1 Warm-up

Exercise Consider the solid formed when you rotate the region $y = 1/x$, where $1 < x$, about the x-axis. Compute its volume.

2 Improper Integrals

Definition We define $\int_a^\infty f(x)\,dx$ to be $\lim_{b\to\infty} \int_a^b f(x)\,dx$ if this limit exists. In this case, we say $\int_a^\infty f(x)\,dx$ **converges**. If this limit does not exist, we say it **diverges**. (Make sure to read the definition box (especially part (c)) on page 414.)

Exercises

1. Consider $\int_1^\infty \frac{1}{t}\,dt$. Does this integral converges?
2. Does the integral $\int_1^\infty \frac{1}{t^2}\,dt$ converge?

3. For which values of the constant p does the integral $\int_1^\infty \frac{1}{t^p}\,dt$ converge? Explain. [Hint: You will need to consider three cases: $p > 1$, $p = 1$, and $p < 1$.]

4. On the interval $[1, \infty]$, we notice that $y = e^{-t}$ is always (greater than/less than) $y = e^{-t^2}$. Use this to decide whether $\int_1^\infty e^{-t^2}\,dt$ converges. Could you answer this questions by finding an antiderivatives of e^{-t^2} and then evaluating the integral like we did before?

5. On the interval $[2, \infty]$, we notice that $y = \frac{1}{\ln x}$ is always (greater than/less than) $y = \frac{1}{x}$. Use this to decide whether $\int_2^\infty \frac{1}{\ln x}\,dx$ converges.

6. Suppose $0 \le f(t) \le g(t)$ for $t \ge a$. Complete the following sentences using converges or diverges:

   If $\int_a^\infty f(t)\,dt$ ______________________________, then $\int_a^\infty g(t)\,dt$ ______________________________.

   If $\int_a^\infty g(t)\,dt$ ______________________________, then $\int_a^\infty f(t)\,dt$ ______________________________.

7. Does $\int_0^\infty xe^{-x}\,dx$ converge? If so, to what does it converge?

8. Use $f(x) = x^3$ to show that it is <u>not</u> true that

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty} \int_{-t}^t f(x)\,dx.$$

9. What is “wrong” with the following integrals? What happens if you try to compute Riemann sums near $x = 0$?

$$\int_0^1 \frac{dx}{\sqrt{x}} \qquad\qquad \int_{-1}^1 \frac{dx}{x^2}$$

10. Define $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0}\int_b^1 \frac{dx}{\sqrt{x}}$. Compute the value.

11. (a) For which values of p does $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converge?

    (b) For which values of p does $\int_0^1 \frac{dx}{x^p}$ converge?

    (c) For which values of p does $\displaystyle\int_0^\infty \frac{dx}{x^p}$ converge?

12. Find $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$.

13. As before, sometimes a comparison can help us determine whether or not an integral converges or diverges. With that said, does $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2-1}}\,dx$ converge or diverge? [Hint: Compare with $\displaystyle\int_0^1 \frac{1}{x}\,dx$.]

14. Decide whether each of the following improper integrals converge or diverge.

    (a) $\displaystyle\int_0^{\infty} \frac{1}{e^2+2^x}\,dx$

(b) $\displaystyle\int_1^\infty \frac{1+\sin\theta}{\theta^2}\,d\theta$

(c) $\displaystyle\int_2^\infty \frac{dx}{x(\ln(x))^2}$

(d) $\displaystyle\int_1^2 \frac{dx}{x(\ln(x))^2}$

(e) $\displaystyle\int_0^1 \frac{dx}{x(\ln(x))^2}$