A warm-up exercise

Cain and Abel by Peter Paul Rubens

On a piece of paper, write down the definition of simple fair division problem for *n* players. Think how many algorithms can you solve this problem now.

Envy rears its ugly head

- Simple fair division for *n* players

Everyone is guaranteed 1/n of the cake by their own assessment.

- Envy-free division for n players

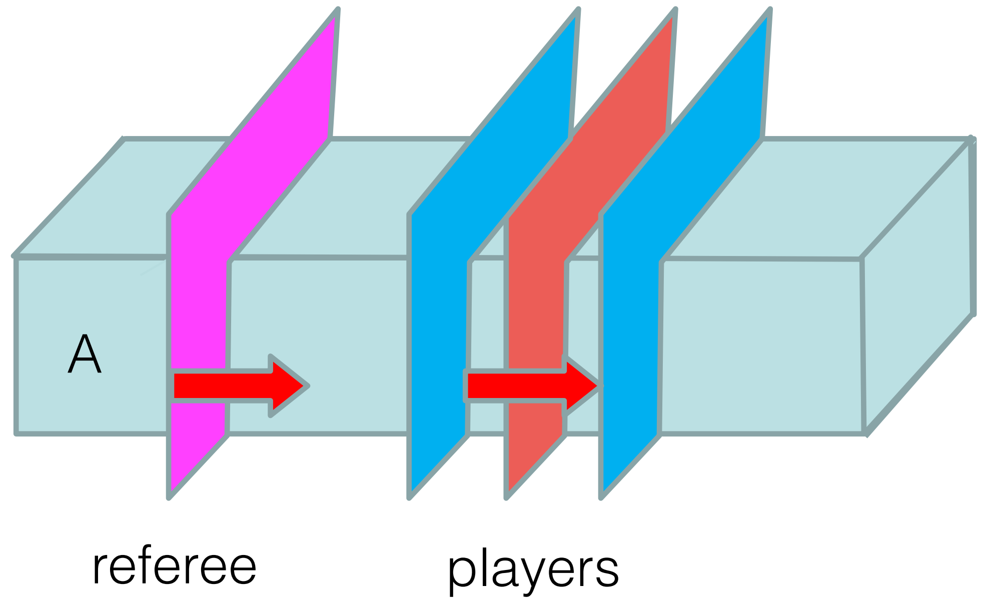
no player feels another has a strictly larger piece

- A mathematical translation:

- Is any of the methods we learned so far envy-free?

Envy-free for 3 players

4 Moving Knives by Stromquist

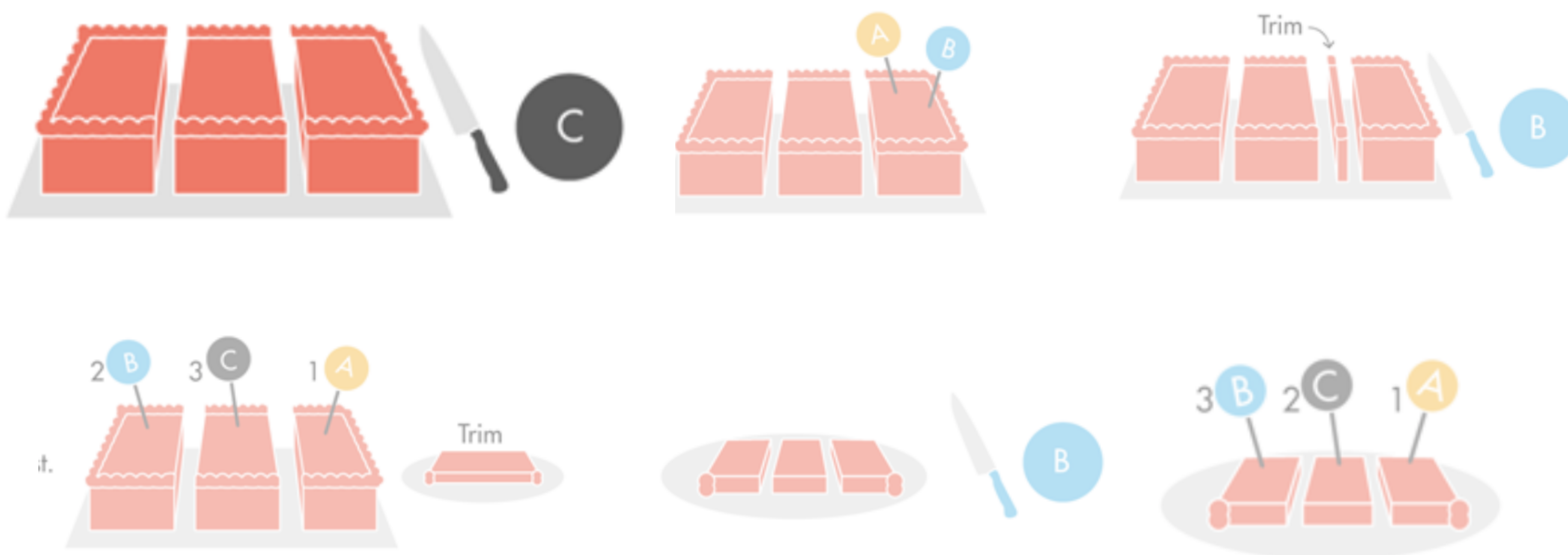


Trick: When any player shouts “cut”, the cake is cut by the referee knife and the knife in the middle.

Important! The player who says “cut” knew what all three pieces would be.

