AERO 557
Homework #4

Problem 1: Optimal Orbit to maximize the final altitude

Consider the case of a transfer in the two-body force field with polar coordinates where a is the non-dimensional applied acceleration and beta is the thrust angle. The spacecraft is initially in a circular orbit with radius of 1.05. A constant acceleration of 0.1 is then applied for a period of 4.0. The objective is to place the spacecraft into a final circular orbit with maximum possible altitude. Determine the orbit transfer. Equations of motion are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2} + a\sin\beta$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a\cos\beta$$

Provide:
-The problem formulation including the state and co-state, boundary conditions, and the control law.
-The plot of the thrust angle history
-A plot of the transfer trajectory
-The initial and final values of the state and co-state variables.
-Value of the performance index

Problem 2: Optimal Orbit with Minimum Fuel ($-m_f$)

Given the equations of motion for two-body force field, determine the orbit that accomplishes a transfer from an initially circular orbit of 1.05 and a final circular orbit of 2.0 using minimum fuel (assuming the transfer is a burn-coast-burn). The maximum thrust is 0.1 and the exhaust velocity is 0.9. The arrival point on the final circular orbit is unconstrained. EOMs are:

$$\ddot{x} = -\frac{x}{r^3} + \frac{T}{m}u_x$$

$$\ddot{y} = -\frac{y}{r^3} + \frac{T}{m}u_y$$

$$\dot{m} = -\frac{T}{v_e}$$

Provide:
-The problem formulation including the state and co-state, boundary conditions, and the control law.
-The plot of the mass history
-A plot of the transfer trajectory
-The initial and final values of the state and co-state variables.
-The switching times
-The value of the performance index

Orbital Trajectory Optimization in Two Dimensions

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Problem 1
Please reference the trajectory and acceleration angle plots on the following pages.

State, costate, state derivative, initial and final parameters:

$$\vec{s} = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} \qquad \vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} s_3 \\ s_4 \\ s_1 s_4^2 - s_1^{-2} + a \sin c_1 \\ s_1^{-1}\left(a \cos c_1 - 2 s_3 s_4\right) \end{bmatrix}$$

$$\vec{s}_0 = \begin{bmatrix} 1.05 \\ 0 \\ 0 \\ 0.9294 \end{bmatrix} \qquad \vec{s}_f = \begin{bmatrix} 1.5376 \\ 2.8230 \\ 0 \\ 0.5245 \end{bmatrix} \qquad \vec{\lambda}_0 = \begin{bmatrix} -4.1867 \\ 0 \\ -0.9804 \\ -2.4997 \end{bmatrix} \qquad \vec{\lambda}_f = \begin{bmatrix} -2.5424 \\ 0 \\ 1.6689 \\ -3.0143 \end{bmatrix}$$

$$J = -r_f = -1.5376$$

Problem formulation:

$$H = \lambda_1 s_3 + \lambda_2 s_4 + \lambda_3 \left(s_1 s_4^2 - s_1^{-2} + a \sin c_1\right) + \lambda_4 \left(s_1^{-1}\left(a \cos c_1 - 2 s_3 s_4\right)\right)$$

$$G = -s_{1f} + \omega_1 s_{3f} + \omega_2 \left(s_{1f}^3 s_{4f}^2 - 1\right)$$

$$\frac{\partial H}{\partial c} = 0 \rightarrow \cos c_1 = -\frac{\lambda_4}{\sqrt{s_1^2 \lambda_3^2 + \lambda_4^2}},\ \sin c_1 = -s_1 \frac{\lambda_3}{\sqrt{s_1^2 \lambda_3^2 + \lambda_4^2}}$$

$$\begin{aligned} \dot{\vec{\lambda}} &= -H_s^{\intercal} \\ &= \begin{bmatrix} -\lambda_3\left(s_4^2 + 2 s_1^{-3}\right) + \lambda_4 s_1^{-2}\left(a \cos c_1 - 2 s_3 s_4\right) \\ 0 \\ -\lambda_1 + 2 s_4 s_1^{-1} \lambda_4 \\ -\lambda_2 - 2 \lambda_3 s_1 s_4 + 2 \lambda_4 s_3 s_1^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -\lambda_3\left(s_4^2 + 2 s_1^{-3}\right) + \lambda_4 s_1^{-2}\left(-a \frac{\lambda_4}{\sqrt{s_1^2 \lambda_3^2 + \lambda_4^2}} - 2 s_3 s_4\right) \\ 0 \\ -\lambda_1 + 2 s_4 s_1^{-1} \lambda_4 \\ -\lambda_2 - 2 \lambda_3 s_1 s_4 + 2 \lambda_4 s_3 s_1^{-1} \end{bmatrix} \end{aligned}$$

