Normalisers in Quasipolynomial Time and the Category of Permutation Groups

Sergio Siccha

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Lehr- und Forschungsgebiet Algebra, RWTH Aachen

Introduction

Goal

Theorem

Let $G = \langle X \rangle \leq \operatorname{Sym}\Omega$ be a primitive group of PA type. The normaliser $N_{\operatorname{Sym}\Omega}(G)$ can be computed in quasipolynomial time $O(n^3 \cdot 2^{2\log n \log\log n} \cdot |X|)$.

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Joint work with Prof. Colva Roney-Dougal.

Recursion for Normalisers

	Intransitive
Mun See Chang	$\updownarrow$
	Transitive
	$\updownarrow$
	Primitive
Me	$\updownarrow$
	Simple

Conventions

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- All groups and sets are finite!
- Ω, Δ denote sets, G, H, T denote groups.
  - T *always* denotes a finite non-abelian simple group.
- Functions act from the left $f(x)$ but groups from the right: $\alpha^g = g(\alpha)$.

Complexity and Computational Group Theory

Complexity Classes

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$$\begin{aligned} &\text{Polynomial Time:} && f \in O(n^c) \\ &\text{Quasipolynomial Time:} && f \in 2^{O((\log n)^c)} \\ &\text{Simply Exponential Time:} && f \in 2^{O(n)} \end{aligned}$$

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Normaliser and Subproblems

Simply Exponential

Normalisers of arbitrary groups

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Normalisers of groups with
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Quasipolynomial

Normalisers of primitive groups

Why restrict to $N_{\operatorname{Sym}\Omega}(G)$?

PA Type Groups and How To Normalise Them

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$$f(\alpha^g) = f(\alpha)^{\varphi(g)}.$$

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Let G be a group. The *socle* of G, denoted $\operatorname{soc} G$, is the group generated by all minimal normal subgroups of G.

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- $\operatorname{soc} G$ is non-abelian simple and non-regular.

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Definition

Let $H \leq \operatorname{Sym}\Delta$ and $K \leq S_\ell$. The base group H^ℓ acts component-wise on Δ^ℓ. The top group K acts on Δ^ℓ by permuting the components.

Wreath Products (2)

Definition

Let $H \leq \operatorname{Sym} \Delta$ and $K \leq S_\ell$. The base group H^ℓ acts component-wise on Δ^ℓ. The top group K acts on Δ^ℓ by permuting the components. This yields the *product action of $H \wr K$ on* Δ^ℓ.

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Lemma

Let $T^\ell \leq \operatorname{Sym}\Delta^\ell$ act component-wise, transitively, and non-regularly. Then

$$N_{\operatorname{Sym}\Delta^\ell}(T^\ell) = N_{\operatorname{Sym}\Delta}(T) \wr S_\ell.$$

The Key Idea ...

Construct $N_{\mathrm{Sym}\,\Omega}(\mathrm{soc}\, G)$!

… And Why It Works …

Lemma

Let $G \leq \operatorname{Sym}\Omega$ *be primitive of type PA. Then*

$$[N_{\operatorname{Sym}\Omega}(\operatorname{soc} G) : \operatorname{soc} G] \leq \sqrt{n} \cdot 2^{\log n \log\log n}.$$

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Let $G = \langle X \rangle \leq \operatorname{Sym}\Omega$ *be primitive of type PA. Let* $N_{\operatorname{Sym}\Omega}(\operatorname{soc} G) = \langle Y \rangle$ *be known.*
Then $N_{\operatorname{Sym}\Omega}(G)$ *can be computed in time*

$$O(n^3 \cdot 2^{2\log n \log\log n} \cdot |X|).$$

… And How To Do It

Compute:

$$\operatorname{soc} G \circlearrowright \Omega \xrightarrow{\sim} T^{\ell} \circlearrowright \Delta^{\ell}$$

The Category of Permutation Groups

Permutation Homomorphisms (1)

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$$\begin{tikzcd} \Omega \arrow[r, "g"] \arrow[d, "f"] & \Omega \arrow[d, "f"] \\ \Delta \arrow[r, "\varphi(g)"] & \Delta \end{tikzcd}$$

