Guidelines and Rules:

1. You can attempt this project in a group of three members. Please restrict your collaborations within your group. You are allowed to collaborate in problem formulation, approach and code development.

2. Each group will turn in one preliminary report and one final report in a pdf format online at CANVAS. The report will contain problem formulation, solution methodology and discussion of results.

    - Preliminary Report (40%): contains solution to Parts a), b) and c). In addition to this, discuss the solution methodology for parts d) and e) including an outline of Matlab code (i.e., psuedo code) for Parts d) and e).
    - Final Report (20%): Discussion of results corresponding to Parts d) and e).
    - Matlab Code (40%): Working Matlab codes for Parts d) and e).

3. **Due Dates:**

    - Preliminary Report+ Matlab codes pertaining to Parts a), b) and c): February 26th, 2025.
    - Final Report and Matlab Code: March 7th, 2025.

4. You will get zero for discussion of results if your submitted Matlab code does not work.

5. Each report should clearly indicate the contribution of each group member in percentage. Only group members contributing 33% or more will get full earned points on the report.

6. Each group member should sign the honor statement indicating that **they have neither given nor received assistance on this project.**

7. The minimum penalty for the academic integrity violation will be ZERO in the Project. Most frequent examples of academic integrity violation includes but not limited to sharing your report or code with other group, writing the Matlab code for other group, copying the results figures and/or result discussion from other group students, etc.

1. The objective of this project is to use concepts learned in this class to study the circular restricted three body problem (CR3BP). The CR3BP plays an important role in many space missions including the James Webb Telescope mission in the Earth-Sun system and the planned Artemis missions to the Moon. Unlike the two body problem, the CR3BP does not have an analytical solution. Through this project, we will gain crucial insights about the motion of a spacecraft (third body) as it experiences the gravitational forces of two large bodies (e.g. Earth and Moon or Earth and Sun).

   Let us consider two primary bodies with masses m_1 and m_2 revolving around their center of mass in circular orbits under the influence of their mutual gravitational attractions (Figure 1). The mass m_3 of the third body is much smaller than either m_1 or m_2. The third body is assumed to move in the plane defined by the two primary bodies. The positions for the primary bodies from the center of mass (i.e, origin) are:

$$\bar{r}_1 = -\frac{m_2 R}{m_1 + m_2}\hat{b}_1, \quad \bar{r}_2 = \frac{m_1 R}{m_1 + m_2}\hat{b}_1$$

   where, R is the distance between the two primary bodies. To study this problem, we define two frames: Inertial frame $\mathcal{N}$ consisting of directions $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ and rotating frame $\mathcal{B}$ consisting of directions $\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$. Your tasks are as follows:

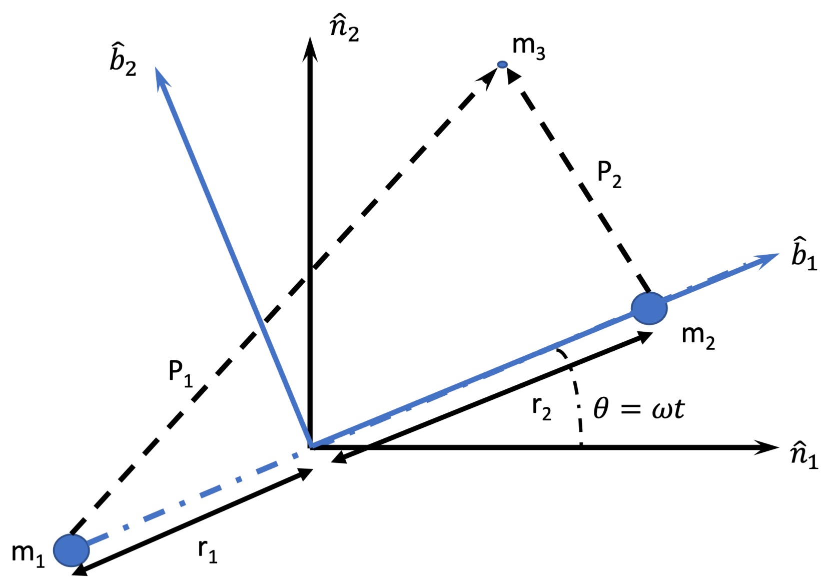

Figure 1: A Schematic of CR3BP.

