X-ray tomography and filtered back-projection

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PhD Winter School 2023

Advanced methods for mathematical image analysis

Bologna, Italy

January 23, 2023

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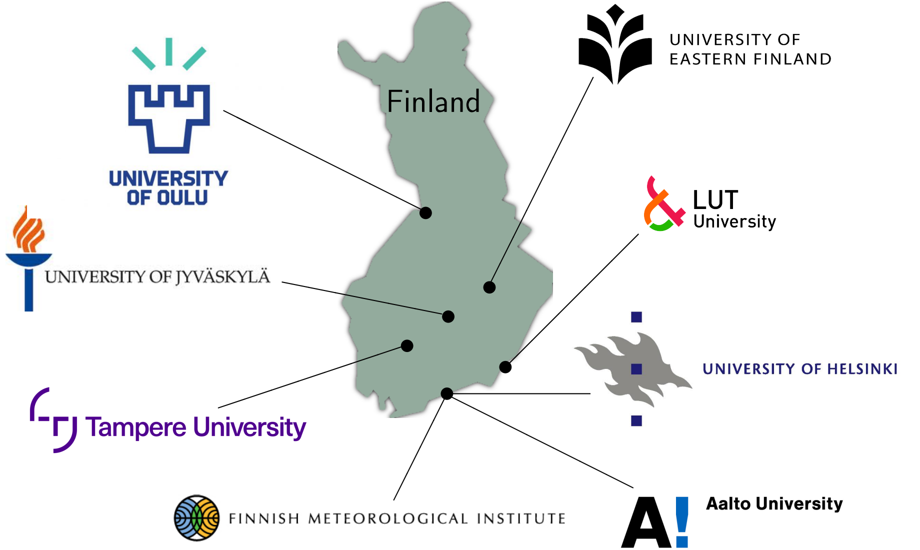
Finnish Centre of Excellence in Inverse Modelling and Imaging

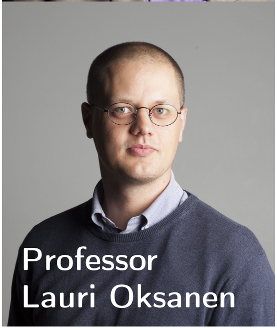
2018-2025

CENTRES OF EXCELLENCE IN RESEARCH



We have a research group of roughly 30 people at University of Helsinki


Professor
Matti Lassas


Professor
Lauri Oksanen

This my industrial-academic background

2009: Professor, University of Helsinki, Finland

2006: Professor, Tampere University of Technology, Finland

2005: R&D scientist at Palodex Group

2004: R&D scientist at GE Healthcare

2002: Postdoc at Gunma University, Japan

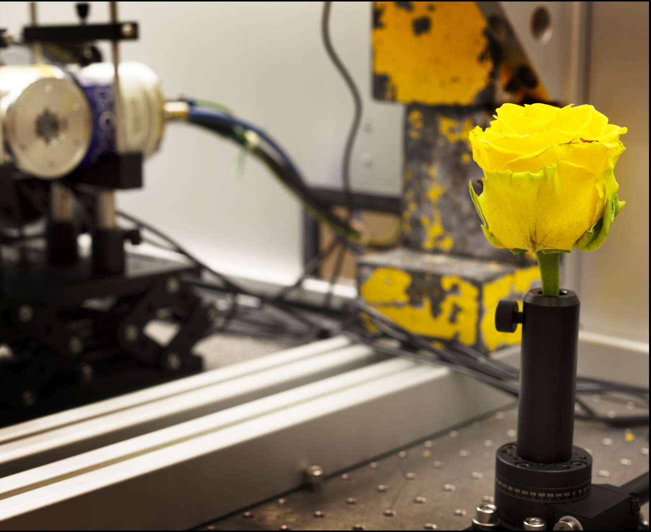
2000: R&D scientist at Instrumentarium Imaging

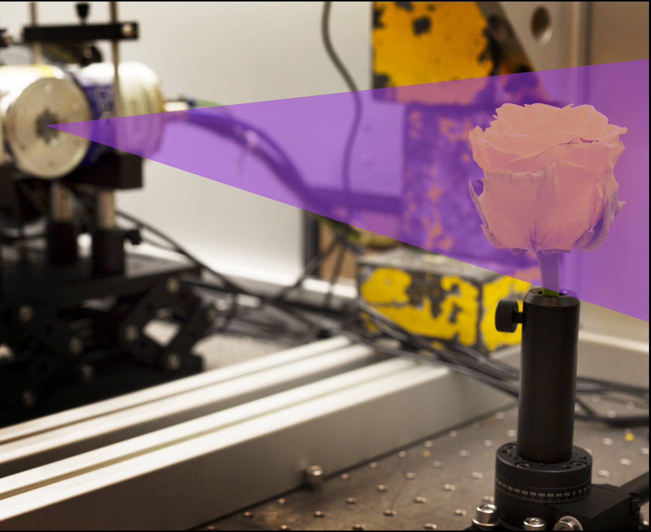
1999: PhD, Helsinki University of Technology, Finland

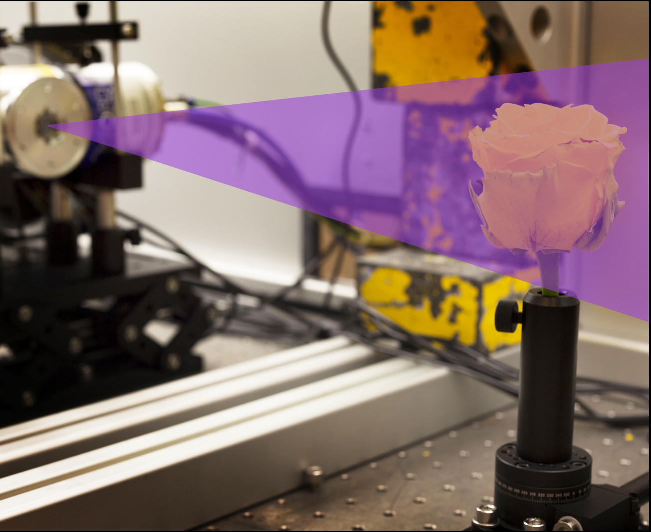
You can find all the slides and codes of this course in GitHub

https://github.com/ssiltane/BolognaWinterSchool2023

Outline








HAMAMATSU
DIGITAL OUT
EXT.TRG.
POWER

X-ray images are very useful for doctors. For example, they can see fractures.

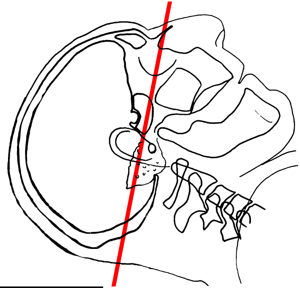
Nevit Dilmen, Wikimedia commons

Outline

Here is a 2D slice through a human head

Andrew Ciscel, Wikimedia commons

Here is a 2D slice through a human head



Andrew Ciscel, Wikimedia commons

Modern CT scanners look like this

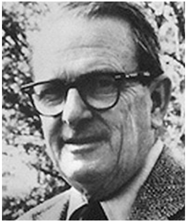
Wikimedia commons

Modern scanners rotate at high speed

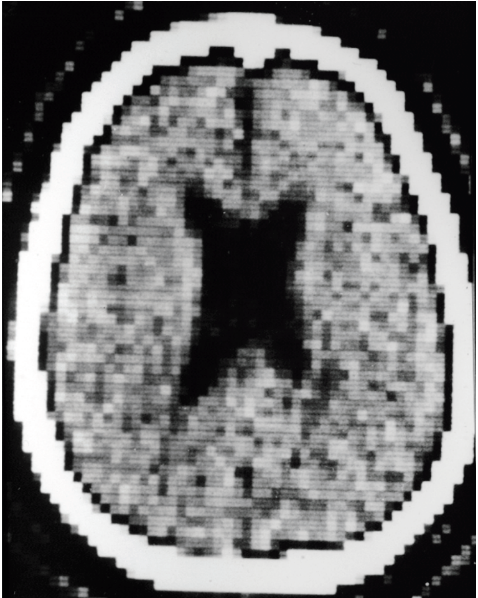
Couch unit for EMI brain scanner.

Image: Science Museum Group.

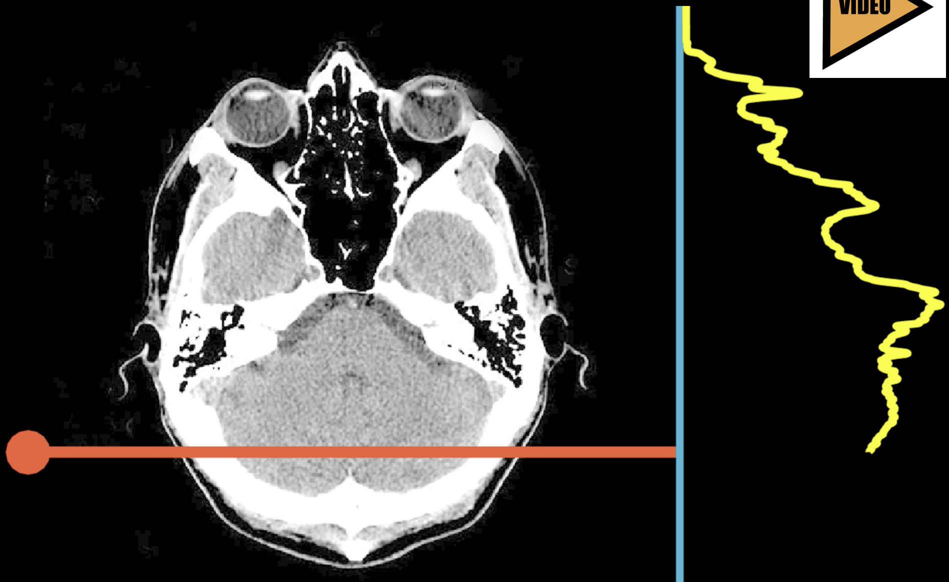
Godfrey Hounsfield and Allan McLeod Cormack developed X-ray tomography



Hounsfield (top) and Cormack received Nobel prizes in 1979.

