Memorandum

To: Dr. Faysal Kolkailah

From: Bryce Ipson, Steven Sharp

Date: June 3, 2022

Subject: Material Selection for an Experimental Landing Gear Design

Introduction:

Spacecraft landing gear must withstand large compressive forces when supporting the weight of the lander and impact with a celestial body's surface. We modeled landing gear for a new interplanetary lander as a system of beams with one rigid beam, one elastic beam, and one spring, and used both exact methods and approximations to determine the critical load so that we may select the optimum material for the landing gear to avoid buckling of the component beams. The lander is still in early development, and will be sized based on the determination of the critical load for the selected material.

Since this is research into a conceptual design, we have decided to include the usage of several materials while leaving the size and shape of the structure as a constant in our work. We also determined the critical load for a few practical values of the spring constant considering the limitations of manufacturing tolerances, tooling cost, and mass.

Nomenclature:

E - modulus of elasticity (psi)

I - area moment of inertia (in^4)

k - spring elastic constant (lbf/in)

L - beam length (in)

P - load (lbf)

Subscripts:

c - critical

Methodology:

We are testing various landing gear structures and the lander will have multiple landing gear assemblies, so for ease of testing we will construct a test facility with a pin connection at the ground for each landing gear structure. The final design will have the contact surface be extremely rough, with friction preventing the landing gear from sliding on the surface, allowing the pin connection to simulate the actual flight conditions. Further, we used a solid rod for the uppermost beam, and a thick-walled hollow structure for the elastic beam, so that buckling of the elastic beam occurs before the uppermost beam experiences elastic deformation, and the displacement due to the rigid beam's deformation is negligible compared to the displacement of the elastic beam. Finally, we assume that internal crippling of the beam does not occur, even though realistically internal crippling would likely precede buckling for the hollow cross-section.

Our schematic is displayed below, where L is the length of the beam at 108 inches:

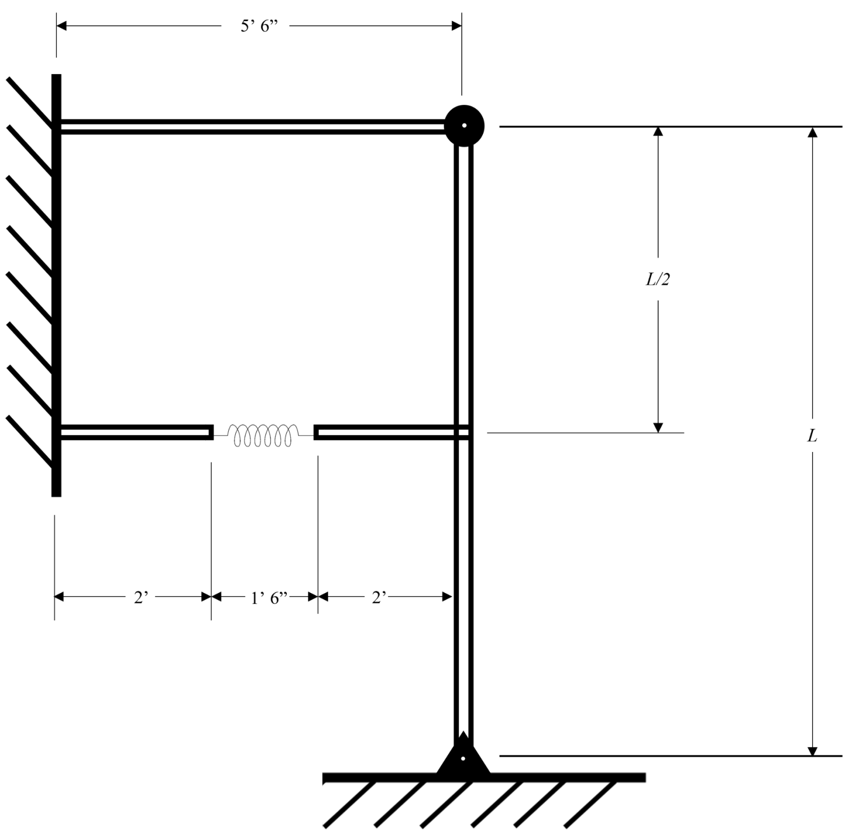

Figure 1: *Conceptual Landing Gear Design*

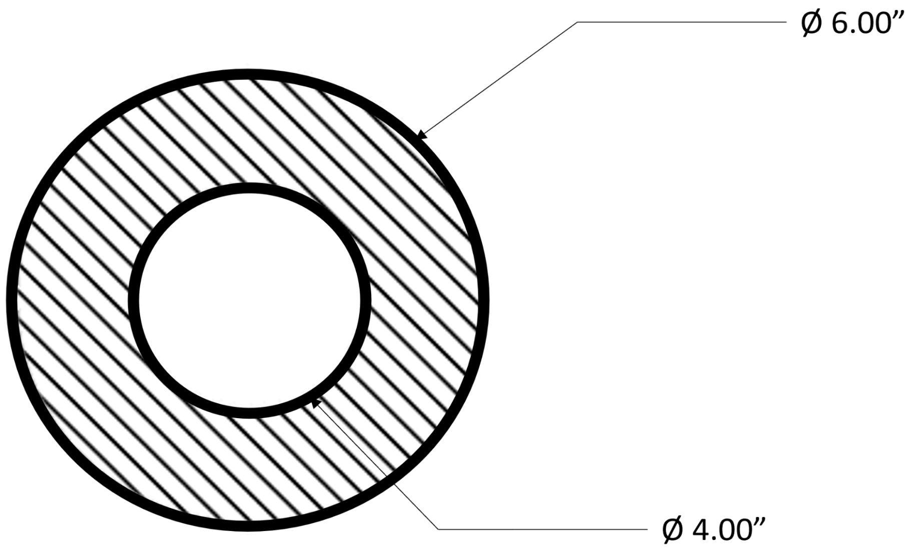

Figure 2: *Dimensions of Tubular Beam (Vertical Beam in Fig. 1)*

We conducted three analyses to find the critical loads for this structure:

- Exact method with no spring
- Approximate method with spring - polynomial displacement
- Approximate method with spring - trigonometric displacement

We used a factor of safety of 1.5, which is reduced from the tensile loading project because the lander will have multiple landing gear that could be used as backup in case of buckling of a single landing gear.

<u>Exact Method</u>

The addition of springs along an elastic beam results in a fourth-order differential equation with a non-zero constant term, which provides a displacement function with 5 constants. We may use the four boundary conditions to eliminate four of the constants, but this leaves the solution in terms of a single arbitrary constant, requiring use of the Rayleigh approximation with respect to the aforementioned constant or assumption of a specific deformation curve. The addition of a single point spring also makes the solution uncertain, as the ability to cancel out the virtual displacement is removed by the term for the spring energy, which is outside of the integral containing the fourth-order differential equation. Therefore, in order to facilitate an exact solution, and also to determine the most conservative value for the critical load, we connect the center attachment point of the column to an elastic cable so that the elastic beam is free to deform without interference. This system has four boundary conditions, with two displacement and two curvature conditions. For the analysis, we introduce an arbitrary virtual displacement and use the geometry of the system to produce a differential equation for the displacement, which is then solved with the boundary conditions to produce an exact expression for the critical load at all displacement modes. The exact analysis is continued in the appendix and results in the following Euler load:

$$P_c = \frac{\pi^2 EI}{L^2}$$

Approximate Solutions

Other than the maximum number of constants determined from the boundary consistent, and the boundary conditions themselves, a primary concern for the selection of the displacement function is symmetry about the point spring. For the trigonometric approximation, we set the coordinate axes such that the beam coordinates extend from 0 to L, and assume a trigonometric displacement function:

$$y = A\sin\left(\frac{\pi x}{L}\right) + B\cos\left(\frac{\pi x}{L}\right)$$

For the polynomial approximation, we move the location of the origin to the center of the beam so that we may generate a polynomial function that is symmetric and will simplify to a reasonable expression for the displacement of the spring:

$$y = A + Bx^2 + Cx^4 + Dx^6$$

The analysis of these scenarios is continued in the appendix.

Material Selection

The concept for our design was inspired by the landing gear on the lunar module (LM) of the Apollo 11 mission [2]. By no means is our design meant to be a replica of the landing gear on that mission. We instead designed a landing gear system which has some characteristics of the LM and fit the parameters of our project.

Many different materials are used in the aerospace industry from aluminum to carbon fiber to kapton. For a load bearing structure such as a landing gear, several different materials can be used and as such we have included results and recommendations based on the properties of different materials.

In texts related to the LM, we found the most common materials used were aluminum alloys [2] so we have included this possibility in our research. It was noted this material was mainly used for its lightness and its ability to absorb shocks if constructed properly in the event of a rough landing. However in the history of space exploration, titanium has been used in many applications as well especially in high radiation environments [3]. Thus we have included results for this metal as well. We also understand though that ferrous metals are often reliable, cheap substitutes due to their commonness. So we chose stainless steel as our third main metal to include in this study.