$$\begin{aligned}\vec{\lambda}_f &= G_{s_f}\\ &= \begin{bmatrix}-1+3\omega_2 s_{1f}^2 s_{4f}^2\\ 0\\ \omega_1\\ 2\omega_2 s_{1f}^3 s_{4f}\end{bmatrix}\end{aligned}$$

$$\lambda_{1f} = -1 + 3\omega_2 s_{1f}^2 s_{4f}^2 \to \omega_2 = \frac{\lambda_{1f}+1}{3s_{1f}^2 s_{4f}^2}$$

$$\lambda_{4f} = 2\omega_2 s_{1f}^3 s_{4f} \to \omega_2 = \frac{\lambda_{4f}}{2s_{1f}^3 s_{4f}}$$

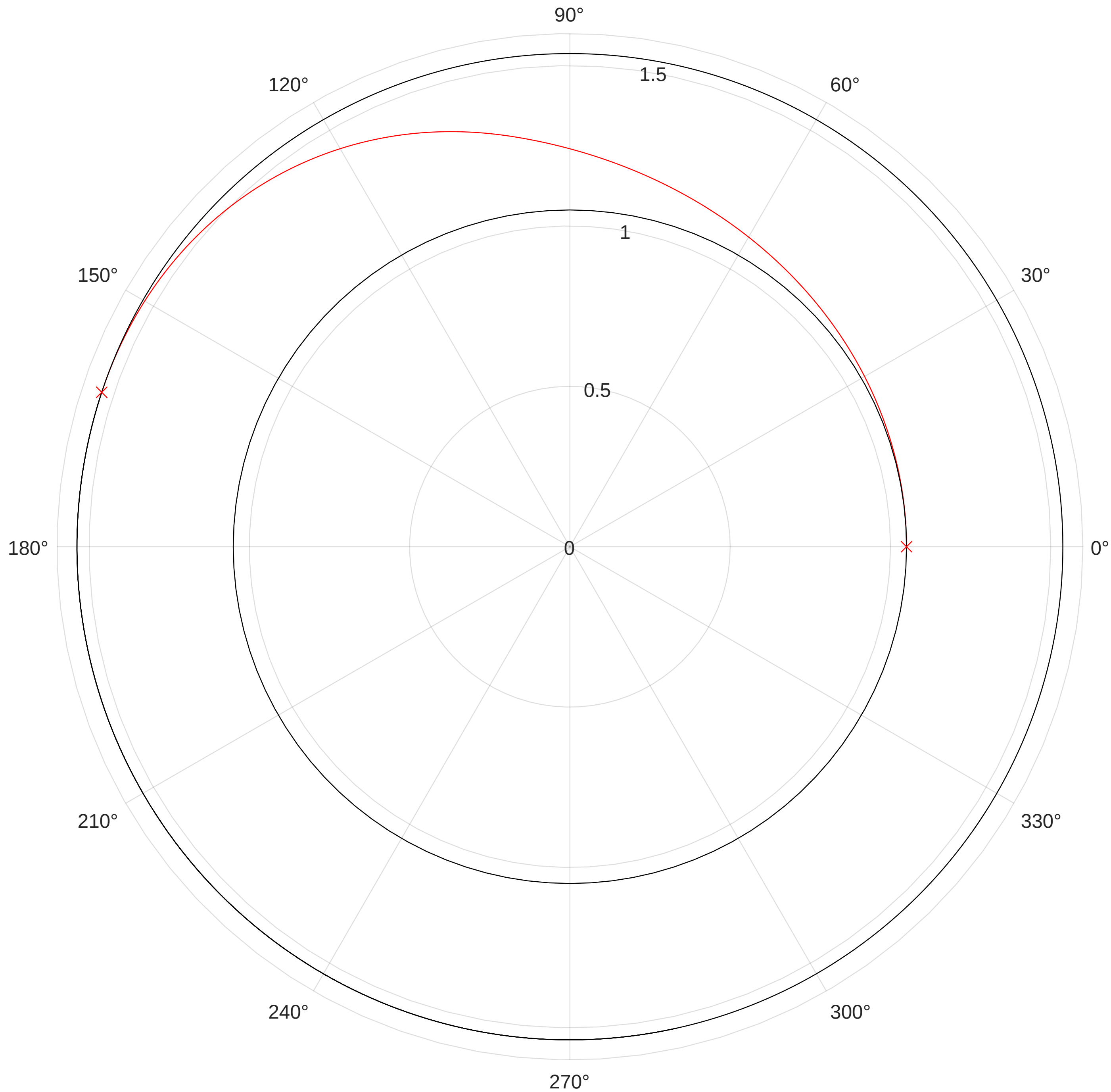
$$\Rightarrow \frac{\lambda_{1f}+1}{3s_{1f}^2 s_{4f}^2} - \frac{\lambda_{4f}}{2s_{1f}^3 s_{4f}} = 0$$