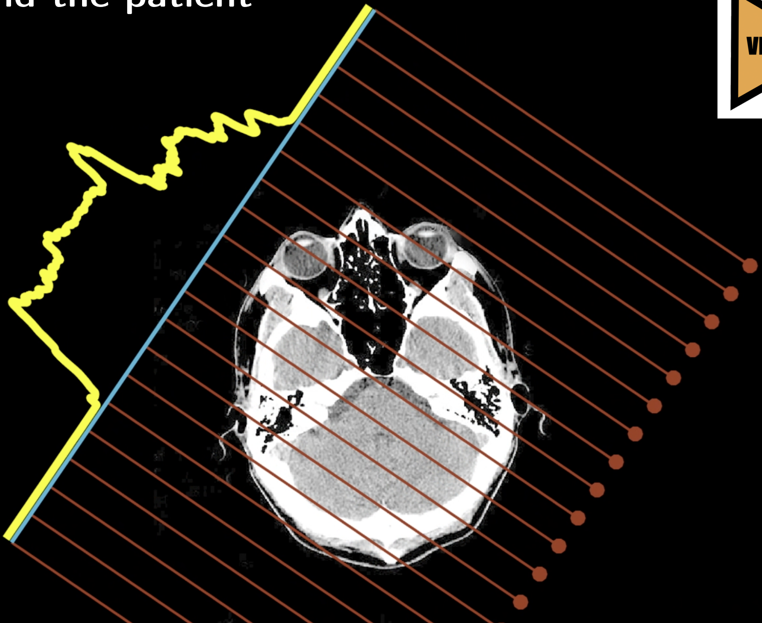


This sweeping movement is the data collection mode of first-generation CT scanners

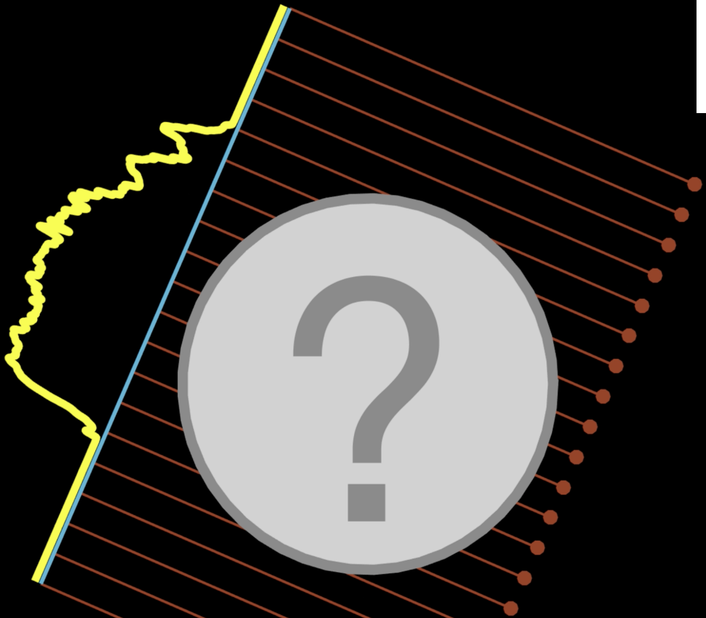



SCANNER

Data is collected by rotating the system around the patient



This is the inverse problem of tomography:



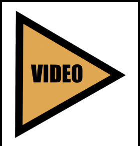
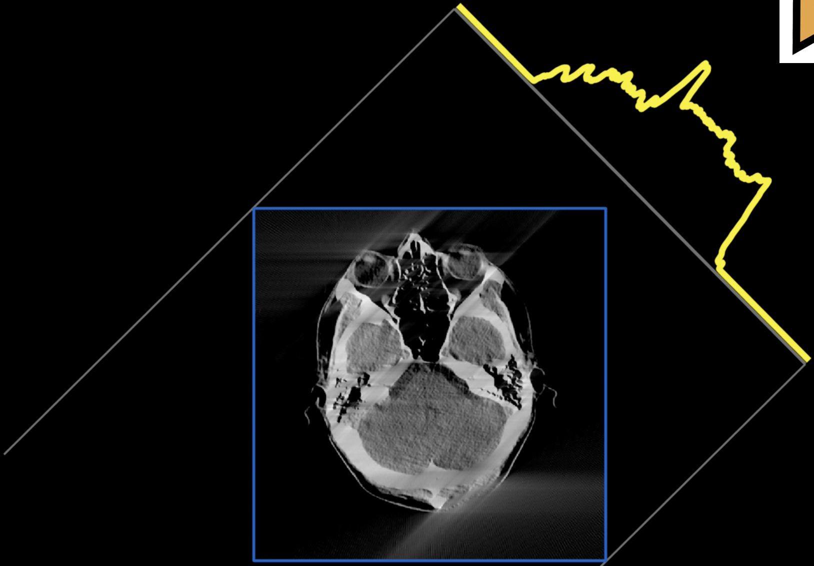
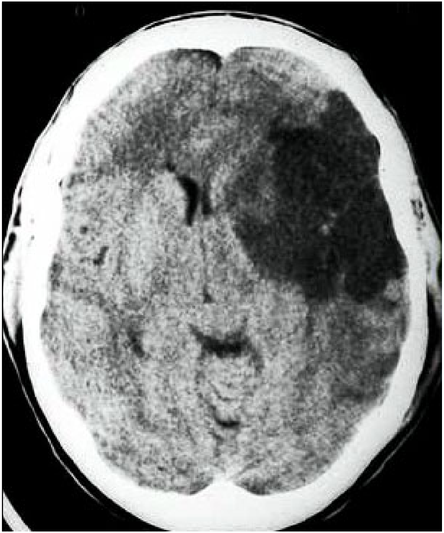


Illustration of the standard reconstruction by filtered back-projection (FBP)



Diagnosing stroke with X-ray tomography

Ischemic stroke



CT image from Jansen 2008

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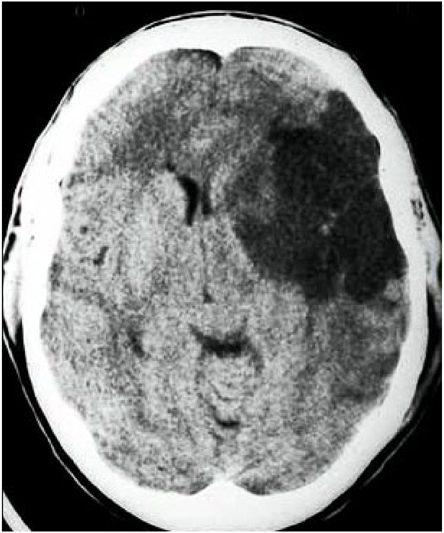
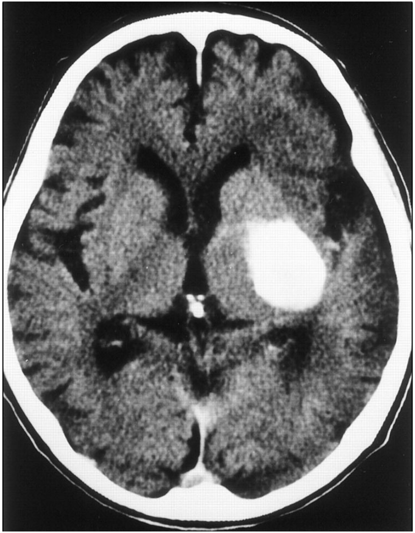


