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~~Presen~~ Quiz 2

Q1. $|z-1|=1$ is a circle centered at $z_0=1$ and of unit radius.

$\Rightarrow C: z(t) = 1+e^{it}$, $t \in [0, 2\pi]$

$= 1+\cos t + i\sin t$

- For the denominator: $|1+z| = |2+e^{it}| = \cancel{|4t}$ $\cancel{\boxed{|z_1+z_2| \quad \text{where } z_1 = 2,\ z_2 = e^{it}}}$

$= |2+\cos t+i\sin t|$

We know, for a complex no. w, $|w| \geq \mathrm{Re}(w)$

$\Rightarrow |2+\cos t+i\sin t| \geq 2+\cos t \geq 1$ —— ①

- For the numerator: $|e^z| = |e^{\mathrm{Re}(z)+i\mathrm{Im}(z)}| = |e^{\mathrm{Re}(z)} \cdot e^{i\mathrm{Im}(z)}|$

since $|e^{i\theta}| = 1$

$\Rightarrow |e^z| = |e^{\mathrm{Re}(z)}|$

From ① we know $\mathrm{Re}(z) = 1+\cos t \leq 2$ (and ≥ 0)

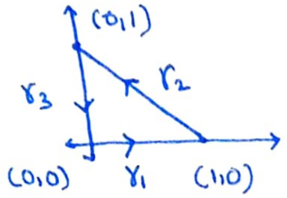
$\Rightarrow |e^z| \leq e^2$

$$\Rightarrow M = \left|\frac{e^z}{1+z}\right| = \frac{|e^z|}{|1+z|} \leq \frac{e^2}{1} = e^2$$

$\because C$ is a circle $\Rightarrow L = 2\pi$

$\Rightarrow$ From ML estimate we can say that $\left|\oint_{|z-1|=1} \frac{e^z}{z+1}\,dz\right| \leq 2\pi e^2$

Q2.



we split γ such that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$\gamma_1 = [0,1]$ $\quad \gamma_1 : z_1(t) = t \quad t \in [0,1]$

$\Rightarrow \dot{z}_1(t) = 1$

$\gamma_2 = [1,i]$ $\quad \gamma_2 : z_2(t) = 1 - t + it \quad t \in [0,1]$

$\Rightarrow \dot{z}_2(t) = -1 + i$

$\gamma_3 = [i,0]$ $\quad \gamma_3 : i(1-t) = z_3(t) \quad t \in [0,1]$

$\Rightarrow \dot{z}_3(t) = -i$

$$\Rightarrow \int_\gamma z\,dz = \int_{\gamma_1} f(z_1(t)) \cdot \dot{z}_1(t)\,dt + \int_{\gamma_2} f(z_2(t)) \cdot \dot{z}_2(t)\,dt + \int_{\gamma_3} f(z_3(t)) \cdot \dot{z}_3(t)\,dt$$

$$= \int_0^1 t\,dt + \int_0^1 (1-t+it)(-1+i)\,dt + \int_0^1 i(1-t)(-i)\,dt$$

$$= \frac{1}{2} + (-1+i)\left[t - \frac{t^2}{2} + \frac{it^2}{2}\right]_0^1 + \left[t - \frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{2} + (-1+i)\frac{(1+i)}{2} + \frac{1}{2} = 1 - \left(\frac{1+1}{2}\right) = 0$$

$$\boxed{\Rightarrow \int_\gamma z\,dz = 0}$$

Q3. $\gamma : z(t) = e^{it}$, $t \in [0, 2\pi]$

$\Rightarrow \dot{z}(t) = i \cdot e^{it}$

$$\Rightarrow \int_\gamma \bar{z}^m\,dz = \int_0^{2\pi} \overline{(e^{it})}^m \cdot ie^{it}\,dt = \int_0^{2\pi} i\,e^{it(1-m)}\,dt$$

$$= \frac{i}{i(1-m)} e^{it(1-m)}\Big|_0^{2\pi} = \frac{1}{1-m}\left(e^{i2\pi(1-m)} - 1\right)$$

- for $m \neq 1$, $\int_\gamma \bar{z}^m\,dz = 0$
- for $m = 1$, $\int_\gamma \bar{z}^m\,dz = i\int_0^{2\pi} dt = 2\pi i$

Q6. We know $\cosh z = \cos(iz)$

$\cos(iz) = 0$ when $iz = \frac{\pi}{2} + k\pi \Rightarrow z = -i\left(\frac{\pi}{2} + k\pi\right)$ where $k \in \mathbb{Z}$

$\Rightarrow$ zeros of $\cosh z = i\pi/2 + il\pi$, where $l \in \mathbb{Z}$

Similarly, since $\sinh z = -i\sin(iz)$

& $\sin(iz) = 0$ when $z = -i\pi k \Rightarrow$ zeros of $\sinh z = i\pi l$, where $k, l \in \mathbb{Z}$

