CS529
Fundamentals of Game Development

Lecture 9a

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Questions?

- Space Partitioning?

Overview

- 2D Transformations
- Homogeneous Coordinates and Matrix Representation
- Composition of 2D Transformations

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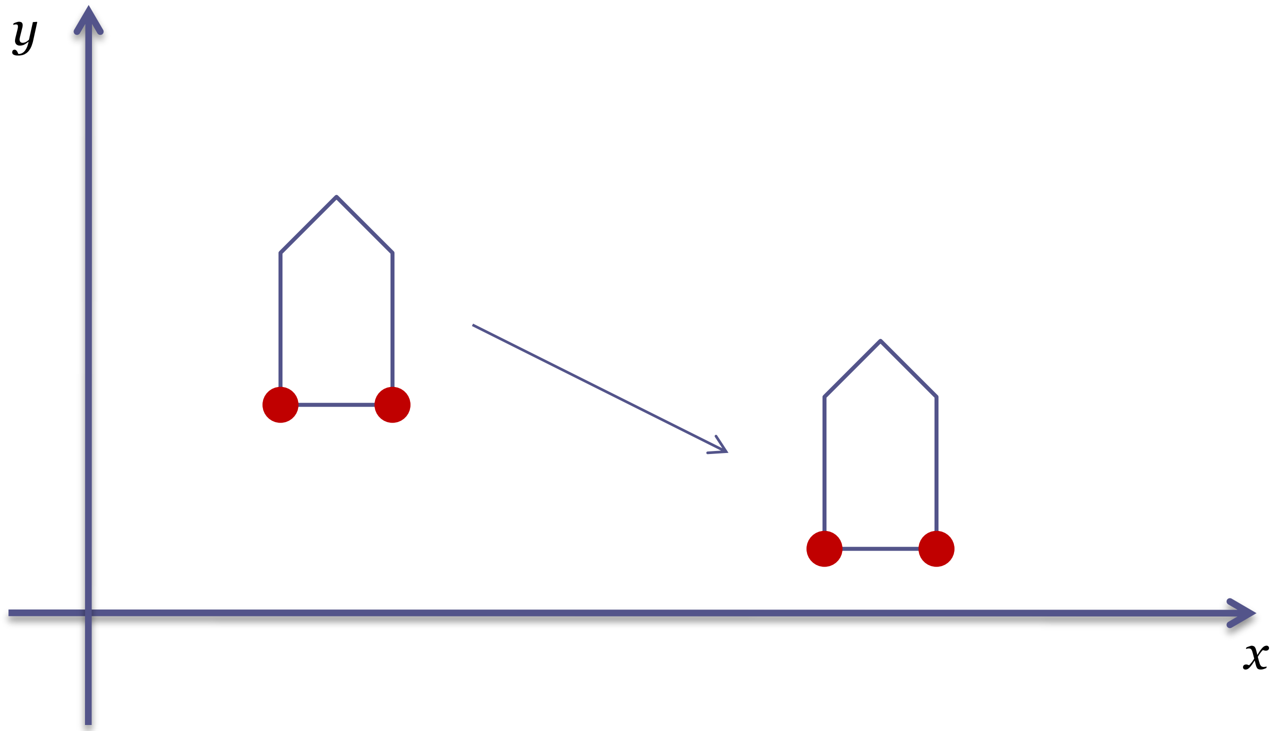
Translation (1/2)

- For each $P(x, y)$ to be moved by d_x, d_y units parallel to the x and y axis we get a new point $P'(x', y')$

$$x' = x + d_x \qquad y' = y + d_y$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix}, \; P' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \; T = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \Rightarrow P' = P + T$$

Translation (2/2)



Scaling (1/2)

