where we recall that $\phi$ is the roundtrip phase shift in a single ring resonator. For ring resonators made of single mode fibers, this phase shift can be written

$$\phi = \frac{\omega}{c} n_{eff} d = \frac{2\pi}{\lambda} n_{eff} d \tag{4.8-45}$$

where $d$ is the circumference of the ring resonator and $n_{\text{eff}}$ is the effective index of the mode of propagation in the fiber.

According to Equation (4.8-13), the magnitude of the coupling constant $|\kappa_{12}|$ is always less than (or equal to) unity ($0 \leq |\kappa_{12}| \leq 1$). As a result, the Bloch wavenumber $K$ can be complex in some spectral regimes according to Equation (4.8-44). A Bloch wave with a complex wavenumber cannot propagate in the periodic medium. According to Equation (4.8-42), the field amplitudes decay exponentially in the spectral regimes where the Bloch wavenumber is complex. These spectral regimes are known as the photonic bandgaps. If we plot the dispersion relationship ($\omega$ vs $K$) using Equation (4.8-44), we find that photonic bandgaps occur in the spectral regime where the roundtrip phase shift $\phi$ is an integral number of $2\pi$. The centers of the photonic bandgaps occur at exactly $\phi = 2m\pi$ ($m = 1, 2, 3 \ldots$). The Bloch wavenumbers at the centers of the photonic bandgaps are given by

$$K\Lambda = m\pi \pm i \cosh^{-1}\left(\frac{1}{|\kappa_{12}|}\right) \tag{4.8-46}$$

We note that the magnitude of the imaginary part of the Bloch wavenumber depends on the cross-coupling constant of the waveguide coupler. In Chapter 12, we will discuss more periodic structures that are also employed in photonics and optical electronics.

## 4.9 MULTICAVITY ETALONS

In the previous section, we described the transmission properties of single ring resonators as well as coupled ring resonators. Resonance occurs when the round-trip propagation phase shift inside the resonator is an integral multiple of $2\pi$. For ring resonators made of single-mode fibers or other single-mode waveguides, propagation attenuation due to bending loss can be significant, particularly for ring resonators with circumferences in the range of micrometers. In this section, we discuss coupled etalons made of thin films or multilayer structures. Propagation attenuation is negligible for thin films with thicknesses in the range of micrometers. There are situations in optical electronics when multicavity etalons are needed for spectral filtering or dispersion compensation applications. The transmission and reflection properties of these multicavity etalons can be analyzed by using the results obtained in Sections 4.1 and 4.8. Particularly, the amplitude transmission and reflection coefficients of Equations (4.1-31) and (4.1-32), or equivalently Equations (4.8-4) and (4.8-5) can be very useful. Referring to Figure 4.27, we consider the transmission of light through an etalon consisting of two mirrors separated by a distance $d$. The transmission and reflection coefficients are written in a more general form as, according to Equations (4.1-31) and (4.1-32),

$$r = r_{12} + \frac{t_{12}t_{21}r_{23}e^{-i2\phi}}{1 - r_{21}r_{23}e^{-i2\phi}} = \frac{r_{12} + (t_{12}t_{21} - r_{12}r_{21})r_{23}e^{-i2\phi}}{1 - r_{21}r_{23}e^{-i2\phi}} \tag{4.9-1}$$

$$t = \frac{t_{12}t_{23}e^{-i\phi}}{1 - r_{21}r_{23}e^{-i2\phi}} \tag{4.9-2}$$

with

$$2\phi = 2k_x d = 2nd \cos\theta(\omega/c) \tag{4.9-3}$$

where $d$ is the thickness of the cavity medium, $n$ is the refractive index of the cavity medium, and $k_x$ is the component of the wavevector along the normal of the mirror surface. The transmission and the reflection coefficients of the mirrors are defined as:

$t_{12}$ = transmission coefficient of mirror 1 with incidence from medium 1
$t_{21}$ = transmission coefficient of mirror 1 with incidence from medium 2
$t_{23}$ = transmission coefficient of mirror 2 with incidence from medium 2
$t_{32}$ = transmission coefficient of mirror 2 with incidence from medium 3
$r_{12}$ = reflection coefficient of mirror 1 with incidence from medium 1
$r_{21}$ = reflection coefficient of mirror 1 with incidence from medium 2
$r_{23}$ = reflection coefficient of mirror 2 with incidence from medium 2
$r_{32}$ = reflection coefficient of mirror 2 with incidence from medium 3

The above definitions are also illustrated in Figure 4.27.

