SANTOSH SHEDBALKAR

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a) i) $i = \sqrt{-1}$, $\Rightarrow ii = -1$

ii) $ijk = -1 \Rightarrow kijk = -k \Rightarrow -1\, ij = -k \Rightarrow ij = k$

iii) $\cancel{jk = -1}$ $ii = -1 \Rightarrow iij = -j \Rightarrow ik = -j$ [$\because ij = k$ (from ii)]

~~iv) $jj = -1 \Rightarrow jjk = -k \Rightarrow$~~

iv) $ijk = -1 \Rightarrow iijk = -i \Rightarrow -1\, jk = -i \Rightarrow jk = i$

~~v) $ijk = -1 \Rightarrow ijjk = -j \Rightarrow i \cdot -1 k = -j \Rightarrow ik = j$~~

v) $jj = -1 \Rightarrow jjk = -k \Rightarrow \cancel{jk}\; ji = -k$ [$\because jk = i$ (from iv)]

vi) $ji = -k$ (from v) $\Rightarrow iji = i(-k) \Rightarrow ki = -ik$ ($\because ij = k$ (from ii)

$\Rightarrow ki = -(-j)$ [from (iii)] $\Rightarrow ki = j$

vii) $kk = -1 \Rightarrow kki = -i \Rightarrow kj = -i$ [$\because ki = j$ (from vi)]

b) i) $LHS = (ij)k = k(k) = -1$ [$\because ij = k$

$RHS = i(jk) = i(i) = -1$ [$jk = i$

$LHS = RHS$

ii) $LHS = (ij)i = ki = j$ [$\because (ij) = k$ & $ki = j$

$RHS = i(ji) = i(-k) = -ik = -(-j) = j$ [$\because (ji) = -k$ and $(ik) = -j$

$LHS = RHS$

iii) $LHS = (ii)i = -1i = -i$

$RHS = i(ii) = i(-1) = -i$

$LHS = RHS$

c) $a = 0 + a_1 i + a_2 j + a_3 k$, $b = 0 + b_1 i + b_2 j + b_3 k$

$ab = (0 + a_1 i + a_2 j + a_3 k)(0 + b_1 i + b_2 j + b_3 k)$

$= \cancel{i(a_1 + b_1) +}\; a_1 b_1 i^2 + a_1 b_2 ij + a_1 b_3 ik + a_2 b_1 ji + a_2 b_2 j^2$

$+ a_2 b_3 jk + a_3 b_1 ki + a_3 b_2 kj + a_3 b_3 k^2$

$= -a_1 b_1 - a_2 b_2 - a_3 b_3 + a_1 b_2 k - a_1 b_3 j - a_2 b_1 k + a_2 b_3 i$

$+ a_3 b_1 j - a_3 b_2 i$

$= -(\vec{a}\cdot\vec{b}) + i(a_2b_3 - b_2a_3) + j(a_3b_1 - b_3a_1) + k(a_1b_2 - b_1a_2)$

$= -(\vec{a}\cdot\vec{b}) + \vec{a}\times\vec{b}$

$\therefore ab = (-(\vec{a}\cdot\vec{b}), \vec{a}\times\vec{b})$

d) Let $q = (a, \vec{v}) = (a, b, c, d)$

then $|q|^2 = a^2 + b^2 + c^2 + d^2$

Let $\bar{q}$ be conjugate of $q \Rightarrow \bar{q} = (a, -\vec{v}) = (a, -b, -c, -d)$

WKT for 2 quaternions $A = (s, \vec{a})$ and $B = (t, \vec{b})$

$A\cdot B = (st - \vec{a}\cdot\vec{b},\ s\vec{b} + t\vec{a} + \vec{a}\times\vec{b})$

so, $q\bar{q} = (a, b, c, d)\cdot(a, -b, -c, -d)$

$= (a^2 + b^2 + c^2 + d^2 - abi - acj - adk + abi + acj + adk$

$+ i(-cd - (-c)(d)) + j(d(-b) - (-d)(b)) + k(b(-c) - (-b)c))$

$= a^2 + b^2 + c^2 + d^2 + 0\cdot i + 0j + 0\cdot k = |q|^2$

$\therefore$ $q\bar{q} = |q|^2$

$q^{-1} = \frac{1}{q} = \frac{1}{q}\frac{\bar{q}}{\bar{q}} = \frac{\bar{q}}{q\bar{q}} = \frac{\bar{q}}{|q|^2}$ $[\because q\bar{q} = |q|^2$

e) For a quaternion $q = (a, b, c, d)$, $|q|^2 = a^2 + b^2 + c^2 + d^2$

Let $q = (\cos(\theta/2), \sin(\theta/2)\vec{v})$ where $\vec{v} = (a, b, c)$ and $\sqrt{a^2+b^2+c^2} = 1$

$|q|^2 = \cos^2(\theta/2) + \sin^2(\theta/2)(a^2 + b^2 + c^2)$

$= \cos^2(\theta/2) + \sin^2(\theta/2)\cdot 1$ $[\because a^2+b^2+c^2 = (\sqrt{a^2+b^2+c^2})^2 = 1^2 = 1$

$= 1$ $[\because \cos^2\theta + \sin^2\theta = 1$

Hence if $\vec{v}$ is a unit quaternion, $(\cos(\theta/2), \sin(\theta/2)\vec{v})$ is a unit quaternion