Duke University
Math 106L: Laboratory Calculus and Functions II

Exam 1

- Do not open this test booklet until you are directed to do so.
- You will have 105 minutes to complete the exam. When the exam is over, you must stop writing immediately, without exception.
- This exam is closed book (no reference to any people, written materials, or electronic devices including calculators), and all work must be your own. However, you may use one side of a letter-sized piece of paper with your own **hand written** notes.
- Throughout the test, in order to obtain appropriate credit, you must explain your reasoning in full. Please write in full sentences and define all variables you used.
- If you write an answer anywhere nontraditional (i.e. on the back of a page, on a different page), be sure to indicate this extremely clearly. If you want something not to be graded, X through it.
- The pages of this test have problems on both the front and the back; make sure you answer all of the questions.

Problem	Points	Grade
1	11	
2	10	
3	16	
4	10	
5	15	
6	14	
7	14	
8	10	
Bonus	6	
Total	100	

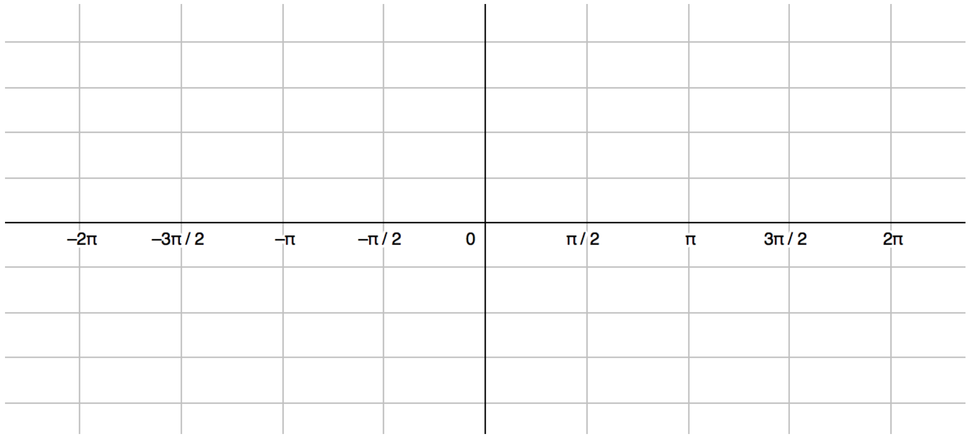
I have adhered to the Duke Community Standard in completing this assignment.

Name: ______________________________

Signed: ______________________________

1 Overture: Something we have done before

(a) (3pt) Draw a graph of $\cos(x)$ with domain $[-2\pi, 2\pi]$ using the given axis below.



period information [1pt], amplitude [1pt], shape [1pt]

(b) (2pt) Explain why $\cos(x)$ is not invertible. $\cos(x)$ is not one-to-one on $[-2\pi, 2\pi]$: $\cos(0) = \cos(2\pi)$. So, it does not pass the horizontal line test.

(c) (2pt) What domain (containing positive numbers as close to zero as possible) should you restrict $f(x)$ to in order to make it invertible?

Fill in the blank: $\boxed{[0, \pi]}$

(d) (4pt) Let's call the inverse of $\cos(x)$ on that domain $\cos^{-1}(x)$, or $\arccos(x)$.

The domain of $\arccos(x)$ is $\boxed{\text{[-1,1]}}$.

The range of $\arccos(x)$ is $\boxed{[0, \pi]}$.

2 Allemande: Let's carry this a little further

(a) (2pt) Fill in the blank: if $y = \arccos(x)$, then $x = \boxed{\cos(y)}$.