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Let $G \leq \operatorname{Sym}\Omega$ *and* $f : \Omega \to \Delta$*. There exists a permutation hom.* (f, φ) *if and only if*

$$\left\{ f^{-1}(\{x\}) \;\middle|\; x \in \operatorname{Im} f \right\}$$

is G*-invariant.*

Permutation Homomorphisms (3)

$$\begin{tikzcd} \Omega \arrow[r, "g"] \arrow[d, "f"] & \Omega \arrow[d, "f"] \\ \Delta \arrow[r, "\varphi(g)"] & \Delta \end{tikzcd}$$

Remark

Let $G \leq \operatorname{Sym} \Omega$ *and* $H \leq \operatorname{Sym} \Delta$. *A surjective map* $f : \Omega \twoheadrightarrow \Delta$

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Remark

Let $G \leq \operatorname{Sym}\Omega$ *and* $H \leq \operatorname{Sym}\Delta$*. A surjective map* $f : \Omega \twoheadrightarrow \Delta$ *uniquely determines, if it exists, a permutation homomorphism* (f, φ)*.*

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- permutation homomorphisms as morphisms.

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Let $G \leq \operatorname{Sym}\Omega$ *and* $H \leq \operatorname{Sym}\Delta$ *be permutation groups. Then* $(G \times H, \Omega \times \Delta)$ *with*

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$$G \xleftarrow{\pi_1} G \times H \xrightarrow{\pi_2} H$$

$$\Omega \xleftarrow{p_1} \Omega \times \Delta \xrightarrow{p_2} \Delta$$

is a product in **PermGrp**.

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Lemma

A family $(f_i)_{i\in I}$ is a cartesian decomposition of X if and only if X with $(f_i)_{i\in I}$ forms a product in $\mathcal{C}$.

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$\rightsquigarrow$ Compute a strictly homogeneous cartesian decomposition of $\operatorname{soc} G$!

Constructing the Normaliser of the Socle

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Note that T^ℓ has *exactly* ℓ minimal normal subgroups.

The Algorithm - Str. Hom. Cartesian Decomposition

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$\rightsquigarrow N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$.

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  - construct the normaliser of the socle in polynomial time,
  - compute the normaliser in quasipolynomial time.

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- Category Theory makes (some) algorithms nicer.
- Let G be a primitive group of PA type. We can
    - construct the normaliser of the socle in polynomial time,
    - compute the normaliser in quasipolynomial time.
      (maybe even in polynomial time?)

Thank you!

Food for Thought

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  $\rightsquigarrow$ Universal property?

Complexity Overview

Polynomial:

Base & SGS, Composition Series, Socle

Complexity Overview

Quasipolynomial:

Graph-Iso

Polynomial:

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Complexity Overview

Quasipolynomial:

String-Iso, Intersection, Centraliser

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Complexity Overview

Simply Exponential:

Normaliser

Quasipolynomial:

String-Iso, Intersection, Centraliser

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Polynomial:

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Universal Property of Wreath Products

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$$H^\Gamma \longrightarrow G \longleftarrow K$$

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$$\begin{tikzcd} H^\Gamma \arrow[r] & G \arrow[r] & K \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \Delta \times \Gamma \arrow[r] & \Delta \times \Gamma \arrow[r] & \Gamma \arrow[l, bend left] \end{tikzcd}$$

Combinatorial Cartesian Decompositions (1)

Definition

Let Ω be a set. For each $\gamma \in \Gamma$ let Δ_γ be a partition of Ω with $|\Delta_\gamma| \geq 2$. We say that $\{\Delta_\gamma\}_{\gamma \in \Gamma}$ is a *(combinatorial) cartesian decomposition of* Ω if

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Lemma

There is a one-to-one correspondence between unordered cartesian decompositions and combinatorial cartesian decompositions.

Combinatorial Cartesian Decompositions (2)

Theorem (Praeger, Schneider)
A group $G \leq \operatorname{Sym}\Omega$ leaves a homogeneous combinatorial cartesian decomposition invariant if and only if G embeds into a product action wreath product $\operatorname{Sym}\Delta \wr \operatorname{Sym}\Gamma$.