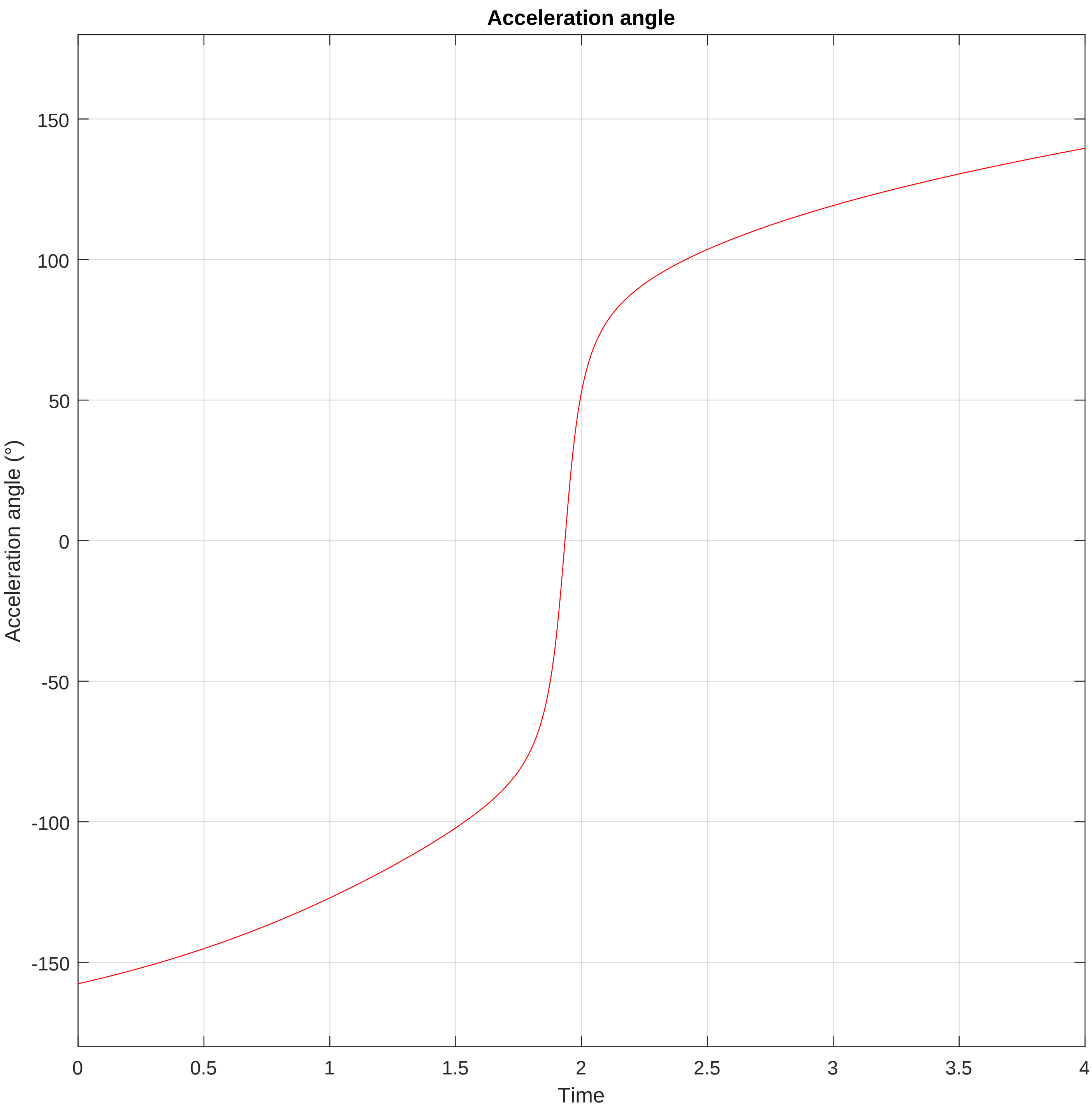
Circular final orbit: $r_f^3 \dot{\theta}_f^2 = s_{1f}^3 s_{4f}^2 = 1$, $\dot{r}_f = s_{3f} = 0$