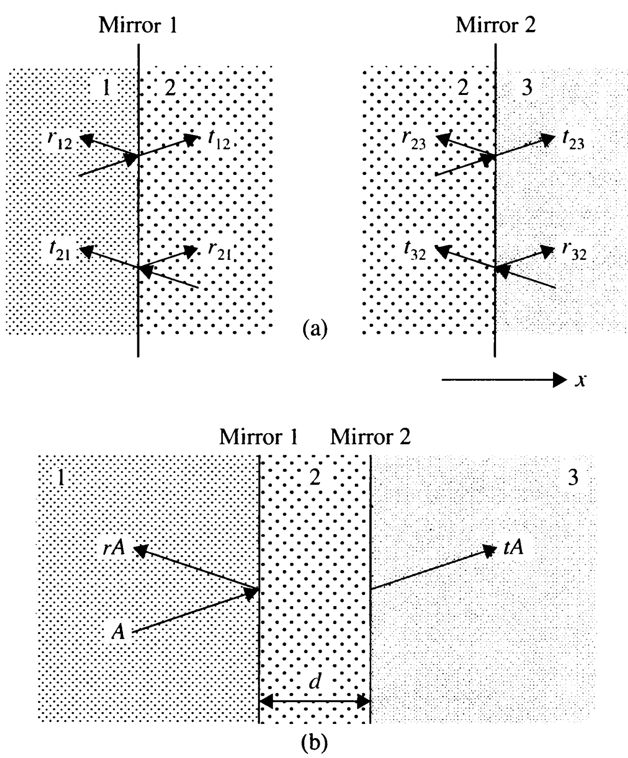



**Figure 4.27** Definition of transmission and reflection coefficients. (a) Definition of the mirror transmission and reflection coefficients. (b) Transmission and reflection coefficients of the etalon formed by the two mirrors.

The four coefficients associated with each mirror are in general complex numbers. As discussed earlier, they are related by fundamental principles of physics, including reciprocity and time-reversal symmetry. Specifically, we have (see Problem 4.16)

$$\begin{aligned} t_{12}t_{21}^* + r_{21}r_{21}^* &= 1, \quad t_{23}t_{32}^* + r_{32}r_{32}^* = 1 \\ t_{12}r_{12}^* + r_{21}t_{12}^* &= 0, \quad t_{23}r_{23}^* + r_{32}t_{23}^* = 0 \end{aligned} \tag{4.9-4}$$

These equations are known as the Stokes relationships [6].

It is important to note that $t_{12}$ and $t_{21}$ are identical only when the refractive indices of media 1 and 2 are the same. For lossless media with different refractive indices, these two sets of transmission coefficients are related by

$$\frac{t_{12}}{k_{1x}} = \frac{t_{21}}{k_{2x}}, \quad \frac{t_{23}}{k_{2x}} = \frac{t_{32}}{k_{3x}} \tag{4.9-5}$$

where $k_{1x}$, $k_{2x}$, and $k_{3x}$ are $x$ components of the wavevectors in media 1, 2, and 3, respectively,

$$\begin{aligned} k_{1x} &= n_1 \cos\theta_1(\omega/c) \\ k_{2x} &= n_2 \cos\theta_2(\omega/c) \\ k_{3x} &= n_3 \cos\theta_3(\omega/c) \end{aligned} \tag{4.9-6}$$

where $\omega$ is the frequency of the incident beam of light, $n_1$, $n_2$, and $n_3$ are the refractive indices, and $\theta_1$, $\theta_2$, and $\theta_3$ are the ray angles in the media, respectively. Equation (4.9-5) is consistent with the reciprocity of transmission of energy. The factors $k_{1x}$, $k_{2x}$, and $k_{3x}$ are essential to account for the different speeds of light in the media.

To illustrate the use of the transmission and reflection formulas, we consider the following example.

### EXAMPLE: THIN FILM SANDWICHED BETWEEN TWO DIFFERENT MEDIA

Referring to Figure 4.28, we consider a thin film of thickness $d_2$ and refractive index $n_2$. The film is sandwiched between two media with refractive indices $n_1$ and $n_3$.

The interface reflection and transmission coefficients are simply the Fresnel reflection and transmission coefficients. They are given by [6]

$$r_{12} = \begin{cases} \dfrac{k_{1x} - k_{2x}}{k_{1x} + k_{2x}} \\[2ex] \dfrac{n_1^2 k_{2x} - n_2^2 k_{1x}}{n_1^2 k_{2x} + n_2^2 k_{1x}} \end{cases} = \begin{cases} \dfrac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} & (s\text{-wave}) \\[2ex] \dfrac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} & (p\text{-wave}) \end{cases} \tag{4.9-7}$$

