1 Review

Fundamental Theorem of Calculus I: If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Extreme Value Theorem: If $f(t)$ is continuous on the closed interval $[a, b]$, then it has a maximum and a minimum on that interval.

Question If $m \leq f(t) \leq M$ for $a \leq t \leq b$, then

$$m(b-a) = \int_a^b m\,dt \leq \int_a^b f(t)dt \leq \int_a^b M\,dt = M(b-a)$$

2 Fundamental Theorem of Calculus

deals with functions defined by an equation of the form $g(x) = \int_a^x f(t)dt$.

Let $g(x) = \int_0^x f(t)dt$ for a continuous function $f(t)$.

1. What is $g(1)$ geometrically?

   area under the graph of f from 0 to 1

2. What is $g(2)$ geometrically?

   area under the graph of f from 0 to 2

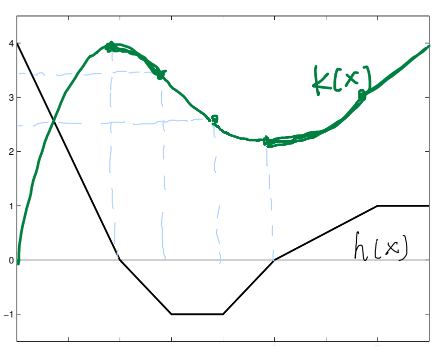
3. What is $g(x)$ geometrically?

   area under the graph of f from 0 to x.

$x \Rightarrow \boxed{\text{differentiation}} \Rightarrow f'(x)$

$x \Rightarrow \boxed{\int_a^x f(t)dt} \Rightarrow g(x)$

Below is a graph of $h(x)$. Sketch a graph of $k(x) = \int_0^x h(t)dt$. Do you see another relationship between $h(x)$ and $k(x)$?



$h(x) = k'(x)$

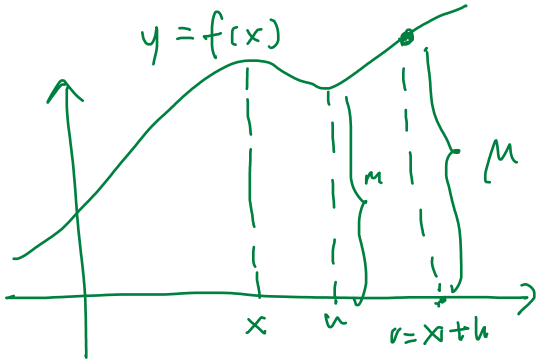
Suppose $f(t)$ is a continuous function and $g(x) = \int_a^x f(t)dt$. Then

$$\frac{g(x+h) - g(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

Since $f(t)$ is continuous on $[x, x+h]$, Extreme Value theorem tells us it attains a smallest value m and a largest value M. Then,

$$m \cdot h \leq \int_x^{x+h} f(t)dt \leq M \cdot h$$

Let's draw the picture.



Going back, dividing everything by h, we get

$$m = f(u) \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq f(v) = M$$

As $h \to 0$, what happens to the interval $[x, x+h]$?

$u \to x, \quad v \to x$

As $h \to 0$, what happens to m and M?

m and M go to f(x) by the squeeze Theorem.

Therefore, $\frac{d}{dx}[g(x)] = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \underline{\quad f(x) \quad}$.

The Second Fundamental Theorem of Calculus
Let f be continuous on an interval. Then for x and a in that interval

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Exercise

1. Write down an antiderivative for e^{-x^2} and check it.

$$\int_a^x e^{-t^2}dt$$

Since e^{-x^2} is continuous on the real line, use SFTC, we have $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

2. Find the following derivative in two different ways: $\frac{d}{dx}\int_2^x \cos t dt$

$$\frac{d}{dx}\int_2^x \cos t\, dt = \cos x.$$

3. Let $g(x) = \int_1^x \sqrt{1+t^2}\,dt$.

   What is $g'(x)$?

   $\sqrt{1+x^2}$

   What is $g(x^3)$?

