |\Pi|\}$, for every $\theta' \in \pi^{n'}$, $y^\theta \leq y^{\theta'}$.[13]

It turns out that the two conditions are sufficient to deal with the way actions shift when, starting from $\Pi_0$, we will recursively construct an (IR) and (IC) partition by merging types adequately. To this end, we shall keep conditions (P1) and (P2) at every step of the algorithm. Moreover, as the next example illustrates, once a partition $\Pi$ fully satisfies (P1), (IR) and (IC), property (P2) cannot be modified to include types whose receiver's constraint is binding (such that $y^\theta = \min X^\theta$). Thus, (P1) and (P2) can be seen as a minimal requirement concerning the relationship between the order of types and the order of cells, as induced by convex combinations of the associated type-contingent actions.

**Example 2.** There are three types $\theta_1$, $\theta_2$, $\theta_3$, with $\theta_1 < \theta_2 < \theta_3$. The utility functions and cell-contingent optimal actions are as depicted in Figure 3. The initial partition is $\Pi_0 =$

[13]Note that property (P2) is meaningful as long as property (P1) is satisfied by partition $\Pi$.

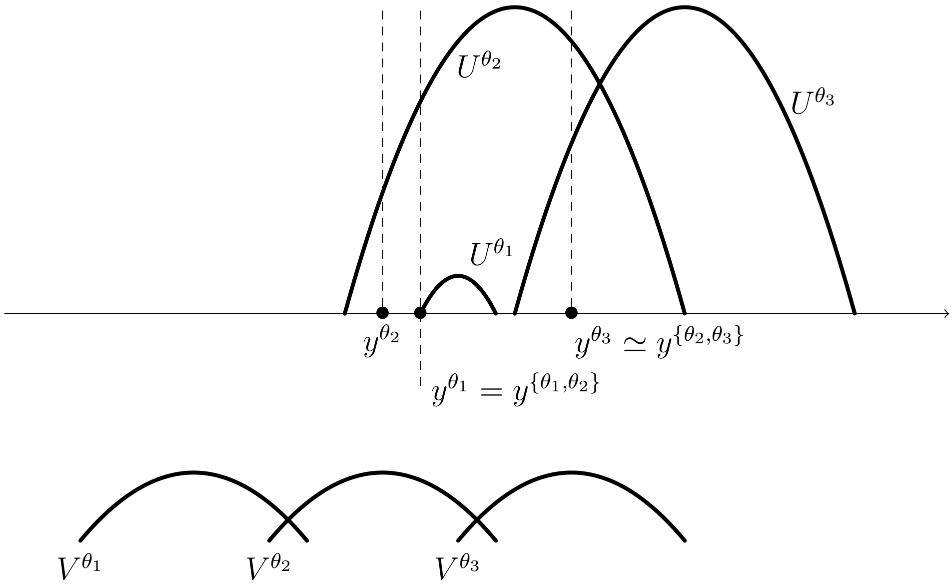


Figure 3: Non-monotonic receiver's constrained optimal actions

$\{\{\theta_2\}, \{\theta_1\}, \{\theta_3\}\}$. The only partition satisfying (IR) and (IC) is $\Pi = \{\pi^1, \pi^2\}$ with $\pi^1 = \{\theta_1\}$, $\pi^2 = \{\theta_2, \theta_3\}$.[14] This partition $\Pi$ is such that $y^{\pi^1} < y^{\pi^2}$, but $\theta_1 \in \pi^1$ (with binding receiver's constraint), $\theta_2 \in \pi^2$ and $y^{\theta_1} > y^{\theta_2}$.

## 5.4 Recursive procedure

Proposition 1 results from the following claim, which we establish in Section 5.7.

**Claim.** Let $\Pi_r$ be a partition which satisfies (P1), (P2), (IR) and (L-IC). If $\Pi_r$ does not satisfy (IC), then there exists a partition $\Pi_{r+1}$, obtained by moving some type $\tilde{\theta}$ from some cell $\pi^n \in \Pi_r$ to some succeeding cell $\pi^{n'}$, $n' > n$, and possibly by merging cells, such that $\Pi_{r+1}$ satisfies conditions (P1), (P2), (IR) and (L-IC).

The claim ultimately guarantees the existence of a partition satisfying (IR) and (IC) because the moves considered in the claim cannot be repeated indefinitely. Indeed, there are finitely many types and finitely many cells. Furthermore, every type move is always directed to a succeeding cell. In other words, starting from $\Pi_0$, and constructing $\Pi_{r+1}$ from $\Pi_r$ according to the claim, we necessarily reach a partition that satisfies (IR) and (IC).

[14]When considering the fully revealing partition $\Pi = \{\{\theta_2\}, \{\theta_1\}, \{\theta_3\}\}$, or the partially revealing partition $\Pi = \{\{\theta_2, \theta_1\}, \{\theta_3\}\}$, type $\theta_2$ prefers $y^{\theta_3}$ to $y^{\theta_2}$ or to $y^{\{\theta_2,\theta_1\}}$ respectively, and if we consider the non revealing partition $\Pi = \{\{\theta_2, \theta_1, \theta_3\}\}$ or the partially revealing partition $\Pi = \{\{\theta_2\}, \{\theta_1, \theta_3\}\}$, the receiver may not satisfy the participation constraint of the sender.

As will be clear below, the algorithm used to establish the claim will be flexible regarding the choice of the cell of the type $\tilde{\theta}$ to be moved. Hence, different runs of the algorithm will be conceivable, possibly leading – a priori – to different IR and IC partitions. Given our primary goal, namely, to establish the existence of an equilibrium without exit, we do not constrain the algorithm unless further specification makes it simpler.

## 5.5 Choices for the moving type and the destination cell

According to the above procedure, we need to establish the existence of a partition $\Pi_{r+1}$ that satisfies conditions (IR) and (L-IC), among the partitions that can be reached from $\Pi_r$, by using the moves prescribed in the above claim.

Keeping (IR) satisfied from $\Pi_r$ to $\Pi_{r+1}$ is easily obtained if only appropriate cells are merged (cells $\pi$ and $\pi'$ such that $X^{\pi} \cap X^{\pi'} \neq \varnothing$), and if, as the following lemma (established in the appendix) states, type moves are restricted to moving a type to a cell that it prefers.

**Lemma 7.** *Let $L, L' \subset \Theta$ with $X^L \neq \varnothing$, $X^{L'} \neq \varnothing$, and suppose that there exists $\theta \in L'$, $\theta \notin L$, such that $U^\theta(y^L) > U^\theta(y^{L'})$. Then we have $X^{L \cup \{\theta\}} \neq \varnothing$.*

Keeping (L-IC) satisfied is more demanding because the considered moves from $\Pi_r$ to $\Pi_{r+1}$ shift some induced actions, and thereby impact condition (L-IC) for *every* type. To that end, we need to carefully monitor every cell-contingent induced action $y^{\pi_r}$, $\pi_r \in \Pi_r$, during the process. As a first step, we will specify the way in which type $\tilde{\theta}$ is moved from its cell $\pi^n$ to a succeeding cell $\pi^{n'}$. Before this specification, let us introduce further notations.

Set, for every $y \in \mathbb{R}$,

$$X^\theta(y) = \{x \in \mathbb{R} : U^\theta(x) \geq U^\theta(y)\}.$$

Since $U^\theta$ is single-peaked, we have:

$$\begin{aligned}
&\text{if } y < x^*(\theta), \text{ then } \min X^\theta(y) = y; \\
&\text{if } y > x^*(\theta), \text{ then } \max X^\theta(y) = y; \\
&\text{if } y = x^*(\theta), \text{ then } X^\theta(y) = \{x^*(\theta)\}.
\end{aligned}$$

Then we can write

$$X^\theta(y) = [x^\theta_-(y), x^\theta_+(y)],$$

with $y = x^\theta_-(y)$ or $y = x^\theta_+(y)$ conditional on $y \leq x^*(\theta)$ and $y \geq x^*(\theta)$ respectively. Note that from Lemma 2, for every $\theta \in \Theta$, $x^\theta_-(y^\theta) = y^\theta$.

### 5.5.1 Destination cell

Concerning the destination cell, if the chosen type $\tilde{\theta}$ to be moved is an element of $\pi^n$, we show that we can w.l.o.g. choose $\pi^{n'}$ with $n' = n + 1$.

Let $\tilde{\theta} \in \pi^n$, for some $n \in \{1, \ldots, |\Pi| - 1\}$, that prefers the action induced in *some* succeeding cell $\pi^{n'}$, $n' > n$, *i.e.* such that $y^{\pi^{n'}} \in X^{\tilde{\theta}}(y^{\pi^n})$. Since $X^{\tilde{\theta}}(y^{\pi^n})$ is an interval, and since we also have $y^{\pi^n} \in X^{\tilde{\theta}}(y^{\pi^n})$, we obtain $[y^{\pi^n}, y^{\pi^{n'}}] \subseteq X^{\tilde{\theta}}(y^{\pi^n})$, where $y^{\pi^n} < y^{\pi^{n'}}$ results from (P1). From (P1), we also get $y^{\pi^{n+1}} \in [y^{\pi^n}, y^{\pi^{n'}}]$. Therefore, we have $y^{\pi^{n+1}} \in X^{\tilde{\theta}}(y^{\pi^n})$, *i.e.* type $\tilde{\theta}$ prefers $y^{\pi^{n+1}}$ to $y^{\pi^n}$.

### 5.5.2 Moving type

We choose, among the types in $\pi^n$ who prefer $y^{\pi^{n+1}}$, the type $\theta$ with the greatest associated action $y^{\theta}$. We set

$$\tilde{\theta} = \theta_{i_\ell},$$

where we ordered $\pi^n = \{\theta_{i_1}, \ldots, \theta_{i_{|\pi^n|}}\}$ in such a way that $y^{\theta_{i_1}} \leq \ldots \leq y^{\theta_{i_{|\pi^n|}}}$, and set

$$\ell = \max\{j : U^{\theta_j}(y^{\pi^{n+1}}) > U^{\theta_j}(y^{\pi^n})\}.$$

The next lemma allows us to characterize $\tilde{\theta}$ within $\pi^n$.

**Lemma 8.** *Either $\ell = |\pi^n|$ or, if $\ell < |\pi^n|$, then for every $j \in \{\ell + 1, \ldots, |\pi^n|\}$, $y^{\theta_{i_j}} = \min X^{\theta_{i_j}}$. In that latter case, $y^{\pi^n} = y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}$.*

## 5.6 Action shifts

The choice of $\tilde{\theta}$ as a maximal element of $\pi^n$ and the choice of moving it to the next succeeding cell $\pi^{n+1}$ allow us to characterize how the induced actions $y^{\pi^n}$ and $y^{\pi^{n+1}}$ shift after the type move.

**Lemma 9.**

- *If $\pi^n \setminus \{\tilde{\theta}\} \neq \varnothing$, then*

$$y^{\pi^n \setminus \{\tilde{\theta}\}} \leq y^{\pi^n}, \tag{1}$$

*and, if moreover $\ell < |\pi^n|$,*

$$y^{\pi^n \setminus \{\tilde{\theta}\}} = y^{\pi^n}. \tag{1a}$$

- $$y^{\tilde{\theta}} \leq y^{\pi^{n+1}\cup\{\tilde{\theta}\}} \leq y^{\pi^{n+1}} \tag{2}$$

- *If $\ell = |\pi^n|$, then*
$$y^{\pi^n} \leq y^{\pi^{n+1}\cup\{\tilde{\theta}\}}. \tag{3}$$

## 5.7 Partition $\Pi_{r+1}$

Finally, we are now able to define partition $\Pi_{r+1}$ from a partition $\Pi = \Pi_r$ that satisfies (IR), (L-IC), (P1) and (P2). We distinguish the following cases.

