p1 Ex 1) $\sum_{k=1}^{\infty} \frac{1}{k}$ harmonic series diverges

2) $\sum_{k=1}^{\infty} \frac{1}{e^k} = \sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$ geometric series w/ $a=1, r=\frac{1}{e}$

Since $e>1$, we know $|r| = \left|\frac{1}{e}\right| < 1$ and so converges to

$$\frac{1}{1-r} = \frac{1}{1-\frac{1}{e}} = \frac{1}{\frac{e-1}{e}} = \frac{e}{e-1}$$

3) $\sum_{k=1}^{\infty} \frac{k}{k+1}$

$\lim_{k\to\infty} \frac{k}{k+1} \overset{LH}{=} \lim_{k\to\infty} \frac{1}{1} = 1 \neq 0$

So, by the Divergence Test, the series diverges.

p3 Ex 1) $\sum_{k=1}^{\infty} \frac{1}{e^k}$

$\int_1^{\infty} \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{e^x}\,dx = \lim_{b\to\infty} \left(-e^{-x}\Big|_1^b\right)$

$= \lim_{b\to\infty} \left(\frac{1}{e} - \frac{1}{e^b}\right) = \frac{1}{e}$

Integral converges, so by the Integral Test, the series converges.

2) $\sum_{k=1}^{\infty} \frac{1}{k}$

$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} (\ln b - \ln 1)$

$= \lim_{b\to\infty} \ln b = \infty$ diverges so series diverges

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3) $\sum_{k=1}^{\infty} \frac{1}{k^{1.01}}$

$\int_1^{\infty} \frac{1}{x^{1.01}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^{1.01}}\,dx = \lim_{b\to\infty}\left(-\frac{x^{-0.01}}{0.01}\Big|_1^b\right)$

$= \lim_{b\to\infty}\left(\frac{1}{0.01} - \frac{1}{0.01\,b^{0.01}}\right) = 100$

converges so series converges

4) $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$

$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\left(\ln(\ln x)\Big|_2^b\right)$

$= \lim_{b\to\infty}\left(\ln(\ln(b)) - \ln(\ln 2)\right) = \infty$

diverges so series diverges

5) $\sum_{k=1}^{\infty} \sin(k)$

$\int_1^{\infty} \sin x\,dx = \lim_{b\to\infty} \int_1^b \sin x\,dx = \lim_{b\to\infty}\left(-\cos x\Big|_1^b\right)$

$= \lim_{b\to\infty}(\cos 1 - \cos b) = \text{DNE} \Rightarrow$ diverges

since $\lim_{b\to\infty}\cos b$ DNE so the series diverges

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6) $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p>1$

$\int_1^{\infty} \frac{1}{x^p}dx$ converges for $p>1$ so series converges

7) $\sum_{k=1}^{\infty} \frac{1}{k^p}$ if $p<1$

$\int_1^{\infty} \frac{1}{x^p}dx$ diverges for $p<1$, so series diverges

8) $\sum_{k=2}^{\infty} \frac{k}{e^k}$

$\int_2^{\infty} \frac{x}{e^x}dx = \lim_{b\to\infty} \int_2^b \frac{x}{e^x}dx = \lim_{b\to\infty}\left(-\frac{x+1}{e^x}\Big|_2^b\right)$

$= \lim_{b\to\infty}\left(\frac{3}{e^2} - \frac{b+1}{e^b}\right) = \frac{3}{e^2}$

Since $\lim_{b\to\infty} \frac{b+1}{e^b} \overset{LH}{=} \lim_{b\to\infty} \frac{1}{e^b} = 0$. Since integral converges, the series converges.

p4 Ex 1) $\int_1^{\infty} f(x)dx = \sum_{k=1}^{\infty} a_k$?

Let $f(x) = \frac{1}{e^x}$ Then $a_k = f(k) = \frac{1}{e^k}$.

$\int_1^{\infty} \frac{1}{e^x}dx = \frac{1}{e}$ by #1 above

not equal

$\sum_{k=1}^{\infty} \frac{1}{e^k} = \sum_{k=1}^{\infty}\left(\frac{1}{e}\right)^k = \frac{e}{e-1}$ by p1 Ex#1.

p5 Ex 1) upper bound on error of $\sum_{k=1}^{50} \frac{1}{k^3}$ to estimate $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

$$\left| \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{50} \frac{1}{k^3} \right| = \sum_{k=51}^{\infty} \frac{1}{k^3} \le \int_{50}^{\infty} \frac{1}{x^3}\,dx$$

$$\int_{50}^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_{50}^{b} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left(-\frac{x^{-2}}{2} \Big|_{50}^{b} \right)$$

$$= \lim_{b\to\infty} \left(\frac{1}{2(50^2)} - \frac{1}{2b^2} \right) = \frac{1}{2(50^2)} = \frac{1}{5000}$$

2) $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges by Integral Test as

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \left(\arctan x \Big|_1^b \right)$$

$$= \lim_{b\to\infty} (\arctan b - \arctan 1) = \lim_{b\to\infty} \arctan b = \frac{\pi}{2}$$

$$S - S_n = \sum_{k=1}^{\infty} \frac{1}{k^2+1} - \sum_{k=1}^{n} \frac{1}{k^2+1} = \sum_{k=n}^{\infty} \frac{1}{k^2+1}$$

$$\le \int_n^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_n^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \left(\arctan x \Big|_n^b \right)$$

$$= \lim_{b\to\infty} (\arctan b - \arctan(n))$$

$$= \frac{\pi}{2} - \arctan(n) \quad < 0.005$$

$$\arctan(n) > \frac{\pi}{2} - 0.005$$

$$\therefore n > \tan\left(\frac{\pi}{2} - 0.005\right) = 199.99833$$

so $n \geq 200$