MATH 539 lecture notes: Ultrafilters and Tychonoff 's compactness theorem [1]

Gregoire Fournier

March 3, 2025

1 Filters and ultrafilters

Definition 1 (filter). Suppose I is a non-empty set, $\mathcal{F} \subseteq \mathcal{P}(I)$ is called a filter on I if :

(i) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.

(ii) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

(iii) If $A \subseteq B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$.

Some well known examples are:

- $\{X : \mathbb{R} \setminus X$ has Lebesgue measure zero$\}$
- If $|I| \geq \kappa$, $\{X : |I \setminus X| < \kappa\}$.
  In particular, the set of all cofinite (i.e complement is finite) subsets of I forms a filter called Frechet filter.
- A filter is called principal if it is generated by one element of I, which means of the form $\{A \subseteq I : x \in A\}$ for some $x \in I$.

Note that the sets of filter can be ordered by the inclusion property. Such a poset satisfies the conditions of Zorn's lemma as we can define for each chain a maximal element.

Definition 2 (ultrafilter). An ultrafilter is a maximal filter (through the poset of filters).

An important consequence of maximality is the following:

Lemma 3. $\mathcal{U}$ is an ultrafilter iff $\forall A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

This property is equivalent to: If $\bigcup_{i=1}^{n} A_i \in \mathcal{U}$, then some $A_i \in \mathcal{U}$.

Proof.

- $\Leftarrow$: Suppose $\mathcal{U}$ is not a maximal, i.e there exists $\mathcal{F} \supsetneq \mathcal{U}$ filter. Take $A \in \mathcal{F} \setminus \mathcal{U}$, by assumption we must have $I \setminus A \in \mathcal{U}$, but since $I \in \mathcal{U}$ and $\mathcal{U}$ is closed under $\cap$, $A \in \mathcal{U}$, which is a contradiction.
- $\Rightarrow$: Given $\mathcal{U}$ an ultrafilter and $A \notin \mathcal{U}$ define $\mathcal{F}_A = \{X \in I : Y \setminus A \subseteq X$ for some $Y \in \mathcal{U}\}$.
  It is not too hard to check that $\mathcal{F}_A$ is a filter on I that contains $I \setminus A$ and $\mathcal{U}$.
  Therefore $I \setminus A \in \mathcal{F}_A = \mathcal{U}$.

$\square$

2 Compactness and Tychonoff's theorem

The compactness of a topological space can be restated in terms of convergence of ultrafilters.

- For (X, τ) a topological space and $\mathcal{F}$ a filter on it, we say that $lim\ \mathcal{F} = x$ if every neighbourhood of x has a non-empty intersection with all of the elements of $\mathcal{F}$.

- Furthermore if $\mathcal{F}$ is an ultrafilter, every set that intersects every element of $\mathcal{F}$ must lie in $\mathcal{F}$. Then, $lim\ \mathcal{F} = x$ iff every open neighbourhood of x is contained in $\mathcal{F}$.

If every two distinct points in X can be separated by disjoint open neighbourhoods, the space is called Hausdorff, and the limit is unique.

Lemma 4. *X is compact if and only if every ultrafilter on X converges to a point in X.*

Proof.
- $\Rightarrow$: Suppose X is compact, and let $\mathcal{U}$ be an ultrafilter on X. If $\mathcal{U}$ does not converge to any point in X, then for any point $x \in X$, x contained in $U_x \in \tau$, with $U_x \notin \mathcal{U}$. Thus, $\{U_x\}_{x \in X}$ forms an open cover of X, which must contain a finite subcover, $U_1, \dots, U_n$. But $\bigcup_j^n U_j = X \subset \mathcal{U}$, so one U_i must be in $\mathcal{U}$ by lemma 3, a contradiction.

