De-tumble Phase Gain Selection

We assume fully modulated thrusters for this problem, so that any thrust vector may be applied at any given time. Further, we use direct velocity feedback for the commanded torque $\vec{\boldsymbol{T}}$ as follows:

$$\vec{\boldsymbol{T}} = \vec{\mathscr{F}}_b^T k \boldsymbol{\omega}_b$$

where k is an arbitrary scalar. In order to select a gain that is appropriate to the problem, we must determine which values of k, if any, produce an asymptotically stable system.

The rotational dynamics of the spacecraft is defined by

$$\boldsymbol{T}_b = \boldsymbol{J}_b \dot{\boldsymbol{\omega}}_b + (\boldsymbol{\omega}_b)^x \boldsymbol{J}_b \boldsymbol{\omega}_b$$

The inertia matrix for the spacecraft during the de-tumble phase about the principal axes of inertia is

$$\boldsymbol{J}_b = \begin{bmatrix} 426.\bar{6} & 0 & 0 \\ 0 & 426.\bar{6} & 0 \\ 0 & 0 & 426.\bar{6} \end{bmatrix} kg \cdot m^2$$

The inertia matrix is a diagonal matrix and thus

$$(\boldsymbol{\omega}_b)^x \boldsymbol{J}_b \boldsymbol{\omega}_b = \boldsymbol{0}$$

$$\rightarrow \; \boldsymbol{T}_b = \boldsymbol{J}_b \dot{\boldsymbol{\omega}}_b$$
$$\rightarrow \; \dot{\boldsymbol{\omega}}_b = \boldsymbol{J}_b^{-1} \boldsymbol{T}_b$$

Let

$$a = 426.\bar{6}\, kg \cdot m^2$$

Then the inertia matrix can be expressed as

$$\boldsymbol{J}_b = a \mathbb{1}$$

Substituting into the rotational dynamics equation yields

$$\dot{\boldsymbol{\omega}}_b = (a \mathbb{1})^{-1} \boldsymbol{T}_b$$
$$\rightarrow \; \dot{\boldsymbol{\omega}}_b = \frac{1}{a} \boldsymbol{T}_b$$

We then substitute the direct velocity feedback value for the commanded torque to obtain the following relationship:

$$\dot{\boldsymbol{\omega}}_b = \frac{k}{a} \boldsymbol{\omega}_b$$

We can expand this expression to reveal the differential equations

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \frac{k}{a} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\rightarrow \quad \dot{\omega}_x = \frac{k}{a} \omega_x$$

$$\rightarrow \quad \dot{\omega}_y = \frac{k}{a} \omega_y$$

$$\rightarrow \quad \dot{\omega}_z = \frac{k}{a} \omega_z$$

We may represent this system as

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{f}(\boldsymbol{x}) = \frac{k}{a} \boldsymbol{x} \end{cases}$$

We define the equilibrium condition for the system as

$$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$

We define a deviation from this equilibrium condition as

$$\Delta \boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^* \quad \forall t \geq 0$$

The time derivative of this deviation is

$$\begin{aligned} \frac{d}{dt}(\Delta \boldsymbol{x}) &= \frac{d}{dt}(\boldsymbol{x}) \\ &= \boldsymbol{f}(\boldsymbol{x}) \\ &= \boldsymbol{f}(\Delta \boldsymbol{x} + \boldsymbol{x}^*) \end{aligned}$$

Assuming that the deviation is small, i.e. $\|\Delta \boldsymbol{x}\| \ll 1$, we expand and neglect higher order terms to obtain

$$\frac{d}{dt}(\Delta \boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}^*) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{x = x^*} \Delta \boldsymbol{x}$$

Let

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{x = x^*}$$

Recall the definition of the equilibrium condition

$$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$

Thus,

$$\frac{d}{dt}(\Delta\boldsymbol{x}) \approx \boldsymbol{A}\,\Delta\boldsymbol{x}$$

We may expand the function into its components and define each sub-function:

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\omega_x) \\ f_2(\omega_y) \\ f_3(\omega_z) \end{bmatrix} = \frac{k}{a}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Note that

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

We may then compute the matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x} = \boldsymbol{x}^*}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}_{\boldsymbol{x} = \boldsymbol{x}^*}$$

$$= \begin{bmatrix} \frac{k}{a} & 0 & 0 \\ 0 & \frac{k}{a} & 0 \\ 0 & 0 & \frac{k}{a} \end{bmatrix}$$

$$= \frac{k}{a}\mathbb{1}$$

Our desired angular velocity at the equilibrium condition is

$$\boldsymbol{\omega}_b^* = \mathbf{0}$$

Therefore, the deviation from the equilibrium condition is

$$\begin{aligned} \Delta \boldsymbol{x} &= \boldsymbol{x} - \boldsymbol{x}^* \\ &= \boldsymbol{\omega}_b - \boldsymbol{\omega}_b^* \\ &= \boldsymbol{\omega}_b \end{aligned}$$

Since $\boldsymbol{A} \in \mathbb{R}^{\mathrm{n \times n}}$, the matrix exponential of $\boldsymbol{A}t$ is defined as

$$e^{\boldsymbol{A}t} = \mathbb{1} + \boldsymbol{A}t + \boldsymbol{A}^2 \frac{t^2}{2!} + \boldsymbol{A}^3 \frac{t^3}{3!} \ldots$$

and the deviation can be approximated by

$$\Delta \boldsymbol{x}(t) \approx e^{\boldsymbol{A}t} \Delta \boldsymbol{x}(0)$$

Let $\boldsymbol{M}$ be some vector and $\boldsymbol{\Omega}$ be some matrix such that

$$\boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{\Omega}$$

The matrix exponential of $\boldsymbol{A}t$ has the useful property

$$\begin{aligned} e^{\boldsymbol{A}t} &= e^{\boldsymbol{M}\boldsymbol{\Omega}\boldsymbol{M}^{-1}t} \\ &= \boldsymbol{M} e^{\boldsymbol{\Omega}t} \boldsymbol{M}^{-1} \end{aligned}$$

Thus,

$$\Delta \boldsymbol{x}(t) \approx \boldsymbol{M} e^{\boldsymbol{\Omega}t} \boldsymbol{M}^{-1} \Delta \boldsymbol{x}(0)$$

In order for the system to be asymptotically stable, we must have

$$\lim_{t \to \infty} \Delta \boldsymbol{x}(t) = \mathbf{0}$$

Since $\boldsymbol{M}$ and $\Delta \boldsymbol{x}(0)$ are constant,

$$\begin{aligned} \lim_{t \to \infty} \Delta \boldsymbol{x}(t) &= \lim_{t \to \infty} \boldsymbol{M} e^{\boldsymbol{\Omega}t} \boldsymbol{M}^{-1} \Delta \boldsymbol{x}(0) \\ &= \lim_{t \to \infty} e^{\boldsymbol{\Omega}t} \\ &= \mathbf{0} \end{aligned}$$

Recall

$$\boldsymbol{A} = \frac{k}{a}\mathbb{1}$$

and

$$a = 426.\overline{6}\, kg \cdot m^2$$

It is obvious that $\boldsymbol{A}$ has only one unique eigenvalue:

$$\boldsymbol{\Omega} = \frac{k}{a}$$

Therefore,

$$\begin{aligned} \lim_{t \to \infty} e^{\boldsymbol{\Omega}\, t} &= \lim_{t \to \infty} e^{\frac{\boldsymbol{k}}{a} t} \\ &= 0 \end{aligned}$$

Since $a > 0$,

$$\lim_{t \to \infty} e^{\frac{k}{a} t} = 0$$

if and only if

$$k < 0$$

QED