Metal	Young's Modulus (10^6 psi)
Aluminum	10.01
Aluminum Bronze	17.4
Titanium	16
Titanium Alloy	15.23 - 17.4
Stainless Steel (AISI 302)	26.11

Results:

The trigonometric approximation produces the following expression for the critical load, including the contribution of the spring in excess of the Euler load found above:

$$P_c = \frac{\pi^2}{L^2} EI + \frac{2L}{\pi^2} k$$

The polynomial approximation produces the following estimation for the critical load results in the following simultaneous linear equations:

$$\left(\frac{24}{5}EI-\frac{17}{35}L^2P+\frac{25}{256}L^3k\right)C+\left(\frac{24}{7}L^2EI-\frac{29}{84}L^4P+\frac{35}{512}L^5k\right)D=0$$
$$\left(\frac{24}{7}L^2EI-\frac{29}{84}L^4P+\frac{35}{512}L^5k\right)C+\left(\frac{5}{2}L^4EI-\frac{303}{1232}L^6P+\frac{49}{1024}L^7k\right)D=0$$

Solving these simultaneous linear equations symbolically would be extremely tedious and result in an extremely unwieldy expression for the critical load. Thus, for each design we will input the beam length, modulus of elasticity, area moment of inertia, and the spring constant and solve the system of equations using a computational solver.

We found the moment of inertia to be 51.051 in^4 for this beam.

We decided against using the spring with spring constant of 5000 lbf/in, as it offered relatively minor performance gains for a spring that would be heavy and costly.

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The critical loads for each situation are tabulated in the appendix for each material, approximation method, and spring constant. Selected trends are displayed on the following pages.

