Normalisers in Quasipolynomial Time and the Category of Permutation Groups

Sergio Siccha

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Lehr- und Forschungsgebiet Algebra, RWTH Aachen

Conventions

- $\log := \log_2$.
- All groups and sets are finite!
- Ω, Δ denote sets, G, H, T denote groups.
  - T *always* denotes a finite non-abelian simple group.
  - If $T \leq \operatorname{Sym}\Delta$ it acts transitively and non-regularly on Δ.
- Functions act from the left $f(x)$ but groups from the right: $\alpha^g = g(\alpha)$.

Introduction

Goal

Theorem

Let $G = \langle X \rangle \leq \mathrm{Sym}\,\Omega$ *be a primitive group of PA type. The normaliser* $N_{\mathrm{Sym}\,\Omega}(G)$ *can be computed in quasipolynomial time* $O(n^3 \cdot 2^{2\log n \log\log n} \cdot |X|)$.

Joint work with Prof. Colva Roney-Dougal.

Recursion for Normalisers

$$\begin{array}{rc} & \text{Intransitive} \\ \text{Mun See Chang} & \updownarrow \\ & \text{Transitive} \\ & \updownarrow \\ & \text{Primitive} \\ \text{Me} & \updownarrow \\ & \text{Simple} \end{array}$$

Complexity and Computational Group Theory

Complexity Classes

We use big O notation.

$$\begin{array}{ll} \text{Polynomial Time:} & f \in O(n^c) \\ \text{Quasipolynomial Time:} & f \in 2^{O((\log n)^c)} \\ \text{Simply Exponential Time:} & f \in 2^{O(n)} \\ \text{Exponential Time:} & f \in 2^{O(n^c)} \end{array}$$

We say a problem A is *polynomial time reducible* to a problem B if there exists a polynomial time algorithm that transforms

- instances of A into instances of B, and
- solutions of B into solutions of A.

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- instances of A into instances of B, and
- solutions of B into solutions of A.

1. properly explain why quasipolynomial is not polynomial!
2. changing input size to $\log n$ jumps up two classes
3. A is easier than B
   OR
   A can be embedded into B

Complexity Overview

Simply Exponential:

Normaliser

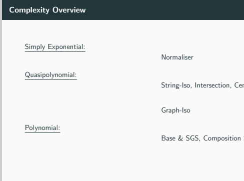
Quasipolynomial:

String-Iso, Intersection, Cer

Graph-Iso

Polynomial:

Base & SGS, Composition



FIXME: USE UNCOVER OR ONLY ALTERNATIVE

https://tex.stackexchange.com/questions/13793/beamer-alt-command-like-visible-instead-of-like-only

Normaliser and Subproblems

Simply Exponential

Normalisers of arbitrary groups

Polynomial

Normalisers of groups with restricted composition factors

Normalisers of simple groups

Quasipolynomial

Normalisers of primitive groups

Simply Exponential

Normalisers of arbitrary groups

Polynomial

Normalisers of groups with restricted composition factors

Normalisers of simple groups

Quasipolynomial

Normalisers of primitive groups

Explain how normaliser of simple works in poly time.

PA Type Groups and How To Normalise Them

Definition
Let $G \leq \operatorname{Sym}\Omega$ be transitive. G is called *imprimitive* if there exists a non-trivial G-invariant partition of Ω. Otherwise it is called *primitive*.

Definition
Let $G \leq \operatorname{Sym}\Omega$ and $H \leq \operatorname{Sym}\Delta$ be permutation groups. We call a pair (f, φ) with $f\colon \Omega \to \Delta$ and $\varphi\colon G \to H$ a *permutation isomorphism* if for all $g \in G$ and $\alpha \in \Omega$ holds $f(\alpha^g) = f(\alpha)^{\varphi(g)}$.

Fundamentals

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- Explain perm iso:
  - Relabel points and how to map $G \to H$ accordingly.
  - $G, H \leq \operatorname{Sym}\Omega$ perm iso iff. exists $\sigma \in \operatorname{Sym}\Omega$ with $G^\sigma = H$.
  - $f : \Omega \xrightarrow{\sim} \Delta$ induces unique group hom $\operatorname{Sym}\Omega \xrightarrow{\sim} \operatorname{Sym}\Delta$.
- perm iso can be extended to the whole symmetric group $\rightsquigarrow$ The normaliser of G over Ω is mapped to the normaliser of H over Δ.

Socles

Definition

Let G be a group. The *socle* of G, denoted $\operatorname{soc} G$, is the group generated by all minimal normal subgroups of G.

