Normalisers in Quasipolynomial Time and the Category of Permutation Groups

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Introduction

Conventions

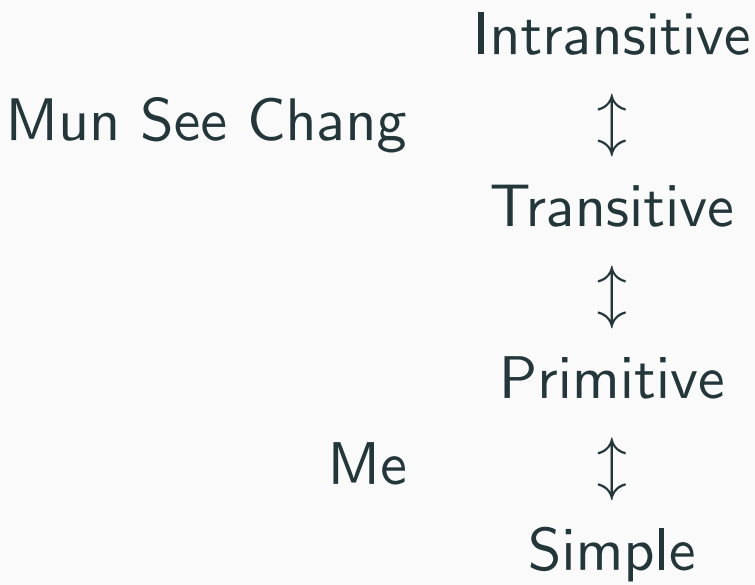
- $\log = \log_2$.
- All groups and sets are finite!
- Capital greek letters denote sets, Capital latin letters denote groups. Lower case letters denote elements or functions.
- Functions from the left $f(x)$ but group actions from the right: $\alpha^g = g(\alpha)$.
  - G acts on functions $\Omega \to \Delta$ via $f^g = f \circ g^{-1}$.
- T *always* denotes a finite non-abelian simple group. If $T \leq \operatorname{Sym} \Delta$ it acts transitively and non-regularly on Δ.

Goal

Theorem

Let $G = \langle X \rangle \leq \operatorname{Sym} \Omega$ *be a primitive group of PA type. The normaliser* $N_{\operatorname{Sym} \Omega}(G)$ *can be computed in quasipolynomial time* $O(n^3 \cdot 2^{2 \log n \log \log n} \cdot |X|)$.

Recursion


Intransitive
Mun See Chang
Transitive
Primitive
Me
Simple

Some Problems in Computational Group Theory

Complexity Classes

Big O Notation

Complexity Classes

Big O Notation

Polynomial Time:	$f \in O(n^c)$
Quasipolynomial Time:	$f \in 2^{O((\log n)^c)}$
Simply Exponential Time:	$f \in 2^{O(n)}$
Exponential Time:	$f \in 2^{O(n^c)}$

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We say a problem A is *polynomial time reducible* to a problem B if there exists a polynomial time algorithm that transforms

- instances of A into instances of B, and
- solutions of B into solutions of A.

1. changing input size to $\log n$ jumps up two classes
2. A is easier than B

   OR

   A can be embedded into B

Complexity Classes

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Complexity Overview

Polynomial:

Base & SGS, Composition Series, Socle

Complexity Overview

Quasipolynomial:

Graph-Iso

Polynomial:

Base & SGS, Composition Series, Socle

Complexity Overview

Quasipolynomial:

String-Iso, Intersection, Centraliser

Graph-Iso

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Complexity Overview

Simply Exponential:

Permutation-Iso, Normaliser,
Canonical Labeling

Quasipolynomial:

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FIXME: USE UNCOVER OR ONLY ALTERNATIVE

https://tex.stackexchange.com/questions/13793/beamer-alt-command-like-visible-instead-of-like-only

Normaliser and Subproblems

Simply Exponential

Normalisers of arbitrary groups

Polynomial

Normalisers of groups with restricted composition factors

Quasipolynomial

Normalisers of primitive groups

PA Type Groups and How To Normalise Them

Fundamentals

Definition

FIXME primitive

Definition

FIXME perm iso

Remark

FIXME $f : \Omega \xrightarrow{\sim} \Delta$ *induces unique group hom* $\operatorname{Sym}\Omega \xrightarrow{\sim} \operatorname{Sym}\Delta$.

Explain perm iso: FIXME

Definition
FIXME primitive

Definition
FIXME perm iso

Remark
FIXME $f : \Omega \xrightarrow{\sim} \Delta$ *induces unique group hom* $\operatorname{Sym}\Omega \xrightarrow{\sim} \operatorname{Sym}\Delta$.

Socles

Definition

FIXME Socle

Theorem

The socle of a primitive group is characteristically simple.

Theorem (O'Nan-Scott)

Let $G \leq \operatorname{Sym}\Omega$ *be primitive. All possible permutational isomorphism types of* $\operatorname{soc} G$ *and* $N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$ *are known.*

Wreath Products (1)

Definition

FIXME Abstract Wreath Product

Theorem

$\mathrm{Aut}(T^{\ell}) \cong \mathrm{Aut}(T) \wr S_{\ell}$

Wreath Products (2)

Definition

FIXME Imprimitive and product action

Theorem

FIXME H non-regular, K finite. $H \wr K$ primitive iff. H primitive and K transitive.

Explain WP actions via

base

top

Definition
FIXME Imprimitive and product action

Theorem
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The PA Type

Definition

FIXME AS

Definition

FIXME PA

Lemma

FIXME Properties of PA type

Definition
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Definition
FIXME PA

Lemma
FIXME Properties of PA type

The PA Type

Mention $G \leq$ norm of socle

The Key Idea ...