CT image from Jansen 2008

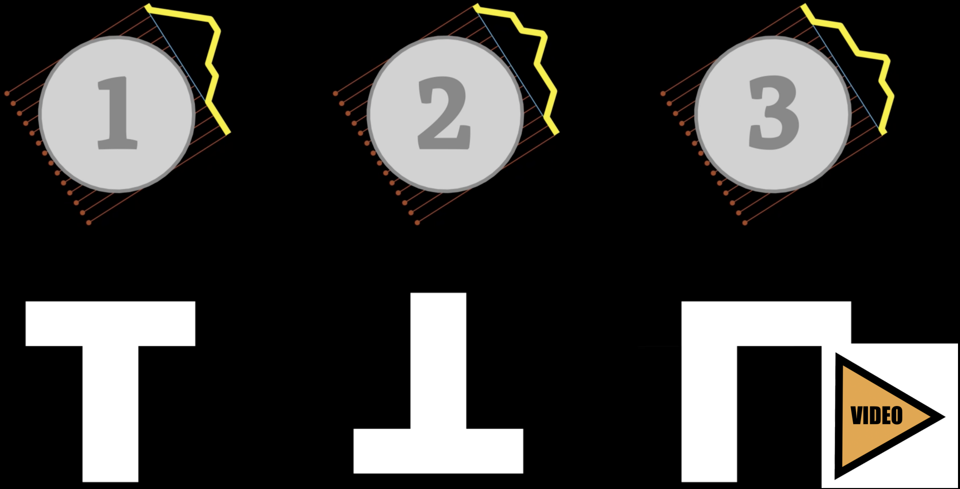
Hemorrhagic stroke



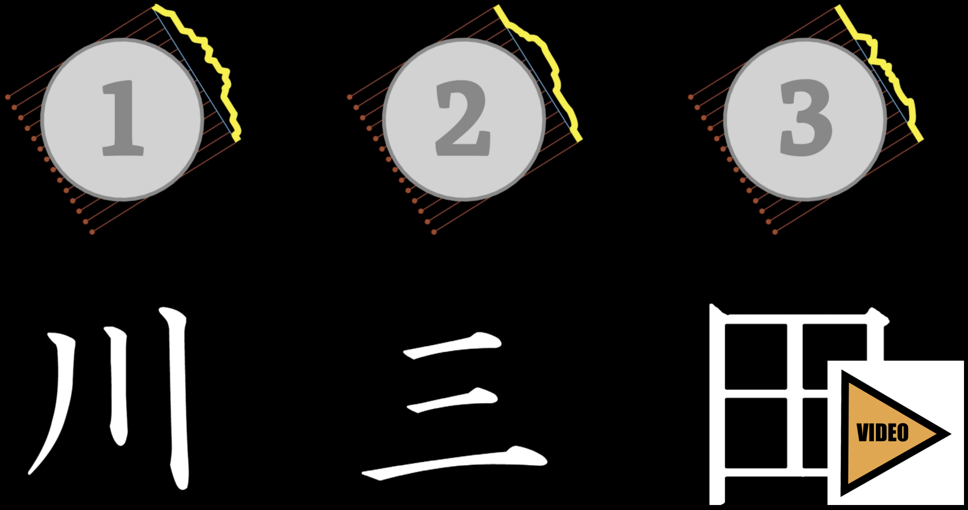
CT image from Nakano *et al.* 2001

Outline

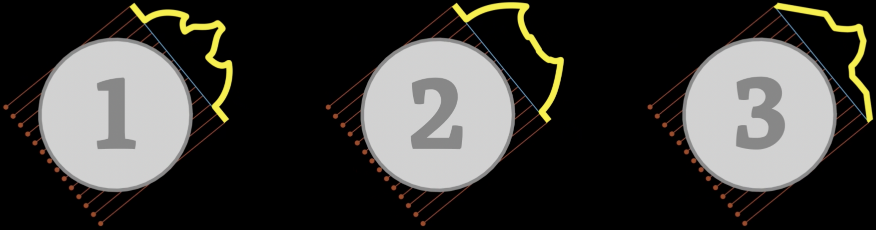
Level 1 tomographic mystery



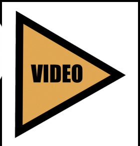
Level 2 tomographic mystery



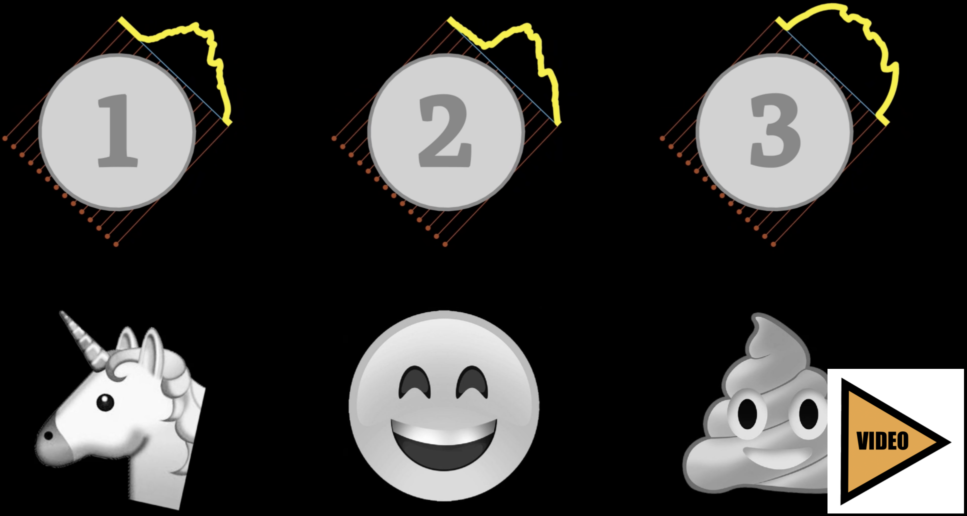
Level 3 tomographic mystery



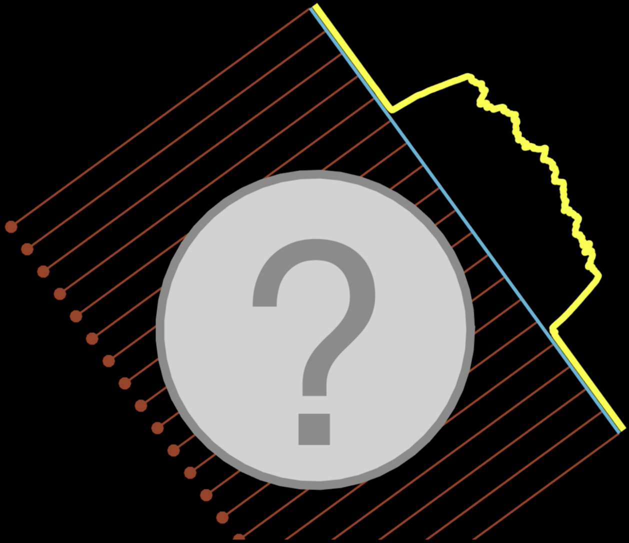
G R 8

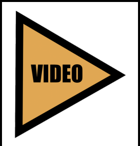


Level 4 tomographic mystery



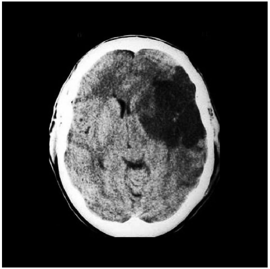
Level 5: can you guess the image?





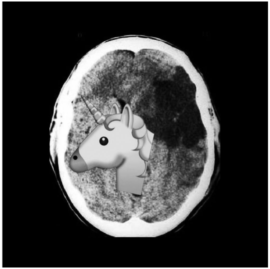
Alternatives

(a)



(b)

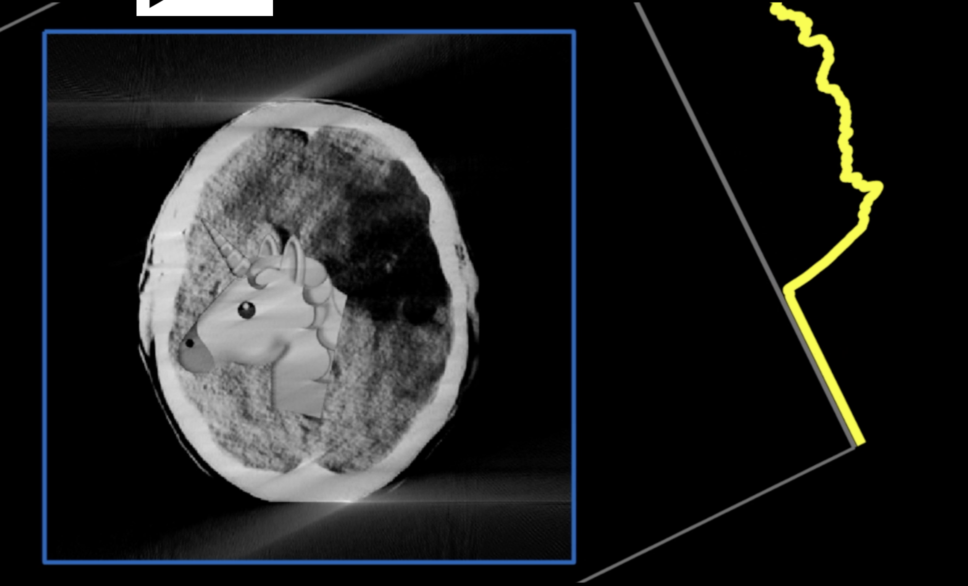
(c)



Solution

VIDEO

VIDEO



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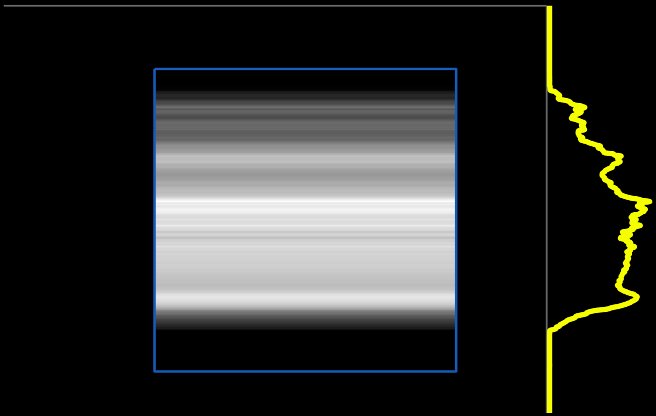
X-ray slice imaging

Are You a Natural Tomographer?