$$\vec{F} = \begin{bmatrix} s_{1f}^3 s_{4f}^2 - 1\\ s_{3f}\\ \lambda_{2f}\\ \frac{\lambda_{1f}+1}{3s_{1f}^2 s_{4f}^2} - \frac{\lambda_{4f}}{2s_{1f}^3 s_{4f}}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}\dot{\vec{s}}\\ \dot{\vec{\lambda}}\end{bmatrix} = \begin{bmatrix} s_3\\ s_4\\ s_1 s_4^2 - s_1^{-2} - a\frac{\lambda_3 s_1}{\sqrt{s_1^2\lambda_3^2+\lambda_4^2}}\\ -s_1^{-1}\left(a\frac{\lambda_4}{\sqrt{s_1^2\lambda_3^2+\lambda_4^2}} + 2s_3 s_4\right)\\ -\lambda_3\left(s_4^2 + 2s_1^{-3}\right) - \lambda_4 s_1^{-2}\left(a\frac{\lambda_4}{\sqrt{s_1^2\lambda_3^2+\lambda_4^2}} + 2s_3 s_4\right)\\ 0\\ -\lambda_1 + 2s_4 s_1^{-1}\lambda_4\\ -\lambda_2 - 2\lambda_3 s_1 s_4 + 2\lambda_4 s_3 s_1^{-1}\end{bmatrix}$$

Optimal Trajectory





Acceleration angle
150
100
50
0
-50
-100
-150
Acceleration angle (°)
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Problem 2
Please reference the trajectory, thrust angle, and switching function plots on the following pages The thruster is on during the solid portions of the trajectory and off during the dashed portion.

State, costate, state derivative, initial and final parameters:

$$\vec{s} = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \\ m \end{bmatrix} \qquad \vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} s_3 \\ s_4 \\ -\frac{s_1}{r^3} + \frac{c_3}{s_5}c_1 \\ -\frac{s_2}{r^3} + \frac{c_3}{s_5}c_2 \\ -\frac{c_3}{v_e} \end{bmatrix} \quad r = \sqrt{s_1^2 + s_2^2} \quad \lambda_v = \sqrt{\lambda_3^2 + \lambda_4^2} \quad \vec{c} = \begin{bmatrix} -\frac{\lambda_3}{\lambda_v} \\ -\frac{\lambda_4}{\lambda_v} \\ T \end{bmatrix}$$

$$\vec{s}_0 = \begin{bmatrix} 1.05 \\ 0 \\ 0 \\ 0.9759 \\ 1 \end{bmatrix} \qquad \vec{s}_f = \begin{bmatrix} -1.3972 \\ -1.4310 \\ 0.5059 \\ -0.4940 \\ 0.7469 \end{bmatrix} \qquad \vec{\lambda}_0 = \begin{bmatrix} -0.7500 \\ 0.0143 \\ 0.0154 \\ -0.8287 \\ -0.7460 \end{bmatrix} \qquad \vec{\lambda}_f = \begin{bmatrix} 0.1964 \\ 0.2036 \\ -0.5904 \\ 0.5832 \\ -1 \end{bmatrix}$$