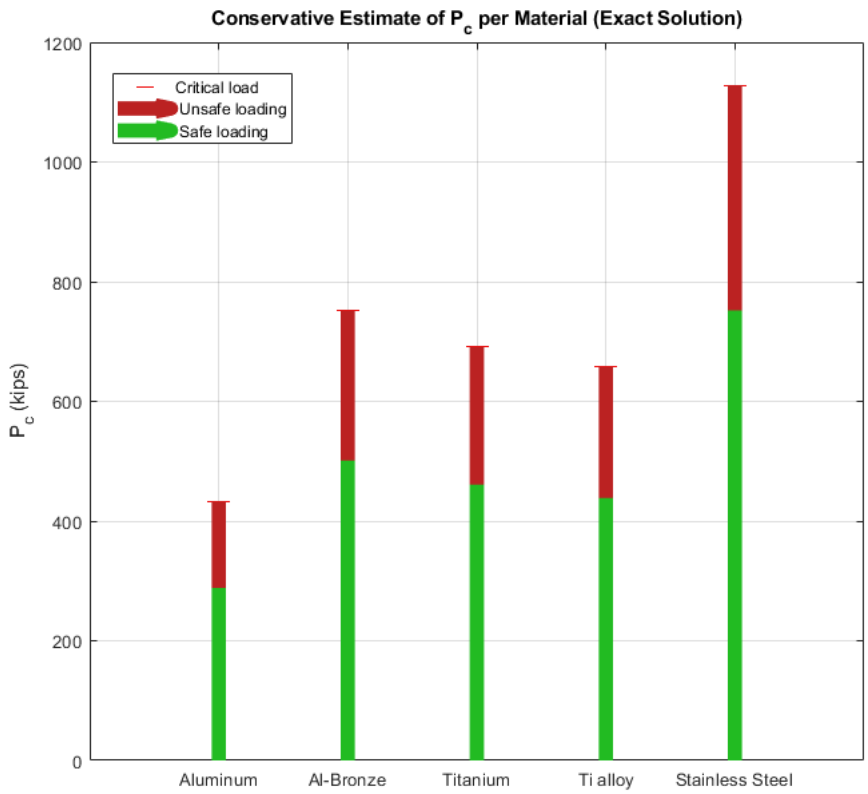

Figure 3: *Exact Critical Load per Material*

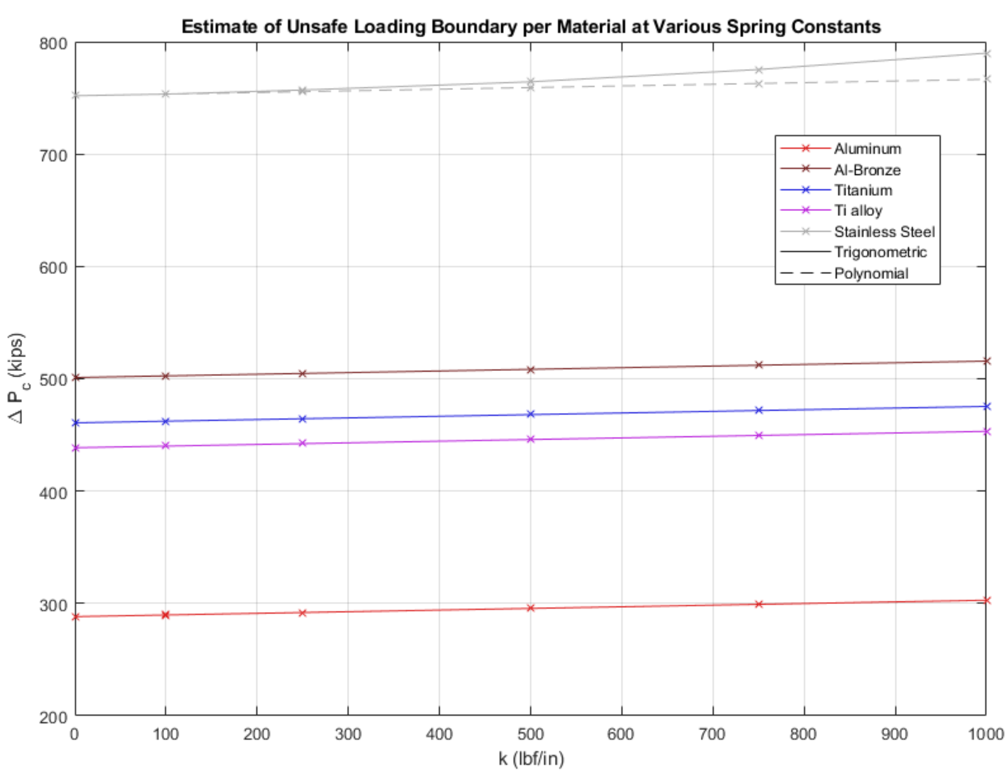

Figure 4: *Critical Load Approximations*

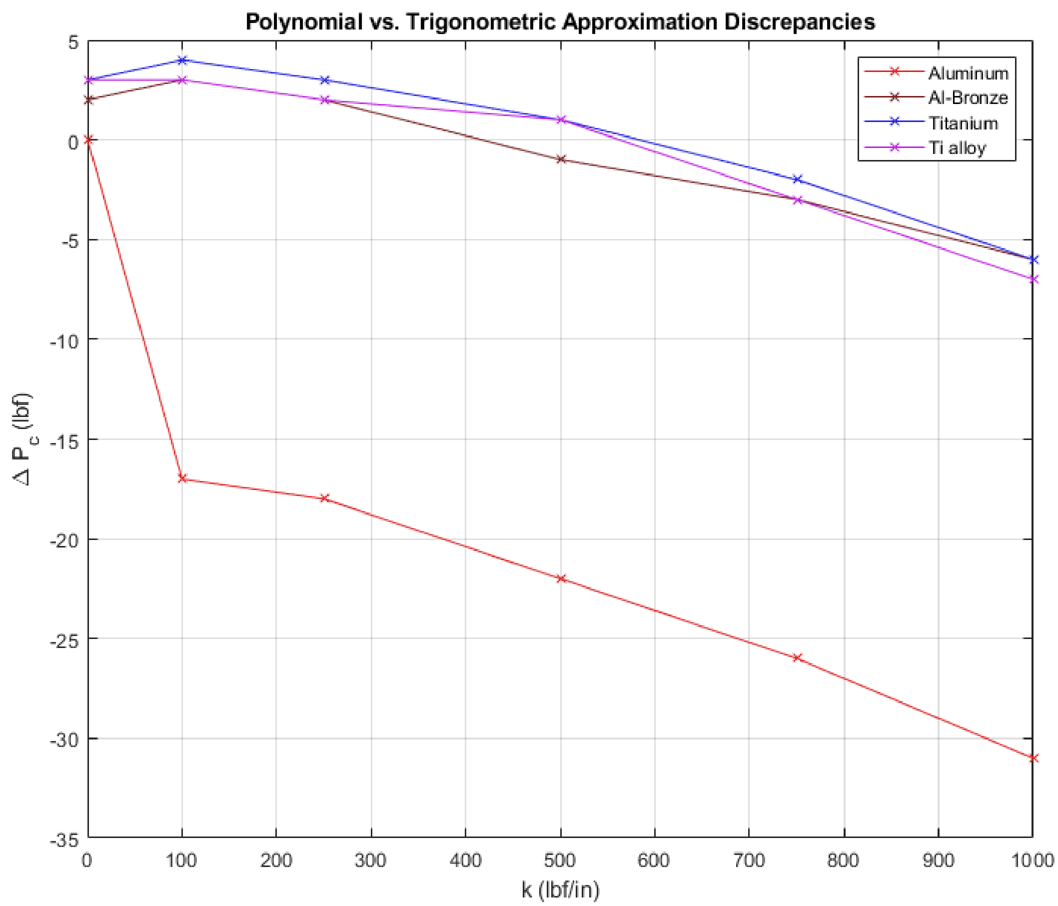

Figure 5: *Approximation Error for Aluminum, Titanium and Alloys*

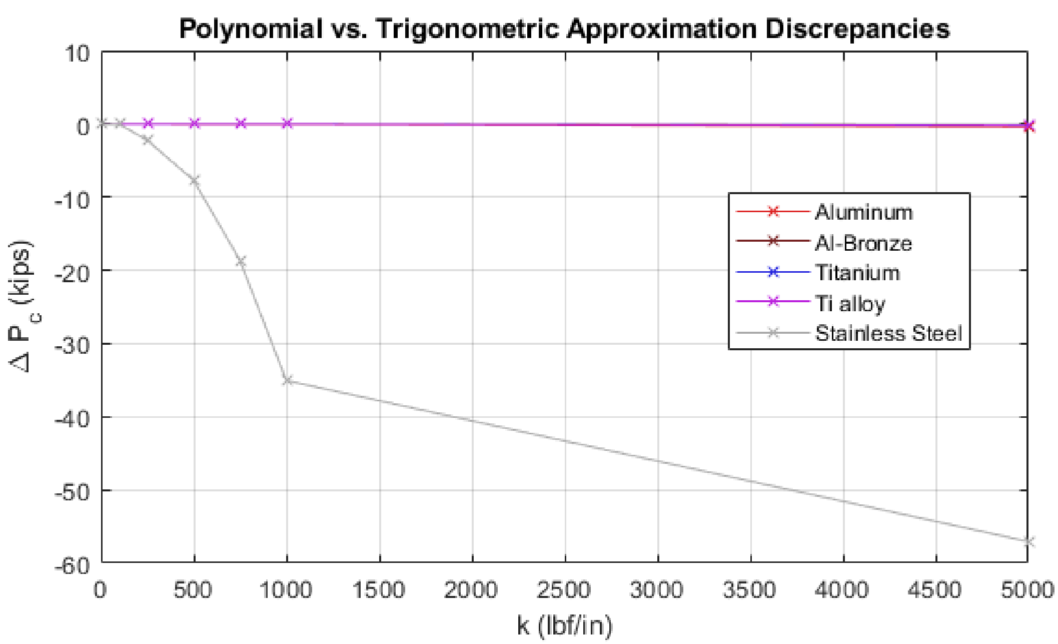

Figure 6. *Approximation Error for Stainless Steel*

The stainless steel approximation error is a major outlier; the accuracy of the polynomial approximation seems to degrade rapidly for values of EI larger than 10^9 lbf in^2 but begins to taper off for larger spring constants.

Conclusion:

The stainless steel beam can withstand the highest loading, but also has the highest density of any material and thus would be costly to launch in an upper stage due to the increased mass. The aluminum-bronze alloy shows a surprisingly high strength compared to the other materials, but would be susceptible to the harsh space environment. The aluminum beam has the lowest critical load, but would be the least expensive option. Finally, since titanium and titanium alloy have similar loading characteristics, we would opt for the titanium alloy as it retains the resistance to the space environment of pure titanium, and would have a lower mass and cost.

One important lesson we learned over the course of this project is how slight changes in the structure can change the manner by which we can determine critical loads. The exact solution contains several intermediate simplifications which rely on specific values to be zero, which are often nonzero for any case other than the most simple loading and constraints. However, the approximate solutions were virtually identical to the exact solution for the