Envy-free for 3 players

A finite algorithm by J. Conway, R. Guy, J. Selfridge

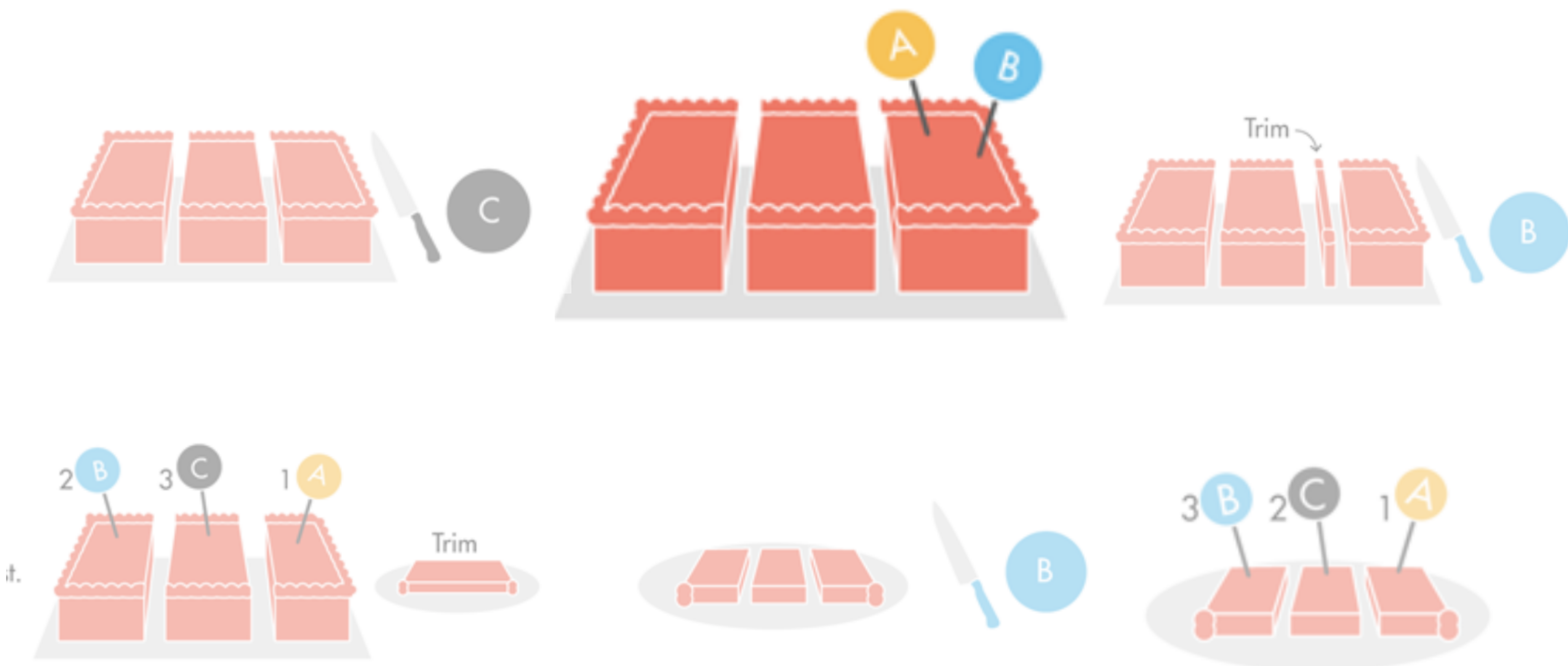


1. C cuts the cake into three equal slices.

Adapted from Quanta Magazine
https://www.quantamagazine.org/new-algorithm-solves-cake-cutting-problem-20161006/

Envy-free for 3 players

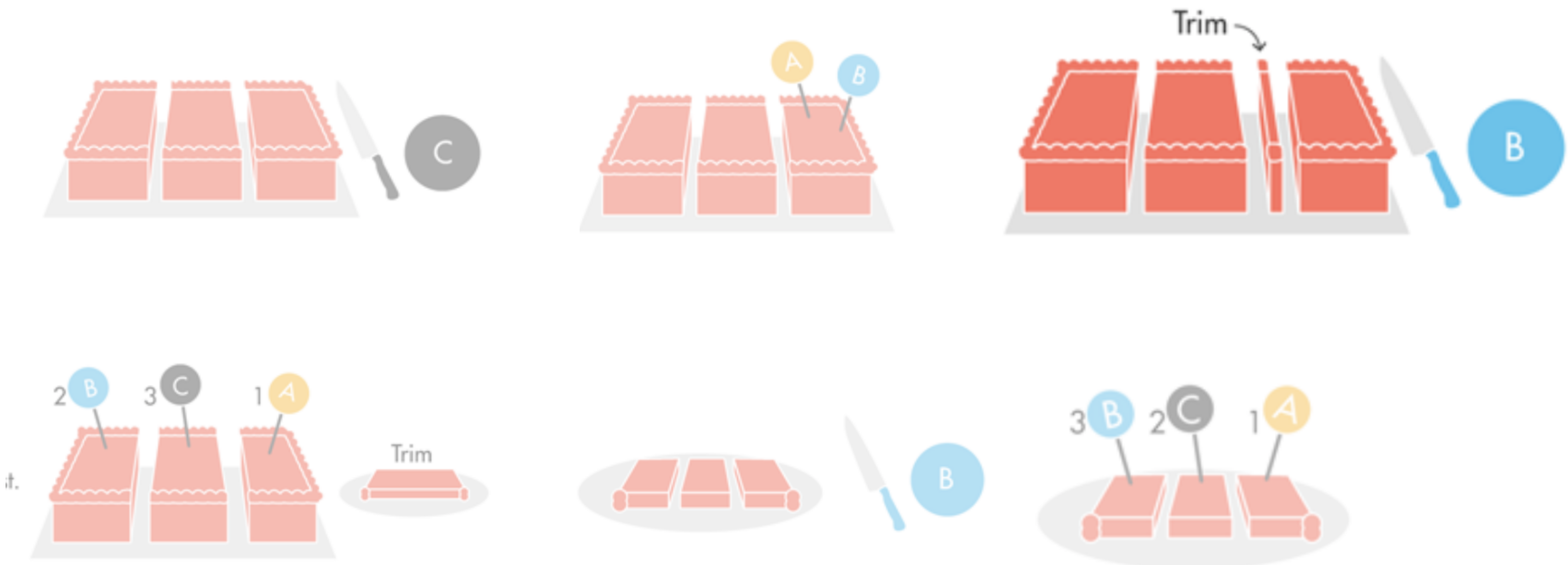
A finite algorithm by J. Conway, R. Guy, J. Selfridge