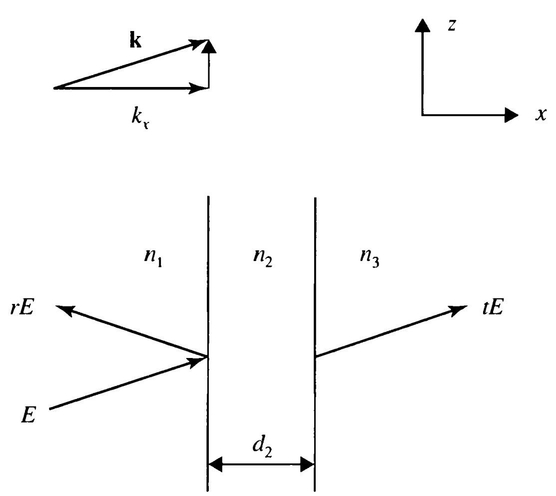



**Figure 4.28** The transmission and reflection of light through and from a thin film. The $x$ axis is perpendicular to the film surface.

$$t_{12} = \begin{cases} \dfrac{2k_{1x}}{k_{1x} + k_{2x}} \\ \\ \dfrac{2n_1 n_2 k_{1x}}{n_1^2 k_{2x} + n_2^2 k_{1x}} \end{cases} = \begin{cases} \dfrac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} & (s\text{-wave}) \\ \\ \dfrac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} & (p\text{-wave}) \end{cases} \tag{4.9-8}$$

where $k_{1x}$ and $k_{2x}$ are the $x$ components of the wavevectors in media 1 and 2, respectively. $\theta_1$ and $\theta_2$ are ray angles in media 1 and 2, respectively. The ray angles are related to the $x$ components of the wavevectors by the relationships (4.9-6). For an $s$-wave, the electric field vector is perpendicular to the plane of incidence ($xz$ plane), whereas for a $p$-wave, the electric field vector is parallel to the plane of incidence. A similar set of transmission and reflection coefficients for the interface between media 2 and 3 can be obtained from the above by replacing $2 \rightarrow 3$ and $1 \rightarrow 2$. We note that the Fresnel reflection and transmission coefficients satisfy the following relationships:

$$\begin{aligned} t_{12}t_{21} - r_{12}r_{21} &= 1 \\ r_{21} &= -r_{12} \end{aligned} \tag{4.9-9}$$

For the purpose of illustrating the concept, we consider the special case of normal incidence. At normal incidence, $\theta_1 = \theta_2 = 0$, and the $s$-wave and $p$-wave are identical. The Fresnel transmission and reflection become

$$\begin{aligned} r_{12} &= \frac{n_1 - n_2}{n_1 + n_2}, \quad r_{21} = -r_{12} \\ r_{23} &= \frac{n_2 - n_3}{n_2 + n_3}, \quad r_{32} = -r_{23} \\ t_{12} &= \frac{2n_1}{n_1 + n_2}, \quad t_{23} = \frac{2n_2}{n_2 + n_3} \\ t_{21} &= \frac{2n_2}{n_1 + n_2}, \quad t_{32} = \frac{2n_3}{n_2 + n_3} \end{aligned} \tag{4.9-10}$$

For lossless media, these coefficients are all real. The reflection and transmission coefficients for the thin film shown in Figure 4.28 can now be written, according to Equations (4.9-1), (4.9-2), and (4.9-10),

$$r = \frac{r_{12} + r_{23}e^{-i2\phi}}{1 - r_{21}r_{23}e^{-i2\phi}} = \frac{\left(\dfrac{n_1 - n_2}{n_1 + n_2}\right) + \left(\dfrac{n_2 - n_3}{n_2 + n_3}\right)e^{-i2\phi}}{1 + \left(\dfrac{n_1 - n_2}{n_1 + n_2}\right)\left(\dfrac{n_2 - n_3}{n_2 + n_3}\right)e^{-i2\phi}} \tag{4.9-11}$$

$$t = \frac{t_{12}t_{23}e^{-i\phi}}{1 - r_{21}r_{23}e^{-i2\phi}} = \frac{\left(\dfrac{2n_1}{n_1 + n_2}\right)\left(\dfrac{2n_2}{n_2 + n_3}\right)e^{-i\phi}}{1 + \left(\dfrac{n_1 - n_2}{n_1 + n_2}\right)\left(\dfrac{n_2 - n_3}{n_2 + n_3}\right)e^{-i2\phi}} \tag{4.9-12}$$

where $\phi = k_2 d_2 = n_2 d_2(\omega/c) = 2\pi n_2 d_2/\lambda$.

