This document is prepared based on *An Introduction to Abstract Mathematics* by Robert J. Bond and William J. Keane.

1 Induction: a method of proof

Induction is a method of proving statements about the positive integers. Let $P(n)$ be such a statement. $P(n)$ may be a formula, such as "the sum of the first n positive integers is $n(n+1)/2$" or "the sum of the first n odd positive integer is perfect square." Or $P(n)$ may be a statement such as "every polynomial of degree n with real coefficients has at most n zeros."
The purpose of an induction proof is to show that a statement $P(n)$ is true for every positive integer n. The first step is to verify $P(1)$ is true: that is the statement is true when $n = 1$. Once $P(1)$ is established to be true, we want to verify $P(2), P(3), P(4)$, and so on. Since there are infinitely many of these statements, they cannot all be verified separately. In an induction proof, we show that, given a positive integer k for which $P(k)$ is true, it follows that the statement $P(k+1)$ is true. This establishes that in the sequence of statements $P(1), P(2), P(3), \ldots$, whenever one statement in this sequence is true, then the next statement in the sequence must also be true.

Theorem First principle of mathematical induction. Let $P(n)$ be a statement about the positive integer n. Suppose that

1. $P(1)$ is true.
2. Whenever k is a positive integer for which $P(k)$ is true, then $P(k+1)$ is also true.

Then $P(n)$ is true for every positive integer n.

Example Prove the formula for the sum of the first n positive integers

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

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Since the formula is true for $n = 1$, $P(1)$ is true. Therefore, condition 1 of the induction principle holds. Suppose now that k is a positive integer for which $P(k)$ is true. Then, the induction hypothesis is

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

. We now want to establish that $P(k+1)$ is true, which is equivalent to showing that

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

. Adding $k+1$ to both sides of the induction hypothesis, we get

$$\begin{aligned}
1 + 2 + 3 + \cdots + k + (k+1) &= \frac{(k+1)(k+2)}{2} + (k+1) \\
&= \frac{k^2+k}{2} + \frac{2k+2}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Hence $P(k+1)$ is true. It now follows by induction that $P(n)$ is true for every positive integer n.

Exercise Prove the formula for the sum of the first n positive integers square.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

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