

Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 2.3) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where both $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$** . We met some limits of this type in Chapter 2. For rational functions, we can cancel common factors:

$$\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{x + 1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as *l'Hospital's Rule*, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

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$$\lim_{x \rightarrow \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \rightarrow \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞ ; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

where both $f(x) \rightarrow \infty$ (or $-\infty$) and $g(x) \rightarrow \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type ∞/∞** . We saw in Section 2.5 that this type of limit can be evaluated for certain functions, including rational functions, by dividing

L'Hospital

L'Hospital's Rule is named after the French mathematician, the Marquis de l'Hospital, who first published it in 1696. However, the rule was discovered by the Swiss mathematician, Johann Bernoulli, around 1690. In fact, Bernoulli sometimes sees l'Hospital's name as L'Hopital. Bernoulli spelled his own name as Bernoulli. The name L'Hospital is often used in the 17th century. In the example, we see the rule. See the historical note at the end of the chapter for more information.

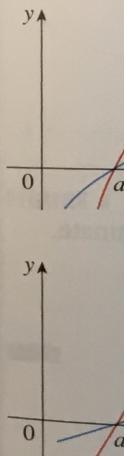


FIGURE 1

Figure 1 suggests that the rule might be useful for differentiable functions whose graphs approach a horizontal asymptote. If we zoom in toward the point where the curve intersects the asymptote, we would start to see what the function is doing near the asymptote. For example, if the function is increasing and concave up, it will cross the asymptote from below. If the function is decreasing and concave down, it will cross the asymptote from above. This is what happens in the top graph. In the bottom graph, the function crosses the asymptote from above.

which is the same as the limit of the ratio of the derivatives, suggests that

line

numerator and denominator by the highest power of x that occurs in the denominator. For instance,

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as (2), but l'Hospital's Rule also applies to this type of indeterminate form.

L'Hospital

L'Hospital's Rule is named after a French nobleman, the Marquis de l'Hospital (1661–1704), but was discovered by a Swiss mathematician, John Bernoulli (1667–1748). You might sometimes see l'Hospital spelled as l'Hôpital, but he spelled his own name l'Hospital, as was common in the 17th century. See Exercise 69 for the example that the Marquis used to illustrate his rule. See the project on page 299 for further historical details.

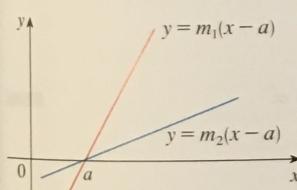
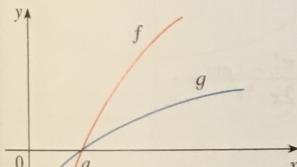


FIGURE 1

Figure 1 suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g , each of which approaches 0 as $x \rightarrow a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually were linear, as in the second graph, then their ratio would be

$$\frac{m_1(x - a)}{m_2(x - a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

$$\text{or that} \quad \lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Note 1: L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

Note 2: L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, " $x \rightarrow a$ " can be replaced by any of the symbols $x \rightarrow a^+$, $x \rightarrow a^-$, $x \rightarrow \infty$, or $x \rightarrow -\infty$.

Note 3: For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} &= \frac{f'(a)}{g'(a)} = \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \end{aligned}$$

The general version of l'Hospital's Rule is more difficult; its proof can be found in more advanced books.

V EXAMPLE 1 An indeterminate form of type $0/0$ Find $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$.

SOLUTION Since

$$\lim_{x \rightarrow 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \rightarrow 1} (x - 1) = 0$$

we can apply l'Hospital's Rule:

Q Notice that when using l'Hospital's Rule we differentiate the numerator and denominator separately. We do *not* use the Quotient Rule.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\ln x}{x - 1} &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \rightarrow 1} \frac{1/x}{1} \\ &= \lim_{x \rightarrow 1} \frac{1}{x} = 1 \end{aligned}$$

The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 63.

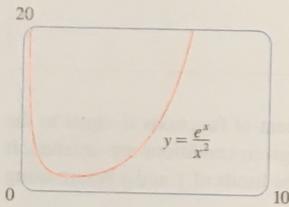


FIGURE 2

The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it isn't surprising that this ratio approaches 0 as $x \rightarrow \infty$. See also Exercise 64.

