

X-ray tomography and filtered back-projection

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PhD Winter School 2023

Advanced methods for mathematical image analysis

Bologna, Italy

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Instagram:
@samuntiede
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@Samuntiedekanava

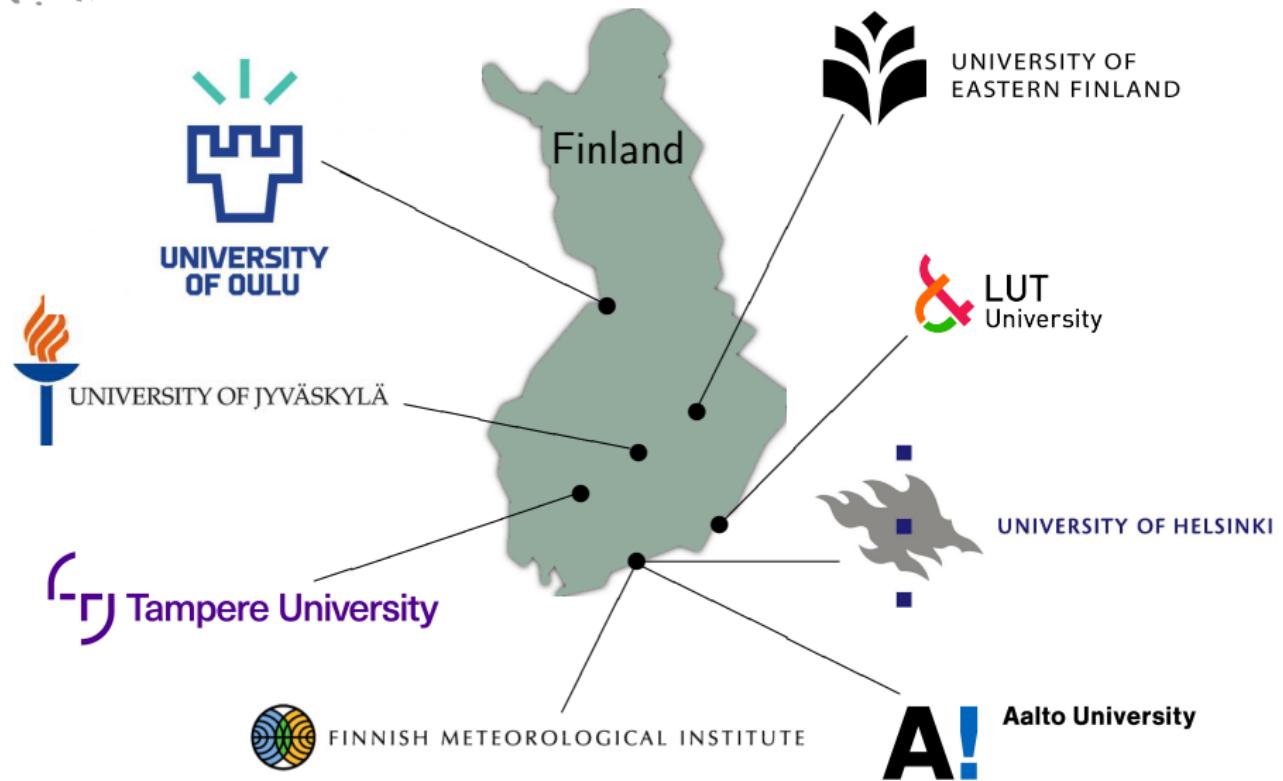


Finnish Centre of Excellence in Inverse Modelling and Imaging

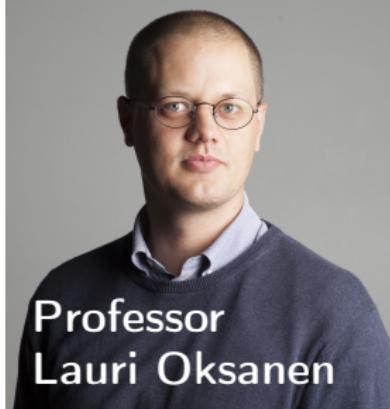
2018-2025



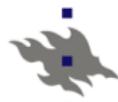
CENTRES OF EXCELLENCE
IN RESEARCH



We have a research group of roughly 30 people
at University of Helsinki



This my industrial-academic background



2009: Professor, University of Helsinki, Finland



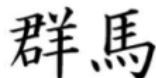
2006: Professor, Tampere University of Technology, Finland



2005: R&D scientist at Palodex Group



2004: R&D scientist at GE Healthcare



2002: Postdoc at Gunma University, Japan



2000: R&D scientist at Instrumentarium Imaging



1999: PhD, Helsinki University of Technology, Finland

You can find all the slides and codes of this course in GitHub

<https://github.com/ssiltane/BolognaWinterSchool2023>



Outline

What is an X-ray image?

X-ray slice imaging

Are You a Natural Tomographer?

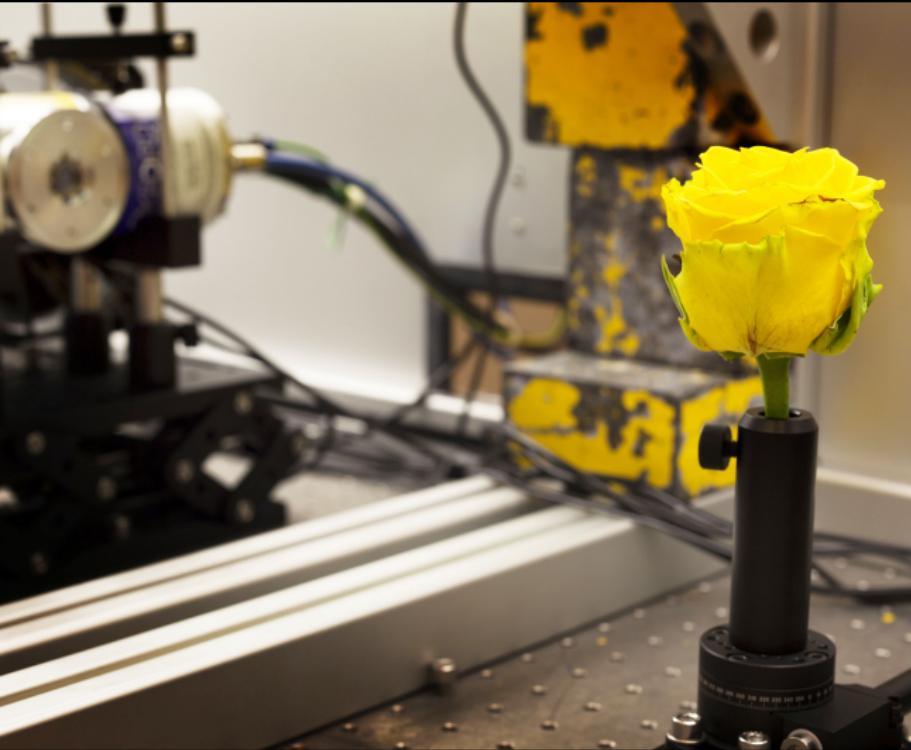
Filtered back-projection and the Radon transform

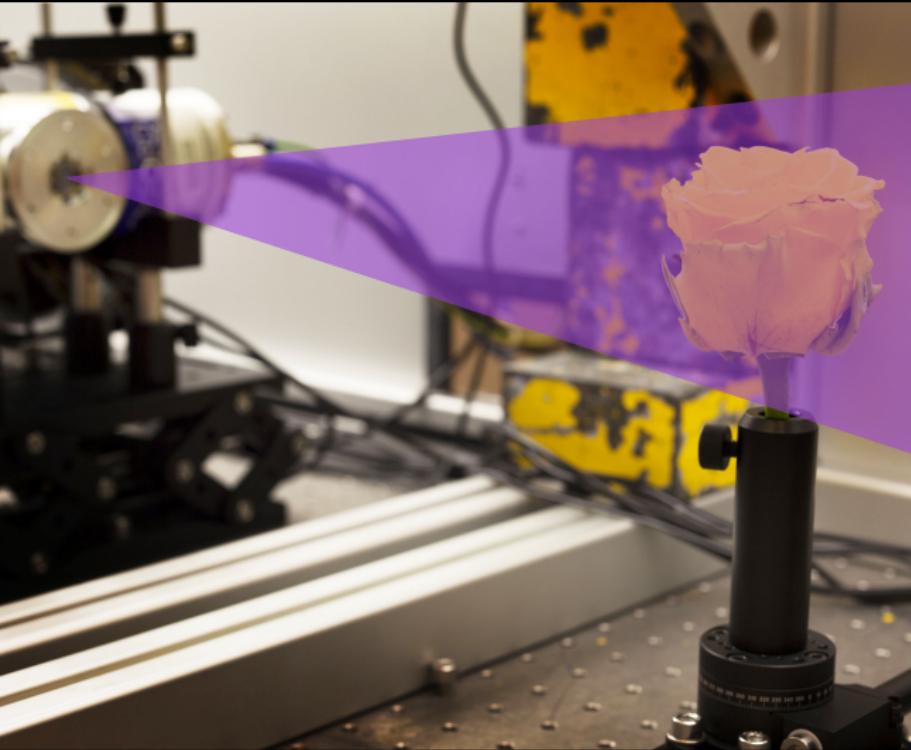
Tomography without X-rays

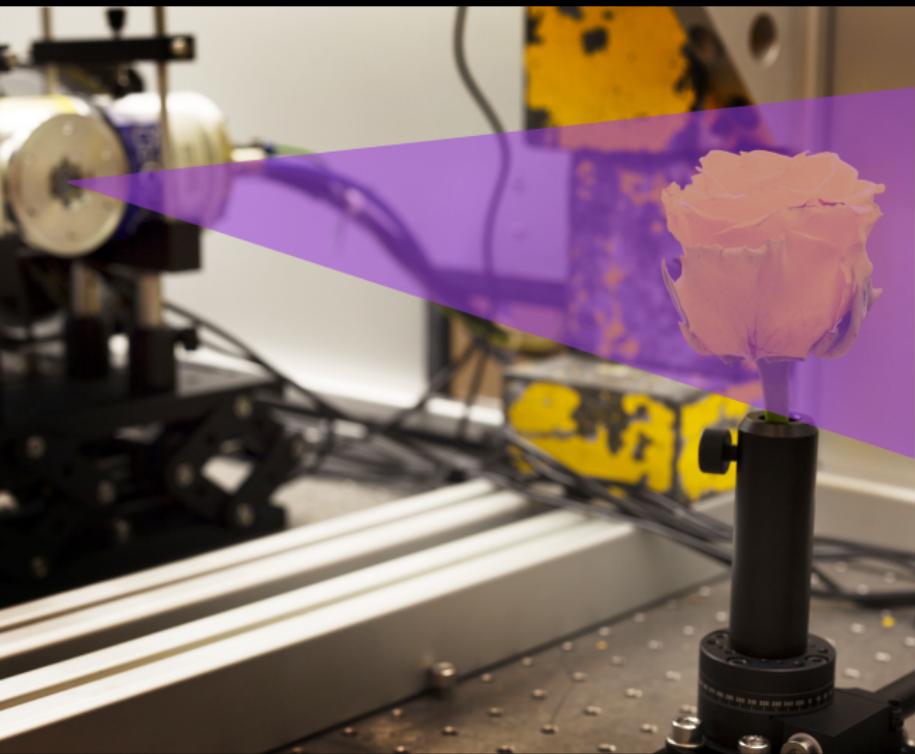
Visible light

Electrons

Neutrons









**X-ray images are very useful for doctors.
For example, they can see fractures.**



Nevit Dilmen,
Wikimedia
commons

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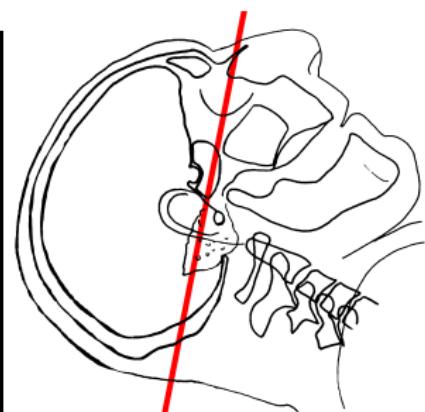
Neutrons

Here is a 2D slice through a human head



Andrew Cisel,
Wikimedia
commons

Here is a 2D slice through a human head

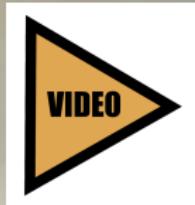


Andrew Cisel,
Wikimedia
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Modern CT scanners look like this



Modern scanners rotate at high speed



Couch unit for EMI brain scanner.

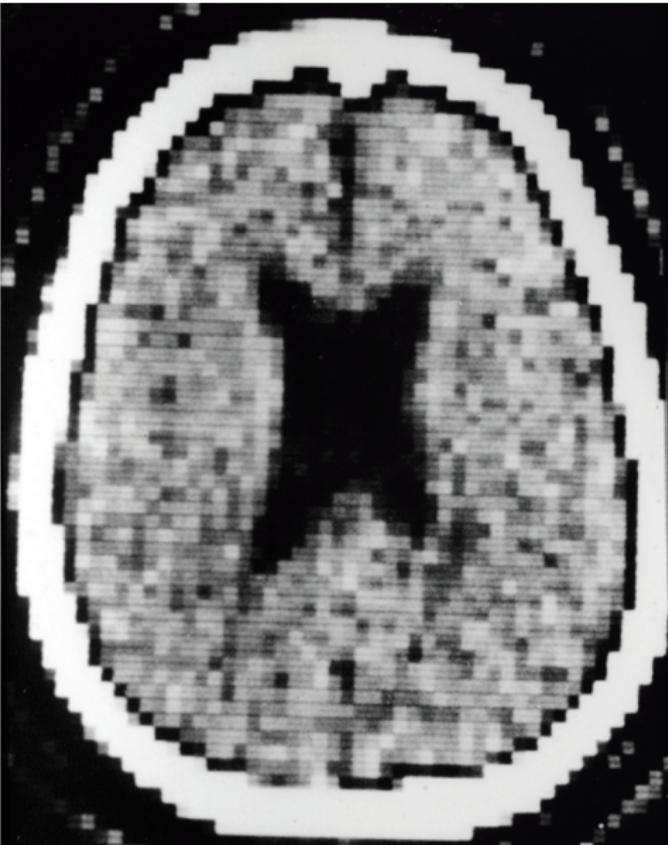


Image: Science Museum Group.

