CS241 Homework 1

For this problem we restrict ourselves to 3x3 matrices as transformations of points in $\Re^3$. No fourth row, fourth column, or fourth (homogeneous) coordinate. A useful subset of all such 3x3 transformations, the so called **rigid transformations** (also known as **orthonormal transformations** in mathematics), are defined here, and some of their properties are investigated.

Definition: A 3x3 matrix is called a **rigid transformation** if it satisfies

Property **R**: $M^{-1} = M^T$

1. Show that another way to state property **R** is $M M^T = I$ **.**

2. Show that the product of any two matrices which satisfy property **R**, also satisfies **R**.

3. Show that all three of our basic 3D rotation matrices satisfy property **R**. (This and #2 taken together mean that any complex rotation matrix built from a product of simple rotation matrices satisfies property **R**.)

4. Let the columns of a matrix M, which satisfies property **R**, be considered as (column) vectors m_1, m_2, m_3. What can you say about the dot product of any two of them. (Considered all possibilities: $m_1 \cdot m_1$, $m_1 \cdot m_2$, ...)

5. Show that such a matrix, when used to transform vectors, preserves both the length of vectors and angles between vectors. (**Note:** It is from this property that the name rigid transformation comes.) **Hint:** Both length of, and angle between, vectors can be written as dot products, and dot products can be written as matrix products, and matrix multiplication satisfies various properties. See hints below.

Hints: Here is a reminder of some well known properties of dot products and matrix multiplication, which you should already know, and may need to use:

- Inverse of a matrix product: $(AB)^{-1} = B^{-1} A^{-1}$
- Transpose of a matrix product: $(AB)^T = B^T A^T$
- Dot product as a matrix product: $a \cdot b = a^T b$
- Matrix product as dot products: Each element m_{ij} of a matrix product, $M = AB$, can be considered as a dot product $m_{ij} = a_i \cdot b_j$ where a_i is A's i^{th} row and b_j is B's j^{th} column.
- Length (squared) of a vector as a dot product: $\|A\|^2 = A \cdot A$
- Angle α between two vectors as a dot product: $\cos\alpha = \dfrac{A \cdot B}{\|A\| \ \|B\|}$.

A note about proofs: The word “show” in these problems implies that you must prove the statement is true for **ALL** possible input values. Don't get caught in an attempt to “prove by example”. Running a specific example through one of these statements is a fine way to gain an understanding of a problem, but it's not a solution to the general problem. (Unless you can enumerate **ALL** possible inputs – something that's not likely when there are infinitely many such.) If you are adept at formal proofs, then all of these problems can be solved with simple algebraic manipulations starting with Property **R**. If not, then long-winded English sentences can be used to explain why a statement is true for any input values. (I'll accept either, but can you guess which I'd prefer?)