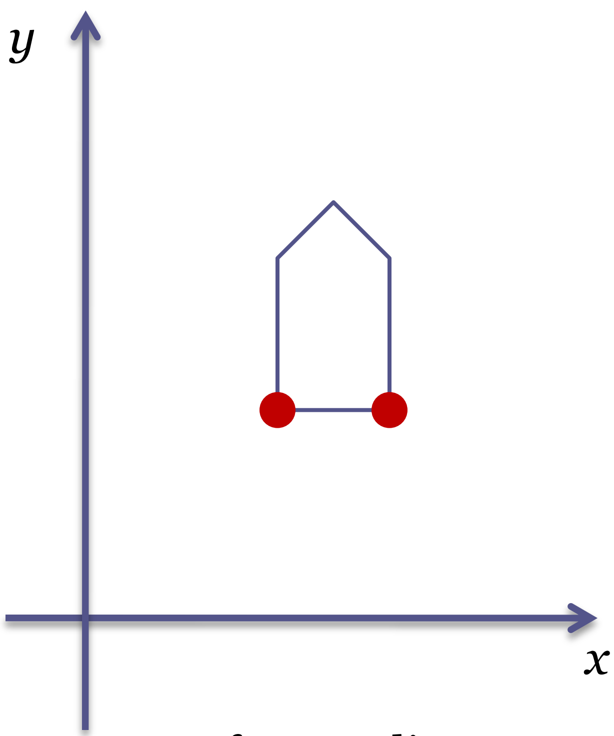
- Points can be stretched by S_x and S_y along the *x-axis* and *y-axis* respectively into new points by the multiplications

$$x' = S_x \cdot x \qquad y' = S_y \cdot y$$

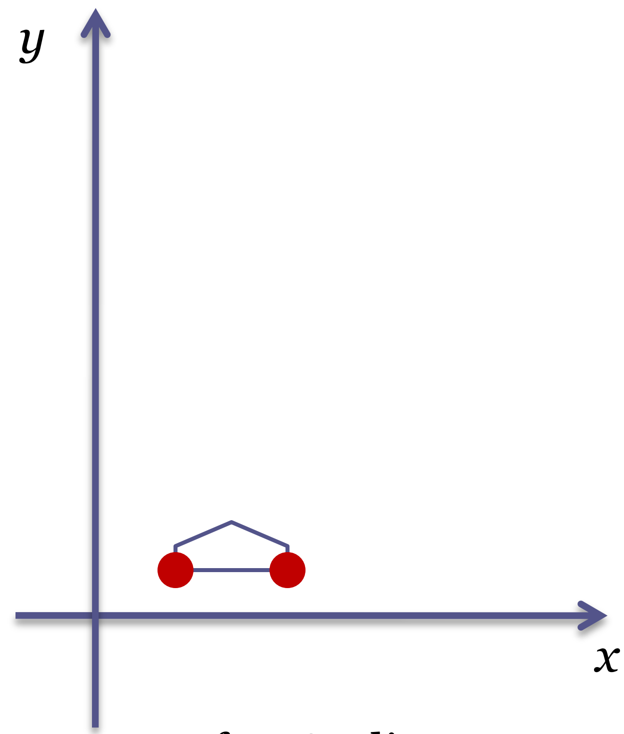
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad P' = S \cdot P$$

Scaling (2/2)

- Scaling about the origin



Before Scaling



After Scaling

Rotation (1/5)

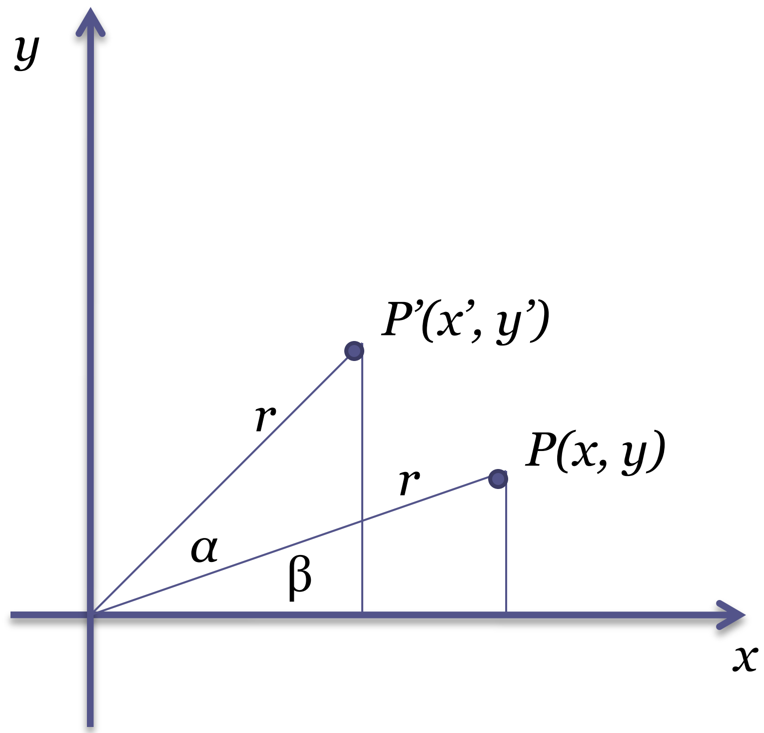
- Points can be rotated through an angle α about the origin, defined as