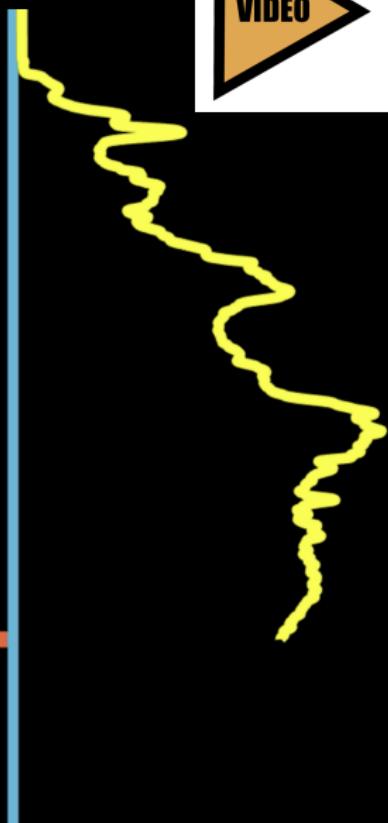
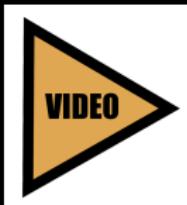
Godfrey Hounsfield and Allan McLeod Cormack developed X-ray tomography

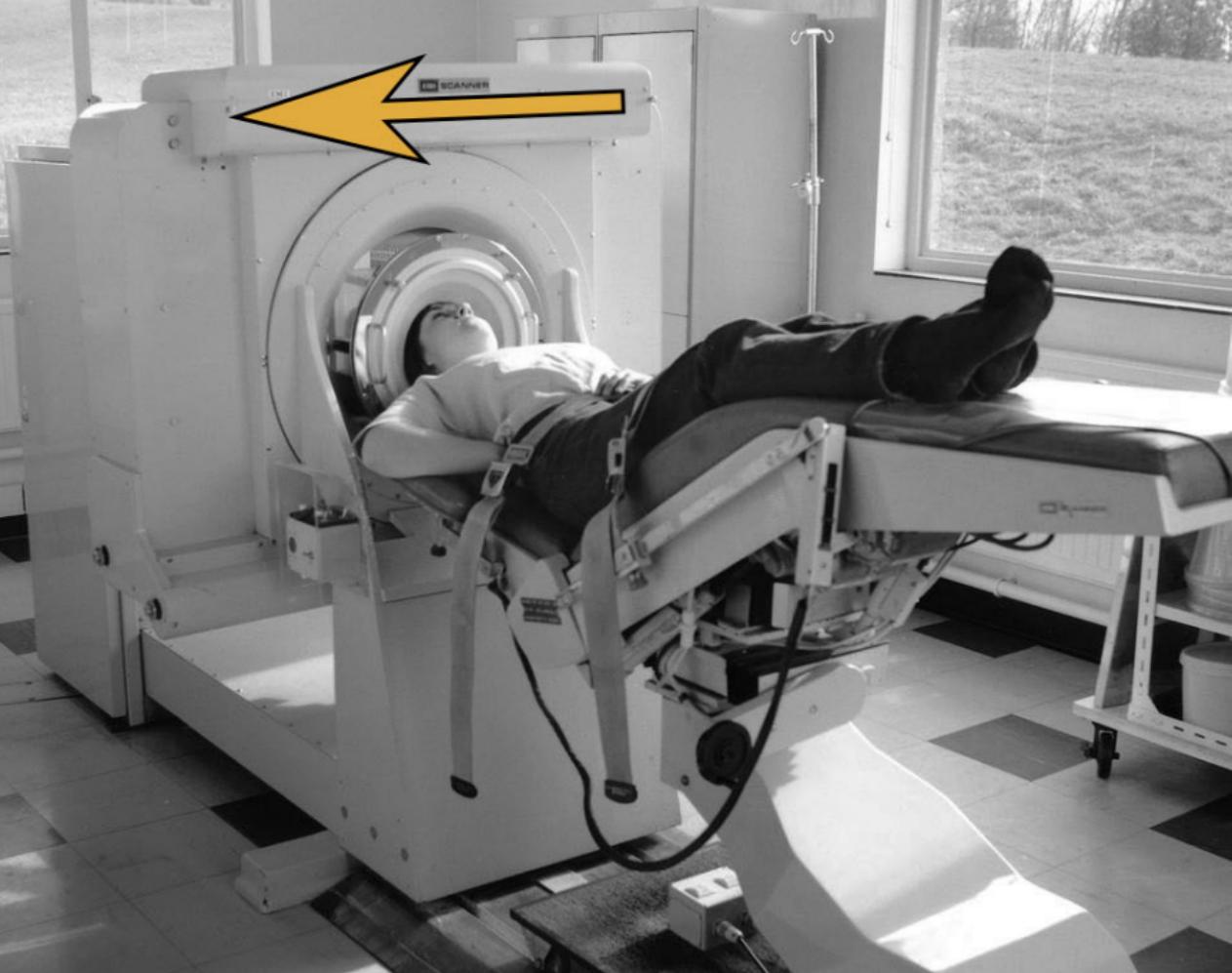


Hounsfield (top) and Cormack received Nobel prizes in 1979.



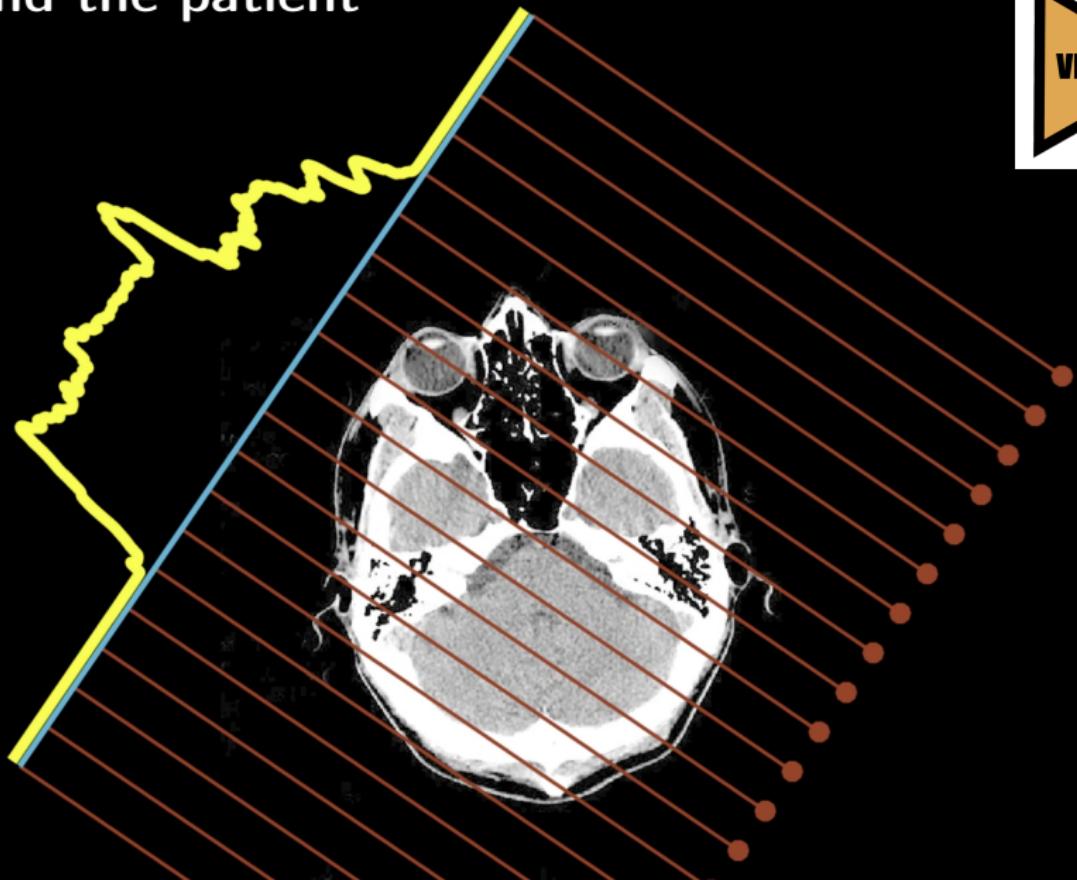
This sweeping movement is the data collection mode of first-generation CT scanners







Data is collected by rotating the system around the patient



This is the inverse problem of tomography:

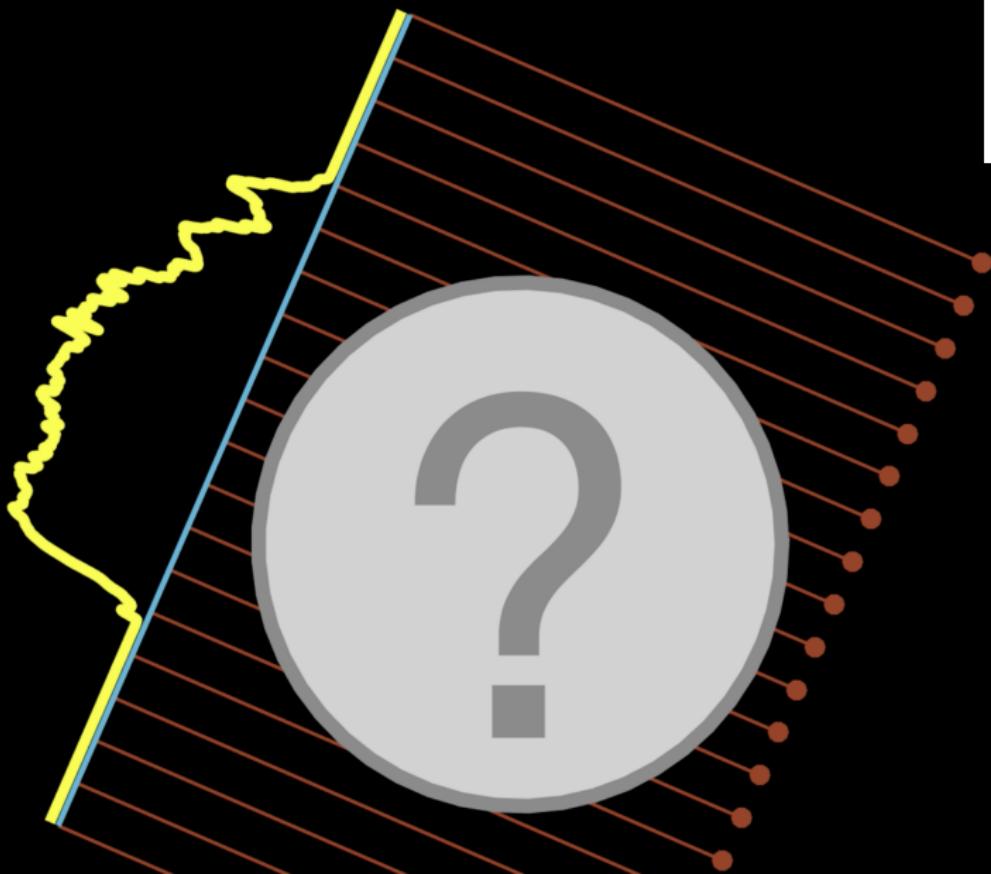
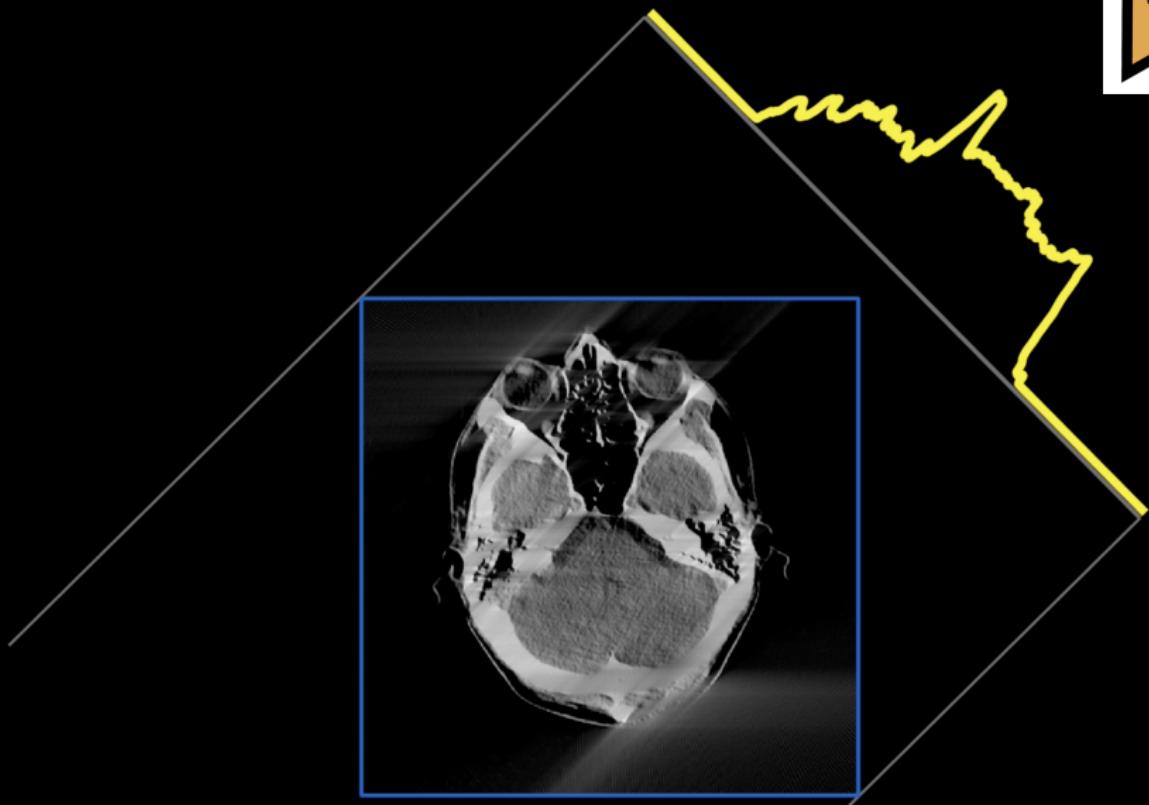
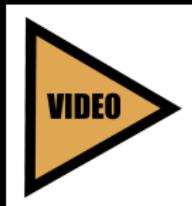
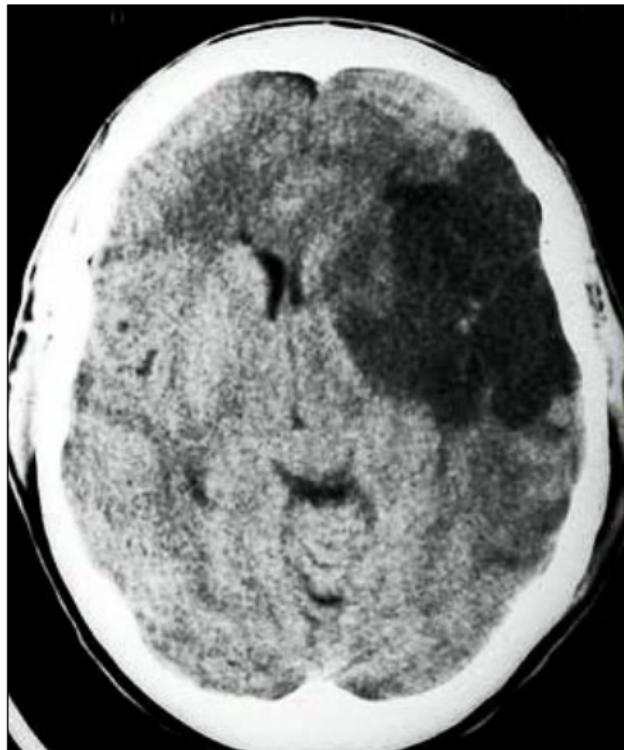