case without spring loading. While the accuracy of the approximations decreased with stiffer springs and beam materials, they vastly simplified the load determination process so that we may simply substitute in problem parameters and compute the critical load for each case. In fact, this process could be easily automated and applied to much more advanced systems by modeling a number of smaller elements with approximate solutions for each element. Thus, it is efficient for engineers to use approximate methods for determination of critical loads in complex structures.

With additional resources, it would be interesting to perform this approximation while treating all beams in the structure to be elastic and include the movement of the beam-beam connections. We could also employ a more advanced design for the spring connector, using telescoping beams for additional stability and control. We would also include the possibility of the landing gear sliding on the surface instead of being fixed at a certain point. Finally, we would determine the cause of the extreme approximation error for the stainless steel material and perhaps generate a more accurate polynomial function.

References:

[1] BesTech, “Modulus of Elasticity - Young Modulus for some common Materials,” BesTech, https://www.bestech.com.au/wp-content/uploads/Modulus-of-Elasticity.pdf

[2] NASA, “Lunar Module - Quick Reference Data,” NASA, https://www.hq.nasa.gov/alsj/LM04_Lunar_Module_ppLV1-17.pdf

[3] NASA, “Spacecraft Materials and the Chemistry of Space Exploration,” Jet Propulsion Laboratory, https://www.jpl.nasa.gov/edu/teach/activity/spacecraft-materials-and-the-chemistry-of-space-exploration/

Appendix:

Tabulated Critical Loads (kips)