Filtered back-projection and the Radon transform

Tomography without X-rays
- Visible light
- Electrons
- Neutrons

Summing all the back-projections results in a blurred reconstruction



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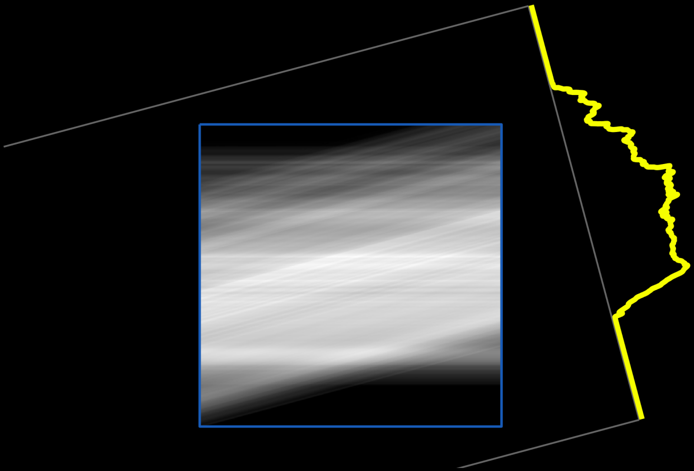

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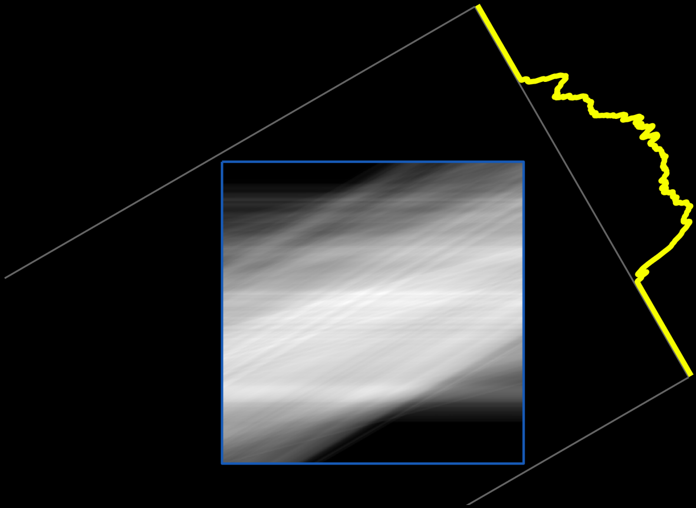

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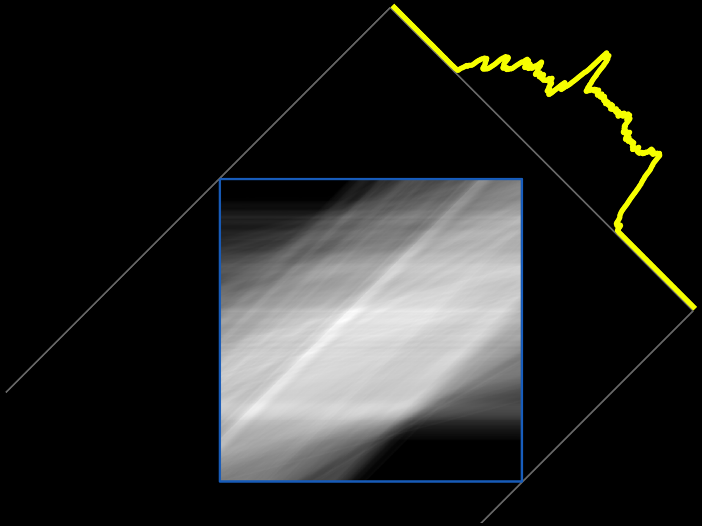

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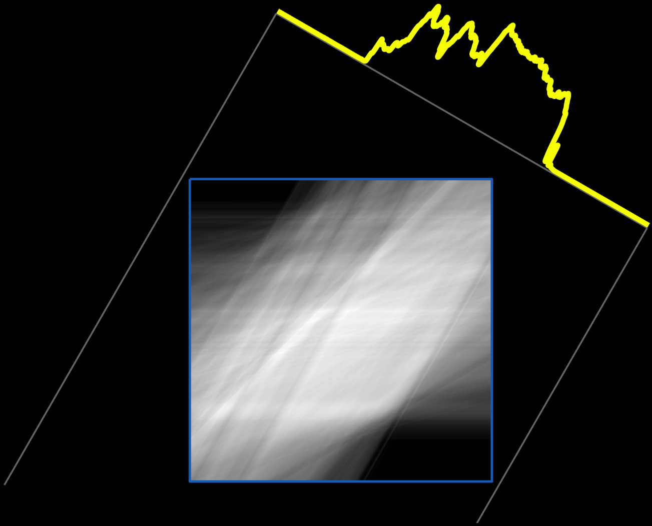

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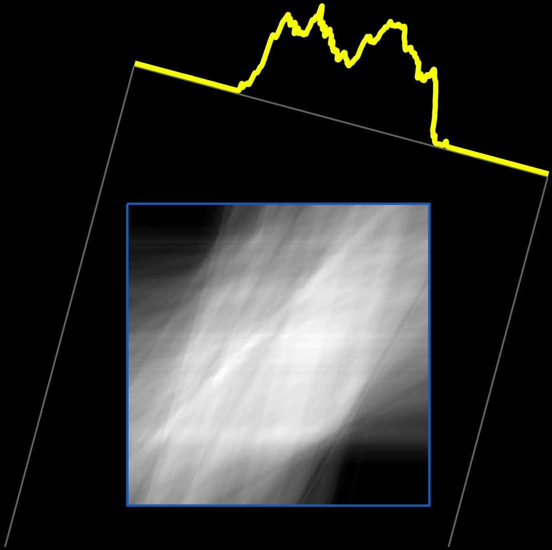

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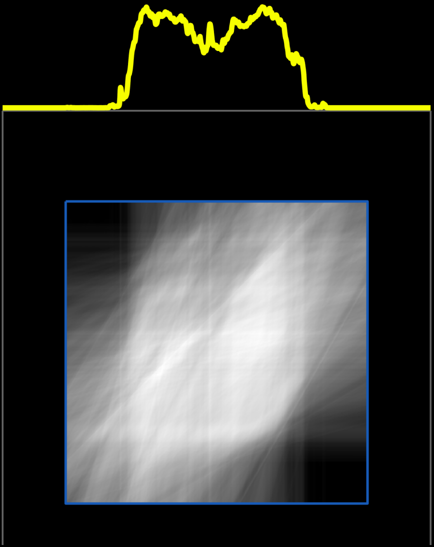

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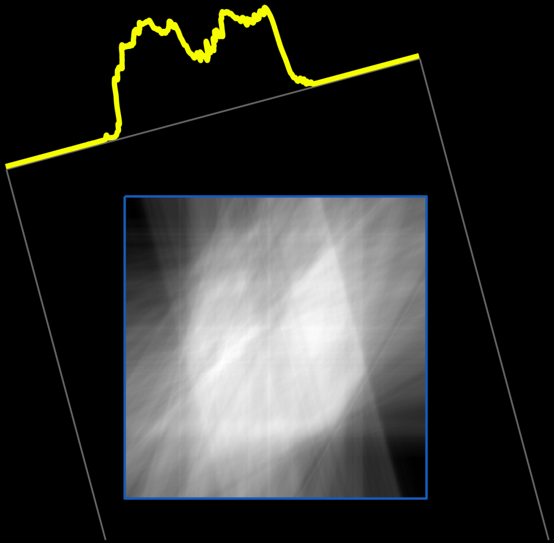

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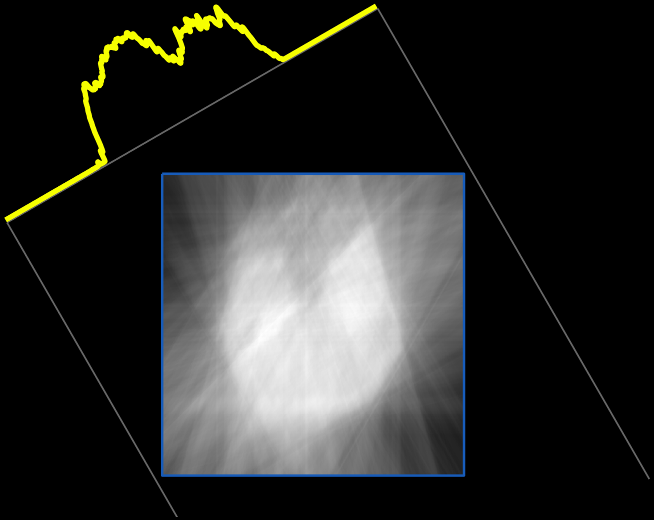

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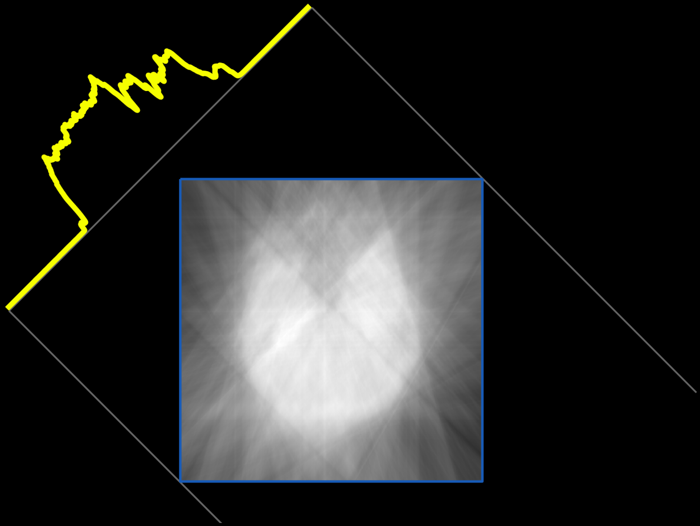