Illustration of the standard reconstruction by filtered back-projection (FBP)



Diagnosing stroke with X-ray tomography

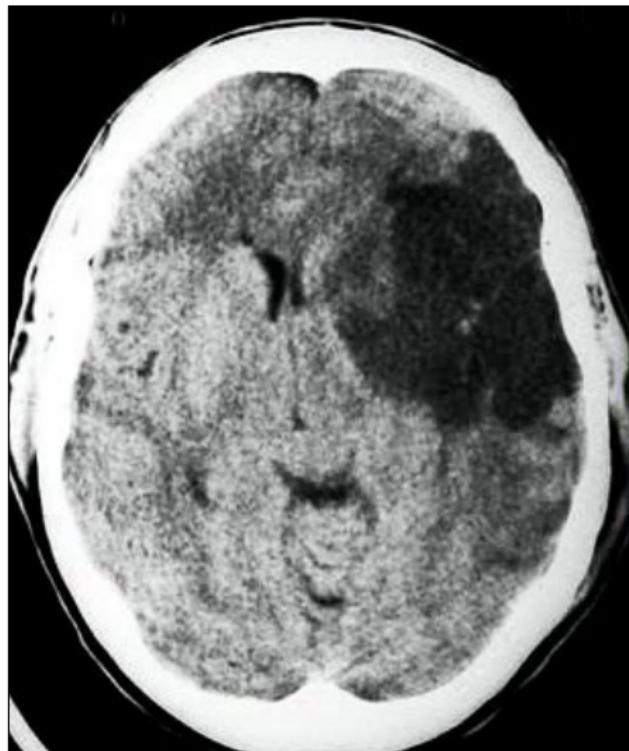
Ischemic stroke



CT image from Jansen 2008

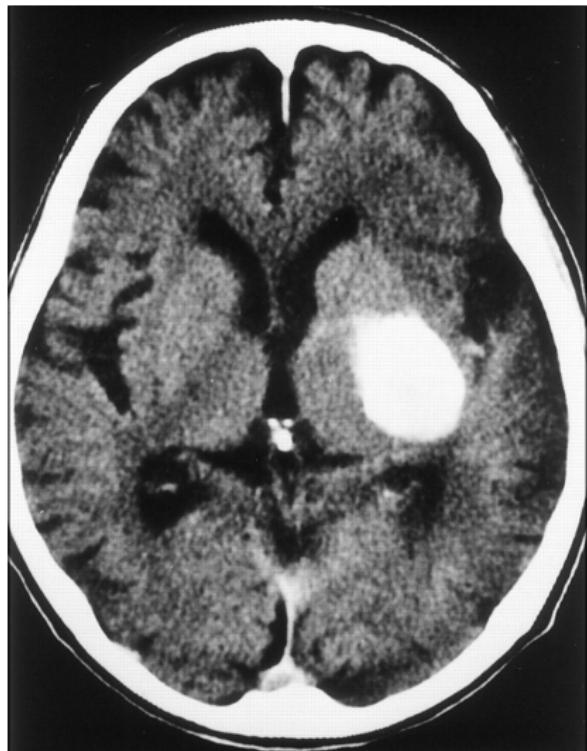
Diagnosing stroke with X-ray tomography

Ischemic stroke



CT image from Jansen 2008

Hemorrhagic stroke



CT image from Nakano et al. 2001

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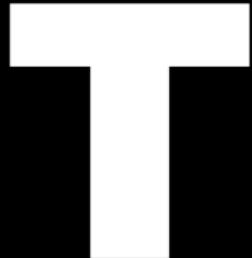
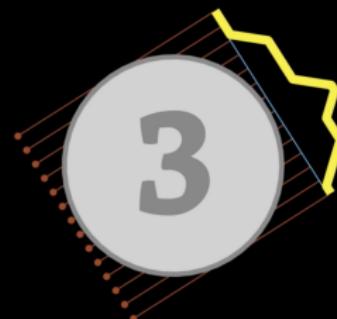
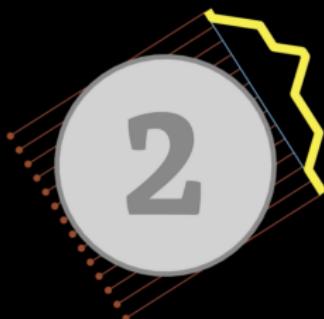
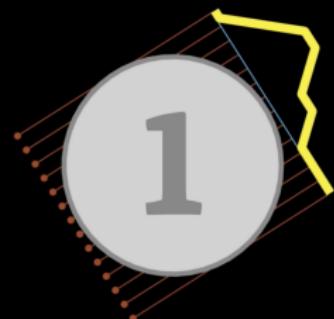
Tomography without X-rays

Visible light

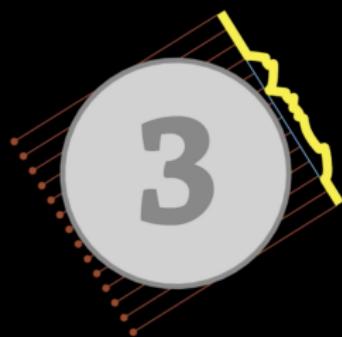
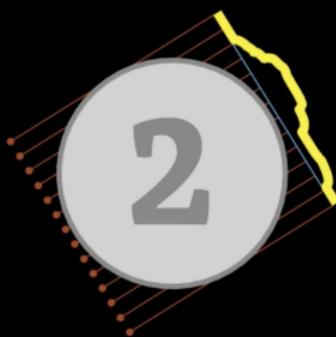
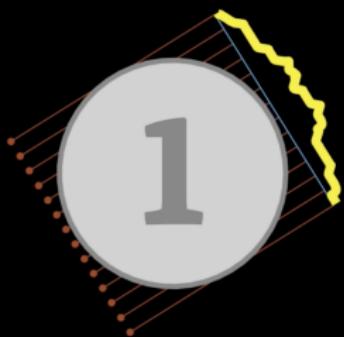
Electrons

Neutrons

Level 1 tomographic mystery

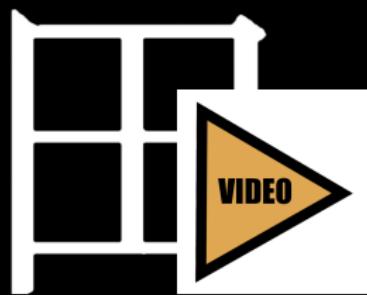


Level 2 tomographic mystery

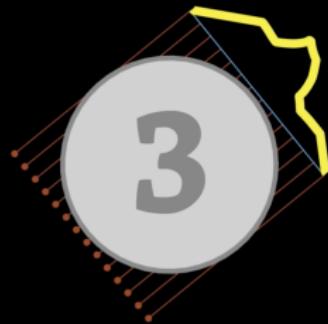
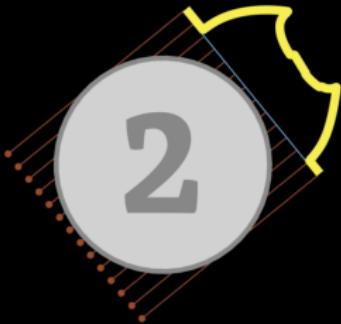
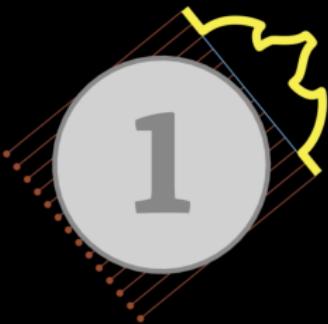


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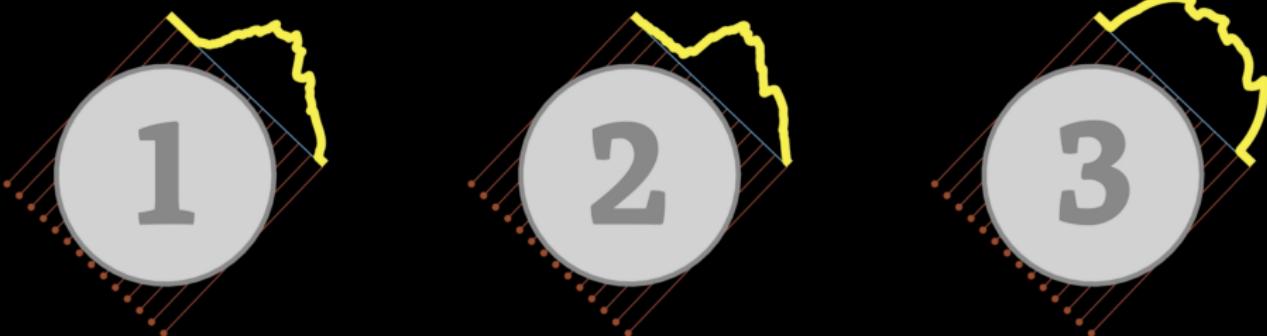


Level 3 tomographic mystery

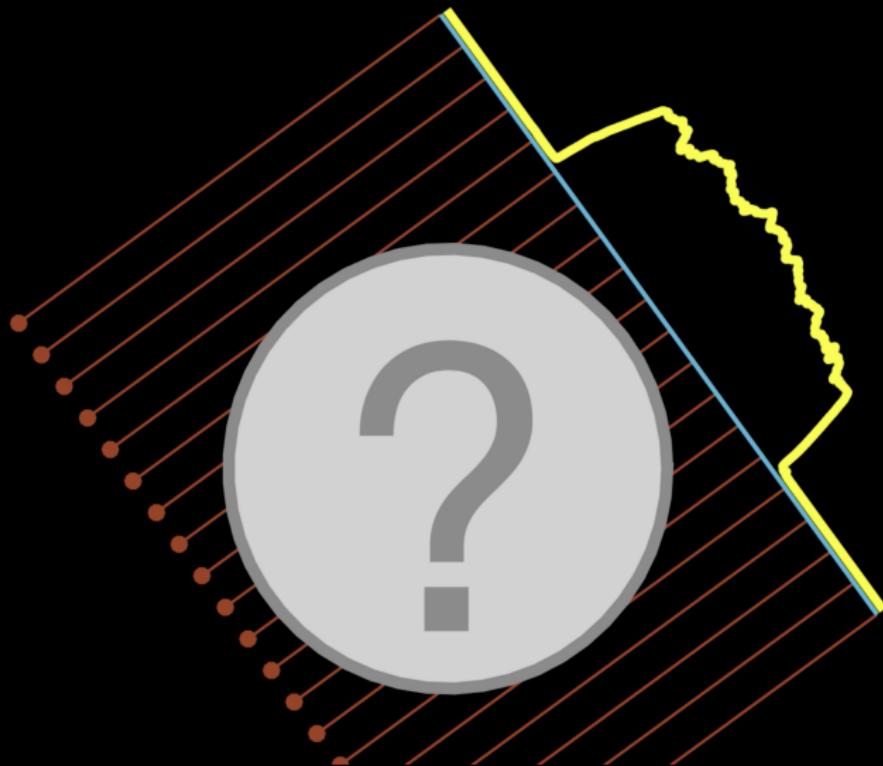


G R 8 

Level 4 tomographic mystery

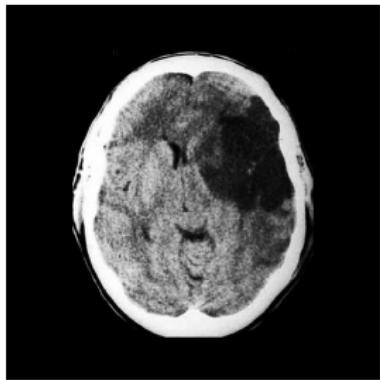


Level 5: can you guess the image?

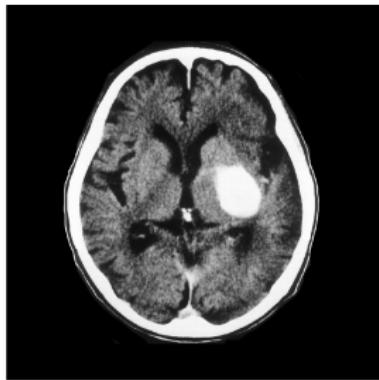


Alternatives

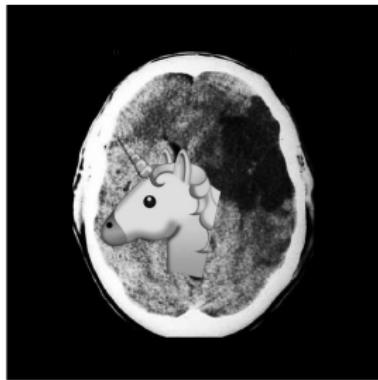
(a)



(b)

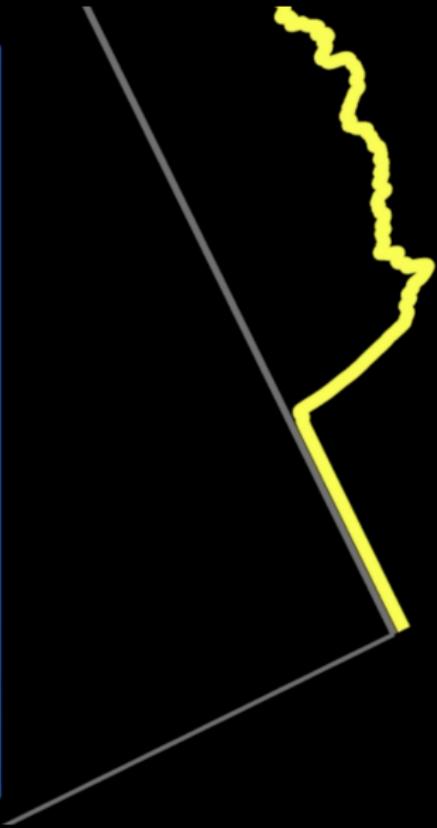
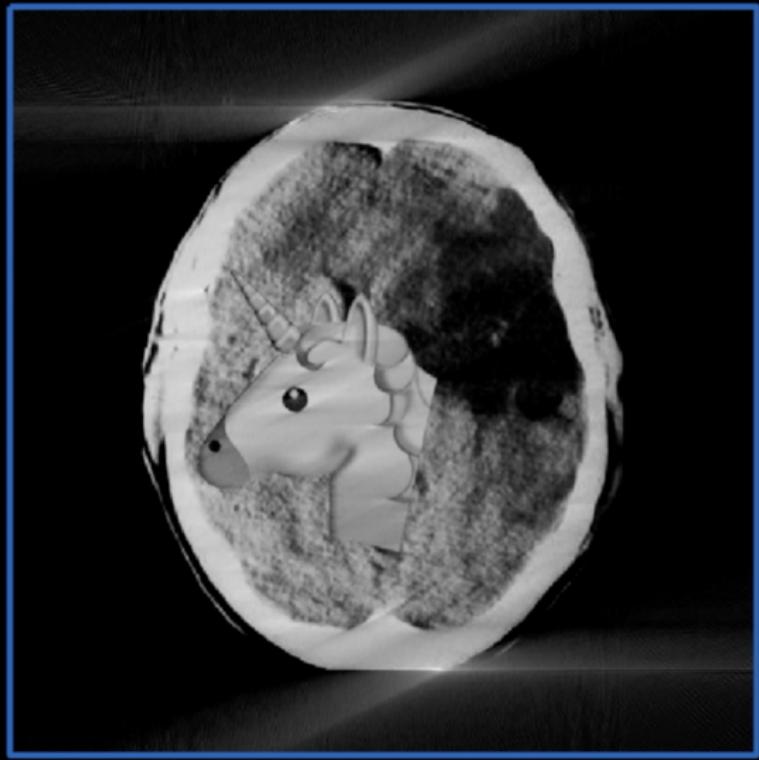