2. A and B identify their first choices.

Envy-free for 3 players

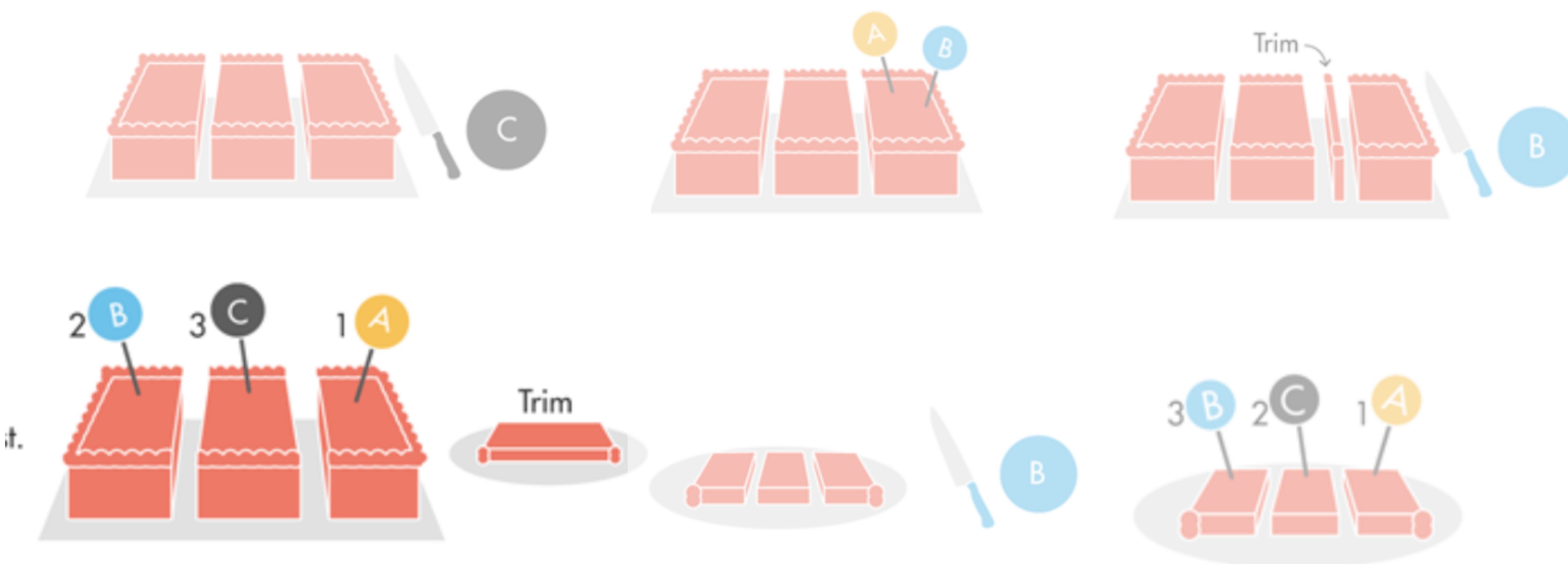
A finite algorithm by J. Conway, R. Guy, J. Selfridge



3. If A and B identify the same choice, B trims her first choice to match her second favorite piece.

Envy-free for 3 players

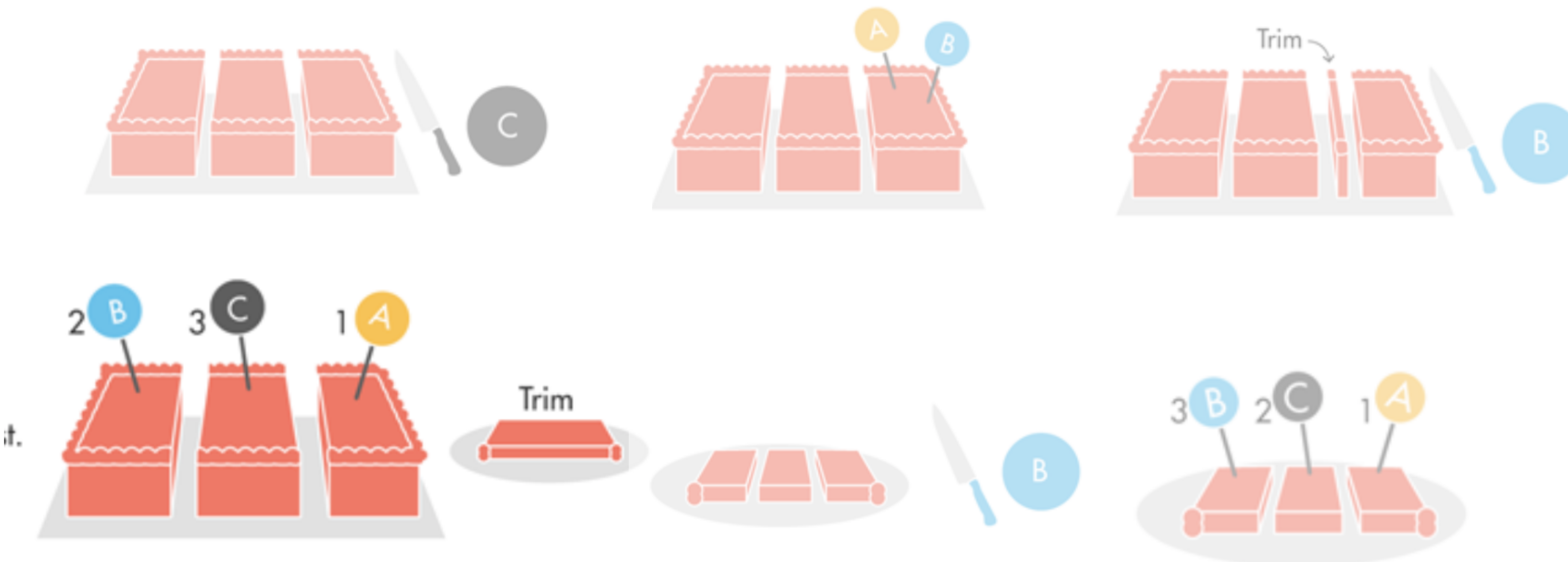
A finite algorithm by J. Conway, R. Guy, J. Selfridge