Theorem

The socle of a primitive group is characteristically simple.

Theorem (O'Nan-Scott)

Let $G \leq \operatorname{Sym}\Omega$ be primitive. All possible permutational isomorphism types of $\operatorname{soc} G$ and $N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$ are known.

Wreath Products (1)

Definition

Let $H \leq \operatorname{Sym} \Delta$ and $K \leq S_\ell$. K acts on the components of H^ℓ. The semidirect product $H \wr K = H^\ell \rtimes K$ is called the *wreath product of* H *with* K. H^ℓ is called the *base group*. K is called the *top group*.

Theorem

$\operatorname{Aut}(T^\ell) \cong \operatorname{Aut}(T) \wr S_\ell$

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Make sure to explain intuition behind wreath products

Wreath Products (2)

Definition

Let $H \leq \operatorname{Sym}\Delta$ and $K \leq S_\ell$. The base group H^ℓ acts component-wise on Δ^ℓ. The top group K acts on the components of Δ^ℓ. This yields an action of $H \wr K$ on Δ^ℓ which we call the *product action of $H \wr K$*.
We call the permutation group on Δ^ℓ induced by $H \wr K$ the *product action wreath product of H wirh K* and also denote it by $H \wr K$.

Theorem

Let $H \leq \operatorname{Sym}\Delta$ and $K \leq S_\ell$. $H \wr K$ in product action is primitive if and only if H is primitive and non-regular and K is transitive.

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Let $H \leq \operatorname{Sym}\Delta$ *and* $K \leq S_\ell$. $H \wr K$ *in product action is primitive if and only if* H *is primitive and non-regular and* K *is transitive.*

Explain WP actions via

base

top

The AS Type

Definition

Let $G \leq \operatorname{Sym}\Omega$ be a primitive group.
We say G is a group of *AS type* if $\operatorname{soc} G = T$ is non-abelian simple and G is almost simple.

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Explain AS via Normaliser of T.

The PA Type

Definition

Let $G \leq \operatorname{Sym}\Omega$ be a primitive group.

We say G is a group of *PA type* if it is permutation isomorphic to a group $\widehat{G} \leq \operatorname{Sym}\Delta^\ell$ with:

- $\operatorname{soc}\widehat{G} = T^\ell$,
- $\widehat{G} \leq N_{\operatorname{Sym}\Delta}(T) \wr S_\ell$.

Lemma

$N_{\operatorname{Sym}\Delta^\ell}(T^\ell) = N_{\operatorname{Sym}\Delta}(T) \wr S_\ell.$

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- PA: an example is our running example!!
- Überleitung to key idea:
  $\operatorname{soc} G$ char G
  $\operatorname{soc} G \trianglelefteq N(G)$
  Thus contained in normaliser of socle

The Key Idea ...

Construct $N_{\mathrm{Sym}\,\Omega}(\mathrm{soc}\,G)$!

Lemma

Let $G \leq \operatorname{Sym}\Omega$ *be primitive of type PA. Then*

$$[N_{\operatorname{Sym}\Omega}(\operatorname{soc} G) : \operatorname{soc} G] \leq \sqrt{n} \cdot 2^{\log n \log\log n}.$$

Lemma

Let $G = \langle X \rangle \leq \operatorname{Sym}\Omega$ *be primitive of type PA. Furthermore let a generating set for* $N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$ *be known.*

Then $N_{\operatorname{Sym}\Omega}(G)$ *can be computed in time*

$$O(n^3 \cdot 2^{2\log n \log\log n} \cdot |X|).$$

... And Why It Works ...

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Explain:

$|\operatorname{Out} T| \leq \sqrt{n}$

$|S_\ell| \leq \ell^\ell$

… And How To Do It

Compute:

$$\operatorname{soc} G \circlearrowright \Omega \xrightarrow{\sim} T^\ell \circlearrowright \Delta^\ell$$

Then:

$$\begin{array}{rcll} G & \hookrightarrow & N_{\operatorname{Sym}\Delta^\ell}(T^\ell) & = N_{\operatorname{Sym}\Delta}(T) \wr S_\ell \\ N_{\operatorname{Sym}\Omega}(\operatorname{soc} G) & \xleftarrow{\sim} & N_{\operatorname{Sym}\Delta^\ell}(T^\ell) & \end{array}$$

... And How To Do It

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- equal and not only isomorphic
- PA WP is a very very special group!