(A) If $[\pi^n \setminus \{\tilde{\theta}\} \neq \varnothing$ and $y^{\pi^{n+1}\cup\{\tilde{\theta}\}} \leq y^{\pi^n\setminus\{\tilde{\theta}\}}]$, we set:

$$\begin{cases} \pi^1_{r+1} = \pi^1_r, \dots, \pi^{n-1}_{r+1} = \pi^{n-1}_r, \\ \pi^n_{r+1} = \pi^n_r \cup \pi^{n+1}_r, \\ \pi^{n+1}_{r+1} = \pi^{n+2}_r, \dots, \pi^{|\Pi_r|-1}_{r+1} = \pi^{|\Pi_r|}_r. \end{cases} \tag{4}$$

(B) If $[\pi^n \setminus \{\tilde{\theta}\} \neq \varnothing$ and $y^{\pi^{n+1}\cup\{\tilde{\theta}\}} > y^{\pi^n\setminus\{\tilde{\theta}\}}]$,

(B.1) if $[n = 1$ or $y^{\pi_r^{n-1}} < y^{\pi_r^n\setminus\{\tilde{\theta}\}}]$, we set

$$\begin{cases} \pi^1_{r+1} = \pi^1_r, \dots, \pi^{n-1}_{r+1} = \pi^{n-1}_r \text{ if } n > 1, \\ \pi^n_{r+1} = \pi^n_r \setminus \{\tilde{\theta}\}, \\ \pi^{n+1}_{r+1} = \pi^{n+1}_r \cup \{\tilde{\theta}\}, \\ \pi^{n+2}_{r+1} = \pi^{n+2}_r, \dots, \pi^{|\Pi_r|}_{r+1} = \pi^{|\Pi_r|}_r. \end{cases} \tag{5}$$

(B.2) if $[n > 1$ and $y^{\pi_r^{n-1}} \geq y^{\pi_r^n\setminus\{\tilde{\theta}\}}]$, we set:

$$\begin{cases} \pi^1_{r+1} = \pi^1_r, \dots, \pi^{n-2}_{r+1} = \pi^{n-2}_r, \\ \pi^{n-1}_{r+1} = \pi^{n-1}_r \cup \left(\pi^n_r \setminus \{\tilde{\theta}\}\right), \\ \pi^n_{r+1} = \pi^{n+1}_r \cup \{\tilde{\theta}\}, \\ \pi^{n+1}_{r+1} = \pi^{n+2}_r, \dots, \pi^{|\Pi_r|-1}_{r+1} = \pi^{|\Pi_r|}_r. \end{cases} \tag{6}$$

(C) If $[\pi^n \setminus \{\tilde{\theta}\} = \varnothing]$. we set:

$$\begin{cases} \pi^1_{r+1} = \pi^1_r, \dots, \pi^{n-1}_{r+1} = \pi^{n-1}_r, \\ \pi^n_{r+1} = \pi^{n+1}_r \cup \{\tilde{\theta}\}, \\ \pi^{n+1}_{r+1} = \pi^{n+2}_r, \dots, \pi^{|\Pi_r|-1}_{r+1} = \pi^{|\Pi_r|}_r. \end{cases} \tag{7}$$

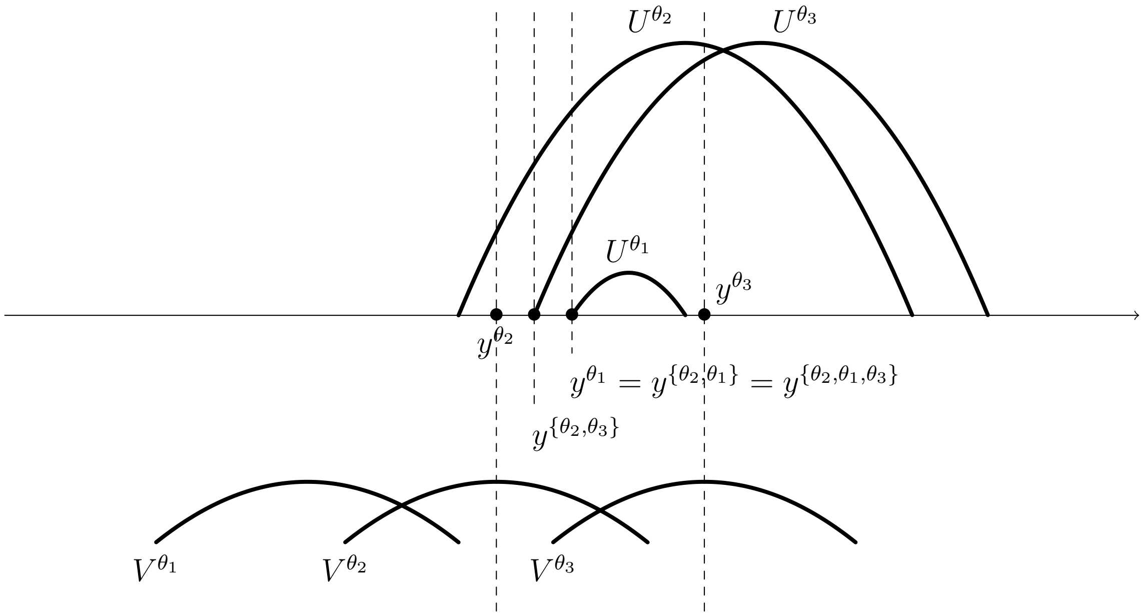


Figure 4: Illustration of Case A.

Case C is the simplest one. It occurs for instance when $\pi^n$ is a cell of the initial partition $\Pi_0$, in particular, if $\Pi_0$ does not satisfy (IC). We provide an illustration of the construction of $\Pi_{r+1}$ in cases A and B in examples 3 and 4 respectively.

**Example 3.** There are three types $\theta_1$, $\theta_2$, $\theta_3$, with $\theta_1 < \theta_2 < \theta_3$. The utility functions and cell-contingent optimal actions are as depicted in Figure 4. In particular, $y^{\theta_2} < y^{\theta_1} < y^{\theta_3}$. Set $\Pi_0 = \{\{\theta_2\}, \{\theta_1\}, \{\theta_3\}\}$ accordingly. Then type $\theta_2$ prefers $y^{\theta_1} = y^{\pi_0^2}$ to $y^{\theta_2} = y^{\pi_0^1}$. Set $\Pi_1 = \{\{\theta_2, \theta_1\}, \{\theta_3\}\}$. Then type $\theta_2$ prefers $y^{\theta_3} = y^{\pi_1^2}$ to $y^{\{\theta_2,\theta_1\}} = y^{\pi_1^1}$. When $\tilde{\theta} = \theta_2$ moves from $\pi_1^1 = \{\theta_2, \theta_1\}$ to $\pi_1^2 = \{\theta_3\}$, we obtain $y^{\pi_1^2 \cup \{\tilde{\theta}\}} = y^{\{\theta_2,\theta_3\}} \leq y^{\theta_1} = y^{\pi_1^1 \setminus \{\tilde{\theta}\}}$. This is case A. In words, moving a type from a cell to a *succeeding* cell may reverse the order of the shifted cell-contingent actions. Hence the resulting cell-contingent actions do not satisfy (P1). However, as we shall show, in such a case it is always possible to merge the considered cells while keeping condition (IR) satisfied. In the present example, the next step is to set $\Pi_2 = \{\pi_1^1 \cup \pi_1^2\} = \{\{\theta_2, \theta_1, \theta_3\}\}$ which does satisfy (IR) and (P1).

**Example 4.** There are four types $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, with $\theta_1 < \theta_2 < \theta_3 < \theta_4$. The utility functions and cell-contingent optimal actions are as depicted in Figure 5. In particular, $y^{\theta_2} < y^{\theta_1} < y^{\theta_3} < y^{\theta_4}$. Set $\Pi_0 = \{\{\theta_2\}, \{\theta_1\}, \{\theta_3\}, \{\theta_4\}\}$ accordingly. Then type $\theta_2$ prefers $y^{\theta_1} = y^{\pi_0^2}$ to $y^{\theta_2} = y^{\pi_0^1}$. Set $\Pi_1 = \{\{\theta_2, \theta_1\}, \{\theta_3\}, \{\theta_4\}\}$. Now type $\theta_2$ prefers $y^{\theta_3} = y^{\pi_1^2}$ to $y^{\{\theta_2,\theta_1\}} = y^{\pi_1^1}$. Set $\Pi_2 = \{\{\theta_1\}, \{\theta_2, \theta_3\}, \{\theta_4\}\}$. Then $y^{\theta_1} < y^{\{\theta_2,\theta_3\}} < y^{\theta_4}$, so that the order on the cell-contingent

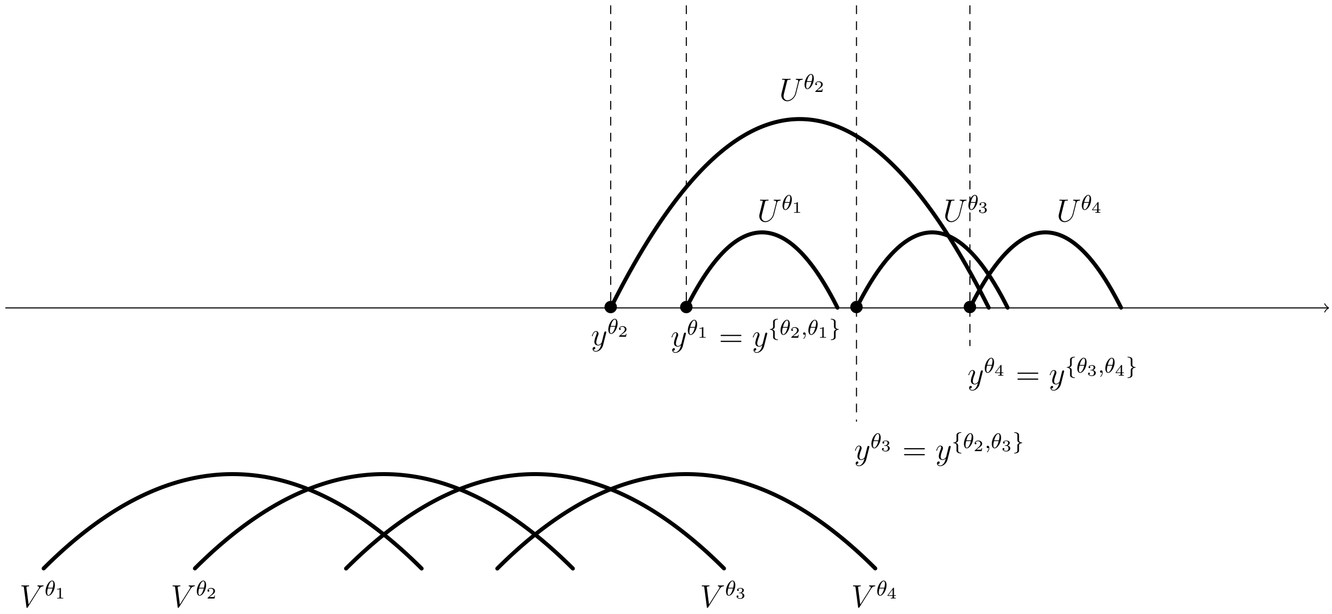


Figure 5: Illustration of Case B

actions does not change from $\Pi_1$ to $\Pi_2$. Accordingly, condition (L-IC) is satisfied. This is case B.1. Now let us go on with the algorithm in order to illustrate case B.2. At $\Pi_2$, $\theta_3$ prefers $y^{\theta_4} = y^{\pi_2^3}$ to $y^{\{\theta_2,\theta_3\}} = y^{\pi_2^2}$. When $\tilde{\theta} = \theta_3$ moves from $\pi_2^2 = \{\theta_2, \theta_3\}$ to $\pi_2^3 = \{\theta_4\}$, then $y^{\pi_2^1} = y^{\theta_1} \geq y^{\theta_2} = y^{\pi_2^2 \setminus \{\tilde{\theta}\}}$. This is case B.2. In words, moving a type from a cell $\pi^n$ to the next succeeding cell may reverse the order of the cell-contingent actions of the $n$th cell (which becomes $\pi^n \setminus \{\tilde{\theta}\}$) and its *preceding* cell $\pi^{n-1}$. In particular, the resulting cell-contingent actions do not satisfy (P1). However, as we shall show, in that case it is always possible to merge $\pi^n \setminus \{\tilde{\theta}\}$ and $\pi^{n-1}$ in such a way that the resulting partition satisfies (IR) and (P1). In the present example, the next step is to set $\Pi_3 = \{\pi_2^1 \cup \pi_2^2 \setminus \{\tilde{\theta}\}, \pi_2^3 \cup \{\tilde{\theta}\}\} = \{\{\theta_2, \theta_1\}, \{\theta_3, \theta_4\}\}$.

