Sequences, Series, and Probability Project

Due Date: If your lab takes place on Thursday, this is due in class on Thursday, October 13. If your lab takes place on Friday, this is due in class on Friday, October 14.

Instructions:

- You are to work on this project with the members of your lab group and no one else. Do not ask for help from other groups, or in the help room. This should be the work of only your group.
- You may ask your instructor (or any 122L instructor) questions. An easy way to do so is by asking a private question on Piazza (you can write a private note by selecting who you post to at the top).
- Only one project is to be handed in per group (and signed by all contributing members).
- All answers must be written in complete sentences. You will be graded on completeness as well as clarity.
- Make sure to show all mathematical justification in an organized way.
- Any work done in Maple should be attached.

1. Suppose $-1 < r < 1$.
   (a) In lab, we derived a formula for $\displaystyle\sum_{k=0}^{n-1} r^k$. Use this to find a formula for $\displaystyle\sum_{k=a}^{n} r^k$ where a is a whole number greater than 0. The use this to find a formula for $\displaystyle\sum_{k=a}^{\infty} r^k$. Simplify your answer as much as possible.
   (b) Suppose you approximate an infinite series $\displaystyle\sum_{k=1}^{\infty} r^k$ by finding its nth partial sum $\displaystyle\sum_{k=1}^{n} r^k$. Find the error in this approximation. Simplify your answer as much as possible.
2. (a) State the definition of convergence for a series $\displaystyle\sum_{k=1}^{\infty} a_k$.
   (b) Use the **definition** of convergence to prove that $\displaystyle\sum_{k=1}^{\infty} 2\left(\frac{1}{4}\right)^k$ converges, and find the value to which it converges.
3. Consider the repeating decimal $\overline{.999}$.
   (a) Write $\overline{.999}$ as a geometric series.
   (b) Prove that $\overline{.999} = 1$.
   (c) Use Maple to plot the first 100 terms of your series in (a). Once again, feel free to use the Maple help menu, or Google in order to learn more about the seq() command.

(d) Now, for the terms a_n of the series you wrote in (a), type the following commands in Maple:
L:=ListTools:-PartialSums([seq(a_n,$n = 1..100$)]):
plots:-listplot(L,style=point);

This gives you a plot of the first 100 partial sums of the series in (a). What is the connection between this plot and your answer in part (b)?

4. Your friend has a coin. Suppose the coin is fair (meaning each side is equally likely), with two different sides, heads and tails, which (s)he will toss. If it comes up heads, (s)he will pay you \$1. If it comes up tails, (s)he will toss the coin again. If, on the second toss, it comes up heads, (s)he will pay you \$2, and if it comes up tails again (s)he will toss it again. On the third toss, if it comes up heads, (s)he will pay you \$3, and if it comes up tails again, (s)he will toss it again... How much should you be willing to pay to play this game? You may use Maple to evaluate any sums, but you need to show all work in getting to that point.

5. You are suspicious that your friend's coin is not fair, so you play a new game. You alternate turns flipping the coin ((s)he starts), and the first person to get heads wins the game. Suppose the probability that the coin comes up heads is p.

(a) What is the probability that you win the game on your third turn?

(b) What is the probability that you win the game?

(c) Suppose you observe that your actual probability of winning the game is $\frac{4}{9}$. What must p be?

6. Consider the sum $\sum_{k=1}^{\infty} kr^{k-1}$, where $|r| < 1$.

(a) Show that this sum is equal to $\sum_{k=0}^{\infty} \frac{d}{dr}\left(r^k\right)$.

(b) We will see later in this course that $\frac{d}{dr}\left(\sum_{k=0}^{\infty} r^k\right) = \sum_{k=0}^{\infty} \frac{d}{dr}\left(r^k\right)$. Use this fact to find a formula for $\sum_{k=1}^{\infty} kr^{k-1}$, where $|r| < 1$.

(c) Let X be the number of rolls it takes for you to roll a six, when rolling a fair, six-sided die. Use your formula from (b) to find $\mathbb{E}[X]$ (without using Maple).

7. For this question, your solutions and work should **not** involve Maple. Consider $\int_0^{100} 2^x dx$.

(a) If you approximate this integral by a Riemann sum using n rectangles, find Δx and x_i, where $1 \leq i \leq n$.

(b) Set up, in Σ-notation, a right-hand Riemann sum with n rectangles. Find this sum without using Maple.

(c) Find $\int_0^{100} 2^x dx$ using the definition of the definite integral.

(d) If you were to approximate $\int_0^{100} 2^x dx$ using the midpoint rule, would it be an overestimate or an underestimate? Why?

(e) Using the Fundamental Theorem of Calculus, evaluate this definite integral.

8. As a New Year's resolution, Adam hopes to improve his time in running a mile. On January 1st, he runs a 10 minute mile, and then will run a mile every day after that. Suppose that each day, his time in running the mile is 99% of his time the day before.

   (a) How many days will it take before he can run a mile in under 9 minutes?

   (b) At the end of January, how much time will he have devoted to running?

   (c) Why is it not realistic that this scheme could go on forever?