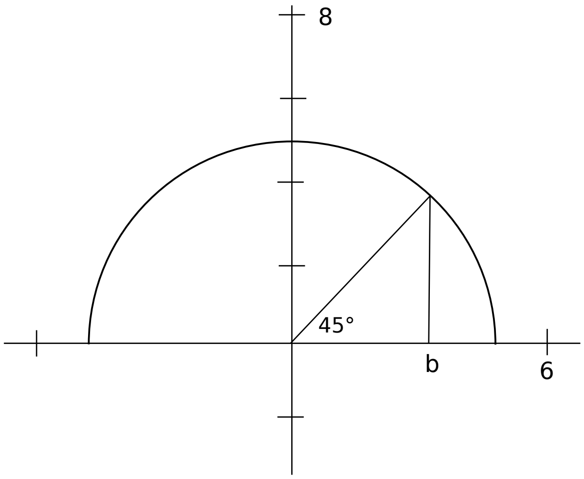
1 Warm-up

Consider the graph of $f(x) = \sqrt{25 - x^2}$ shown below:



1. What is the value of b?

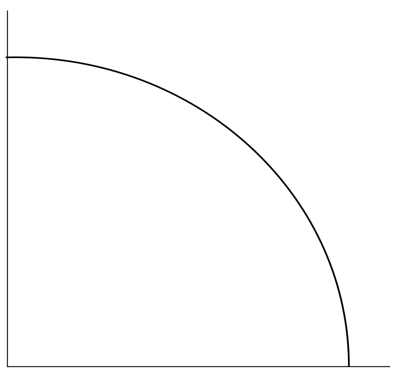
2. Use geometry to find the exact area of the region under $f(x)$ between $x = 0$ and $x = b$.

2 Riemann Sums

Our goal is to approximate this area using vertical rectangles with equal width. Suppose we want to use 5 rectangles.

1. What is the width of each rectangle?

2. If we label the x-coordinates of edges of the rectangles by $x_0, x_1, x_2, x_3, x_4, x_5$, then what would their values be? Mark them on the following figure.



$x_0 =$

$x_1 =$

$x_2 =$

$x_3 =$

$x_4 =$

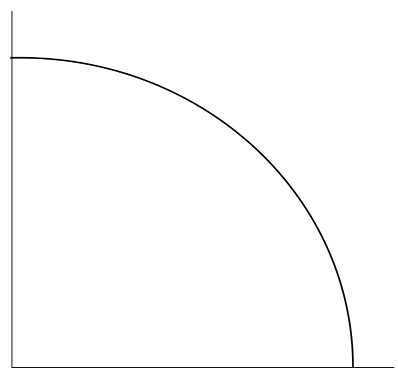
$x_5 =$

3. How do we decide the height of each rectangle?

4. For each rectangle, choose it's height to be where it's left edge intersects with $f(x)$. Draw these rectangles into the figure above. This is called the **Left Hand Sum** with five rectangles, denoted LHS(5). Why does this name make sense?

5. To approximate the area under $f(x)$ from $x = 0$ to $x = b$, add up the area of these 5 rectangles.

6. Now suppose we choose each rectangle's height to be where it's right edge intersects with $f(x)$. Draw these rectangles into the figure below. This is called the **Right Hand Sum** with five rectangles, denoted RHS(5). Why does this name make sense?



7. To approximate the area under $f(x)$ from $x = 0$ to $x = b$, add up the area of these 5 rectangles.

3 Sigma Notation

Sigma notation is used to concisely write a sum:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Here

- i is the indexing variable or index and a_i is the i-th term
- $i = 1$ indicates that we start with that value
- n indicates the last value of i

Examples

1. $\sum_{i=1}^{5} i =$

2. $\sum_{k=1}^{4} k^3 =$

3. $\sum_{m=1}^{4} 3^m =$

4. $\sum_{n=6}^{8} n =$

5. $\sum_{r=1}^{4} (-1)^r =$

This notation is especially useful when writing a long sum, such as

$$\sum_{i=1}^{100} i = 1 + 2 + \cdots + 100,$$

or for sums of variable length, such as

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2.$$

3.1 Properties of Σ-Notation

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

Examples Show that the properties above hold for each of the following sums.

1. $\sum_{i=1}^{n} 2i =$

2. $\sum_{i=1}^{n} (i + i^2) =$

There is more than one way to represent a sum in Σ-notation. For example, the following are all ways to write $1 + 2 + \cdots + 10$ in Σ-notation.

$$\sum_{i=1}^{10} i \qquad \sum_{i=0}^{9} (i+1) \qquad \sum_{2}^{11} (i-1)$$

Convince yourself that this is the case. Give two more ways.

Examples These are a few very important examples. Be very careful when writing out what they mean. (Don't compute their values.)

1. $\sum_{i=1}^{10} i =$
2. $\sum_{k=1}^{10} k =$
3. $\sum_{i=1}^{10} k =$
4. What did you notice about these problems? What makes them important?

Examples Write each of the following sums in Σ-notation.

1. $2 + 4 + 6 + 8 + 10 + \cdots + 100 =$
2. $-3 - 2 - 1 - 0 + 1 + 2 + 3 + 4 + 5 =$
3. $f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) =$
4. $f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x =$
5. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1000} =$
6. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} =$

4 Back to Riemann Sums

Suppose you want to approximate the area under the curve $f(x)$ from $x = a$ to $x = b$ using n rectangles with equal width. The width of the rectangles is then $\Delta x = \frac{b-a}{n}$. Let x_k represent the x-values at the right edge of the k-th rectangle. Then $x_k = a + k\Delta x$. Convince yourself of this last fact.

Then

$$LHS(n) = (f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}))\Delta x = \sum_{k=0}^{n-1} f(x_k)\Delta x$$

$$RHS(n) = (f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n))\Delta x = \sum_{k=1}^{n} f(x_k)\Delta x$$