Once $\Pi_{r+1}$ is constructed from $\Pi_r$, we have to show that it also satisfies (IR), (L-IC), (P1) and (P2). The construction is greatly simplified when $y^\theta$ is increasing w.r.t. the initial order of types, a property that does not hold in examples 3 and 4. It holds when participation constraints are not binding for individual types (i.e., $y^\theta = y^*(\theta)$). Then only two cases (B.1 and C) can arise, the initial order of types is maintained all along the procedure and at every step, properties (P1) and (P2) are satisfied in a straightforward way. As an illustration, the proof of case C is given below. We give the proofs of cases A and B in the appendix.

**The proof of case C.** We suppose $\pi_r^n \setminus \{\tilde{\theta}\} = \varnothing$, *i.e.* $\pi_r^n = \{\tilde{\theta}\}$, and set

$$\begin{cases} \pi_{r+1}^1 = \pi_r^1, \ldots, \pi_{r+1}^{n-1} = \pi_r^{n-1}, \\ \pi_{r+1}^n = \pi_r^{n+1} \cup \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+2}, \ldots, \pi_{r+1}^{|\Pi_r|-1} = \pi_r^{|\Pi_r|}. \end{cases} \tag{7}$$

By construction, $\Pi_{r+1}$ satisfies (IR).

We show that $\Pi_{r+1}$ satisfies (P1). Since $\Pi_r$ satisfies (P1), we only have to show:

$$y^{\pi_r^{n-1}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} < y^{\pi_r^{n+2}}.$$

From $\pi_r^n = \{\tilde{\theta}\}$, $y^{\pi_r^n} = y^{\tilde{\theta}}$. From $y^{\pi_r^n} < y^{\pi_r^{n+1}}$, we get $y^{\tilde{\theta}} < y^{\pi_r^{n+1}}$ and thus, from Lemma 6, $y^{\tilde{\theta}} \leq y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}}$, that is:

$$y^{\pi_r^n} \leq y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} < y^{\pi_r^{n+1}}. \tag{8}$$

Then $y^{\pi_r^{n-1}} < y^{\pi_r^n}$ and $y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}}$ gives the result.

Now we show that $\Pi_{r+1}$ satisfies (P2). Let $n_1 \in \{1, \ldots, |\pi_{r+1}| - 1 = |\pi_r| - 2\}$ and let $\theta \in \pi_{r+1}^{n_1}$. Suppose that there exists $n_2 \geq n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^{\theta}$. We show that necessarily $y^{\theta} = \min X^{\theta}$.

- If $n_2 \leq n - 1$, then $n_1 \leq n - 2$ and $\theta \in \pi_r^{n_1}$, $\theta' \in \pi_r^{n_2}$ with $n_2 \geq n_1 + 1$, so that we have $y^{\theta} = \min X^{\theta}$ because $\Pi_r$ satisfies (P2).
- If $n_2 = n$, then either $\theta' \in \pi_r^{n+1}$ or $\theta' = \tilde{\theta} \in \pi_r^n$. Moreover, $n_1 \leq n_2 - 1 = n - 1$ implies $\theta \in \pi_r^{n_1}$ with $n_1 < n$. The result also follows from (P2) applied to $\Pi_r$.
- If $n_2 \geq n + 1$, then $\theta' \in \pi_r^{n_2+1}$.
  - If $n_1 \geq n + 1$ too, then $\theta \in \pi_r^{n_1+1}$ and the result follows by the same argument as above.
  - If $n_1 \leq n$, then either $\theta \in \pi_r^{n+1}$, or $\theta = \tilde{\theta} \in \pi_r^n$ or $\theta \in \pi_r^{n_1}$. The result is similarly obtained in all three cases.

Finally, we show that $\Pi_{r+1}$ satisfies (L-IC). Let $n_2' \in \{2, \ldots, |\pi_{r+1}|\}$, $n_1' \in \{1, \ldots, n_2' - 1\}$ and $\theta \in \pi_{r+1}^{n_2'}$. We want to show that $y^{\pi_{r+1}^{n_1'}} \notin X^{\theta}(y^{\pi_{r+1}^{n_2'}})$.

We first consider the case $\theta = \tilde{\theta}$. In that case, $\pi_{r+1}^{n_2'} = \pi_r^{n+1} \cup \{\tilde{\theta}\}$ and we have to show $y^{\pi_{r+1}^{n_1'}} \notin X^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}})$, *i.e.*

$$U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) > U^{\tilde{\theta}}(y^{\pi_{r+1}^{n_1'}}).$$

From $n_1' < n_2' = n$, we have $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^{n_1'}}$. Since $\Pi_{r+1}$ satisfies (P1), we have $y^{\pi_r^{n_1'}} < y^{\pi_r^n}$, and from (8), we obtain the inequalities

$$y^{\pi_r^{n_1'}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}}.$$

If $U^{\tilde{\theta}}$ increases at $y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}$, then the first inequality gives the result. Otherwise, the second inequality gives $U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) \geq U^{\tilde{\theta}}(y^{\pi_r^{n+1}})$. Now by construction, we have $U^{\tilde{\theta}}(y^{\pi_r^{n+1}}) > U^{\tilde{\theta}}(y^{\pi_r^n})$, and since $\Pi_r$ satisfies (L-IC), we also have $U^{\tilde{\theta}}(y^{\pi_r^n}) > U^{\tilde{\theta}}(y^{\pi_r^{n_1'}})$. This gives the result.

Next we suppose $\theta \neq \tilde{\theta}$. In that case, we use the following lemma.

**Lemma 10.** *Let $\Pi = \{\pi^1, \ldots, \pi^{|\Pi|}\}$ be a partition of $\Theta$ that satisfies (IR), (P1) and (L-IC). Let $n_2 \in \{2, \ldots, |\Pi|\}$, $n_1 \in \{1, \ldots, n_2 - 1\}$ and let $\theta \in \pi^{n_2}$. For every $y_1, y_2 \in \mathbb{R}$, if $y_1 < y_2$, $y_1 \leq y^{\pi^{n_1}}$ and $y_2 \leq y^{\pi^{n_2}}$, then $y_1 \notin X^\theta(y_2)$.*

According to that lemma, partition $\Pi_{r+1}$ satisfies (L-IC) if $y_1 = y^{\pi_{r+1}^{n_1'}}$ and $y_2 = y^{\pi_{r+1}^{n_2'}}$ are such that $y_1 < y_2$ (guaranteed because $\Pi_{r+1}$ satisfies (P1)) and such that their exist $n_2 \in \{2, \ldots, |\pi_r|\}$ and $n_1 \in \{1, \ldots, n_2 - 1\}$ with $\theta_2 \in \pi_r^{n_2}$, $y_1 \leq y^{\pi_r^{n_1}}$ and $y_2 \leq y^{\pi_r^{n_2}}$. The result is obtained by the following case by case analysis.

- If $n_2' \leq n - 1$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} = \pi_r^{n_2}$ with $n_2 = n_2'$, so that $y_2 = y^{\pi_r^{n_2'}} = y^{\pi_r^{n_2}}$. From $n_1 \leq n_2' - 1 \leq n - 2$, we also have $y_1 = y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$. This gives the result.

- If $n_2' = n$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} \cup \{\tilde{\theta}\}$, and since $\theta \neq \tilde{\theta}$, we have $\theta \in \pi_r^{n_2'} = \pi_r^{n_2}$ with $n_2 = n_2'$, and, according to (2), $y_2 \leq y^{\pi_r^{n_2'}} = y^{\pi_r^{n_2}}$. From $n_1' \leq n_2' - 1 = n - 1$, we have $y_1 = y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$, which gives the result.

- If $n_2' \geq n + 1$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'+1} = \pi_r^{n_2}$ with $n_2 = n_2' + 1$, and then $y_2 = y^{\pi_r^{n_2'+1}} = y^{\pi_r^{n_2}}$ with $n_2 = n_2' + 1 \geq n + 2$.

  - If $n_1' \geq n + 1$ or $n_1' \leq n - 1$, then $\pi_{r+1}^{n_1'} = \pi_r^{n_1'} = \pi_r^{n_1}$ with $n_1 = n_1'$, and thus $y_1 = y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$, which gives the result.
  - If $n_1' = n$, inequality (2) gives $y_1 \leq y^{\pi_r^{n+1}} = y^{\pi_r^{n_1}}$ with $n_1 = n + 1$, and the result follows.

# 6 Concluding remarks

We have established the existence of a perfect Bayesian equilibrium without exit in a class of games of information transmission with sender's approval. To do so, we have proposed a step by step algorithm which achieves an IR and IC partition of the sender's type. In the equilibrium associated with the partition, the sender reveals the cell containing his type and for every cell, the receiver makes the optimal decision subject to participation of all types in the cell.

The algorithm starts by ordering the types so that no type envies a lower one (L-IC). The initial partition of singletons, which is trivially IR, is gradually modified into a not finer partition satisfying (L-IC), as long as some type envies a higher one. Our construction is applicable in every game of the class under consideration. As noted above, our algorithm allows for some flexibility, at every step $r$, to pick the cell $\pi^n \in \Pi_r$ containing an envying type; then, a specific envying type $\tilde{\theta}$ is selected, namely, the maximal element of $\pi^n$ and $\tilde{\theta}$ is moved to the next cell $\pi^{n+1}$. As a consequence, it can happen that the algorithm has several distinct runs.

In any case, our algorithm guarantees that the sender-receiver game with sender's exit option has a partitional equilibrium without exit. Since, in our model, exit gives an infinitely low payoff to the receiver, our existence result implies that the problem of maximizing the equilibrium payoff of the receiver (interpreted as a principal) has a finite solution. A natural topic for future research would be to identify conditions guaranteeing that the receiver's best equilibrium is partitional, as may happen in models like the one we consider, with finitely many types for the sender, when participation constraints do not matter (see Frug (2016)). Under such conditions, one might hope that (variants of) our algorithm converge to this best partitional equilibrium.

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# A Proof of the lemmas

**Lemma 1.** *For every $\theta \in \Theta$, $X^\theta = [\min X^\theta, \max X^\theta]$.*

*Proof.* The proof is elementary analysis. Let us give the arguments. Function $U^\theta$ is concave with respect to $x$, with a maximum value at $x = x^*(\theta)$, where it is strictly positive. Since $U^\theta(x^*(\theta)) > 0$, there exists $\tilde{x} < x^*(\theta)$ such that $U^\theta(\tilde{x}) > 0$. By concavity, on $(-\infty, \tilde{x}]$, the graph of $x \mapsto U^\theta(x)$ is below its tangent line at $\tilde{x}$. Since $x \mapsto U^\theta(x)$ is increasing on $(-\infty, x^*(\theta))$, the tangent line has a negative slope. Therefore $U^\theta(x) < 0$ if $x$ is sufficiently small. Since $U^\theta$ is strictly positive at $\tilde{x}$, continuous and strictly increasing on $(-\infty, \tilde{x}]$, the Intermediate Value Theorem ensures the existence of a unique $\underline{x}(\theta) < \tilde{x}$ at which $U^\theta(x)$ vanishes. Then $U^\theta(x) < 0$ on $(-\infty, \underline{x}(\theta))$, and $U^\theta(x) > 0$ on $(\underline{x}(\theta), x^*(\theta)]$. Similarly, we can define $\overline{x}(\theta)$ that annihilates $U^\theta$ and that is such that $U^\theta(x) > 0$ on $(x^*(\theta), \overline{x}(\theta)))$, and $U^\theta(x) < 0$ on $(\overline{x}(\theta), +\infty)$. Then $\min X^\theta = \underline{x}(\theta)$, $\max X^\theta = \overline{x}(\theta)$ and $X^\theta = [\min X^\theta, \max X^\theta]$. □

**Lemma 2.** *For every $\theta \in \Theta$, $y^\theta < x^*(\theta) < \max X^\theta$. In particular, $U^\theta$ is increasing at $x = y^\theta$.*

*Proof.* Let $\theta \in \Theta$. From $y^*(\theta) < x^*(\theta)$, function $x \mapsto V^\theta(x)$ decreases on $[x^*(\theta) - \varepsilon, \max X^\theta] \subset X^\theta$ for some $\varepsilon > 0$. Therefore its maximal argument $y^\theta$ on $X^\theta$ is lower than $x^*(\theta)$. □

**Lemma 3.** *For every $\theta \in \Theta$, $y^*(\theta) \leq y^\theta$, with equality iff $y^*(\theta) \geq \min X^\theta$. In particular, $V^\theta$ is not increasing at $x = y^\theta$.*

*Proof.* Let $\theta \in \Theta$. Function $x \mapsto V^\theta(x)$ increases on $(-\infty, y^*(\theta))$ and decreases on $(y^*(\theta), +\infty)$. From Lemma 2, $V^\theta$ is not increasing at $x = y^\theta$. Therefore $y^*(\theta) \leq y^\theta$.