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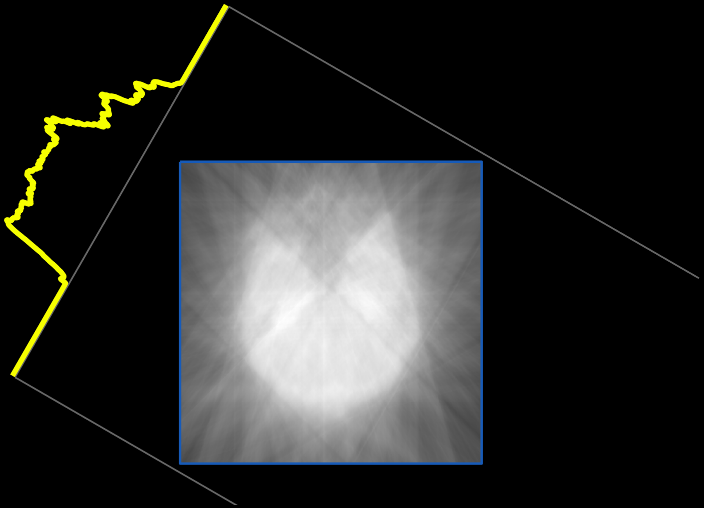

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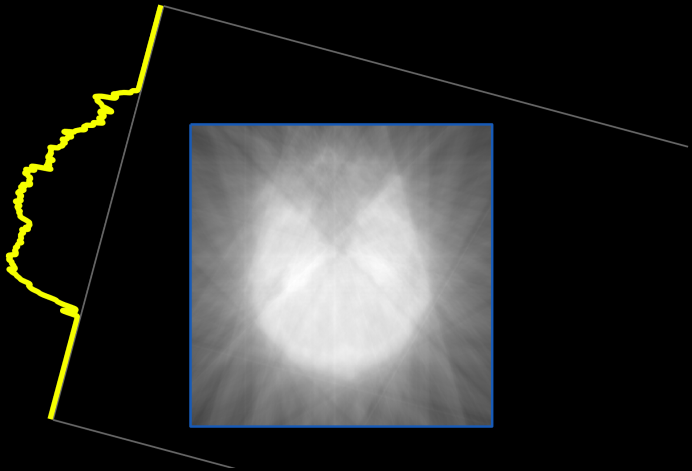
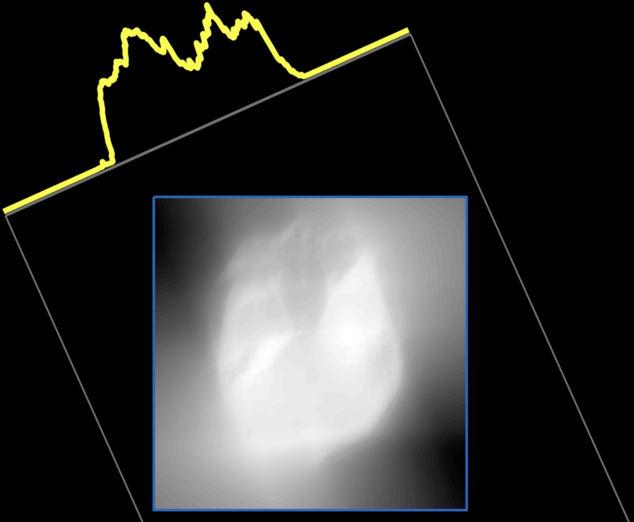
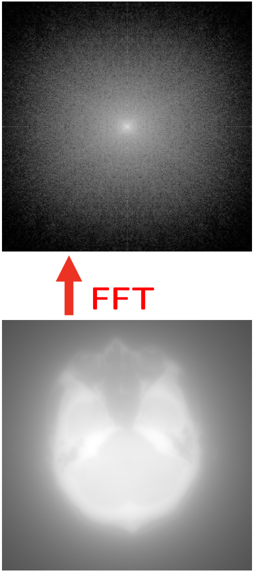


Here we use more directions, so the back-projected image quality is higher



VIDEO

Final reconstruction involves high-pass filtering on top of the back-projection



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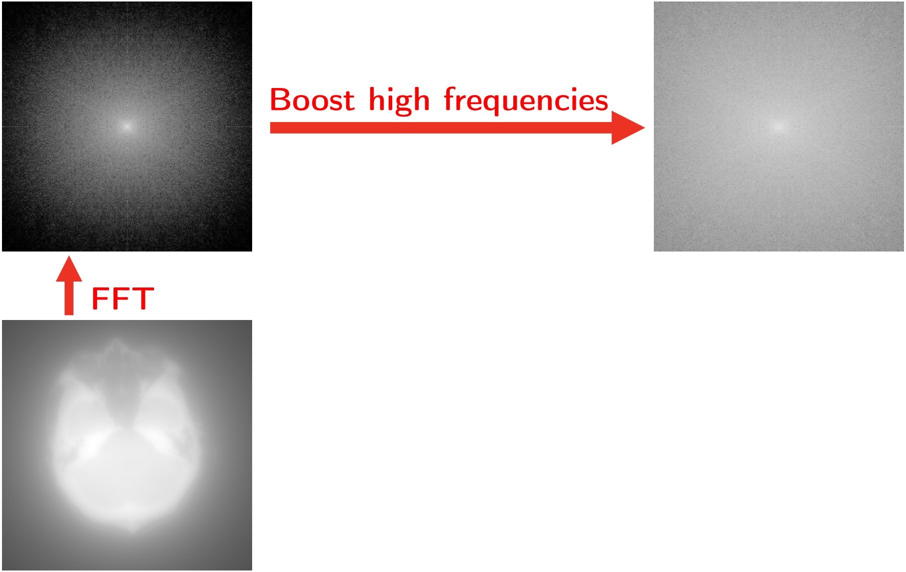

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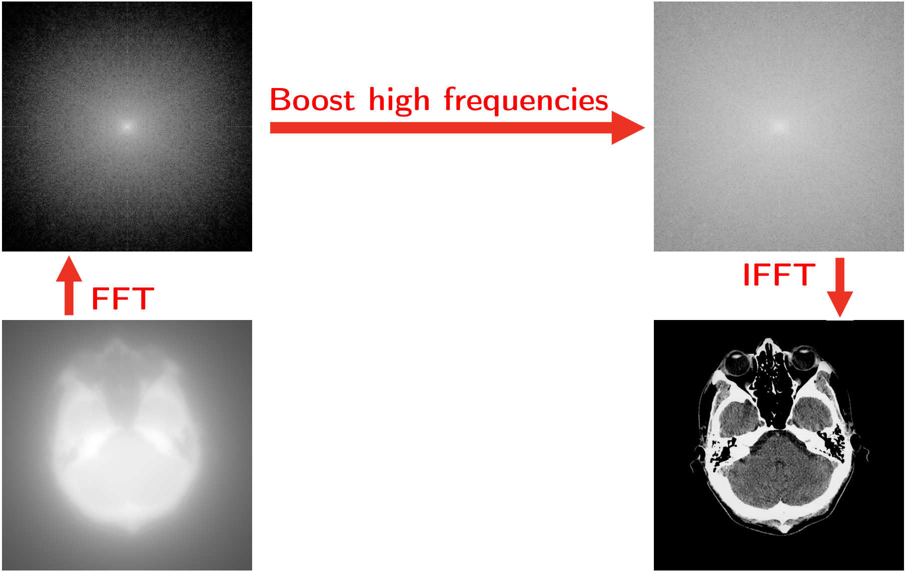

Let's observe the Fourier transform by dividing it into frequency bands

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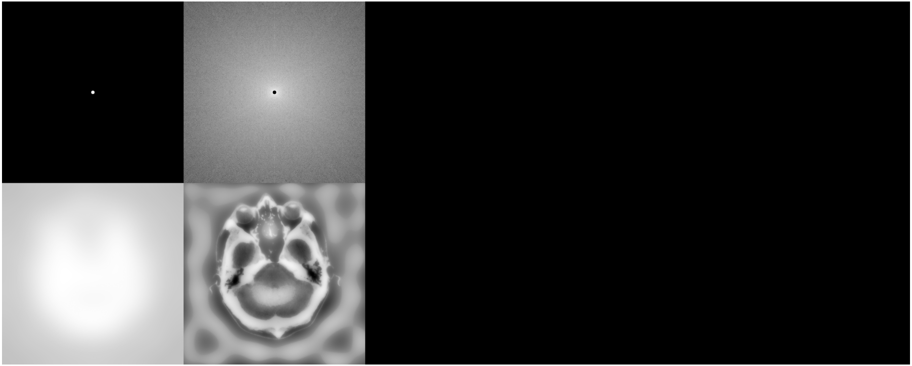