4. Put the trim to one side and choose in the following order: A > B > C.

Envy-free for 3 players

A finite algorithm by J. Conway, R. Guy, J. Selfridge



So far, it's envy free.

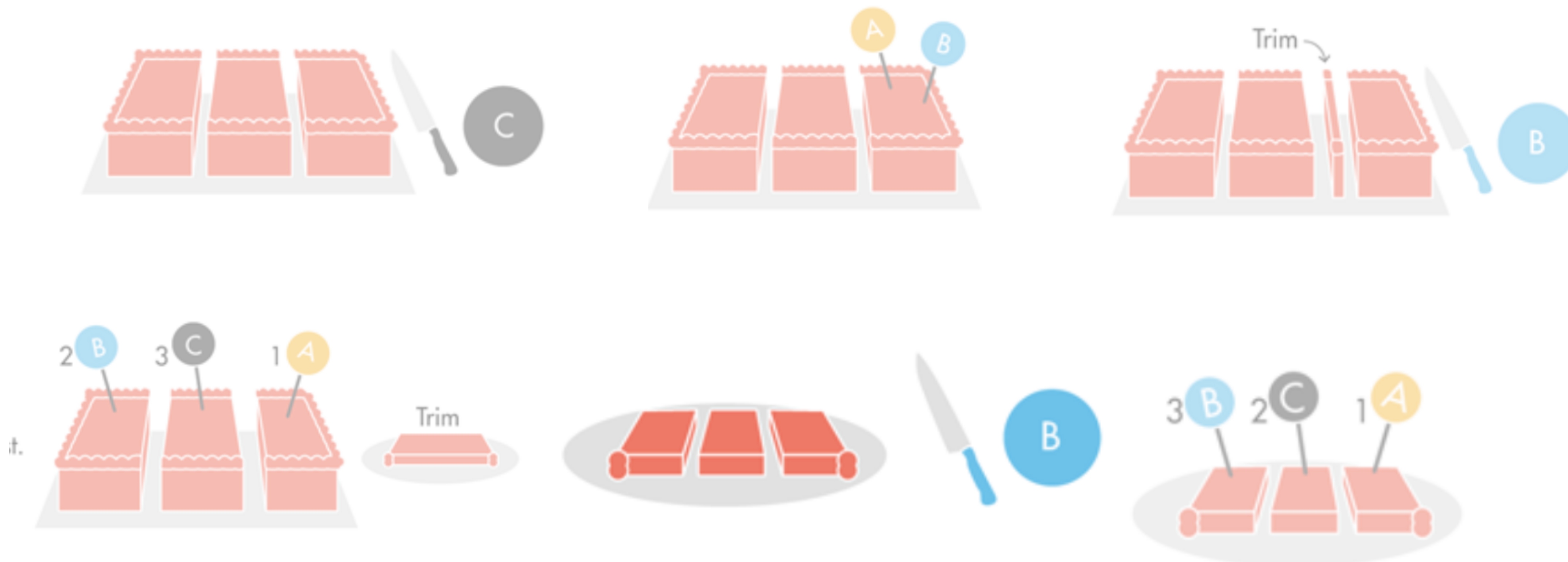
A:

B:

C:

Envy-free for 3 players

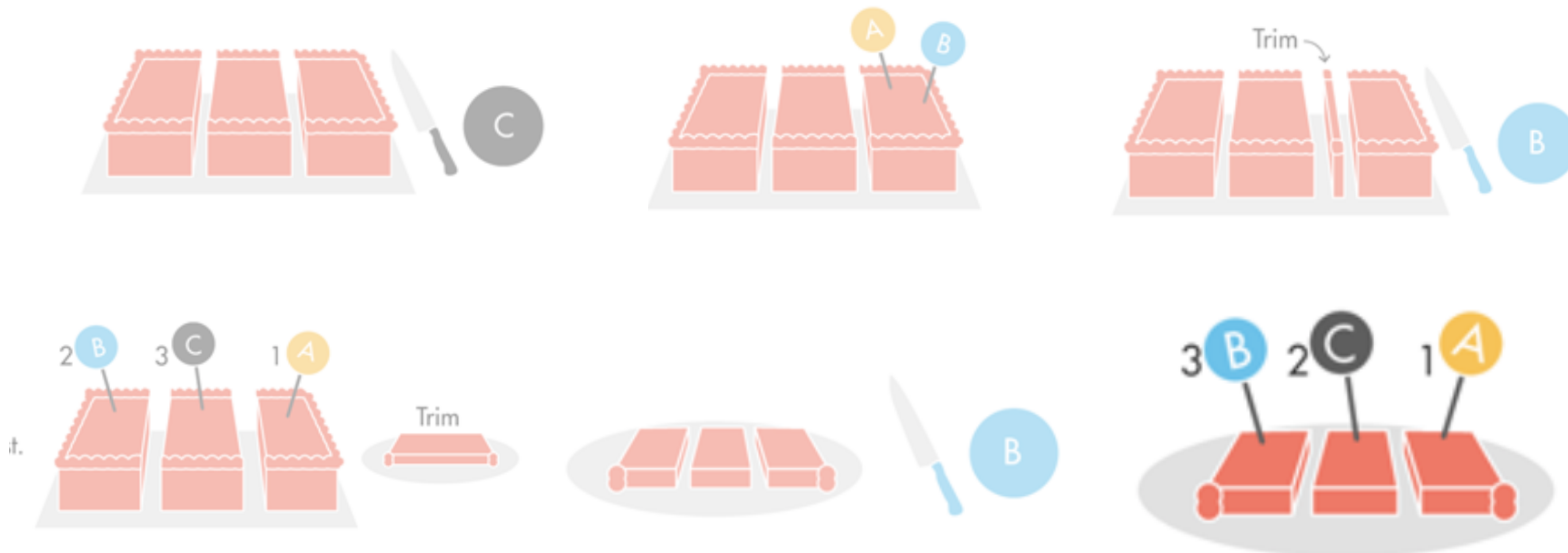
A finite algorithm by J. Conway, R. Guy, J. Selfridge



