DEPARTMENT OF ELECTRONICS AND
ELECTRICAL COMMUNICATION
ENGINEERING, IIT KHARAGPUR

Experiment 5: Power Spectrum Estimation

Samyak Sheersh, Anubhav Mitra

Roll Numbers: 22EC30045, 22EC30007
Group Number: 24

22 October 2024

1 Objectives

1. To generate a random signal $x[n]$ with a known Power Spectral Density(PSD) by passing it through a filter $H(z)$.

2. We then use two techniques to find out the PSD, and in the process measure their accuracy and other properties. In particular, we will use:

    (a) The Welch Non-parametric method: Averaging a modified periodogram.

    (b) The Yule-Walker AR model: A parametric method of power spectrum estimation.

2 Procedure

2.1 Task 1: Generating the signal

We generate a random normally distributed signal $r_w[n] \sim N(\mu = 0, \sigma^2)$ and pass it through a digital filter $H(z)$ where:

$$H(z) = \frac{1}{(1 - 0.9z^{-1})(1 - 0.9jz^{-1})(1 + 0.9jz^{-1})} \tag{1}$$

to get $x[n]$. Thus ensuring that the Power spectral density of $x[n]$ is

$$P_{xx}(f) = |H(e^{j2\pi f})|^2 \cdot \sigma^2 \tag{2}$$

2.2 Task 2: The Welch Non-Parametric method

We know that the PSD of $x[n]$ is:

$$P_{xx}(f) = \lim_{N \to \infty} \frac{1}{2N+1} E\left\{ \left| \sum_{n=-N}^{N} x[n]e^{-j2\pi fn} \right|^2 \right\} \tag{3}$$

for a periodogram, we make a the sum over a finite 0 to $N-1$(N samples from $t = 0$), and drop the estimate sign, to get the expression:

$$\hat{P}_{xx}(f) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n]e^{-j2\pi fn} \right|^2 \tag{4}$$

where the $\hat{P}$ is used to denote that this is an estimator for the true P. In this case, this is the Naive estimator $\hat{X} = X_i$

It can be shown that this is an asymptotically unbiased estimator, but the variance doesn't decay to 0 as $N \to \infty$. However, to overcome this (i.e. to reduce the variance), we can use the result from Central Limit Theorem that the variance of the sample mean reduces by $1/n$ as the number of samples(in this case, the number of periodograms we generate) n increases. So we generate

a lot of periodograms from different parts of the signal and take their mean to come to an estimate.

The Welch method also introduces a windowing function and overlapping segments:

1. We divide $x[n]$ into L(typically $L = 8$) overlapping blocks of length M and an overlap of D between consecutive blocks.Such that for the ith block:

$$x_i[n] = x[n + i(M - D)] \tag{5}$$

where $0 \leq n < M$ and $0 \leq i < L$.

This also gives us a way to calculate M in terms of the N, L and D as:

$$x[M - 1 + (L - 1)(M - D)] = x[N - 1] \tag{6}$$

$$\Rightarrow M = \left\lceil \frac{N + (L - 1)D}{L} \right\rceil \tag{7}$$

Since we know D as a fraction of M($D = dM$, where d is the percentage of overlap), we can write:

$$\Rightarrow M = \left\lceil \frac{N}{L - (L - 1)d} \right\rceil \tag{8}$$

2. We then obtain the estimated PSD of the ith block as:

$$\hat{P}_{xx}^{(i)}(f) = \frac{1}{MU} \left| \sum_{n=0}^{M-1} x_i(n)w(n)e^{-j2\pi fn} \right|^2 ; 0 \leq i < L \tag{9}$$

where $w(n)$ is a (usually Hamming) Window function of length M, and U is the normalisation factor(the energy of the window function):

$$U = \frac{1}{M} \sum_{n=0}^{M-1} w(n)^2 \tag{10}$$

3. We then obtain the Welch spectrum estimate by the average of these modified periodograms:

$$P_{xx}^W(f) = \frac{1}{L} \sum_{i=0}^{L-1} \hat{P}_{xx}^{(i)}(f) \tag{11}$$

2.3   Task 3: The Yule-Walker AR Method

We can do a much better job of getting an estimated PSD if we know how the signal was generated(i.e. a prior knowledge regarding the source of data generation). We also don't require any Windowing in this method since all

samples would be assumed to come from the same process. This is particularly useful for generating PSDs of signals of which we only have a short segment, since we'll only have to converge to a few parameters. This is the approach behind Yule-Walker AR Method.

