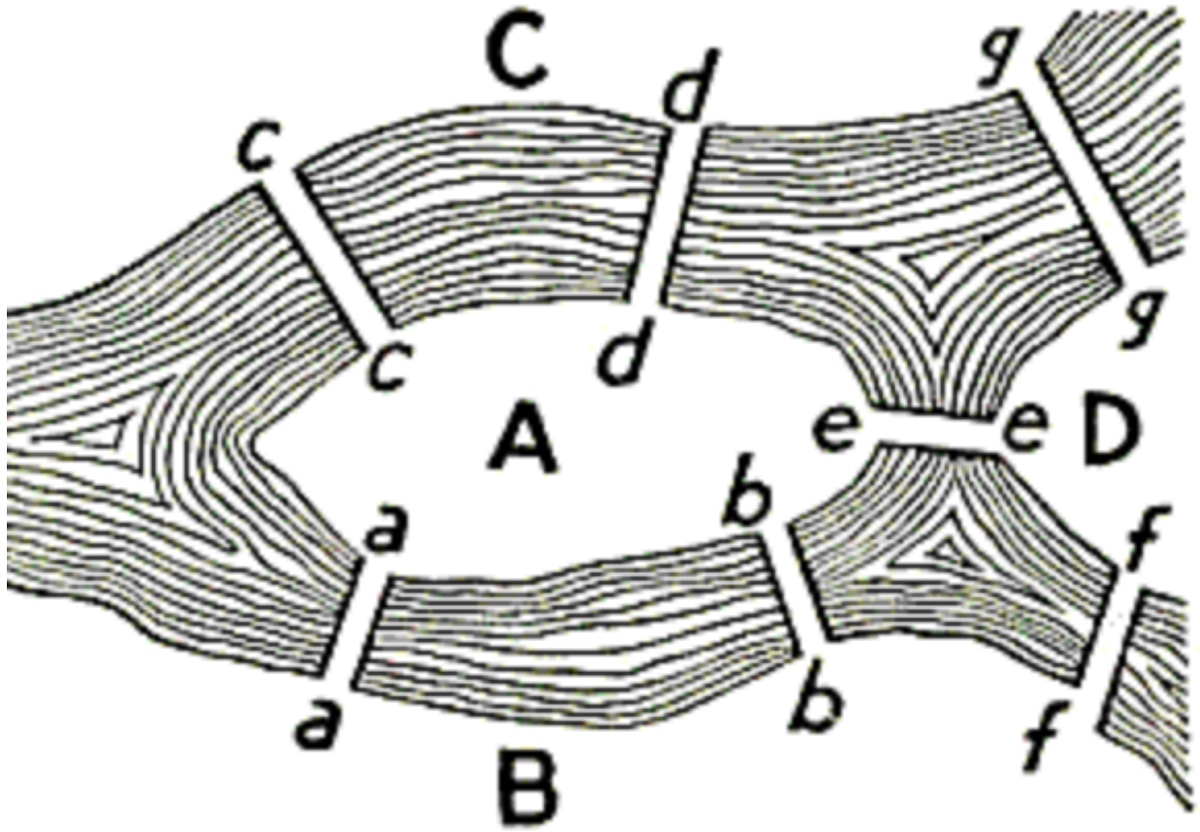
A warm-up exercise



A map of Koenigsberg

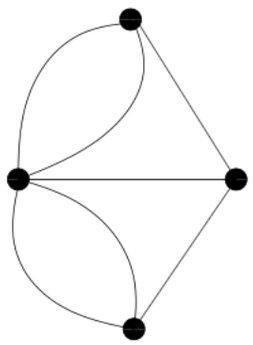
In the 18th century, the city of Koenigsberg consisted of islands where two branches of the river Pregel joined. Seven bridges connected various islands as above. Is it possible to walk through town, starting and ending at the same place but using each bridge exactly once?

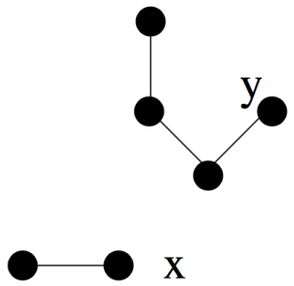
Graph

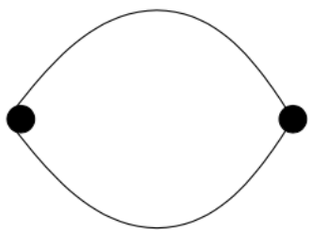
The diagram we draw is a mathematical object called ***graph***. Euler's paper on the Koenigsberg problem is considered the birth of *graph theory.*

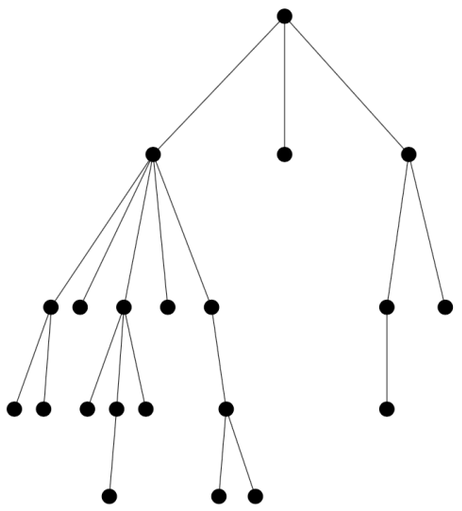
Definition (Graph). A graph is an assembly of two kinds of things, *vertices* and *edges*. The only rule is that each edge starts at a vertex and ends at a vertex.

Examples









Simple fair division in graph setting

Recall the original language:
<u>simple fair division for *n* players</u>
Let *S* be the cake to be divided. Let $P_1, P_2, \ldots, P_n$ be *n* players with measures $u_1, u_2 \ldots, u_n$. We want to divide *S* into $s_1, s_2 \ldots, s_n$ *such that* $u_i(s_i)$ is greater or equal to1/n.

Construct a graph G(V, E)
Assume *S* has been somehow cut into *n* pieces.

- V: $P_1, P_2, \ldots, P_n, s_1, s_2 \ldots, s_n$
- E: $P_i s_j$ is an edge if and only if $u_i(s_j)$ is greater or equal to1/n.

The fairness graph

Can you draw a graph that represents the information given in the following table from lecture 01?

	s1	**s2**
Unicorn	0.3	0.7
Lion	0.6	0.4

Bipartite graph and Matching

A bipartite graph has 2 subsets of vertices, and every edge joins a vertex in each subset.

A matching is set of edges with no common endvertex.

A perfect matching is a matching that covers all vertices.

Matching in fair division

Goal: write fair division problem in graph theory language.

- Matching: any assignment of **some** of the pieces to the players such that each player gets at most one acceptable piece.

- Perfect matching: all of the pieces get assigned

- Fair division = find a perfect matching in the fairness bipartite graph

Is there a perfect matching?

Hall's Theorem

- Not always.

- Hall's theorem.

Summary

- A graph theory setting of fair division problem
- Hall's theorem
- The matching algorithm
- Key: existence proof vs. finding a procedure

To do:

- A worksheet on Hall's theorem

Reference:
Cake-Cutting Algorithms: be fair if you can by Robertson and Webb [Chapter 6]