(b) (4pt) Use part (a) and implicit differentiation to show that

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{(1-x^2)}}.$$

Since $x = \cos(y)$, differentiating on both sides give $\frac{d}{dx}x = \frac{d}{dx}\cos(y)$, and thus $1 = -\sin(y)\frac{dy}{dx}$. So we have

$$\frac{dy}{dx} = -1\frac{1}{\sin(y)}.$$

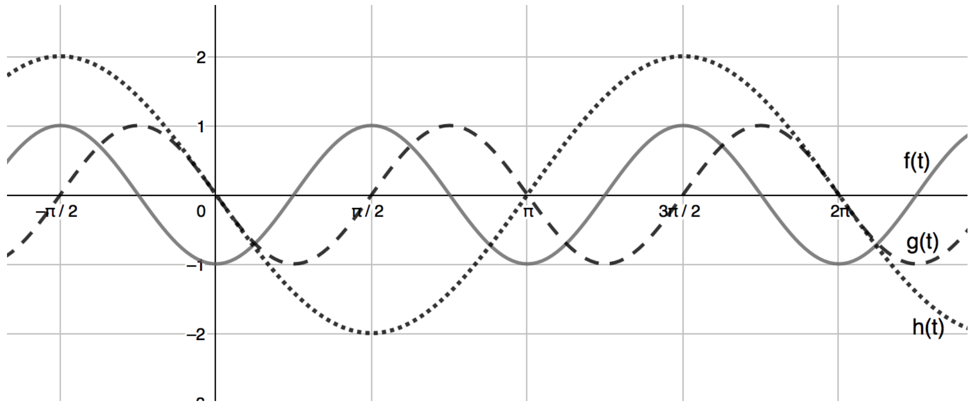
Since $\sin^2(y) + \cos^2(y) = 1$, $\sin(y) = \sqrt{(1-\cos^2(y)} = \sqrt{(1-x^2}$. Thus, $\frac{dy}{dx} = -\frac{1}{\sqrt{(1-x^2)}}$.

(c) (4pt) The domain of $\frac{d}{dx}\arccos(x)$ is $\boxed{(-1, 1)}$.

The range of $\frac{d}{dx}\arccos(x)$ is $\boxed{(-\infty, \infty)}$.

3 Sarabande: Let's get back to sin and arcsin

(a) (6pt) Match the formulas with the graphs in the following figure. Write $f(t), g(t)$ or $h(t)$ in the box.



$y = \sin(2(t - \frac{\pi}{2}))$: $\boxed{\text{g(t)}}$ $y = \sin(2t - \frac{\pi}{2})$: $\boxed{\text{f(t)}}$ $y = -2\sin(t)$: $\boxed{\text{h(t)}}$

(b) (4pt) Compute the following limit.

$$\begin{aligned}\lim_{h \to 0} \frac{\arcsin(1+h) - \arcsin(1)}{h} &= \frac{d}{dx} \arcsin(1) \\ &= \frac{1}{\sqrt{(1-1^2)}} \\ &= \infty\end{aligned}$$

The limit doesn't exist.

(c) (2pt) Find arcsin(1).
arcsin(1) is $\frac{\pi}{2}$.

(d) (2pt) Find x such that $\sin(x) = 1$. [Think: why is this different from (c).]
$\frac{\pi}{2} + 2k\pi$ for any integer k.

(e) (2pt) Compute the period of $\sin(4\pi x)$. The period is $\boxed{\frac{1}{2}}$.

4 Aria: Another important limit

(a) (2pt) Fill in the blank: $\lim_{x\to 0} \frac{\sin(x)}{x} = \boxed{1}$

(b) (6pt) Use part (a) to show that $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$. The first step has been done for you:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x} &= \lim_{x\to 0} \left(\frac{1-\cos(x)}{x} \frac{1+\cos(x)}{1+\cos(x)}\right) \\
&= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} \\
&= 1 \cdot 0
\end{aligned}$$

(c) (2pt) Compute $\lim_{x\to 0} \frac{\csc(x)-\cot(x)}{x\csc(x)}$. You may use part (b).

$$\lim_{x\to 0} \frac{\csc(x)-\cot(x)}{x\csc(x)} = \lim_{x\to 0} \frac{1-\frac{\cot(x)}{\csc(x)}}{x} = \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