Material		Aluminum	Al-Bronze	Titanium	Ti-alloy	Stainless Steel
Exact Solution		432.405	751.633	691.157	657.895	1127.881
Approximate Solutions						
k = 0 lbf/in.	Trigonometric	432.405	751.633	691.157	657.895	1127.881
	Polynomial	432.405	751.635	691.160	657.898	1127.846
k = 100 lbf/in.	Trigonometric	434.593	753.821	693.345	660.083	1130.070
	Polynomial	434.576	753.824	693.349	660.086	1130.035
k = 250 lbf/in.	Trigonometric	437.876	757.104	696.628	663.366	1135.541
	Polynomial	437.858	757.106	696.631	663.368	1133.318
k = 500 lbf/in.	Trigonometric	443.348	762.576	702.099	668.837	1146.484
	Polynomial	443.326	762.575	702.100	668.838	1138.787
k = 750 lbf/in.	Trigonometric	448.819	768.047	707.571	674.309	1162.898
	Polynomial	448.793	768.044	707.569	674.306	1144.256
k = 1000 lbf/in.	Trigonometric	454.290	773.518	713.042	679.780	1184.783
	Polynomial	454.259	773.512	713.036	679.773	1149.725
k = 5000 lbf/in.	Trigonometric	541.832	861.060	800.583	767.322	1294.210
	Polynomial	541.487	860.872	800.381	767.108	1237.144

Exact Solution

We integrate and neglect higher order terms of the virtual displacement given its proportion to the actual displacement to obtain

$$U_{beam}=\frac{1}{2}\int_0^L EI(y'')^2 dx$$

$$\begin{aligned}
\rightarrow \delta U_{beam} &= \frac{1}{2}EI\int_0^L (y''+\delta y'')^2 dx-\frac{1}{2}EI\int_0^L (y'')^2 dx \\
&= \frac{1}{2}EI\int_0^L \left((y'')^2+(\delta y'')(y'')+(\delta y'')^2-(y'')^2\right) dx \\
&= \frac{1}{2}EI\int_0^L \left((\delta y'')(y'')+(\delta y'')^2\right) dx \\
&= \frac{1}{2}EI\int_0^L (\delta y'')(y'') dx
\end{aligned}$$

To find the potential, we must determine the axial displacement of the beam, similarly neglecting higher order differential terms:

$$\Delta L=\int_0^L (ds-dx)$$

$$\begin{aligned}
ds &= \sqrt{dx^2+dy^2} \\
&= dx\sqrt{1+(y')^2} \\
&= dx\left(1+\frac{1}{2}(y')^2+\ldots\right) \\
&= dx+dx(y')^2
\end{aligned}$$

$$\rightarrow ds-dx=\frac{1}{2}(y')^2$$

$$\rightarrow \Delta L=\frac{1}{2}\int_0^L (y')^2 dx$$

$$\begin{aligned}
V &= -P\Delta L \\
&= -\frac{P}{2}\int_0^L (y')^2 dx
\end{aligned}$$

Neglecting the higher order virtual displacement terms, we obtain

$$\delta V = -\frac{P}{2}\int_0^L (y'+\delta y')^2 dx + \frac{P}{2}\int_0^L (y')^2 dx$$
$$= -\frac{P}{2}\int_0^L \left((y')^2+(\delta y')(y')+(\delta y')^2-(y')^2\right) dx$$
$$= -\frac{P}{2}\int_0^L \left((\delta y')(y')+(\delta y')^2\right) dx$$
$$= -\frac{P}{2}\int_0^L (\delta y')(y') dx$$

Using the First Minimum Principle, we obtain the following equality:

$$\delta U_{beam} + \delta V = 0$$
$$\rightarrow \frac{1}{2} EI \int_0^L (\delta y'')(y'') dx - \frac{P}{2}\int_0^L (\delta y')(y') dx = 0$$
$$\rightarrow EI \int_0^L (\delta y'')(y'') dx - P\int_0^L (\delta y')(y') dx = 0$$
$$\rightarrow \int_0^L (\delta y)(EI\, y''''+P\, y'') dx = 0$$

Since the virtual displacement must be non-zero at at least one point on the beam, it follows that

$$EI\, y'''' + Py'' = 0$$

Let

$$v = y''$$

This produces a well-known second-order linear differential equation form:

$$v'' + \frac{P}{EI} v = 0$$

Assuming that the load is nonzero, the solution to this differential equation is

$$v = A\sin\left(\sqrt{\frac{P}{EI}}x\right) + B\cos\left(\sqrt{\frac{P}{EI}}x\right)$$