The Category of Permutation Groups

Permutation Homomorphisms (1)

Definition

Let $G \leq \operatorname{Sym}\Omega$ and $H \leq \operatorname{Sym}\Delta$ be permutation groups. Let $f\colon \Omega \to \Delta$ be a map and $\varphi\colon G \to H$ be a group hom..
The pair (f, φ) is called a *permutation hom. from* (G, Ω) *to* (H, Δ) if for all $g \in G$ holds:

$$\begin{array}{ccc} \Omega & \xrightarrow{g} & \Omega \\ \Big\downarrow f & & \Big\downarrow f \\ \Delta & \xrightarrow{\varphi(g)} & \Delta \end{array}$$

Permutation Homomorphisms (2)

$$\begin{tikzcd} \Omega \arrow[r, "g"] \arrow[d, "f"] & \Omega \arrow[d, "f"] \\ \Delta \arrow[r, "\varphi(g)"] & \Delta \end{tikzcd}$$

Remark

Let $G \leq \operatorname{Sym}\Omega$ and $H \leq \operatorname{Sym}\Delta$. The map $f : \Omega \twoheadrightarrow \Delta$ uniquely determines, if it exists, a permutation homomorphism (f, φ).

Permutation Homomorphisms (3)

Lemma

Let $G \leq \operatorname{Sym}\Omega$ *and* $f : \Omega \to \Delta$. *There exists a permutation hom.* (f, φ) *if and only if*

$$\left\{ f^{-1}(\{x\}) \;\middle|\; x \in \operatorname{Im} f \right\}$$

is G*-invariant.*

Lemma
Let $G \leq \operatorname{Sym}\Omega$ and $f : \Omega \to \Delta$. There exists a permutation hom. (f, φ) if and only if

$$\{ f^{-1}(\{x\}) \mid x \in \operatorname{Im} f \}$$

is G-invariant.

Give examples:

map to orbits

factor out block systems

if f is bijection such a φ always exists.

PermGrp

Definition

The *category of permutation groups*, denoted **PermGrp**, consists of all pairs (G, Ω) with $G \leq \operatorname{Sym} \Omega$ as objects with permutation homomorphisms as morphisms.

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equiv to cat of (G, Ω, ρ)

Product in PermGrp

Lemma

Let $G \leq \operatorname{Sym}\Omega$ *and* $H \leq \operatorname{Sym}\Delta$ *be permutation groups. Then* $(G \times H, \Omega \times \Delta)$ *with* (p_1, π_1) *and* (p_2, π_2) *is a product in* **PermGrp**.

Cartesian Decompositions

Definition

Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. A family of morphisms $(f_i)_{i \in I}$ with $f_i \colon X \to X_i$ is called a *cartesian decomposition of X* if

$$\prod_{i \in I} f_i \colon X \to \prod_{i \in I} X_i$$

is an isomorphism.

Lemma

A family $(f_i)_{i \in I}$ is a cartesian decomposition of X if and only if X with $(f_i)_{i \in I}$ forms a product in C.

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- explain cartesian decomposition of sets and of perm groups
- mention combinatorial cartesian decompositions by Laszlo Kovacs, Cheryl Praeger and Csaba Schneider!
  their theorem
  cartesian decompositions are to PA WPs what block systems are to IMP WPs.

Homogeneous Cartesian Decompositions

Definition

Let $(f_i)_{i \in I}$ be a cartesian decomposition of X. We call $(f_i)_{i \in I}$ a *homogeneous cartesian decomposition of* X if
for all $i, j \in I$ we have $f_i(X) \cong f_j(X)$.

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Let $(f_i)_{i \in I}$ be a cartesian decomposition of X. We call $(f_i)_{i \in I}$ a *strongly homogeneous cartesian decomposition of* X if
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$\rightsquigarrow$ Compute a strongly homogeneous cartesian decomposition of the permutation group $\operatorname{soc} G$!

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$\rightsquigarrow$ Compute a strongly homogeneous cartesian decomposition of the permutation group $\operatorname{soc} G$!

Explain hom cartesian decomposition and str hom cartesian decomposition of Ω and then of $\operatorname{soc} G$.

Constructing the Normaliser of the Socle

The Algorithm - Input

Let $G = \langle\, X \,\rangle \leq \operatorname{Sym}\Omega$ be a primitive group of PA type.

Note that T^ℓ has *exactly* ℓ minimal normal subgroups.

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Note that T^ℓ has *exactly* ℓ minimal normal subgroups.

T^ℓ has exactly one "basis"!

