Optimal Information Provision for Strategic Hybrid Workers

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***Abstract*— We focus on the information provision problem of a strategic central planner that seeks to publicly reveal signals into an uncertain infectious risk parameter. Public belief over the parameter is updated with the signal and agents then decide whether to work remotely or in-person according to Nash equilibrium. The central planner maintains a set of desirable outcomes for each possible value of the infectious risk parameter and seeks to maximize the probability that agents choose an acceptable outcome for the true parameter. We distinguish between stateless and stateful objectives; in the former the set of desirable outcomes does not change as a function of the risk parameter, whereas in the latter it does. For stateless objectives, we reduce the problem to maximizing the probability of inducing mean beliefs that lie in intervals computable from the description of the desirable sets. We derive the principal's optimal signalling mechanism and show that generally these mechanisms use at most two signals and partition the parameter domain into at most two intervals with the signals generated according to an interval-specific distribution. We also consider stateful outcome sets that enforce in-person work capacity limits that progressively get more stringent as the risk parameter increases. In this case, we show that the optimal signalling mechanism can be obtained by solving a linear program. We numerically verify the improvement in achieving desirable outcomes relative to signalling benchmarks.**

I. Introduction

A. Motivation

The COVID-19 pandemic sparked tremendous interest in practical tools to mitigate disease spread [1], [2], [3], [4], [5], [6]. We distinguish between two types of non-pharmaceutical interventions central planners use: (i) hard and (ii) soft.

Hard interventions are rigorously enforced measures such as lockdowns, capacity limits or mask mandates. They prohibit certain actions that risk infectious spread. Balancing these interventions with their negative economic impact has recently been an active area of research [7], [8], [9]. These measures are indeed effective at flattening the contagion curve, particularly in the early phases of a pandemic when cures and vaccines are unavailable [10]. However, these measures alone are not viable long-term as they remain too economically and socially costly.

On the other hand, *soft interventions* aim to *influence* agents to choose less risky actions. For example, a central planner can influence agents' choices by incentivizing safer alternative actions or penalizing riskier actions. Soft interventions are generally much less costly to implement than hard interventions, however their design is a much more nascent area of study [11], [12], [13]. In this paper, we focus on a central planner's use of strategic information provision: the generation of stochastic, possibly coarse signals into the true value of an uncertain parameter that factors into agent utilities. The signals strategically reshape agents' beliefs over this parameter and thereby influence their chosen actions. For disease spread, the signals practically could correspond to public reporting on the virulence of a disease strain or reporting case counts to a particular granularity.

In this paper, we focus on how to optimally provision information to strategic hybrid workers. We consider the following problem described in Section II. A workforce has K groups of non-atomic agents, each deriving value from working in-person. However, there is a stochastic cost associated with becoming infected that increases with the number of in-person contacts and an unknown continuous stochastic parameter, $\theta \in \Theta$, measuring the disease's infectious risk. The central planner is symmetrically informed over θ, but can publicly generate signals into the true value of θ using a stochastic mechanism. Agents all observe the signal; public belief over θ is updated and agents simultaneous choose where to work according to Nash equilibrium. We consider a central planner that is strategic and designates a set of agent outcomes as desirable based on the true value of θ and some arbitrary pre-existing criteria. The novelty of our information provision problem is that this criteria need only admit a desirable set of outcomes for each parameter value. We distinguish between stateless and stateful objectives; in the former the set of desirable outcomes does not change as a function of the risk parameter, whereas in the latter it does. The central planner's objective is to maximize the probability that a desirable outcome is achieved. We investigate how to optimally provision information about the infectious risk and evaluate its effectiveness to further the central planner's objective.

In Section III, we derive the optimal signalling mechanisms for stateless objectives. We first characterize the equilibrium agents will achieve as a function of their beliefs over θ in Proposition 1. At equilibrium, we show that if there are agents working remotely from a group deriving higher benefit from in-person work than another group, all agents in the other group are working remotely. This restricts the set of possible equilibria to a one-dimensional manifold in the K-dimensional simplex. The intersection of this manifold and the desirable set of outcomes becomes the set of outcomes the central planner can hope to induce via information provision with high probability. In Lemmas 3.1, we characterize the mapping between posterior mean beliefs over

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θ and all achievable equilibria. Finally, in Theorem 3.1, we derive optimal signalling mechanisms. The structure of the mechanism depends on the relative position of the prior mean belief over θ relative to the subset of posterior mean beliefs in Θ that induce desirable outcomes. These mechanisms use at most two signals and partition the parameter domain into at most two intervals with the signals generated according to an interval-specific distribution. This result adds to a growing literature on the optimality of monotone partitional signalling mechanisms which partition Θ into intervals and generate signals symmetrically in each interval [14], [15], [16].

We next consider stateful outcome sets that enforce in-person capacity limits that progressively get more stringent as θ increases in Section IV. In Proposition 2, we consider Θ to be discrete and identify a linear program that solves for the optimal objective achievable with a signalling mechanism.

We then present in Section V a numerical comparison of the optimal signalling mechanisms we derived against two information benchmarks: the central planner either publicly (i) revealing the true θ (full information) and (ii) revealing nothing (no information). The derived optimal signalling mechanisms significantly improve on these benchmarks for both stateless and stateful capacity objectives.

B. Related Literature

We are centrally focused on the design of information provision mechanisms to control infectious spread. Information design has been active area of research in the economics community spawned by the seminal results shown by [17] (see [18], [19], [20] for useful reviews).

Our technical contribution adds to a recent literature on information provision for multiple agents over a continuous unknown parameter. The authors of [21] motivate the equivalence between signalling mechanisms over continuous parameters and mean-preserving contractions of the parameter's prior distribution. We exploit this insight to reduce the complexity of our search over signalling mechanisms. The authors in [22] build on this idea to characterize an optimal recommendation system for a discrete number of agents that are uncertain about a system state and must reveal a privately held type. Their results characterize how the designer must choose his recommended actions to maximize his quasilinear objective over these actions while retaining incentive compatibility. Our results instead focus on a setting with non-atomic agents and a broader class of objectives that need only be represented by sets of outcomes.

With regard to the design of soft interventions to mitigate disease spread, the authors of [12] consider the optimal design of rotation schemes where in-person workers alternate on some schedule. Related to information provision, [13] present an activity-based model where the principal informs agents of the infection rate and identify when full disclosure maximizes society's expected welfare. Most closely related to our own work, [11] presents a macro-perspective on how central planners seek to provision information about unknown stochastic risk during a pandemic. Our model is similar to theirs where agents must choose between isolating and an economically viable action. However our setting is more general, as their central planner seeks to globally optimize over an objective function that takes a weighted average of functional measures of the economic health and infectious health. Again, our model allows for a larger class of objectives that can incorporate other constraints or factors. Our results also technically go beyond a binary distribution over the infectious risk by considering the unknown parameter as being drawn from any bounded, continuous distribution.

There is also significant recent literature of infection dynamics in mean-field games [23], [24], [25], however the results of our paper focus predominantly on information design. Finally, we highlight several works that aim to contain infectious spread through hard interventions [7], [9], [26]. Critically these works establish that to optimally balance maintaining economic activity and reducing infectious cost, there should be a type and location dependence on where to institute lockdowns. Likewise, testing is a prominent hard intervention that seeks to recategorize infected individuals to quarantined individuals in SIQR models. The optimal allocation of tests across groups has been of recent interest with many results confirming the importance of type and location dependence in implementing the testing intervention [8], [27], [28].

II. Model and Problem Formulation

We consider a model where strategic hybrid workers (agents) must choose whether or not to work at a shared workspace amidst a pandemic of a communicable disease. Prior to making their choice, agents face uncertainty over the infectiousness of the disease (parameter) and thus over their likelihood of incurring the cost associated with becoming ill. In Sec. II-A, we describe agents' utilities and their dependence on the unknown parameter.

A central planner (principal) overseeing the agents will maintain preferences over the outcomes that the agents can achieve. The preferences may or may not depend on the true parameter. Fixing a set of desirable agent outcomes for each possible value of the parameter, the principal seeks to maximize the probability that the equilibrium outcome lies in the set corresponding to the true parameter. Despite facing the same uncertainty over the parameter as the agents, the principal seeks to implement a mechanism that stochastically generates a publicly disclosed signal into the parameter's true value. The signal is revealed before agents make their choices, thereby influencing public belief over the parameter and affecting the equilibrium that is achieved. In Sec. II-B, we formalize this class of signalling mechanisms and identify the principal's problem of designing the optimal mechanism.

A. Agent Behavior and Infectious Risk

We consider a unit mass of risk-neutral, Bayesian-rational agents. Each agent simultaneously commits to either working in-person at a shared workspace (ℓ_O) or working remotely (ℓ_R). Each agent belongs to a group $k \in [K] := \{1, .., K\}$. We represent the population mass in each group using the mass vector $\mathbf{x} := (x_1, .., x_K) \in \mathbb{R}_+^K$ with the mass of agents belonging to each group k, x_k, being common knowledge. Upon choosing their actions, we represent the *remote mass* vector by $\mathbf{y} := (y_1, .., y_K)$ where $0 \leq y_k \leq x_k$ agents from group k elected to work remotely. The *in-person mass* vector is then represented by $\mathbf{x}-\mathbf{y}$ with $x_k - y_k \geq 0$ agents working in-person. Letting $\|\cdot\|$ correspond to the L^1-norm, note that $\|\mathbf{x}\| = 1 = \|\mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\|$ as the in-person and remote masses sum to the initial unit mass.

Any agent in any group k who works remotely faces no infectious risk, but also receives no benefit from the work environment. Consequently, their utility $u_k(\ell_R, \mathbf{y}; \theta) = 0$ independent of θ and $\mathbf{y}$.

Agents from group k that work in-person, however, each receive a commonly-known benefit v_k where $v_k > 0$ without loss of generality. The benefit encapsulates the personal value agents derive from working in-person net any travel costs. Without loss of generality, we assume that v_k are distinct and we index groups so that v_k is strictly decreasing in k.