5 Passepied: Choose your own score

In this problem, you can choose you your own score. Here is the rule, you can pick any number you like in $[-\pi, \pi]$, and your score will be $10\sin(x) - 5\cos(2x)$. Think you can optimize your score. We know the maximum occurs at these critical points. Let $f(x)$ denote $10\sin(x) - 5\cos(2x)$. Since

$$f'(x) = 10\cos(x) + 10\sin(2x) = 10\cos(x) + 20\sin(x)\cos(x),$$

when $f'(x) = 0$ we have $10\cos(x) + 20\sin(x)\cos(x) = 0$. This yields to

$$\cos(x)(1 + 2\sin(x)) = 0$$

Either $\cos(x)$ or $\sin(x) = -1/2$. On $[-\pi, \pi]$, $\cos(x) = 0$ when $x = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. Plugging in to $f(x)$, we have $f(\frac{\pi}{2}) = 15$ and $f(-\frac{\pi}{2}) = -5$. On $[-\pi, \pi]$, there are two values of x that yield $\sin(x) = -1/2$, but both of them are negative. Since $-5\cos(2x)$ is at most 5, $10(-\frac{1}{2}) + 5$ is at most 0. So you want to choose $x = \frac{\pi}{2}$.

6 Menuet: Now let's launch a rocket

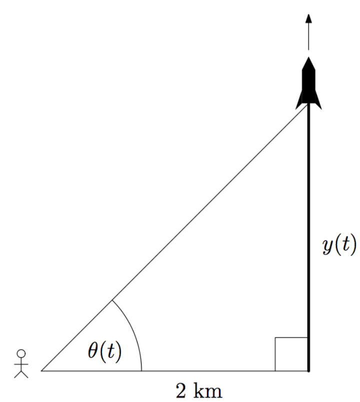
An observer watches a rocket launch from a distance of 2 kilometers. The angle of elevation θ is increasing at 3° per second at the instant when $\theta = 45°$. How fast is the rocket climbing at that instant? The following questions are guiding you to solve this problem.

(a) (3pt) We first notice that the angle measure was in degrees. Convert the two angles 3° and 45° in radians.

$$3° = 3° \times \frac{\pi \text{ radians}}{180°} = \frac{\pi}{60} \text{ radians}$$
$$45° = 45° \times \frac{\pi \text{ radians}}{180°} = \frac{\pi}{4} \text{ radians}$$

(b) (3pt) Draw a well-labeled picture of this situation, and introduce all variables that you will be using.

The figure below illustrates the situation. Let $y(t)$ denote the height of the rocket at time t. Let $\theta(t)$ denote the angle of elevation at time t.



(c) (8pt) Now answer how fast is the rocket climbing at that instant.

We know that when $\theta = \frac{\pi}{4}$ radians, $\frac{d}{dt}(\theta(t)) = \frac{\pi}{60}$ radians per second. We want to find out what is $\frac{d}{dt}y(t)$ at that instant. To answer this, we need to understand how y is related to θ. First we know that

$$\tan\theta(t) = \frac{y(t)}{2} \text{ or } y(t) = 2\tan\theta(t)$$

Taking the derivative with respect to t of both sides of the last equation in the previous step, and with chain rule, we have

$$\frac{d}{dt}y(t) = \frac{d}{dt}[2\tan\theta(t)]$$
$$\frac{dy}{dt} = 2\frac{1}{\cos^2\theta(t)}\frac{d\theta}{dt}$$

Now at the instant when $\theta = \frac{\pi}{4}$,

$$\frac{dy}{dt} = 2\frac{1}{\cos^2(\frac{\pi}{4})}\frac{\pi}{60}$$
$$= 2\frac{1}{(1/\sqrt{(}2)^2}$$
$$= \frac{\pi}{15} \text{ km/s}$$

Conclusion: the rocket is climbing at $\frac{\pi}{15}$ kilometers per second when θ is 45°.

7 Burlesca: The moving man now on the moon

An astronaut is standing on the moon. On the moon the acceleration due to gravity is 5 ft/sec^2. Imagine the astronaut jumping straight up into the air with an initial velocity A ft/sec, for some constant A.