Now we show that $y^*(\theta) = y^\theta$ *iff* $y^*(\theta) \geq \min X^\theta$. If $y^*(\theta) < \min X^\theta$, then $V^\theta$ decreases on $X^\theta$, so that its maximal argument on $X^\theta$, that is $y^\theta$, is $\min X^\theta$, and thus $y^*(\theta) < y^\theta$. This shows that $y^*(\theta) \geq y^\theta$ (and in particular, $y^*(\theta) = y^\theta$) implies $y^*(\theta) \geq \min X^\theta$. Conversely, if $y^*(\theta) \geq \min X^\theta$, then from $x^*(\theta) < \max X^\theta$ and $y^*(\theta) < x^*(\theta)$ we obtain $y^*(\theta) \in X^\theta$. Since $y^*(\theta)$ maximizes $V^\theta$ on $\mathbb{R}$, it maximizes $V^\theta$ on $X^\theta$ and therefore $y^*(\theta) = y^\theta$. □

**Lemma 4.** *Let $L \subseteq \Theta$ be such that $X^L \neq \varnothing$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \leq y^{\theta'}$. Then $y^\theta \leq y^L$.*

*Proof.* We distinguish the four cases: (i) $y^\theta = \min X^\theta$; (ii) $y^\theta > \min X^\theta$ and $y^L = \min X^L$; (iii) $y^\theta > \min X^\theta$ and $y^L = \max X^L$; (iv) $y^\theta > \min X^\theta$ and $\min X^L < y^L < \max X^L$.

**(i) $y^\theta = \min X^\theta$.** Then $y^L \geq \min X^L = \max\{\min X^{\theta'}, \theta' \in L\} \geq \min X^\theta = y^\theta$.

**(ii) $y^\theta > \min X^\theta$ and $y^L = \min X^L$.** Let $\theta'$ be such that $\min X^L = \min X^{\theta'}$. Then $y^{\theta'} = \min X^{\theta'}$. Thus $y^L = y^{\theta'} \geq y^\theta$

**(iii) $y^\theta > \min X^\theta$ and $y^L = \max X^L$.** Let $\theta'$ be such that $\max X^L = \max X^{\theta'}$. Then $y^L = \max X^L = \max X^{\theta'} > y^{\theta'} \geq y^\theta$,

**(iv) $y^\theta > \min X^\theta$ and $\min X^L < y^L < \max X^L$.** If the derivative of $V^L$ is positive at $y^\theta$, then the result follows. Otherwise, there is some $\theta' \in L$ such that $V^{\theta'}$ is decreasing at $x = y^\theta$. Then $V^{\theta'}$ is also decreasing at $x = y^{\theta'} \geq y^\theta$. This implies $y^{\theta'} = \min X^{\theta'}$. Then $y^L > \min X^L \geq \min X^{\theta'} = y^{\theta'} \geq y^\theta$. □

**Lemma 5.** *Let $L \subseteq \Theta$ be such that $X^L \neq \varnothing$. Let $\theta \in L$ be such that for every $\theta' \in L$, $y^\theta \geq y^{\theta'}$. Then $y^\theta \geq y^L$.*

*Proof.* We distinguish the two cases: (i) $y^L = \min X^L$; and (ii) $y^L > \min X^L$.

**(i) $y^L = \min X^L$.** Let $\theta'$ be such that $\min X^L = \min X^{\theta'}$. Then $y^{\theta'} \geq \min X^{\theta'} = \min X^L = y^L$. The result follows from $y^\theta > y^{\theta'}$.

**(ii)** $y^L > \min X^L$. Function $V^L$ vanishes at $y^L$. For every $\theta' \in L$, $V^{\theta'}$ is decreasing on the right side of $x = y^{\theta'}$. Thus it decreases at $x = y^{\theta}$. Therefore $V^L$ decreases at $x = y^{\theta}$. Hence $y^{\theta} > y^L$. $\square$

**Lemma 6.** *Let $L_1, L_2 \subseteq \Theta$ be such that $X^{L_1 \cup L_2} \neq \varnothing$. If $y^{L_1} \geq y^{L_2}$ then $y^{L_1} \geq y^{L_1 \cup L_2} \geq y^{L_2}$.*

*Proof.* First we show $y^{L_1 \cup L_2} \geq y^{L_2}$ by distinguishing the cases (i) $\min X^{L_1} \geq y^{L_2}$; (ii) $\min X^{L_1} < y^{L_2}$ and $y^{L_2} = \min X^{L_2}$; and (iii) $\min X^{L_1} < y^{L_2}$ and $y^{L_2} > \min X^{L_2}$.

**(i)** $\min X^{L_1} \geq y^{L_2}$. From $\min X^{L_1} \geq y^{L_2}$ and $y^{L_2} \geq \min X^{L_2} \geq \min X^{L_1 \cup L_2}$, we obtain $\min X^{L_1} \geq \min X^{L_1 \cup L_2}$. But since $X^{L_1 \cup L_2} \subseteq X^{L_1}$, we have $\min X^{L_1} \leq \min X^{L_1 \cup L_2}$. Therefore $\min X^{L_1 \cup L_2} = \min X^{L_1}$, and then $\min X^{L_1 \cup L_2} \geq y^{L_2}$. The result follows from $y^{L_1 \cup L_2} \geq \min X^{L_1 \cup L_2}$.

**(ii)** $\min X^{L_1} < y^{L_2}$ **and** $y^{L_2} = \min X^{L_2}$. From $\min X^{L_1} < y^{L_2} = \min X^{L_2}$, we obtain $\min X^{L_1} < \min X^{L_2}$. Then $\min X^{L_1 \cup L_2} = \max\{\min X^{L_1}, \min X^{L_2}\} = \min X^{L_2}$. The result follows from $y^{L_1 \cup L_2} \geq \min X^{L_1 \cup L_2} = \min X^{L_2}$ and assumption $y^{L_2} = \min X^{L_2}$.

**(iii)** $\min X^{L_1} < y^{L_2}$ **and** $y^{L_2} > \min X^{L_2}$. From $y^{L_1} \geq y^{L_2}$, we have that $V^{L_1}$ is non decreasing at $x = y^{L_2}$. From $y^{L_2} > \min X^{L_2}$, so is $V^{L_2}$, and thus so is $V^{L_1 \cup L_2} = V_1^L + V^{L_2}$. This implies $y^{L_1 \cup L_2} \geq y^{L_2}$.

Next we show $y^{L_1 \cup L_2} \leq y^{L_1}$ by distinguishing the cases (i) $\max X^{L_2} \leq y^{L_1}$; (ii) $\max X^{L_2} > y^{L_1}$ and $y^{L_1} = \max X^{L_1}$; and (iii) $\max X^{L_2} > y^{L_1}$ and $y^{L_1} < \max X^{L_1}$.

**(i)** $\max X^{L_2} \leq y^{L_1}$. From $\max X^{L_2} \leq y^{L_1}$ and $y^{L_1} \leq \max X^{L_1}$, we obtain $\max X^{L_2} \leq \max X^{L_1}$. Therefore $\max X^{L_1 \cup L_2} = \min\{\max X^{L_1}, \max X^{L_2}\} = \max X^{L_2}$. Then the result follows from $y^{L_1 \cup L_2} \leq \max X^{L_1 \cup L_2} = \max X^{L_2}$ and assumption $\max X^{L_2} \leq y^{L_1}$.

**(ii)** $\max X^{L_2} > y^{L_1}$ **and** $y^{L_1} = \max X^{L_1}$. From $\max X^{L_2} > y^{L_1} = \max X^{L_1}$, we obtain $\max X^{L_1 \cup L_2} = \min\{\max X^{L_1}, \max X^{L_2}\} = \max X^{L_1}$. Then the result follows from $y^{L_1 \cup L_2} \leq \max X^{L_1 \cup L_2} = \max X^{L_1}$ and assumption $\max X^{L_1} = y^{L_1}$.

**(iii)** $\max X^{L_2} > y^L$ **and** $y^{L_1} < \max X^{L_1}$. From $y^{L_2} \leq y^{L_1}$ and $\max X^{L_2} > y^{L_1}$, we have $y^{L_1} \in [y^{L_2}; \max X^{L_2})$. Since $y^{L_2}$ is the maximal argument of $V^{L_2}$ on $X^{L_2}$, it is also the maximal argument of $V^{L_2}$ on $[y^{L_2}; \max X^{L_2}) \subset X^{L_2}$. Thus from $y^{L_1} \in [y^{L_2}; \max X^{L_2})$, we have

$V^{L_2}(y^{L_2}) \geq V^{L_2}(y^{L_1})$. From that, we get that $V^{L_2}$ is non increasing on $[y^{L_2}, +\infty)$. In particular, it is non increasing at $x = y^{L_1} \geq y^{L_2}$. Therefore $V^{L_1 \cup L_2} = V^{L_1} + V^{L_2}$ is non increasing at $x = y^{L_1}$. This implies $y^{L_1 \cup L_2} \leq y^{L_1}$. □

**Lemma 7.** *Let $L, L' \subset \Theta$ with $X^L \neq \varnothing$, $X^{L'} \neq \varnothing$, and suppose that there exists $\theta \in L'$ with $\theta \notin L$. If $U^\theta(y^L) > U^\theta(y^{L'})$, then $X^{L \cup \{\theta\}} \neq \varnothing$.*

*Proof.* By definition, $y^{L'} \in X^{L'}$, and since $X^{L'} = X^{L' \setminus \{\theta\}} \cap X^\theta \subseteq X^\theta$, we have $y^{L'} \in X^\theta$. Therefore $U^\theta(y^{L'}) \geq 0$, so that $U^\theta(y^L) > U^\theta(y^{L'})$ implies $U^\theta(y^L) > 0$. Hence $y^L \in X^\theta$. Since $y^L \in X^L$, we obtain $y^L \in X^L \cap X^\theta = X^{L \cup \{\theta\}}$, and in particular $X^{L \cup \{\theta\}} \neq \varnothing$. □

**Lemma 8.** *Either $\ell = |\pi^n|$ or, if $\ell < |\pi^n|$, then for every $j \in \{\ell+1, \ldots, |\pi^n|\}$, $y^{\theta_{i_j}} = \min X^{\theta_{i_j}}$. In that latter case, $y^{\pi^n} = y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}$.*

*Proof.* If $\ell \neq |\pi^n|$, let $j \in \{\ell + 1, ..., |\pi^n|\} \neq \varnothing$. First, we show

$$y^{\pi^{n+1}} \geq x_+^{\theta_{i_j}}(y^{\pi^n}). \tag{9}$$

Since $j > \ell$, $U^{\theta_{i_j}}(y^{\pi^{n+1}}) \leq U^{\theta_{i_j}}(y^{\pi^n})$. This means that either (9) holds, or $y^{\pi^{n+1}} \leq x_-^{\theta_{i_j}}(y^{\pi^n})$. However this latter inequality cannot occur. Indeed, from (P1), we have $y^{\pi^{n+1}} > y^{\pi^n}$. And from $y^{\pi^n} \in X^{\theta_{i_j}}(y^{\pi^n})$, we have $y^{\pi^n} \geq x_-^{\theta_{i_j}}(y^{\pi^n})$. We obtain $y^{\pi^{n+1}} > x_-^{\theta_{i_j}}(y^{\pi^n})$.