$$J = -m_f = -0.7469$$

Burn 1: 1.3425 TU Coast: 4.8274 TU Burn 2: 0.9355 TU

Time of Flight: 7.1054 TU

Problem formulation:

$$H = \lambda_1 s_3 + \lambda_2 s_4 + \lambda_3 \left(-\frac{s_1}{r^3} + \frac{c_3}{s_5} c_1 \right) + \lambda_4 \left(-\frac{s_2}{r^3} + \frac{c_3}{s_5} c_2 \right) - \lambda_5 \frac{c_3}{v_e}$$

$$G = -s_{5f} + \omega_1 \left(\sqrt{s_{1f}^2 + s_{2f}^2} - 2 \right) + \omega_2 \left(\sqrt{s_{3f}^2 + s_{4f}^2} - \frac{\sqrt{2}}{4} \right) + \omega_3 \left(s_{4f} s_{1f} - s_{3f} s_{2f} - \frac{\sqrt{2}}{2} \right)$$

$$\frac{\partial H}{\partial c} = 0 \rightarrow s_5^{-1} (\lambda_3 c_1 + \lambda_4 c_2 + (\lambda_3 + \lambda_4) c_3) - \frac{\lambda_5}{v_e} = 0$$

$$\$ = \sqrt{\lambda_3^2 + \lambda_4^2} v_e + \lambda_5 s_5$$

$$\begin{aligned} \dot{\vec{\lambda}} &= -H_s^{\intercal} \\ &= \begin{bmatrix} r^{-5} \left(\lambda_3 \left(r^2 - 3s_1^2 \right) - 3\lambda_4 s_1 s_2 \right) \\ r^{-5} \left(\lambda_4 \left(r^2 - 3s_2^2 \right) - 3\lambda_3 s_1 s_2 \right) \\ -\lambda_1 \\ -\lambda_2 \\ c_3 s_5^{-2} (c_1 + c_2) \end{bmatrix} \\ &= \begin{bmatrix} r^{-5} \left(\lambda_3 \left(r^2 - 3s_1^2 \right) - 3\lambda_4 s_1 s_2 \right) \\ r^{-5} \left(\lambda_4 \left(r^2 - 3s_2^2 \right) - 3\lambda_3 s_1 s_2 \right) \\ -\lambda_1 \\ -\lambda_2 \\ -\lambda_v c_3 s_5^{-2} \end{bmatrix} \end{aligned}$$