5. B cuts the trim into three equal pieces.

Envy-free for 3 players

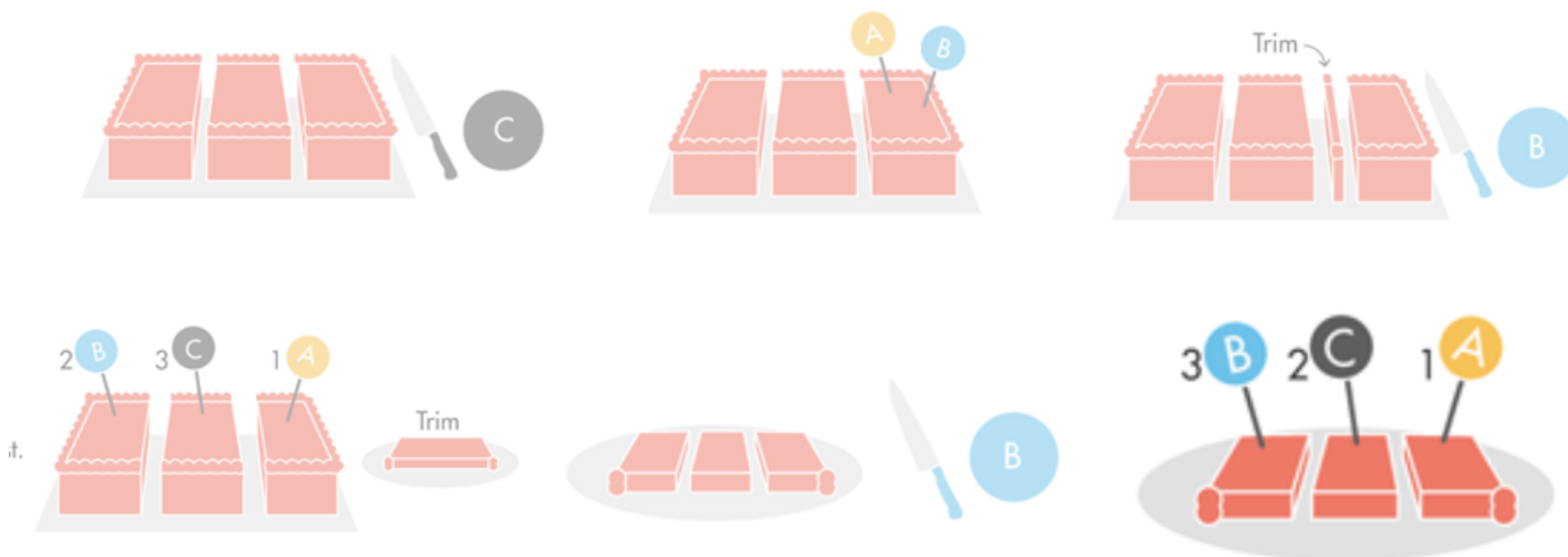
A finite algorithm by J. Conway, R. Guy, J. Selfridge



6. Choose in the order: A > C > B.

Envy-free for 3 players

A finite algorithm by J. Conway, R. Guy, J. Selfridge



It's envy-free.