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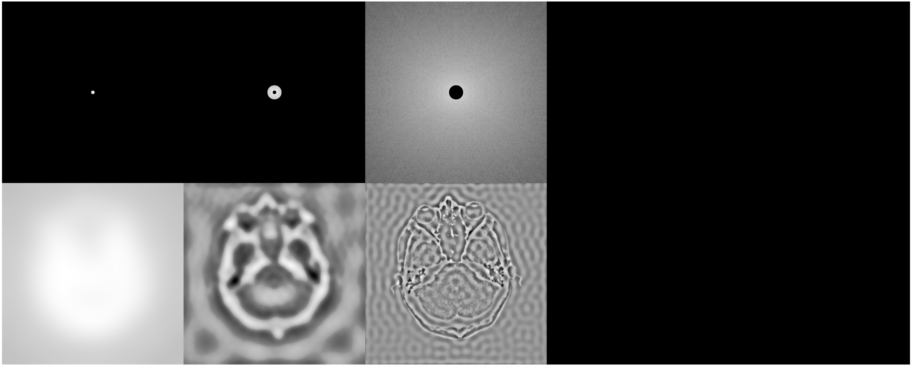

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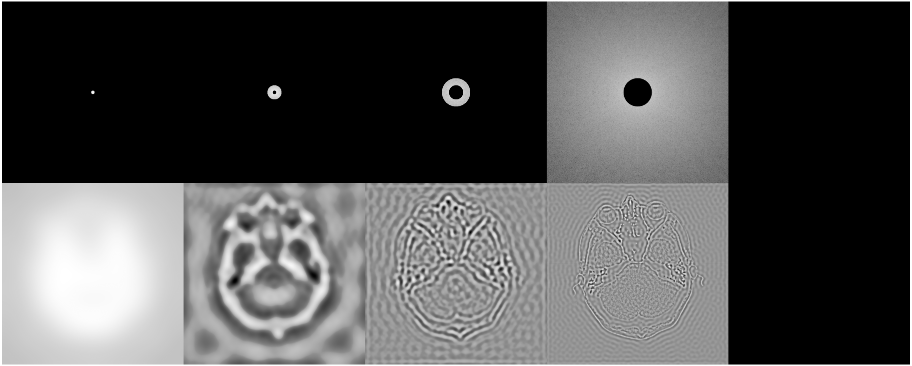

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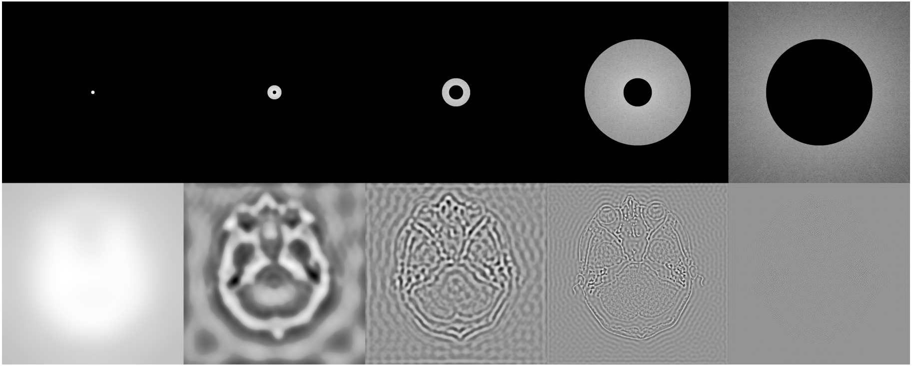

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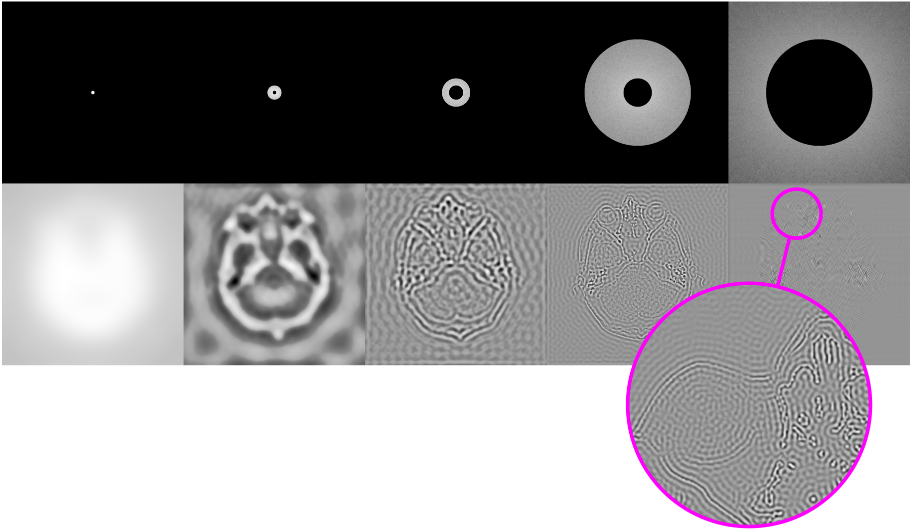
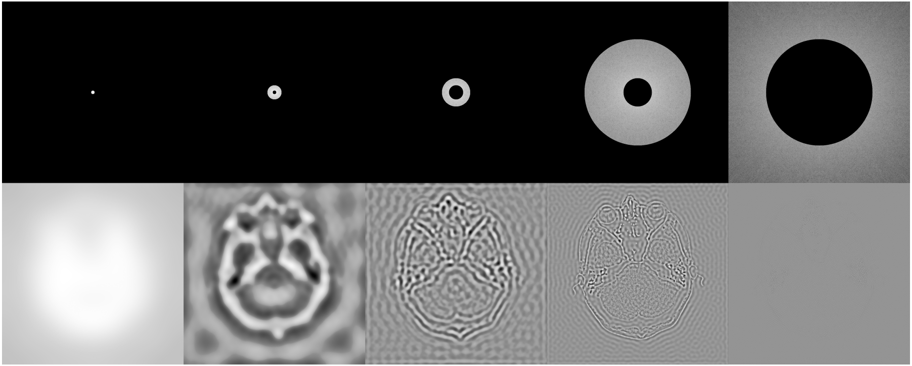
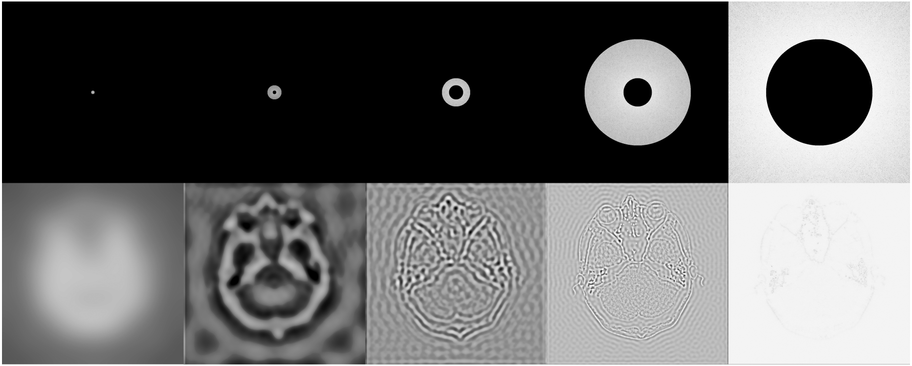


Filtering means reducing the role of some frequency bands, and boosting some of them



Reduce a lot	Reduce little	Keep as is	Boost little	Boost a lot

Filtering means reducing the role of some frequency bands, and boosting some of them



Reduce a lot Reduce little Keep as is Boost little Boost a lot

Fa So La
HELLO KITTY


Fa So La
HELLO KITTY

A  of Continuous Theory

Let us start with a crucial calculation

To reconstruct f at a point x, the most obvious data related to $f(x)$ are the integrals over lines passing through x. Let us sum them all together, call the result $Tf(x)$ and see what we get by introducing polar coordinates:

$$
\begin{aligned}
Tf(x) &= \int_0^{\pi} \int_{-\infty}^{\infty} f(x+t\theta)dtd\theta \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{f(x+t\theta)}{t} tdtd\theta \\
&= \int_{\mathbb{R}^2} \frac{f(x+y)}{|y|} dy \\
&= \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \\
&= (f(y) * \frac{1}{|y|})(x),
\end{aligned}
$$

where $*$ stands for convolution.

