Get into a group of four and play among yourselves.

1 Some notations

S is the entire cake. You guys are P_1, P_2, P_3, P_4. So far we have a partition $\{X_1, X_2, X_3, X_4, R\}$ of S.

Definition (α advantage) Given notations as above, we say P_1 holds an α advantage over P_1 if

$$\mu_1(X_1) - \mu_1(X_2) = \alpha > 0.$$

Definition (R advantage) Given notations as above, we say P_1 holds an R advantage over P_1 if

$$\mu_1(X_1) \geq \mu_1(X_2 \cup R).$$

2 Brahms-Taylor's envy-free algorithm for 4 players

As we said in class, there are two basic algorithms that are used repeatedly in Brahms and Taylor's method. These basics algorithms help us to divide part of the cake envy-free among the four players, and drive the remaining portion small. Let's first have a look on these two basics ones.

2.1 Algorithm I

Given a piece of cake A, 4 players and an $\epsilon > 0$, a subset B of A can be divided among the four players in an envy-free way so that the remaining portion $R = A - B$ has value smaller than ϵ for all four players.

1. P_1 cuts A into 5 equal pieces.
2. P_2 identifies 3 largest pieces, and trims the larger two so that 3 equal pieces result. Set the trims aside.
3. P_3 identifies the largest 2 pieces, and trims the larger one so that 2 equal pieces result. Set the trims aside.
4. Choose in the order of P_4, P_3, P_2, P_1 with restriction that both P_2 and P_3 must choose a piece they trimmed if it is available.
5. Repeat on the reminder and let each player perform the role of P_1¿

Why envy-free? P_4 is the first to choose. P_2, P_3 get one of their largest. P_1 get an untrimmed piece.
Now think why the remaining portion R can be less than ϵ for any players and any given ϵ?

2.2 Algorithm II

Given two pieces of cake A, B. If P_1 views A and B as equal, but P_2 views B as larger, then P_1 can be assigned part of A, P_2 can be assigned part of B, and P_3 and P_4 can be assigned part from $A \cup B$ so that the division of the four pieces is envy-free. Furthermore, P_1 and P_2 will both hold an $\alpha > 0$ advantage over each other for some α.
Assume that $\frac{\mu_2(B)}{\mu_2(A)} > \frac{10^k+10}{10^k}$ for some positive integer k. Suppose that $\mu_2(B) > 10^k + 10, \mu_2(A) < 10^k$.

1. P_1 cut A into $10^k + 2$ equal pieces. Then there exists A_1, A_2, A_3 from A such that $\mu_2(A_1) \leq \mu_2(A_2) \leq \mu_2(A_3) < 1$. Why?
   If there are at least 10^k pieces that are larger than 1, then $\mu_2(A) \geq 10^k$. Contradiction.
2. P_1 cut B into $10^k + 3$ equal pieces. Then there exists B_1, B_2, B_3 from B such that $1 < \mu_2(B_3) \leq \mu_2(A_3) \leq \mu_2(A_1)$. Why?
   Order pieces in B as $\mu_2(B_1) \geq \mu_2(B_2) \geq \mu_2(B_3) \ldots \mu_2(B_{10^k+3})$. If $\mu_2(B_3) > 1$, then we are done. If $\mu_2(B_3) \leq 1$, then $\mu_2(B_1) > 3$. Now P_2 cuts B_1 into 3 equal pieces, $B_1' \cup B_2' \cup B_3'$. Note P_1 views pieces from A as equal and larger than or equal to pieces in B.

3. P_3 identifies the largest 2 pieces among the six, and trims the larger one so that 2 equal pieces result. Set the trims aside.

4. Choose in the order of P_4, P_3, P_2, P_1 with restriction that P_3 must choose the piece she trimmed if it is available.

Why envy-free? P_2 choose from B's and P_1 choose from A's.
Now talk to your group what is α?

2.3 Scramble-it-up

1. P_1 cuts the entire cake into 4 equal pieces.

2. If there is a total agreement, we are done. Else, apply algorithm II to P_1 and P_2 with α_1 advantage among P_1, P_2

3. Apply algorithm I to the remaining cake until the remainder R has value less than α_1 for all players.

4. P_1 cuts R into 12 equal pieces.

5. Each of the other players declares to be
   type I: if she agrees all the pieces are the same size
   type II: if she disagrees

6. If type II players have R advantage over type I players. Then give the 12 pieces to type I players.

7. Otherwise, choose any pair that are non dominant to each other and repeat the process

Now talk to your group why 12 equal pieces? Convince yourself this is envy-free.