1 Review

Definition Given a function $f(x)$ on $[a,b]$. We have the following:

$$\text{RHS}_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

$$\text{LHS}_n = \sum_{k=0}^{n-1} f(x_k)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$.

2 Definite Integral

Definition Let $f(x)$ be a continuous function on $[a,b]$ or [if f has only a finite # of jump discontinuities.]. Then the **definite integral** of f from a to b is defined as

integral sign intro. by Leibniz. $\rightarrow$ integrand

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

if the limit exists, where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$, and $x_k^* \in [x_{k-1}, x_k]$. The definite integral is the area under the graph of f from a to b.

Example Give three other ways to equivalently define the integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \quad (\text{by RHS})$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x \quad (\text{by LHS})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{x_k + x_{k-1}}{2}\right)\Delta x \quad (\text{by Midpoint rule})$$

Exercises

1. Use the definition of definite integral to compute $\displaystyle\int_a^b c\,dx$ for constants $a < b$ and c. Check your answer geometrically.

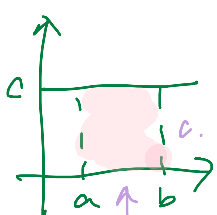
$$\int_a^b c\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} c\cdot\Delta x \quad \text{since } f(x_k) = c \text{ for all } x_k.$$

$$= \lim_{n\to\infty} c\cdot \sum_{k=1}^{n} \Delta x \quad (\text{by scaling properties of } \Sigma)$$

$$= c \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \quad [\text{limit properties}]$$

$$= c\cdot \lim_{n\to\infty} \Delta x \cdot n = c\cdot \lim_{n\to\infty} \frac{b-a}{\cancel{n}}\cdot \cancel{n} = c(b-a)$$



$(b-a)$ Area $= c.(b-a)$

2. Use the definition of definite integral to simplify $\int_a^b cf(x)\,dx$ for constant c.

$$\int_a^b c f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{a} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} c f(x_k)\Delta x$$

$$= \lim_{n\to\infty} c \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= c \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x}_{\int_a^b f(x)\,dx} = c\int_a^b f(x)\,dx.$$

3. Use the definition of definite integral to show that

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

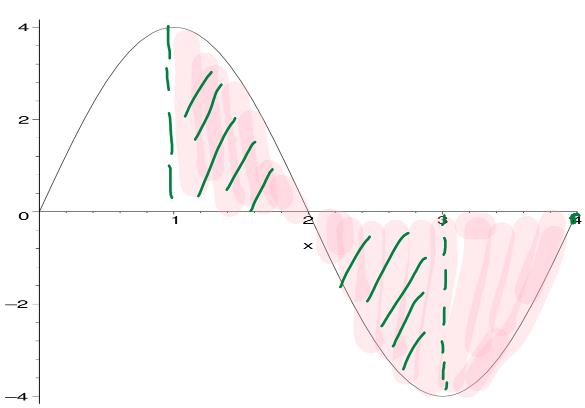
$$\int_a^b f(x) + g(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[f(x_k) + g(x_k)\right]\Delta x$$

distributative properties
$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(x_k) + \sum_{k=1}^{n} g(x_k)\right]\Delta x$$

properties of limit
$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x + \lim_{n\to\infty} \sum_{k=1}^{n} g(x_k)\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

4. Use the "obvious" symmetries to find $\int_1^4 f(x)\,dx$, where the graph of $f(x)$ is given below.



$$\int_1^4 f(x)\,dx$$

$$= \int_3^4 f(x)\,dx.$$

5. Evaluate $\int_0^3 (2x)dx$.

$$\int_0^3 (2x)dx = 2\int_0^3 x\,dx = 2\cdot\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x, \text{ where } \Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

$$x_k = a + k\Delta x = k\cdot\Delta x = k\frac{3}{n}$$

$$= 2\cdot\lim_{n\to\infty}\sum_{k=1}^{n} x_k\cdot\Delta x$$

$$= 2\cdot\lim_{n\to\infty}\sum_{k=1}^{n} k\cdot\frac{9}{n^2}$$

$$= 2\lim_{n\to\infty}\frac{9}{n^2}\cdot\left(\sum_{k=1}^{n} k\right) \leftarrow \frac{(1+n)\cdot n}{2}$$

$$= 2\lim_{n\to\infty}\frac{9}{2}\frac{(1+n)\cancel{n}}{n^{\cancel{2}}} = 2\lim_{n\to\infty}\frac{9}{2}\left(\frac{1}{n}+1\right)$$

$$= 9\lim_{n\to\infty}\left(\frac{1}{n}+1\right) = 9\left(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{0} + \underbrace{\lim_{n\to\infty} 1}_{=1}\right) = 9$$