(c)



Solution





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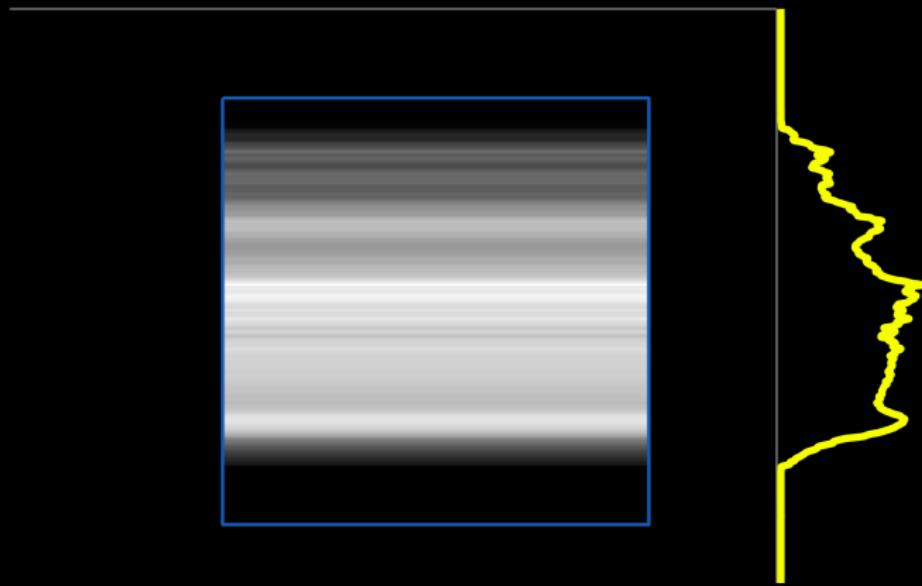
Tomography without X-rays

Visible light

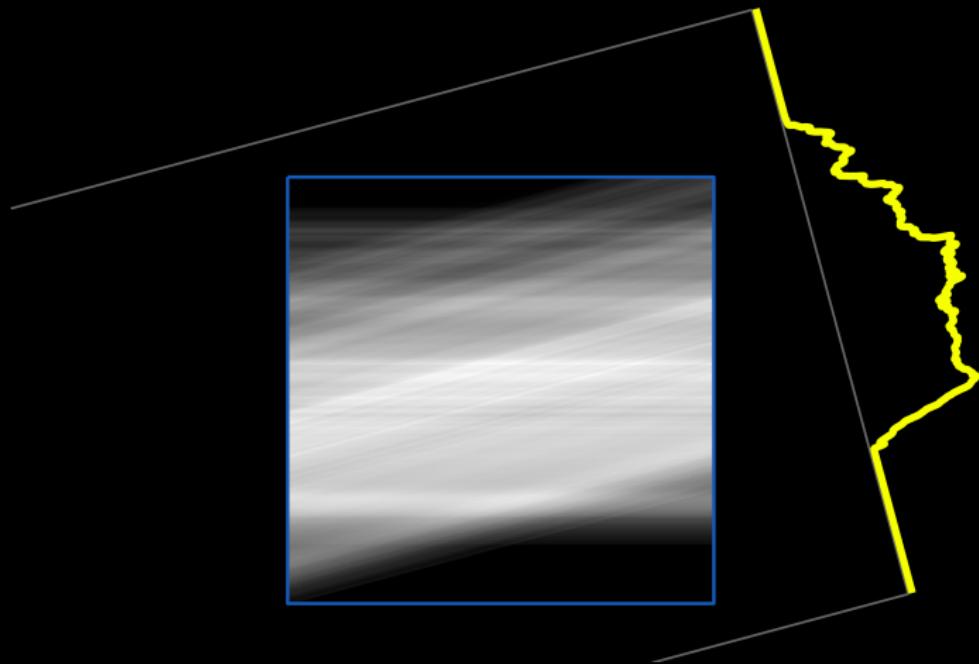
Electrons

Neutrons

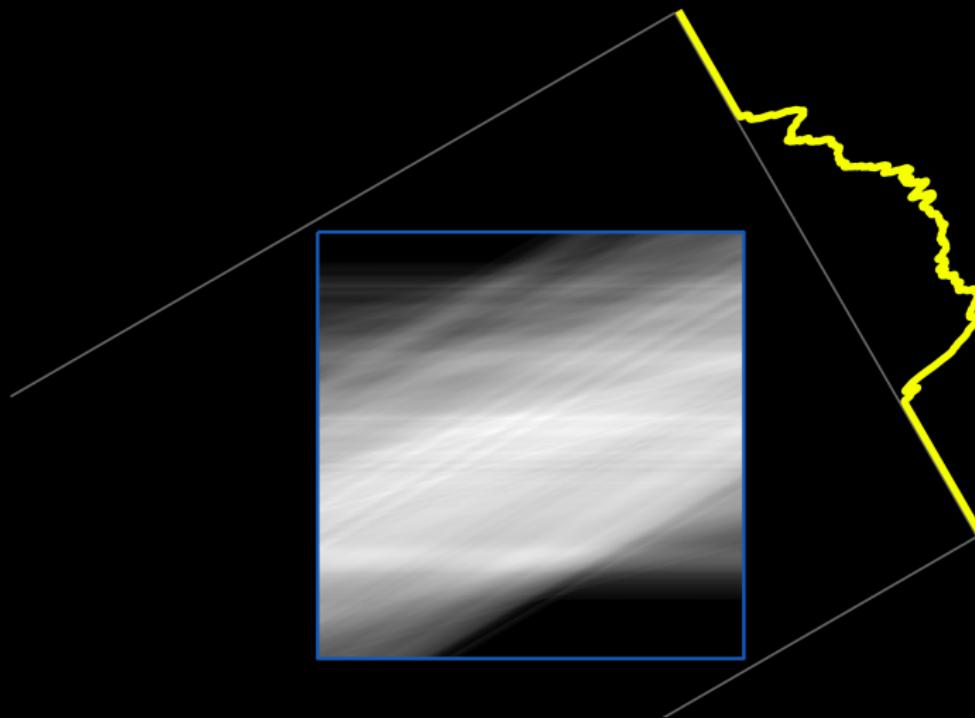
Summing all the back-projections
results in a blurred reconstruction