Agents that work in-person face both an uncertain risk of contracting the disease per infectious contact and also face uncertain costs in the event they do become ill. Both the transmissivity and severity of the disease are natural sources of public uncertainty as they cannot be easily estimated individually. The overall risk agents can expect to incur also directly depends on the in-person mass in a known way as the increased frequency of contacts and likelihood of infected individuals being present can be easily estimated. Since agents are risk neutral, we capture the product of the uncertain risk per contact and uncertain costs into an uncertain *infectiousness parameter* $\theta \in \Theta := [0, M]$ for some $M \in \mathbb{R}_+$. Critically, we assume that the agents cannot perceive the value of the parameter θ that was realized for this specific disease, but that the agents and principal all commonly believe a prior distribution F over Θ from which $\theta \sim F$.

We model the (expected) stochastic infectious cost $\beta(\theta, \mathbf{y})$ to be linear in θ and decreasing in the remote mass (increasing for in-person mass). Specifically, for commonly-known, positive, differentiable functions $c_1, c_2 : [0, 1] \to \mathbb{R}_+$ with c_1 decreasing, c_2 non-increasing and $c_1(1) = c_2(1) = 0$:

$$\beta(\theta, \mathbf{y}) := \theta c_1(\|\mathbf{y}\|) + c_2(\|\mathbf{y}\|) \tag{1}$$

We motivate this form of the infectious cost in greater detail in Appendix VII-A. Notice that given the actions of all agents $\mathbf{y}$ and the true parameter θ, the utility of agents of group k working in-person is:

$$u_k(\ell_O, \mathbf{y}; \theta) = v_k - \beta(\theta, \mathbf{y}) \tag{2}$$

B. Signalling and Central Planner Objective

The principal can implement a *signalling mechanism* which publicly provisions information into the true realization of θ to all agents. Specifically, we assume that principal publicly commits to and discloses a mechanism $\pi = \langle \{g_\theta\}_{\theta \in \Theta}, \mathcal{I} \rangle$ where $\{g_\theta\}_{\theta \in \Theta}$ are a set of probability distributions with each g_θ distributed over the set of *signals* $\mathcal{I}$. If the true realization of the infectious risk is $\tilde{\theta} \in \Theta$, then a signal $i \in \mathcal{I}$ is randomly generated with probability $g_{\tilde{\theta}}(i)$. The principal critically does not observe the true parameter value before committing to π and we assume the mechanism's generation of these public signals is a black-box process.

Prior to choosing their actions, all agents view $i \in \mathcal{I}$ and symmetrically update their belief over θ to a posterior distribution F_i where:

$$\begin{aligned} F_i(t) &= \mathbb{P}[\theta \leq t | i] \\ &= \frac{\int_0^t g_\theta(i) dF(\theta)}{\int_0^M g_\theta(i) dF(\theta)} \end{aligned} \tag{3}$$

For any given π, it is useful to represent the corresponding probability with which each signal $i \in \mathcal{I}$ is generated and the posterior mean of θ upon viewing each signal $i \in \mathcal{I}$. If agents' strategies only depend on their mean beliefs over Θ, this representation is equivalent to a *direct mechanism* where the principal performs the update on the agents' behalf and simply shares the updated posterior mean with all agents. We denote such a mechanism by the set of tuples $\mathcal{T}_\pi = \{(q_i, \theta_i)\}_{i \in \mathcal{I}}$ where:

$$q_i := \int_0^M g_\theta(i) dF(\theta) \tag{4}$$

$$\theta_i := \frac{\int_0^M \theta g_\theta(i) dF(\theta)}{\int_0^M g_\theta(i) dF(\theta)} \tag{5}$$

A particularly relevant subclass of mechanisms is one in which Θ is partitioned into intervals and the parameters in each interval collectively map to a single interval-specific signal. Namely, we call a mechanism π *monotone partitional* if there exists a finite partition of Θ, $\mathcal{P} := \{\Theta_i\}_{i=1}^m = \{[t_{i-1}, t_i]\}_{i=1}^m$ for some m with $t_0 = 0, t_m = M$ and some increasing sequence $\{t_i\}_{i=1}^{m-1} \subseteq \Theta$ such that $\mathcal{I} = [m]$ and for all θ, $g_\theta(i) = \mathbb{I}\{\theta \in [t_{i-1}, t_i]\}$. Observe from (4) and (5) that for these mechanisms $q_i = F(t_i) - F(t_{i-1})$ and $\theta_i = \frac{\int_{t_{i-1}}^{t_i} \theta dF(\theta)}{q_i}$. The monotone partitional structure is important in optimal information provision for continuous parameters as these interval-based signalling mechanisms correspond to the vertices of the polytope containing all feasible signalling mechanisms over the parameter [14], [16], [22], [29].

Each instance of our game for a particular initial mass vector $\mathbf{x}$ and public belief G over Θ can be expressed in normal form $\Gamma(\mathbf{x}, G) = \langle \mathcal{N}, \mathcal{A}, \mathbf{u} \rangle$:

- $\mathcal{N}$: mass of players across groups, $\mathcal{N} = \mathbf{x}$
- $\mathcal{A}$: set of actions (same across players), $\mathcal{A} = \{\ell_O, \ell_R\}$
- $\mathbf{u}$: group-specific utility functions
  $\mathbf{u} = \{\mathbb{E}_{\theta \sim G}[u_k(\cdot, \cdot; \theta)]\}_{k \in K}$

We consider the notion of a (Nash) equilibrium where no agents have an incentive to deviate from where they decided to work. Hence, agents of group k at an equilibrium $\mathbf{y}^*$ should always be choosing an action that generates utility equal to the best alternative of working remotely or in-person, $\arg\max_{\ell \in \{\ell_O, \ell_R\}} \mathbb{E}[u_k(\ell, \mathbf{y}^*; \theta)]$. Observe that in an equilibrium, a group's agents mix across both actions if and only if this group's agents are indifferent between ℓ_O and ℓ_R. The following definition exploits this insight:

Definition 1: The remote mass vector $\mathbf{y}^*$ is a Nash equilibrium of our game $\Gamma(\mathbf{x}, G)$ if and only if both:

(i) $\mathbb{E}_{\theta\sim G}[u_k(\ell_O, \mathbf{y}^*; \theta)] > 0 \implies y_k^*(i) = 0$

(ii) $\mathbb{E}_{\theta\sim G}[u_k(\ell_O, \mathbf{y}^*; \theta)] < 0 \implies y_k^*(i) = x_k$

Once signal i is publicly revealed with the mechanism π, a signal-specific public belief F_i over the risk parameter (computed using (3)) is induced and we denote the equilibrium remote mass that agents choose in response by $\mathbf{y}_\pi^*(i) \in \mathbb{R}_+^K$. In our model, the principal is interested in maximizing the probability that the equilibrium that is achieved by the agents belongs to a set of desirable outcomes $\mathcal{Y}(\theta) \subset \mathbb{R}^K$ that depends on the true parameter θ. The significance of this set for which we seek to maximize its reachability can arise directly out of notions like capping the stochastic infection cost (i.e. the set $\mathcal{Y}(\theta) := \{\mathbf{y} : \beta(\theta, \mathbf{y}) \leq \epsilon\}$) or maximizing total social surplus (i.e. the set $\mathcal{Y}(\theta) := \{\mathbf{y} : \sum_{i=1}^K v_i(x_i - y_i) - (1 - \|\mathbf{y}\|)\beta(\theta, \mathbf{y}) \geq \epsilon\}$).

In this paper, we focus explicitly on two classes of objectives. The first is what we denote as *stateless* objectives where $\mathcal{Y}(\theta)$ does not depend on θ. Hence, the objective set can be represented by a set $\mathcal{Y}$ independent of θ.

The second is a *stateful* objective wherein we restrict the shapes of the objective sets to be nested as θ increases. Particularly, we focus on a *scaled capacity* objective where $\mathcal{Y}(\theta) = \{\mathbf{y} : \|\mathbf{y}\| \geq b(\theta)\}$ for some increasing function $b(\cdot)$. This stateful objective mimics capacity restrictions that are imposed on the shared workspace which are gradually scaled down as the infectiousness decreases.

We can express the (expected) objective $V(\pi)$ of the principal using π by:

$$V(\pi) = \mathbb{P}\{\mathbf{y}_\pi^*(i) \in \mathcal{Y}(\theta)\} \tag{6}$$

For a stateless objective parameterized by $\mathcal{Y}$ this corresponds to $V(\pi) = \mathbb{P}\{\mathbf{y}_\pi^*(i) \in \mathcal{Y}\}$. The principal seeks to design π^* so that:

$$\pi^* \in \underset{\pi:\langle\{g_\theta\}_{\theta\in\Theta}, \mathcal{I}\rangle}{\arg\max} \; V(\pi) \tag{7}$$

III. Stateless Information Design

In this section, we focus on designing π^* for a *stateless* principal objective $\mathcal{Y}$. In Sec. III-A, we compute the exact form of the equilibrium $\mathbf{y}_\pi^*(i)$ as a function of agent's mean beliefs θ_i. We then derive the structure of the optimal mechanism used by the principal in Sec. III-B.

A. Equilibrium Characterization

Recall from Equation (2) that agents' utilities have a linear dependence on θ. Since agents are risk-neutral, it is clear from Definition 1 that the equilibrium outcomes $\mathbf{y}_\pi^*(i)$ should only vary as a function of the posterior means θ_i. Observe that agents that work in-person incur the same expected costs but may derive different benefits, so the agents deriving higher value from working in-person should do so in an equilibrium. The following proposition formalizes these insights by computing the equilibrium in-person mass $m(\theta_i) := 1 - \|\mathbf{y}_\pi^*(i)\|$ and equilibrium mass vector $\mathbf{y}_\pi^*(i)$. The proofs are deferred to the appendix.

