4-2

1) $\int_2^{10} \frac{1}{x^2-1}\,dx$

$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$

$1 = A(x-1) + B(x+1)$

$= \int_2^{10} -\frac{1}{2(x+1)} + \frac{1}{2(x-1)}\,dx$

$= (A+B)x + B - A$

$A+B=0 \qquad B-A=1$

$= \frac{1}{2}\left(-\int_2^{10}\frac{1}{x+1}dx + \int_2^{10}\frac{1}{x-1}dx\right)$

$A=-B \qquad B-(-B)=1$

$2B=1$

$= \frac{1}{2}\Big[-\ln|x+1| + \ln|x-1|\Big]_2^{10}$

$A=-\frac{1}{2} \qquad B=\frac{1}{2}$

$= \frac{1}{2}\left(-\ln 11 + \ln 9 + \ln 3 - \ln 1\right)$

$= \frac{1}{2}\left(-\ln 11 + 2\ln 3 + \ln 3\right)$

$= \frac{1}{2}\left(3\ln 3 - \ln 11\right)$

2) $\int \frac{3}{x^2+x}\,dx$

$\frac{3}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$

$3 = A(x+1) + Bx$

$= \int \frac{3}{x} - \frac{3}{x+1}\,dx$

$= (A+B)x + A$

$A=3 \qquad A+B=0$

$= 3(\ln|x| - \ln|x+1|) + C$

$B=-A=-3$

3) $\int \frac{x}{x^2-5x+6}\,dx$

$$\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$x = A(x-3)+B(x-2)$$

$$= (A+B)x - 3A - 2B$$

$$1 = A+B \qquad 0 = -3A-2B$$

$$= A - \tfrac{3}{2}A \qquad 2B = -3A$$

$$= -\tfrac{1}{2}A \qquad B = -\tfrac{3}{2}A$$

$$-2 = A \qquad = 3$$

$$= \int -\frac{2}{x-2} + \frac{3}{x-3}\,dx$$

$$= -2\ln|x-2| + 3\ln|x-3| + C$$

4) $\int \frac{1}{x^2+9}\,dx$

Hint: $\int \frac{1}{x^2+1}\,dx = \arctan x + C$

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Ex: $\int \frac{x^3-4}{x^2-4}\,dx$

$$\begin{array}{r} x \\ x^2-4 \,\big)\, x^3-4 \\ \underline{x^3-4x} \\ 4x-4 \end{array}$$

$$= \int x + \frac{4x-4}{x^2-4}\,dx$$

$$= \frac{x^2}{2} + C + \int \frac{4x-4}{x^2-4}\,dx$$

$$\frac{4x-4}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$4x-4 = A(x-2)+B(x+2)$$

$$= (A+B)x - 2A + 2B$$

$$4 = A+B \qquad -4 = -2A+2B$$

$$-2 = -A+B$$

$$2 = 2B \quad B = 1 \quad A = 3$$

$$= \frac{x^2}{2} + \int \frac{3}{x+2} + \frac{1}{x-2}\,dx + C$$

$$= \frac{x^2}{2} + 3\ln|x+2| + \ln|x-2| + C$$

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1) $\int z\sqrt{z-1}\,dz$ $\qquad u = z-1 \qquad z = u+1$

$du = dz$

$= \int (u+1)\sqrt{u}\,du$

$= \int u^{3/2} + u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(z-1)^{5/2} + \frac{2}{3}(z-1)^{3/2} + C$

$= \frac{2}{15}(z-1)^{3/2}(2+3z) + C$

2) $\int_0^1 \tanh(t)\,dt = \int_0^1 \frac{e^t - e^{-t}}{e^t + e^{-t}}\,dt$ $\qquad u = e^t + e^{-t}$

$du = (e^t - e^{-t})\,dt$

$= \int_2^{e+\frac{1}{e}} \frac{1}{u}\,du$

$= \ln|u| \Big|_2^{e+\frac{1}{e}}$

$= \ln(e + \frac{1}{e}) - \ln 2$

3) $\int_{-x}^{x} \cos^2 t\,dt = 2\int_0^x \cos^2 t\,dt = 2\int_0^x \frac{1+\cos 2t}{2}\,dt$

$u = 2t \quad du = 2dt$

$= \int_0^x 1 + \cos 2t\,dt = t\Big|_0^x + \int_0^x \cos 2t\,dt = x + \frac{1}{2}\int_0^{2x} \cos u\,du$

$= x + \frac{1}{2}\sin u\Big|_0^{2x} = x + \frac{1}{2}\sin 2x$

4-2

4) $\int \tan^3 x \sec^2 x \, dx$ $\quad u = \tan x$

$du = \sec^2 x \, dx$

$= \int u^3 \, du$

$= \frac{u^4}{4} + C$

$= \frac{\tan^4 x}{4} + C$

5) $\int \frac{2x}{1+x^4} \, dx$ $\quad u = x^2$

$du = 2x \, dx$

$= \int \frac{1}{1+u^2} \, du$

$= \arctan u + C$

$= \arctan x^2 + C$

6) $\int \frac{2}{4-x^2} \, dx$ $\quad \frac{2}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$

$2 = A(2+x) + B(2-x)$

$= \frac{1}{2} \int \frac{1}{2-x} + \frac{1}{2+x} \, dx$ $\quad = (2A+2B) + (A-B)x$

$0 = A - B$ $\quad 2 = 2A + 2B$

$= \frac{1}{2} (\ln|2-x| + \ln|2+x|) + C$ $\quad A = B$ $\quad 1 = A + B$

$= 2A$

$A = \frac{1}{2} = B$

4-2

7) $\int \frac{dx}{1+\sqrt{x}} = \int \frac{1}{1+u} \cdot 2u\,du$ $\quad u=\sqrt{x} \quad x=u^2$

$dx = 2u\,du$

$= 2\int \frac{u}{1+u}\,du$ $\quad v=1+u \quad u=v-1$

$dv = du$

$= 2\int \frac{v-1}{v}\,dv$

$= 2\int 1 - \frac{1}{v}\,dv$

$= 2v - 2\ln|v| + C$ $\quad v=1+u = 1+\sqrt{x}$

$= 2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + C$

$= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + C$