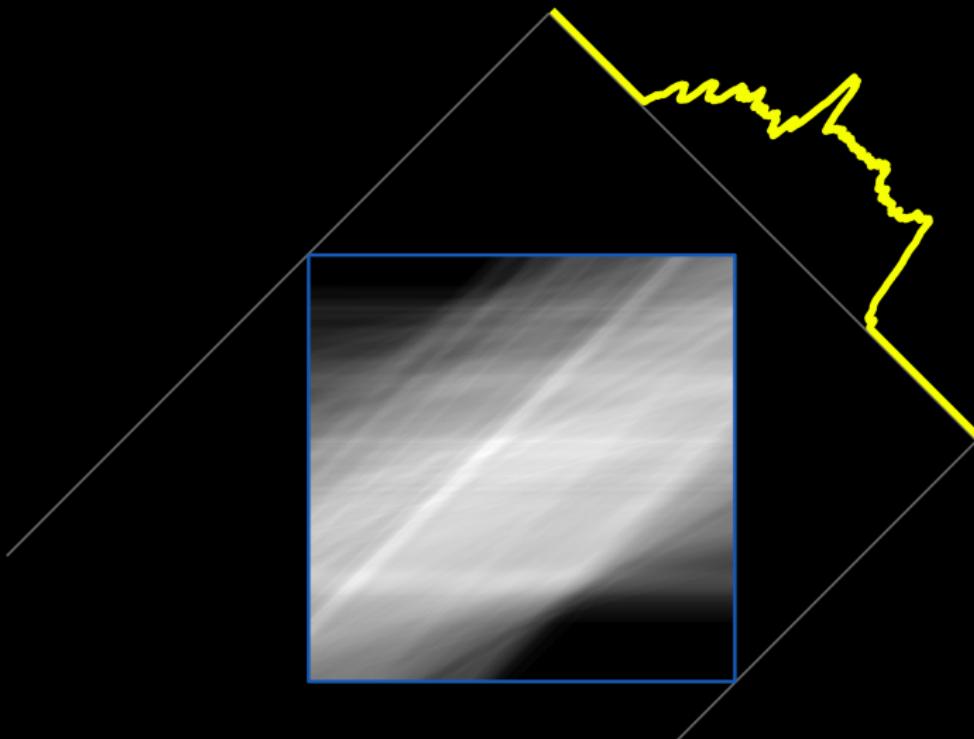
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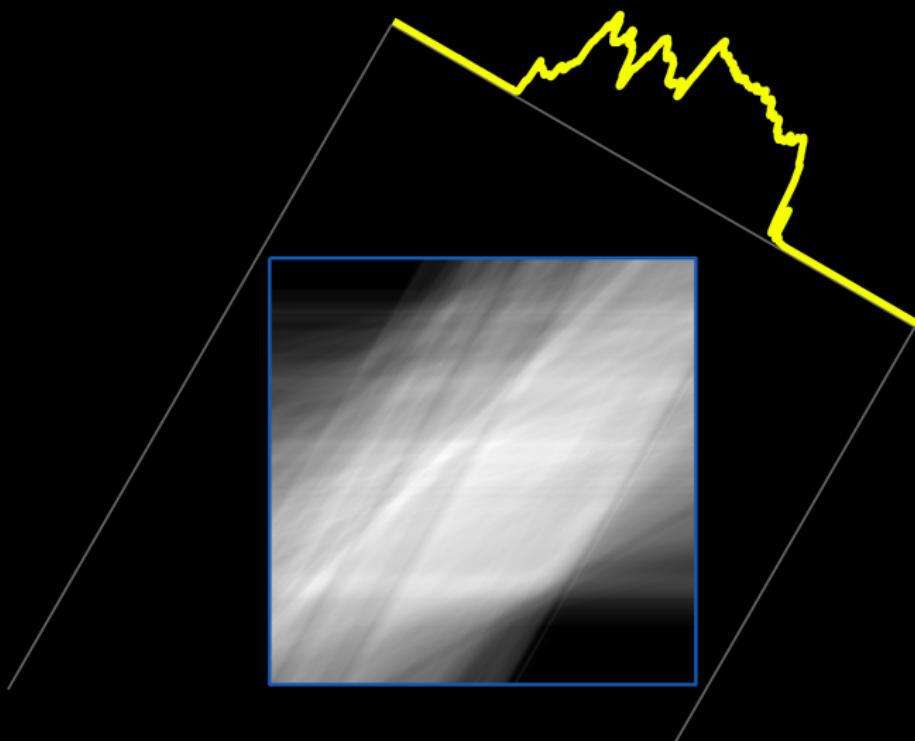
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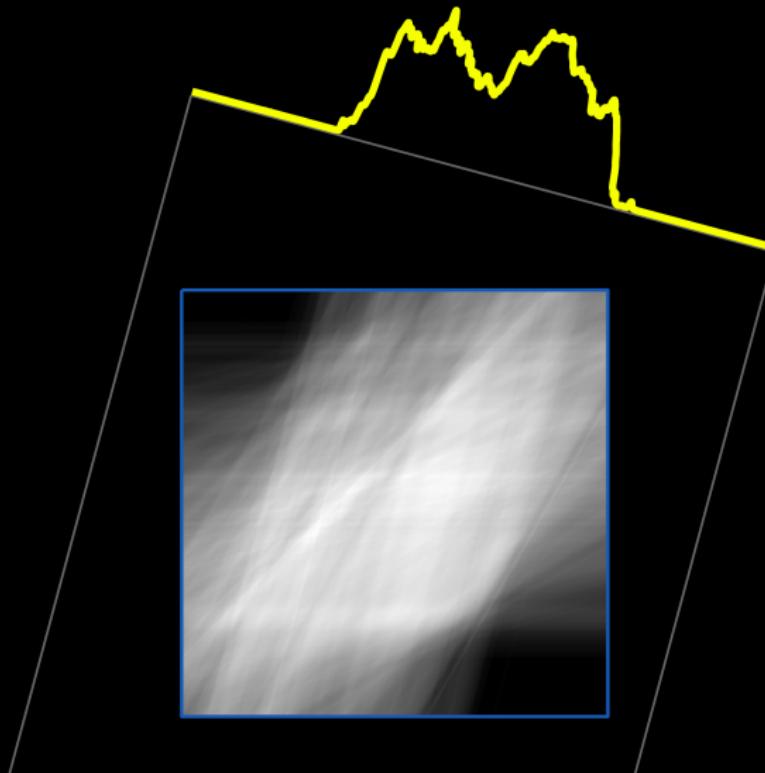
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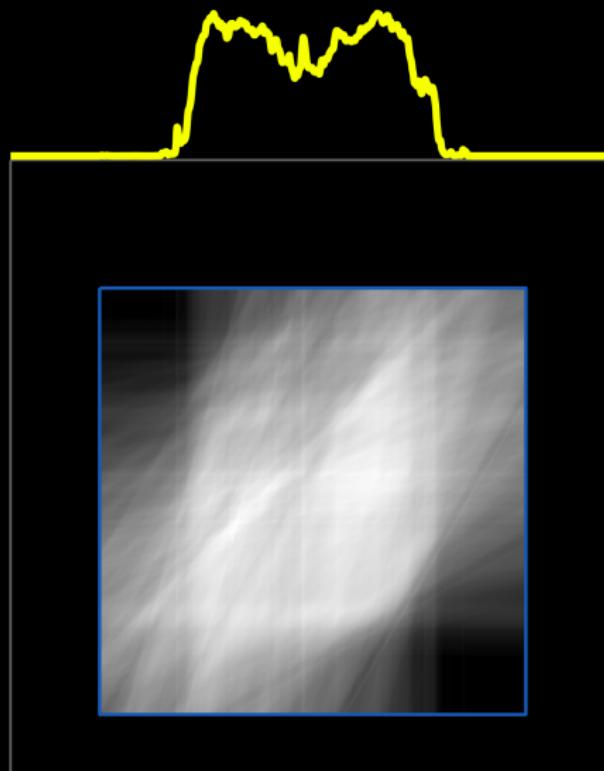
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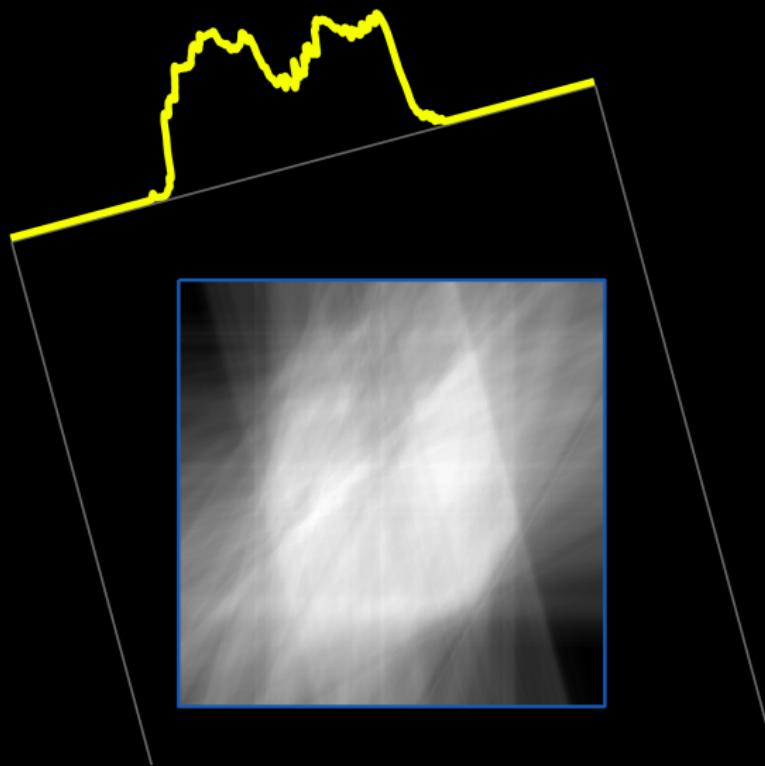
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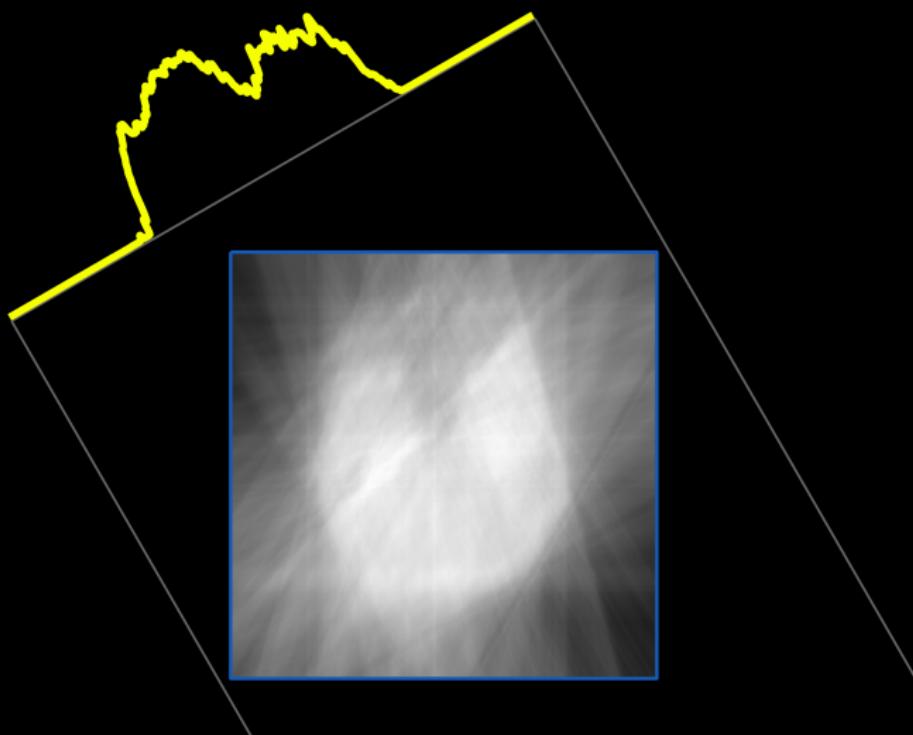
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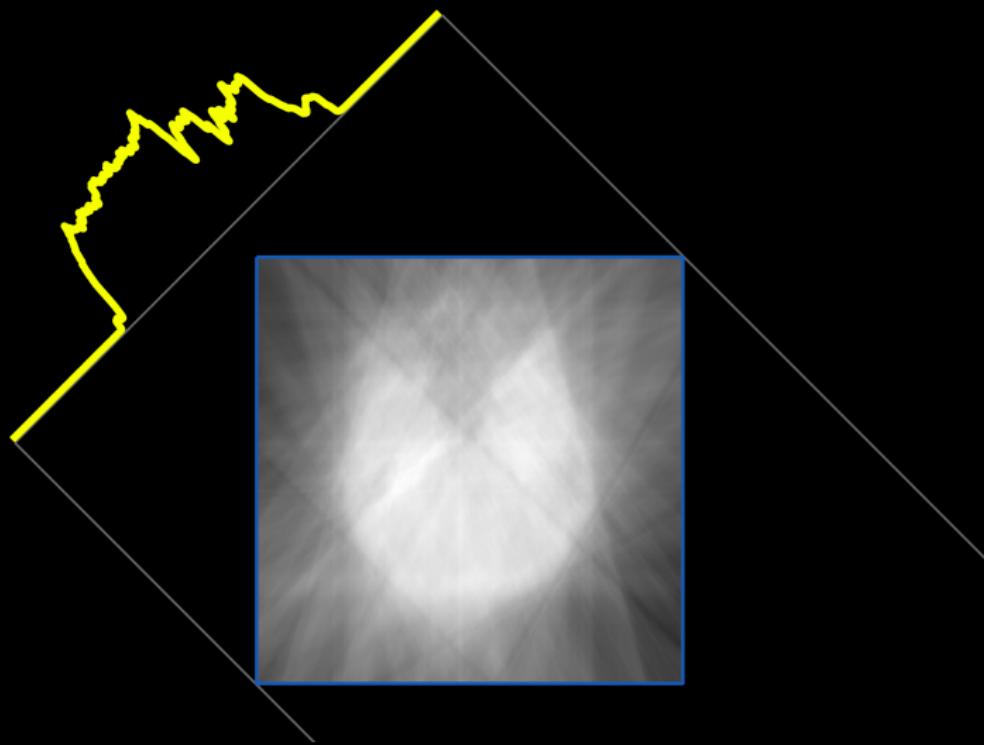
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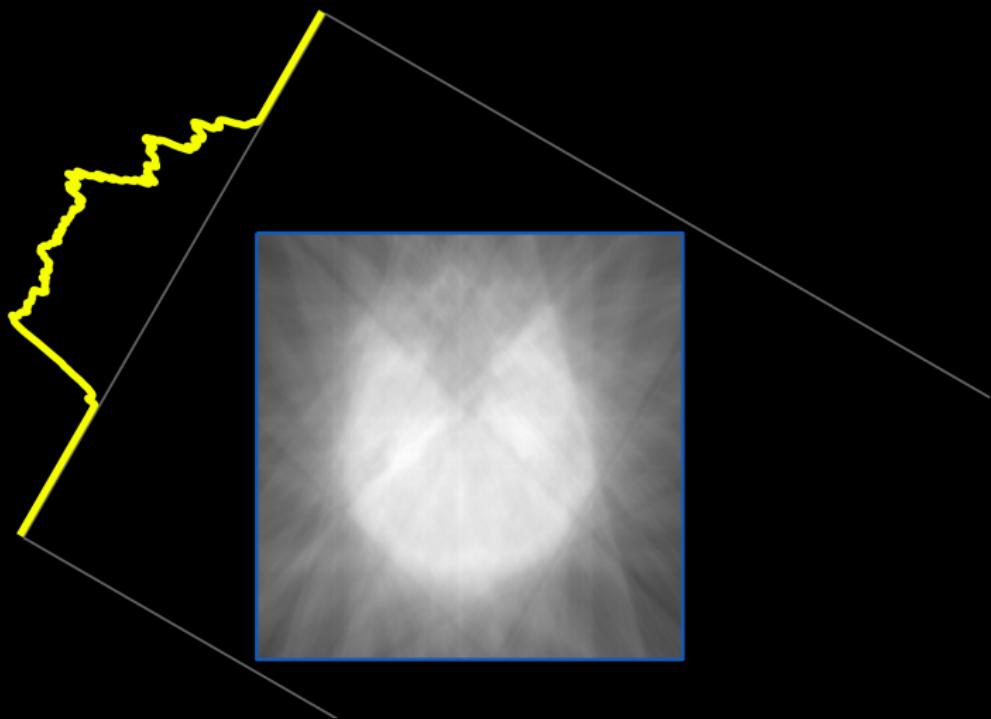
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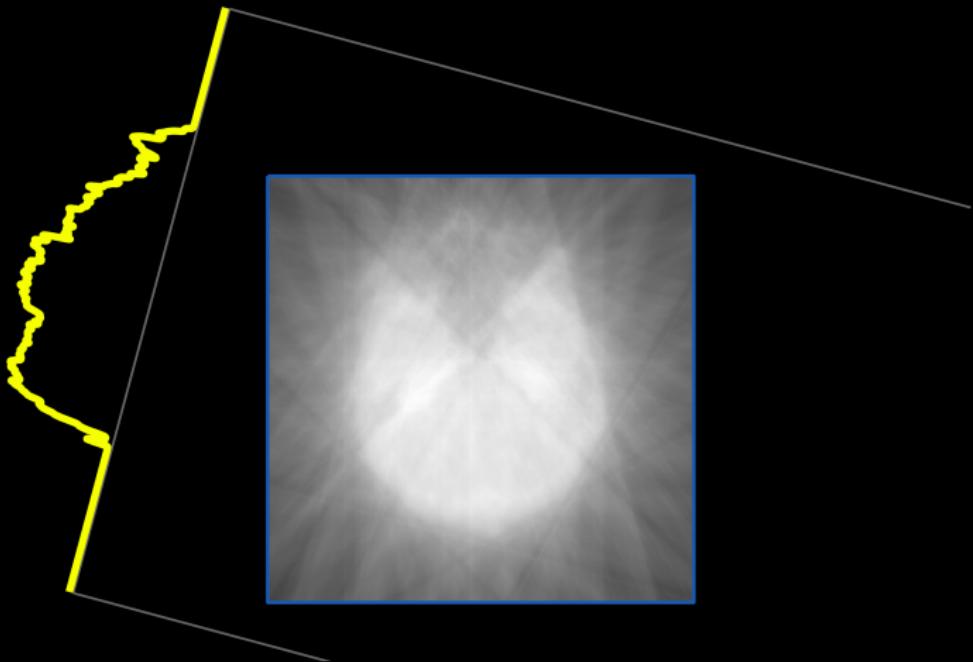
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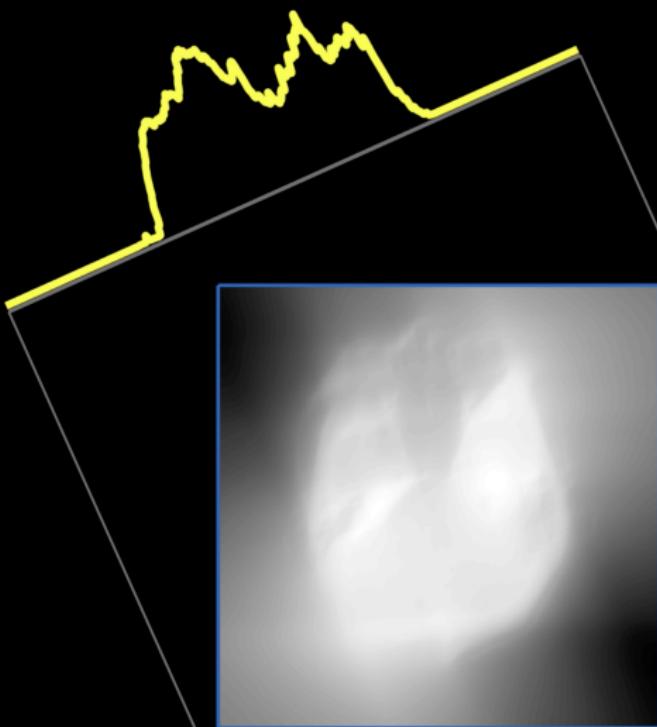
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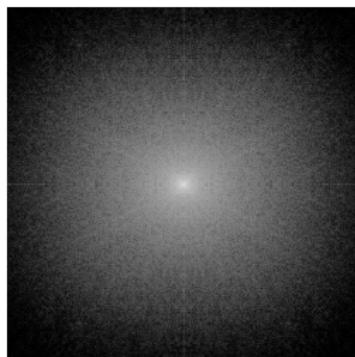
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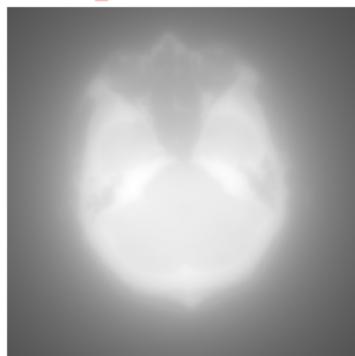
Here we use more directions, so the back-projected image quality is higher



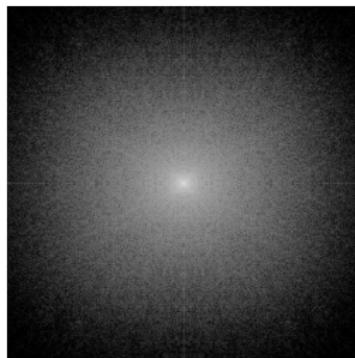
Final reconstruction involves high-pass filtering
on top of the back-projection



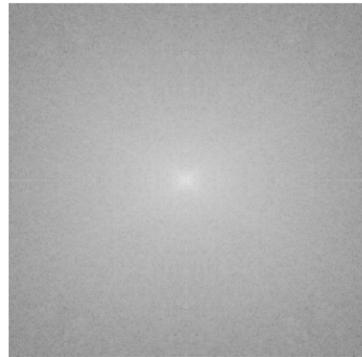
↑ FFT



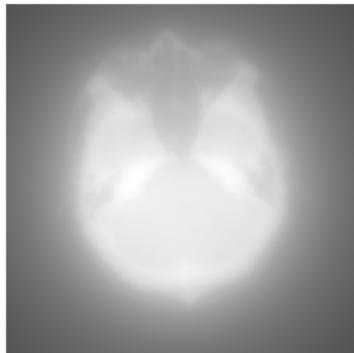
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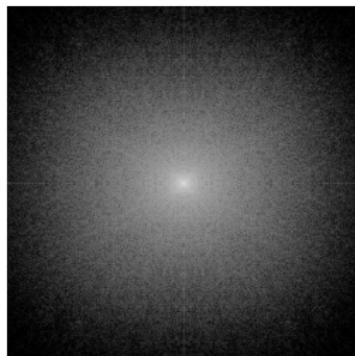
Boost high frequencies



