# EXERCISE SET 3.1

1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$, and find the actual error.

   **a.** $f(x) = \cos x$ **b.** $f(x) = \sqrt{1+x}$

   **c.** $f(x) = \ln(x+1)$ **d.** $f(x) = \tan x$

2. Use Theorem 3.3 to find an error bound for the approximations in Exercise 1.

3. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

   **a.** $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

   **b.** $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

   **c.** $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$

   **d.** $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$

4. Use Neville's method to obtain the approximations for Exercise 3.

5. Use Neville's method to approximate $\sqrt{3}$ with the function $f(x) = 3^x$ and the values $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$.

6. Use Neville's method to approximate $\sqrt{3}$ with the function $f(x) = \sqrt{x}$ and the values $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 5$. Compare the accuracy with that of Exercise 5.

7. The data for Exercise 3 were generated using the following functions. Use the error formula to find a bound for the error, and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.

   **a.** $f(x) = x \ln x$

   **b.** $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$

   **c.** $f(x) = x \cos x - 2x^2 + 3x - 1$

   **d.** $f(x) = \sin(e^x - 2)$

8. Let $f(x) = \sqrt{x - x^2}$ and $P_2(x)$ be the interpolation polynomial on $x_0 = 0$, $x_1$ and $x_2 = 1$. Find the largest value of $x_1$ in $(0, 1)$ for which $f(0.5) - P_2(0.5) = -0.25$.

9. Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$, and $(2, 2)$. Find $y$ if the coefficient of $x^3$ in $P_3(x)$ is 6.

10. Use the Lagrange interpolating polynomial of degree three or less and four-digit chopping arithmetic to approximate $\cos 0.750$ using the following values. Find an error bound for the approximation.

    $$\cos 0.698 = 0.7661 \quad \cos 0.733 = 0.7432 \quad \cos 0.768 = 0.7193 \quad \cos 0.803 = 0.6946$$

    The actual value of cos 0.750 is 0.7317 (to four decimal places). Explain the discrepancy between the actual error and the error bound.

11. Use the following values and four-digit rounding arithmetic to construct a third Lagrange polynomial approximation to $f(1.09)$. The function being approximated is $f(x) = \log_{10}(\tan x)$. Use this knowledge to find a bound for the error in the approximation.

    $$f(1.00) = 0.1924 \quad f(1.05) = 0.2414 \quad f(1.10) = 0.2933 \quad f(1.15) = 0.3492$$

# EXERCISE SET 4.3

1. Approximate the following integrals using the Trapezoidal rule.

   a. $\int_{0.5}^{1} x^4\,dx$ b. $\int_{0}^{0.5} \frac{2}{x-4}\,dx$

   c. $\int_{1}^{1.5} x^2 \ln x\,dx$ d. $\int_{0}^{1} x^2 e^{-x}\,dx$

   e. $\int_{1}^{1.6} \frac{2x}{x^2-4}\,dx$ f. $\int_{0}^{0.35} \frac{2}{x^2-4}\,dx$

   g. $\int_{0}^{\pi/4} x \sin x\,dx$ h. $\int_{0}^{\pi/4} e^{3x} \sin 2x\,dx$

2. Find a bound for the error in Exercise 1 using the error formula, and compare this to the actual error.
3. Repeat Exercise 1 using Simpson's rule.
4. Repeat Exercise 2 using Simpson's rule and the results of Exercise 3.
5. Repeat Exercise 1 using the Midpoint rule.
6. Repeat Exercise 2 using the Midpoint rule and the results of Exercise 5.
7. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?
8. The Trapezoidal rule applied to $\int_0^2 f(x)\,dx$ gives the value 5, and the Midpoint rule gives the value 4. What value does Simpson's rule give?
9. Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

10. Let $h = (b-a)/3$, $x_0 = a$, $x_1 = a + h$, and $x_2 = b$. Find the degree of precision of the quadrature formula

$$\int_a^b f(x)\,dx = \frac{9}{4}hf(x_1) + \frac{3}{4}hf(x_2).$$

11. The quadrature formula $\int_{-1}^{1} f(x)\,dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$ is exact for all polynomials of degree less than or equal to 2. Determine $c_0$, $c_1$, and $c_2$.
12. The quadrature formula $\int_0^2 f(x)\,dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than an equal to 2. Determine $c_0$, $c_1$, and $c_2$.
13. Find the constants $c_0$, $c_1$, and $x_1$ so that the quadrature formula

$$\int_0^1 f(x)\,dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

14. Find the constants $x_0$, $x_1$, and $c_1$ so that the quadrature formula

$$\int_0^1 f(x)\,dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

An important property shared by all the composite integration techniques is a stability with respect to roundoff error. To demonstrate this, suppose we apply the Composite Simpson's rule with $n$ subintervals to a function $f$ on $[a, b]$ and determine the maximum bound for the roundoff error. Assume that $f(x_i)$ is approximated by $\tilde{f}(x_i)$ and that

$$f(x_i) = \tilde{f}(x_i) + e_i, \quad \text{for each } i = 0, 1, \ldots, n,$$

where $e_i$ denotes the roundoff error associated with using $\tilde{f}(x_i)$ to approximate $f(x_i)$. Then the accumulated error, $e(h)$, in the Composite Simpson's rule is

$$\begin{aligned} e(h) &= \left| \frac{h}{3}\left[ e_0 + 2\sum_{j=1}^{(n/2)-1} e_{2j} + 4\sum_{j=1}^{n/2} e_{2j-1} + e_n \right]\right| \\ &\le \frac{h}{3}\left[ |e_0| + 2\sum_{j=1}^{(n/2)-1} |e_{2j}| + 4\sum_{j=1}^{n/2} |e_{2j-1}| + |e_n| \right]. \end{aligned}$$

If the roundoff errors are uniformly bounded by $\varepsilon$, then

$$e(h) \le \frac{h}{3}\left[\varepsilon + 2\left(\frac{n}{2} - 1\right)\varepsilon + 4\left(\frac{n}{2}\right)\varepsilon + \varepsilon\right] = \frac{h}{3} 3n\varepsilon = nh\varepsilon.$$

But $nh = b - a$, so

$$e(h) \le (b - a)\varepsilon,$$

a bound independent of $h$ (and $n$). This means that, even though we may need to divide an interval into more parts to ensure accuracy, the increased computation that is required does not increase the roundoff error. This result implies that the procedure is stable as $h$ approaches zero. Recall that this was not true of the numerical differentiation procedures considered at the beginning of this chapter.

## EXERCISE SET 4.4

1. Use the Composite Trapezoidal rule with the indicated values of $n$ to approximate the following integrals.

   **a.** $\displaystyle\int_1^2 x \ln x \, dx, \quad n = 4$

   **b.** $\displaystyle\int_{-2}^2 x^3 e^x \, dx, \quad n = 4$

   **c.** $\displaystyle\int_0^2 \frac{2}{x^2 + 4} \, dx, \quad n = 6$

   **d.** $\displaystyle\int_0^{\pi} x^2 \cos x \, dx, \quad n = 6$

   **e.** $\displaystyle\int_0^2 e^{2x} \sin 3x \, dx, \quad n = 8$

   **f.** $\displaystyle\int_1^3 \frac{x}{x^2 + 4} \, dx, \quad n = 8$

   **g.** $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2 - 4}} \, dx, \quad n = 8$

   **h.** $\displaystyle\int_0^{3\pi/8} \tan x \, dx, \quad n = 8$

2. Use the Composite Simpson's rule to approximate the integrals in Exercise 1.
3. Use the Composite Midpoint rule with $n + 2$ subintervals to approximate the integrals in Exercise 1.