## EXAMPLE: ANTIREFLECTION (AR) COATING

We consider an example of a thin film coating to eliminate the Fresnel reflection at $\lambda = 1.5\ \mu\text{m}$. Let medium 1 be air with a refractive index of $n_1 = 1$, and medium 3 be a semiconductor with a refractive index of $n_3 = 3.5$. Equation (4.9-11) can be employed to investigate the reflection as a function of the film thickness and its index of refraction. According to Equation (4.9-11), the reflection coefficient vanishes when

$$r_{12} + r_{23}e^{-i2\phi} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) + \left(\frac{n_2 - n_3}{n_2 + n_3}\right)e^{-i2\phi} = 0 \tag{4.9-13}$$

The above equation can easily be satisfied by choosing

$$n_2 = \sqrt{n_1 n_3} \quad \text{and} \quad \phi = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2, \ldots \tag{4.9-14}$$

In other words, a thin film with a quarter-wave thickness and a refractive index equal to the geometric average of the bounding media can lead to zero reflectance. Since $\phi = k_2 d_2 = n_2 d_2(\omega/c) = 2\pi n_2 d_2/\lambda$, the quarter thickness is

$$d_2 = \frac{\lambda}{4n_2} = \frac{\lambda}{4\sqrt{n_1 n_3}} \tag{4.9-15}$$

Using $n_1 = 1$ and $n_3 = 3.5$, the refractive index of the thin film must be

$$n_2 = \sqrt{n_1 n_3} = 1.87$$

and the thickness can be

$$d_2 = \frac{\lambda}{4n_2} = \frac{\lambda}{4\sqrt{n_1 n_3}} = 0.2\ \mu\text{m}$$

or an odd integral multiple of it. It is important to note that the antireflection coating described above is only limited to the wavelength at $\lambda = 1.5\ \mu\text{m}$. Equation (4.9-9) can be employed to investigate the reflectivity at wavelength around $\lambda_0 = 1.5\ \mu\text{m}$. Figure 4.29 shows the reflectance ($|r|^2$) as a function of the phase $\phi$ for various indices of refraction of the thin film coating. The reflectance is a periodic function of the phase with minima occurring at $\phi = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2, \ldots$ for $n_2 < n_3$. When the film index is greater than $n_3$, the reflectance is higher and the minima occur at $\phi = \pi, 2\pi, 3\pi, \ldots$

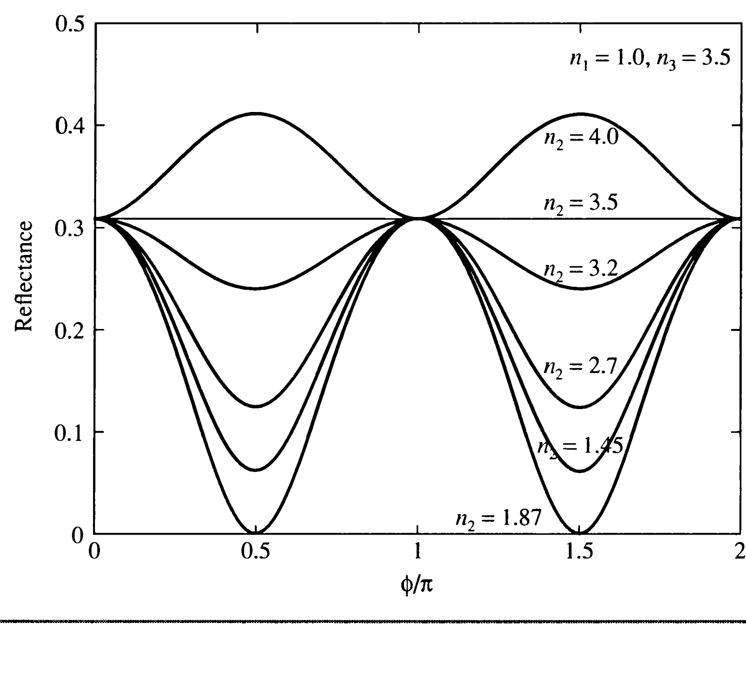



**Figure 4.29** Reflectance versus $\phi$ for $n_1 = 1$, $n_3 = 3.5$, and various values of $n_2$.

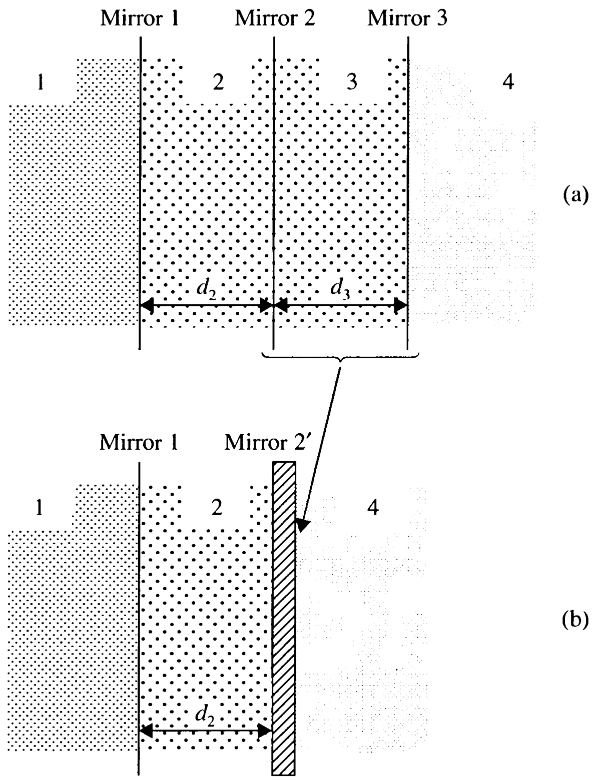



**Figure 4.30** Transmission and reflection problem of a two-cavity etalon. The second etalon consisting of mirrors 2 and 3 separated by a distance $d_3$ is equivalent to a mirror (mirror 2′). This reduces the two-cavity etalon problem to a one-cavity etalon problem.