↑ FFT

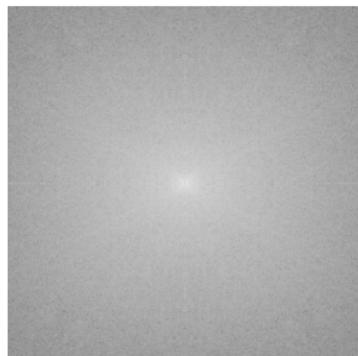


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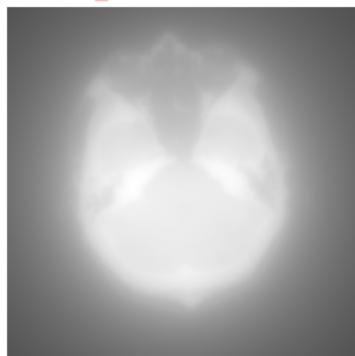


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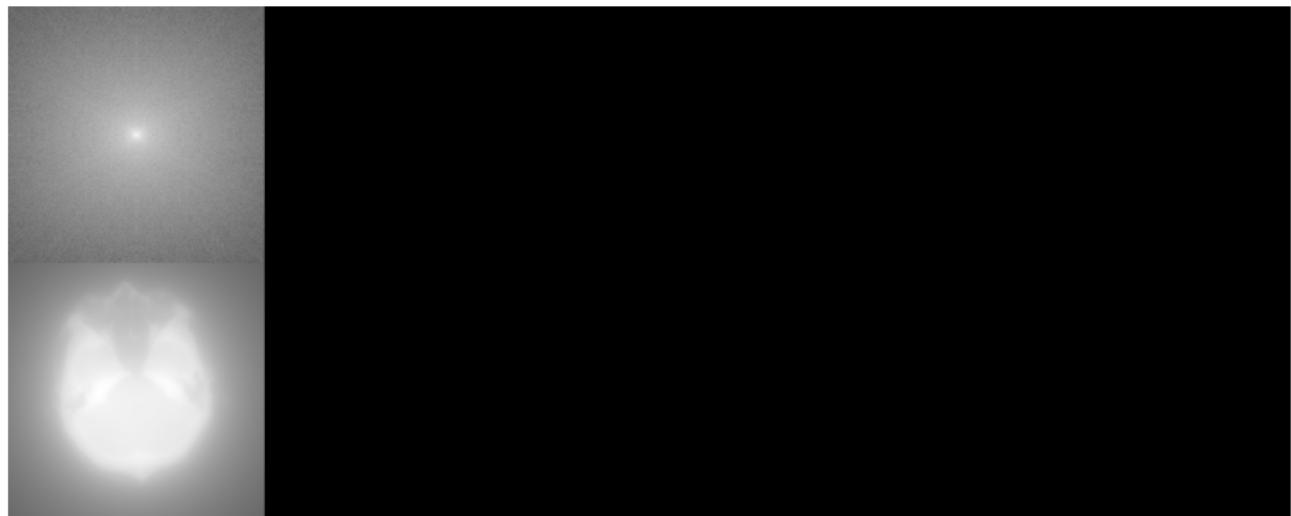
Boost high frequencies



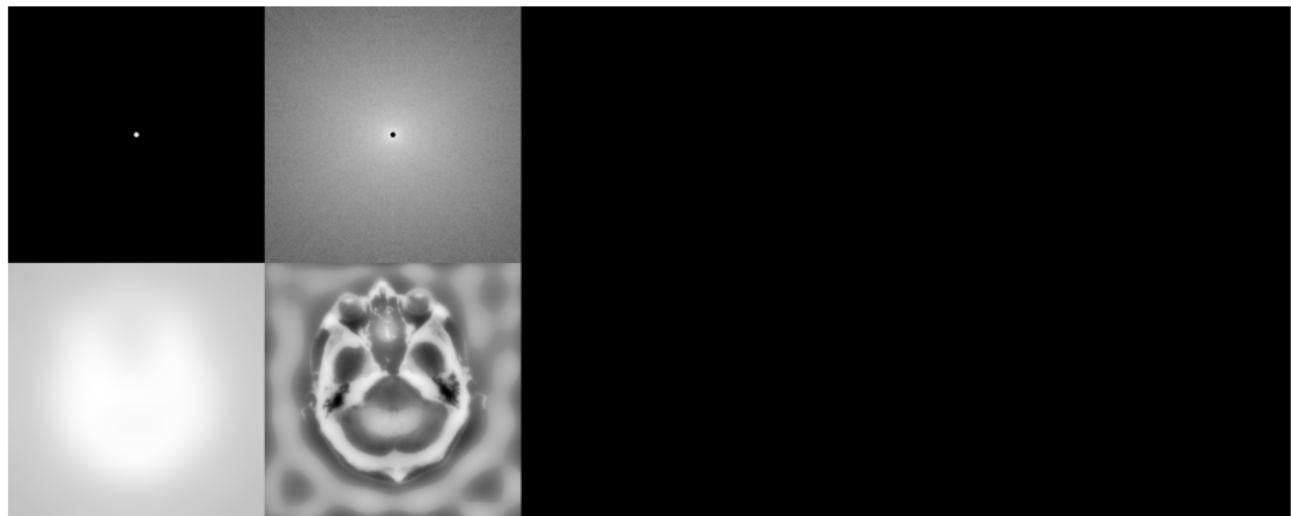
IFFT ↓



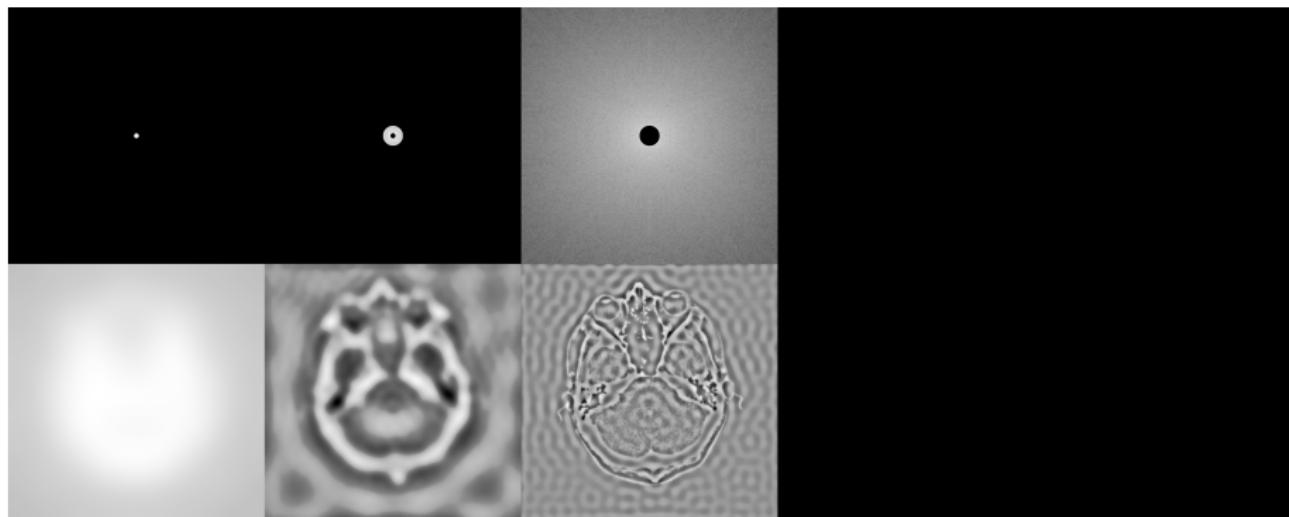
Let's observe the Fourier transform
by dividing it into frequency bands



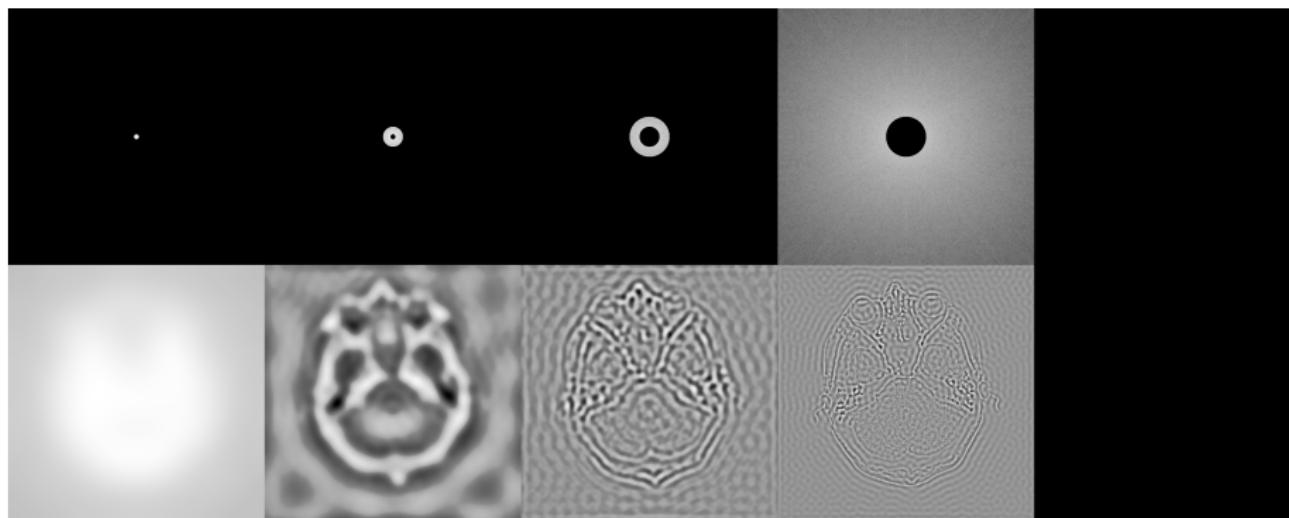
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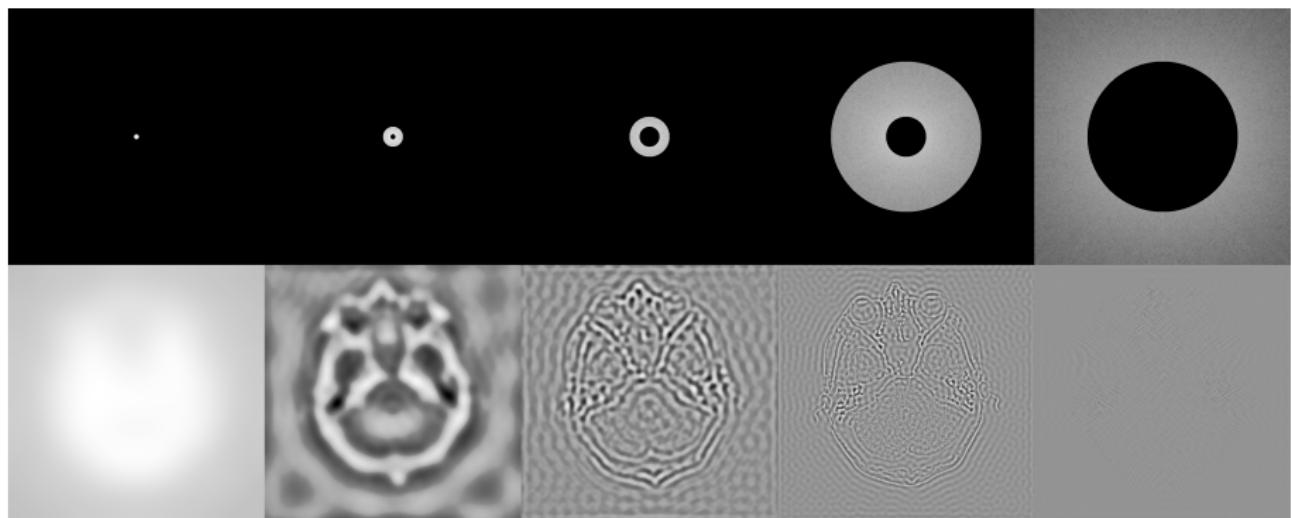
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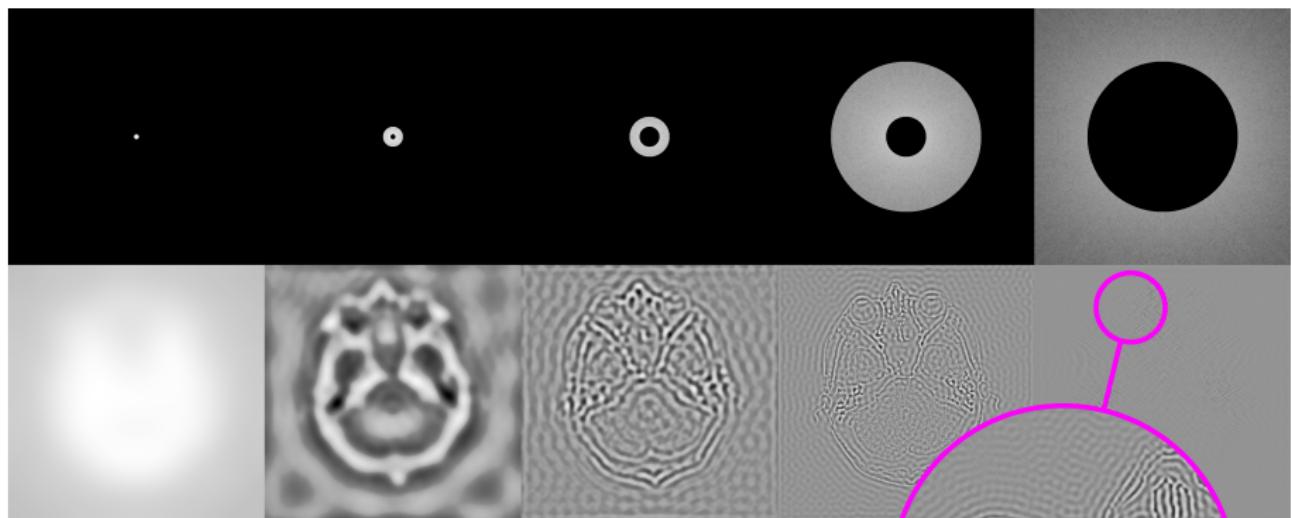
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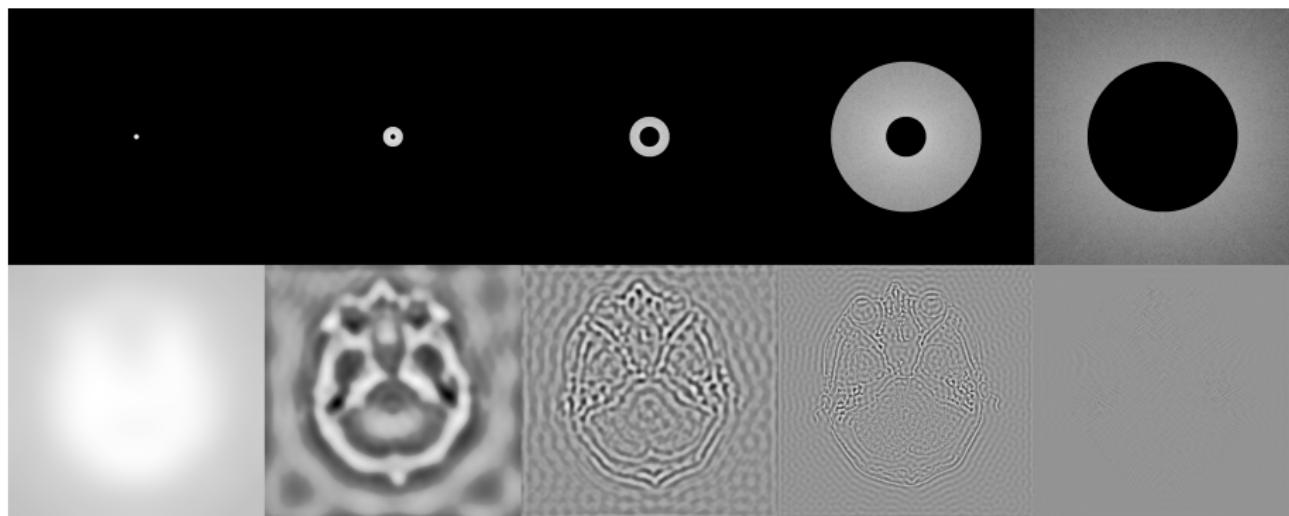
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Let's observe the Fourier transform
by dividing it into frequency bands



