1.

For which values of x does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv. abs ?

i.e For which values of x does $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ conv.?

Apply the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \Big/ \frac{x^n}{n} \right|$$

$$= \lim_{n\to\infty} |x| \cdot \left|\frac{n}{n+1}\right| = |x|$$

If $|x| < 1$, then $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ conv.

If $|x| > 1$, then $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ div.

If $|x| = 1$, we cannot conclude from Ratio Test

but $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right|$ div.

Hence for $|x| < 1$, $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ conv.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv. abs.

For which values of x does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv. conditionally

i.e For which values of x does $\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv.

but $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ div ?

Apply Ratio's test to conclude about $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \Big/ \frac{x^n}{n} \right|$$

$$= \lim_{n\to\infty} |x| \cdot \left|\frac{n}{n+1}\right| = |x|.$$

If $|x| > 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ diverges

so $\sum_{n=1}^{\infty} \frac{x^n}{n}$ cannot conv. cond.

If $|x| < 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

and $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ converges.

so $\sum_{n=1}^{\infty} \frac{x^n}{n}$ cannot be conv. cond.

If $|x|=1$

two cases

① $x=1$. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ diverges

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ cannot conv. conditionally.

② $x=-1$. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges

$\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$ diverges.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ conv. conditionally.

#2. Assuming $|x|<1$. Find a series whose sum is $\frac{1}{x^2+1}$.

Let $r = -x^2$.

then

$$\sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)}$$

since $|x|<1$

$|r| = x^2 < 1$.

#3. For which values of x does $\sum_{n=0}^{\infty} \frac{x^n}{5^n}$ conv.?

Method 1: Geometric series

observe that

$$\sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$$ is a geometric series

if $\left|\frac{x}{5}\right| < 1$, i.e. $|x| < 5$

then $\sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$ conv. to $\frac{1}{1-\frac{x}{5}}$.

Method 2: (Ratio Test).

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}}\Big/\frac{x^n}{5^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{5}\right| = \left|\frac{x}{5}\right|$$

If $\left|\frac{x}{5}\right|<1$ i.e $|x|<5$. then series conv.

if $\left|\frac{x}{5}\right|>1$ i.e $|x|>5$. then series div.

if $\left|\frac{x}{5}\right|=1$, i.e $|x|=5$. Then.

we have $\sum_{n=0}^{\infty}\frac{x^n}{5^n} = \sum_{n=0}^{\infty} 1 = \infty$. div.

Hence $\sum_{n=0}^{\infty}\frac{x^n}{5^n}$ conv for $|x|<5$.