The Calderón operator Λ is the inverse of T

Recall that Fourier transform converts convolution to multiplication ($\widehat{g * h} = \hat{g}\hat{h}$) and that

$$\widehat{\frac{1}{|y|}}(\xi) = \frac{1}{|\xi|}.$$

Furthermore, define the Calderón operator Λ by

$$\Lambda f(x) := \mathcal{F}^{-1}|\xi|\hat{f}(\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}|\xi|\hat{f}(\xi)d\xi,$$

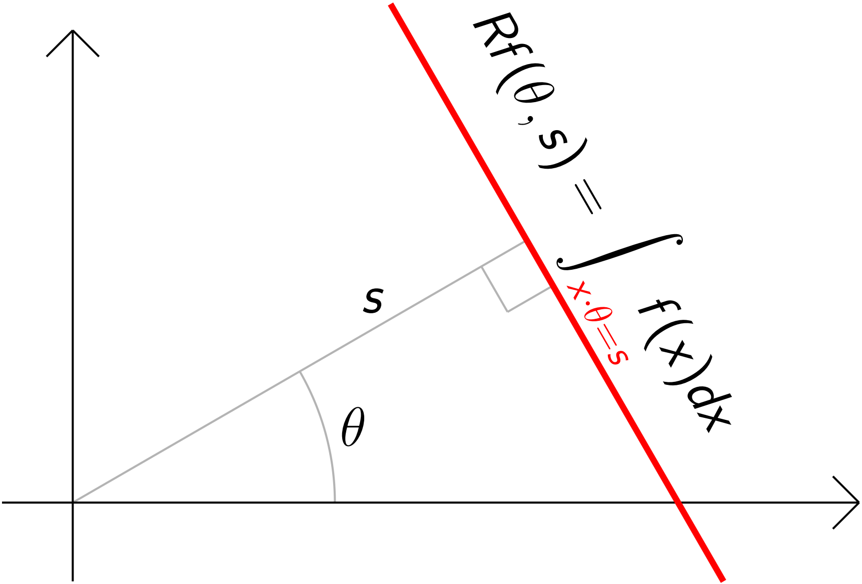
where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Note that Λ can be thought of as a high-pass filter. Now we see that

$$\widehat{Tf}(\xi) = \frac{\hat{f}(\xi)}{|\xi|},$$

and thus

$$\Lambda Tf = f.$$

Radon transform organizes sets of line integrals



Definition of the Radon transform

Let $f(x) = f(x_1, x_2)$ be the X-ray attenuation coefficient. The classical model for tomographic data is the Radon transform

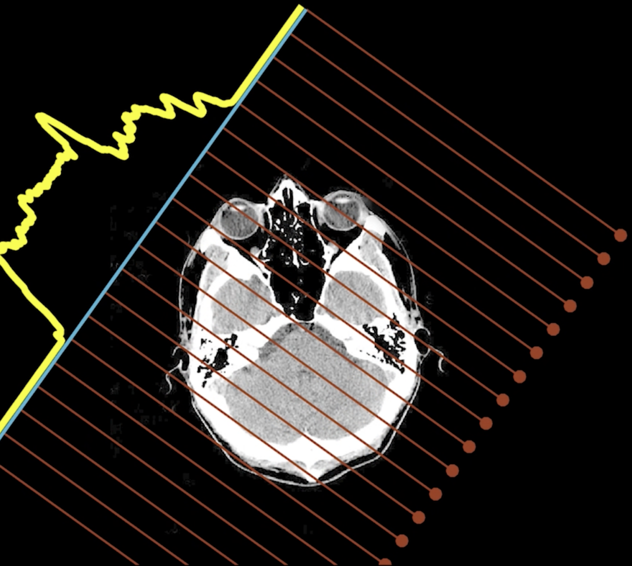
$$Rf(\theta, s) = \int_{x\cdot\vec{\theta}=s} f(x)dx = \int_{-\infty}^{\infty} f(s\vec{\theta}+\tau\vec{\theta}^{\perp})d\tau, \qquad \vec{\theta} \in S^1, s \in \mathbb{R},$$

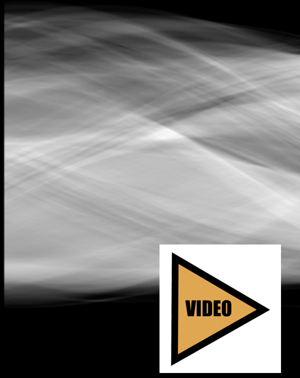
where S^1 is the unit circle, $\vec{\theta}^{\perp}$ is a unit vector perpendicular to the unit vector $\vec{\theta} = (\cos\theta, \sin\theta)$, and $x \cdot \vec{\theta}$ denotes vector inner product.

Note that f is defined on $\mathbb{R}^2$ and Rf is defined on $S^1 \times \mathbb{R}^1$.

Also: $Rf(\theta, s) = Rf(\theta + \pi, -s)$, or in vector notation $Rf(\vec{\theta}, s) = Rf(-\vec{\theta}, -s)$.

Radon transform as sinogram





Adjoint of the Radon transform R: the back-projection operator R^*

Use change of variables $x = s\vec{\theta} + \tau\vec{\theta}^{\perp}$; then $dx = ds d\tau$. Calculate

$$
\begin{aligned}
& \int_0^{2\pi}\int_{-\infty}^{\infty} Rf(\theta,s)\, g(\theta,s)\, ds d\theta \\
= & \int_0^{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(s\vec{\theta}+\tau\vec{\theta}^{\perp}) d\tau\Big) g(\theta,s)\, ds d\theta \\
= & \int_0^{2\pi}\left(\int_{\mathbb{R}^2} f(x)\, g(\theta, x\cdot\vec{\theta})\, dx\right) d\theta \\
= & \int_{\mathbb{R}^2} f(x)\left(\int_0^{2\pi} g(\theta, x\cdot\vec{\theta}) d\theta\right) dx \\
=: & \int_{\mathbb{R}^2} f(x)\, R^*g(x)\, dx
\end{aligned}
$$

Both f and R^*g are defined on $\mathbb{R}^2$. Note that $Tf = \frac{1}{2}R^*Rf$.

We arrive at the modern way of writing the filtered back-projection reconstruction formula

Johann Radon (1887-1956)

$$f = \frac{1}{2}\Lambda R^* R f$$

Radon's 1917 formulation:

$$f(P) = -\frac{1}{\pi}\int_0^\infty \frac{d\overline{F_p}(q)}{q}$$

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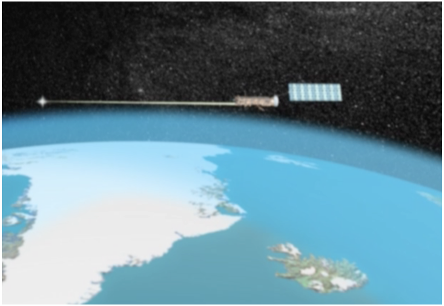
Filtered back-projection and the Radon transform

Tomography without X-rays

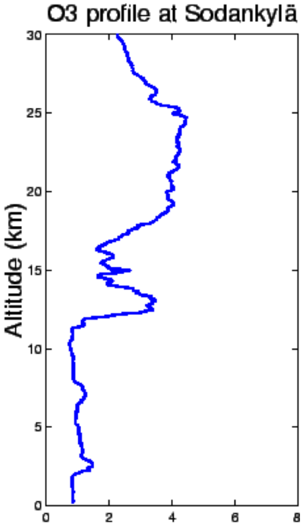
- Visible light
- Electrons
- Neutrons

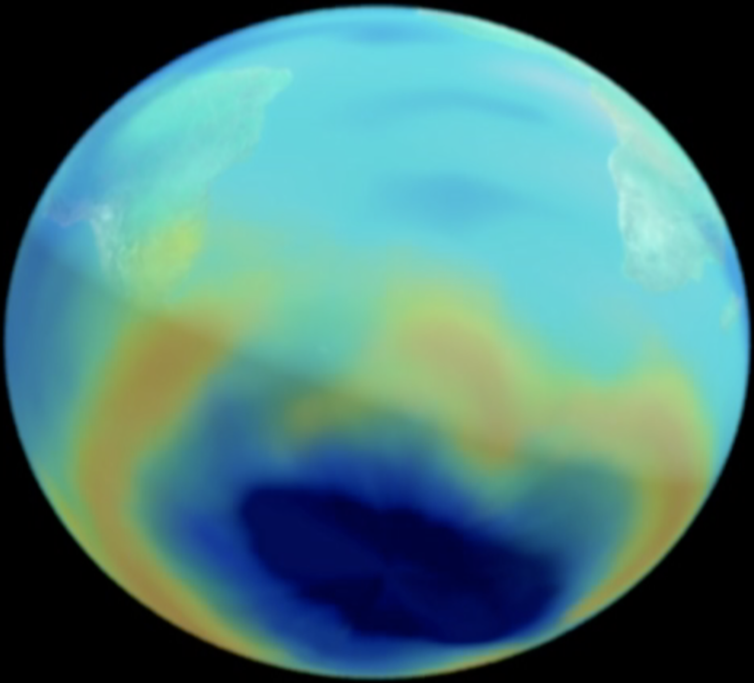
Outline

The mathematics of X-ray tomography can be used for recovering the ozone layer



European Space Agency
Finnish Meteorological Institute
Envisat and GOMOS projects
Thanks to **Johanna Tamminen!**





Outline

What is an X-ray image?

X-ray slice imaging

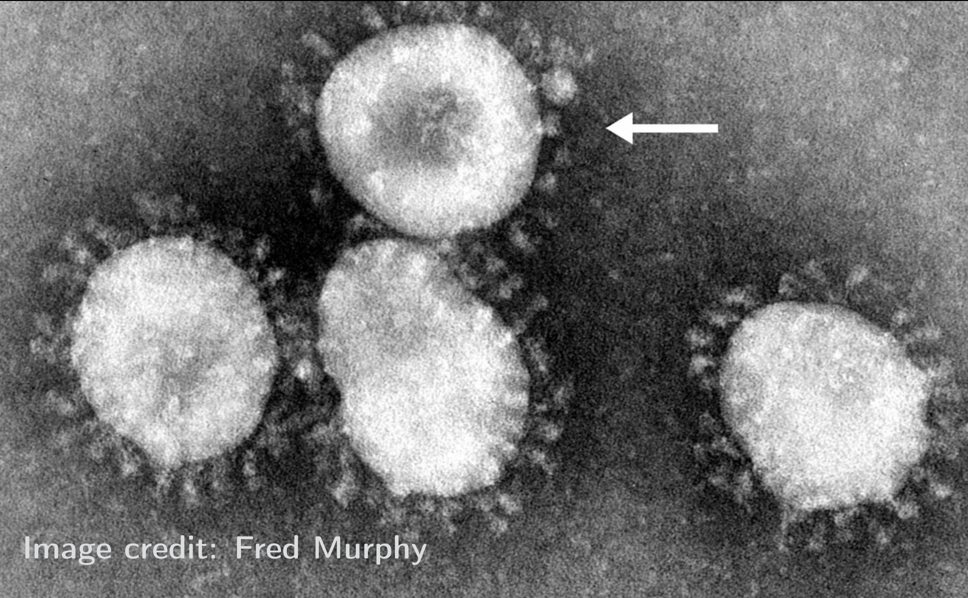
Are You a Natural Tomographer?

