Proof Note that Eq. (5.16) can be rewritten

$$y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2}f'(t_i, y_i) - \cdots - \frac{h^n}{n!}f^{(n-1)}(t_i, y_i) = \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)),$$

for some ξ_i in (t_i, t_{i+1}). So the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)),$$

for each $i = 0, 1, \ldots, N-1$. Since $y \in C^{n+1}[a, b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i = O(h^n)$, for each $i = 1, 2, \ldots, N$. ■ ■ ■

EXERCISE SET 5.3

1. Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.25$
2. Repeat Exercise 1 using Taylor's method of order four.
3. Use Taylor's method of order two and four to approximate the solution for each of the following initial-value problems.
   a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 1.2, \quad y(1) = 1,$ with $h = 0.1$
   b. $y' = \sin t + e^{-t}, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$
   c. $y' = 1/t(y^2 + y), \quad 1 \le t \le 3, \quad y(1) = -2,$ with $h = 0.5$
   d. $y' = -ty + 4t/y, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.25$
4. Use the Taylor method of order two with $h = 0.1$ to approximate the solution to
$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \quad y(0) = 0.$$
5. Given the initial-value problem
$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$
with exact solution $y(t) = t^2(e^t - e)$:
   a. Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
   b. Use the answers generated in part (a) and linear interpolation to approximate y at the following values, and compare them to the actual values of y.
      i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$

Euler's method with $h = 0.025$, the Midpoint method with $h = 0.05$, and the Runge-Kutta fourth-order method with $h = 0.1$ are compared at the common mesh points of these methods 0.1, 0.2, 0.3, 0.4, and 0.5. Each of these techniques requires 20 functional evaluations to determine the values listed in Table 5.8 to approximate $y(0.5)$. In this example, the fourth-order method is clearly superior. ■

Table 5.8

t_i	Exact	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta Order Four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
0.3	1.0150706	1.0089334	1.0147254	1.0150701
0.4	1.2140877	1.2056345	1.2136079	1.2140869
0.5	1.4256394	1.4147264	1.4250141	1.4256384

EXERCISE SET 5.4

1. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.
   a. $y' = te^{3t} - 2y$, $0 \le t \le 1$, $y(0) = 0$, with $h = 0.5$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.
   b. $y' = 1+(t-y)^2$, $2 \le t \le 3$, $y(2) = 1$, with $h = 0.5$; actual solution $y(t) = t+\frac{1}{1-t}$.
   c. $y' = 1+y/t$, $1 \le t \le 2$, $y(1) = 2$, with $h = 0.25$; actual solution $y(t) = t\ln t+2t$.
   d. $y' = \cos 2t + \sin 3t$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.25$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.
2. Repeat Exercise 1 using Heun's method.
3. Repeat Exercise 1 using the Midpoint method.
4. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.
   a. $y' = y/t - (y/t)^2$, $1 \le t \le 2$, $y(1) = 1$, with $h = 0.1$; actual solution $y(t) = t/(1 + \ln t)$.
   b. $y' = 1 + y/t + (y/t)^2$, $1 \le t \le 3$, $y(1) = 0$, with $h = 0.2$; actual solution $y(t) = t\tan(\ln t)$.
   c. $y' = -(y + 1)(y + 3)$, $0 \le t \le 2$, $y(0) = -2$, with $h = 0.2$; actual solution $y(t) = -3 + 2(1 + e^{-2t})^{-1}$.
   d. $y' = -5y + 5t^2 + 2t$, $0 \le t \le 1$, $y(0) = \frac{1}{3}$, with $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$.
5. Use the results of Exercise 4 and linear interpolation to approximate values of $y(t)$, and compare the results to the actual values.
   a. $y(1.25)$ and $y(1.93)$
   b. $y(2.1)$ and $y(2.75)$
   c. $y(1.3)$ and $y(1.93)$
   d. $y(0.54)$ and $y(0.94)$

6. Repeat Exercise 4 using Heun's method.

7. Repeat Exercise 5 using the results of Exercise 6.

8. Repeat Exercise 4 using the Midpoint method.

9. Repeat Exercise 5 using the results of Exercise 8.

10. Repeat Exercise 1 using the Runge-Kutta method of order four.

11. Repeat Exercise 4 using the Runge-Kutta method of order four.

12. Use the results of Exercise 11 and Cubic Hermite interpolation to approximate values of $y(t)$, and compare the approximations to the actual values.

a. $y(1.25)$ and $y(1.93)$ **b.** $y(2.1)$ and $y(2.75)$

c. $y(1.3)$ and $y(1.93)$ **d.** $y(0.54)$ and $y(0.94)$

13. Show that the Midpoint method, the Modified Euler method, and Heun's method give the same approximations to the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1,$$

for any choice of h. Why is this true?

14. Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2 \sqrt{2g} \frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft/s^2, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)$ ft^3.

a. Compute the water level after 10 min with $h = 20$ s.

b. Determine, to within 1 min, when the tank will be empty.

15. The irreversible chemical reaction in which two molecules of solid potassium dichromate ($K_2Cr_2O_7$), two molecules of water (H_2O), and three atoms of solid sulfur (S) combine to yield three molecules of the gas sulfur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two molecules of solid chromic oxide (Cr_2O_3) can be represented symbolically by the *stoichiometric equation*:

$$2K_2Cr_2O_7 + 2H_2O + 3S \longrightarrow 4KOH + 2Cr_2O_3 + 3SO_2.$$

If n_1 molecules of $K_2Cr_2O_7$, n_2 molecules of H_2O, and n_3 molecules of S are originally available, the following differential equation describes the amount $x(t)$ of KOH after time t:

$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2 \left(n_2 - \frac{x}{2}\right)^2 \left(n_3 - \frac{3x}{4}\right)^3,$$

where k is the velocity constant of the reaction. If $k = 6.22 \times 10^{-19}$, $n_1 = n_2 = 2 \times 10^3$, and $n_3 = 3 \times 10^3$, how many units of potassium hydroxide will have been formed after 0.2 s?

16. Show that the difference method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + a_1 f(t_i, w_i) + a_2 f(t_i + \alpha_2, w_1 + \delta_2 f(t_i, w_i)),$$

for each $i = 0, 1, \ldots, N-1$, cannot have local truncation error $O(h^3)$ for any choice of constants a_1, a_2, α_2, and δ_2.