Filtering means reducing the role of some frequency bands, and boosting some of them



Reduce a lot

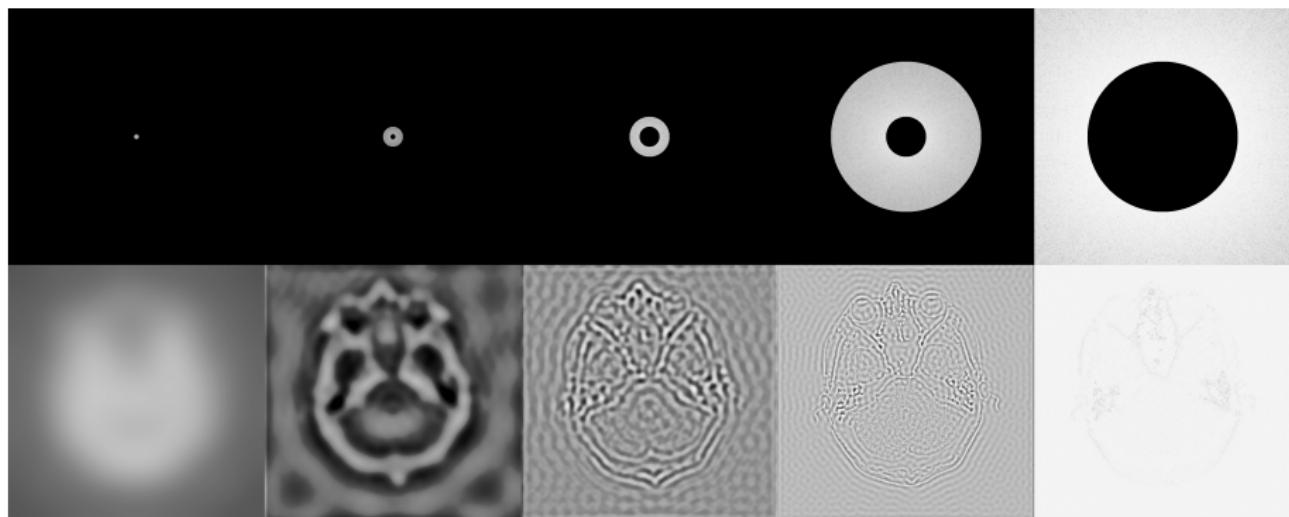
Reduce little

Keep as is

Boost little

Boost a lot

Filtering means reducing the role of some frequency bands, and boosting some of them



Reduce a lot

Reduce little

Keep as is

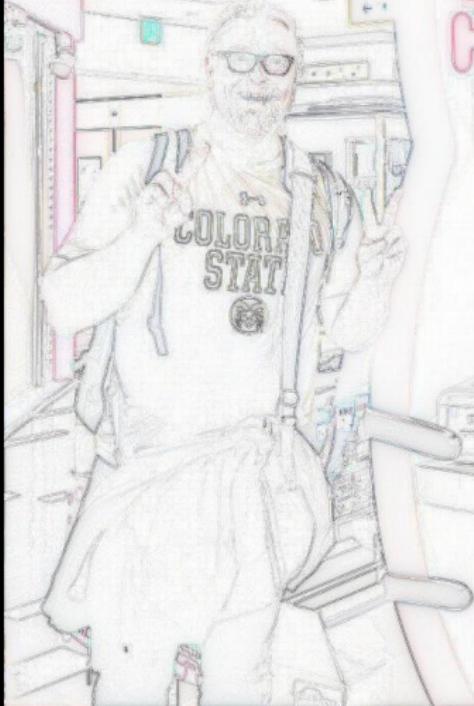
Boost little

Boost a lot

Fa So La
HELLO KITTY



Fa So La
HELLO KITTY





A
of
Continuous
Theory

Let us start with a crucial calculation

To reconstruct f at a point x , the most obvious data related to $f(x)$ are the integrals over lines passing through x . Let us sum them all together, call the result $Tf(x)$ and see what we get by introducing polar coordinates:

$$\begin{aligned} Tf(x) &= \int_0^\pi \int_{-\infty}^\infty f(x + t\theta) dt d\theta \\ &= \int_0^{2\pi} \int_0^\infty \frac{f(x + t\theta)}{t} t dt d\theta \\ &= \int_{\mathbb{R}^2} \frac{f(x + y)}{|y|} dy \\ &= \int_{\mathbb{R}^2} \frac{f(y)}{|x - y|} dy \\ &= (f(y) * \frac{1}{|y|})(x), \end{aligned}$$

where $*$ stands for convolution.

The Calderón operator Λ is the inverse of T

Recall that Fourier transform converts convolution to multiplication ($\widehat{g * h} = \hat{g}\hat{h}$) and that

$$\widehat{\frac{1}{|y|}}(\xi) = \frac{1}{|\xi|}.$$

Furthermore, define the Calderón operator Λ by

$$\Lambda f(x) := \mathcal{F}^{-1} |\xi| \hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} |\xi| \hat{f}(\xi) d\xi,$$

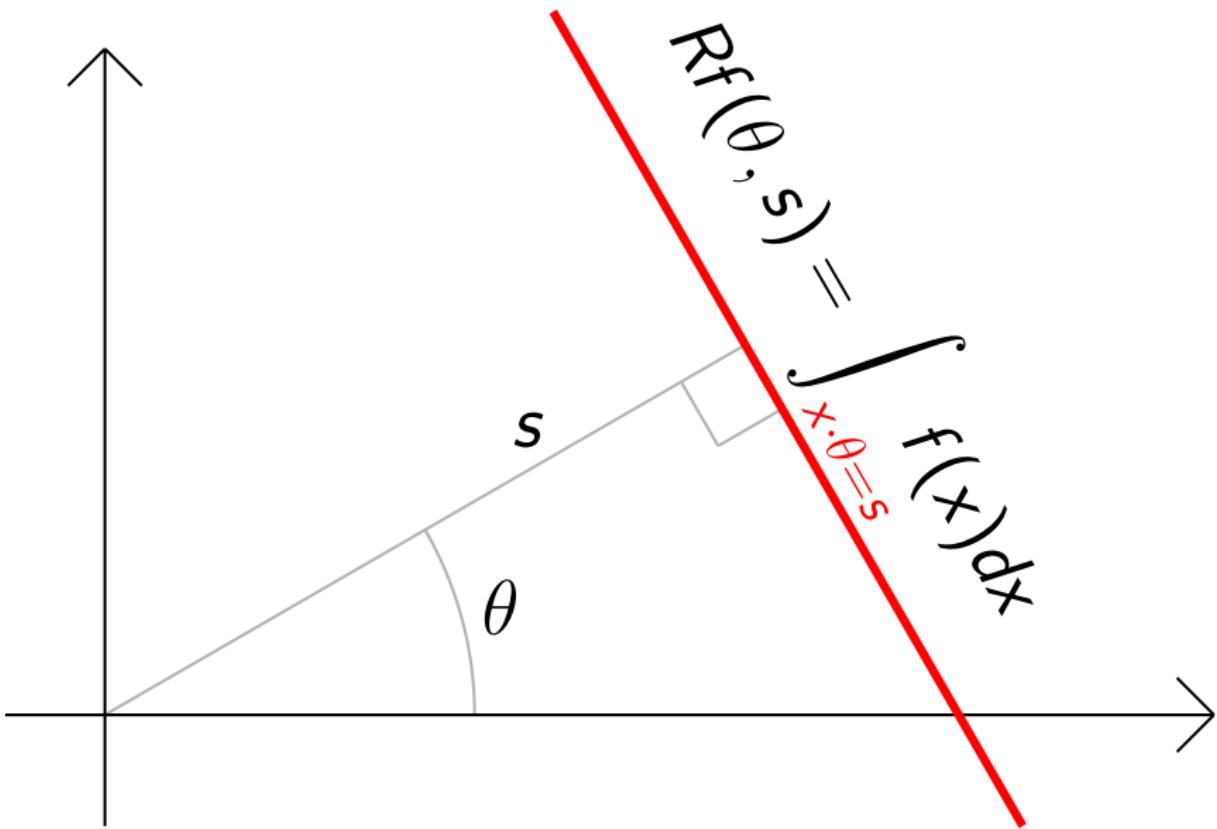
where \mathcal{F}^{-1} is the inverse Fourier transform. Note that Λ can be thought of as a high-pass filter. Now we see that

$$\widehat{Tf}(\xi) = \frac{\hat{f}(\xi)}{|\xi|},$$

and thus

$$\Lambda Tf = f.$$

Radon transform organizes sets of line integrals



Definition of the Radon transform

Let $f(x) = f(x_1, x_2)$ be the X-ray attenuation coefficient. The classical model for tomographic data is the **Radon transform**

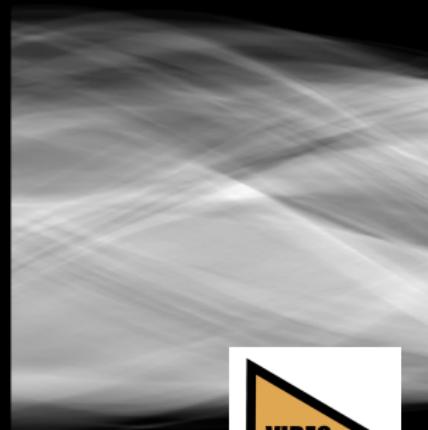
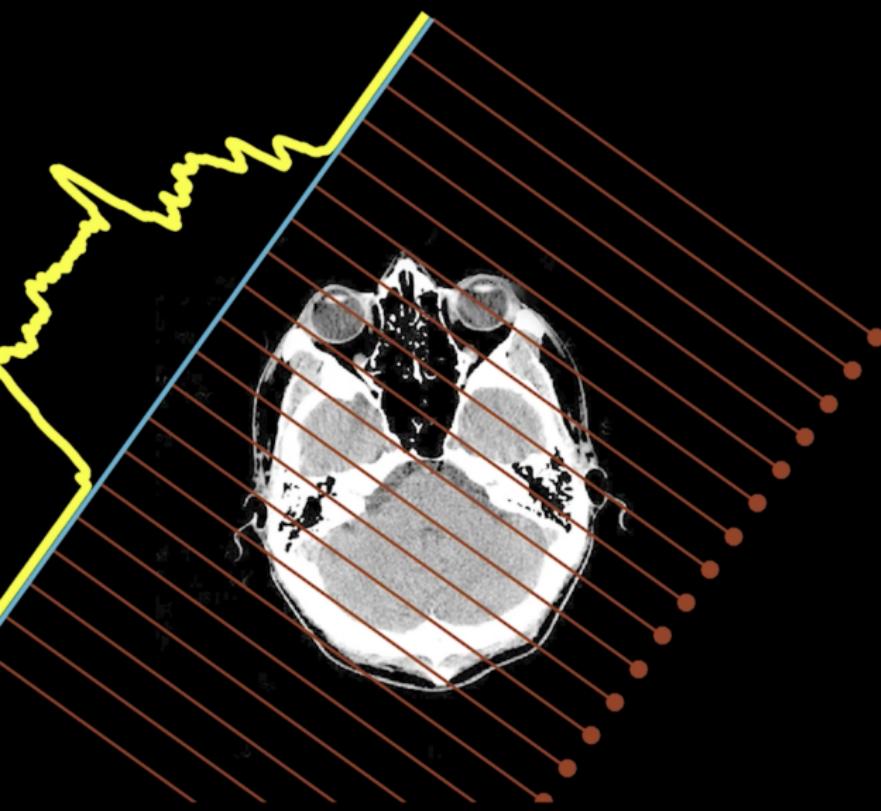
$$Rf(\theta, s) = \int_{x \cdot \vec{\theta} = s} f(x) dx = \int_{-\infty}^{\infty} f(s\vec{\theta} + \tau\vec{\theta}^\perp) d\tau, \quad \vec{\theta} \in S^1, s \in \mathbb{R},$$

where S^1 is the unit circle, $\vec{\theta}^\perp$ is a unit vector perpendicular to the unit vector $\vec{\theta} = (\cos \theta, \sin \theta)$, and $x \cdot \vec{\theta}$ denotes vector inner product.

Note that f is defined on \mathbb{R}^2 and Rf is defined on $S^1 \times \mathbb{R}^1$.

Also: $Rf(\theta, s) = Rf(-\theta, -s)$.

Radon transform as sinogram



Adjoint of the Radon transform R : the back-projection operator R^*

Use change of variables $x = s\vec{\theta} + \tau\vec{\theta}^\perp$; then $dx = dsd\tau$. Calculate

$$\begin{aligned}& \int_0^{2\pi} \int_{-\infty}^{\infty} Rf(\theta, s) g(\theta, s) ds d\theta \\&= \int_0^{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s\vec{\theta} + \tau\vec{\theta}^\perp) d\tau \right) g(\theta, s) ds d\theta \\&= \int_0^{2\pi} \left(\int_{\mathbb{R}^2} f(x) g(\theta, x \cdot \vec{\theta}) dx \right) d\theta \\&= \int_{\mathbb{R}^2} f(x) \left(\int_0^{2\pi} g(\theta, x \cdot \vec{\theta}) d\theta \right) dx \\&=: \int_{\mathbb{R}^2} f(x) R^*g(x) dx\end{aligned}$$

Both f and R^*g are defined on \mathbb{R}^2 . Note that $Tf = \frac{1}{2}R^*Rf$.

We arrive at the modern way of writing the filtered back-projection reconstruction formula



Johann Radon (1887-1956)

$$f = \frac{1}{2} \Lambda R^* R f$$

Radon's 1917 formulation:

$$f(P) = -\frac{1}{\pi} \int_0^\infty \frac{d\overline{F}_p(q)}{q}$$

Outline

What is an X-ray image?