Second, we show

$$y^{\pi^n} = x_-^{\theta_{i_\ell}}(y^{\pi^n}). \tag{10}$$

If (10) does not hold, then $y^{\pi^n} = x_+^{\theta_{i_\ell}}(y^{\pi^n})$. If this holds, then from $U^{\theta_{i_\ell}}(y^{\pi^{n+1}}) > U^{\theta_{i_\ell}}(y^{\pi^n})$, we obtain $y^{\pi^{n+1}} < x_+^{\theta_{i_\ell}}(y^{\pi^n}) = y^{\pi^n}$. But from (P1), we have $y^{\pi^n} < y^{\pi^{n+1}}$.

Third, we show

$$\theta_{i_j} < \theta_{i_\ell}. \tag{11}$$

From (9), $x_+^{\theta_{i_j}}(y^{\pi^n}) \leq y^{\pi^{n+1}}$ and from $U^{\theta_{i_\ell}}(y^{\pi^{n+1}}) > U^{\theta_{i_\ell}}(y^{\pi^n})$, we have $y^{\pi^{n+1}} < x_+^{\theta_{i_\ell}}(y^{\pi^n})$. Then we obtain $x_+^{\theta_{i_\ell}}(y^{\pi^n}) > x_+^{\theta_{i_j}}(y^{\pi^n})$. This implies $x_+^{\theta_{i_\ell}}(y^{\pi^n}) \notin X^{\theta_{i_j}}(y^{\pi^n})$. Since $y^{\pi^n} \in X^{\theta_{i_j}}(y^{\pi^n})$, we get

$$U^{\theta_{i_j}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_j}}(y^{\pi^n}) < 0. \tag{12}$$

From (10), $U^{\theta_{i_\ell}}(x_-^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_\ell}}(y^{\pi^n}) = 0$, and since $U^{\theta_{i_\ell}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) = U^{\theta_{i_\ell}}(x_-^{\theta_{i_\ell}}(y^{\pi^n}))$, we get

$$U^{\theta_{i_\ell}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_\ell}}(y^{\pi^n}) = 0.$$

Therefore, according to (12),

$$U^{\theta_{i_j}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_j}}(y^{\pi^n}) < U^{\theta_{i_\ell}}(x_+^{\theta_{i_\ell}}(y^{\pi^n})) - U^{\theta_{i_\ell}}(y^{\pi^n}). \tag{13}$$

Now from (10), $x_+^{\theta_{i_\ell}}(y^{\pi^n}) > y^{\pi^n}$ so that from the single-crossing condition (A1), (13) holds only if (11) holds.

Fourth, we show

$$y^{\theta_{i_j}} = \min X^{\theta_{i_j}}. \tag{14}$$

Inequality (11) implies that either $y^{\theta_{i_j}} = \min X^{\theta_{i_j}}$ or $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$, since otherwise, $y^{\theta_{i_j}} > \min X^{\theta_{i_j}}$ and $y^{\theta_{i_\ell}} > \min X^{\theta_{i_\ell}}$, which implies, $y^*(\theta_{i_j}) = y^{\theta_{i_j}}$ and $y^*(\theta_{i_\ell}) = y^{\theta_{i_\ell}}$. But in that case, from the single-crossing assumption on $V^\theta(x)$, $y^*(\theta_{i_j}) < y^*(\theta_{i_\ell})$ which gives $y^{\theta_{i_j}} < y^{\theta_{i_\ell}}$, *i.e.* $j < \ell$, a contradiction. Hence if $y^{\theta_{i_\ell}} > \min X^{\theta_{i_\ell}}$ then necessarily (14) holds. It remains to show that $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$ also implies (14). If $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$ then $y^*(\theta_{i_\ell}) \le \min X^{\theta_{i_\ell}}$. From (11) and the single-crossing condition, $y^*(\theta_{i_j}) < y^*(\theta_{i_\ell})$ and thus $y^*(\theta_{i_j}) < \min X^{\theta_{i_\ell}} = y^{\theta_{i_\ell}}$. Then from $y^{\theta_{i_\ell}} \le y^{\theta_{i_j}}$, we obtain $y^*(\theta_{i_j}) < y^{\theta_{i_j}}$, which precisely implies (14).

Finally, consider $j = |\pi^n|$. Since for every $\theta \in \pi^n$, $y^\theta \le y^{\theta_{i_j}}$, according to Lemma 5, we have, on the one hand,

$$y^{\pi^n} \le y^{\theta_{i_j}}.$$

On the other hand, from $y^{\pi^n} \in \bigcap_{\theta \in \pi^n} X^\theta$, we also have $y^{\pi^n} \ge \min X^{\theta_{i_j}}$. That is, according to (14), $y^{\pi^n} \ge y^{\theta_{i_j}}$. Hence we obtain

$$y^{\pi^n} = y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}.$$

□

**Lemma 9.** • *If $\pi^n \setminus \{\tilde{\theta}\} \neq \varnothing$, then*

$$y^{\pi^n \setminus \{\tilde{\theta}\}} \le y^{\pi^n}, \tag{1}$$

*and, if moreover $\ell < |\pi^n|$,*

$$y^{\pi^n \setminus \{\tilde{\theta}\}} = y^{\pi^n}. \tag{1a}$$

•

$$y^{\tilde{\theta}} \le y^{\pi^{n+1} \cup \{\tilde{\theta}\}} \le y^{\pi^{n+1}} \tag{2}$$

• *If $\ell = |\pi^n|$, then*

$$y^{\pi^n} \le y^{\pi^{n+1} \cup \{\tilde{\theta}\}}. \tag{3}$$

*Proof of* (1) *and* (1a). According to Lemma 8, either $\ell = |\pi^n|$, or $[\ell < |\pi^n|$ and $y^{\pi^n} = \min X^{\theta_{i_{|\pi^n|}}} = y^{\theta_{i_{|\pi^n|}}}]$. If $\ell = |\pi^n|$, then for every $\theta \in \pi^n$, $y^\theta \leq y^{\theta_{i_\ell}}$. According to Lemma 6, this implies

$$y^{\pi^n \setminus \{\theta_{i_\ell}\}} \leq y^{\left(\pi^n \setminus \{\theta_{i_\ell}\}\right) \cup \{\theta_{i_\ell}\}} = y^{\pi^n}.$$

If instead $\ell < |\pi^n|$ and $y^{\pi^n} = \min X^{\theta_{i_{|\pi^n|}}} = y^{\theta_{i_{|\pi^n|}}}$, then from $\theta_{i_{|\pi^n|}} \in \pi^n \setminus \{\theta_{i_\ell}\}$, we have $y^{\pi^n \setminus \{\theta_{i_\ell}\}} \in X^{\theta_{i_{|\pi^n|}}}$ and then $y^{\pi^n \setminus \{\theta_{i_\ell}\}} \geq \min X^{\theta_{i_{|\pi^n|}}} = y^{\pi^n}$. Moreover, since for every $\theta \in \pi^n$, $y^\theta \leq y^{\theta_{i_{|\pi^n|}}}$, we also have for every $\theta \in \pi^n \setminus \{\theta_{i_\ell}\}$, $y^\theta \leq y^{\theta_{i_{|\pi^n|}}}$. Therefore, according to Lemma 5, $y^{\pi^n \setminus \{\theta_{i_\ell}\}} \leq y^{\theta_{i_{|\pi^n|}}} = \min X^{\theta_{i_{|\pi^n|}}}$. Thus, we obtain (1a). □

*Proof of* (2). According to Lemma 6, inequalities (2) hold whenever

$$y^{\theta_{i_\ell}} \leq y^{\pi^{n+1}} \tag{2a}$$

holds. To show (2a), we use (P2).

Suppose that $\theta_{i_\ell}$ is such that $y^{\theta_{i_\ell}} > \min X^{\theta_{i_\ell}}$. Then from (P2), for every $\theta' \in \pi^{n+1}$, $y^{\theta_{i_\ell}} \leq y^{\theta'}$. Then (2a) derives from Lemma 4.

If instead $\theta_{i_\ell}$ is such that $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$, then from $y^{\pi^n} \in X^{\theta_{i_\ell}}$, we have $y^{\pi^n} \geq \min X^{\theta_{i_\ell}} = y^{\theta_{i_\ell}}$. Then from $y^{\pi^{n+1}} > y^{\pi^n}$, we obtain $y^{\pi^{n+1}} > y^{\theta_{i_\ell}}$, which implies (2a). □

*Proof of* (3). If $\ell = |\pi^n|$, then for every $\theta \in \pi^n \setminus \{\theta_{i_\ell}\}$, $y^\theta \leq y^{\theta_{i_\ell}}$. Then From Lemma 5, $y^{\pi^n \setminus \{\theta_{i_\ell}\}} \leq y^{\theta_{i_\ell}}$, and then from Lemma 6,

$$y^{\pi^n} = y^{\left(\pi^n \setminus \{\theta_{i_\ell}\}\right) \cup \{\theta_{i_\ell}\}} \leq y^{\theta_{i_\ell}}. \tag{15}$$

Now suppose that $\theta_{i_\ell}$ is such that $y^{\theta_{i_\ell}} > \min X^{\theta_{i_\ell}}$. Then from (P2) we have: for every $\theta' \in \pi^{n+1}$, $y^{\theta_{i_\ell}} \leq y^{\theta'}$. Then from Lemma 4, we have $y^{\theta_{i_\ell}} \leq y^{\pi^{n+1}}$, and from Lemma 6, we deduce $y^{\theta_{i_\ell}} \leq y^{\pi^{n+1} \cup \{\theta_{i_\ell}\}}$. Then from (15), we obtain $y^{\pi^n} \leq y^{\theta_{i_\ell}} \leq y^{\pi^{n+1} \cup \{\theta_{i_\ell}\}}$, so that (3) holds. If instead $y^{\theta_{i_\ell}} = \min X^{\theta_{i_\ell}}$, then from $y^{\pi^{n+1} \cup \{\theta_{i_\ell}\}} \in X^{\theta_{i_\ell}}$, we necessarily have $y^{\pi^{n+1} \cup \{\theta_{i_\ell}\}} \geq \min X^{\theta_{i_\ell}} = y^{\theta_{i_\ell}}$, which implies (3) using (15). □

**Lemma 10.** *Let $\Pi = \pi^1 \cup \ldots \cup \pi^{|\Pi|}$ be a partition of $\Theta$ that satisfy (IR), (P1) and (L-IC). Let $n_2 \in \{2, \ldots, |\Pi|\}$, $n_1 \in \{1, \ldots, n_2 - 1\}$ and let $\theta \in \pi^{n_2}$. For every $y_1, y_2 \in \mathbb{R}$, if $y_1 < y_2$, $y_1 \leq y^{\pi^{n_1}}$ and $y_2 \leq y^{\pi^{n_2}}$, then $y_1 \notin X^\theta(y_2)$.*

*Proof.* Since $\Pi$ satisfies (L-IC), given $n_2 \in \{2, \ldots, |\Pi|\}$, $n_1 \in \{1, \ldots, n_2 - 1\}$ and $\theta \in \pi^{n_2}$, we have $y^{\pi^{n_1}} \notin X^\theta(y^{\pi^{n_2}})$. Therefore either $y^{\pi^{n_1}} < x_-^\theta(y^{\pi^{n_2}})$ or $y^{\pi^{n_1}} > x_+^\theta(y^{\pi^{n_2}})$. Since $\Pi$ satisfies

(P1), we have $y^{\pi^{n_1}} < y^{\pi^{n_2}} \leq x^{\theta}_{+}(y^{\pi^{n_2}})$, so that necessarily

$$y^{\pi^{n_1}} < x^{\theta}_{-}(y^{\pi^{n_2}}). \tag{16}$$

Let us first suppose $y_2 \in X^{\theta}(y^{\pi^{n_2}})$. This means $U^{\theta}(y_2) \geq U^{\theta}(y^{\pi^{n_2}})$, which implies $X^{\theta}(y_2) \subseteq X^{\theta}(y^{\pi^{n_2}})$. In particular, $x^{\theta}_{-}(y^{\pi^{n_2}}) \leq x^{\theta}_{-}(y_2)$. This gives, with $y_1 \leq y^{\pi^{n_1}}$ and (16),

$$y_1 \leq y^{\pi^{n_1}} < x^{\theta}_{-}(y^{\pi^{n_2}}) \leq x^{\theta}_{-}(y_2).$$

In particular, $y_1 \notin X^{\theta}(y_2)$.