Proposition 1: Let $s_j = \sum_{i=1}^j x_i$ for all $j \in \{0, 1, .., K\}$, and $v(u) = v_j$ if $s_{j-1} \leq u < s_j$ for all $u \in [0, 1]$. In response to signal i under mechanism π, the equilibrium in-person mass $m(\theta_i) = \sup\{u : v(u) \geq c_1(1-u)\theta_i + c_2(1-u)\}$ and:

$$(\mathbf{y}_\pi^*(i))_k = \begin{cases} 0 & \text{if } s_k \leq m(\theta_i) \\ m(\theta_i) - s_{k-1} & \text{if } s_{k-1} < m(\theta_i) < s_k \\ x_k & \text{if } s_{k-1} \geq m(\theta_i) \end{cases}$$

Intuitively, the proposition sets the in-person mass $m(\theta_i)$ by having as many agents work in-person as possible until the remaining agents find the infectious cost too high. The non-increasing function $v(u)$ corresponds to the highest benefit amongst the mass of $1-u$ agents with the smallest benefits v_i. The computation merely increases u until this no longer exceeds the expected infectious costs.

Proposition 1 also identifies a threshold-based structure that the equilibrium must obey. Specifically, we can define a critical group $k^*(i) = \inf\{k : s_k > m(\theta_i)\}$ where all agents in groups k for $k < k^*(i)$ all work in-person and agents in groups k for $k > k^*(i)$ all work remotely. This characteristic determines a strict subset of all possible remote mass vectors $\mathbf{y}$ that are achievable in equilibrium. Hence, from Proposition 1, we can conclude that for any π, any equilibrium $\mathbf{y}_\pi^*(i)$ must lie on a one-dimensional manifold $\mathcal{Z}(\mathbf{x}) := \{c\mathbf{e}_k + \sum_{i=k+1}^K x_i\mathbf{e}_i | k \in [K], c \in [0, x_k]\}$ where $\mathbf{e}_i \in \mathbb{R}^K$ with $(\mathbf{e}_i)_k = 1$ if $i = k$ and 0 otherwise; see Fig. 1.

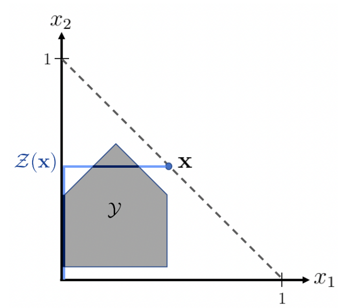


Fig. 1: Example of $\mathcal{Z}(\mathbf{x})$ (solid curve in blue) when $K = 2$.

Moreover, as the posterior mean belief of the infectiousness θ_i increases, agents will be more cautious as the infectious cost increases on expectation. The following lemma establishes that the equilibrium in-person mass is monotone and smooth in θ_i.

Lemma 3.1: $m(\theta)$ is non-increasing, bounded and continuous in θ.

Using Proposition 1 and Lemma 3.1, we conclude that as the posterior mean θ_i increases, the equilibrium mass vector moves further from the initial point $\mathbf{x}$ and closer towards the origin along $\mathcal{Z}(\mathbf{x})$. As a result, the design of the optimal signalling mechanism can be viewed as the problem of maximizing the probability that posterior means generated by the mechanism correspond to points $\mathbf{y} \in \mathcal{Z}(\mathbf{x})$ that the principal deems desirable (i.e. $\mathbf{y} \in \mathcal{Y}$).

B. Optimal Information Design

We now use the equilibrium characterization to solve the principal's problem of choosing an optimal signalling mechanism π.

The manifold $\mathcal{Z}(\mathbf{x})$ contains all possible equilibria $\mathbf{y}^*_\pi(i)$ that could possibly be achieved for any π and any realized signal i. However, the manifold need not intersect with the goal set $\mathcal{Y}$ at all. In such cases, no π can ever induce an equilibrium $\mathbf{y}^*_\pi(i) \in \mathcal{Y}$. Consequently, there exists a geometric relationship between $\mathcal{Y}$ and $\mathcal{Z}(\mathbf{x})$ that must be incorporated into this design problem.

More generally, in relation to well-behaved sets $\mathcal{Y}$, our next result shows that the equilibria in $\mathcal{Z}(\mathbf{x}) \cap \mathcal{Y}$ corresponds to a finite number of intervals of in-person masses. Specifically, we make the following assumption on our goal sets $\mathcal{Y}$:

(A1) $\mathcal{Y}$ is closed and convex.

Moreover, we denote $k(\mathbf{x}, u) = \min\{j > 0 : s_j > u\}$ as the lowest group that has not completely chosen in-person work after the mass of u agents with highest benefit choose in-person work. Likewise, $z(\mathbf{x}, u)$ corresponds to the point on $\mathcal{Z}(\mathbf{x})$ after u agents choose in-person work or equivalently $z(\mathbf{x}, u) = (u - \sum_{k=1}^{k(\mathbf{x},u)-1} x_k)\mathbf{e}_{k(\mathbf{x},u)} + \sum_{k=k(\mathbf{x},u)+1}^{K} x_k \mathbf{e}_k$. Proposition 1 implies that $z(\mathbf{x}, 1 - \|\mathbf{y}^*_\pi(i)\|) = \mathbf{y}^*_\pi(i)$ for all i.

Lemma 3.2: Under Assumption A1, there exists $\tilde{K} \leq K$ closed and disjoint intervals $\Omega_i = [\omega^1_i, \omega^2_i] \subset [0, 1]$ for all $i \in [\tilde{K}]$ with $\mathcal{Y} \cap \mathcal{Z}(\mathbf{x}) = \cup_{i=1}^{\tilde{K}} \{z(\mathbf{x}, u) : u \in \Omega_i\}$.

Observe that if $\mathcal{Y} \cap \{z(\mathbf{x}, u) : s_{k-1} \leq u \leq s_k\}$ is empty, then $\omega^1_k > \omega^2_k$. Trivially, notice that if $\mathcal{Y}$ is contained in the interior of the positive orthant of $\mathbb{R}^K$, then there is at most only one non-empty segment of intersection Ω_1 since $\{z(\mathbf{x}, u) : s_{k-1} \leq u \leq s_k\} \cap \mathrm{int}(\mathbb{R}^K_+) = \emptyset$ for all $k > 1$. We note that our results are generalizable beyond Assumption A1. Namely, if $\mathcal{Y}$ is nonconvex, our results are valid as long as the $\mathcal{Y} \cap \mathcal{Z}(\mathbf{x})$ is able to be represented by a finite number of intersecting line segments Ω_i as in Lemma 3.2.

Using Lemma 3.2, we can now re-express (7) to instead depend on whether the remote masses lie in one of the intervals Ω_i:

$$\begin{aligned}
V^* &= \max_{\pi:\langle\{g_\theta\}_{\theta\in\Theta},\mathcal{I}\rangle} \mathbb{P}\{\mathbf{y}^*_\pi(i) \in \mathcal{Y}\} \\
&= \max_{\pi:\langle\{g_\theta\}_{\theta\in\Theta},\mathcal{I}\rangle} \mathbb{P}\{\mathbf{y}^*_\pi(i) \in \mathcal{Y} \cap \mathcal{Z}(\mathbf{x})\} \\
&= \max_{\pi:\langle\{g_\theta\}_{\theta\in\Theta},\mathcal{I}\rangle} \sum_{k=1}^{\tilde{K}} \mathbb{P}\{1 - \|\mathbf{y}^*_\pi(i)\| \in \Omega_i\} \\
&= \max_{\pi:\langle\{g_\theta\}_{\theta\in\Theta},\mathcal{I}\rangle} \sum_{k=1}^{\tilde{K}} \mathbb{P}\{\omega^1_k \leq m(\theta_i) \leq \omega^2_k\} \qquad (8)
\end{aligned}$$

From Lemma 3.1, we note that since m is monotone, bounded and continuous, the preimage of this mapping over the closed interval $[\omega^1_k, \omega^2_k]$ is another closed interval $[\underline{\theta}_k, \bar{\theta}_k]$ where $m(\theta_i) \in \Omega_i$ if and only if $\theta_i \in [\underline{\theta}_k, \bar{\theta}_k]$. Without loss of generality, since the intervals Ω_i are disjoint, we let $\underline{\theta}_k$ be increasing in k which also implies that $\bar{\theta}_k$ is increasing in k. Exploiting this fact and using a mechanism π's direct counterpart $\mathcal{T}_\pi$ as computed in (4) and (5), we can express the principal's program as:

$$V^* = \max_{\mathcal{T}_\pi:\{(q_i,\theta_i)\}_{i\in\mathcal{I}}} \sum_{k=1}^{\tilde{K}} q_i \mathbb{I}\{\underline{\theta}_k \leq \theta_i \leq \bar{\theta}_k\} \qquad (9)$$

We make some immediate observations about this result. Firstly, observe that if the objective set $\mathcal{Y}$ does not intersect with the manifold $\mathcal{Z}(\mathbf{x})$, then all the Ω_i are empty and hence the objective in (8) is necessarily equal to 0. By contrast,if $\mathbb{E}_{\theta\sim F}[\theta] := \mu \in [\underline{\theta}_l, \bar{\theta}_l]$ for some l, then a non-informative mechanism which is achieved with any π where $\mathcal{I}$ is a singleton, has $\mathcal{T}_\pi = \{(1, \mu)\}$. Hence, the principal achieves the maximal objective of 1 with any non-informative mechanism. However, if $\mu \notin [\underline{\theta}_l, \bar{\theta}_l]$ for any l, then a non-informative mechanism achieves an objective of 0. Consequently, we find it useful to categorize distributions F based on the positioning of the prior mean μ relative to the intervals $\{[\underline{\theta}_k, \bar{\theta}_k]\}_{k\in[\tilde{K}]}$ and solve for the optimal mechanism in each regime. Letting $\underline{s}(t) = \mathbb{E}[\theta|\theta \leq t]$ and $\bar{s}(t) = \mathbb{E}[\theta|\theta \geq t]$, we introduce the following regimes and subregimes on the prior distribution F:

(R1): $\exists k$ such that $\mu \in [\underline{\theta}_k, \bar{\theta}_k]$

(R2): $\exists k$ such that $\mu \in (\bar{\theta}_k, \underline{\theta}_{k+1})$

(R2a): $\exists \ell, k \in [\tilde{K}], t \in \Theta, \alpha, \lambda \in [0, 1]$

s.t. $\dfrac{\underline{s}(t)\lambda F(t) + \bar{s}(t)\alpha(1 - F(t))}{\lambda F(t) + \alpha(1 - F(t))} \in [\underline{\theta}_k, \bar{\theta}_k]$

and $\dfrac{\underline{s}(t)(1-\lambda)F(t) + \bar{s}(t)(1-\alpha)(1 - F(t))}{(1-\lambda)F(t) + (1-\alpha)(1 - F(t))} \in [\underline{\theta}_\ell, \bar{\theta}_\ell]$

(R3): $\mu > \bar{\theta}_{\tilde{K}}$

(R4): $\mu < \underline{\theta}_1$

Before solving for an optimal mechanism in each regime, note that any mechanism π that achieves the best-possible objective by solving the optimization in (9) will not be unique. This is immediate as any signal $i \in \mathcal{I}$ can be forked into two signals i_1, i_2 uniformly at random and induce symmetric posterior means $\theta_i = \theta_{i_1} = \theta_{i_2}$ with probability $\frac{q_i}{2}$ each and hence achieve the same objective V^*. To eliminate such multiplicity, we arrive at the following lemma: it shows that we need not consider mechanisms that use a set of signals $\mathcal{I}$ of size $|\mathcal{I}| > \tilde{K}+1$ since we require at most one signal to correspond to a posterior mean in each of the $\tilde{K}$ intervals and one extra signal whose posterior mean resides outside these sets.