   (a) **Equations of Motion:** Let the position vector of mass m_3 is $\bar{r} = x\hat{b}_1 + y\hat{b}_2$. Write down the kinetic energy of the spacecraft and the Lagrangian given that the gravitational potential energy is given as:

$$V = -G\frac{m_3 m_1}{p_1} - G\frac{m_3 m_2}{p_2} \tag{1}$$

   Use the Lagrangian approach as learned in the class to derive the following equations of motion for m_3 along $\hat{b}_1$ and $\hat{b}_2$ directions:

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = -\omega^2 \frac{(1-\mu)(x+\mu)}{p_1^3} - \omega^2 \frac{\mu(x-1+\mu)}{p_2^3} \tag{2}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\omega^2 \frac{(1-\mu)y}{p_1^3} - \omega^2 \frac{\mu y}{p_2^3} \tag{3}$$

You are given that the angular velocity of the rotating frame is $\omega^2 = G\frac{(m_1+m_2)}{(r_1+r_2)^3}$. The distance between two primaries has been normalized to one, i.e., $R = r_1 + r_2 = 1$ distance unit. The mass ratio μ is defined as: $\mu = \frac{m_2}{m_1+m_2}$ and hence $\bar{r}_1 = -\mu\hat{b}_1$ and $\bar{r}_2 = (1-\mu)\hat{b}_1$. Furthermore, if we scale the time variable $t = \frac{1}{\omega}\tau$, then show that we can re-write equations of motion as:

$$x'' - 2y' - x = -\frac{(1-\mu)(x+\mu)}{p_1^3} - \frac{\mu(x-1+\mu)}{p_2^3} \tag{4}$$

$$y'' + 2x' - y = -\frac{(1-\mu)y}{p_1^3} - \frac{\mu y}{p_2^3} \tag{5}$$

$$\tag{6}$$

where $(.)' = \frac{d(.)}{d\tau}$ and $(.)'' = \frac{d^2(.)}{d\tau^2}$.

Finally, show that the equations of motion can be written as:

$$x'' - 2y' = \frac{\partial U}{\partial x} = U_x \tag{7}$$

$$y'' + 2x' = \frac{\partial U}{\partial y} = U_y \tag{8}$$

where U is the effective potential defined as $U = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{p_1} + \frac{\mu}{p_2}$. Note that the gradient of U provides gravitational and centrifugal forces.

(b) **Equilibrium Points:** Euler and Lagrange showed that there are five equilibrium points for the CR3BP. These points are also known as the *Lagrange Points.* Lagrange points are assumed to be at rest in the rotating frame, i.e., $x' = y' = x'' = y'' = 0$. Euler showed that three equilibrium points are collinear along the line $y = 0$. These points are labeled as L_1, L_2 and L_3 as shown in Figure 2, and are listed in Table 1.

Lagrange later discovered that the other two Lagrange points form equilateral triangles with the two primary bodies. Hence, they can be found by letting $p_1 = p_2 = 1$. Your task is to find the location of L_4 and L_5 Lagrange points for different CR3BP systems listed in Table 1.

Table 1: Lagrange Points for Different CR3BP Systems.

System	μ	L_1	L_2	L_3	L_4	L_5
Sun-Earth	3.0039×10^{-6}	0.9900261	1.0100345	-1.0000012	...	...
Earth-Moon	1.2151×10^{-2}	0.836915	1.15568	-1.00506	...	...
Saturn-Titan	2.366×10^{-4}	0.9575	1.0425	-1.0001	...	...

(c) **Stability of Lagrange Points:** Let (x_0, y_0) be the coordinates of the Lagrange points. Now, the points in the neighborhood of Lagrange points can be written as:

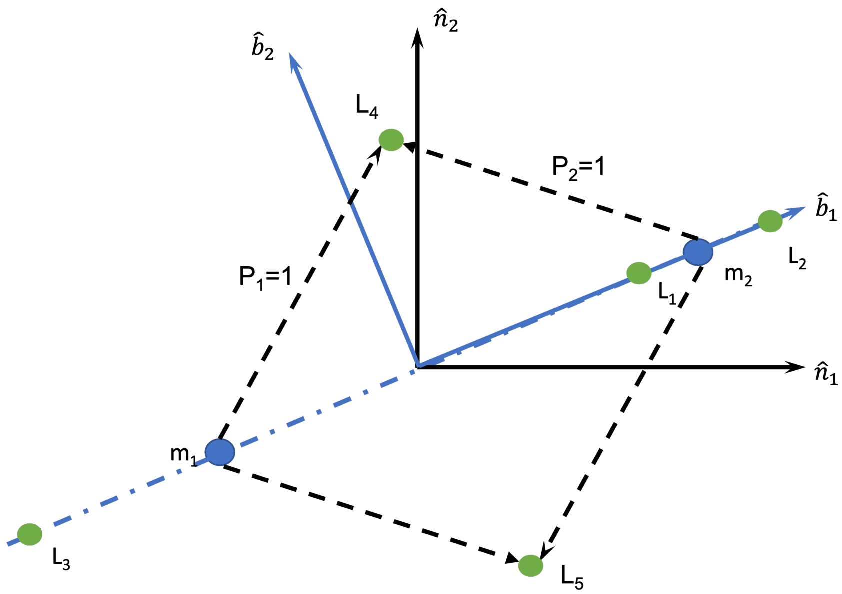

Figure 2: A Schematic of Lagrange Points.

$$x = x_0 + \delta x, \;\; y = y_0 + \delta y \tag{9}$$
$$x' = 0 + \delta x', \;\; y' = 0 + \delta y' \tag{10}$$

Assuming δx and δy to be small, linearize equations of motion given by (7)-(8). Show that the linearized equations of motion in the neighborhood of a Lagrange point can be written as:

$$\delta\bar{x}' = A\delta\bar{x}, \quad \delta\bar{x} = \{\delta x \;\, \delta y \;\, \delta x' \;\, \delta y'\}^T \tag{11}$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{bmatrix}, \quad U_{xx} = \frac{\partial^2 U}{\partial x^2}, \; U_{yy} = \frac{\partial^2 U}{\partial y^2}, \; U_{xy} = \frac{\partial^2 U}{\partial x \partial y} \tag{12}$$

Compute the eigenvalues of the A matrix for each Lagrange point for the different CR3BP systems listed in Table 1 and comment on the stability of these Lagrange points.

(d) **Simulating Orbits in the Earth-Moon System around L_2 point:** Integrate your equations of motion given by (7)-(8) numerically to solve for spacecraft motion in a Lyapunov orbit around L_2. The initial conditions, $\bar{x}(0) = \{x(0), y(0), x'(0), y'(0)\}^T$ corresponding to the Lyapunov orbit and simulation time (T) are given in file *EM_L2-304P1.mat*. We call this solution to be the nominal orbit solution,

$$\bar{x}_N(t) = \{x_N(t), y_N(t), \dot{x}_N(t), \dot{y}_N(t)\}^T$$

Plot $x_N(t)$ vs. $y_N(t)$ to visualize this orbit. Convert body frame coordinates $x_N(t)$ and $y_N(t)$ to inertial frame coordinates $X_N(t)$ and $Y_N(t)$. Plot $X_N(t)$ vs. $Y_N(t)$.

Let us perturb our initial condition by $\delta\bar{x}(0) = \{\delta x(0), \delta y(0), \delta x'(0), \delta y'(0)\}^T$. The initial condition perturbation $\delta\bar{x}(0)$ is given by the variable *perturbation* in file *EM_L2-304P1.mat*. Integrate your equations of motion given by (7)-(8) numerically for the perturbed initial condition, $\bar{x}(0) + \delta\bar{x}(0)$. We denote this solution by $\bar{x}(t)$. Compute the departure motion at each time, i..e, $\delta\bar{x}(t) = \bar{x}(t) - \bar{x}_N(t)$. Plot $\delta\bar{x}(t)$ (both position and velocity) versus time.

Solve the linearized equations of motion (given by (11)) in the neighborhood of L_2 Lagrange point with initial condition $\delta\bar{x}(0)$ to find $\delta\bar{x}_L(t)$. Compare $\delta\bar{x}_L(t)$ with $\delta\bar{x}(t)$. Comment on your findings.

(e) **Simulating Orbits in the Earth-Moon System around L_4 point:** Repeat the previous step for the variables given in file *EM_L4-304P1.mat*