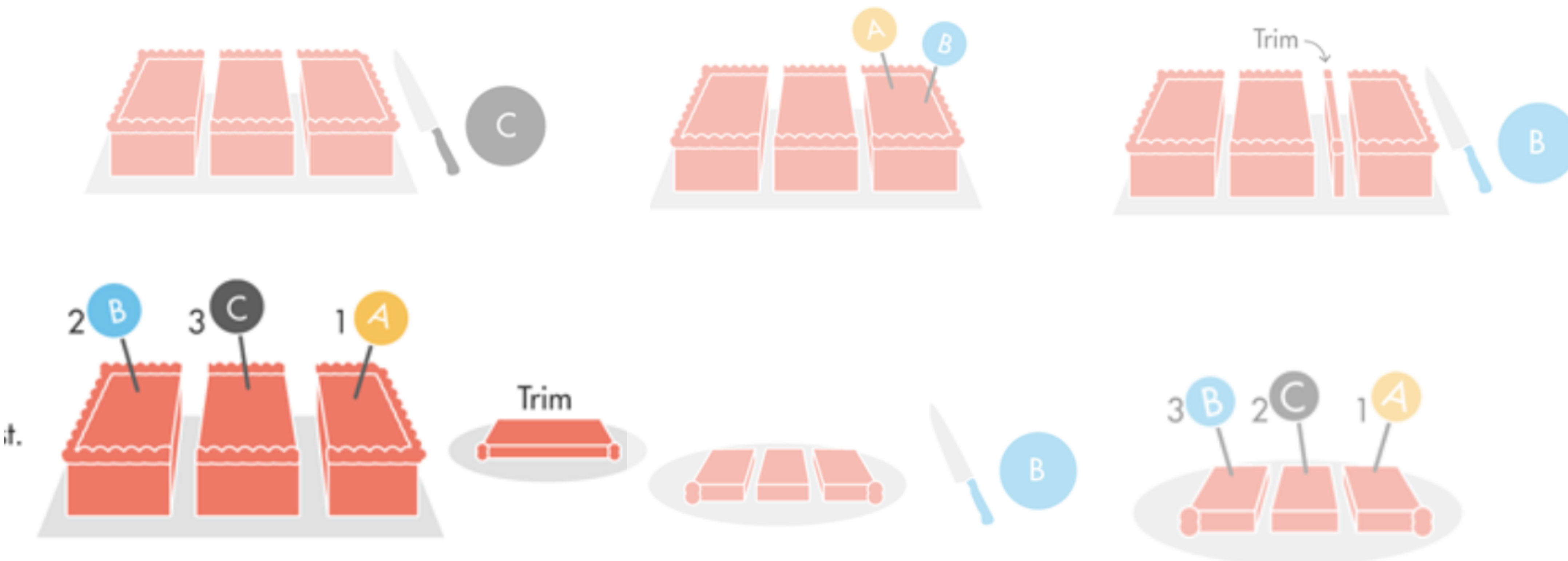
A:

B:

C:

Envy-free for 3 players

A finite algorithm by J. Conway, R. Guy, J. Selfridge



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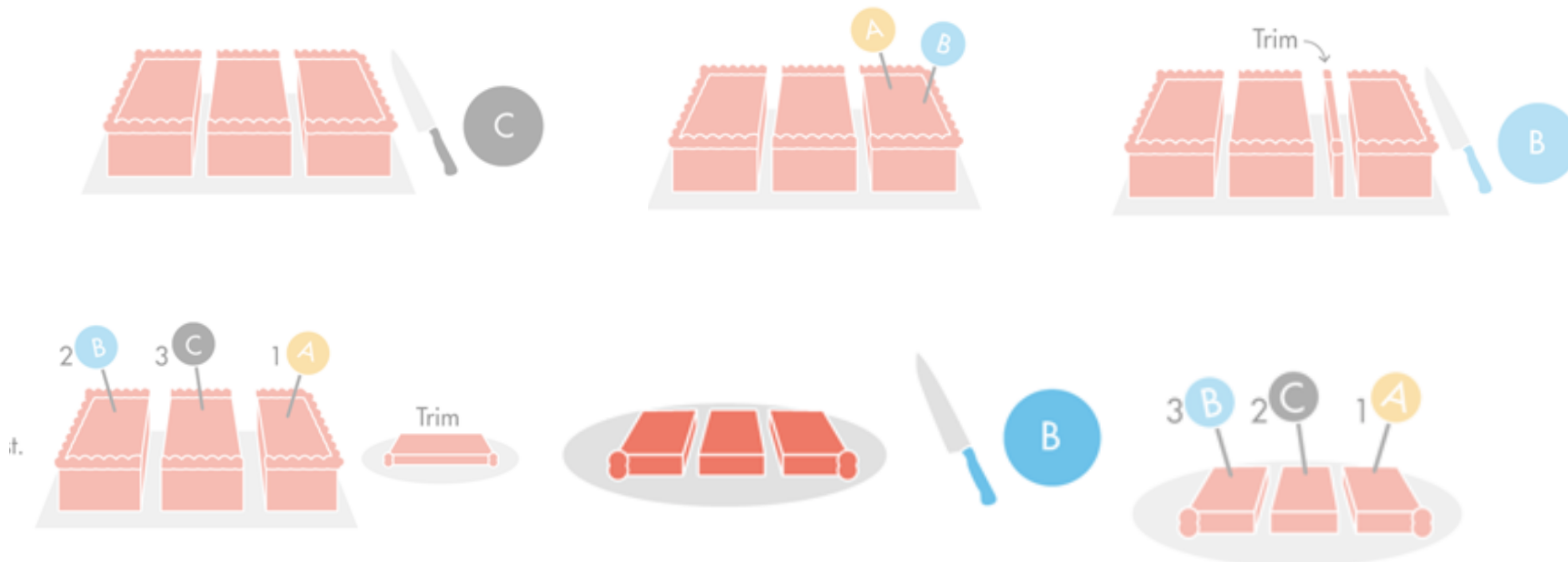
Envy-free for 3 players