Lemma 3.3: There exists an optimal mechanism $\pi^* = \langle\{g^*_\theta\}_{\theta\in\Theta}, [\tilde{K}+1]\rangle$ where $V(\pi^*) = V^*$ such that for each $k \in [\tilde{K}]$, $\theta_k \in [\underline{\theta}_k, \bar{\theta}_k]$ and $\theta_{\tilde{K}+1} \notin [\underline{\theta}_k, \bar{\theta}_k]$ for all $k \in [\tilde{K}]$.

The insight given by Lemma 3.3 consequently reduces the search complexity over all possible mechanisms π to those that use at most $\tilde{K}+1$ signals and can be represented by a tuple $\mathcal{T}_\pi$ with at most $\tilde{K}+1$ tuples. Using this reduction, we can without loss modify the optimization in (9) to only go over the set of all possible $\mathcal{T}_\pi$ with less than $\tilde{K}+1$ sets. Notice that if we simply specify a $\{(q_i,\theta_i)\}_{i\in[\tilde{K}+1]}$ instead of deriving it from a particular π, we are possibly searching over tuples that cannot actually be implemented by a mechanism π. Particularly, we will require that the distribution over the posterior means G specified by the tuple with cumulative distribution $G(t)=\sum_{i=1}^{\tilde{K}+1} q_i\mathbb{I}\{\theta_i\le t\}$ is a mean-preserving contractions of F. The equivalence between the distributions G that are *mean-preserving contraction* of F and the distributions over posterior means that are implementable through a mechanism has been well-established [30], [21]. Formally, $G\preccurlyeq F$ is a mean-preserving contraction if and only if $\forall t\in\Theta$, $\int_0^t F(t)dt\le\int_0^t G(t)dt$.

We exploit the above reduction and solve for the optimal mechanism that solves (9). Define the increasing function $h(\theta):=\sup\{s:\int_0^s F^{-1}(t)dt\le s\theta\}$. The following theorem characterizes the optimal objective and optimal mechanism across R1, R2a, R3 and R4 (we comment on the optimal direct signalling mechanism for R2 in the appendix).

Theorem 3.1: The optimal objective V^* and a corresponding optimal mechanism can be computed as follows for each regime:

(R1): $V^*=1$ and π^* is monotone partitional with $t_0=0$ and $t_1=M$

(R2a): $V^*=1$ and π^* has $\mathcal{I}=\{1,2\}$, $g_\theta(1)=\lambda\mathbb{I}\{\theta\in[0,t]\}+\alpha\mathbb{I}\{\theta\in[t,M]\}$ and $g_\theta(1)=(1-\lambda)\mathbb{I}\{\theta\in[0,t]\}+(1-\alpha)\mathbb{I}\{\theta\in[t,M]\}$.

(R3): $V^*=q_1^*:=\min\{h(\bar{\theta}_{\tilde{K}}),\frac{M-\mu}{M-\theta_{\tilde{K}}}\}$ and π^* is monotone partitional with $t_0=0$, $t_1=F^{-1}(q_1^*)$ and $t_2=M$.

(R4): $V^*=1-q_2^*$ where $q_2^*:=\inf\{q\ge\frac{\underline{\theta}_1-\mu}{\underline{\theta}_1}:q\le h(\underline{\theta}_1-\frac{\underline{\theta}_1-\mu}{q})\}$ and π^* is monotone partitional with $t_0=0$, $t_1=F^{-1}(q_2^*)$ and $t_2=M$.

Notice that in subcase (R1), an optimal mechanism is monotone partitional with a single signal; in Figure 2, the entire probability mass over Θ maps to a single signal. In (R3) and (R4) an optimal mechanism is also monotone partitional with two signals, where a threshold is specified by the theorem and the Θ is partitioned into a "low" and "high" regime, where each regime corresponds to a signal. In (R2b), we find a threshold and then fork the signals according to an interval-specific distribution for both the intervals below and above this threshold.However, for both the left and right of this threshold, we use the same fixed set of two signals. The analysis of these cases show that simple signalling mechanisms that use at most two signals are sufficient to achieve the optimum.

IV. Scaled Capacity Objective

In this section, we consider a stateful scaled capacity objective $\{\mathcal{Y}(\theta)\}_{\theta\in\Theta}$ that is characterized by a collection of sets $\mathcal{Y}(\theta)=\{\mathbf{y}:\|\mathbf{y}\|\ge b(\theta)\}$ for some increasing function b. Sets of this form reflect a capacity limit on in-person work that gets more stringent as the infectious risk increases; see Fig. 3(a). Observe that in this setting we cannot reduce the search over π by searching over the corresponding direct mechanisms $\mathcal{T}_\pi$ since a tuple (q_i,θ_i) for a signal i conceals vital information about the exact values of θ that are used to generate this signal. Hence, we cannot evaluate the desirability of posterior means θ_i alone since we cannot express the exact sets $\mathcal{Y}(\theta)$ with which the posterior mean is sufficient to enter.

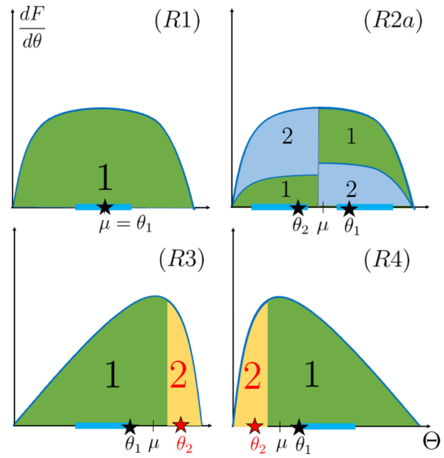


Fig. 2: The pdf for θ and the intervals $[\underline{\theta}_k,\bar{\theta}_k]$ (blue) with the solution for the four cases from Theorem 3.1 marked by the induced posterior means θ_1,θ_2 (star). The posteriors are induced by mapping the probability mass to signals as marked (green, violet, yellow).

To make analysis more tenable, for some $N>0$ we restrict $\Theta:=\{\nu_1,\nu_2,..,\nu_N\}$ with $0<\nu_1<..<\nu_N$. We denote $\mathbb{P}[\theta=\nu_i]=p_i$ for all $i\in[N]$. To simplify notation, we also denote $b(\nu_i)=b_i$ for all $i\in[N]$ where b_i is strictly increasing in i.

Using Equation (6), we can express the objective of the principal by:

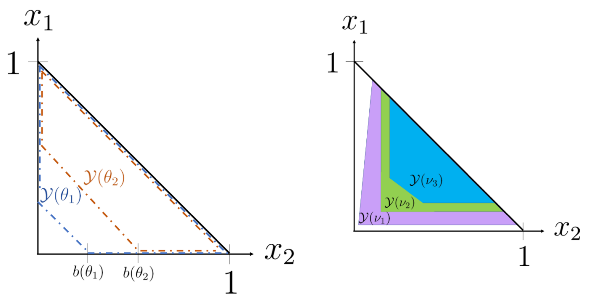
$$V^*=\max_{\langle\{g_\nu\}_{\nu\in\Theta},\mathcal{I}\rangle}\mathbb{P}\{\mathbf{y}_\pi^*(i)\in\mathcal{Y}(\nu)\}\tag{10}$$

$$=\max_{\langle\{g_\nu\}_{\nu\in\Theta},\mathcal{I}\rangle}\sum_{j=1}^{N}\sum_{i=1}^{|\mathcal{I}|}p_j g_\nu(i)\mathbb{I}\{\mathbf{y}_\pi^*(i)\in\mathcal{Y}(\nu_j)\}\tag{11}$$

As with the stateless design, $\mathbf{y}_\pi^*(i)$ only depends on the induced posterior mean θ_i and $\mathbf{y}_\pi^*(i)\in\mathcal{Y}(\nu_j)$ if and only $\mathbf{y}_\pi^*(i)\in\mathcal{Y}(\nu_j)\cap\mathcal{Z}(\mathbf{x})$. The latter follows if and only if $b_j\le\|\mathbf{y}_\pi^*(i)\|\le 1$ or $0\le m(\theta_i)\le 1-b_j$. Using Proposition 1 and Lemma 3.1, observe that this corresponds to $\theta_i\in[\gamma_j,\infty)$ for some constants γ_j where γ_j are strictly increasing in j. For completeness, we denote $\gamma_0=0$ and $\gamma_{N+1}=\infty$. We can then reformulate (11) accordingly:

$$V^*=\max_{\langle\{g_\nu\}_{\nu\in\Theta},\mathcal{I}\rangle}\sum_{j=1}^{N}\sum_{i=1}^{|\mathcal{I}|}p_j g_{\nu_j}(i)\mathbb{I}\{\gamma_j\le\theta_i<\infty\}\tag{12}$$

Observe that our analysis generalizes beyond the scaled capacity objective. Suppose that the desirable sets satisfy Assumption A1 and are also (i) nested with $\mathcal{Y}(\nu_{i+1})\subset\mathcal{Y}(\nu_i)$ for all $i\in[N-1]$, and (ii) half-bounded with $\mathbf{x}\in\mathcal{Y}(\nu_i)$ for all initial mass vectors $\mathbf{x}$ and all $i\in[N]$; see Fig. 3(b). Then observe that we arrive at an identical formulation to (12) as



(a) Goal sets for stateful objective when $K = N = 2$ $(\theta_1 < \theta_2)$.