One of the simplest models is an autoregressive model of pth order. The equation looks like:

$$y[n] = \phi_1 y[n-1] + \phi_2 y[n-2] + \ldots + \phi_p y[n-p] + \epsilon[n] \tag{12}$$

where $\epsilon[n]$ is random noise. If we treat this as a system with the random noise as an input into it. The transfer function is:

$$Y(z)(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \ldots - \phi_p z^{-p}) = X(z) \tag{13}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - \phi_1 z^{-1} - \phi_2 z^{-2} - \ldots - \phi_p z^{-p})} \tag{14}$$

$$\Rightarrow H(z) = \frac{1}{1 - \sum_{n=1}^{p} \phi_n z^{-n}} = \frac{1}{1 + \sum_{n=1}^{p} a_n z^{-n}} \tag{15}$$

where $a_i = -\phi_i$, $X(z)$ is the input signal and $Y(z)$ is the output signal. To obtain the PSD of our sequence $x[n]$, we can apply the Yule-Walker AR method to estimate the $H(z)$ that generates $x[n]$ and then plot $\sigma^2 |H(e^{j2\pi f})|^2$

1. We calculate the autocorrelation function(+ve lags only) as:

$$R_{xx}[n] = \frac{1}{N} \sum_{m=0}^{N-n-1} x[m]x[m+n]; 0 \leq n < N \tag{16}$$

2. We then solve the Yule-Walker AR model to get a_i:

$$\begin{pmatrix} R_{xx}(0) & R_{xx}(1) & \ldots & R_{xx}(p-1) \\ R_{xx}(1) & R_{xx}(0) & \ldots & R_{xx}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(p-1) & R_{xx}(p-2) & \ldots & R_{xx}(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = - \begin{pmatrix} R_{xx}(1) \\ R_{xx}(2) \\ \vdots \\ R_{xx}(p) \end{pmatrix} \tag{17}$$

where $A_{ij} = R_{xx}(|i-j|)$

3. We then estimate the variance of $x[n]$ as:

$$\sigma_{rp}^2 = R_{xx}[0] + \sum_{i=1}^{p} a_i R_{xx}[i] \tag{18}$$

4. We then construct the transfer function $\hat{H}(z)$ as:

$$\hat{H}(z) = \frac{1}{1 + \sum_{i=1}^{p} a_i z^{-i}} \tag{19}$$

and thus the estimated PSD as such:

$$P_{xx}^{YW}(f) = \sigma_{rp}^2 |\hat{H}(e^{j2\pi f})|^2 = \frac{\sigma_{rp}^2}{|1 + \sum_{k=1}^{p} a_k e^{-j2\pi fk}|^2} \tag{20}$$
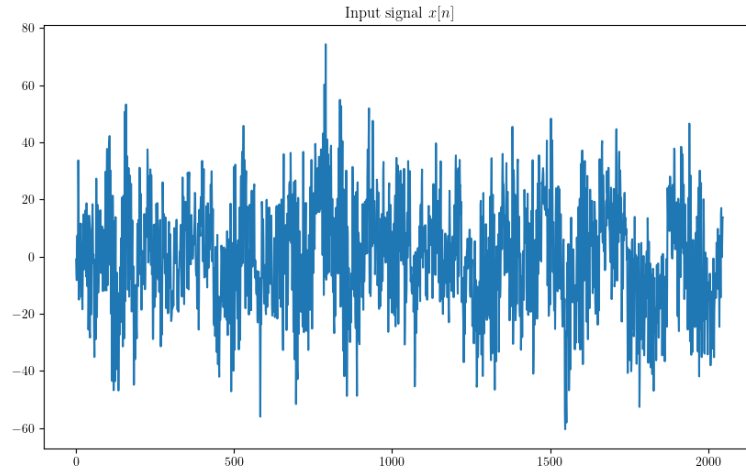
3 Results