Let

$$k = \sqrt{\frac{P}{EI}}$$

Substituting the displacement back into the solution gives another differential equation:

$$y'' = A\sin(kx) + B\cos(kx)$$

We integrate this differential equation twice to get the general form of the displacement function:

$$y' = A\sin(kx) + B\cos(kx) + C$$
$$y = A\sin(kx) + B\cos(kx) + Cx + D$$

We use the boundary conditions to simplify:

$$y(0) = y(L) = 0$$
$$y''(0) = y''(L) = 0$$

$$y(0) = B + D = 0$$
$$\rightarrow y = A\sin(kx) - D\cos(kx) + Cx + D$$
$$y''(0) = Dk^2 = 0$$
$$\rightarrow y = A\sin(kx) + Cx$$
$$y(L) = A\sin(kL) + CL = 0$$
$$\rightarrow C = -A\frac{\sin(kL)}{L}$$
$$\rightarrow y = A\sin(kx) - A\frac{\sin(kL)}{L}$$
$$y''(L) = -Ak^2\sin(kL) = 0$$
$$\rightarrow -A\sin(kL) = 0$$

The arbitrary constant is nonzero for all modes except for the trivial case of a straight beam. Therefore, it is true for these other modes, including the Euler loading, that

$$\sin(kL) = 0$$
$$\rightarrow kL = n\pi$$
$$\rightarrow \sqrt{\frac{P}{EI}} = n\frac{\pi}{L}$$
$$\rightarrow \frac{P}{EI} = n^2\frac{\pi^2}{L^2}$$
$$\rightarrow P = n^2\frac{\pi^2 EI}{L^2}$$

Since there are no additional constraints on the beam, buckling will occur at the Euler load in the first buckling mode, so that the critical load is

$$P = \frac{\pi^2 EI}{L^2}$$

Trigonometric Approximation

$$y = A\sin\left(\frac{\pi x}{L}\right) + B\cos\left(\frac{\pi x}{L}\right)$$

Boundary conditions:

$$y(0) = y(L) = 0$$
$$y''(0) = y''(L) = 0$$

From the displacement boundary condition, we eliminate the constant B:

$$y(0) = A\sin(0) + B\cos(0)$$
$$= B$$
$$\rightarrow B = 0$$

This results in a displacement as a function of only one constant:

$$y = A\sin\left(\frac{\pi x}{L}\right)$$

Differentiation:

$$y = A\sin\left(\frac{\pi x}{L}\right)$$

$$y' = \frac{\pi}{L}A\cos\left(\frac{\pi x}{L}\right)$$
$$y'' = -\left(\frac{\pi}{L}\right)^2 A\sin\left(\frac{\pi x}{L}\right)$$

Energy analysis:

$$U_{beam} = \frac{EI}{2}\int_0^L (y'')^2 dx \qquad V = -\frac{P}{2}\int_0^L (y')^2 dx$$

$$= \frac{EI}{2}\int_0^L \left(\frac{\pi}{L}\right)^4 \left(A\sin\left(\frac{\pi x}{L}\right)\right)^2 dx \qquad = -\frac{P}{2}\int_0^L \left(\frac{\pi}{L}\right)^2 \left(A\cos\left(\frac{\pi x}{L}\right)\right)^2 dx$$

$$= \frac{EI\,\pi^4 A^2}{2L^4}\int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \qquad = -\frac{P\,\pi^2 A^2}{L^2}\int_0^L \cos^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{EI\,\pi^4 A^2}{4L^3} \qquad = -\frac{P\,\pi^2 A^2}{4L}$$

$$U_{spring} = \frac{1}{2}k\,y^2$$

$$= \frac{1}{2}k\,y^2\left(\frac{L}{2}\right)$$

$$= \frac{1}{2}k\,A^2 \sin^2\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{2}k\,A^2$$

Using the Rayleigh approximation, we differentiate the energy functions with respect to the arbitrary constant to find the critical load:

$$\frac{\partial U + \partial V}{\partial A} = \frac{EI\,\pi^4 A}{2L^3} + k\,A - \frac{P\,\pi^2 A}{2L} = 0$$

$$\rightarrow \frac{\pi^2}{2L}P = \frac{\pi^4}{2L^3}EI + k$$

$$\rightarrow P = \frac{\pi^2}{L^2}EI + \frac{2L}{\pi^2}k$$

Polynomial approximation

$$y = A + Bx^2 + Cx^4 + Dx^6$$

The boundary conditions are

$$y\left(\frac{L}{2}\right) = y\left(-\frac{L}{2}\right) = 0$$

$$y''\left(\frac{L}{2}\right) = y''\left(-\frac{L}{2}\right) = 0$$

Using these boundary conditions, we obtain an expression for the displacement in terms of two constants, C and D:

$$y''\left(\frac{L}{2}\right) = 2B + 12C\frac{L^2}{4} + 30D\frac{L^4}{64} = 0$$

$$\rightarrow B = -\frac{3}{2}CL^2 - \frac{15}{16}DL^4$$

$$\rightarrow y = A + \left(-\frac{3}{2}CL^2 - \frac{15}{16}DL^4\right)x^2 + Cx^4 + Dx^6$$