The formulas for transmission and reflection coefficients of the etalon, Equations (4.9-1) and (4.9-2), can be generalized to treat multicavity etalons. To illustrate this, we consider a two-cavity etalon as shown in Figure 4.30. To find the transmission and reflection coefficients, we first find the transmission and reflection coefficients of the second etalon formed by medium 3 sandwiched between mirrors 2 and 3. This is done by removing mirror 1 and medium 1. The formula in Equations (4.9-1) and (4.9-2) can be employed for this purpose. We then consider a one-cavity problem formed by mirror 1 and an equivalent mirror (mirror 2′). The equivalent mirror (mirror 2′) consists of the second etalon formed by mirrors 2 and 3, sandwiched between media 2 and 4. The equivalent one-cavity etalon is illustrated in Figure 4.30b. The transmission and reflection coefficients can then be obtained by employing again the general formulas of Equations (4.9-1) and (4.9-2). The method described here can be generalized to treat the transmission and reflection problem of multicavity etalons.

## 2 × 2 Matrix Method

In addition to the use of an equivalent mirror, the problem of transmission and reflection properties of multicavity etalons can be treated by using a 2 × 2 matrix method. The matrix method is so general that it can be applied to treat the transmission and reflection properties of a general multilayer structure. To introduce the matrix method, we consider a general case of plane wave propagation on both sides of a dielectric interface (see Figure 4.31). We can write the electric field as

$$E = \begin{cases} (A_1 e^{-ik_{1x}x} + B_1 e^{+ik_{1x}x})e^{-i\beta z} & \text{medium 1} \\ (A_2 e^{-ik_{2x}x} + B_2 e^{+ik_{2x}x})e^{-i\beta z} & \text{medium 2} \end{cases} \tag{4.9-16}$$

where $A_1$ and $A_2$ are the amplitudes of the right-traveling wave (along $+x$ direction), $B_1$ and $B_2$ are the amplitudes of the left-traveling wave (along $-x$ direction), $\beta$ is the $z$ component of

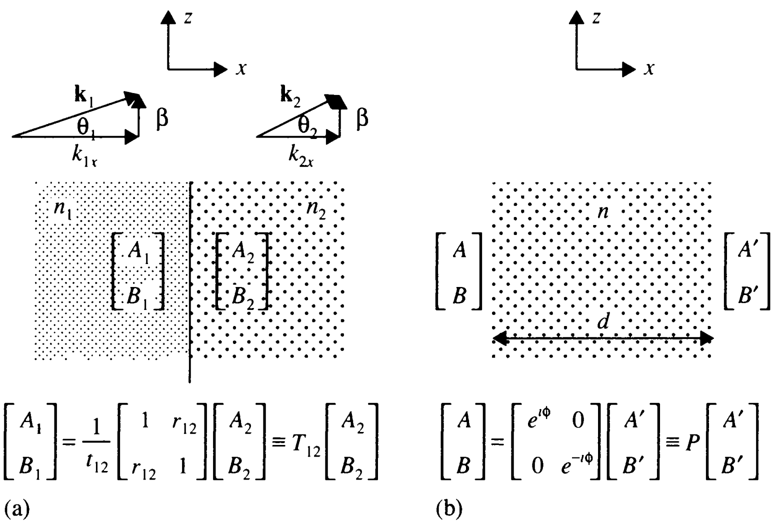



**Figure 4.31** (a) Transition matrix for a dielectric interface between two media. (b) Propagation matrix for a homogeneous medium.

the wavevector, and $k_{1x}$ and $k_{2x}$ are $x$ components of the wavevector. Note that $\beta$ is the same in both media as the structure is homogeneous in the $z$ direction. These components of the wavevectors are related by the following equations:

$$\begin{aligned} k_{1x}^2 + \beta^2 &= (n_1\omega/c)^2 \\ k_{2x}^2 + \beta^2 &= (n_2\omega/c)^2 \end{aligned} \tag{4.9-17}$$

