7 Homework (Hilbert Spaces)

1. On $\mathbb{R}^n$ we consider the function that for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is defined as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Prove that this defines an inner product on $\mathbb{R}^n$.

2. Prove that on $L^2 0, 1]$ the function

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx.$$

defines an inner product.

3. On $L^2[-\pi, \pi]$ we consider the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx.$$

a) Consider $f(x) = \cos x$ and $g(x) = \sin x$. Calculate $\langle f, g \rangle$.
b) Consider $f(x) = \sin x$ and $g(x) = \sin 2x$. Calculate $\langle f, g \rangle$.
c) Find the norm $\|f\|_2$, with $f(x) = \sin x$.

4. On $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices we consider the function

$$\langle A, B \rangle = tr(A^t B),$$

where $tr(C) = \sum_{i=1}^{n} a_{ii}$ is the trace of the matrix C.

a) Prove that the given function defines an inner product, so $M_{m,n}(\mathbb{R})$ becomes a Hilbert space with this inner rpoduct.
b) Find the norm induced by this inner product?
c) Find $\left\| \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \right\|$ given in this norm.

5) Prove that in any inner product space we have

$$\langle x, y \rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4}.$$

6) Prove that in any inner product space we have

$$\|\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle\| \leq \|x_1 - x_2\| \|y_1\| + \|x_2\| \|y_1 - y_2\|.$$