6. Evaluate $\int_0^3 (3x+7)dx$.

$$\int_0^3 (3x+7)dx$$

$$= \int_0^3 3x\,dx + \int_0^3 7\,dx$$

$$= 3\int_0^3 x\,dx + \int_0^3 7\,dx$$

$$= 3\cdot\frac{9}{2} + 21$$

7. Consider the Riemann sum

$$\sum_{k=0}^{499} \underbrace{[5-(2+.006k)^2]}_{f(x_k)}\Delta x.$$

$x_k = a + 0.006k$, $a = 2$, $\Delta x = 0.006$

Let's figure out a definite integral this Riemann sum approximates. To do so, fill in the following:

$n =$ 500

$[a,b] =$ $[2,5]$

$\Delta x =$ 0.006

since $\Delta x = \frac{b-a}{n}$

$n\Delta x = b-a$

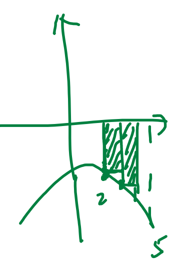
$b = n\Delta x + a$

$b = 500\times 0.006 + 2 = 5$

So what definite integral are we approximating?

$$\int_2^5 5-x^2\,dx$$

Is this an over or under approximation of the integral you gave? Or, can we tell?



Over approximation, since $5-x^2$ is decreasing on $[2,5]$, and LHS over approximates decreasing functions.

It turns out this is not the only possible definite integral it could approximate. With that said, what is another possible function and interval we could be looking at?

This question is removed

8. For each of the following, write the integral which is equal to it.

(a) $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2\ln(1+\frac{2k}{n})}{n} = \lim_{n\to\infty}\sum_{k=1}^{n} 2\ln(\underbrace{1+\tfrac{2k}{n}}_{a+k\cdot\Delta x})\cdot\frac{1}{n}$

$a = 1$

$\Delta x = \frac{2}{n}$.

$b = n\cdot\Delta x + a = 3$

$\displaystyle = \lim_{n\to\infty}\sum_{k=1}^{n}\ln(x_k)\cdot\Delta x$

$\displaystyle = \int_1^3 \ln(x)\,dx$

(b) $\displaystyle \lim_{k\to\infty}\sum_{i=0}^{k-1}\frac{1}{k\sqrt{1-\left(\frac{i}{k}\right)^2}} = \lim_{k\to\infty}\sum_{i=0}^{k-1}\frac{1}{\sqrt{1-(\frac{i}{k})^2}}\cdot\frac{1}{k}$

LHS

$x_k = \frac{i}{k}$.

$x_k = a + \Delta x\cdot i$

$= 0 + \frac{i}{k}$.

$\Delta x = \frac{1}{k}$

$b = k\cdot\Delta x$

$= 1$.

$\displaystyle = \lim_{k\to\infty}\sum_{i=0}^{k-1}\frac{1}{\sqrt{1-(x_k)^2}}\cdot\Delta x$

$\displaystyle = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

(c) $\displaystyle \lim_{m\to\infty}\sum_{k=1}^{m}\frac{\pi\sin\left(\frac{\pi k}{2m}\right)}{2m} = \lim_{n\to\infty}\sum_{k=1}^{m}\frac{\pi\sin(\overset{x_k}{\frac{\pi\cdot k}{2m}})}{2m}$

RHS.

$\Delta x = \frac{\pi}{2m}$

$x_k = \Delta x\cdot k$.

$a = 0$

$b = 0 + \frac{\pi}{2\pi}$

$= \frac{1}{2}$.

$\displaystyle = \lim_{n\to\infty}\sum_{k=1}^{m}\sin(x_k)\cdot\Delta x$.

$\displaystyle = \int_0^{\frac{1}{2}}\sin(x)\,dx$.