Based on the definition of the transmission and reflection coefficients associated with the interface, the amplitudes are related by the following equations:

$$\begin{aligned} B_1 &= A_1 r_{12} + B_2 t_{21} \\ A_2 &= A_1 t_{12} + B_2 r_{21} \end{aligned} \tag{4.9-18}$$

where $r_{12}$, $r_{21}$, $t_{12}$, and $t_{21}$ are the Fresnel reflection and transmission coefficients discussed earlier. Solving for $A_1$ and $B_1$ in terms of $A_2$ and $B_2$, we obtain

$$\begin{aligned} A_1 &= \frac{1}{t_{12}} A_2 - \frac{r_{21}}{t_{12}} B_2 \\ B_1 &= \frac{r_{12}}{t_{12}} A_2 + \left( t_{21} - \frac{r_{12} r_{21}}{t_{12}} \right) B_2 \end{aligned} \tag{4.9-19}$$

Using Equation (4.10-9), the above can be rewritten

$$\begin{aligned} A_1 &= \frac{1}{t_{12}} A_2 + \frac{r_{12}}{t_{12}} B_2 \\ B_1 &= \frac{r_{12}}{t_{12}} A_2 + \frac{1}{t_{12}} B_2 \end{aligned} \tag{4.9-20}$$

or, equivalently, in a matrix form

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \frac{1}{t_{12}} \begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \equiv T_{12} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \tag{4.9-21}$$

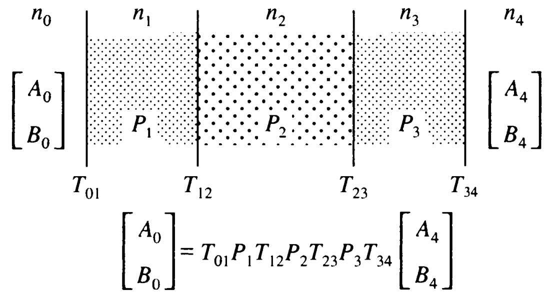



**Figure 4.32** Schematic drawing showing the matrix method of treating an example of multilayer structures (with $N = 3$).

where $T_{12}$ is defined as the transition matrix between layers 1 and 2. We notice the strong resemblance between the transition matrix $T_{12}$ and the scattering matrix $S_{12}$ of Equation (4.8-24) for ring resonators. Physically, $t_{12}$ is equivalent to the cross-coupling coefficient $\kappa_{12}$, and $r_{12}$ is equivalent to the straight-through coupling coefficient $\kappa_{11}$. It is important to note that the reflection and transmission coefficients of the interface, $r_{12}$ and $t_{12}$, depend on the state of polarization (either $s$ or $p$). Thus the problem of transmission and reflection is often treated separately for each of the polarization states. We note the transition matrix $T_{12}$ is a symmetric matrix.

The transition matrix relates the field column vector on both sides of a dielectric interface. We now need a matrix for propagation from one end of a homogeneous layer to the other end of the layer. Referring to Figure 4.31b, we consider propagation of plane waves in a homogeneous medium with thickness $d$ and refractive index $n$. Using the column vector for the field representation, we obtain

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} \equiv P \begin{bmatrix} A' \\ B' \end{bmatrix} \tag{4.9-22}$$

where $A$ and $B$ are the field amplitudes on the left end of the medium, whereas $A'$ and $B'$ are the field amplitudes on the right end of the medium. The phase shift is given by

$$\phi = k_x d = nd \cos\theta(\omega/c) \tag{4.9-23}$$

where $k_x$ is the $x$ component of the wavevector in the medium and $\theta$ is the ray angle (angle between the wavevector and the $x$ axis). The diagonal matrix $P$ is defined as the propagation matrix. For lossless media, the propagation matrix is unitary. The transition matrix $T$ and the propagation matrix $P$ can now be employed to treat a general multilayer structure. This is illustrated in Figure 4.32 for the case of a stack of three layers ($N = 3$) sandwiched between semi-infinite media of refractive indices $n_0$ and $n_4$. The column vector of the plane wave amplitudes $(A_0, B_0)$ in medium 0 is related to the column vector of the plane wave amplitudes $(A_{N+1}, B_{N+1})$ by the following relationship:

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix} \equiv M \begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix} \tag{4.9-24}$$

where $M$ is the $2 \times 2$ matrix that can be obtained by multiplying the transition matrices and the propagation in sequence:

$$M = T_{01}P_1T_{12}P_2T_{23}P_3 \ldots P_{N-1}T_{N-1,N}P_NT_{N,N+1} \tag{4.9-25}$$

We note the propagation matrix is independent of the state of polarization, whereas the transition matrix depends on the state of polarization. Using the symmetric property of the transition matrix $T_{12}$ and the unitary property of the propagation matrix, it can easily be shown that

$$M_{21} = M_{12}^*, \qquad M_{22} = M_{11}^* \tag{4.9-26}$$

for lossless multilayer structures. Using Equations (4.9-21) and (4.9-22) as well as the explicit forms of the Fresnel reflection and transmission coefficients (4.9-7) and (4.9-8), it can be shown that the determinant is given by

$$|M| = \frac{k_{N+1x}}{k_{0x}} \tag{4.9-27}$$

where $k_{N+1x}$ and $k_{0x}$ are the $x$ components of the wavevectors in media 0 and $N + 1$, respectively. Equation (4.9-27) is valid even if loss is present in the layered structure. If the two media are identical (i.e., $n_0 = n_{N+1}$), then the matrix $M$ is unimodular.