   $\int_1^{x^3} \sqrt{1+t^2}\,dt$

   What is $\frac{d}{dx}g(x^3)$?

   $\frac{d}{dx} g(x^3) = g'(x^3) \cdot 3x^2 = \sqrt{1+(x^3)^2} \cdot 3x^2$

   $= \sqrt{1+x^6} \cdot 3x^2$

4. Find a function $g(x)$ such that $g'(x) = \sqrt{1+x^2}$ and $g(2) = 0$.

   By 2nd FTC: $g(x) = \int_a^x \sqrt{1+t^2}\,dt$ for some constant a

   $g(2) = \int_a^2 \sqrt{1+t^2}\,dt = 0$ choose $a = 0$ $g(x) = \int_2^x \sqrt{1+t^2}\,dt$

5. Find a function $g(x)$ such that $g'(x) = \sqrt{1+x^2}$ and $g(2) = 10$.

   $g(x) = \int_a^x \sqrt{1+t^2}\,dt + C$, for some constant a.

   $g(2) = \int_a^2 \sqrt{1+x^2}\,dt + C = 10$ So choose $a = 2$. then let $C = 10$.

6. Let $g(x) = \int_0^x f(t)\,dt$ where f is continuous.

   If $f(t) > 0$ and $x > 0$, then $g(x)$ is increasing. and positive. .

   If $f(t) > 0$ and $x < 0$, then $g(x)$ is increasing and positive. .

   If $f(t) < 0$ and $x > 0$, then $g(x)$ is decreasing and negative .

7. Find $\frac{d}{dx}\int_{x^2}^3 \frac{\sin t}{t}\,dt$.

   let $g(x) = \int_3^x \frac{\sin t}{t}\,dt$

   $g(x^2) = \int_3^{x^2} \frac{\sin t}{t}\,dt$

   $g'(x) = \frac{\sin x}{x}$

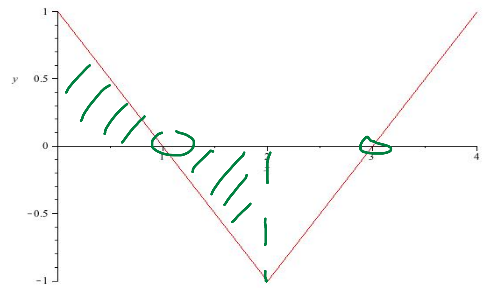
   $\frac{d}{dx}\int_{x^2}^3 \frac{\sin t}{t}\,dt = \frac{d}{dx}\left[-\int_3^{x^2} \frac{\sin t}{t}\,dt\right]$

   $= -\frac{d}{dx}\left[g(x^2)\right] = -g'(x^2)\cdot 2x$

   $= -\frac{\sin(x^2)}{x^2} \cdot 2x$

   $= \frac{-2\sin(x^2)}{x}$

8. Consider the function $f(x)$, defined by the graph below. Let $F(x) = \int_0^x f(t)\,dt$ for $x \in [0,4]$.



(a) Find $F(0), F(1), F(2), F(3,), $ and $F(4)$.

$F(0) = 0$

$F(1) = \int_0^1 f(t)dt = \frac{1}{2}$

$F(2) = \int_0^2 f(t)\,dt = 0$

$F(3) = \int_0^3 f(t)dt = -\frac{1}{2}$

$F(4) = 0$

x	$F(x)$
0	0
1	$\frac{1}{2}$
2	0
3	$-\frac{1}{2}$
4	0

(b) Find the average value of $f(x)$ on the interval $[0,4]$.

$\frac{1}{4}\int_0^4 f(x) = \frac{1}{4}F(4) = 0$

(c) Find the critical points of $F(x)$ on the interval $(0,4)$ making sure to label each as either a maximum or a minimum.

Set $F'(x) = f(x) = 0$

then $x = 1$ or 3.

$F''(x) = f'(x)$

then $f'(1) = -1$ ∩ 1 is a local max

$f'(3) = 1$ ∪ 1 is a local min