(b) A more general nested structure of goal sets.

Fig. 3

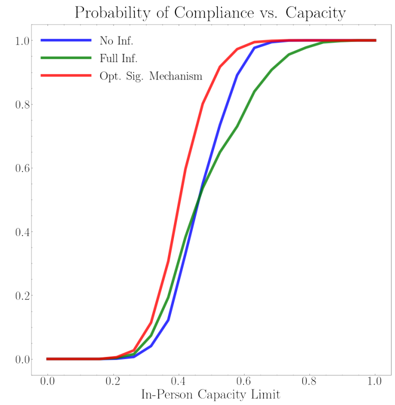


Fig. 4: Comparison between non-informative, fully-informative and optimal signalling mechanism in enforcing a 50% capacity rule. $(K = 10, \mathbf{x} \sim Unif(L_1), v_i \sim Unif[0, 10], \theta \sim Unif[5, 20])$

the minimal posterior mean beliefs for the agent outcomes to enter $\mathcal{Y}(\nu_j)$ is an increasing sequence γ_j.

Using a similar reduction in the set of signals as in stateless design, we observe that different posterior means in each interval $[\gamma_{i-1}, \gamma_i)$ for $i \in [N+1]$ induce equilibrium outcomes that lie in $\mathcal{Y}(\nu_j)$ if and only $j \geq i-1$. This implies that an optimal mechanism exists that uses at most $N+1$ signals. In the following proposition we make use of the following linear program to identify an optimal signalling mechanism for this stateful scaled capacity objective.

$$
\begin{aligned}
V^* = \max_{\{\mathbf{z}\}} & \sum_{j=1}^{N} \sum_{i=j+1}^{N+1} z_{ji} \\
\text{s.t.} & \sum_{i=1}^{N+1} z_{ji} = p_j, \qquad \forall j \in [N] \\
& 0 \leq z_{ji} \leq p_j, \qquad \forall i \in [N+1], j \in [N] \\
& \gamma_{i-1} \sum_{j=1}^{N} z_{ji} \leq \sum_{j=1}^{N} \nu_j z_{ji}, \qquad \forall i \in [N+1] \\
& \sum_{j=1}^{N} \nu_j z_{ji} \leq \gamma_i \sum_{j=1}^{N} z_{ji}, \qquad \forall i \in [N+1]
\end{aligned}
$$

Proposition 2: The solution V^* to the above linear program yields the optimal achievable objective possible under the stateful scaling capacity objective. For any solution $\{z_{ji}^*\}_{j \in [N], i \in [N+1]}$ that maximizes the program, we can generate an optimal signalling mechanism $\pi = \langle \{g_\nu\}_{\nu \in \Theta}, \mathcal{I} \rangle$ by choosing $\mathcal{I} = [N+1]$ and setting $g_{\nu_j}(i) = \frac{z_{ji}^*}{p_j}$ for all $i \in \mathcal{I}, j \in [N]$.

The above proposition says... Observe that if under a stateless objective the θ_i all converge to a single value so...

Contrast this to no information where $V(j) = 1$ if and only if $\mathbb{E}[\theta] \geq c_j$. Hence $V(j) = 1$ for all $j \leq \hat{j}_{NI} := \max\{i :\}$ and $V(j) = 0$ otherwise. and full information where $V(j) = 1$ if and only if $\theta_j \geq c_j$. Hence $V(j) = 1$ for all $j \leq \hat{j}_{FI} := \max\{i :\}$ and $V(j) = 0$ otherwise. Since signalling is optimal and the no information and full information mechanisms are encompassed in optimization problem, $\hat{j} > \hat{j}_{NI}, \hat{j}_{FI}$.

- Observe that as θ collapse, this becomes stateful problem as - do without proof
- We do not show rigorous proof, but for $N = 2$, the policy is equivalent to stateless objective using $\mathcal{Y}(\theta_2)$.

V. Numerical Experiments

In the following section we present some numerical experiments to motivate the improvement our optimally designed signalling mechanisms from Theorem 3.1 achieve relative to given benchmarks. In subsection V-A, we consider a stateless capacity model where the capacity the designer seeks to implement does not depend on θ. We vary the capacity limit to demonstrate how the relative improvement to the benchmarks changes.

In subsection V-B, we consider a scaling capacity model motivated by infectious cost and compare the performance of the optimal mechanism characterized in Proposition 2 against the same benchmarks.

A. Stateless Objective Simulations

We implement an experiment where $\mathcal{Y} = \{\mathbf{y} : \|\mathbf{y}\| \geq b\}$ where we vary the constants c between 0 and 1. The number of groups is fixed at $K = 10$ and we draw $\mathbf{x} \sim Unif(L_1), v_i \sim Unif[0, 10], \theta \sim Unif[5, 20]$. We implement the optimal mechanism as chosen by Theorem 3.1 against two benchmarks. The first benchmark is the non-informative mechanism where agents receive no added information and consequently have a posterior mean belief of μ over Θ. The second benchmark is the fully-informative mechanism, where the true value of θ is directly revealed so agents maintain posterior mean beliefs θ' where θ' is sampled from F. The resulting objective that measures the probability of compliance is averaged across simulations and plotted in Figure 4. Observe that the optimal signalling mechanism strictly improves relative to both benchmarks. This is as expected since both benchmarks correspond to signalling mechanisms that the optimal design necessarily improves. The improvement deteriorates as $b \nearrow 1$ as the

Information Scheme	V^*	$V^*\|\nu_1$	$V^*\|\nu_2$	$V^*\|\nu_3$
No Information	0.300	1	0	0
Full Information	0.000	0	0	0
Stateful Design	0.425	1	0.417	0

Fig. 5: $N = 3, \mathbf{p} = (0.3, 0.3, 0.4), \nu = (0.4, 0.6, 1), \gamma = (0.5, 0.9, 1.2)$

region of intersection between the outcomes achievable in equilibria and the outcomes the principal deems acceptable dwindles. Naturally, the mechanism ability to influence agents to a set of outcomes decreases as the set of outcomes becomes smaller. Likewise, as $b \searrow 0$, observe that the set of acceptable outcomes grows to encompass all outcomes. Hence, the mechanism approaches full compliance, but so do simpler mechanisms. Consequently, the relative value of this optimally-designed signalling mechanism deteriorates as the set of outcomes the principal is willing to accept grows.

B. Stateful Scaling Capacity

In Table **??**, we refer In Fig. **??**, we see that the optimal stateful mechanism outperforms the non-informative and fully-informative settings and . Moreover, we see that ...

VI. Concluding Remarks

In this paper, we introduced a model to study information provision for strategic hybrid workers. The central planner sought to control the mass of in-person workers across each group present in the equilibrium outcome. Our model captures two key features: (a) a general objective that simply aims to maximize the probability that particular sets of equilibria are achieved ; (b) workers making game-theoretic decisions in response to signals. We provided a complete description of the equilibria of the game in response to the signals and derived the optimal signalling mechanism that the principal can employ.

These results suggest guidelines for the design and deployment of signalling mechanisms as hard intervention measures become infeasible as pandemics progress. In future work, we hope to extend our results to more sophisticated stateful objectives beyond scaled capacity.

References

[1] T. Ji, H.-L. Chen, J. Xu, L.-N. Wu, J.-J. Li, K. Chen, and G. Qin, “Lockdown Contained the Spread of 2019 Novel Coronavirus Disease in Huangshi City, China: Early Epidemiological Findings,” *Clinical Infectious Diseases*, vol. 71, no. 6, pp. 1454–1460, Sep. 2020.

[2] C. Nowzari, V. M. Preciado, and G. J. Pappas, “Analysis and Control of Epidemics: A Survey of Spreading Processes on Complex Networks,” *IEEE Control Systems Magazine*, vol. 36, no. 1, pp. 26–46, Feb. 2016, conference Name: IEEE Control Systems Magazine.

[3] M. U. G. Kraemer, C.-H. Yang, B. Gutierrez, C.-H. Wu, B. Klein, D. M. Pigott, Open COVID-19 Data Working Group, L. du Plessis, N. R. Faria, R. Li, W. P. Hanage, J. S. Brownstein, M. Layan, A. Vespignani, H. Tian, C. Dye, O. G. Pybus, and S. V. Scarpino, “The effect of human mobility and control measures on the COVID-19 epidemic in China,” *Science (New York, N.Y.)*, vol. 368, no. 6490, pp. 493–497, May 2020.

[4] M. Gatto, E. Bertuzzo, L. Mari, S. Miccoli, L. Carraro, R. Casagrandi, and A. Rinaldo, “Spread and dynamics of the COVID-19 epidemic in Italy: Effects of emergency containment measures,” *Proceedings of the National Academy of Sciences*, vol. 117, no. 19, pp. 10 484–10 491, May 2020, publisher: Proceedings of the National Academy of Sciences.

[5] K. Drakopoulos, A. Ozdaglar, and J. N. Tsitsiklis, “An Efficient Curing Policy for Epidemics on Graphs,” *IEEE Transactions on Network Science and Engineering*, vol. 1, no. 2, pp. 67–75, Jul. 2014, conference Name: IEEE Transactions on Network Science and Engineering.

[6] V. Chernozhukov, H. Kasaha, and P. Schrimpf, “Causal Impact of Masks, Policies, Behavior on Early Covid-19 Pandemic in the U.S,” *Journal of Econometrics*, vol. 220, no. 1, pp. 23–62, Jan. 2021, arXiv: 2005.14168.

[7] D. Acemoglu, V. Chernozhukov, I. Werning, and M. D. Whinston, “Optimal Targeted Lockdowns in a Multi-Group SIR Model,” National Bureau of Economic Research, Working Paper 27102, May 2020, series: Working Paper Series.