$$x' = x \cdot \cos\alpha - y \cdot \sin\alpha$$

$$y' = x \cdot \sin\alpha + y \cdot \cos\alpha$$

- These two equations are derived from the following

Rotation (2/5)



Rotation (3/5)

- Rotation by α transforms $P(x, y)$ into $P'(x', y')$
- Rotation along the origin, meaning the distances from the origin to P and P' is r

$$x = r \cdot \cos\beta, \qquad y = r \cdot \sin\beta$$

$$x' = r \cdot \cos(\alpha + \beta) = r \cdot \cos\beta \cdot \cos\alpha - r \cdot \sin\beta \cdot \sin\alpha$$

$$y' = r \cdot \sin(\alpha + \beta) = r \cdot \cos\beta \cdot \sin\alpha + r \cdot \sin\beta \cdot \cos\alpha$$

Rotation (4/5)

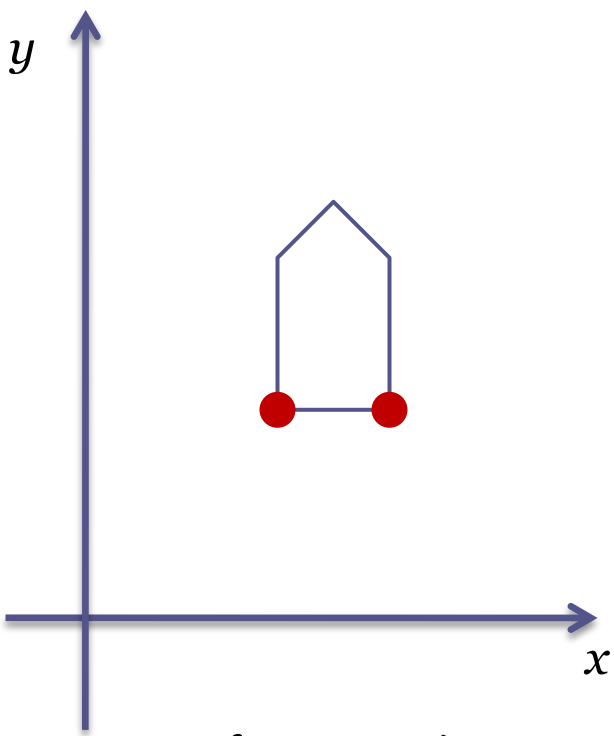
- The rotation matrix would be

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad P' = R \cdot P$$

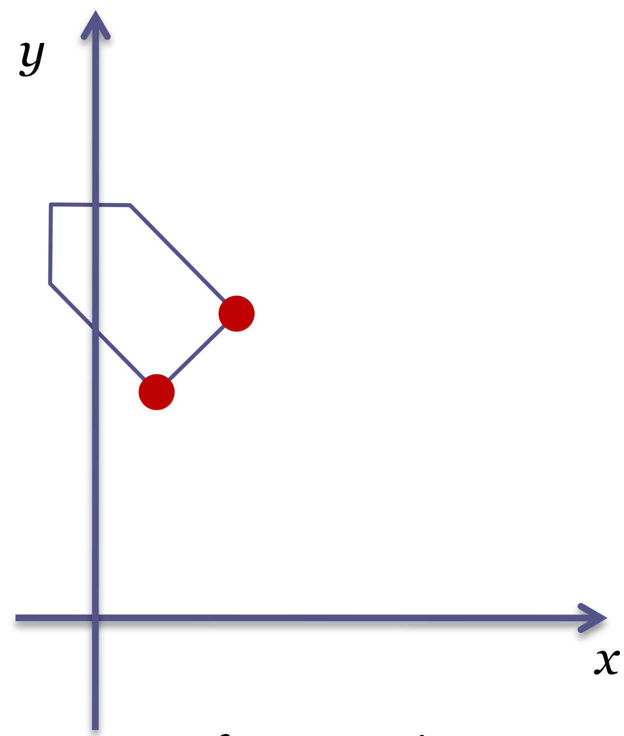
- Where positive angles are measured counterclockwise from x towards y

$$\cos(-\alpha) = \cos\alpha \qquad \sin(-\alpha) = -\sin\alpha$$

Rotation (5/5)