3.1 Task 1
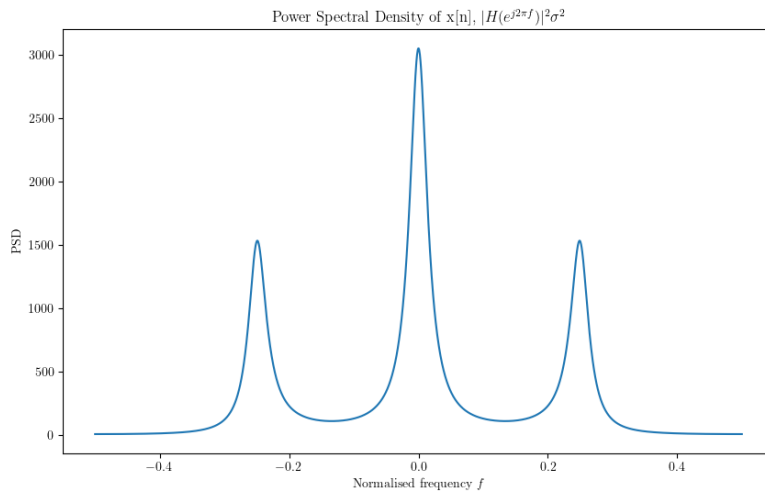

Figure 1: Signal $x[n]$ with $N = 2048$

Figure 2: Theoretical PSD of $x[n]$, $N = 2048$
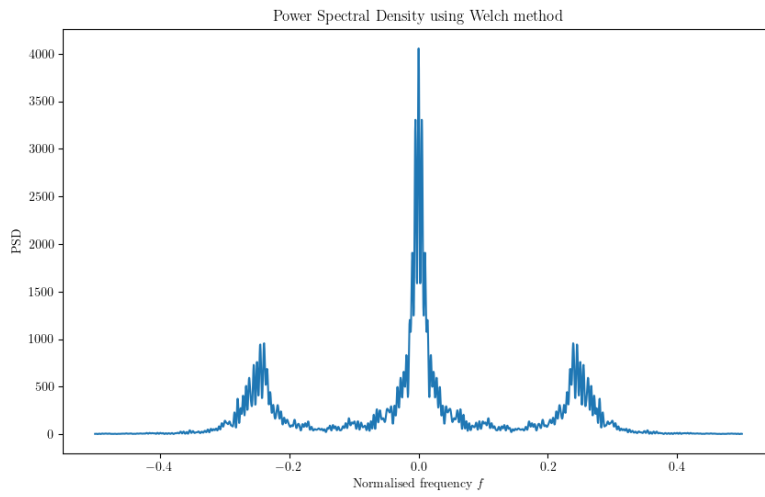
3.2   Task 2

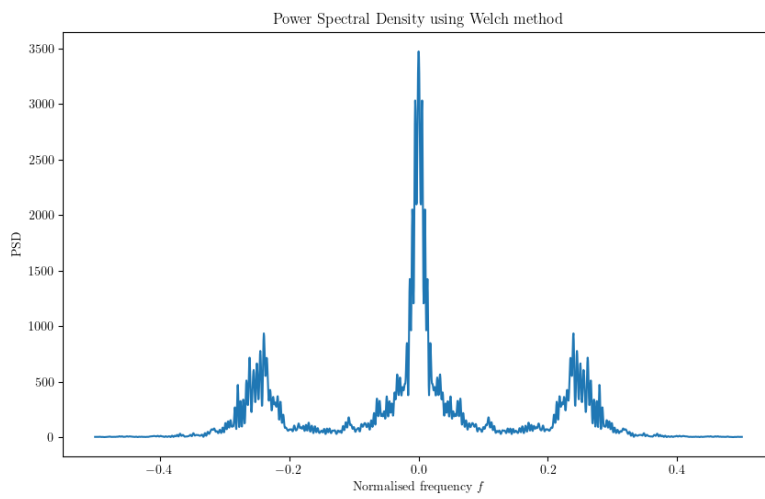Figure 3: PSD using Welch method with $D = 0$