[8] D. Acemoglu, A. Fallah, A. Giometto, D. Huttenlocher, A. Ozdaglar, F. Parise, and S. Pattathil, “Optimal adaptive testing for epidemic control: combining molecular and serology tests,” *arXiv:2101.00773 [physics, q-bio]*, Jan. 2021, arXiv: 2101.00773.

[9] J. R. Birge, O. Candogan, and Y. Feng, “Controlling Epidemic Spread: Reducing Economic Losses with Targeted Closures,” Social Science Research Network, Rochester, NY, SSRN Scholarly Paper ID 3590621, May 2020.

[10] R. M. Anderson, H. Heesterbeek, D. Klinkenberg, and T. D. Hollingsworth, “How will country-based mitigation measures influence the course of the COVID-19 epidemic?” *The Lancet*, vol. 395, no. 10228, pp. 931–934, Mar. 2020, publisher: Elsevier.

[11] F. de Véricourt, H. Gurkan, and S. Wang, “Informing the Public About a Pandemic,” *Management Science*, vol. 67, no. 10, pp. 6350–6357, Oct. 2021, publisher: INFORMS.

[12] J. Ely, A. Galeotti, and J. Steiner, “Rotation as Contagion Mitigation,” *Management Science*, vol. 67, no. 5, pp. 3117–3126, May 2021, publisher: INFORMS.

[13] A. Hernandez-Chanto, C. Oyarzun, and J. Hedlund, “Contagion Management through Information Disclosure,” Social Science Research Network, Rochester, NY, SSRN Scholarly Paper ID 3988157, Dec. 2021.

[14] P. Dworczak and G. Martini, “The simple economics of optimal persuasion,” *Journal of Political Economy*, vol. 127, no. 5, pp. 1993–2048, 2019, publisher: The University of Chicago Press Chicago, IL.

[15] A. Kolotilin, “Optimal information disclosure: A linear programming approach,” *Theoretical Economics*, vol. 13, no. 2, pp. 607–635, 2018, _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.3982/TE1805.

[16] Y. Guo and E. Shmaya, “The Interval Structure of Optimal Disclosure,” *Econometrica*, vol. 87, no. 2, pp. 653–675, 2019, _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.3982/ECTA15668.

[17] E. Kamenica and M. Gentzkow, “Bayesian Persuasion,” *American Economic Review*, vol. 101, no. 6, pp. 2590–2615, Oct. 2011.

[18] O. Candogan, “Information Design in Operations,” in *Pushing the Boundaries: Frontiers in Impactful OR/OM Research*, ser. INFORMS TutORials in Operations Research. INFORMS, Nov. 2020, pp. 176–201, section: 8.

[19] D. Bergemann and S. Morris, “Information Design: A Unified Perspective,” *Journal of Economic Literature*, vol. 57, no. 1, pp. 44–95, Mar. 2019.

[20] E. Kamenica, “Bayesian Persuasion and Information Design,” *Annual Review of Economics*, vol. 11, no. 1, pp. 249–272, 2019, _eprint: https://doi.org/10.1146/annurev-economics-080218-025739.

[21] M. Gentzkow and E. Kamenica, “A Rothschild-Stiglitz Approach to Bayesian Persuasion,” *American Economic Review*, vol. 106, no. 5, pp. 597–601, May 2016.

[22] O. Candogan and P. Strack, “Optimal Disclosure of Information to Privately Informed Agents,” Social Science Research Network, Rochester, NY, SSRN Scholarly Paper ID 3773326, Jan. 2021.

[23] S. Y. Olmez, S. Aggarwal, J. W. Kim, E. Miehling, T. Başar, M. West, and P. G. Mehta, “Modeling Presymptomatic Spread in Epidemics via Mean-Field Games,” Nov. 2021.

[24] A. Aurell, R. Carmona, G. Dayanıklı, and M. Laurière, “Finite State Graphon Games with Applications to Epidemics,” *Dynamic Games and Applications*, vol. 12, no. 1, pp. 49–81, Mar. 2022.

[25] D. La Torre, D. Liuzzi, R. Maggistro, and S. Marsiglio, "Mobility Choices and Strategic Interactions in a Two-Group Macroeconomic–Epidemiological Model," *Dynamic Games and Applications*, vol. 12, no. 1, pp. 110–132, Mar. 2022.

[26] L. Cianfanelli, F. Parise, D. Acemoglu, G. Como, and A. Ozdaglar, "Lockdown interventions in SIR model: Is the reproduction number the right control variable?" *arXiv:2112.06546 [physics]*, Dec. 2021, arXiv: 2112.06546.

[27] M. Wu, D. Shelar, R. Gopalakrishnan, and S. Amin, "Optimal Testing Strategy for Containing COVID-19: A Case-Study on Indian Migrant Worker Population," Social Science Research Network, Rochester, NY, SSRN Scholarly Paper ID 3703429, Oct. 2020.

[28] G. Perakis, D. Singhvi, O. Skali Lami, and L. Thayaparan, "COVID-19: A Multipeak SIR Based Model for Learning Waves and Optimizing Testing," Feb. 2021.

[29] M. Ivanov, "Optimal Signals in Bayesian Persuasion Mechanisms," 2015.

[30] D. Blackwell and M. Girshick, "Theory of Games and Statistical Decisions." New York: John Willey and Sons, 1954.

[31] A. R. Hota and K. Gupta, "A Generalized SIS Epidemic Model on Temporal Networks with Asymptomatic Carriers and Comments on Decay Ratio," in *2021 American Control Conference (ACC)*. New Orleans, LA, USA: IEEE, May 2021, pp. 3176–3181.

[32] L. J. S. Allen, "An Introduction to Stochastic Epidemic Models," in *Mathematical Epidemiology*, ser. Lecture Notes in Mathematics, F. Brauer, P. van den Driessche, and J. Wu, Eds. Berlin, Heidelberg: Springer, 2008, pp. 81–130.

[33] O. Candogan, "Optimality of Double Intervals in Persuasion: A Convex Programming Framework," Social Science Research Network, Rochester, NY, SSRN Scholarly Paper ID 3452145, Sep. 2019.

[34] "Univariate Stochastic Orders," in *Stochastic Orders*, ser. Springer Series in Statistics, M. Shaked and J. G. Shanthikumar, Eds. New York, NY: Springer, 2007, pp. 3–79.

[35] M. Ivanov, "Optimal monotone signals in Bayesian persuasion mechanisms," *Economic Theory*, vol. 72, no. 3, pp. 955–1000, Oct. 2021.

VII. Appendix

A. A-SIYS Dynamics

To briefly motivate the form of the infectious cost presented in Equation (1), consider the A-SIYS activity-driven epidemic model introduced in [31], [32] on a complete graph with M nodes (agents) over two periods. Each agent begins at $t = 0$ in one of the three possible states: susceptible (S), asymptomatic (X) and symptomatic (Y). Under this model both the asymptomatic and symptomatic agents are infectious. We can denote the infectious state of each agent i at time t by $\chi_i(t) \in \{S, X, Y\}$. Assuming the symptomatic individuals are required to self-isolate, the remaining $N < M$ agents with $\chi_i(0) \neq Y$ are subject to the decision-making process we consider. However, the remaining agents cannot discern their existing state and conditioned on $\chi_i(0) \neq Y$, independently each agent i has $\chi_i(0) = S$ with probability p. Letting the action of agent i be $a_i \in \{\ell_O, \ell_R\}$, the number of agents working in person is $m = \sum_{i=1}^{N} \mathbb{I}\{a_i = \ell_O\}$ and the number of asymptomatic agents working in-person is $m_I = \sum_{i=1}^{N} \mathbb{I}\{a_i = \ell_O\}\mathbb{I}\{\chi_i(0) = X\}$. In accordance with the A-SIYS dynamics, for this unknown parameter θ, any initially susceptible agent i with $\chi_i(0) = S$ and $a_i = \ell_O$ transitions to being infected $\chi_i(1) \in \{X, Y\}$ with probability θm_I for small θ. Only susceptible agents will pay an incremental infectious cost as the remaining agents were already infected. Specifically agents i incur a cost γ if and only if $\chi_i(0) = S$ and $\chi_i(1) \in \{X, Y\}$. Hence, if $a_i = \ell_R$, agent i has no contact with other individuals so his infectious cost on expectation is 0. If $a_i = \ell_O$, then agent i pays on expectation $\beta(\theta, m)$:

$$\begin{aligned}
\beta(\theta, m) &= \mathbb{E}[\gamma \mathbb{I}\{\chi_i(0) = S, \chi_i(1) \in \{X, Y\}\}] \\
&= \gamma \mathbb{P}[\chi_i(0) = S]\mathbb{P}[\chi_i(1) \in \{X, Y\}\}|\chi_i(0) = S] \\
&= \gamma(1 - p)\theta \mathbb{E}[m_I] \\
&= \gamma p(1 - p)\theta m
\end{aligned}$$

The above computation suggests the expected infectious cost has a linear dependence on θ and the mass of agents working in-person m (this is $1 - \|\mathbf{y}\|$ in our non-atomic setting). Observe this is consistent with a functional form of infectious costs we consider if $c_1(u) = 1 - u$ and $c_2(u) = 0$. As the network structure underlying the infection dynamics becomes specialized or other diseases become intermingled, the associated c_i may be better estimated through some other functions. However, we consider $\beta(\theta, \mathbf{y}) = \theta(1 - \|\mathbf{y}\|)$ as a running example for numerical experiments.

B. Proof of Proposition 1

By construction, the groups of agents are implicitly ordered by their preference for working in-person. The following lemma shows that *under any equilibrium* $\mathbf{y}^*_\pi(i)$ as defined in Definition 1, there is a critical group $k^*(i) \in [K]$ such that all agents of groups l with smaller benefits $v_l < v_{k^*(i)}$ work remotely, and all agents of groups l with larger benefits $v_l > v_{k^*(i)}$ work in-person.

Lemma 7.1: For any equilibrium $\mathbf{y}^*_\pi(i)$, there exists a critical group $k^*(i) \in [K]$ such that for all $l < k^*(i)$, $y^*_l(i) = 0$ and all $l > k^*(i)$ $y^*_l(i) = x_l$.