Before Rotation



After Rotation

Overview

- 2D Transformations
- Homogeneous Coordinates and Matrix Representation
- Composition of 2D Transformations

Homogeneous (1/3)

$$\boldsymbol{P' = P + T}$$

$$\boldsymbol{P' = S \cdot P}$$

$$\boldsymbol{P' = R \cdot P}$$

- Problem: Translation can not be represented as 2x2 matrix
- Solution: Use **homogeneous coordinates** by adding a third coordinate w

Homogeneous (2/3)

- Instead of points represented by pairs *(x, y)* we use triples *(x, y, w)*
- Points *(x, y, w)* and *(x', y', w')* are considered equal (represent the same 2D point) if one is scalar multiple of the other:

$$\textbf{e.g.}\quad \mathbf{(4, 2, 1) \equiv (8, 4, 2) \equiv (16, 8, 4)}$$

Homogeneous (3/3)

- Divide by w to obtain a 2D point:

$$(24,16,4) \equiv \left(\frac{24}{4}, \frac{16}{4}, \frac{4}{4}\right) = (6,4,1)$$

- For **points** we use $w = 1$

$$(6,4) \mapsto (6,4,1)$$

- For **direction vectors** we use $w = 0$ (points at ∞)

Translation

- In the 3x3 matrix form for homogeneous coordinates, the translation equation is:

$$P' = T(d_x, d_y) \cdot P$$

$$T(d_x, d_y) = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise

- What happens if a point P is translated by $T(d_{x1},d_{y1})$ to P' and then translated by $T(d_{x2}, d_{y2})$ to P'' ?

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Scaling

$$P' = S(S_x, S_y) \cdot P$$

$$S(S_x, S_y) = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- What about successive scaling?

Rotation

$$P' = R(\alpha) \cdot P$$

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- What about successive rotations?

Identity

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rigid-Body Transformations

- Has the form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & d_x \\ \sin\alpha & \cos\alpha & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

- The upper 2x2 submatrix is orthogonal
  - Preserves angles and lengths
- A unit square after transformation will remain a unit square

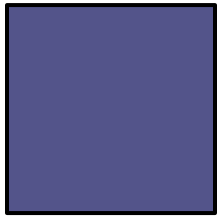
Affine Transformations (1/2)

- What about the product of an arbitrary sequence of rotation, translation and scale matrices?

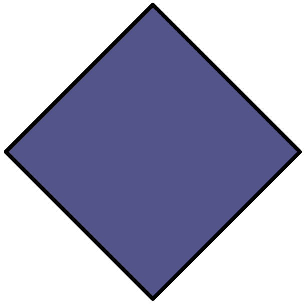
  Answer: Affine transformations

- Affine transformations, have the property of preserving parallelism of lines but **not** lengths and angles
  - That also goes for a sequence of rotation, scale and translation operations

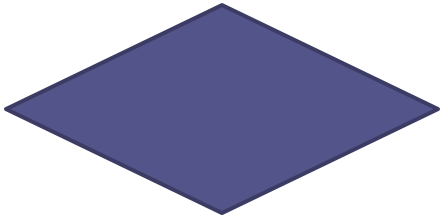
Affine Transformations (2/2)



Unit Cube



Rotation about 45°



Scale in x, not in y

Overview

- 2D Transformations
- Homogeneous Coordinates and Matrix Representation
- Composition of 2D Transformations