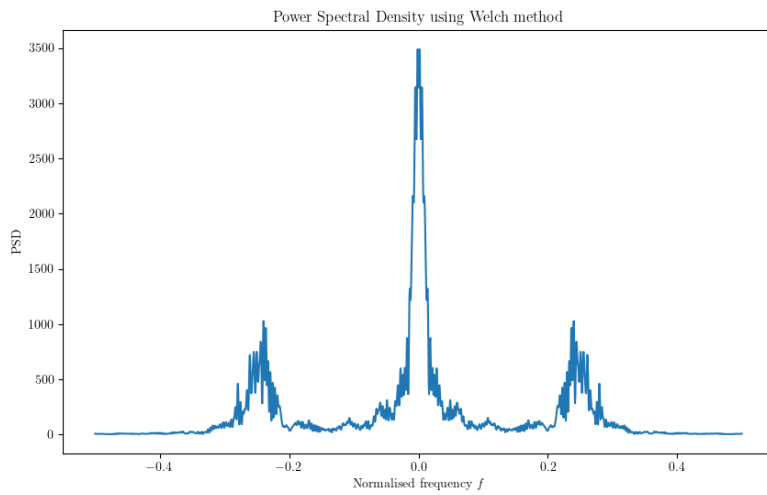Figure 4: PSD using Welch method with 10% overlap

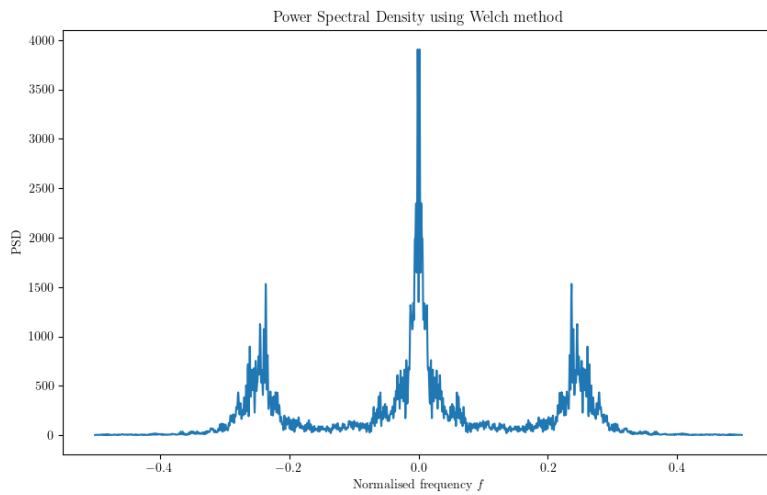Figure 5: PSD using Welch method with 25% overlap