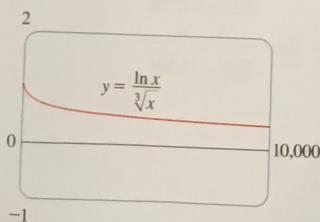


FIGURE 3

The graph in Figure 4 shows the result of Example 2. It is too far, however, to draw the graph because it is small. See Exer-

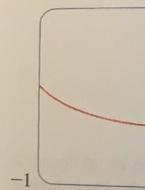


FIGURE 4

V EXAMPLE 2 An indeterminate form of type ∞/∞ Calculate $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x \rightarrow \infty} e^x = \infty$ and $\lim_{x \rightarrow \infty} x^2 = \infty$, so l'Hospital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

Since $e^x \rightarrow \infty$ and $2x \rightarrow \infty$ as $x \rightarrow \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

V EXAMPLE 3 Calculate $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \rightarrow \infty$ and $\sqrt[3]{x} \rightarrow \infty$ as $x \rightarrow \infty$, l'Hospital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x \rightarrow \infty} \frac{3}{\sqrt[3]{x}} = 0$$

EXAMPLE 4 Using l'Hospital's Rule three times Find $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$.

SOLUTION Noting that both $\tan x - x \rightarrow 0$ and $x^3 \rightarrow 0$ as $x \rightarrow 0$, we use l'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x}$$

Because $\lim_{x \rightarrow 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \rightarrow 0} \sec^2 x \cdot \lim_{x \rightarrow 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sec^2 x}{1} = \frac{1}{3} \end{aligned}$$

The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because $\tan x$ is close to x when x is small. See Exercise 30(d) in Section 2.2.

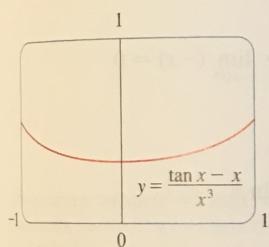


FIGURE 4

V EXAMPLE 5 Find $\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\textcircled{O} \quad \lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \rightarrow \pi^-} \frac{\cos x}{-\sin x} = -\infty$$

This is **wrong!** Although the numerator $\sin x \rightarrow 0$ as $x \rightarrow \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π and the denominator is nonzero there:

$$\lim_{x \rightarrow \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0$$

Example 5 shows what can go wrong if you use l'Hospital's Rule without thinking. Other limits *can* be found using l'Hospital's Rule but are more easily found by other methods. (See Examples 3 and 5 in Section 2.3, Example 5 in Section 2.5, and the discussion at the beginning of this section.) So when evaluating any limit, you should consider other methods before using l'Hospital's Rule.

Indeterminate Products

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \rightarrow a} f(x)g(x)$, if any, will be. There is a struggle between f and g . If f wins, the limit will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$** . We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

V EXAMPLE 6 Evaluate $\lim_{x \rightarrow 0^+} x \ln x$. Use the knowledge of this limit, together with information from derivatives, to sketch the curve $y = x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \rightarrow 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \rightarrow \infty$ as $x \rightarrow 0^+$, so l'Hospital's Rule gives

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0$$

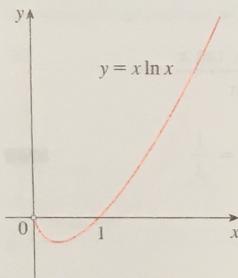


FIGURE 5

If $f(x) = x \ln x$, then

$$f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

so $f'(x) = 0$ when $\ln x = -1$, which means that $x = e^{-1}$. In fact, $f'(x) > 0$ when $x > e^{-1}$ and $f'(x) < 0$ when $x < e^{-1}$, so f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$. Thus, by the First Derivative Test, $f(1/e) = -1/e$ is a local (and absolute) minimum. Also, $f''(x) = 1/x > 0$, so f is concave upward on $(0, \infty)$. We use this information, together with the crucial knowledge that $\lim_{x \rightarrow 0^+} f(x) = 0$, to sketch the curve in Figure 5.

Although forms indeterminate,
(See Exercise 7)

Note: In solving Example 6 another possible option would have been to write

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type $0/0$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

Indeterminate Differences

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then the limit

$$\lim_{x \rightarrow a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$** . Again there is a contest between f and g . Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .

EXAMPLE 7 An indeterminate form of type $\infty - \infty$ Compute $\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \rightarrow \infty$ and $\tan x \rightarrow \infty$ as $x \rightarrow (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\begin{aligned}\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x) &= \lim_{x \rightarrow (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \\ &= \lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \rightarrow (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0\end{aligned}$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \rightarrow 0$ and $\cos x \rightarrow 0$ as $x \rightarrow (\pi/2)^-$.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \rightarrow a} [f(x)]^{g(x)}$$

1. $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ type 0^0
2. $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = 0$ type ∞^0
3. $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$ type 1^∞

Although forms of the type 0^0 , ∞^0 , and 1^∞ are indeterminate, the form 0^∞ is not indeterminate. (See Exercise 72.)

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \text{ then } \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 8 An indeterminate form of type 1^∞ Calculate $\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \rightarrow 0^+$, we have $1 + \sin 4x \rightarrow 1$ and $\cot x \rightarrow \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$

so l'Hospital's Rule gives

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{4 \cos 4x}{1 + \sin 4x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y . To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^4$$

V EXAMPLE 9 An indeterminate form of type 0^0 Find $\lim_{x \rightarrow 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \rightarrow 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x} = e^0 = 1$$

The graph of the function $y = x^x$, $x > 0$, is shown in Figure 6. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \rightarrow 0^+$. This confirms the result of Example 9.

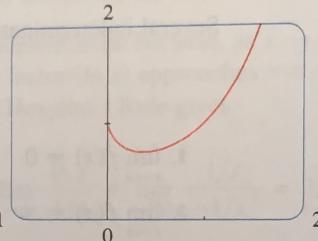


FIGURE 6

45 Exercises

1–4 Given that

$$\begin{aligned}\lim_{x \rightarrow a} f(x) &= 0 & \lim_{x \rightarrow a} g(x) &= 0 & \lim_{x \rightarrow a} h(x) &= 1 \\ \lim_{x \rightarrow a} p(x) &= \infty & \lim_{x \rightarrow a} q(x) &= \infty\end{aligned}$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ (b) $\lim_{x \rightarrow a} \frac{f(x)}{p(x)}$ (c) $\lim_{x \rightarrow a} \frac{h(x)}{p(x)}$
- (d) $\lim_{x \rightarrow a} \frac{p(x)}{f(x)}$ (e) $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$
2. (a) $\lim_{x \rightarrow a} [f(x)p(x)]$ (b) $\lim_{x \rightarrow a} [h(x)p(x)]$
- (c) $\lim_{x \rightarrow a} [p(x)q(x)]$
3. (a) $\lim_{x \rightarrow a} [f(x) - p(x)]$ (b) $\lim_{x \rightarrow a} [p(x) - q(x)]$
- (c) $\lim_{x \rightarrow a} [p(x) + q(x)]$
4. (a) $\lim_{x \rightarrow a} [f(x)]^{g(x)}$ (b) $\lim_{x \rightarrow a} [f(x)]^{p(x)}$ (c) $\lim_{x \rightarrow a} [h(x)]^{p(x)}$
- (d) $\lim_{x \rightarrow a} [p(x)]^{f(x)}$ (e) $\lim_{x \rightarrow a} [p(x)]^{q(x)}$ (f) $\lim_{x \rightarrow a} \sqrt[q(x)]{p(x)}$

5–46 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

5. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - x}$
6. $\lim_{x \rightarrow 1} \frac{x^a - 1}{x^b - 1}$
7. $\lim_{x \rightarrow (\pi/2)^+} \frac{\cos x}{1 - \sin x}$
8. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\tan 5x}$
9. $\lim_{t \rightarrow 0} \frac{e^t - 1}{t^3}$
10. $\lim_{t \rightarrow 0} \frac{e^{3t} - 1}{t}$
11. $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$
12. $\lim_{\theta \rightarrow \pi/2} \frac{1 - \sin \theta}{\csc \theta}$
13. $\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$
14. $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x}$
15. $\lim_{x \rightarrow 0} \frac{\sqrt{1 + 2x} - \sqrt{1 - 4x}}{x}$
16. $\lim_{x \rightarrow 1} \frac{\ln x}{\sin \pi x}$
17. $\lim_{t \rightarrow 0} \frac{5^t - 3^t}{t}$
18. $\lim_{u \rightarrow \infty} \frac{e^{u/10}}{u^3}$
19. $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$
20. $\lim_{x \rightarrow 0} \frac{\cos mx - \cos nx}{x^2}$

Graphing calculator or computer with graphing software required

Computer algebra system required

1. Homework Hints available in TEC