This document is prepared based on *An Introduction to Abstract Mathematics* by Robert J. Bond and William J. Keane.

1 The notion of a set

A *set* is a collection of objects[1].
The objects in a set are called *elements*. We use the notation $a \in A$ to express that the object a is an element of the set A[2].

Examples

1. The elements may simply be listed. For example, $\{1, 2, 3, 4, 5\}$ is the set consisting of the first five positive integers. Putting the elements of a set in brackets $\{\}$ is a standard notation.
2. $\mathbb{R}$ is the set of real numbers. $\mathbb{Z}$ is the set of all integers, which may also be written $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
3. $\{1, 2, 3, 4, 5\}$ may also be written $\{n \in \mathbb{Z} \mid 1 \leq n \leq 5\}$. The symbol $|$ is read "such that". The expression before the symbol $|$ should introduce the variables used to represent the elements of the set and the expression after the $|$ symbol should be an open sentence. For example, $\{x \in \mathbb{Z} \mid x \text{is even}\}$ is the set of even integers and here the open sentence is "x is even".
4. Can you think of other examples of a set?

Exercise In class today, we said a cake is a set.

1. Suppose a cake can be represented as a unit interval. Can you write down its bracket notation?
2. Suppose a cake is represented as a rectangular box in 3 dim. Can you write down its bracket notation?
3. In general, when we say "Let S be a cake to be divided," S is an arbitrary set.

2 Size of a set

An important property of set is its *order*, which is the number of elements in the set.

Definition Let A be a set. The order of A is the number of elements in A. Note that the order can be infinite.

Exercise Write down the order of each set in the examples above.

3 Subsets

Definition Let A and B be sets. We say that A is a subset of B and write $A \subseteq B$, if every element of A is also an element of B.
If A is a subset of B and $A \neq B$, we write $A \subset B$ and say A is a proper subset of B.

[1] We will not give a formal definition of set. Instead, this word will be a basic, undefined term, part of our intuitively understood vocabulary

[2] You should realize that this idea of membership in a set is also undefined, part of the concept of set

Examples

1. $\{1, 2, 3\} \subset \{0, 1, 2, 3, 4\}$.
2. $\mathbb{Z} \subset \mathbb{R}$.
3. $\{x \in \mathbb{Z} \mid x \text{is even}\} \subset \mathbb{Z}$.
4. Let S be a cake to be divided. The lion cuts it into two slices s_1, s_2. s_1 and s_2 are both sets. Furthermore, $s_1 \subset S$ and $s_2 \subset S$.

4 Combining Sets: unions and intersections

It is possible to create new sets out of two or more given sets. The two most common ways of doing this are given in the following definition.

Definition Let A and B be sets. The *union* of A and B, denoted $A \cup B$, is $\{x \mid x \in A \text{or} x \in B\}$. The *intersection* of A and B, denoted $A \cap B$, is $\{x \mid x \in A \text{and} x \in B\}$.

Examples

1. $\{1, 2, 3\} \cup \{4, 5\} = \{1, 2, 3, 4, 5\}$.
2. $\mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$.
3. Let S, s_1 and s_2 be given as above. $S = s_1 \cup s_2$.

Exercise Suppose we now divided the cake S into n slices, $s_1, s_2, \ldots, s_n$. How can you write S in terms of $s_1, s_2 \ldots, s_n$?

5 Power Set

An important example of a set of sets is the *power set* of a set A.

Definition Let A be a set. The power set of A, written as $\mathbf{P}(A)$, is $\{X \mid X \subseteq A\}$.

Exercise Let $A = \{1, 2, 3\}$. Write down the power set of A.

Exercise (Challenge yourself) Let A be a set of order n. What's the order of its power set? In other words, how many subsets does A have?

6 Partition

Definition A partition of a set divides the set into different nonempty subsets so that every element of the set is in one of the subsets and no element is in more than one.

Exercise Given S, s_1, s_2 as before, is $\{s_1, s_2\}$ a partition of S?

7 A weird set

If you want something really fun, google "Cantor set" and see what you find.