Figure 6: PSD using Welch method with 50% overlap

3.3   Task 3
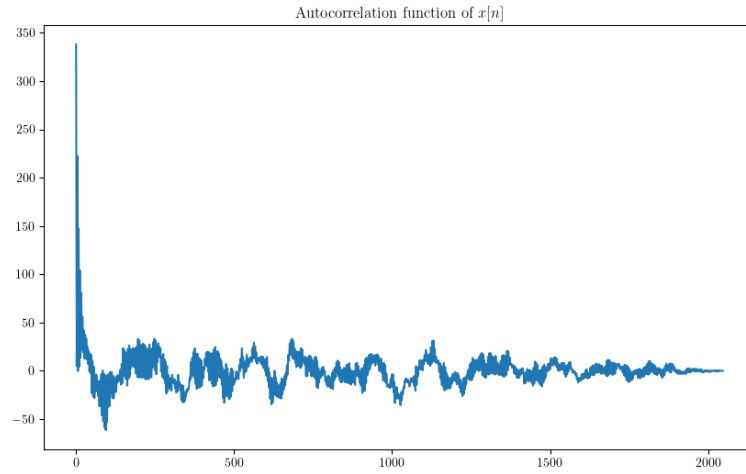

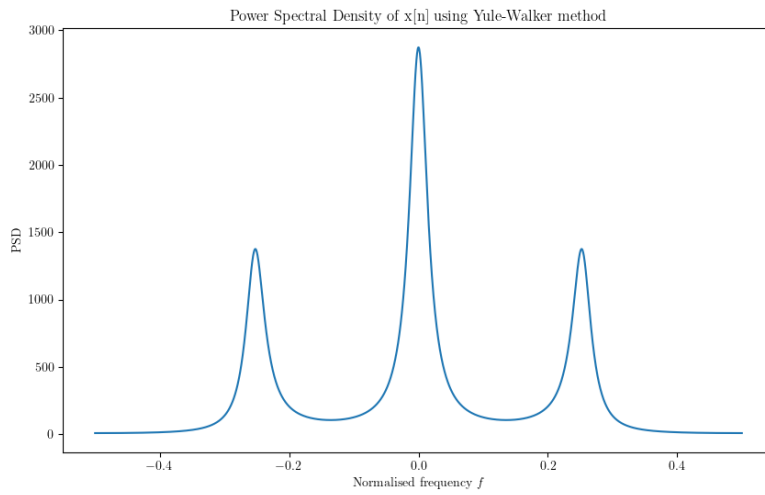Figure 7: Autocorrelation function of $x[n]$

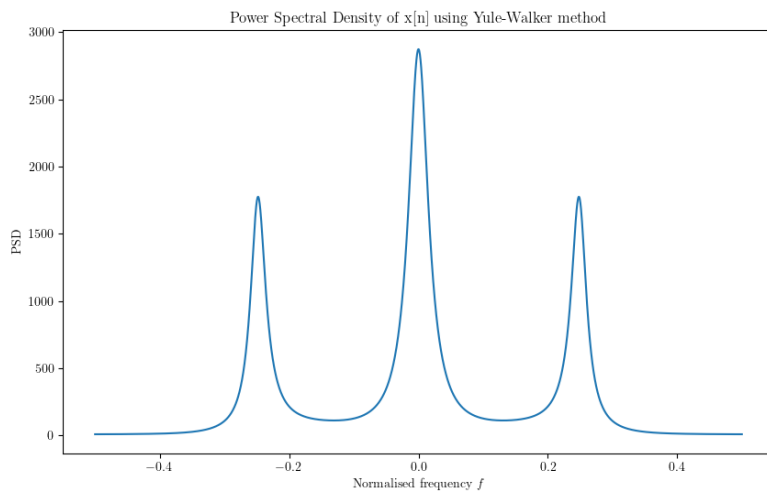Figure 8: PSD using Yule-Walker method with $p = 3^{th}$ AR model

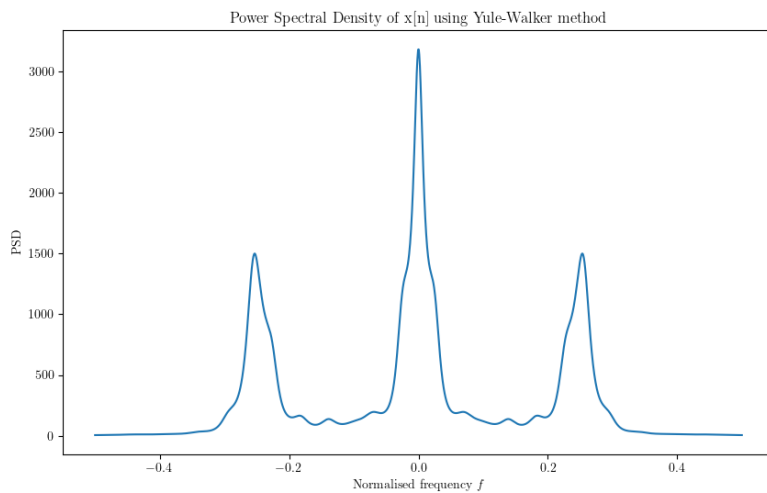Figure 9: PSD using Yule-Walker method with $p = 10^{th}$ AR model