- $\Leftarrow$: Let $\mathcal{C} = \{U_i : i \in I\}$ be an open cover of X. Suppose that it contains no finite subcover, i.e any finite intersection $(X \setminus U_1) \cap \dots \cap (X \setminus U_n)$ is non-empty.
Therefore $\mathcal{F} = \{F \subset X : X \setminus U \subset F$, for some $U \in \mathcal{C}\}$ is a filter. Consider $\mathcal{U}$ an ultrafilter containing $\mathcal{F}$, by assumption, let $x = lim\ \mathcal{U}$. Since $\mathcal{C}$ is a cover, there is $U_i \in \mathcal{C}$ with $x \in U_i$. Finally, $X \setminus U_i \in \mathcal{U}$ and since $x = lim\ \mathcal{U}$, $U_i \in \mathcal{U}$, a contradiction.

$\square$

Let $\{(X_\gamma, \tau_\gamma)\}_{\gamma \in \Gamma}$ be a collection of topological spaces. The Cartesian product $X = \prod_{\gamma \in \Gamma} X_\gamma$ is the set of functions $x : \Gamma \to \bigcup_{\gamma \in \Gamma} X_\gamma$ st. $x(\gamma) \in X\gamma$. We write $x(\gamma) = x_\gamma$ and $x = \{x_\gamma\}_{\gamma \in \Gamma}$. Let $\pi_\gamma : X \to X_\gamma$ be the usual projection map.
We define the product topology on the Cartesian product X to be the topology generated by the sets $\pi_\gamma^{-1}(U_\gamma)$, where $U_\gamma \in \tau_\gamma$. It is interesting to note that:

$$\{\bigcap_{i \in F} \pi_{\gamma_i}^{-1}(U_i) : \text{ where } F \text{ is finite and the } U_i \text{ are open in } X_{\gamma_i}\}$$

is a base for the product topology.
The product topology is the minimal topology on X in which all the projection maps are continuous.
In the same way, the weak* topology is the minimal topology on X^* in which the $J(x)(x^*) = x^*(x)$ are continuous and the weak topology is the minimal topology on X in which sets of finitely many elements of X^* are uniformly continuous.

Theorem 5 (Tychonoff's compactness theorem). *If all $X_\gamma, \gamma \in \Gamma$, are compact, then the Cartesian product $X = \prod_{\gamma \in \Gamma} X_\gamma$ is compact in the product topology.*

Proof. Let $\mathcal{U}$ be an ultrafilter on X. For any $\gamma \in \Gamma$, consider $\mathcal{U}_\gamma = \pi_\gamma(\mathcal{U})$. It is easy to check that $\mathcal{U}_\gamma$ is an ultrafilter on X_γ. Since X_γ is compact, there exists a limit $x_\gamma = lim\ \mathcal{U}_\gamma$. Let us show that $x = \{x_\gamma\}_{\gamma \in \Gamma} = lim\ \mathcal{U}$.
Take U an open neighbourhood of x in the product topology. Then along with x, U contains a finite intersection of basis sets $\bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i)$. We have $x_{\gamma_i} \in U_i$, and hence $U_i \in \mathcal{U}_{\gamma_i}$, which means there exist $A_i \in \mathcal{U}$ such that $\pi_{\gamma_i}(A_i) = U_i$. Then $A_i \subset \pi^{-1}\gamma_i(U_i)$, which implies that $\pi_{\gamma_i}^{-1}(U_i) \in \mathcal{U}$, for each i, and finally $\bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i) \in \mathcal{U}$.
Since U contains that intersection, U must be in the ultrafilter $\mathcal{U}$. $\square$

3 Application to Banach Alaoglu's theorem

Recall we defined the weak*-open neighbourhood of $f \in X^*$ to be:

$$U^{x_1,\dots,x_n}_{\varepsilon_1,\dots,\varepsilon_n}(x) = \{g \in X^* : |g(x_i) - f(x_i)| < \varepsilon_i, \forall i = [|n|]\}$$

The collection of all weakly*-open sets forms a topology, and the weak*-topology is the minimal topology on X^* in which the $J(x)(x^*) = x^*(x)$ are continuous.

Theorem 6 (Banach-Alaoglu). The closed unit ball of a dual space is compact in the weak*-topology.