Now suppose $y_2 \notin X^{\theta}(y^{\pi^{n_2}})$. This means $U^{\theta}(y_2) < U^{\theta}(y^{\pi^{n_2}})$, so that $y^{\pi^{n_2}} \in X^{\theta}(y_2)$. In particular,

$$y^{\pi^{n_2}} \leq x^{\theta}_{+}(y_2).$$

Let us consider the cases $y^{\pi^{n_2}} < x^{\theta}_{+}(y_2)$ and $y^{\pi^{n_2}} = x^{\theta}_{+}(y_2)$ respectively. If $y^{\pi^{n_2}} < x^{\theta}_{+}(y_2)$, then from $y_2 \leq y^{\pi^{n_2}}$, we obtain $y_2 < x^{\theta}_{+}(y_2)$. Then necessarily $y_2 = x^{\theta}_{-}(y_2)$. Then from $y_1 < y_2$ we have $y_1 < x^{\theta}_{-}(y_2)$, which implies $y_1 \notin X^{\theta}(y_2)$. If $y^{\pi^{n_2}} = x^{\theta}_{+}(y_2)$ then $X^{\theta}(y_2) = X^{\theta}(y^{\pi^{n_2}})$, and in particular $x^{\theta}_{-}(y^{\pi^{n_2}}) = x^{\theta}_{-}(y_2)$. This gives, with $y_1 \leq y^{\pi^{n_1}}$ and (16),

$$y_1 \leq y_1^{\pi^{n_1}} < x^{\theta}_{-}(y^{\pi^{n_2}}) = x^{\theta}_{-}(y_2),$$

and thus $y_1 \notin X^{\theta}(y_2)$. □

# B Proofs of cases A and B

**Case A.** Suppose $\pi^n_r \setminus \{\tilde{\theta}\} \neq \varnothing$ and

$$y^{\pi^{n+1}_r \cup \{\tilde{\theta}\}} \leq y^{\pi^n_r \setminus \{\tilde{\theta}\}}. \tag{17}$$

In that case, partition $\Pi_{r+1}$ is obtained by merging cells $\pi^n$ and $\pi^{n+1}$. From (17), (1) and $y^{\pi^n_r} < y^{\pi^{n+1}_r}$ respectively, we have

$$y^{\pi^{n+1}_r \cup \{\tilde{\theta}\}} \leq y^{\pi^n_r \setminus \{\tilde{\theta}\}} \leq y^{\pi^n_r} \leq y^{\pi^{n+1}_r}. \tag{18}$$

Note that $y^{\pi^{n+1}_r \cup \{\tilde{\theta}\}} \in \bigcap_{\theta \in \pi^{n+1}_r \cup \{\tilde{\theta}\}} X^{\theta}$ and $y^{\pi^{n+1}_r} \in \left( \bigcap_{\theta \in \pi^{n+1}_r} X^{\theta} \right) \cap X^{\tilde{\theta}} = \bigcap_{\theta \in \pi^{n+1}_r \cup \{\tilde{\theta}\}} X^{\theta}$. This means that for every $\theta \in \pi^{n+1}_r \cup \{\tilde{\theta}\}$, we have $\left\{ y^{\pi^{n+1}_r \cup \{\tilde{\theta}\}}, y^{\pi^{n+1}_r} \right\} \subset X^{\theta}$. However, $X^{\theta}$ is an interval. Therefore for every $\theta \in \pi^{n+1}_r \cup \{\tilde{\theta}\}$, we have $[y^{\pi^{n+1}_r \cup \{\tilde{\theta}\}}, y^{\pi^{n+1}_r}] \subset X^{\theta}$. Then from (18), we have that for every $\theta \in \pi^{n+1}_r \cup \{\tilde{\theta}\}$, $y^{\pi^n_r \setminus \{\tilde{\theta}\}} \in X^{\theta}$. But this is also true for every $\theta \in \pi^n_r \setminus \{\tilde{\theta}\}$.

Therefore, for every $\theta \in \left(\pi_r^n \setminus \{\tilde{\theta}\}\right) \cup \left(\pi_r^{n+1} \cup \{\tilde{\theta}\}\right) = \pi_r^n \cup \pi_r^{n+1}$, we have $y^{\pi_r^n \setminus \{\tilde{\theta}\}} \in X^\theta$. In particular,

$$\bigcap_{\theta \in \pi_r^n \cup \pi_r^{n+1}} X^\theta \neq \varnothing.$$

Thus $y^{\pi_r^n \cup \pi_r^{n+1}}$ is well defined. Let us show that

$$y^{\pi_r^n \cup \pi_r^{n+1}} = y^{\pi_r^n}. \tag{19}$$

- If $\ell = |\pi_r^n|$, from (3) we have $y^{\pi_r^n} = y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}$, so that (18) gives $y^{\pi_r^n \cup \{\tilde{\theta}\}} = y^{\pi_r^n \setminus \{\tilde{\theta}\}} = y^{\pi_r^n}$. Then (19) derives from $\pi_r^n \cup \pi_r^{n+1} = (\pi_r^n \cup \{\tilde{\theta}\}) \cup (\pi_r^n \setminus \{\tilde{\theta}\})$.

- If $\ell < |\pi_r^n|$, from Lemma 8, $y^{\pi_r^n} = y^{\theta_{i_{|\pi_r^n|}}} = \min X^{\theta_{i_{|\pi_r^n|}}}$. Since $\theta_{i_{|\pi_r^n|}} \in \pi_r^n \cup \pi_r^{n+1}$, we then have $y^{\pi_r^n \cup \pi_r^{n+1}} \geq \min X^{\theta_{i_{|\pi_r^n|}}} = y^{\pi_r^n}$. But from Lemma 6, we also have $y^{\pi_r^n} \leq y^{\pi_r^n \cup \pi_r^{n+1}}$. This gives (19).

We set

$$\begin{cases} \pi_{r+1}^1 = \pi_r^1, \ldots, \pi_{r+1}^{n-1} = \pi_r^{n-1}, \\ \pi_{r+1}^n = \pi_r^n \cup \pi_r^{n+1}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+2}, \ldots, \pi_{r+1}^{|\Pi_r|-1} = \pi_r^{|\Pi_r|}. \end{cases} \tag{4}$$

Partition $\Pi_{r+1}$ satisfies (IR) by construction. Since $\Pi_r$ satisfies (P1), from (19), $\Pi_{r+1}$ also satisfies (P1).

Let us show that it also satisfies (P2). Let $n_1 \in \{1, \ldots, |\Pi_{r+1}| - 1 = |\Pi_r| - 2\}$ and let $\theta \in \pi_r^{n_1}$. Suppose that there exists $n_2 \geq n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^\theta$. Let us show that $y^\theta = \min X^\theta$.

- If $n_1 \leq n - 1$, then from $\pi_{r+1}^{n_1} = \pi_r^{n_1}$, we have $\theta \in \pi_r^{n_1}$. Then $n_2 \geq n$, so that $\theta' \in \pi_r^{n_2'}$ for some $n_2'$ such that $n_2' = n_2$ or $n_2' = n_2 + 1$. In particular $n_2' \geq n_1$. Since $\Pi_r$ satisfies (P2), we then get $y^\theta = \min X^\theta$.

- If $n_1 = n$, then from $n_2 \geq n_1 + 1 = n + 1$, we have $\pi_{r+1}^{n_2} = \pi_r^{n_2+1}$ and thus $\theta' \in \pi_r^{n_2+1}$ with $n_2 + 1 \geq n + 2$. We have either $\theta \in \pi_r^n$ or $\theta \in \pi_r^{n+1}$. Thus $\theta \in \pi_r^{n_1'}$ with $n_1' = n$ or $n_1' = n + 1$. In both cases, $n_2 + 1 \geq n + 2 \geq n_1'$. Since $\Pi_r$ satisfies (P2), we get $y^\theta = \min X^\theta$.

- If $n_1 \geq n+1$ then from $n_2 \geq n_1+1 \geq n+2$ we have $\theta \in \pi_r^{n_1'}$ and $\theta' \in \pi_r^{n_2'}$, with $n_1' = n_1+1$ and $n_2' = n_2 + 1$ so that $n_2' = n_2 + 1 \geq (n_1 + 1) + 1 = n_1' + 1$. Since $\Pi_r$ satisfies (P2), we get $y^\theta = \min X^\theta$.

Finally, we show that $\Pi_{r+1}$ satisfies (L-IC). Let $n_2' \in \{2, \ldots, |\Pi_{r+1}|\}$, $n_1' \in \{1, \ldots, n_2' - 1\}$ and $\theta \in \pi_{r+1}^{n_2'}$. We want to show that $y^{\pi_{r+1}^{n_1'}} \notin X^\theta(y^{\pi_{r+1}^{n_2'}})$.

- If $n_1' \neq n$ and $n_2' \neq n$, then either $\pi_{r+1}^{n_1'} = \pi_r^{n_1'}$ and $\pi_{r+1}^{n_2'} = \pi_r^{n_2'}$, or $\pi_{r+1}^{n_1'} = \pi_r^{n_1'}$, and $\pi_{r+1}^{n_2'} = \pi_r^{n_2'+1}$, or $\pi_{r+1}^{n_2'} = \pi_r^{n_2'+1}$ and $\pi_{r+1}^{n_2'} = \pi_r^{n_2'+1}$. Since $\Pi_r$ satisfies (L-IC), $y^{\pi_r^{n_1'}} \notin X^\theta(y^{\pi_r^{n_2'}})$, and $y^{\pi_r^{n_1'}} \notin X^\theta(y^{\pi_r^{n_2'+1}})$, and $y^{\pi_r^{n_1'+1}} \notin X^\theta(y^{\pi_r^{n_2'+1}})$, so that $y^{\pi_{r+1}^{n_1'}} \notin X^\theta(y^{\pi_{r+1}^{n_2'}})$ in the three cases.

- If $n_1' = n$, then from (19), $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^n}$, and from $n_2' \geq n_1' + 1 = n + 1$, $y^{\pi_{r+1}^{n_2'}} = y^{\pi_r^{n_2'+1}}$, with $\theta \in \pi_r^{n_2'+1}$. Since $\Pi_r$ satisfies (L-IC), $y^{\pi_r^n} \notin X^\theta(y^{\pi_r^{n_2'+2}})$ and the result follows.

- If $n_2' = n$, then from (19), $y^{\pi_{r+1}^{n_2'}} = y^{\pi_r^n}$, and from $n_1' \leq n_2' - 1 = n - 1$, $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^{n_1'}}$. We have $\theta \in \pi_r^n$ or $\theta \in \pi_r^{n+1}$.

  - If $\theta \in \pi_r^n$, then $y^{\pi_r^{n_1'}} \notin X^\theta(y^{\pi_r^n})$ because $\Pi_r$ satisfies (L-IC).
  - If $\theta \in \pi_r^{n+1}$, let us set $y_1 = y^{\pi_r^{n_1'}}$ and $y_2 = y^{\pi_r^n}$. Then $y_1 < y_2$ (because $n_1' \leq n-1$ and $\Pi_r$ satisfies (P1)), $y_1 \leq y^{\pi_r^{n_1'}}$ and $y_2 \leq y^{\pi_r^{n+1}}$ (because $\Pi_r$ satisfies (P1)), and Lemma 10 gives the result.

**Case B.** Suppose $\pi_r^n \setminus \{\tilde{\theta}\} \neq \varnothing$ and

$$y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} > y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \tag{20}$$

In this case, we distinguish cases (B.1) [$n = 1$ or $y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}}$], and (B.2) [$n > 1$ and $y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}$].

**Case B.1.** Suppose that $n = 1$, or

$$y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \tag{21}$$

In that case, the move of $\tilde{\theta}$ from $\pi_r^n$ to $\pi_r^{n+1}$ keeps the order of the corresponding cell-contingent actions $y^\pi$.

