FAST AVERAGE CASE CONNECTIVITY ALGORITHMS ON RANDOM GRAPHS

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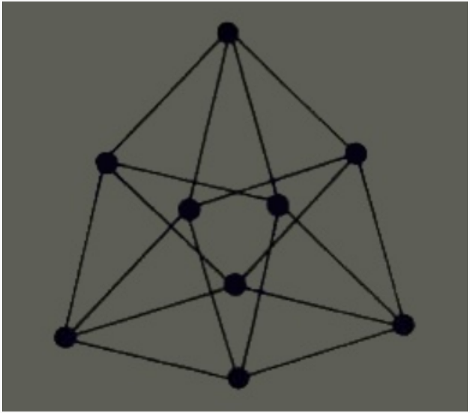
PRELIMINARIES



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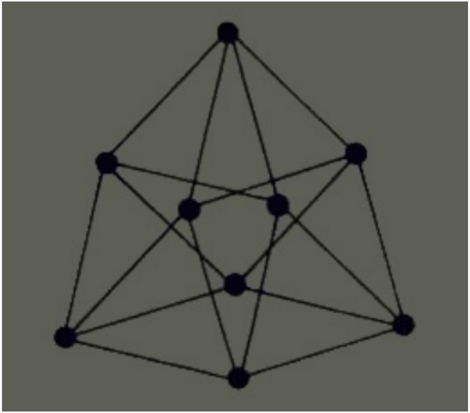

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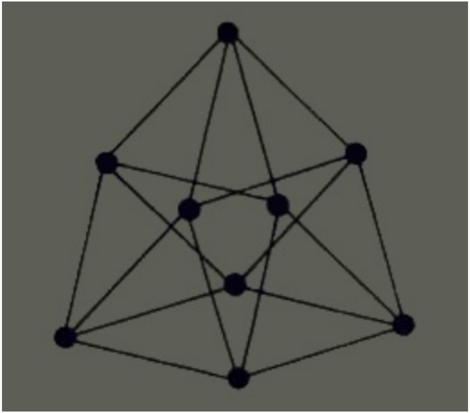

- Probability distribution over Graphs
- Two major models, $G(n, m)$ and $G(n, p)$
- Entities(vertices and edges) independent of each other

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- Else, the algorithm picks the top vertex u from the stack and scans T. On finding the first neighbor v of u in T, algorithm removes u from T and pushes it on top of U
- If u does not have a neighbor in T, pop u from U and insert it into S
- Query all remaining edges after $U \cup T = \phi$

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We can study the component structure using the properties of $\bar{X} = (X_i)_{i=1}^{N}$

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  Giant component is formed in this continuous interval.

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KARP'S CONNECTED COMPONENTS ALGORITHM

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- Second, elements not in the giant component grouped into components

Algorithm 1 Connected Components

```
procedure STAGE I(G)
    Let E_0 = ϕ
    while E_0 ≠ E do
        Let D = ϕ
        if |E − E_0| < n then
            D = E − E_0
        else
            D ← n distinct randomly selected edges from E − E_0
        end if
        Use DFS to find connected components of G_0 = (V, E_0)
        if There exists a connected component in G_0 of size > θn then
            STAGE II
            return
        end if
    end while
    return
end procedure
```

Algorithm 2 Connected Components

```
procedure STAGE II(G)
    Mark vertices of 'Giant component' as 'giant' and remaining as unmarked
    for v ∈ σ do
        if v is unmarked then
            Start DFS from vertex v
            if An edge leading to a giant vertex is found then
                Abort dfs and mark all vertices explored as 'giant'
            else
                Mark all vertices visited as a new component
            end if
        end if
    end for
end procedure
```

AVERAGE CASE RUNTIME ANALYSIS

Theorem 9b in [2]

Let $\varrho_{n,N}$ denote the size of the greatest component of $\Gamma_{n,N}$. If $N(n) \sim cn$ where $c > \frac{1}{2}$ we have for any $\eta > 0$

$$\lim_{n \to +\infty} \mathcal{P}\left(\left| \frac{\varrho_{n,N(n)}}{n} - G(c) \right| < \eta \right) = 1$$

where $G(c) = 1 - \frac{x(c)}{2c}$ and $x(c) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (2ce^{-2c})^k$ is the solution satisfying satisfying $0 < x(c) < 1$ of the equation $x(c)e^{-x(c)} = 2ce^{-2c}$.