Proof: Our proof proceeds by construction. Suppose that $\mathbf{y}^*_\pi(i)$ is an equilibrium and let $k^*(i) = \min\{i : y^*_i(i) > 0\}$. Suppose, by contradiction, that there exists $l > k^*(i)$ such that $y^*_l(i) < x_l$. Then, $v_l \geq c_1(\|\mathbf{y}\|)\theta_i + c_2(\|\mathbf{y}\|)$. However, since $v_{k^*(i)} > v_l$, this implies that $v_{k^*(i)} > c_1(\|\mathbf{y}\|)\theta_i + c_2(\|\mathbf{y}\|)$ and by Definition 1, $y_{k^*(i)}(i) = 0$. This is a contradiction, so it is proven that for all $l > k^*(i)$, $y^*_l(i) = x_l$ and hence $k^*(i)$ satisfies the conditions of the critical group. ■

The next lemma precisely computes the in-person equilibrium mass in response to signal i. Given the restriction on $\mathbf{y}^*_\pi(i)$ established in the previous lemma, this precisely identifies the equilibrium mass vector $\mathbf{y}^*_\pi(i)$ that the agents will achieve as in the proposition statement.

Lemma 7.2: In any equilibrium $\mathbf{y}^*_\pi(i)$, the equilibrium in-person mass $m(\theta_i) \coloneqq 1 - \|\mathbf{y}^*_\pi(i)\| = \sup\{u : v(u) \geq c_1(1 - u)\theta_i + c_2(1 - u)\}$.

Proof: Suppose $m(\theta_i) = z$ with $s_{j-1} < z < s_j$ for some j, then by Definition 1, $v_j = c_1(1 - z)\theta_i + c_2(1 - z)$. Notice, $v_j = v(z) = c_1(1 - z)\theta_i + c_2(1 - z)$ and that $v(u)$ is non-increasing and $c_1(1 - u)\theta_i + c_2(1 - u)$ is strictly increasing in u, so $z = \sup\{u : v(u) \geq c_1(1 - u)\theta_i + c_2(1 - u)\}$.

Suppose $m(\theta_i) = z$ with $z = s_j$ for some j. Then $v(z) = v_{j+1}$ and for all $\epsilon < x_j$, $v(z - \epsilon) = v_j$. But by Definition 1, $v(z) = v_{j+1} \leq c_1(1 - z)\theta_i + c_2(1 - z)$ and for all $\epsilon < x_j$, $v(z - \epsilon) = v_j \geq c_1(1 - z + \epsilon)\theta_i + c_2(1 - z + \epsilon)$. But again since $c_1(1-u)\theta_i + c_2(1-u)$ is strictly increasing in u, this implies that $z = \sup\{u : v(u) \geq c_1(1-u)\theta_i + c_2(1-u)\}$. ■

C. Proof of Lemma 3.1

Observe that $m(\theta) \leq 1$ since for all $u > 1$, $v(u) = 0$ and $c_1(1-u)\theta + c_2(1-u) > 0$. Similarly, since $v(0) \geq 0 = c_1(1)\theta + c_2(1)$, $m(\theta) \geq 0$ so clearly $m(\theta)$ is bounded.
Similarly, letting $f(u) = \frac{v(u)-c_2(1-u)}{c_1(1-u)}$, observe $m(\theta) = \sup\{u : f(u) \geq \theta\}$. Since $c_1(1-u)$ is a strictly increasing function in u and $v(u) - c_2(1-u)$ is a non-increasing function, $f(\cdot)$ is monotonically decreasing. For any $\theta' \leq \theta''$ notice $\{u : f(u) \geq \theta''\} \subseteq \{u : f(u) \geq \theta'\}$, so $m(\theta'') \leq m(\theta')$ and hence m is non-increasing. Moreover, applying Berge's Maximum Principle, we can determine that $m(\theta)$ is continuous.■

D. Proof of Lemma 3.2

The proof is quite immediate since notice that $\mathcal{Z}(\mathbf{x}) = \cup_{i=1}^{K}\{z(\mathbf{x}, u) : s_{k-1} \leq u \leq s_k\}$. Consequently, $\mathcal{Y} \cap \mathcal{Z}(\mathbf{x}) = \cup_{i=1}^{K}\big(\mathcal{Y} \cap \{z(\mathbf{x}, u) : s_{k-1} \leq u \leq s_k\}\big)$. However, notice $\{z(\mathbf{x}, u) : s_{k-1} \leq u \leq s_k\}$ is a line segment in $\mathbb{R}^K$ so its intersection with a convex, compact set $\mathcal{Y} \subset \mathbb{R}^K$ is just another line segment $\{z(\mathbf{x}, u) : \tilde{\omega}_k^1 \leq u \leq \tilde{\omega}_k^2\}$ for some constants $\tilde{\omega}_k^1, \tilde{\omega}_k^2$. Eliminating empty intervals and condensing the intervals if $\tilde{\omega}_j^2 = \tilde{\omega}_{j+1}^1$ for any j yields the result.

E. Proof of Lemma 3.3

The proof follows directly from Lemma 1 in [33].

F. Proof of Theorem 3.1

(R1) The proof for distributions F in regime 1 follows from construction as π induces $\mathcal{T}_\pi = \{(1, \mu)\}$ and from Equation (9) this implies that $V^* = 1$ which is maximal.

(R2a) The proof for distributions F in regime 2a follows from construction as π induces $\mathcal{T}_\pi = \{(\lambda F(t) + \alpha(1 - F(t)), \frac{\underline{s}(t)\lambda F(t) + \bar{s}(t)\alpha(1-F(t))}{\lambda F(t) + \alpha(1-F(t))}), ((1-\lambda)F(t) + (1-\alpha)(1-F(t)), \frac{\underline{s}(t)(1-\lambda)F(t) + \bar{s}(t)(1-\alpha)(1-F(t))}{(1-\lambda)F(t) + (1-\alpha)(1-F(t))})\}$ and from Equation (9) this implies that $V^* = 1$ which is maximal.

(R3-4)
The proofs for R3 and R4 are symmetric (can modify all objects in Θ by going from θ to $M - \theta$ and the proof is analogous) so we only prove for R3. Moreover, without loss of generality, assume $M = 1$ as an optimal signalling mechanism is invariant to linear scaling.
Observe that we aim to solve:

$$V^* = \max_{\pi : \langle \{g_\theta\}_{\theta \in \Theta}, \mathcal{I} \rangle} \sum_{k=1}^{\tilde{K}} q_i \mathbb{I}\{\underline{\theta}_k \leq \theta_i \leq \bar{\theta}_k\}$$

However, using the established properties of signalling mechanisms, we can search over the distributions of the posterior means G by searching over the space of all mean-preserving contractions of F, $G \succcurlyeq F$. Recall that $G \succcurlyeq F$ if and only if $\int_0^x (1 - G(t))dt \geq \int_0^x (1 - F(t))dt$ for all $x \in [0, 1]$ with equality at $x = 1$. In the quantile space, this is equivalent to $\int_0^x G^{-1}(t)dt \geq \int_0^x F^{-1}(t)dt$ for all $x \in [0, 1]$ with equality at $x = 1$. We will refer to this constraint as the MPC constraint.
Moreover, using Lemma 3.3 that we can further restrict G to be a discrete distribution using $\tilde{K} + 1$ mass points (each corresponding to a posterior mean) with $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$ with probability q_i and $\theta_{\tilde{K}+1}$ with probability $q_{\tilde{K}+1}$. Since the posterior mean θ_i is induced with probability q_i $\mu > \bar{\theta}_{\tilde{K}}$ and $\sum_{i=1}^{\tilde{K}+1} q_i\theta_i = \mu$. However, since $\theta_i < \bar{\theta}_{\tilde{K}} < \mu$ for all $i \in [\tilde{K}]$, this implies that $\theta_{\tilde{K}+1} > \mu$ and $q_{\tilde{K}+1} > 0$. Hence, we can re-express the objective as follows:

$$\begin{aligned} V^* &= \max_{G \succcurlyeq F; \forall i \in [\tilde{K}] \mathbb{P}_{\theta \sim G}[\theta = \theta_i] = q_i > 0 \text{ for unique } \theta_i \in [\underline{\theta}_i, \bar{\theta}_i]} \sum_{k=1}^{\tilde{K}} q_i \\ &= \min_{G \succcurlyeq F; \forall i \in [\tilde{K}] \mathbb{P}_{\theta \sim G}[\theta = \theta_i] = q_i > 0 \text{ for unique } \theta_i \in [\underline{\theta}_i, \bar{\theta}_i]} q_{\tilde{K}+1} \end{aligned}$$

Observe that G is parameterized by $(q_i, \theta_i)_{i \in [\tilde{K}+1]}$, so we can equivalently express the condition that $G \succcurlyeq F$ as in Lemma 4 of [22] by adding constraints that enforce the mean-preserving contraction results:

$$\min q_{\tilde{K}+1} \tag{13}$$
$$\text{s.t.} \quad \sum_{i=1}^{\tilde{K}+1} q_i = 1 \tag{14}$$
$$\sum_{i=1}^{\tilde{K}+1} q_i\theta_i = \mu \tag{15}$$
$$\underline{\theta}_l \leq \theta_l \leq \bar{\theta}_l \qquad \forall l \in [\tilde{K}] \tag{16}$$
$$\mu \leq \theta_{\tilde{K}+1} \leq 1 \tag{17}$$
$$\sum_{j=1}^{m} q_j\theta_j \geq \int_0^{\sum_{j=1}^m q_j} F^{-1}(t)dt \quad \forall m \in [\tilde{K}] \tag{18}$$

We can reduce the enforcement of the MPC constraints in Equation (18) from $\int_0^s F^{-1}(t)dt \leq \int_0^s G^{-1}(t)dt$ for all $x \in [0, 1]$ to merely checking for $x \in \mathcal{S} := \{\sum_{j=1}^m q_j : m \in [\tilde{K}+1]\}$. This is done without loss since $\int_0^s F^{-1}(t)dt$ is convex and $\int_0^s G^{-1}(t)dt$ is piecewise-linear with breakpoints at these $x \in \mathcal{S}$ so the remaining constraints are implied.
Observe that we can sum in order of index in Equation (18) since θ_i is increasing for all $i \in [\tilde{K} + 1]$. Notice that by using all $\tilde{K} + 1$ possible placements of $\theta_{\tilde{K}+1}$ relative to $\{\theta_i\}_{i \in [\tilde{K}]}$, we can (R2) in general, this program will solve for the optimal direct signalling mechanism by taking the resultant $\{(q_i^*, \theta_i^*)\}_{i \in [\tilde{K}+1]}$.
Recall that $q_{\tilde{K}+1}^* > 0$ and next suppose that $q_j^* > 0$ for some $j < \tilde{K}$. This is a contradiction; consider decreasing q_j^* by $(\frac{\theta_{\tilde{K}+1} - \theta_{\tilde{K}}}{\theta_{\tilde{K}+1} - \theta_j})\epsilon$, increasing $q_{\tilde{K}}^*$ by ϵ, decreasing $q_{\tilde{K}+1}^*$ by $(\frac{\theta_{\tilde{K}} - \theta_j}{\theta_{\tilde{K}+1} - \theta_j})\epsilon$. Then, observe that the objective is strictly improved and all the non-MPC constraints are still obeyed by design. Moreover, notice using the alternative definition of mean-preserving contraction in Theorem 3.A.7. of [34], this

convex stochastic modification creates a mean-preserving contraction of the G^* that implements $\{(q_i^*, \theta_i^*)\}$ so it must be a mean-preserving contraction of F and satisfy the MPC constraints in Equation (18).