$$y\left(\frac{L}{2}\right) = A - \frac{3}{8}CL^4 - \frac{15}{64}DL^6 + \frac{1}{16}CL^4 + \frac{1}{64}DL^6 = 0$$

$$\rightarrow A = \frac{5}{16}CL^4 + \frac{7}{32}DL^6$$

$$\rightarrow y = \frac{5}{16}CL^4 + \frac{7}{32}DL^6 + \left(-\frac{3}{2}CL^2 - \frac{15}{16}DL^4\right)x^2 + Cx^4 + Dx^6$$

$$\rightarrow y = \left(x^4 - \frac{3}{2}L^2x^2 + \frac{5}{16}L^4\right)C + \left(x^6 - \frac{15}{16}L^4x^2 + \frac{7}{32}L^6\right)D$$

Differentiation

$$y' = \left(4x^3 - 3L^2x\right)C + \left(6x^5 - \frac{15}{8}L^4x\right)D$$

$$y'' = \left(12x^2 - 3L^2\right)C + \left(30x^4 - \frac{15}{8}L^4\right)D$$

Energy analysis

$$U_{beam} = \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} EI\left(y''\right)^2 dx$$

$$= \frac{EI}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\left(12x^2 - 3L^2\right)C + \left(30x^4 - \frac{15}{8}L^4\right)D\right)^2 dx$$

$$= EI\left(\frac{12}{5}C^2L^5 + \frac{24}{7}CDL^7 + \frac{5}{4}D^2L^9\right)$$

$$U_{spring} = \frac{1}{2}k\,y(0)^2$$

$$= \frac{1}{2}k\left(\frac{5}{16}L^4C + \frac{7}{32}L^6D\right)^2$$

$$= k\left(\frac{25}{512}C^2L^8 + \frac{35}{512}CDL^{10} + \frac{49}{2048}D^2L^{12}\right)$$

$$V = -\frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} P\left(y'\right)^2 dx$$

$$= -\frac{P}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\left(4x^3 - 3L^2x\right)C + \left(6x^5 - \frac{15}{8}L^4x\right)D\right)^2 dx$$

$$= -P\left(\frac{17}{70}C^2L^7 + \frac{29}{84}CDL^9 + \frac{303}{2464}D^2L^{11}\right)$$

$$\frac{\partial U_{beam}}{\partial C} = EI\left(\frac{24}{5}CL^5 + \frac{24}{7}DL^7\right)$$

$$\frac{\partial U_{beam}}{\partial D} = EI\left(\frac{24}{7}CL^7 + \frac{5}{2}DL^9\right)$$

$$\frac{\partial U_{spring}}{\partial C} = k\left(\frac{25}{256}CL^8 + \frac{35}{512}DL^{10}\right)$$

$$\frac{\partial U_{spring}}{\partial D} = k\left(\frac{35}{512}CL^{10} + \frac{49}{1024}DL^{12}\right)$$

$$\frac{\partial V}{\partial C} = -P\left(\frac{17}{35}CL^7 + \frac{29}{84}DL^9\right)$$
$$\frac{\partial V}{\partial D} = -P\left(\frac{29}{84}CL^9 + \frac{303}{1232}DL^{11}\right)$$

$$\frac{\partial U_{beam} + \partial U_{spring} + \partial V}{\partial C} = EI\left(\frac{24}{5}CL^5 + \frac{24}{7}DL^7\right) + k\left(\frac{25}{256}CL^8 + \frac{35}{512}DL^{10}\right) - P\left(\frac{17}{35}CL^7 + \frac{29}{84}DL^9\right) = 0$$
$$\frac{\partial U_{beam} + \partial U_{spring} + \partial V}{\partial D} = EI\left(\frac{24}{7}CL^7 + \frac{5}{2}DL^9\right) + k\left(\frac{35}{512}CL^{10} + \frac{49}{1024}DL^{12}\right) - P\left(\frac{29}{84}CL^9 + \frac{303}{1232}DL^{11}\right) = 0$$

$$\rightarrow \left(\frac{24}{5}EI - \frac{17}{35}L^2 P + \frac{25}{256}L^3 k\right)C + \left(\frac{24}{7}L^2 EI - \frac{29}{84}L^4 P + \frac{35}{512}L^5 k\right)D = 0$$
$$\rightarrow \left(\frac{24}{7}L^2 EI - \frac{29}{84}L^4 P + \frac{35}{512}L^5 k\right)C + \left(\frac{5}{2}L^4 EI - \frac{303}{1232}L^6 P + \frac{49}{1024}L^7 k\right)D = 0$$