$$\vec{\lambda}_f = G_{s_f}$$

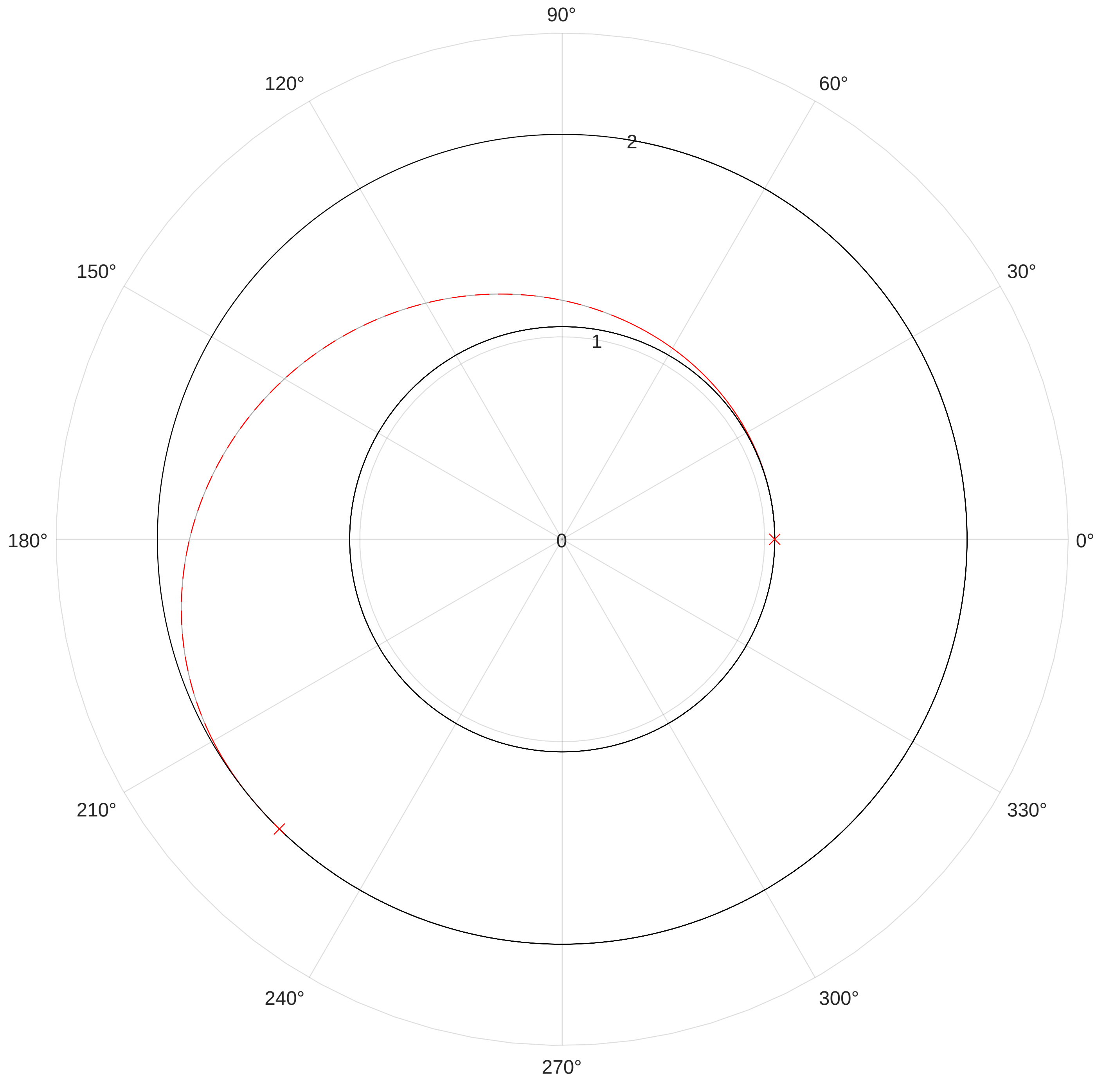
$$= \begin{bmatrix} s_{1f}\omega_1 r_f^{-1} + \omega_3 s_{4f} \\ s_{2f}\omega_1 r_f^{-1} - \omega_3 s_{3f} \\ s_{3f}\omega_2 r_f^{-1} - \omega_3 s_{2f} \\ s_{4f}\omega_2 r_f^{-1} - \omega_3 s_{1f} \\ -1 \end{bmatrix}$$