Renaming $q_{\tilde{K}}$ to q_1 and $q_{\tilde{K}+1}$ to q_2, we can reduce the optimization to:

$$\max q_1 \tag{19}$$
$$\text{s.t. } q_1\theta_1 + q_2\theta_2 = \mu \tag{20}$$
$$q_1 + q_2 = 1 \tag{21}$$
$$q_1\theta_1 \geq \int_0^{q_1} F^{-1}(t)dt \tag{22}$$
$$\underline{\theta}_{\tilde{K}} \leq \theta_1 \leq \bar{\theta}_{\tilde{K}} \tag{23}$$
$$\mu \leq \theta_2 \leq 1 \tag{24}$$

Suppose we find a solution $q_1^*, q_2^*, \theta_1^*, \theta_2^*$ where $\theta_1^* < \bar{\theta}_{\tilde{K}}$. Observe that we can find another solution with the same objective by choosing $q_1' = q_1^*, q_2' = q_2^*$, $\theta_1' = \bar{\theta}_{\tilde{K}}$ and $\theta_2' = \mu + \frac{(\mu-\bar{\theta}_{\tilde{K}})(\theta_2^*-\mu)}{(\mu-\theta_1^*)}$. Observe that this solution obeys all the previous constraints. Consequently, we can restrict $\theta_1 = \bar{\theta}_{\tilde{K}}$ without loss.

Observing that $f(x) = \int_0^x F^{-1}(t)dt$ is convex with $f(0) = 0$ implies that $\theta x \geq f(x)$ for all $x \in [0, h(\theta)]$. Replacing $\theta_{\tilde{K}}$ with $\bar{\theta}_{\tilde{K}}$, q_2 with $1 - q_1$ and then the constraint in (22) with $0 \leq q_1 \leq h(\bar{\theta}_{\tilde{K}})$ results in the following optimization:

$$\max_{q_1,\theta_2} q_1$$
$$\text{s.t. } q_1\bar{\theta}_{\tilde{K}} + (1-q_1)\theta_2 = \mu$$
$$0 \leq q_1 \leq h(\bar{\theta}_{\tilde{K}})$$
$$\mu \leq \theta_2 \leq M$$

We then eliminate θ_2 by using the first constraint as $\theta_2 = \frac{\mu - q_1\bar{\theta}_{\tilde{K}}}{1-q_1}$. Since $\mu \geq \bar{\theta}_{\tilde{K}}$, $\mu \leq \theta_2 \leq M$ simplifies to $q_1 \leq \frac{M-\mu}{M-\bar{\theta}_{\tilde{K}}}$ and we admit the solution that $q_1^* = \max\{h(\bar{\theta}_{\tilde{K}}), \frac{M-\mu}{M-\bar{\theta}_{\tilde{K}}}\}$.

Finally, given the objective value V^* returned, we solve the dual problem of finding the mechanism $\langle\{g_\theta\}_{\theta\in\Theta}, \mathcal{I}\rangle$ that results in V^*. To solve this, we appeal to the results in Remark 1 and Proposition 1 in [21] as well as Theorem 3.A.4 in [34]. Consider the discrete distribution G that places probability q_1^* on $\bar{\theta}_{\tilde{K}}$ and $1 - q_1^*$ on θ_2^*. Then, the corresponding function $g(x) = \int_0^x G(t)dt$ defined by:

$$g(x) = \begin{cases} 0 & x \leq \bar{\theta}_{\tilde{K}} \\ q_1^*(x - \bar{\theta}_{\tilde{K}}) & \bar{\theta}_{\tilde{K}} < x \leq \theta_2^* \\ q_1^*(\theta_2^* - \bar{\theta}_{\tilde{K}}) + (x - \theta_2^*) & x > \theta_2^* \end{cases} \tag{25}$$

It's known that g is convex and that, for $f(x) = \int_0^x F(t)dt$, $g(x) \leq f(x)$ for all $x \in [0, 1]$. Observe that $g'(0) = f'(0)$ and $g'(1) = f'(1)$. From [34] and [35], it is known that if there exists $s \in [\bar{\theta}_{\tilde{K}}, \theta_2^*]$ such that $f(s) = g(s)$, then g is tangent to f at s and therefore is tangent on each linear segment of g. Using the previous results, the mechanism can then be implemented using a monotone partitional signalling mechanism with $t_0 = 0, t_1 = s, t_2 = 1$ and consequently the $s = F^{-1}(q_1^*)$ since the probability signal 1 is induced is equivalently represented by q_1^* and $\mathbb{P}[\theta \in [0, s]] = F(s)$. Suppose by contradiction that for all $s \in [\bar{\theta}_{\tilde{K}}, \theta_2^*]$, $f(s) > g(s)$. Then since $f - g$ is convex over $[\bar{\theta}_{\tilde{K}}, \theta_2^*]$, let $\inf_{t\in[\bar{\theta}_{\tilde{K}},\theta_2^*]} f(t) - g(t) = \epsilon > 0$ with some minimizer $t^* \in [\bar{\theta}_{\tilde{K}}, \theta_2^*]$ such that $f(t^*) - g(t^*) = \epsilon$. Moreover, let $\tilde{\theta}_2^*$ solve $(q_1^* + \epsilon)(x - \bar{\theta}_{\tilde{K}}) = q_1^*(\theta_2^* - \bar{\theta}_{\tilde{K}}) + (x - \theta_2^*)$. Then consider the function $\tilde{g}$ as follows:

$$\tilde{g}(x) = \begin{cases} 0 & x \leq \bar{\theta}_{\tilde{K}} \\ (q_1^* + \epsilon)(x - \bar{\theta}_{\tilde{K}}) & \bar{\theta}_{\tilde{K}} < x \leq \tilde{\theta}_2^* \\ (q_1^* + \epsilon)(\tilde{\theta}_2^* - \bar{\theta}_{\tilde{K}}) + (x - \tilde{\theta}_2^*) & x > \tilde{\theta}_2^* \end{cases} \tag{26}$$

Observe that $\tilde{g}$ is still convex and that $g \leq \tilde{g} \leq f$, so by [21] the distribution over posterior means with $q_1^* + \epsilon$ on $\bar{\theta}_{\tilde{K}}$ and $1 - q_1^* - \epsilon$ on $\tilde{\theta}_2^*$ is implementable through a signalling mechanism. However, this violates the optimality of q_1^* and is a contradiction. Hence, there must exist an $s \in [\bar{\theta}_{\tilde{K}}, \theta_2^*]$ such that $f(s) = g(s)$.

G. Proof of Proposition 2

Observe that we can expand the optimization in (12) as follows:

$$V^* = \max_{\langle\{g_\nu\}_{\nu\in\Theta},\mathcal{I}\rangle} \sum_{j=1}^{N} \sum_{i=1}^{|\mathcal{I}|} p_j g_{\nu_j}(i)\mathbb{I}\{c_j \leq \theta_i \leq \infty\} \tag{27}$$
$$\text{s.t. } \sum_{i=1}^{|\mathcal{I}|} g_{\nu_j}(i) = 1, \forall j \in [N] \tag{28}$$
$$0 \leq g_{\nu_j}(i) \leq 1, \forall i \in |\mathcal{I}|, j \in [N] \tag{29}$$
$$\theta_i = \frac{\sum_{j=1}^N \nu_j p_j g_{\nu_j}(i)}{\sum_{j=1}^N p_j g_{\nu_j}(i)}, \quad \forall i \in |\mathcal{I}| \tag{30}$$

Observe analogous to Lemma 3.3, that if two signals i_1, i_2 are such that there exists $\gamma_j \leq \theta_{i_1}, \theta_{i_2} < \gamma_{j+1}$, we can merge the signals without loss. Therefore, for each interval $[\gamma_{i-1}, \gamma_i)$ with $i \in [N+1]$, at most one posterior mean need be present so $\mathcal{I} = [N + 1]$ with $\theta_i \in [\gamma_{i-1}, \gamma_i)$ for all i. Reducing by observing that $\theta_i \geq \gamma_j$ for all $i > j$:

$$V^* = \max_{\{g_{\nu_j}(i)\}} \sum_{j=1}^{N} \sum_{i=j+1}^{N+1} p_j g_{\nu_j}(i) \tag{31}$$
$$\text{s.t. } \sum_{i=1}^{N+1} g_{\nu_j}(i) = 1, \quad \forall j \tag{32}$$
$$0 \leq g_{\nu_j}(i) \leq 1, \quad \forall i, j \tag{33}$$
$$\gamma_{i-1} \leq \frac{\sum_{j=1}^N \nu_j p_j g_{\nu_j}(i)}{\sum_{j=1}^N p_j g_{\nu_j}(i)} \leq \gamma_i, \quad \forall i \in [N+1] \tag{34}$$

By renaming $z_{ji} = p_j g_{\nu_j}(i)$, we arrive at the linear program.