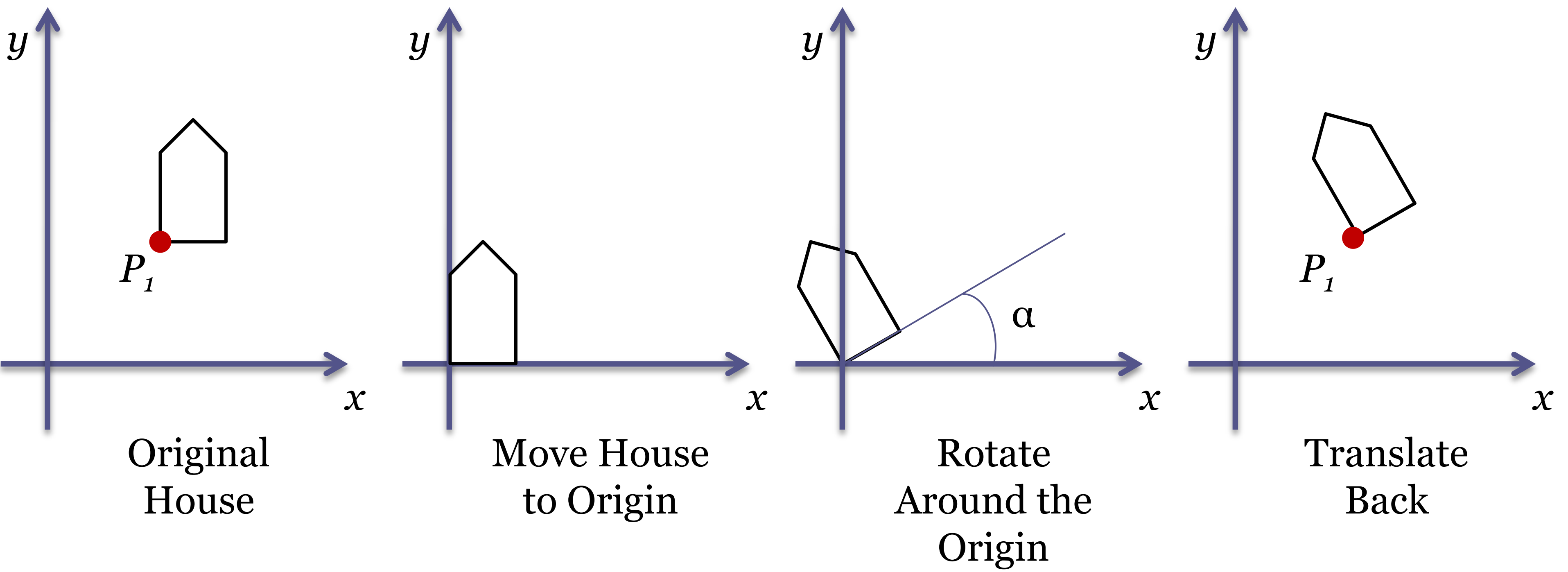
Composition of 2D Transformations

- Purpose:
  - Gain efficiency by applying a single composed transformation to a point, rather than a series of them

Rotation About Some Arbitrary Point (1/3)

- To rotate about an arbitrary point $P_1(x_1, y_1)$ we need a sequence of transformations:
  1. Translate P_1 by $(-x_1, -y_1)$ (i.e. move to origin)
  2. Rotate by α
  3. Translate P_1 by (x_1, y_1) (i.e. translate back)

Rotation About Some Arbitrary Point (2/3)



Rotation About Some Arbitrary Point (3/3)

- The transformation is

$$T(x_1, y_1) \cdot R(\alpha) \cdot T(-x_1, -y_1) =$$

$$\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & x_1(1-\cos\alpha) + y_1\sin\alpha \\ \sin\alpha & \cos\alpha & y_1(1-\cos\alpha) - x_1\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

Scale About Some Arbitrary Point

- The transformation is

$$T(x_1, y_1) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1) =$$

$$\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} S_x & 0 & x_1(1-S_x) \\ 0 & S_y & y_1(1-S_y) \\ 0 & 0 & 1 \end{bmatrix}$$

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Matrix Multiplications (review)

- Matrix multiplication is associative:

$$\mathbf{C}(\mathbf{B}(\mathbf{A}\mathbf{x})) = \mathbf{C}((\mathbf{B}\mathbf{A})\mathbf{x}) = (\mathbf{C}(\mathbf{B}\mathbf{A}))\mathbf{x}$$
$$= ((\mathbf{C}\mathbf{B})\mathbf{A})\mathbf{x}$$
$$= (\mathbf{C}\mathbf{B})(\mathbf{A}\mathbf{x})$$

