An introduction to fair division and cake-cutting algorithms
Lecture notes

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How to divide a cake fairly among two kids? If you are like my dad, you would have answered without hesitation, “Cut along the diameter of the cake, and give each a half.” Yes, doing so guarantees the same amount of cake for each half. But what if these two kids disagree on the value of the pieces? For example, this cake may be half strawberry and half chocolate. If one kid likes chocolate, she won't be happy if the piece assigned to her is all or mostly strawberry - even if she gets the same amount of cake. Should we give her the entire chocolate piece then? What if the other kid also likes chocolate? Should we cut the cake so that the pieces are exactly the same? Or should we respect each kid's own preference? Which is fair?

You see, fairness is hard to define. People have different opinions on what fairness means. Think about the fairness issues surround us. Was our rent split fairly? Was the property of a divorce settled fairly? Are we fairly taxed? Are congressional districts fairly drawn? Is this international treaty fair to countries involved? Were country borders fairly divided? We hardly agree on anything, among which some ends up with a broken friendship, and some costs life. Can we quantify “fairness” in some way so that it can be studied mathematically and thus leads us to an agreeable ground for all?

This course aims to look at the various mathematical tools (graph theory, complexity, number theory, combinatorial topology, etc) in addressing problems that arise from fair division. The first three days examine one setting, simple fair division that is fairly dividing a cake among n players so that each player thinks she gets at least $1/n$ of the cake. We will see that mathematics not only proves the existence of an agreeable ground, but also helps us find out how to reach it. For the second half of the course, we play with some variations on the theme of fair sharing. We will look at cases when there is an unequal share (for example, one kid only wants a quarter of the cake), when the fairness criteria is different (for example, how to cut the cake so that each player thinks she gets the largest slice), when fair division is applied to solve rental dispute. And there are many more! You will explore some other aspects of the fair division problem yourself in group projects.

I think we are now ready to begin our journey of fair division. Bon voyage!

1 The simple fair division problem

We have seen that what makes fair division difficult (or rich, you may say) is that players disagree on the values of different portions of the cake. In order to reach an agreement, the player must think the division is fair by *her own assessment*. To model a player's assessment, we define a utility function, that assigns to any piece of cake a number that measures its worth to that player. This utility function has a special name - mathematicians like to call them “measure” [1]. Let μ denote the measure for this player. Then, for instance, $\mu(S) = 1$ means the value of the entire cake S in her eyes is 1. Now think what does $\mu(S) = 0.3$ means? We will explore some properties of μ in the group project session later. For now, let's see how can we use this new tool. With this measure, we can properly define the *simple division* problem as follows.

Definition (Simple division). Let S denote the cake to be shared, and P_1, P_2 are two players with measures μ_1, μ_2 respectively. The simple division problem is to divide S into s_1 and s_2 such that $\mu_i(s_i) \geq \frac{1}{2}$, $i = 1, 2$.

The slices s_1 and s_2 from the simple fair division problem are called simple fair shares. We give a formal description as follows.

[1] Later in life, you will find yourself having a full semester course on measure theory. Measures are very fun to play with, and make the rigid Euclidean system a little bit softer.

Definition (Simple fair share) Let P be a player with measure μ, s is a simple fair share to player P if $\mu(s) > \frac{1}{2}$.

Exercise Alice and Bob are sharing a cake. Suppose that the cake is divided into two slices, s_1, s_2. The following table gives each player's measure on each slice.

	s_1	s_2
Alice	0.3	0.7
Bob	0.6	0.4

Which of the two slices is a fair share to Alice? Which of the two slices is a fair share to Bob? Find a simple division of the cake using s_1, s_2.

1.1 Cut and Choose

The exercise above suggests we can check whether a cut is a simple fair division, but it doesn't tell us *how* to get the two slices. The place to start discussing any procedural solution of simple fair division is an ancient method called cut and choose. The method is extremely simple, and we give its formal descriptions as follows.

