2025 USA-NA-AIO Round 1, Problem 2, Part 8

USAAIO Mar 2025

Part 8 (10 points, non-coding task)

This question follows Part 7.

Denote $r = \operatorname{rank}(\mathbf{W})$.

Compute the rank of $\mathbf{W}^\top \mathbf{W}$.

- Reasoning is required.

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Misplaced '#'

Let $\mathbf{W}$ be with the following singular value decomposition:

$$\mathbf{W} = \sum_{i=0}^{r-1} \mathbf{u}_i \sigma_i \mathbf{v}_i^\top.$$

Hence,

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$$\begin{aligned}\mathbf{W}^\top\mathbf{W} &= \left(\sum_{i=0}^{r-1}\mathbf{u}_i\sigma_i\mathbf{v}_i^\top\right)^\top\left(\sum_{j=0}^{r-1}\mathbf{u}_j\sigma_j\mathbf{v}_j^\top\right)\\ &= \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\mathbf{v}_i\sigma_i\mathbf{u}_i^\top\mathbf{u}_j\sigma_j\mathbf{v}_j^\top\\ &= \sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\mathbf{v}_i\sigma_i\sigma_j\delta_{ij}\mathbf{v}_j^\top\\ &= \sum_{i=0}^{r-1}\mathbf{v}_i\sigma_i^2\mathbf{v}_j^\top.\end{aligned}$$

This is the singular value decomposition of $\mathbf{W}\mathbf{W}^\top$. Therefore, the rank of this matrix is also r.

""" END OF THIS PART """

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