2025 USA-NA-AIO Round 1, Problem 1, Part 4

USAAIO Mar 2025

Part 4 (10 points, non-coding task)

In this part, you are asked to prove a general result that holds for any real-valued symmetric matrix.

Let $\mathbf{U} \in \mathbb{R}^{N \times N}$ be a real-valued symmetric matrix.

Consider the following eigenvalue equation

$$\mathbf{U}\mathbf{v} = \lambda \mathbf{v},$$

where $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^{N \times 1}$.

Let λ_i and λ_j be two distinct real eigenvalues and $\mathbf{v}_i, \mathbf{v}_j \in \mathbb{R}^{N \times 1}$ be two eigenvectors associated with them, respectively.

Prove that

$$\mathbf{v}_i^\top \mathbf{v}_j = 0.$$

That is, vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal.

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WRITE YOUR SOLUTION IN THIS TEXT CELL.

We use two approaches to compute

$$\mathbf{v}_i^\top \mathbf{U} \mathbf{v}_j.$$

Approach 1

We have

$$\begin{aligned} \mathbf{v}_i^\top \mathbf{U} \mathbf{v}_j &= \mathbf{v}_i^\top \left(\mathbf{U} \mathbf{v}_j \right) \\ &= \mathbf{v}_i^\top \left(\lambda_j \mathbf{v}_j \right) \\ &= \lambda_j \mathbf{v}_i^\top \mathbf{v}_j. \qquad (1) \end{aligned}$$

Approach 1

We have

$$\begin{aligned} \mathbf{v}_i^\top \mathbf{U} \mathbf{v}_j &= \left(\mathbf{v}_i^\top \mathbf{U} \right) \mathbf{v}_j \\ &= \left(\mathbf{v}_i^\top \mathbf{U}^\top \right) \mathbf{v}_j \\ &= \left(\mathbf{U} \mathbf{v}_i \right)^\top \mathbf{v}_j \\ &= \left(\lambda_i \mathbf{v}_i \right)^\top \mathbf{v}_j \\ &= \lambda_i \mathbf{v}_i^\top \mathbf{v}_j, \qquad (2) \end{aligned}$$

where the second equality follows from the property that $\mathbf{U}$ is a symmetric matrix.

Two approaches should yield the same result. Equating (1) and (2), we have

$$\lambda_j \mathbf{v}_i^\top \mathbf{v}_j = \lambda_i \mathbf{v}_i^\top \mathbf{v}_j.$$

Because $\lambda_i \neq \lambda_j$, we must have

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$$\mathbf{v}_i^\top \mathbf{v}_j = 0.$$

This completes the proof.

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