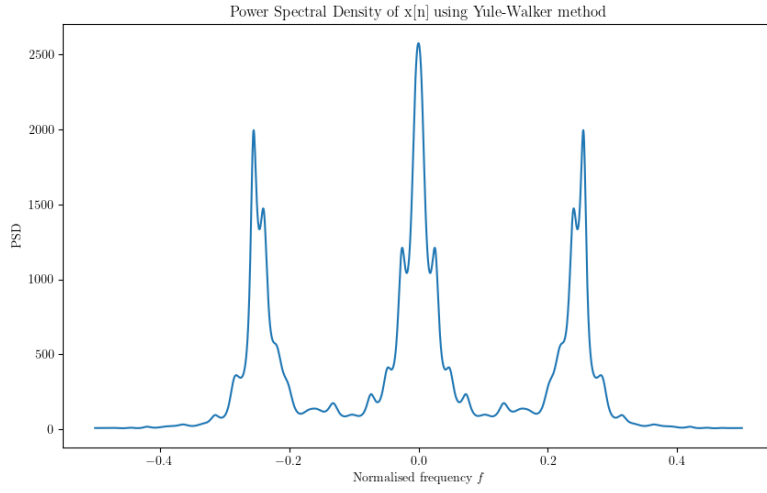Figure 10: PSD using Yule-Walker method with $p = 30^{th}$ AR model

8

Figure 11: PSD using Yule-Walker method with $p = 50^{th}$ AR model

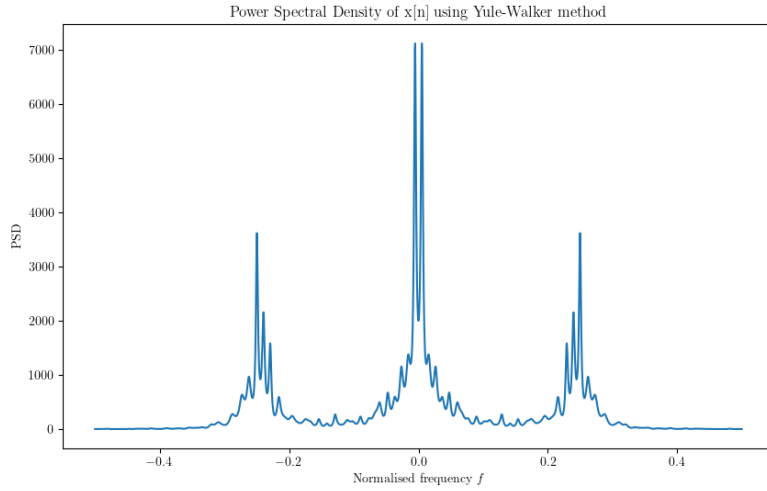Figure 12: PSD using Yule-Walker method with $p = 100^{th}$ AR model

4   Discussion: Samyak Sheersh

1. We saw for the Welch Method, that as the overlap D increased, the estimate of the PSD converged up until 50% overlap(see Figures 3, 4, 5, and 6). After that the accuracy actually started worsening (the lines became thicker and more uncertain), probably because some values in the middle got a much stronger influence on the final PSD due to being considered multiple times.

2. The PSD using Welch method was also more noisy than the one using Yule-Walker models. This was expected since the Welch method has to converge to a whole distribution which has a lot more variables and thus a lot more uncertainity, while the Yule-Walker generates a totally new mdoel from the few parameter it infers.

3. The autocorrelation function(see Figure 7) quickly decayed to 0 for increasing n, as expected, since the signal is not periodic or with a coherent structure

4. As we had put in a random signal through a known PSD, we had known that is was an AR(3) model(the transfer function has a 3 degree denominator polynomial), and so the $p = 3$ Model performed the best, and the values it got for a_i were very close to the actual values ($[-0.913, 0.821, -0.733]$ compared to the actual $[-0.9, 0.81, -0.729]$).

5. However as we increased p (see the figures 8, 9, 10, 11, and 12) the accuracy of the PSD started to drop, and it was starting to overfit to the particular random sequence we sent it through. Thus we need to make sure in the Yule-Walker method that we don't choose too many parameters and end up overfitting the PSD.