2025 USA-NA-AIO Round 1, Problem 1, Part 7

USAAIO Mar 2025

Part 7 (10 points, non-coding task)

Use the spectral decomposition result to derive a closed form of F_n for any $n \in \{0, 1, \cdots\}$.

- Reasoning is required.
- Your answer shall be written in terms of λ_0, λ_1, and F_0 and F_1.

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Misplaced '#'

For $n \geq 1$, we have

$$\begin{aligned} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \mathbf{A}^{n-1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\ &= \left(\mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^\top \right)^{n-1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\ &= \mathbf{Q} \mathbf{\Lambda}^{n-1} \mathbf{Q}^\top \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} . \end{aligned}$$

Thus, for $n \geq 1$, we have

$$
\begin{aligned}
F_n &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{A} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{Q} \mathbf{\Lambda}^{n-1} \mathbf{Q}^\top \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\lambda_0}{\sqrt{1+\lambda_0^2}} & \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} \\ \frac{1}{\sqrt{1+\lambda_0^2}} & \frac{1}{\sqrt{1+\lambda_1^2}} \end{bmatrix} \begin{bmatrix} \lambda_0^{n-1} & 0 \\ 0 & \lambda_1^{n-1} \end{bmatrix} \begin{bmatrix} \frac{\lambda_0}{\sqrt{1+\lambda_0^2}} & \frac{1}{\sqrt{1+\lambda_0^2}} \\ \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} & \frac{1}{\sqrt{1+\lambda_1^2}} \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda_0}{\sqrt{1+\lambda_0^2}} & \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} \end{bmatrix} \begin{bmatrix} \lambda_0^{n-1} & 0 \\ 0 & \lambda_1^{n-1} \end{bmatrix} \begin{bmatrix} \frac{\lambda_0}{\sqrt{1+\lambda_0^2}} & \frac{1}{\sqrt{1+\lambda_0^2}} \\ \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} & \frac{1}{\sqrt{1+\lambda_1^2}} \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda_0^n}{\sqrt{1+\lambda_0^2}} & \frac{\lambda_1^n}{\sqrt{1+\lambda_1^2}} \end{bmatrix} \begin{bmatrix} \frac{\lambda_0}{\sqrt{1+\lambda_0^2}} & \frac{1}{\sqrt{1+\lambda_0^2}} \\ \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} & \frac{1}{\sqrt{1+\lambda_1^2}} \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\lambda_0^{n+1}}{1+\lambda_0^2} + \frac{\lambda_1^{n+1}}{1+\lambda_1^2} & \frac{\lambda_0^n}{1+\lambda_0^2} + \frac{\lambda_1^n}{1+\lambda_1^2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} \\
&= \boxed{\left(\frac{\lambda_0^{n+1}}{1+\lambda_0^2} + \frac{\lambda_1^{n+1}}{1+\lambda_1^2} \right) F_1 + \left(\frac{\lambda_0^n}{1+\lambda_0^2} + \frac{\lambda_1^n}{1+\lambda_1^2} \right) F_0}.
\end{aligned}
$$

For $n = 0$, we can

$$
\frac{\lambda_0^{0+1}}{1+\lambda_0^2} + \frac{\lambda_1^{0+1}}{1+\lambda_1^2} = 0
$$

and

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$$\frac{\lambda_0^0}{1+\lambda_0^2} + \frac{\lambda_1^0}{1+\lambda_1^2} = 1.$$

Therefore, the above answer holds for all $n \in \{0, 1, \cdots\}$.

""" END OF THIS PART """

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