Construct $N_{\mathrm{Sym}\,\Omega}(\mathrm{soc}\,G)$!

... And Why It Works ...

Lemma

Let $G \leq \operatorname{Sym}\Omega$ *be primitive of type PA. Then*

$$[N_{\operatorname{Sym}\Omega}(\operatorname{soc} G) : \operatorname{soc} G] \leq \sqrt{n} \cdot 2^{\log n \log\log n}.$$

Lemma

Let $G = \langle X \rangle \leq \operatorname{Sym}\Omega$ *be primitive of type PA. Furthermore let a generating set for* $N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$ *be known. Then* $N_{\operatorname{Sym}\Omega}(G)$ *can be computed in time* $O(n^3 \cdot 2^{2 \log n \log\log n} \cdot |X|)$.

... And How To Do It

Compute:

$$\operatorname{soc} G \circlearrowright \Omega \xrightarrow{\sim} T^\ell \circlearrowright \Delta^\ell$$

Then:

$$\begin{aligned} G &\hookrightarrow N_{\operatorname{Sym}\Delta^\ell}(T^\ell) \\ &= N_{\operatorname{Sym}\Delta}(T) \wr S_\ell \end{aligned}$$

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Then:

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- equal and not only isomorphic
- PA WP is a very very special group!

The Category of Permutation Groups

Permutation Homomorphisms (1)

Definition

Let $G \leq \operatorname{Sym} \Omega$ and $H \leq \operatorname{Sym} \Delta$ be permutation groups. A tuple (f, φ) with map $f : \Omega \to \Delta$ and group hom. $\varphi : G \to H$ is called a *permutation hom. from* (G, Ω) *to* (H, Δ) if for all $g \in G$ holds

FIXME COMMUTINGDIAGRAM

Permutation Homomorphisms (2)

Lemma

Let $G \leq \operatorname{Sym}\Omega$ and $f : \Omega \to \Delta$. There exist a group $H \leq \operatorname{Sym}\Delta$ and a group hom. $\varphi\colon G \to H$ such that (f, φ) is a permutation hom. if and only if

$$\left\{ f^{-1}(\{x\}) \;\middle|\; x \in \operatorname{Im} f \right\}$$

is G-invariant.

Permutation Homomorphisms (3)

FIXME EXAMPLE

Remark

Let $G \leq \operatorname{Sym} \Omega$ and $H \leq \operatorname{Sym} \Delta$. $f : \Omega \twoheadrightarrow \Delta$ uniquely determines, if it exists, a group hom. $\varphi : G \to H$ such that (f, φ) is a permutation hom.

PermGrp

Definition

FIXME Define **PermGrp**.

PermGrp

Definition
FIXME Define **PermGrp**.

equiv to cat of (G, Ω, ρ)

Product in PermGrp

Lemma

Let $G \leq \operatorname{Sym} \Omega$ *and* $H \leq \operatorname{Sym} \Delta$ *be permutation groups. Then* $(G \times H, \Omega \times \Delta)$ *with* (p_1, π_1) *and* (p_2, π_2) *is a product in* **PermGrp**.

Cartesian Decompositions

Definition

Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. A family of morphisms $(f_i)_{i\in I}$ with $f_i : X \to X_i$ is called a *cartesian decomposition of* X if

$$\prod_{i\in I} f_i : X \to \prod_{i\in I} X_i$$

is an isomorphism.

Lemma

A family $(f_i)_{i\in I}$ is a cartesian decomposition of X if and only if X with $(f_i)_{i\in I}$ is a product in C.

Homogeneous Cartesian Decompositions

Definition

FIXME hom cartesian decomposition. For all $i, j \in I$ have $f_i(X) \cong f_j(X)$

Definition

FIXME strongly hom cartesian decomposition For all $i, j \in I$ have $f_i(X) = f_j(X)$

$\Rightarrow$ Compute a strongly homogeneous cartesian decomposition of soc G!

Combinatorial Cartesian Decompositions

FIXME LEAVE THIS FRAME OUT?

Definition

CCD

Lemma

unordered cd bijection CCD

Theorem (Praeger, Schneider)

G leaves CCD invariant if and only if G embeds into PA WP.

Constructing the Normaliser of the Socle

The Algorithm (1)

Note that T^ℓ has *exactly* ℓ minimal normal subgroups.

The Algorithm (2)

Algorithm

- $\operatorname{soc} G$ $(\cong T_1 \times \ldots \times T_\ell)$.

The Algorithm (2)

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- $g_1, \ldots, g_\ell \in G$ *such that* $T_i^{g_i} = T_1$.

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- $R_i \colon \Delta_i \to \Delta_1,\ \delta \mapsto \delta^{g_i} \quad \Rightarrow \quad \rho_i \colon T_i \to T_1$.
- $P_i := R_i \circ Q_i \colon \Omega \to \Delta_1$

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- $P_i := R_i \circ Q_i \colon \Omega \to \Delta_1 \quad \Rightarrow \quad \varphi_i \colon G \to T_1$.

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- $P_i := R_i \circ Q_i \colon \Omega \to \Delta_1 \quad \Rightarrow \quad \varphi_i \colon G \to T_1$.

Strongly homogeneous cartesian decomposition $((P_i, \varphi_i))_{i \leq \ell}$ of $\operatorname{soc} G$ in polynomial time.

Summary

What To Take Away

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- For primitive groups of PA type we can construct the normaliser of the socle in polynomial time.

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- Category Theory makes (some) algorithms nicer.
- For primitive groups of PA type we can construct the normaliser of the socle in polynomial time.
- For primitive groups of PA type we can compute the normaliser in quasipolynomial (maybe even polynomial?) time.