Periods of $\cosh z$: $2\pi m i$ where $m \in \mathbb{Z}$

$\sinh z$: $2\pi m i$ where $m \in \mathbb{Z}$

$\because \cosh(z + 2\pi m i) = \cos(iz - 2\pi m)$ & $2\pi k$, $k \in \mathbb{Z}$ is the period of $\cos\theta$

Similary $\sinh(z + 2\pi m i) = -i\sin(iz - 2\pi m)$ & $2\pi k$, $k \in \mathbb{Z}$ is the period of $\sin\theta$

Q7. $i^i = e^{i \log i} = e^{i(\log|i| + i\mathrm{Arg}(i) + i2\pi k)} = e^{i(i\pi/2 + i2\pi k)}$

$= e^{-(\pi/2 + 2\pi k)}$ where $k \in \mathbb{Z}$

$i^{i^i} = (i^i)^i = e^{i\log(i^i)} = e^{i[\log(e^{-(\pi/2+2\pi k)})]} = \boxed{\cancel{e^{i\log}}}$

$= e^{i[\log|e^{-(\pi/2+2\pi k)}| + i\mathrm{Arg}(e^{-(\pi/2+2\pi k)}) + i2\pi m]}$

$= e^{i[-(\pi/2+2\pi k) + 0 + i2\pi m]} = e^{-2\pi m} e^{-i(\pi/2+2\pi k)} = -ie^{2\pi m}$

where $m \in \mathbb{Z}$

But $i^{-1} = 1/i = -i$

Q9. Let $\arctan(z) = w$ for some $w \in \mathbb{C}$

$$\Rightarrow z = \tan(w) = \frac{\sin w}{\cos w} = \frac{1/2i\,(e^{iw} - e^{-iw})}{1/2\,(e^{iw} + e^{-iw})}$$

$$\Rightarrow z = \frac{1\,(e^{iw} - e^{iw})}{i\,(e^{iw} + e^{-iw})} \Rightarrow iz(e^{iw} + e^{-iw}) = e^{iw} - e^{-iw}$$

$$\Rightarrow iz(e^{2iw} + 1) = e^{2iw} - 1 \Rightarrow e^{2iw}(iz-1) + (iz+1) = 0$$

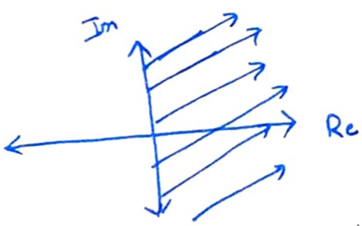
Solving quadratic for e^{iw} we get

$$e^{iw} = \frac{\pm\sqrt{-4(iz-1)(iz+1)}}{2(iz-1)} = \pm\sqrt{\frac{iz+1}{1-iz}}$$

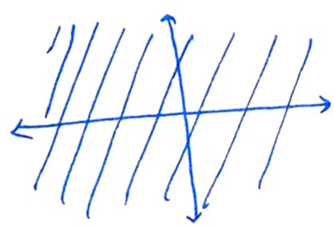
Taking ln on both sides we get

$$iw = \ln\left[\sqrt{\frac{1+iz}{1-iz}}\right] \Rightarrow w = \frac{1}{2i}\ln\left[\frac{1+iz}{1-iz}\right] = \arctan(z)$$

Q8. a)



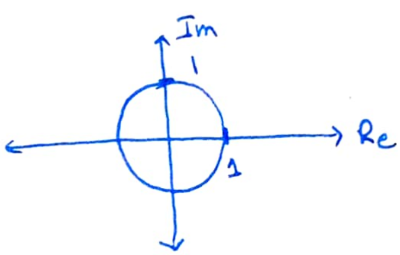
z-plane



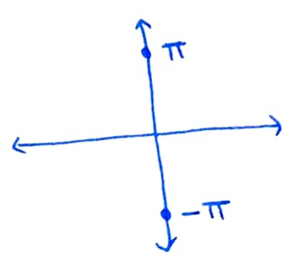
w-plane

entire plane

b)

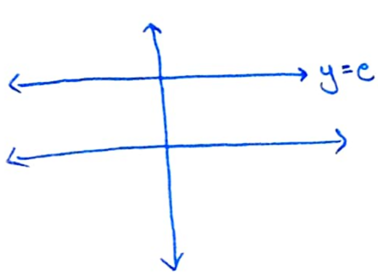


z-plane



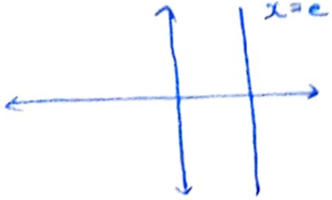
w-plane

c)



z-plane

d)



z-plane