and the initial condition

```
>init:=y(0)=0.5;
```

The names `deq` and `init` are chosen by the user. The command to solve the initial-value problems is

```
>deqsol:=dsolve({deq,init},y(t));
```

The response is

$$\text{deqsol} := y(t) = 1 + t^2 + 2t - \frac{1}{2}e^t$$

To use the solution to obtain $y(1.5)$, we enter

```
>q:=rhs(deqsol); evalf(subs(t=1.5,q));
```

with the result 4.009155465.

The function `rhs` is used to assign the solution of the initial-value problem to the function q, which we then evaluate at $t = 1.5$. The function `dsolve` can fail if an explicit solution to the initial-value problem cannot be found. For example, the command

```
>deqsol2:=dsolve({D(y)(t)=1+t*sin(t*y(t)),y(0)=0},y(t));
```

does not succeed because an explicit solution cannot be found. In this case a numerical method must be used.

EXERCISE SET 5.1

1. Use Theorem 5.4 to show that each of the following initial-value problems has a unique solution, and find the solution.
   a. $y' = y\cos t, \quad 0 \le t \le 1, \quad y(0) = 1.$
   b. $y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0.$
   c. $y' = -\frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = \sqrt{2}e.$
   d. $y' = \frac{4t^3y}{1+t^4}, \quad 0 \le t \le 1, \quad y(0) = 1.$

2. For each choice of $f(t, y)$ given in parts (a)–(d):
   i. Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$?
   ii. Can Theorem 5.6 be used to show that the initial-value problem

   $$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

   is well-posed?

   a. $f(t, y) = t^2y + 1$
   b. $f(t, y) = ty$
   c. $f(t, y) = 1 - y$
   d. $f(t, y) = -ty + \frac{4t}{y}$

where δ_i denotes the roundoff error associated with u_i. Using methods similar to those in the proof of Theorem 5.9, we can produce an error bound for the finite-digit approximations to y_i given by Euler's method.

Theorem 5.10 Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha \tag{5.12}$$

and $u_0, u_1, \ldots, u_N$ be the approximations obtained using (5.11). If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the hypotheses of Theorem 5.9 hold for (5.12), then

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)[e^{L(t_i-a)} - 1] + |\delta_0|e^{L(t_i-a)}, \tag{5.13}$$

for each $i = 0, 1, \ldots, N$. ■

The error bound (5.13) is no longer linear in h. In fact, since

$$\lim_{h\to 0}\left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty,$$

the error would be expected to become large for sufficiently small values of h. Calculus can be used to determine a lower bound for the step size h. Letting $E(h) = (hM/2) + (\delta/h)$ implies that $E'(h) = (M/2) - (\delta/h^2)$.

$$\text{If} \quad h < \sqrt{2\delta/M}, \quad \text{then} \quad E'(h) < 0 \text{ and } E(h) \text{ is decreasing.}$$

$$\text{If} \quad h > \sqrt{2\delta/M}, \quad \text{then} \quad E'(h) > 0 \text{ and } E(h) \text{ is increasing.}$$

The minimal value of $E(h)$ occurs when

$$h = \sqrt{\frac{2\delta}{M}}. \tag{5.14}$$

Decreasing h beyond this value tends to increase the total error in the approximation. Normally, however, the value of δ is sufficiently small that this lower bound for h does not affect the operation of Euler's method.

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with $h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.25$
2. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.

a. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

b. $y(t) = t + \frac{1}{1-t}$

c. $y(t) = t \ln t + 2t$

d. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

3. Use Euler's method to approximate the solutions for each of the following initial-value problems.

a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1, \text{ with } h = 0.1$

b. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0, \text{ with } h = 0.2$

c. $y' = -(y+1)(y+3), \quad 0 \le t \le 2, \quad y(0) = -2, \text{ with } h = 0.2$

d. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3}, \text{ with } h = 0.1$

4. The actual solutions to the initial-value problems in Exercise 3 are given here. Compute the actual error in the approximations of Exercise 3.

a. $y(t) = \frac{t}{1 + \ln t}$

b. $y(t) = t\tan(\ln t)$

c. $y(t) = -3 + \frac{2}{1 + e^{-2t}}$

d. $y(t) = t^2 + \frac{1}{3}e^{-5t}$

5. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$

c. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using Eq. (5.10).

6. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

a. Use Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$

c. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.05$ using eq. (5.10).

7. Given the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 5, \quad y(0) = 1,$$

with exact solution $y(t) = e^{-t} + t$:

a. Approximate $y(5)$ using Euler's method with $h = 0.2$, $h = 0.1$, and $h = 0.05$.

b. Determine the optimal value of h to use in computing $y(5)$, assuming $\delta = 10^{-6}$ and that Eq. (5.14) is valid.