Power,Taylor, and Fourier Series Project

Due Date: If your lab takes place on Thursday, this is due in class on Monday, November 21. If your lab takes place on Friday, this is due in class on Tuesday, November 22.

Instructions:

- You are to work on this project with the members of your lab group and no one else. Do not ask for help from other groups, or in the help room. This should be the work of only your group.
- You may ask your instructor (or any 122L instructor) questions.
- Only one project is to be handed in per group (and signed by all contributing members).
- Each answer must be thoroughly justified and explained, and your work must be nicely organized. You will be graded on completeness as well as clarity.
- Make sure to show all mathematical justification in an organized way.
- Any work done in Maple should be attached.

1. Suppose X is normally distributed with mean 0 and standard deviation 1. Then X has pdf $f(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(\dfrac{-x^2}{2}\right)$.
   (a) Using the Empirical Rule, what is $\mathbf{P}(-1 \leq X \leq 1)$?
   (b) Now we will approximate the probability in (a) using power series. Recall that $\mathbf{P}(-1 \leq X \leq 1) = \displaystyle\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\, dx$. Why can we say here that, in addition, $\mathbf{P}(-1 \leq X \leq 1) = 2\displaystyle\int_{0}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\, dx$?
   (c) Use your knowledge of Taylor series to write $2\displaystyle\int_{0}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\, dx$ as a series (and show all work in getting there).
   (d) Using your knowledge of error bounds for series, find the smallest value of n such that the nth partial sum of your power series in (c) approximates $\mathbf{P}(-1 \leq X \leq 1)$ to within 0.0001. Then find that partial sum.

2. Let $f(x) = (x+1)\ln(x+1)$.
   (a) Derive the second degree Taylor polynomial, $T(x)$, for $f(x)$ centered at $x = 0$. Use Maple to plot these two functions on the same set of axes, from $x = -0.5$ to $x = 0.5$.
   (b) Using Taylor's Remainder Theorem, what is the best upper bound for $|f(x) - T(x)|$ on the interval $[-0.5, 0.5]$?
   (c) Now consider $h(x)$, which is equal to $(x+1)\ln(x+1)$ on $[-0.5, 0.5]$ and has period 1. Find the Fourier series of $h(x)$ through the 5th harmonic, and plot it on the same set of axes as $f(x)$. In finding the coefficients for the fifth harmonic, you should feel free to compute the definite integrals in Maple. Just attach your Maple worksheet.

3. Consider the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $x = \frac{1}{2}$. Suppose we approximate $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ by $\sum_{n=0}^{10} \left(\frac{1}{2}\right)^n$. We will analyze the resulting error in three different ways. Do not use Maple on this question.

   (a) Bound the error above using the Integral Test bounds.

   (b) Bound the error above using Taylor's Theorem.

   (c) Find the error exactly by using your knowledge of geometric series.

   (d) Show that $\sum_{n=0}^{\infty} x^n$ converges for $0 \le x \le 0.5$. Then use Taylor's Remainder Theorem to show that $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ for $0 \le x \le 0.5$.

4. Recall that a recursive sequence $\{a_n\}$ is one in which in order to find a_n, you must know a_{n-1}. In section 8.1, we saw that recursive sequences were more difficult to handle. The Fibonacci sequence $\{F_n\}$ is a recursive sequence defined by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. Now consider the power series whose coefficients are the terms of the Fibonacci sequence:

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$$

   If we can find a closed-form representation of these coefficients, we will have a closed-form representation of the general term of the Fibonacci sequence (as opposed to a recursive representation).

   (a) First rewrite $f(x)$ as $1 + x + \sum_{n=0}^{\infty} F_{n+2}x^{n+2}$, and show that $f(x) = 1 + xf(x) + x^2 f(x)$.

   (b) Using the result in (a), show that $f(x) = \frac{1}{1-x-x^2}$.

   (c) Show that you can factor the denominator from (b) as $1 - x - x^2 = (1-\alpha x)(1-\beta x)$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

   (d) Use partial fraction decomposition to show that $f(x) = \frac{1}{\sqrt{5}}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right)$. These are α and β as defined in (c), but the computation will be easier if you leave it as α and β.

   (e) Now, using the result from (d) and your knowledge about representing functions as power series, show that we can write $f(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\alpha^{n+1} - \beta^{n+1}\right) x^n$. Since we started with $f(x) = \sum_{n=0}^{\infty} F_n x^n$, you have just shown that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

   (f) Now that you have a closed-form representation of F_n, find F_{100}.

5. Consider the function $f(x)$, whose values are given in the table below.

x	$-\pi$	$\frac{-4\pi}{5}$	$\frac{-3\pi}{5}$	$\frac{-2\pi}{5}$	$\frac{-\pi}{5}$	0	$\frac{\pi}{5}$	$\frac{2\pi}{5}$	$\frac{3\pi}{5}$	$\frac{4\pi}{5}$	π
$f(x)$	1	.2	.15	.4	.8	2	1.8	1.5	1.4	1.25	1

(a) Find the Fourier series for $f(x)$ through the fifth harmonic. Again, you may use Maple to do so but you should attach all Maple work and make sure to write out what the Fourier series for $f(x)$ up to the fifth harmonic is.

(b) Plot the function you found in (a).