3D transformations

The primitive building block transformations

$$S(S_X, S_Y, S_Z) = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_Z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(T_X, T_Y, T_Z) = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algorithm for $R_i(\theta)$:

where

i=0,1,2 for the X,Y,Z axes (respectively)

j = (i+0) mod 3

k = (i+2) mode 3

$R_{jj} = \cos\theta$

$R_{kk} = \cos\theta$

$R_{ii} = R_{33} = 1$

$R_{jk} = -\sin\theta$

$R_{kj} = \sin\theta$

$others = 0$

3D Transformation Interactions

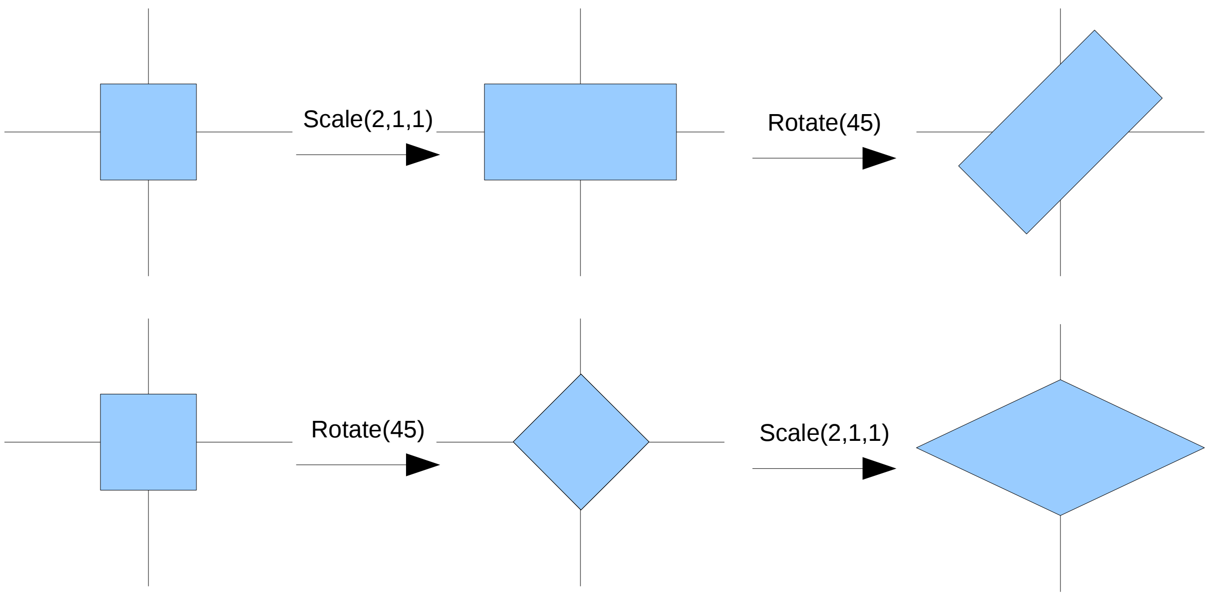
Scale vs Rotate

left[matrix{1# 0# 0# 0## 0# 1# 0# 0## 0# 0# 1# 0## 0# 0# 0# 1} right]

left[matrix{2# 0# 0# 0## 0# 1# 0# 0## 0# 0# 1# 0## 0# 0# 0# 1} right]

left[matrix{cos 45# -sin 45# 0# 0## sin 45# cos 45# 0# 0## 0# 0# 1# 0## 0# 0# 0# 1} right]

left[matrix{1# 0# 0# 0## 0# 1# 0# 0## 0# 0# 1# 0## 0# 0# 0# 1} right]