We set

$$\begin{cases} \pi_{r+1}^1 = \pi_r^1, \ldots, \pi_{r+1}^{n-1} = \pi_r^{n-1} \text{ if } n > 1, \\ \pi_{r+1}^n = \pi_r^n \setminus \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+1} \cup \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+2} = \pi_r^{n+2}, \ldots, \pi_{r+1}^{|\Pi_r|} = \pi_r^{|\Pi_r|}. \end{cases} \tag{5}$$

Then $\Pi_{r+1}$ satisfies (IR) by construction. Concerning (P1), we clearly have $y^{\pi_{r+1}^1} < ... < y^{\pi_{r+1}^{n-1}}$ (if $n > 1$) and $y^{\pi_{r+1}^{n+2}} < \ldots < y^{\pi_{r+1}^{|\Pi_{r+1}|}}$. From (20), we also have $y^{\pi_{r+1}^n} = y^{\pi^n \setminus \{\tilde{\theta}\}} < y^{\pi^{n+1} \cup \{\tilde{\theta}\}} = y^{\pi_{r+1}^{n+1}}$. Moreover, from (2) we have $y^{\pi_{r+1}^{n+1}} = y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \le y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}} = y^{\pi_{r+1}^{n+2}}$. Then $\Pi_{r+1}$ satisfies (P1) whenever $y^{\pi_{r+1}^{n-1}} < y^{\pi_{r+1}^n}$, *i.e.* whenever $y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}}$, *i.e.* (21).

Now we show that it satisfies (P2). Let $n_1 \in \{1, \ldots, |\Pi_{r+1}| - 1 = |\Pi_r| - 1\}$ and let $\theta \in \pi_{r+1}^{n_1}$. Suppose that there exists $n_2 \ge n_1 + 1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^{\theta}$. We show that necessarily $y^\theta = \min X^\theta$.

- If $\theta \neq \tilde{\theta}$ and $\theta' \neq \tilde{\theta}$, then $\theta \in \pi_r^{n_1}$ and $\theta' \in \pi_r^{n_2}$ with $n_2 \ge n_1 + 1$, so that we have $y^\theta = \min X^\theta$ because $\Pi_r$ satisfies (P2).
- If $\theta = \tilde{\theta}$, then $\theta \in \pi_r^n$. Moreover, $n_1 = n + 1$, and from $n_2 \ge n_1 + 1 = n + 2$, we have $\theta' \in \pi_{r+1}^{n_2} = \pi_r^{n_2}$. Since $n_2 \ge n+2 > n$, and since $\Pi_r$ satisfies (P2), we obtain $y^\theta = \min X^\theta$.
- If $\theta' = \tilde{\theta}$, then $\theta' \in \pi_r^n$. Moreover, $n_2 = n + 1$, and from $n_2 \ge n_1 + 1$, we obtain $n + 1 \ge n_1 + 1$, *i.e.* $n_1 \le n$. From $\theta \neq \theta'$, we derive $\theta \in \pi_r^{n_1}$.
  - If $n_1 < n$, since $\Pi_r$ satisfies (P2), we obtain $y^\theta = \min X^\theta$.
  - If $n_1 = n$, from $y^{\theta'} = y^{\tilde{\theta}} < y^\theta$ we have $\theta = \theta_{i_j}$ for some $j > \ell$. In that case, Lemma 8 gives $y^\theta = \min X^\theta$.

Finally, we show that $\Pi_{r+1}$ satisfies (L-IC). Let $n_2' \in \{2, \ldots, |\Pi_{r+1}|\}$, $n_1' \in \{1, \ldots, n_2' - 1\}$ and $\theta \in \pi_{r+1}^{n_2'}$. We want to show that $y^{\pi_{r+1}^{n_1'}} \notin X^\theta(y^{\pi_{r+1}^{n_2'}})$.

We first suppose $\theta = \tilde{\theta}$. In that case, $\pi_{r+1}^{n_2'} = \pi_r^{n+1} \cup \{\tilde{\theta}\}$ and we have to show $y^{\pi_{r+1}^{n_1'}} \notin X^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}})$, *i.e.*

$$U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) > U^{\tilde{\theta}}(y^{\pi_{r+1}^{n_1'}}).$$

- If $n_1' = n_2' - 1 = n$, then $\pi_{r+1}^{n_1'} = \pi_r^n \setminus \{\tilde{\theta}\}$, and thus $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^n \setminus \{\tilde{\theta}\}}$. From (20) and (2), we obtain the two inequalities

$$y^{\pi_r^n \setminus \{\tilde{\theta}\}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \le y^{\pi_r^{n+1}}.$$

  If $U^{\tilde{\theta}}$ increases at $y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}$, then the first inequality gives the result. Otherwise, the second inequality gives $U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) \ge U^{\tilde{\theta}}(y^{\pi_r^{n+1}})$. By construction, we also have $U^{\tilde{\theta}}(y^{\pi_r^{n+1}}) > U^{\tilde{\theta}}(y^{\pi_r^n})$. Since $y^{\pi_r^n} < y^{\pi_r^{n+1}}$, it must be that $U^{\tilde{\theta}}$ increases at $y^{\pi_r^n}$. From (1), *i.e.* $y^{\pi_r^n \setminus \{\tilde{\theta}\}} \le y^{\pi_r^n}$, we then get $U^{\tilde{\theta}}(y^{\pi_r^n}) \ge U^{\tilde{\theta}}(y^{\pi_r^n \setminus \{\tilde{\theta}\}})$, and therefore $U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) > U^{\tilde{\theta}}(y^{\pi_r^n \setminus \{\tilde{\theta}\}})$.

- If instead $n_1' \leq n_2' - 2 = n - 1$ (if $n > 1$), then $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^{n_1'}}$. In that case, since $\Pi_r$ satisfies (P1), and from (21) and (20) respectively,

$$y^{\pi_r^{n_1'}} \leq y^{\pi_r^{n-1}} < y^{\pi_r^n \setminus \{\tilde{\theta}\}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}}.$$

  The same argument as above gives the result.

Next we suppose $\theta \neq \tilde{\theta}$. Set $y_1 = y^{\pi_{r+1}^{n_1'}}$ and $y_2 = y^{\pi_{r+1}^{n_2'}}$. Then according to Lemma 10, the result derives from $y_1 < y_2$ (guaranteed because $\Pi_{r+1}$ satisfies (P1)), and the existence of $n_2 \in \{2, \ldots, |\pi_r|\}$ and $n_1 \in \{1, \ldots, n_2 - 1\}$ such that $\theta \in \pi_r^{n_2}$, $y_1 \leq y^{\pi_r^{n_1}}$ and $y_2 \leq y^{\pi_r^{n_2}}$.

- If $n_2' \leq n - 1$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} = \pi_r^{n_2}$ with $n_2 = n_2'$, so that $y_2 = y^{\pi_r^{n_2}}$. Setting $n_1 = n_1' \leq n_2' - 1 \leq n - 2$, we have $y_1 = y^{\pi_r^{n_1}}$. This gives the result.

- If $n_2' = n$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} \setminus \{\tilde{\theta}\}$, and according to (1), $y_2 \leq y^{\pi_r^{n_2'}} = y^{\pi_r^{n_2}}$ with $n_2 = n_2'$. Setting $n_1 = n_1' \leq n_2' - 1 = n - 1$, we have $y_1 = y^{\pi_r^{n_1}}$. This gives the result.

- If $n_2' = n + 1$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} \cup \{\tilde{\theta}\}$, and according to (2), $y_2 \leq y^{\pi_r^{n_2'}} = y^{\pi_r^{n_2}}$, with $n_2 = n_2'$, and $\theta \in \pi_r^{n_2}$ because $\theta \neq \tilde{\theta}$.
  - If $n_1' = n_2' - 1 = n$, then $\pi_{r+1}^{n_1'} = \pi_r^{n_1'} \setminus \{\tilde{\theta}\}$, and according to (1), $y_1 \leq y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$. This gives the result.
  - If $n_1' < n_2' - 1 = n$, then $\pi_{r+1}^{n_1'} = \pi_r^{n_1'} = \pi_r^{n_1}$ with $n_1 = n_1'$, and then $y_1 = y^{\pi_r^{n_1}}$, which also gives the result.

- If $n_2' \geq n + 2$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} = \pi_r^{n_2}$ with $n_2 = n_2'$, and then $y_2 = y^{\pi_r^{n_2}}$.
  - If $n_1' \geq n + 2$ or $n_1' \leq n - 1$, then $\pi_{r+1}^{n_1'} = \pi_r^{n_1'} = \pi_r^{n_1}$ with $n_1 = n$, and thus $y_1 = y^{\pi_r^{n_1}}$, which gives the result.
  - If $n \leq n_1' \leq n + 1$, inequality (1) (if $n_1' = n$) or inequality (2) (if $n_1' = n + 1$) gives $y_1 \leq y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$, and the result follows.

**Case B.2.** Suppose

$$y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}. \tag{22}$$

In that case, when type $\tilde{\theta}$ is removed from $\pi_r^n$, the $n$th action $y^{\pi_r^n}$ in partition $\Pi_r$ moves to a rank which is possibly strictly lower than $n$. In that case, we merge $\pi_r^n \setminus \{\tilde{\theta}\}$ with $\pi_r^{n-1}$.

We first show that necessarily $\ell = |\pi_r^n|$. Indeed, if $\ell < |\pi_r^n|$, then from Lemma 8, $y^{\pi_r^n} = y^{\theta_{i_{|\pi_r^n|}}} = \min X^{\theta_{i_{|\pi_r^n|}}}$. Then from (1a), we have $y^{\pi_r^n \setminus \{\tilde{\theta}\}} = y^{\pi_r^n}$. Since $y^{\pi_r^{n-1}} < y^{\pi_r^n}$, inequality (22) cannot hold.

Next we show that $X^{\pi_r^{n-1}} \cap X^{\pi_r^n \setminus \{\tilde{\theta}\}} \neq \varnothing$. From (22) and $y^{\pi_r^n} > y^{\pi_r^{n-1}}$ we get

$$y^{\pi_r^n} > y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}.$$

Since $X^{\pi_r^n \setminus \{\tilde{\theta}\}}$ is an interval, from $y^{\pi_r^n} \in X^{\pi_r^n} \subseteq X^{\pi_r^n \setminus \{\tilde{\theta}\}}$ and $y^{\pi_r^n \setminus \{\tilde{\theta}\}} \in X^{\pi_r^n \setminus \{\tilde{\theta}\}}$, we get $[y^{\pi_r^n \setminus \{\tilde{\theta}\}}, y^{\pi_r^n}] \subseteq X^{\pi_r^n \setminus \{\tilde{\theta}\}}$. We obtain $y^{\pi_r^{n-1}} \in X^{\pi_r^n \setminus \{\tilde{\theta}\}}$. Since moreover $y^{\pi_r^{n-1}} \in X^{\pi_r^{n-1}}$, we obtain

$$y^{\pi_r^{n-1}} \in X^{\pi_r^{n-1}} \cap X^{\pi_r^n \setminus \{\tilde{\theta}\}} \neq \varnothing.$$

This allows us to set

$$\pi = \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde{\theta}\}\right),$$

with $\pi$ such that $X^\pi \neq \varnothing$.

Next we show $y^\pi = y^{\pi_r^{n-1}}$. Let $\underline{\theta} \in \pi_r^n \setminus \{\tilde{\theta}\}$ be such that for every $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$, $y^{\underline{\theta}} \leq y^\theta$. Then from Lemma 4 and from (22),

$$y^{\underline{\theta}} \leq y^{\pi_r^n \setminus \{\tilde{\theta}\}} \leq y^{\pi_r^{n-1}}.$$

Now let $\overline{\theta} \in \pi_r^{n-1}$ be such that for every $\theta \in \pi_r^{n-1}$, $y^{\overline{\theta}} \geq y^\theta$. Then from Lemma 5,

$$y^{\overline{\theta}} \geq y^{\pi_r^{n-1}}.$$

Therefore $y^{\overline{\theta}} \geq y^{\underline{\theta}}$, where $\underline{\theta} \in \pi_r^n$ and $\overline{\theta} \in \pi_r^{n-1}$. Since partition $\Pi_r$ satisfies (P2), we get

$$y^{\overline{\theta}} = \min X^{\overline{\theta}}.$$

Consequently, from $y^{\pi_r^{n-1}} \geq \min X^{\pi_r^{n-1}} \geq \min X^{\overline{\theta}}$, we have $y^{\pi_r^{n-1}} \geq y^{\overline{\theta}}$. Hence $y^{\overline{\theta}} \geq y^{\pi_r^{n-1}}$ and $y^{\pi_r^{n-1}} \geq y^{\overline{\theta}}$, and thus

$$y^{\pi_r^{n-1}} = y^{\overline{\theta}} = \min X^{\overline{\theta}}.$$

From Lemma 6 and $y^{\pi_r^{n-1}} \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}$, we have $y^{\pi_r^{n-1}} \geq y^\pi \geq y^{\pi_r^n \setminus \{\tilde{\theta}\}}$, and in particular $y^\pi \leq \min X^{\overline{\theta}}$. But $\overline{\theta} \in \pi$, and necessarily $y^\pi \geq \min X^{\overline{\theta}}$. Thus $y^\pi = \min X^{\overline{\theta}} = y^{\pi_r^{n-1}}$.