Proof. Notice that for any $f \in \overline{B(X^*)}$, and $x \in X$, $f(x) \in [-||x||, ||x||]$. Therefore consider $\overline{B(X^*)}$ as a subset of the product space $T = \prod_{x \in X}[-||x||, ||x||]$. By Tychonoff's theorem, this product space is compact in the product topology.
It suffices to show that $\overline{B(X^*)}$ is closed in T, because convergence of nets in the product topology is equivalent to point-wise convergence, which for elements of $\overline{B(X^*)}$ amounts to weak*-convergence.

To this end, let $\{f_\alpha\}_{\alpha \in A}$ be a net in $\overline{B(X^*)}$ with $lim\ f_\alpha = f \in T$. By linearity of f_α's and the 'point-wise' sense of the limit above, we conclude that:

$$f(\lambda x + \mu y) \leftarrow f_\alpha(\lambda x + \mu y) = \lambda f_\alpha(x) + \mu f_\alpha(y) \to \lambda f(x) + \mu f(y)$$

Thus, f is linear, and since $|f_\alpha(x)| \leq ||x||$, we also have $|f(x)| \leq ||x||$ for all $x \in X$, so $f \in \overline{B(X^*)}$.

□

4 Ultraproducts and Los's theorem [2],[3]

4.1 Some basics of logic

Definition 7 (Language). A language $\mathcal{L}$ is given by specifying:

1. a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
3. a set of constant symbols $\mathcal{C}$.

Definition 8 ($\mathcal{L}$-structure). An $\mathcal{L}$-structure $\mathcal{M}$ is given by:

1. a nonempty set M called the universe, domain, or underlying set of M;

Definition 9 (Terms). The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that:

Definition 10 (Formulas).

Definition 11 (Assignment).

Definition 12 (Satisfaction).

Definition 13 (Ultraproduct). Let $\mathcal{L}$ be a first-order language. Suppose that $\mathcal{M}_i$ is an $\mathcal{L}$-structure for all $i \in I$ with universe M_i. Let $\mathcal{U} \in \mathcal{P}(I)$ be an ultrafilter.
We define E on $\prod_{i \in I} M_i$ as follows:

$$\forall f, g \in \prod_{i \in I} M_i,\ fEg \text{ iff } \{i \in I : f(i) = g(i)\} \in \mathcal{U}$$

It is pretty straightforward to check that E defines an equivalence relation on $\prod_{i \in I} M_i$. We will interpret the symbols of $\mathcal{L}$ in M to construct an $\mathcal{L}$-structure $\mathcal{M}$ as follows:

- The universe of $\mathcal{M}$ is $M = (\prod_{i \in I} M_i)_{/E}$.
- $\forall c \in L^{con}$, $c^{\mathcal{M}}$ is the equivalence class of the sequence given by $f(i) = c^{\mathcal{M}_i}$ for each $i \in I$.
- $\forall F \in L^{fun}$, $g_1, \ldots g_{n_F} \in M$, $F^{\mathcal{M}}(g_{1/E}, \ldots, g_{n_F/E}) = g_{/E}$, where $g : I \to \bigcup_{i \in I} M_i$ defined as $g(i) = F^{\mathcal{M}_i}(g_1(i), \ldots, g_{n_f}(i))$.
- $\forall R \in L^{rel}$, define $R^{\mathcal{M}} \subseteq M^{n_R}$ for any $g_1, \ldots g_{n_R} \in M$, $(g_{1/E}, \ldots, g_{n_R/E}) \in R^{\mathcal{M}}$ iff $\{i \in I : (g_1(i), \ldots, g_{n_R}(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}$.

We write $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$

Theorem 14 (Los theorem). Let $\phi(x_1, ..., x_n)$ be any $\mathcal{L}$-formula, then:

$$\mathcal{M} \Vdash \phi(g_{1/E}, \ldots, g_{n/E}) \text{ iff } \{\ i \in I : \mathcal{M}_i \Vdash \phi(g_{1/E}(i), \ldots, g_{n/E}(i))\} \in \mathcal{U}$$

Proof. We prove this by induction on complexity of formulas. $\square$

References

[1] R. Shvydkoy. Math 539 functional analysis lecture notes, 2022.

[2] J. Nagloo. Math 506 model theory lecture notes, 2022.

[3] D. Marker. Math 502 model theory lecture notes, 2015.