On next frame: mention i always stands for all i from 1 to ℓ

Algorithm

- $\operatorname{soc} G$ $(= T_1 \times \ldots \times T_\ell)$.
- *minimal normal subgroups* T_i *of* $\operatorname{soc} G$.
- *complements* C_i *of the* T_i, *partitions* $\Delta_i = \{\text{orbits of } C_i\}$.
- $Q_i \colon \Omega \to \Delta_i,\ \alpha \mapsto \alpha^{C_i} \quad \Rightarrow \quad \psi_i \colon G \to T_i$.
- $g_1, \ldots, g_\ell \in G$ *such that* $T_i^{g_i} = T_1$.
- $R_i \colon \Delta_i \to \Delta_1,\ \delta \mapsto \delta^{g_i} \quad \Rightarrow \quad \rho_i \colon T_i \to T_1$.
- $P_i := R_i \circ Q_i \colon \Omega \to \Delta_1 \quad \Rightarrow \quad \varphi_i \colon G \to T_1$.

$((P_i, \varphi_i))_{i \leq \ell}$ is strongly homogeneous cartesian decomposition.

The Algorithm - Normaliser of Socle

- $((P_i, \varphi_i))_{i \leq \ell}$ is a strongly homogeneous cartesian decomposition of $\operatorname{soc} G$.
- This yields $\operatorname{soc} G \circlearrowright \Omega \xrightarrow{\sim} T^\ell \circlearrowright \Delta^\ell$.
- Compute $N_{\operatorname{Sym}\Delta}(T)$.
- Construct $N_{\operatorname{Sym}\Delta}(T) \wr S_\ell \leq \operatorname{Sym}\Delta^\ell$.
- Map back into $\operatorname{Sym}\Omega$.

$\rightsquigarrow N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$.

The Algorithm - Normaliser of Socle

- $((P_i, \varphi_i))_{i \le \ell}$ is a strongly homogeneous cartesian decomposition of soc G.
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- Map back into $\operatorname{Sym}\Omega$.

$\rightsquigarrow N_{\operatorname{Sym}\Omega}(\operatorname{soc} G)$.

Use running example to explain Q_i and R_i.

Outlook and Summary

Food for Thought

- $G \hookrightarrow H \wr K \leq N_{\mathrm{Sym}\,\Delta}(T) \wr S_\ell$.
  $\rightsquigarrow$ Normalisers in polynomial time?

- G leaves a combinatorial cartesian decomposition invariant if and only if it can be embedded into a product action wreath product $S_m \wr S_\ell$.
  $\rightsquigarrow$ Universal property?

- Define a tree data structure via permutation homomorphisms to do many normaliser computations "at once".

What To Take Away

- Category Theory makes (some) algorithms nicer.
- Let G be a primitive group of PA type. We can
  - construct the normaliser of the socle in polynomial time,
  - compute the normaliser in quasipolynomial time.
    (maybe even in polynomial time?)

Thank you!

Universal Property of Wreath Products

Let $H \leq \operatorname{Sym}\Delta$, $K \leq \operatorname{Sym}\Gamma$.

$$H^\Gamma \longrightarrow G \longleftarrow K$$

$$\Delta^\Gamma \longrightarrow \Delta^\Gamma \longleftarrow \Gamma$$

$$\begin{tikzcd}
H^\Gamma \arrow[r] & G \arrow[r] & K \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta \times \Gamma \arrow[r] & \Delta \times \Gamma \arrow[r] & \Gamma \arrow[l, bend left]
\end{tikzcd}$$

Combinatorial Cartesian Decompositions (1)

Definition

Let Ω be a set. For each $\gamma \in \Gamma$ let Δ_γ be a partition of Ω with $|\Delta_\gamma| \geq 2$. We say that $\{\Delta_\gamma\}_{\gamma \in \Gamma}$ is a *(combinatorial) cartesian decomposition of* Ω if for any choice of $\delta_\gamma \in \Delta_\gamma$ we have that

$$\bigcap_{\gamma \in \Gamma} \delta_\gamma$$

is a singleton set.

Lemma

There is a one-to-one correspondence between unordered cartesian decompositions and combinatorial cartesian decompositions.

Combinatorial Cartesian Decompositions (2)

Theorem (Praeger, Schneider)
A group $G \leq \mathrm{Sym}\,\Omega$ leaves a homogeneous combinatorial cartesian decomposition invariant if and only if G embeds into a product action wreath product $\mathrm{Sym}\,\Delta \wr \mathrm{Sym}\,\Gamma$.