We set:

$$\begin{cases} \pi_{r+1}^1 = \pi_r^1, \ldots, \pi_{r+1}^{n-2} = \pi_r^{n-2}, \\ \pi_{r+1}^{n-1} = \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde{\theta}\}\right), \\ \pi_{r+1}^n = \pi_r^{n+1} \cup \{\tilde{\theta}\}, \\ \pi_{r+1}^{n+1} = \pi_r^{n+2}, \ldots, \pi_{r+1}^{|\Pi_r|-1} = \pi_r^{|\Pi_r|}. \end{cases} \tag{6}$$

By construction, $\Pi_{r+1}$ satisfies (IR). It satisfies (P1) if $y^{\pi_r^1} < \ldots < y^{\pi_r^{n-2}}$ and $y^{\pi_r^{n+2}} < \ldots < y^{\pi_r^{|\Pi_r|}}$, which hold by assumption (P1) on $\Pi_r$, and if moreover $y^{\pi_r^{n-2}} < y^{\pi_r^{n-1}\cup\left(\pi_r^n\setminus\{\tilde\theta\}\right)} = y^{\pi} = y^{\pi_r^{n-1}}$, which also holds by assumption (P1) on $\Pi_r$, and if moreover

$$y^{\pi_r^{n-1}\cup\left(\pi_r^n\setminus\{\tilde\theta\}\right)} = y^{\pi_r^{n-1}} < y^{\pi_r^{n+1}\cup\{\tilde\theta\}}, \tag{23}$$

and

$$y^{\pi_r^{n+1}\cup\{\tilde\theta\}} < y^{\pi_r^{n+2}}. \tag{24}$$

Inequality (23) results from $y^{\pi_r^{n-1}} < y^{\pi_r^n}$, and, from $\ell = |\pi_r^n|$, inequality (3), that is: $y^{\pi_r^n} < y^{\pi_r^{n+1}\cup\{\tilde\theta\}}$. Inequality (24) results from $y^{\pi_r^{n+1}} < y^{\pi_r^{n+2}}$ and from the second inequality in (2), that is: $y^{\pi_r^{n+1}\cup\{\tilde\theta\}} < y^{\pi_r^{n+1}}$.

Now we show that $\Pi_{r+1}$ satisfies (P2). Note that $|\Pi_{r+1}| = |\Pi_r|-1$. Let $n_1 \in \{1,\ldots,|\Pi_r|-2\}$ and let $\theta \in \pi_{r+1}^{n_1}$. Suppose that there exists $n_2 \geq n_1+1$ and $\theta' \in \pi_{r+1}^{n_2}$ such that $y^{\theta'} < y^{\theta}$. We show that necessarily $y^{\theta} = \min X^{\theta}$.

- If $n_2 < n-1$, then $n_1 \leq n_2 - 1 < n-2$, and then $\theta \in \pi_r^{n_1}$ and $\theta' \in \pi_r^{n_2}$. Since $\Pi_r$ satisfies (P2), $y^{\theta'} < y^{\theta}$ implies $y^{\theta} = \min X^{\theta}$.

- If $n_2 = n-1$, then $n_1 \leq n_2 - 1 = n-2$, and then $\theta \in \pi_r^{n_1}$ and either $\theta' \in \pi_r^{n-1}$, with $n-1 > n_1$, or $\theta' \in \pi_r^n$, with $n \geq n_1+2$, and the same argument holds.

- If $n_2 = n$, then $\theta' \in \pi_r^{n+1}$ or $\theta' = \tilde\theta \in \pi_r^n$. Moreover, either $n_1 \leq n_2 - 2 = n-2$, or $n_1 = n_2 - 1 = n-1$.

  - If $n_1 \leq n-2$ then $\theta \in \pi_r^{n_1}$ with $n_1 < n+1$ and $n_1 < n$ respectively, and the same argument as above holds.

  - If $n_1 = n-1$, then $\theta \in \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde\theta\}\right)$. In particular, either $\theta \in \pi_r^{n-1}$, or $\theta \in \pi_r^n \setminus \{\tilde\theta\}$. If $\theta \in \pi_r^{n-1}$ then we are done. If $\theta \in \pi_r^n \setminus \{\tilde\theta\}$ and $\theta' \in \pi_r^{n+1}$ then we are done too. It remains the case in which $\theta \in \pi_r^n \setminus \{\tilde\theta\}$ and $\theta' = \tilde\theta \in \pi_r^n$. In that case, $y^{\theta'} < y^{\theta}$ guarantees that $\theta = \theta_{i_j}$ for some $j > \ell$, and Lemma 8 gives the result.

- if $n_2 > n$, then $\theta' \in \pi_r^{n_2+1}$. In each of the following cases, $\theta$ is in a cell of $\Pi_r$ whose rank is lower than $n_2+1$. Since $\Pi_r$ satisfies (P2), $y^{\theta'} < y^{\theta}$ implies $y^{\theta} = \min X^{\theta}$.

  - If $n_1 > n$, then $\theta \in \pi_r^{n_1+1}$, with $n_1 + 1 < n_2 + 1$.

– If $n_1 = n$, then either $\theta = \tilde{\theta} \in \pi_r^n$ with $n < n_2 < n_2 + 1$, or $\theta \in \pi_r^{n+1}$ with $n + 1 < n_2 + 1$.

– If $n_1 = n - 1$ and $\theta \neq \tilde{\theta}$, then either $\theta \in \pi_r^{n-1}$ or $\theta \in \pi_r^n$, with $n - 1 < n_2 - 1 < n_2 + 1$ and $n < n_2 < n_2 + 1$ respectively.

– If $n_1 \leq n - 2$, then $\theta \in \pi_r^{n_1}$ with $n_1 < n_2 < n_2 + 1$.

Finally, we show that $\Pi_{r+1}$ satisfies (L-IC). Let $n_2' \in \{2, \ldots, |\Pi_{r+1}|\}$, $n_1' \in \{1, \ldots, n_2' - 1\}$ and $\theta \in \pi_{r+1}^{n_2'}$. We want to show that $y^{\pi_{r+1}^{n_1'}} \notin X^\theta(y^{\pi_{r+1}^{n_2'}})$.

We first consider the case $\theta = \tilde{\theta}$. In that case, $\pi_{r+1}^{n_2'} = \pi_r^{n+1} \cup \{\tilde{\theta}\}$ and we have to show $y^{\pi_{r+1}^{n_1'}} \notin X^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}})$, *i.e.*

$$U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) > U^{\tilde{\theta}}(y^{\pi_{r+1}^{n_1'}}).$$

If $n_1' = n_2' - 1 = n - 1$, then $\pi_{r+1}^{n_1'} = \pi$, and thus $y^{\pi_{r+1}^{n_1'}} = y^\pi = y^{\pi_r^{n-1}}$. If instead $n_1' < n_2' - 1 = n - 1$, then $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^{n_1'}}$. In both cases, $y^{\pi_{r+1}^{n_1'}} = y^{\pi_r^{n'}}$ for some $n' \leq n - 1$. Since $\Pi_r$ satisfies (P1), $y^{\pi_r^{n'}} \leq y^{\pi_r^{n-1}}$ and from (23) and (2), we obtain the two inequalities

$$y^{\pi_r^{n'}} < y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}}.$$

If $U^{\tilde{\theta}}$ increases at $y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}$, then the first inequality gives the result. Otherwise, the second inequality gives $U^{\tilde{\theta}}(y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}}) \geq U^{\tilde{\theta}}(y^{\pi_r^{n+1}})$. Now by construction, we have $U^{\tilde{\theta}}(y^{\pi_r^{n+1}}) > U^{\tilde{\theta}}(y^{\pi_r^n})$, and since $\Pi_r$ satisfies (L-IC), we also have $U^{\tilde{\theta}}(y^{\pi_r^n}) > U^{\tilde{\theta}}(y^{\pi_r^{n'}})$. This gives the result.

Next we suppose $\theta \neq \tilde{\theta}$. As above, we set $y_1 = y^{\pi_{r+1}^{n_1'}}$ and $y_2 = y^{\pi_{r+1}^{n_2'}}$. Then according to Lemma 10, the result derives from $y_1 < y_2$ (guaranteed because $\Pi_{r+1}$ satisfies (P1)), and the existence of $n_2 \in \{2, \ldots, |\Pi_r|\}$ and $n_1 \in \{1, \ldots, n_2 - 1\}$ such that $\theta \in \pi_r^{n_2}$, $y_1 \leq y^{\pi_r^{n_1}}$ and $y_2 \leq y^{\pi_r^{n_2}}$.

- If $n_2' \leq n - 2$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'} = \pi_r^{n_2}$ with $n_2 = n_2'$, so that $y_2 = y^{\pi_r^{n_2}}$, and from $n_1' \leq n_2' - 1 \leq n - 3$, we also have $y_1 = y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$. This gives the result.

- If $n_2' = n - 1$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n-1} \cup \left(\pi_r^n \setminus \{\tilde{\theta}\}\right) = \pi$. In particular, $y_2 = y^\pi$, where we showed $y^\pi = y^{\pi_r^{n-1}}$.

  – If $\theta \in \pi_r^n \setminus \{\tilde{\theta}\}$, setting $n_2 = n$, we obtain $\theta \in \pi_r^n = \pi_r^{n_2}$ and $y_2 \leq y^{\pi_r^n} = y^{\pi_r^{n_2}}$. This gives the result.

  – If $\theta \in \pi_r^{n-1}$, setting $n_2 = n - 1$, we have $y_2 \leq y^{\pi_r^{n_2}}$ and $\theta \in \pi_r^{n_2}$. From $n_1' \leq n_2' - 1 = n - 2$, we have $y_1 = y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$. This also gives the result.

- If $n_2' = n$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n+1} \cup \{\tilde{\theta}\}$. Since $\theta \neq \tilde{\theta}$, we have $\theta \in \pi_r^{n+1} = \pi_r^{n_2}$ with $n_2 = n+1$, and, according to (2), $y_2 = y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}} = y^{\pi_r^{n_2}}$.
  - If $n_1' = n_2' - 1 = n - 1$, then $\pi_{r+1}^{n_1'} = \pi$, and then $y_1 = y^{\pi} = y^{\pi_r^{n-1}} = y^{\pi_r^{n_1}}$ with $n_1 = n - 1$. This gives the result.
  - If $n_1' < n_2' - 1 = n - 1$, then $\pi_{r+1}^{n_1'} = \pi_r^{n_1'}$ and thus $y_1 = y^{\pi_r^{n_1'}} = y^{\pi_r^{n_1}}$ with $n_1 = n_1'$, which also gives the result.
- If $n_2' \geq n+1$, then $\theta \in \pi_{r+1}^{n_2'} = \pi_r^{n_2'+1}$, and $y_2 = y^{\pi_r^{n_2'+1}} = y^{\pi_r^{n_2}}$ with $n_2 = n_2' + 1$.
  - If $n_1' \geq n+1$ or $n_1' \leq n-2$, then $\pi_{r+1}^{n_1'} = \pi_r^{n_1'}$ or $\pi_{r+1}^{n_1'} = \pi_r^{n_1'+1}$, *i.e.* $\pi_{r+1}^{n_1'} = \pi_r^{n_1}$ with $n_1 = n_1'$ or $n_1 = n_1' + 1$ respectively, and the result follows.
  - If $n_1' = n$, then $y_1 = y^{\pi_{r+1}^{n}} = y^{\pi_r^{n+1} \cup \{\tilde{\theta}\}} \leq y^{\pi_r^{n+1}}$ from (2). Setting $n_1 = n+1$ gives the result.
  - If $n_1' = n - 1$, then $y_1 = y^{\pi} = y^{\pi_r^{n-1}} = y^{\pi_r^{n_1}}$ with $n_1 = n - 1$ gives the result.