(a) (3pt) Let $s(t)$ be the distance above the starting point as a function of time. Write down the differential equation and initial conditions that are to be satisfied by $s(t)$.

$$s''(t) = -5$$

This is the differential equation satisfied by $s(t)$. The initial conditions are $s'(0) = A$ and $s(0) = 0$.

(b) (4pt) Solve the initial value problem for the differential equation.

An antiderivative for -5 is $-5t$. Thus, all functions of the form $-5t + b$ are antiderivatives, where b is a constant.

$$s'(t) = -5t + b$$

Substituting $t = 0$ into $s'(t) = -5t + b$, we find $b = A$. Therefore, $s'(t) = -5t + A$.

An antiderivative for $-5t + A$ is $-5/2t^2 + At$. Thus, all functions of the form $-5/2t^2 + At + c$ are antiderivatives, where c is a constant. So

$$s(t) = -5/2t^2 + At + c$$

Plugging in $t = 0$ into $s(t) = -5/2t^2 + At + c$, we have $c = 0$. Therefore, $s(t) = -5/2t^2 + At$.

(c) (4pt) If the astronaut reaches a maximum height of 10ft above its starting point, what is its initial velocity A?

At the maximum height, velocity is zero, i.e. $s'(t_{max}) = -5t_{max} + A = 0$. Here t_{max} denotes the time at which the astronaut reaches the maximum height. Then we have $A = 5t_{max}$. Plugging into the distance equation yields $10 = -5/2t_{max}^2 + 5t_{max}^2 = 5/2t_{max}^2$. So $t_{max} = 2$ seconds, from which we get $A = 5(2) = 10$ ft/sec.

(d) (3pt) Using the velocity you computed from the last part, what's the velocity of the astronaut when he lands on the ground?

When the astronaut is at height $s(t) = 0$, he either just landed or is about to jump. Set $s(t) = 0$ to get $0 = 10t - 5/2t^2$. We have $0 = t(t - 4)$. When $t = 0$ he jumps off, and when $t = 4$ he just landed. The velocity $t = 4$ is given by $s'(4) = -10$ ft/sec.

8 Gigue: True or False

For the following statements, circle T if the statement is true and circle F if the statement is false.

(a) (T) If a function is periodic, then its derivative is also periodic.

(b) (F) For positive A, B, C, the maximum value of the function $y = A\sin(Bx) + C$ is $y = A$.

(c) (F) $\arcsin(\sin(\pi)) = \pi$.

(d) (F) The differential equation $y''(t) = t^106 + 21t + 3$ has order 106.

(e) (T) The function $h(x) = tan(x)$ is invertible on the domain $1.8\pi \leq x \leq 1.9\pi$.

9 Bonus: Let's mix arccos with tan

Let $f(x) = \tan(\arccos(x))$.

(a) What is the domain of $f(x)$? To help you answer this question, fill in the following blanks.

The domain of $\tan(x)$ is $\boxed{(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)}$.

$\arccos(x)$ outputs a number within that domain when x is in $\boxed{(0, \pi] \text{ and } [-\pi, 0)}$

So, the domain of $f(x)$ is $\boxed{(0, \pi] \text{ and } [-\pi, 0)}$.

(b) What is the range of $f(x)$? Explain your answer. The range of $f(x)$ is $(-\infty, \infty)$. Since the range of arccos on the domain in part (a) is $[0, \pi]$ except $\frac{\pi}{2}$, tan takes values from $-\infty$ to ∞ on that domain.

(c) Is $f(x)$ invertible? Explain your answer. Yes. On $(0, \pi]$, $\arccos(x)$ is decreasing and takes values from $[0, \frac{\pi}{2})$. $\tan(x)$ on $[0, \frac{\pi}{2})$ is increasing and takes values from $[0, \infty)$. Similarly, on $[-\pi, 0)$,$\arccos(x)$ is decreasing and takes values from $(-\frac{\pi}{2}), 0]$. $\tan(x)$ on $(-\frac{\pi}{2}), 0]$ is increasing and takes values from $(-\infty, 0]$. Therefore, $f(x)$ is invertible.