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For a random graph G with n vertices and n edges, there is a constant $\theta > 1/2$ such that **whp** there is a component in G with atleast θn vertices

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  - Improve it by ‘remembering‘ the connected component data found in previous round

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$$O\Big(\sum_{i=1}^{\infty} ni^2(1-p_n)^{i-1}\Big) = O\Big(\frac{n}{p_n^3}\Big) = O(n) \text{ as } n \to \infty$$

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- All vertices not equally likely

Lemma

Let C_0 be the giant component vertex set in G_0 found during Stage I of the algorithm. Whenever an entry $w \in adj(v)$ is examined, the probability that $w \in C_0$, given that $\{v, w\} \notin E_0$ is $\geq \theta$.

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- Summing over all $v \in V$, expected number of edges examined in Stage II is $O(n/\theta) = O(n)$

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   Let S denote set of all graphs which satisfy (1) - (3)

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$$\begin{aligned}
\mathcal{P}(w_i | E_0, \dots (v_{i-1}, w_{i-1})) &= \frac{1}{d(v_i, G) - a_i} \\
\Rightarrow \mathcal{P}((v_i, w_i) | \hat{G}, E_0, \dots (v_{i-1}, w_{i-1})) &= \frac{1}{d(v_i, G) - a_i} = \frac{1}{d(v_i, \hat{G}) - a_i + b_i} \\
\Rightarrow p(\alpha | \hat{G}, E_0) &= \prod_{i=1}^{k-1} \frac{1}{d(v_i, \hat{G}) - a_i + b_i} \qquad (1)
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We can now write

$$p(\alpha|\hat{G}, E_0) = \prod_{u \in V - C_0} f_{u,\alpha}(d(u, \hat{G})) \tag{2}$$

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$$p(\hat{G}|\alpha, E_0) = \frac{p(\alpha|\hat{G}, E_0)p(\hat{G}|E_0)}{p(\alpha|E_0)}$$
$$\Rightarrow p(\hat{G}|\alpha, E_0) = c \prod_{u \in V - C_0} f_{u,\alpha}(d(u, \hat{G})) \tag{3}$$

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- Call w_k eligible if $w_k \neq v_k$, $\mathbf{adj}(w_k)$ is not exhausted and $\{v_k, w_k\} \notin E_0$
- Let $x \in C_0$, we'll show $\mathcal{P}(w_k = x | w_k \textit{ is eligible}) \geq 1/(n-1)$

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- Define relation R on $G_{d+} \times G_{d-}$ by $(H_1, H_2) \in R$ iff $H_1 \in G_{d+}, H_2 \in G_{d-}$ and H_2 is obtained from H_1 by deleting edge $\{v_k, x\}$ and adding some other edge $\{v_k, x'\}$ incident to v_k such that x' is eligible.

- If $(H_1, H_2) \in R$ then in going from H_1 to H_2, we remove $\{v_k, x\}$ and add $\{v_k, x'\}$. If $x' \in C_0$ then $p(\hat{G}|\alpha, E_0)$ remains the same. If $x' \notin C_0$ then $d(x', \hat{G})$ increases, decreasing $p(\hat{G}|\alpha, E_0)$. This implies, $p(H_1|\alpha, E_0) \geq p(H_2|\alpha, E_0)$

PROOF OF LEMMA

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Summing over all x in C_0 gives $\mathcal{P}(w_k \in C_0) \geq \theta n/(n-1) \geq \theta$

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It has been shown that on random graphs the first algorithm takes time closer to $O(n^2\sqrt{\log n})$. We believe that further work is possible on this and it can be reduced to $O(n^2)$.

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Thank You !