# 10 Practice Homework for the final exam (do not turn in)

page: 271: 1c

page 280: 1c, 2c, 3c, 10c

1. On $\mathbb{R}^n$ we consider the function that for $x = (x_1, ..., x_n)$ is defined as

$$\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}.$$

Prove that this defines a norm on $\mathbb{R}^n$.

2. On $\mathbb{R}^n$ we consider the function that for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is defined as

$$\langle x, y \rangle = \sum_{i=1}^{n} w_i x_i y_i,$$

where $w_i, i = 1, ..., n$ are positive constants. Prove that this defines an inner product on $\mathbb{R}^n$. What is the norm induced by this inner product.

3. On $L^2[-\pi, \pi]$ we consider the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

a) Consider $f(x) = \cos 2x$ and $g(x) = \sin x$. Calculate $\langle f, g \rangle$.

c) Find the norm $\|f\|_2$, with $f(x) = \sin 2x$.

4) Prove that in a Hilbert space two vectors $x, y$ are orthogonal if and only if

$$\|x + y\|^2 = \|x - y\|^2.$$

5. Let $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi \leq x \leq 0 \\ 0 & \text{if} \quad 0 < x \leq \pi \end{cases}.$$

Find the Fourier coefficients $a_{-2}a_{-1}, a_0, a_1, a_2$.

6. Let $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(x) = |x|.$$

Find the Fourier coefficients $a_{-2}a_{-1}, a_0, a_1, a_2$. Assume that $..., a_{-n}, ..., a_n, ...$ are its Fourier coefficients. Using Parseval's identity, calculate

$$\sum_{n=-\infty}^{\infty} |a_n|^2.$$

7. Consider the Haar wavelet $\psi_{n,k}(x) = 2^{\frac{n}{2}} \psi_{n,k}(2^n x - k)$, $n, k \in \mathbb{Z}$.

a) Prove that $\psi_{2,1}(x)$ and $\psi_{2,0}(x)$ are orthogonal.

b) Prove that $\psi_{1,0}(x)$ and $\psi_{2,0}(x)$ are orthogonal.

b) Consider $f(x) = x$ find the wavelet coefficients $a_{2,0}$, $a_{2,1}$, $a_{2,2}$ and $a_{2,3}$.