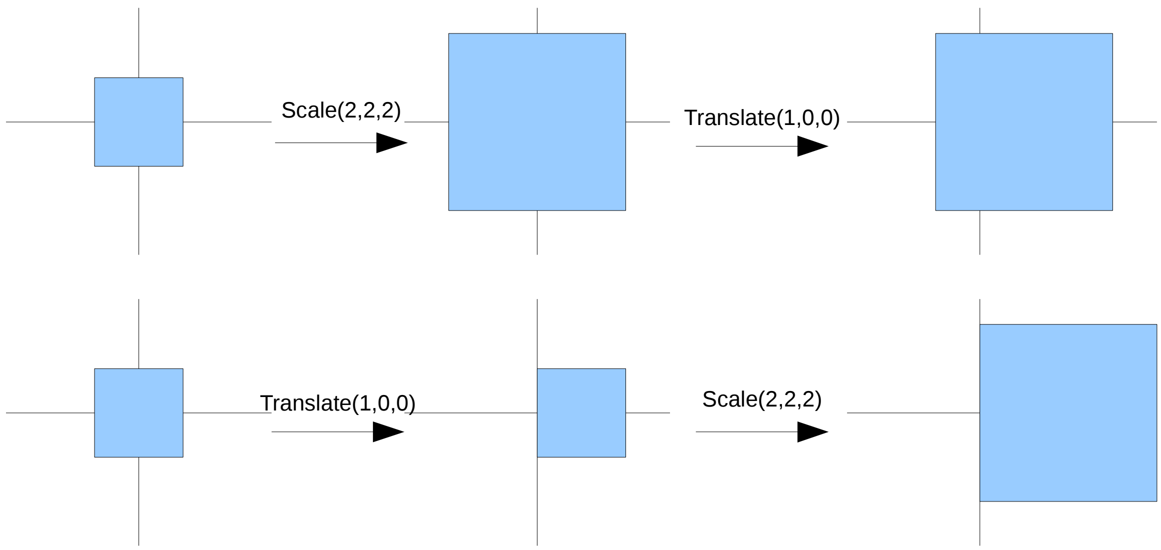


In matrix form, these two series of transformations are:

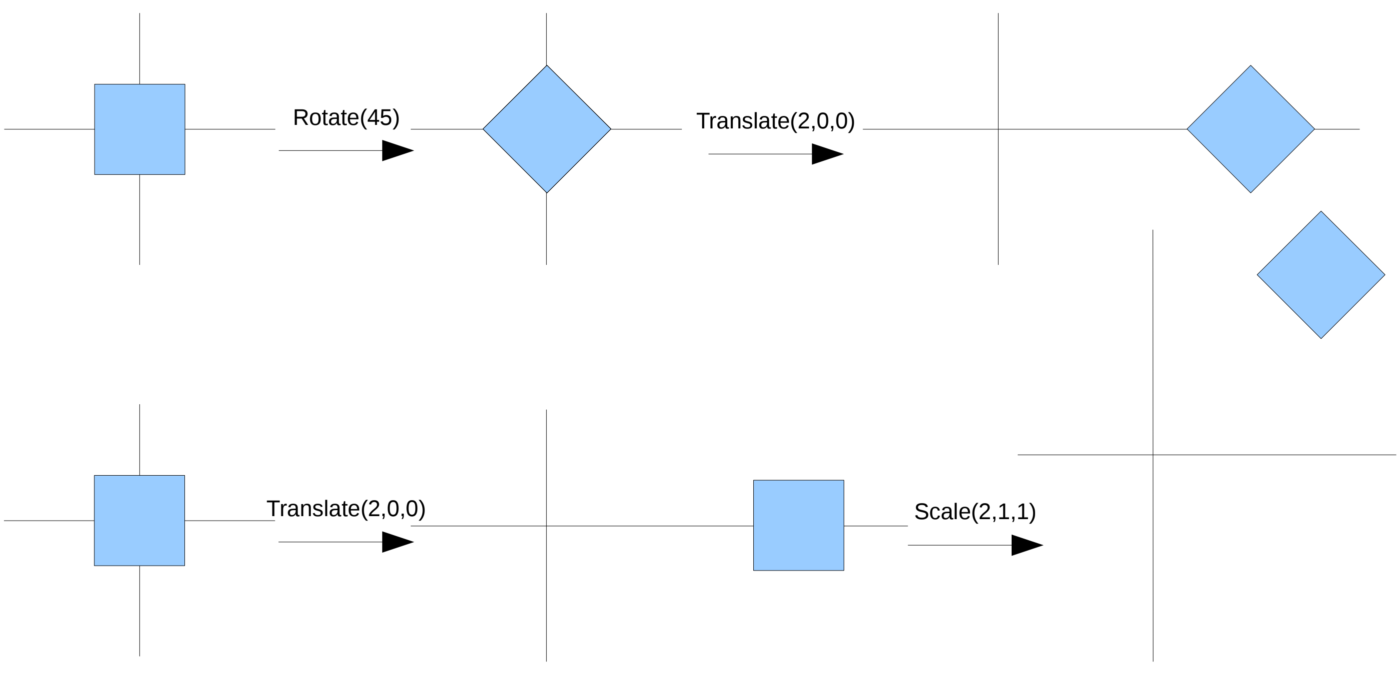
$$\begin{bmatrix} \cos 45 & -\sin 45 & 0 & 0 \\ \sin 45 & \cos 45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cos 45 & -\sin 45 & 0 & 0 \\ 2\sin 45 & \cos 45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45 & -\sin 45 & 0 & 0 \\ \sin 45 & \cos 45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cos 45 & -2\sin 45 & 0 & 0 \\ \sin 45 & \cos 45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