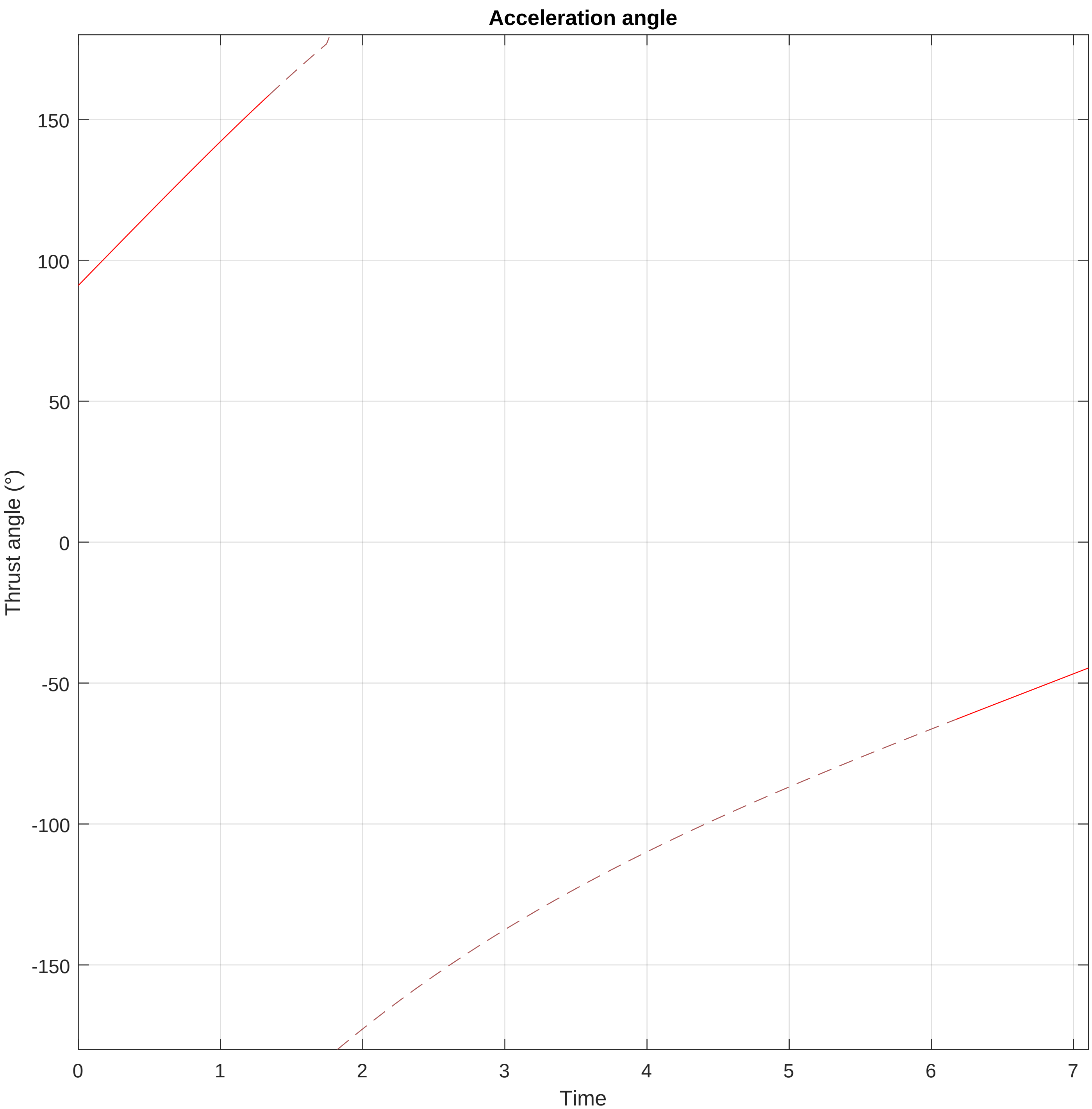
Circular final orbit: $r_f = \sqrt{s_{1f}^2 + s_{2f}^2} = 2$, $v_f = \sqrt{s_{3f}^2 + s_{4f}^2} = \frac{\sqrt{2}}{2}$, $r_f v_f = s_{3f}s_{1f} - s_{3f}s_{2f} = \sqrt{2}$

$$\vec{F} = \begin{bmatrix} \sqrt{s_{1f}^2 + s_{2f}^2} - 2 \\ \sqrt{s_{3f}^2 + s_{4f}^2} - \frac{\sqrt{2}}{2} \\ s_{3f}s_{1f} - s_{3f}s_{2f} - \sqrt{2} \\ -\lambda_{1f}\left(\frac{s_2}{s_1}\right) + \lambda_{2f} - \lambda_{3f}\left(\frac{s_4\left(s_3 + s_2 s_4 s_1^{-1}\right)}{s_3\left(s_1 + s_2 s_4 s_3^{-1}\right)}\right) + \lambda_{4f}\left(\frac{s_3 + s_2 s_4 s_1^{-1}}{s_1 + s_2 s_4 s_3^{-1}}\right) \\ \lambda_{5f} - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

