

2025 USA-NA-AIO Round 1, Problem 1, Part 6

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In the remaining parts of this problem, for all non-coding tasks, when you need to use those eigenvalues,

keep them as λ_0 and λ_1 . No need to apply their formulae.

Part 6 (10 points, non-coding task)

Since matrix A is symmetric, let us do the spectral decomposition of it.

That is, you should write A in the following form:

$$A = Q\Lambda Q^\top,$$

where

- $Q \in \mathbb{R}^{2 \times 2}$ is an orthonormal matrix.
- $\Lambda = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ is a diagonal matrix with $\lambda_0 > \lambda_1$

In this task, you need to compute Q . Reasoning is required.



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Misplaced '#'

To compute an eigenvector \mathbf{x}_i associated with the eigenvalue λ_i for $i \in \{0, 1\}$, we solve the following equation:

$$\begin{bmatrix} 1 - \lambda_i & 1 \\ 1 & -\lambda_i \end{bmatrix} \begin{bmatrix} x_{i0} \\ x_{i1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the determinant of the 2-by-2 matrix on the L.H.S. is 0, the rank of this matrix is 1. Thus, it is sufficient to only solve the following equation:

$$\begin{bmatrix} 1 & -\lambda_i \end{bmatrix} \begin{bmatrix} x_{i0} \\ x_{i1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One solution is

$$\mathbf{x}_i = \begin{bmatrix} x_{i0} \\ x_{i1} \end{bmatrix} = \begin{bmatrix} \lambda_i \\ 1 \end{bmatrix}.$$

Thus, the i th orthonormal vector is

$$\begin{aligned} \mathbf{q}_i &= \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|_2} \\ &= \frac{1}{\sqrt{1 + \lambda_i^2}} \begin{bmatrix} \lambda_i \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore,

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$$\mathbf{Q} = \begin{bmatrix} \frac{\lambda_0}{\sqrt{1+\lambda_0^2}} & \frac{\lambda_1}{\sqrt{1+\lambda_1^2}} \\ \frac{1}{\sqrt{1+\lambda_0^2}} & \frac{1}{\sqrt{1+\lambda_1^2}} \end{bmatrix}.$$

"" END OF THIS PART ""

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