Prove DDPM Q1

problem:

Given $q(x_{1-T}|x_0) = \prod_{t=1}^{T} q(x_t|x_{t-1})$

Show $q(x_{1-T}|x_0) = q(x_T|x_0) \prod_{t=T}^{2} q(x_{t-1}|x_t, x_0)$

Because $x_1, x_2, \cdots x_T$ form a Markov chain when conditioned on x_0, we can rewrite $q(x_t|x_{t-1}) = q(x_t|x_{t-1}, x_0)$, where x_0 term is superfluous due to the Markov property.

$$q(x_{1-T}|x_0) = \prod_{t=1}^{T} q(x_t|x_{t-1})$$

$$= q(x_1|x_0) \prod_{t=2}^{T} q(x_t|x_{t-1}, x_0) \longrightarrow ①$$

Then, according to Bayes rule, we have the following derivation.

$$q(x_t \mid x_{t-1}, x_0) = \frac{q(x_t, x_{t-1}, x_0)}{q(x_{t-1}, x_0)}$$

$$= \frac{q(x_{t-1}, x_t, x_0)}{q(x_{t-1}, x_0)}$$

$$= \frac{q(x_{t-1} \mid x_t, x_0)\, q(x_t, x_0)}{q(x_{t-1}, x_0)}$$

$$= \frac{q(x_{t-1} \mid x_t, x_0)\, q(x_t \mid x_0)\, \cancel{q(x_0)}}{q(x_{t-1} \mid x_0)\, \cancel{q(x_0)}}$$

$$= \frac{q(x_{t-1} \mid x_t, x_0)\, q(x_t \mid x_0)}{q(x_{t-1} \mid x_0)} \longrightarrow ②$$

Now we can substitude ② into ①.

$$q(x_{1-T} \mid x_0) = q(x_1 \mid x_0) \prod_{t=2}^{T} q(x_t \mid x_{t-1}, x_0)$$

$$= q(x_1 \mid x_0) \prod_{t=2}^{T} \frac{q(x_{t-1} \mid x_t, x_0)\, q(x_t \mid x_0)}{q(x_{t-1} \mid x_0)}$$

$$= q(x_1|x_0)\,\frac{q(x_1|x_2,x_0)\,\cancel{q(x_2|0)}}{q(x_1|x_0)}\,\frac{q(x_2|x_3,x_0)\,\cancel{q(x_3|0)}}{\cancel{q(x_2|x_0)}}$$

$$\cdots\cdots\ \frac{q(x_{T-1}|x_T,x_0)\,q(x_T|0)}{\cancel{q(x_{T-1}|x_0)}}$$

$$= \cancel{q(x_1|x_0)}\,\frac{q(x_T|x_0)}{\cancel{q(x_1|x_0)}}\prod_{t=2}^{T} q(x_{t-1}|x_t,x_0)$$

$$= q(x_T|x_0)\prod_{t=2}^{T} q(x_{t-1}|x_t,x_0)\ \#$$