- Matrix multiplication is **not** commutative

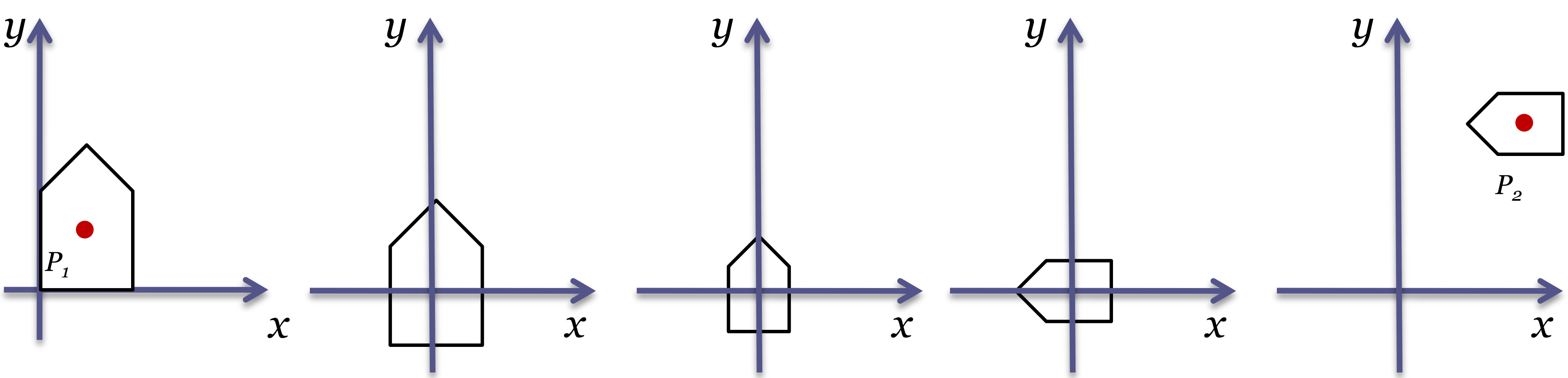
$$\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$$

A rotation followed by a translation is not the same as a translation followed by a rotation

Applying Transformation Matrix (1/3)

- General form:

$$T(x_2, y_2) \cdot R(\alpha) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$



Original House | Translate to Origin | Scale | Rotate | Translate to Final Position P_2

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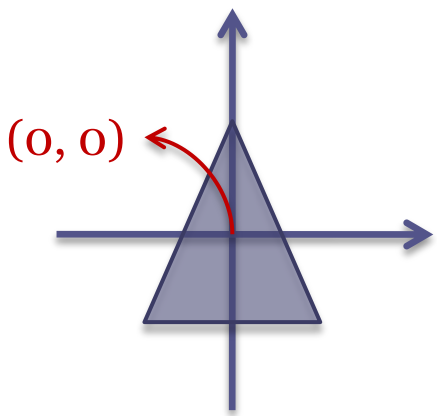
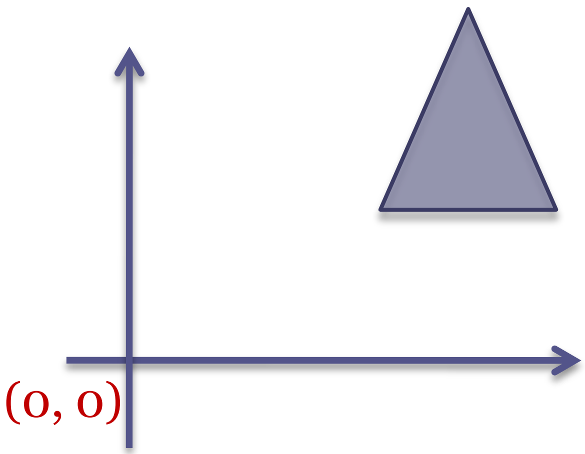
Applying Transformation Matrix (2/3)

- Having our object in its local coordinate system



Sprite's Local Coordinate System



World Coordinate System

Applying Transformation Matrix (3/3)

- We will apply

$$\begin{bmatrix} posX' \\ posY' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} posX \\ posY \\ 1 \end{bmatrix}$$

Snippets

- Transformation
- Scaling
- Rotation
- Translation