Filtered back-projection and the Radon transform

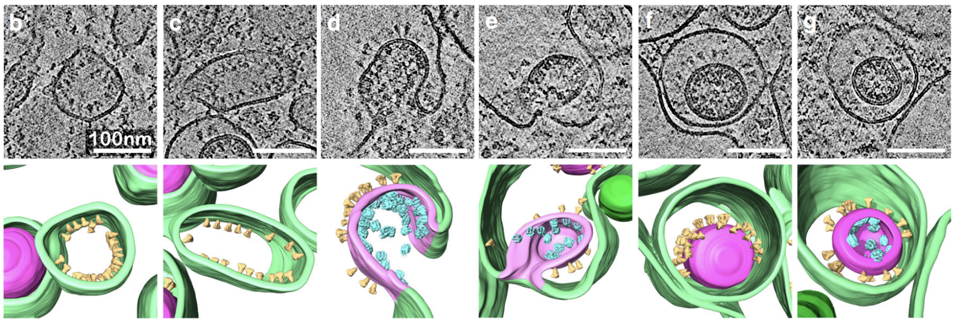
Tomography without X-rays

- Visible light
- Electrons
- Neutrons

Electron microscopy allows us to see things smaller than the wavelength of light, such as viruses

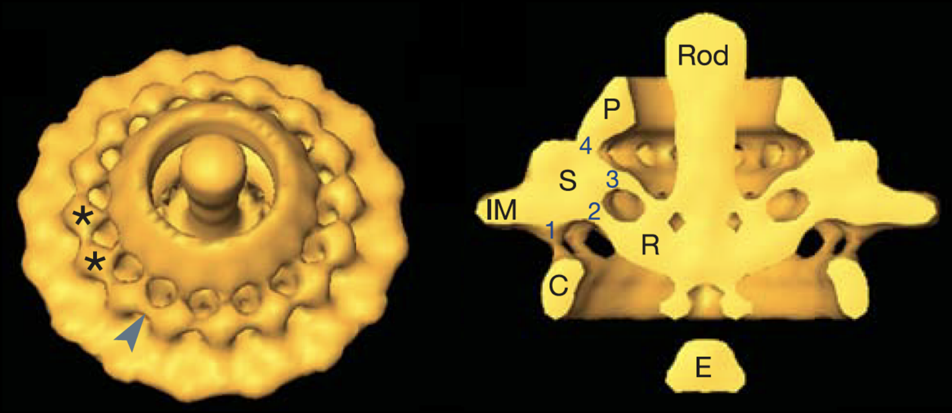


We can observe a cell manufacturing corona viruses



[Klein, Cortese, Winter, Wachsmuth-Melm, Neufeldt, Cerikan, Stanifer, Boulant, Bartenschlager & Chlanda 2021]

Electron transmission cryotomography reveals the swimming engine of *Treponema primitia* bacteria



[Murphy, Leadbetter & Jensen 2016]

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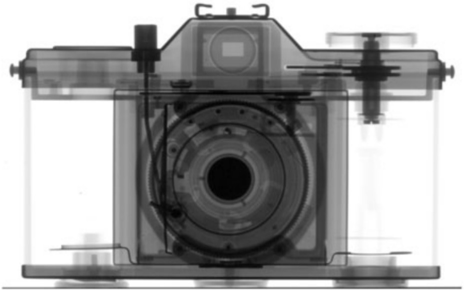
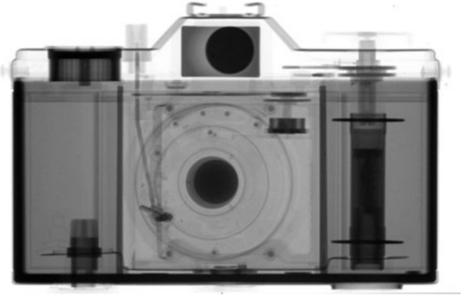
Neutron beams deliver projection images different from X-ray radiographs



X-ray image

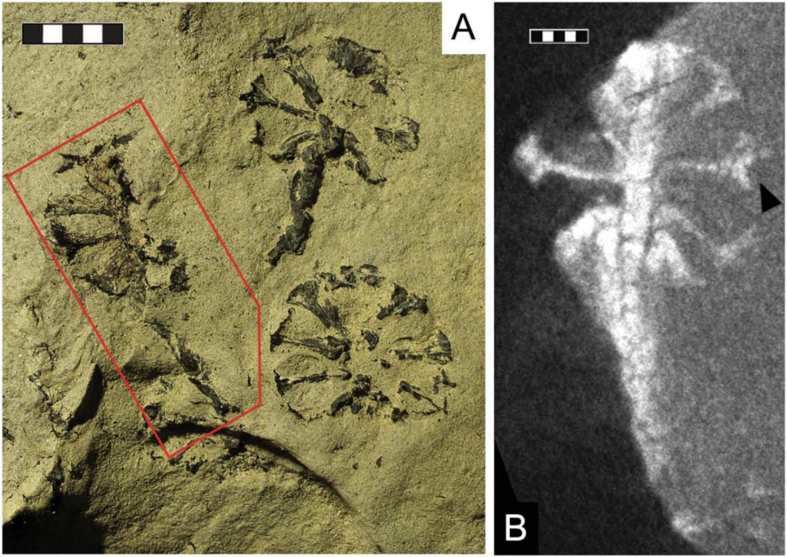


Neutron image

Imaging with neutrons opens up new possibilities as water attenuates but metal is transparent

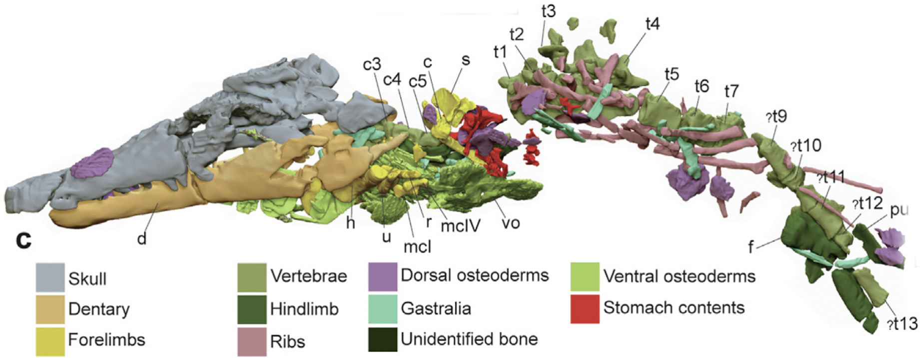
Video:
Anders Kaestner
Neutron Imaging and Activation Group,
Paul Scherrer Institute

Neutron tomography for fossilized cones



Mays, Cantrill, Stilwell & Bevitt 2018

Neutron tomography revealed the lunch of a fossil crocodile *Confractosuchus sauroktonos*

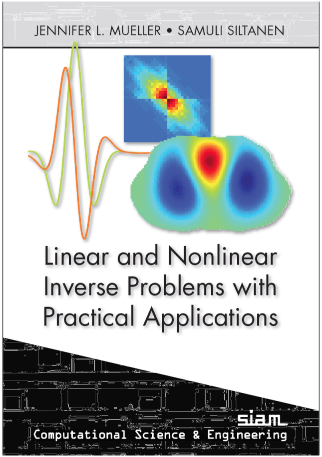


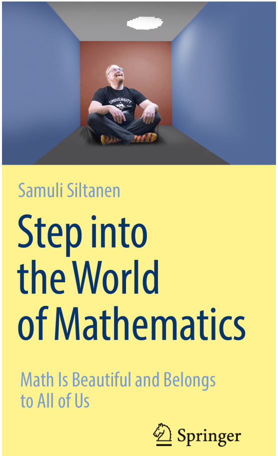
White, Bell, Campione, Sansalone, Brougham, Bevitt, Molnar, Cook, Wroe & Elliott 2022

Bon appetit, *Confractosuchus sauroktonos* **!**

Image credit: Dr. Matt White / Australian age of dinosaurs

In case you liked this material, here are a couple of pointers to further cool stuff





https://blog.fips.fi/

More books recommended for learning the mathematics of practical X-ray tomography

2008 Buzug: Computed Tomography: From Photon Statistics to Modern Cone-Beam CT

2008 Epstein: Introduction to the mathematics of medical imaging

2010 Hansen: Discrete inverse problems

2012 Mueller & S: Linear and Nonlinear Inverse Problems with Practical Applications

2014 Kuchment: The Radon Transform and Medical Imaging

2021 Hansen, Jørgensen, Lionheart (eds.): Computed Tomography: Algorithms, Insight, and Just Enough Theory


Thank you for your attention!
⟵Slime mold called Lycogala conicum