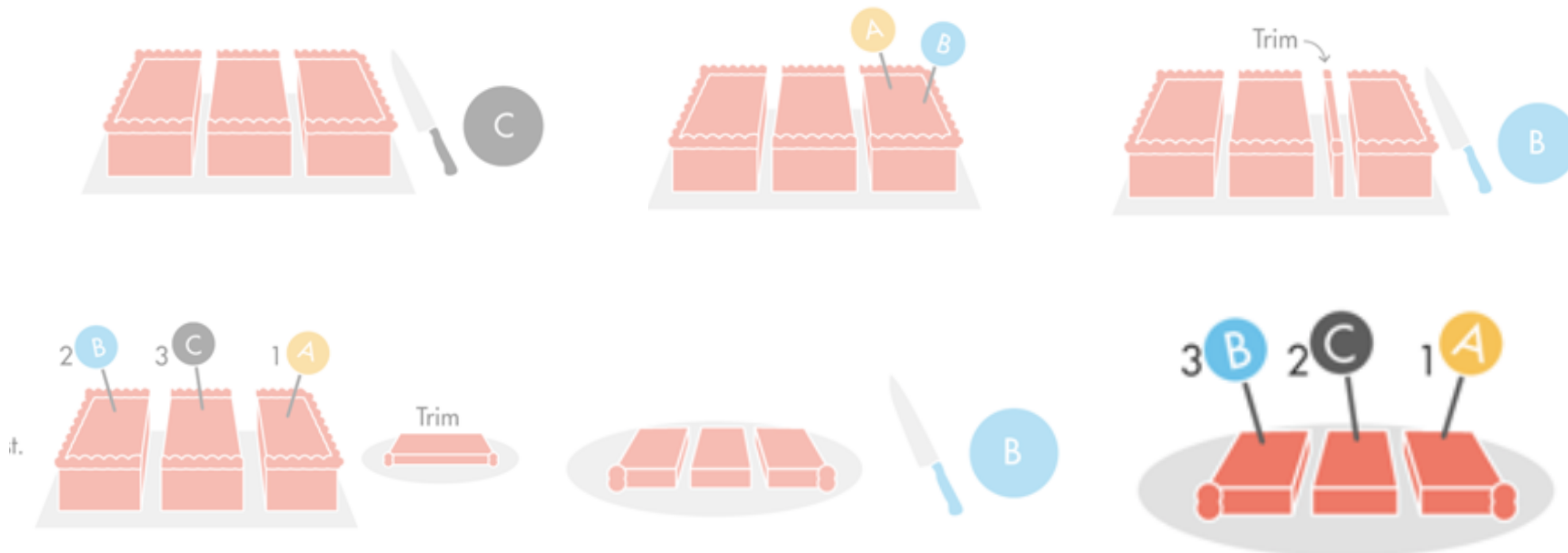
A finite algorithm by J. Conway, R. Guy, J. Selfridge



5. A cuts the trim into three equal pieces.

Envy-free for 3 players

A finite algorithm by J. Conway, R. Guy, J. Selfridge



6. Choose in the order: A > C > B.

Four envy-free glasses of wine

Player A pours (possibly all) the wine equally into glasses G_1 through G_4. B's first choice B_1 is in G_1, B_2 in G_2.
Case 1. C_1 in G_3. B evens off B_1 and B_2 by pouring out of B_1 into a small carafe. The glass that B pours out of is now marked B^* meaning that B takes this if it remains at his turn to choose. C evens off C_1 and C_2 by pouring out and marking C^* on G_3. Now D then C then B choose.
Case 2. C_1 in G_1, C_2 in G_3. Either B evens off B_1 and B_2 or C evens off C_1 and C_2, whichever would pour the most out of G_1. (A pourout by B followed by a pourout by C may be necessary. In cake cutting, it would be better if B and C could agree who would trim the most before actually cutting.) Assume without loss that B evens off and G_1 is marked B^*. Since C_1 is now in G_3, finish this case by using the last two sentences of Case 1.
Case 3. C_1, C_2 are in G_1, G_2 in some order. Either B evens off G_2 and B_3 or C evens off G_2 and C_3, whichever would pour the <u>least</u> out of G_2. (A pourout by B followed by a slight refilling by C may be necessary. In cake cutting, it is especially bad if we can't avoid the larger trim.) Now whoever evened off, evens off G_1 to make three glasses even. Assume without loss that B_1, B_2, and B_3 are even, Mark A^* on B_4 so that B won't get B_4. C_1 is in G_2 or G_1 and C evens off C_1 and C_2 by pouring out of C_1 and marking it C^*. Now D then C then A choose.
No more cases. C_1 in G_2, C_2 in G_3 is Case 1 in disguise.

J. L. Selfridge
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DeKalb, IL 60115

June 11, 1992

Walter: Good to see you last month.

John

envelope said: c/o Rdch. K. Guy, Dept Math & Stat, Univ. Calgary, 2500 Univ Dr. NW, Calgary Alberta Canada T2N 1N4

Envy-free for n players

Very hard problem!

1995 Brams and Taylor

Key: dominance — reduce # of players that we envy

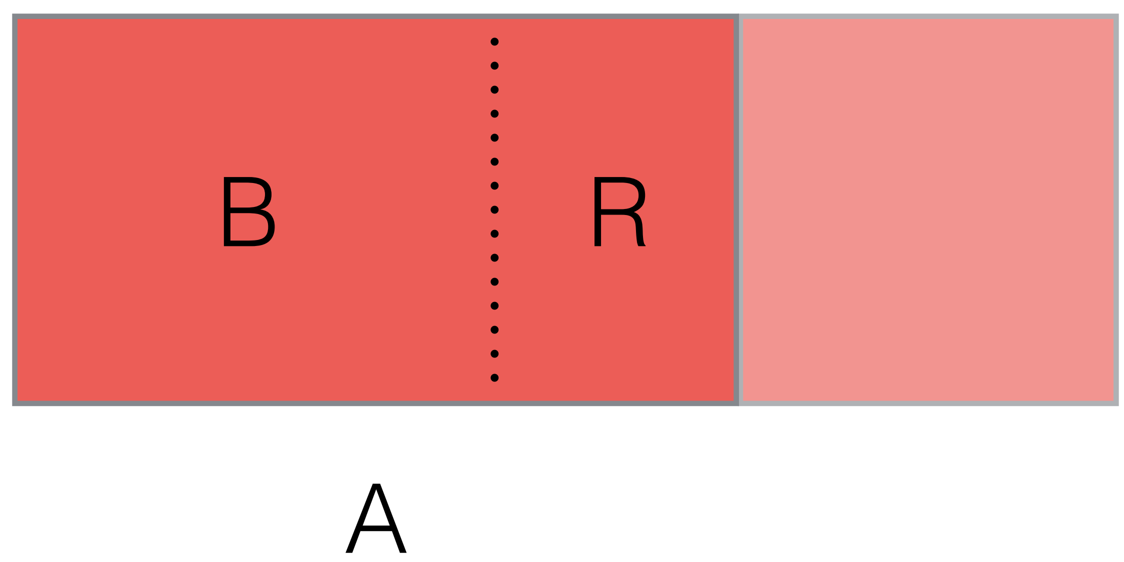
Envy-free for 4 players

Outline:

- Two basic algorithms on part of the cake
    - notation and statement
    - algorithm
- Envy-free for the entire cake

Algorithm I

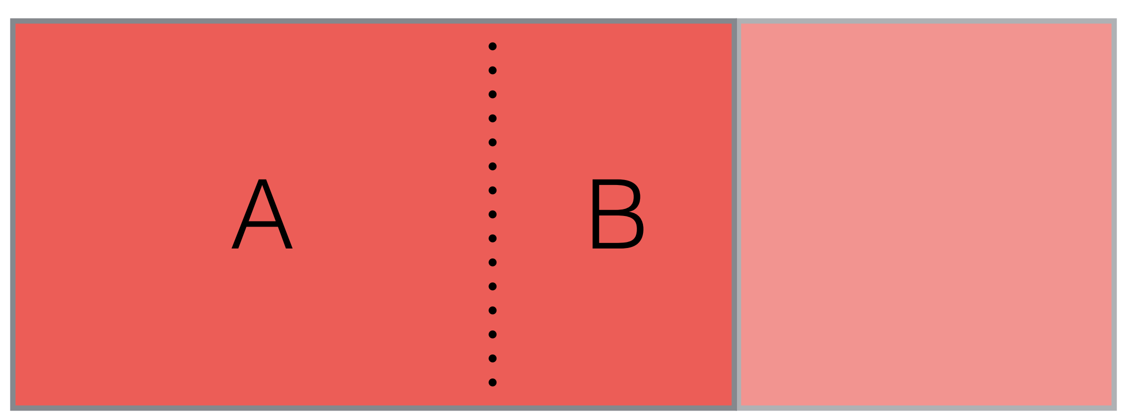
Given a piece of cake *A*, 4 players, and an $e>0$, a subset *B* of *A* can be divided among the four players in an envy-free way so that the remaining portion $R = A - B$ has value smaller than *e* for all four players.



Algorithm II

Given two pieces of cake *A,B.* If P1 views A and B as equal but P2 views B as larger, then P1 can be assigned part of A, P2 can be assigned part of B, and P3 and P4 can be assigned part from AUB so that the division of the four pieces is envy-free.

Furthermore, P1 and P2 will both hold an a>0 advantage over each other.



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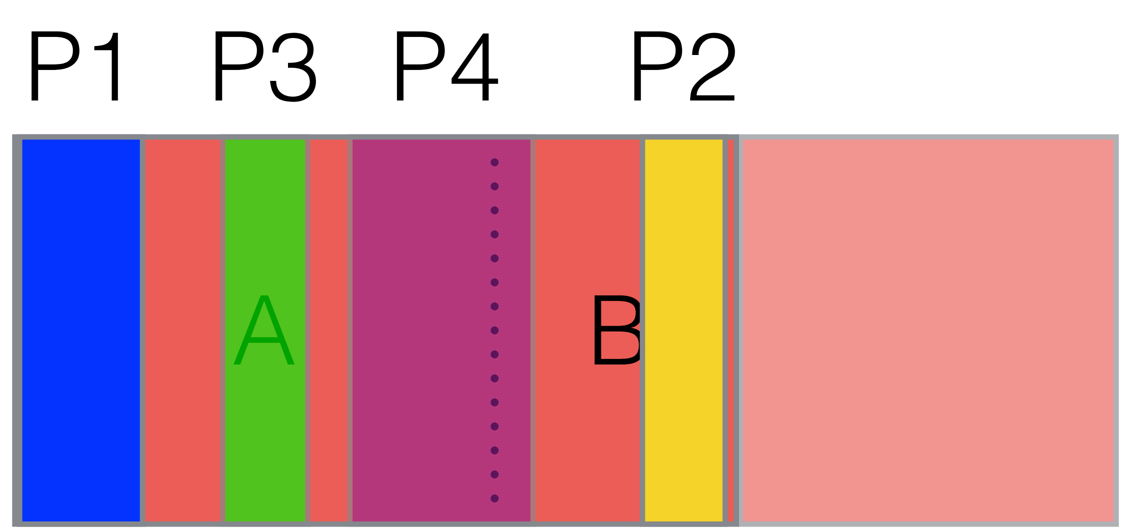

Envy-free for 4 players

1. P1 cuts the entire cake into 4 equal pieces.
2. If there is a total agreement, we are done. Else, apply **algorithm II** to P1 and P2 with a1 advantage among P1, P2.
3. Apply **algorithm I** to the remaining cake until the remainder R has value less than a1 for all players.

So far we have a partition {X1, X2, X3, X4, R}
{X1, X2, X3, X4} is envy free
P1 thinks that X1 is larger than X2 U R, P2 thinks that X2 is larger than X2 U R.
(P1 and P2 are with R advantage)

Envy-free for 4 players

4. P1 cuts R into 12 equal pieces.
5. Each of the other players declares to be
type I: if she agrees all the pieces are the same size
type II: if she disagrees
6. If type II players have R advantage over type I players. Then give the 12 pieces to type I players.
7. Otherwise, choose any pair that are non dominant to each other and repeat the process.

Summary

- Envy-free fair division
- Stromquist's moving knives
- Conway's pouring wine algorithm
- Brahms and Taylor's algorithm

- Key: dominance

Reference:
Cake-Cutting Algorithms: be fair if you can by Robertson and Webb [Chapters 10.3]