Algorithm 1: Cut and Choose Method

Either person cuts what she considers equal halves;
The other person chooses, leaving the other share to the cutter

Exercise There are many interesting questions can be asked here, for example,

1. Why is the cutter guaranteed a fair share in this method? Why is the chooser guaranteed a fair share?
2. Why will the cutter always split S in a way she determines to be 50/50?
3. Would you rather be the cutter or the chooser?
4. Now try to come up with your own questions. What would happen if...?

1.2 Carol comes to visit

You may have written down, "What would happen if there are more than two players?" Let's begin with something easier to bite, what would happen if Carol comes to visit and we have three players to share the cake. The first algorithm, the moving knife method, is due to Dubins and Spanier. This time we assume the cake to be cut is rectangular-shaped.

Algorithm 2: Moving Knife

Referee slides the knife from left to right;
Any one who thinks the left piece has reached 1/3 of the cake says "stop" and gets the left piece;
The other two players use cut and choose method for the remaining piece.

Is everyone guaranteed a fair share using this method? The first person who says "stop" (suppose it's Alice) thinks the slice she gets is worth 1/3 of the cake, so Alice has a fair share. The other two players (Bob and Carol) both think the slice Alice gets is less than 1/3 of the cake, because otherwise they would say "stop" before Alice. Hence, they both think the remaining piece is worth at least 2/3 of the cake. Now using cut and choose, Bob and Carol are guaranteed to have a half of the remaining piece, which is 1/3 of the original cake. Therefore, the moving knife solves the simple fair division problem for three players.

How does the moving knife algorithm differ from the cut-choose method we discussed earlier? One critical difference is that Cut and Choose only requires two decisions - one person decides where a half is and cut it, while the other evaluates the two pieces and chooses the larger. The algorithm that only requires a finite number of decisions is called a *finite* algorithm. Moving Knife, however, is not a finite algorithm. One has to evaluate the left piece at every instant of time while the knife moving continuously from the left to right. Because of this, the Moving Knife method will be called a *continuous* algorithm.

Exercise What else can you do when there are only two players? Should players be honest in this game? Who seems to get a better deal among the three? Can you think of any other interesting questions?

The moving knife and the cut-choose method are the two cornerstones in cake cutting algorithms. Many procedural work in fair division were developed later based on these two fundamental ones. The next algorithm is a modification of the moving knife method that is finite. The goal is to identify where the first person would have said "stop" without moving the knife. This algorithm is called the trimming method and is due to Knaster and Banach in the 1940s.

Algorithm 3: Trimming Algorithm

Player P_1 cuts a slice of size 1/3 from the cake ;

The cut slice is passed to P_2. If P_2 values it more than 1/3, then she trims it so the reduced value is exactly 1/3 ;

The slice (whether trimmed or not) is passed to P_3. P_3 takes it, if she considers it at least 1/3 of the cake. Otherwise the slice is given to the last player who cuts it. The player receiving this piece drops out ;

Use cut and choose on the remaining portion of the cake.

Exercise Is everyone guaranteed a fair share using this method? What other questions do you want to explore?

The last algorithm we will look at today is called the "Successive Pairs Algorithm". It gives a series of pairwise divisions among Alice, Bob and Carol. We first have Alice and Bob divide the cake using the cut and choose method (suppose Alice cuts and Bob chooses). Then Alice and Bob each has at least 1/2 of the cake from their view. Now they need to give up some of their share to Carol. They cut each of their share into 3 equal slices, so each slice is at least 1/6 to their assessment. Carol picks one slice each from both Alice and Bob and Alice and Both keep the rest two slices. We know Alice gets exactly 2/3 of exactly 1/2 by her own assessment. We also know Bob gets exactly 2/3 of at least 1/2 by her own assessment. For Carol, she gets at least 1/3 of Alice's half and Bob's half, whose sum is the original. Hence, Carol also gets at least 1/3.

Exercise Can you give a formal description to the "Successive Pairs Algorithm"? What other questions do you want to explore?

Exercise Can we extend these algorithms to an arbitrary number of players? (Hint: Use mathematical induction.)