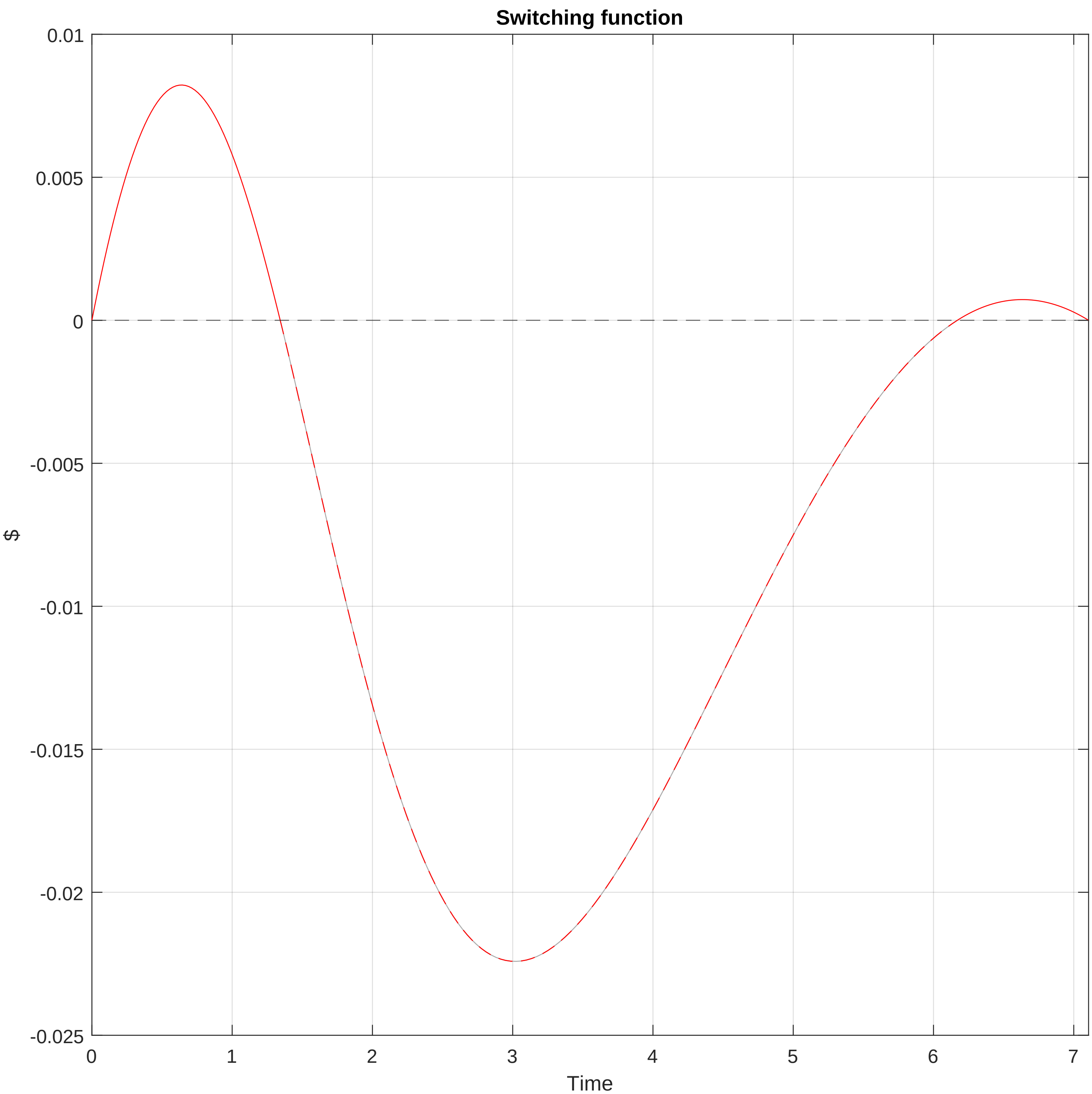
$$\begin{bmatrix} \dot{\vec{s}} \\ \dot{\vec{\lambda}} \end{bmatrix} = \begin{bmatrix} s_3 \\ s_4 \\ -\frac{s_1}{r^3} - \frac{\lambda_3 T}{\lambda_v s_5} \\ -\frac{s_2}{r^3} - \frac{\lambda_4 T}{\lambda_v s_5} \\ -\frac{T}{v_e} \\ r^{-5}\left(\lambda_3\left(r^2 - 3s_1^2\right) - 3\lambda_4 s_1 s_2\right) \\ r^{-5}\left(\lambda_4\left(r^2 - 3s_2^2\right) - 3\lambda_3 s_1 s_2\right) \\ -\lambda_1 \\ -\lambda_2 \\ -\lambda_v \frac{T}{s_5^2} \end{bmatrix}$$

Optimal Trajectory





Acceleration angle
Thrust angle (°)
Time
150
100
50
0
-50
-100
-150
0
1
2
3
4
5
6
7



Switching function
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
$
0
1
2
3
4
5
6
7
Time