The transition matrices for both the $s$-wave and $p$-wave are given as, according to Equations (4.9-7), (4.9-8), and (4.9-21),

$$T_{12} = \begin{cases} \dfrac{1}{2k_{1x}} \begin{bmatrix} k_{1x} + k_{2x} & k_{1x} - k_{2x} \\ k_{1x} - k_{2x} & k_{1x} + k_{2x} \end{bmatrix} & (s\text{-wave}) \\[2ex] \dfrac{1}{2n_1 n_2 k_{1x}} \begin{bmatrix} n_1^2 k_{2x} + n_2^2 k_{1x} & n_1^2 k_{2x} - n_2^2 k_{1x} \\ n_1^2 k_{2x} - n_2^2 k_{1x} & n_1^2 k_{2x} + n_2^2 k_{1x} \end{bmatrix} & (p\text{-wave}) \end{cases} \tag{4.9-28}$$

or equivalently, in terms of ray angles

$$T_{12} = \begin{cases} \dfrac{1}{2n_1 \cos\theta_1} \begin{bmatrix} n_1 \cos\theta_1 + n_2 \cos\theta_2 & n_1 \cos\theta_1 - n_2 \cos\theta_2 \\ n_1 \cos\theta_1 - n_2 \cos\theta_2 & n_1 \cos\theta_1 + n_2 \cos\theta_2 \end{bmatrix} & (s\text{-wave}) \\[2ex] \dfrac{1}{2n_1 \cos\theta_1} \begin{bmatrix} n_1 \cos\theta_2 + n_2 \cos\theta_1 & n_1 \cos\theta_2 - n_2 \cos\theta_1 \\ n_1 \cos\theta_2 - n_2 \cos\theta_1 & n_1 \cos\theta_2 + n_2 \cos\theta_1 \end{bmatrix} & (p\text{-wave}) \end{cases} \tag{4.9-29}$$

The propagation matrix is given by

$$P = \begin{bmatrix} e^{ik_x d} & 0 \\ 0 & e^{-ik_x d} \end{bmatrix} \tag{4.9-30}$$

where $d$ is the thickness of the medium, and $k_x$ is the $x$ component of the wavevector:

$$k_x = \sqrt{(n\omega/c)^2 - \beta^2} = (n\omega/c)\cos\theta \tag{4.9-31}$$

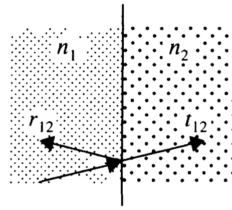
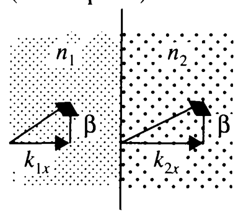
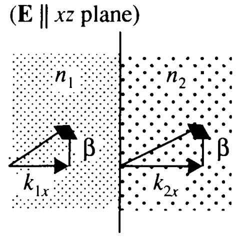
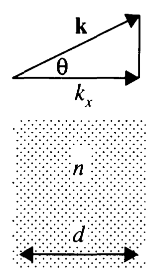
Table 4.1 is a summary of the $2 \times 2$ matrices for both the $s$-wave and $p$-wave. At normal incidence, the $s$-wave and $p$-wave are identical; the transition matrix becomes

$$T_{12} = \frac{1}{2n_1} \begin{bmatrix} n_1 + n_2 & n_1 - n_2 \\ n_1 - n_2 & n_1 + n_2 \end{bmatrix} \quad \text{(normal incidence)} \tag{4.9-32}$$

Once the matrix is obtained, the transmission and reflection coefficients of the multilayer structure can be calculated as follows. Let $A_0$ be the amplitude of the incident beam, $B_0$ be that of the reflected beam, and $A_{N+1}$ be that of the transmitted beam. We set $B_{N+1} = 0$, as the beam is launched from the left side. These amplitudes are related by, according to Equation (4.9-24),

$$\begin{aligned} A_0 &= M_{11} A_{N+1} \\ B_0 &= M_{21} A_{N+1} \end{aligned} \tag{4.9-33}$$

