8 Homework (Fourier Series)

1. Prove that $\cos x$ and $\sin 2x$ are orthogonal as functions in $L^2[-\pi, \pi]$.

2. Let $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } \quad -\pi \leq x \leq 0 \\ 1 & \text{if } \quad 0 < x \leq \pi \end{cases} .$$

a) Find the Fourier coefficients $a_{-2}a_{-1}, a_0, a_1, a_2$.

b) Find the partial Fourier sum: $F(x) = a_{-2}e_{-2}+a_{-1}e_{-1}+a_0e_0+a_1e_1+a_2e_2$.

c) Graph the real part of the function $F(x)$ (using eventually a calculator or software).

3. Let $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(x) = x.$$

a) Find the Fourier coefficients $a_{-2}a_{-1}, a_0, a_1, a_2$.

b) Find the partial Fourier sum: $F(x) = a_{-2}e_{-2}+a_{-1}e_{-1}+a_0e_0+a_1e_1+a_2e_2$.

c) Graph the real part of the function $F(x)$ (using eventually a calculator or software).

4. Let $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(x) = x^2.$$

Assume that $..., a_{-n}, ..., a_n, ...$ are its Fourier coefficients. Using Parseval's identity, calculate

$$\sum_{n=-\infty}^{\infty} |a_n|^2.$$

5. Prove that in the $L^2[-\pi, \pi] \times [-\pi, \pi]$, space with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y)g(x, y)dxdy.$$

the functions $f(x, y) = \sin x \cdot \cos y$ and $g(x, y) = \cos x \cdot \cos y$ are orthogonal.