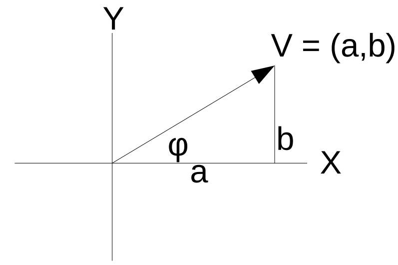
Scale vs Translate



Translate vs Rotate



Complex transformation example



Let's build a rotation by θ around a vector $v = (a, b)$.

Since we know how to rotate around the axes, we must

1. rotate v to the X axis: $R_Z(-\phi)$
2. rotate by the desired angle around X: $R_X(\theta)$
3. rotate v back to its starting direction: $R_Z(\phi)$

Final transformation is

$$R_Z(\phi)\, R_Z(\theta)\, R_Z(-\phi)$$

What about ϕ?

We don't really need ϕ, but only $\cos(\phi)$ and $\sin(\phi)$ to build $R_Z(\phi)$

A little trigonometry gets those two values:

$$\cos(\phi) = a/\sqrt{a^2+b^2}$$
$$\sin(\phi) = b/\sqrt{a^2+b^2}$$

Eye transformation

Specified by

E: Eye position

Z: Direction of view

U: Approximate UP vector

Compute:

$$Y \;=\; U - \frac{U \cdot Z}{Z \cdot Z} Z \text{ as the “real” up vector}$$

$$X \;=\; Z \times Y$$

Normalize:

X, Y, and Z

Note that X, Y, and Z are **orthonormal**

Step 1: translate to origin:

Translate(-E)

Step 2: rotate:

$$R \;=\; \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} \qquad \text{Note that: } \begin{aligned} R\,X^T &= [1,0,0]^T \\ R\,Y^T &= [0,1,0]^T \\ R\,Z^T &= [0,0,1]^T \end{aligned}$$

Why is that called a “rotate?

Orthonormal bases and rigid transformations:

Rows of R, taken as vectors are:

normal: $X \cdot X \;=\; Y \cdot Y \;=\; Z \cdot Z \;=\; 1$

mutually orthogonal: $X \cdot Y \;=\; Y \cdot Z \;=\; Z \cdot X \;=\; 0$

Such transformations are called *rigid* because:

canonical orthonormal vectors transform to orthonormal vectors

Has a matrix interpretation:

Like this: $R R^T = I$,

but: $R R^{-1} = I$,

so also: $R^T = R^{-1}$

This is a feature of all rotates:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R(-\theta) = \left(R(\theta)\right)^{-1} = \left(R(\theta)\right)^{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

And of all products of rotates:

If A, B are rigid transformations, then AB is also:

$A^{-1}=A^{T}$, and $B^{-1}=B^{T}$

so

$(AB)^{-1}=B^{-1}A^{-1}=B^{T}A^{T}=(AB)^{T}$

A more Interactive approach

Model sits on a turntable, with controls:

- C : center of turntable
- α : angle of turntable spin
- β : angle of turntable forward/backward tilt
- γ : perhaps angle of turntable spindle (up) projection
- d : distance of viewing

Transformations:

$$T(0,0,d) \cdot R_z(\gamma) \cdot R_x(\beta) \cdot R_z(\alpha) \cdot T(-C)$$