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2025 USA-NA-AIO Round 2, Problem 3, Part 18

USAAIO May 2025

Part 18 (5 points, non-coding task)

Lemma 1 implies that the projection of $\mathbf{x}$ onto any direction is a standard normal. Therefore, all directions are homogeneous.

Therefore,

$$\mathbb{P}\left(\frac{\mathbf{x}^\top \mathbf{y}}{||\mathbf{x}||_2\,||\mathbf{y}||_2} > \epsilon\right) = \mathbb{P}\left(\frac{\mathbf{x}^\top \mathbf{y}}{||\mathbf{x}||_2\,||\mathbf{y}||_2} > \epsilon \mid \mathbf{x} = \hat{\mathbf{x}}\right), \ \forall\ \hat{\mathbf{x}} \in \mathbb{R}^d.$$

For simplicity, we consider

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^d.$$

Therefore, we only need to bound

$$\mathbb{P}\left(\frac{y_0}{||\mathbf{y}||_2} > \epsilon\right)$$

By symmetry, it is easy to see that

$$\mathrm{E}\left[\frac{y_0}{||\mathbf{y}||_2}\right] = 0.$$

Skip to main content

Hence, we get

$$
\begin{aligned}
\mathbb{P}\left(\frac{y_0}{||\mathbf{y}||_2} > \epsilon\right) &\leq \frac{\operatorname{Var}\left[\frac{y_0}{||\mathbf{y}||_2}\right]}{\epsilon^2} \\
&= \frac{\mathrm{E}\left[\frac{y_0^2}{||\mathbf{y}||_2^2}\right]}{\epsilon^2} \\
&= \frac{1}{\epsilon^2 d}\, \mathrm{E}\left[\frac{y_0^2}{\frac{1}{d}\sum_{i=0}^{d-1} y_i^2}\right]
\end{aligned}
$$

where the first inequality follows from the Chebyshev's inequality.

To prove the theorem, it is equivalent to prove the following lemma.

Lemma 2:

- Let $y_0, \cdots, y_{d-1}$ be identically and independent variables that are all standard normals. Then for large d,

$$
\mathrm{E}\left[\frac{y_0^2}{\frac{1}{d}\sum_{i=0}^{d-1} y_i^2}\right] \approx 1.
$$

In this part, your task is to prove this lemma.

Hint: It is hard to prove this statement in an exact way. You can make any reasonable approximation.

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Skip to main content

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Since y_i is a standard normal, $\mathrm{E}\left[y_i^2\right] = 1$ and $\mathrm{Var}\left[y_i^2\right] = 2$.

For large d, the central limit theorem implies

$$\frac{1}{d} \sum_{i=0}^{d-1} y_i^2 \sim N\left(1, \frac{2}{d}\right).$$

Hence, for large d, $\frac{1}{d} \sum_{i=0}^{d-1} y_i^2$ can be approximated as its mean value, 1.

Therefore, for large d,

$$\begin{aligned} \mathrm{E}\left[\frac{y_0^2}{\frac{1}{d} \sum_{i=0}^{d-1} y_i^2}\right] & \approx \mathrm{E}\left[y_0^2\right] \\ & = 1. \end{aligned}$$

""" END OF THIS PART """

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Skip to main content

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