**TABLE 4.1** $2 \times 2$ Matrices for Dielectric Interface and Homogeneous Layer

| | | |
|---|---|---|
| Dielectric interface between two media with refractive indices $n_1$ and $n_2$ |  | $\frac{1}{t_{12}}\begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix}$ |
| Dielectric interface between two media with refractive indices $n_1$ and $n_2$ | $s$-polarization ($\mathbf{E} \perp xz$ plane)  | $\frac{1}{2k_{1x}}\begin{bmatrix} k_{1x}+k_{2x} & k_{1x}-k_{2x} \\ k_{1x}-k_{2x} & k_{1x}+k_{2x} \end{bmatrix}$ |
| Dielectric interface between two media with refractive indices $n_1$ and $n_2$ | $p$-polarization ($\mathbf{E} \parallel xz$ plane)  | $\frac{1}{2n_1n_2k_{1x}}\begin{bmatrix} n_1^2k_{2x}+n_2^2k_{1x} & n_1^2k_{2x}-n_2^2k_{1x} \\ n_1^2k_{2x}-n_2^2k_{1x} & n_1^2k_{2x}+n_2^2k_{1x} \end{bmatrix}$ |
| Homogeneous medium of thickness $d$ |  | $\begin{bmatrix} e^{ik_xd} & 0 \\ 0 & e^{-ik_xd} \end{bmatrix}$ |

The transmission and reflection coefficients are thus given by

$$t = \left(\frac{A_{N+1}}{A_0}\right)_{B_{N+1}=0} = \frac{1}{M_{11}}$$
$$r = \left(\frac{B_0}{A_0}\right)_{B_{N+1}=0} = \frac{M_{21}}{M_{11}} \qquad (4.9\text{-}34)$$

When the incident beam is launched from the right side, the transmission and reflection coefficients are given by

$$t' = \left(\frac{B_0}{B_{N+1}}\right)_{A_0=0} = \frac{|M|}{M_{11}}$$
$$r' = \left(\frac{A_{N+1}}{B_{N+1}}\right)_{A_0=0} = -\frac{M_{12}}{M_{11}} \qquad (4.9\text{-}35)$$

where $|M|$ is the determinant of the matrix. Let $T$ and $T'$ be the transmittances of the layered structure when light is incident from the left medium (medium 0) and right medium (medium $N + 1$), respectively. These two transmittances are given by

$$T = \frac{k_{N+1x}}{k_{0x}}|t|^2, \quad T' = \frac{k_{0x}}{k_{N+1x}}|t'|^2 \tag{4.9-36}$$

respectively. Using Equations (4.9-34) and (4.9-35) and the expression for $|M|$ in Equation (4.9-27), we obtain

$$T' = T \tag{4.9-37}$$

and

$$t' = |M|t = \frac{k_{N+1x}}{k_{0x}} t \tag{4.9-38}$$

Based on Equation (4.9-37), we note that the transmittances $T'$ and $T$ of a layered structure are identical, regardless of whether the beam is incident from the left side or the right side. This is true even if loss is present in the layered structure. In the event when media 0 and $N + 1$ are identical (i.e., $n_0 = n_{N+1}$), we have

$$t' = t \tag{4.9-39}$$

This is consistent with the principle of reciprocity.

Using Equations (4.9-34) and (4.9-35) for the general expressions of the transmission and reflection coefficients, we obtain

$$tt'^* + rr^* = 1, \quad tr'^* + rt^* = 0 \tag{4.9-40}$$

which are identical to the results obtained from the symmetry argument (time-reversal symmetry) [6]. They are known as the Stokes relationships. It can also be shown that, for a lossless layered structure,

$$R + T = 1 \tag{4.9-41}$$

which is consistent with the conservation of energy.

## 4.10 MODE MATCHING AND COUPLING LOSS

A basic problem of both theoretical and practical interest is how to couple efficiently an incident beam of light with complex amplitude $E_{\text{in}}(x, y)$ to a given mode of an optical resonator or an optical fiber, and also derive a measure of the residual, undesirable, excitation of other resonator modes. In the case of an optical fiber, this process is known as end-fire coupling. The problem arises frequently in optical communications when the light is coupled in and out of a single-mode fiber for wavelength selection, power amplification, or dispersion management purposes, and when the light is coupled in and out of a laser resonator (or amplifier). In optical fibers, a poor mode matching will lead to a significant insertion loss.

Referring to Figure 4.33, we designate the electric field of an incident beam of light at the "input" plane $z_1$ as $E_{\text{in}}(x, y)$ and the wavefunction of modes of the fiber or resonator as $E_{mn}(x, y)$, where $m$, $n$ are the transverse mode integers of the Gaussian beam of an optical resonator, or the LP mode of an optical fiber. The set $E_{mn}(x, y)$ of modes of a fiber or a resonator constitutes a complete orthogonal set of wavefunctions. They satisfy