X-ray slice imaging

Are You a Natural Tomographer?

Filtered back-projection and the Radon transform

Tomography without X-rays

Visible light

Electrons

Neutrons

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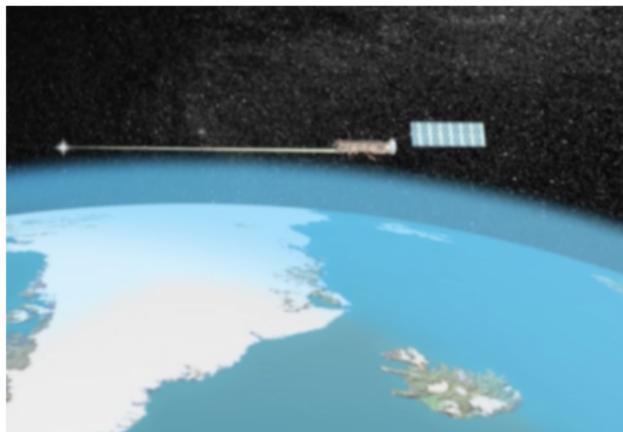
Tomography without X-rays

Visible light

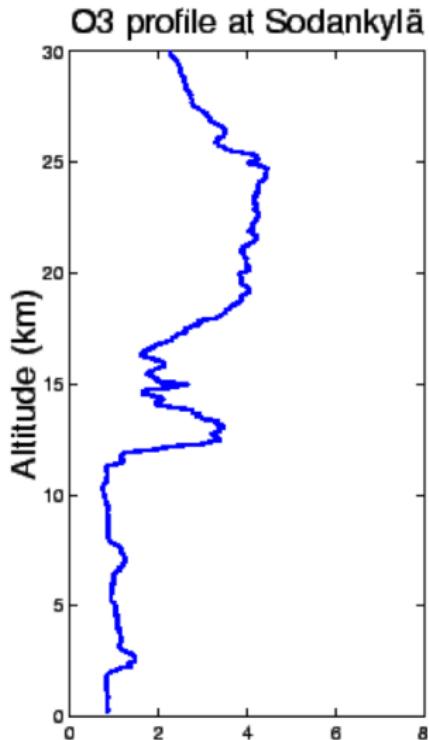
Electrons

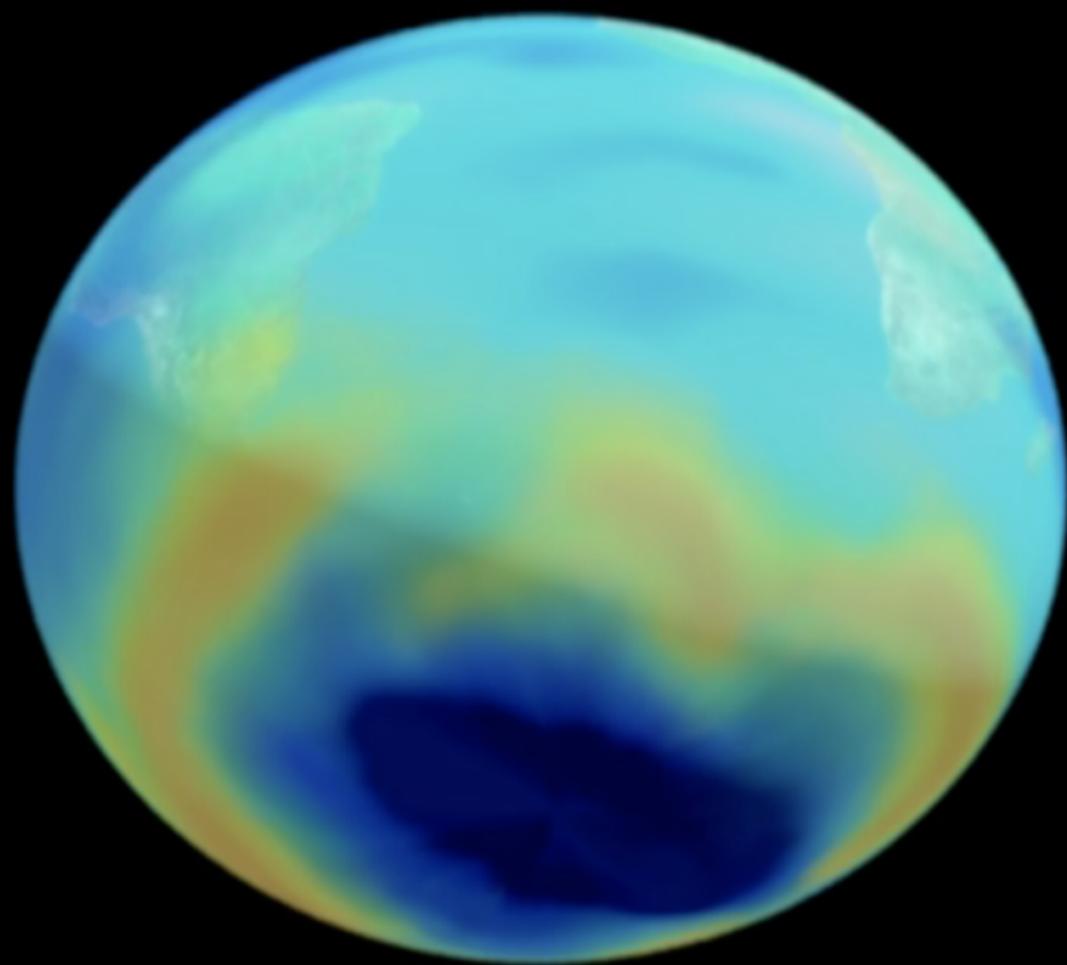
Neutrons

The mathematics of X-ray tomography can be used for recovering the ozone layer



European Space Agency
Finnish Meteorological Institute
Envisat and GOMOS projects
Thanks to Johanna Tamminen!





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Electron microscopy allows us to see things smaller than the wavelength of light, such as viruses

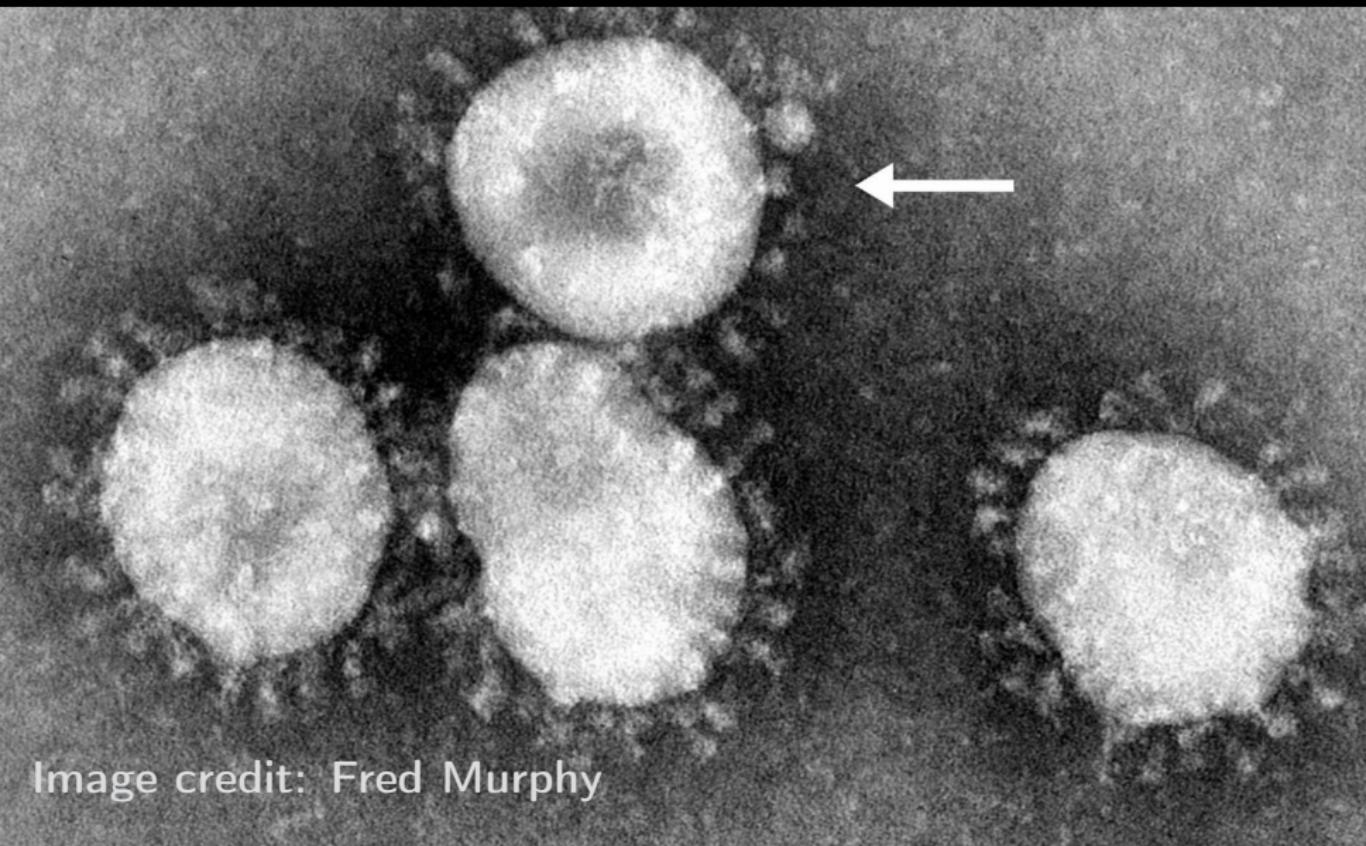
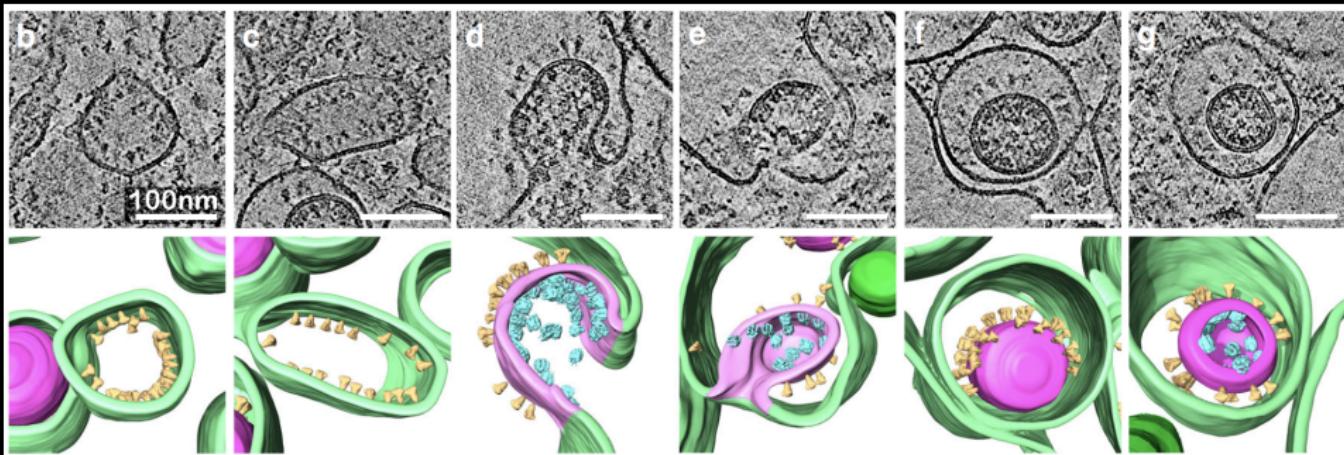


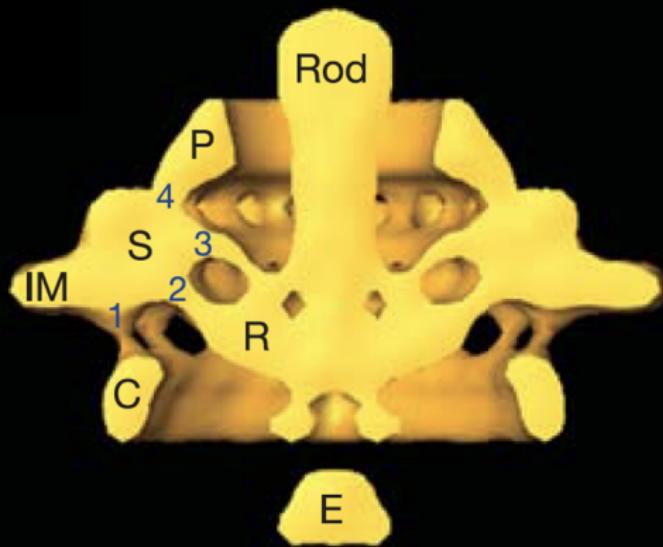
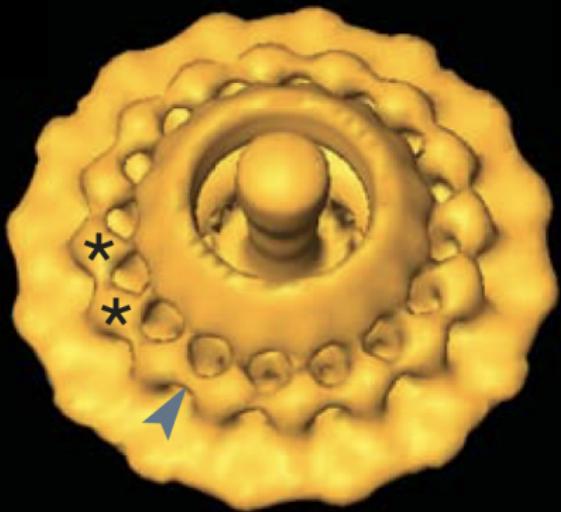
Image credit: Fred Murphy

We can observe a cell manufacturing corona viruses



[Klein, Cortese, Winter, Wachsmuth-Melm, Neufeldt, Cerikan, Stanifer, Boulant, Bartenschlager & Chlanda 2021]

Electron transmission cryotomography reveals the swimming engine of *Treponema primitia* bacteria



[Murphy, Leadbetter & Jensen 2016]

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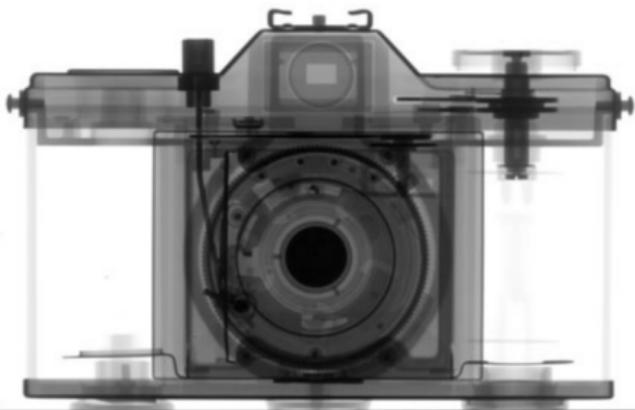
Tomography without X-rays

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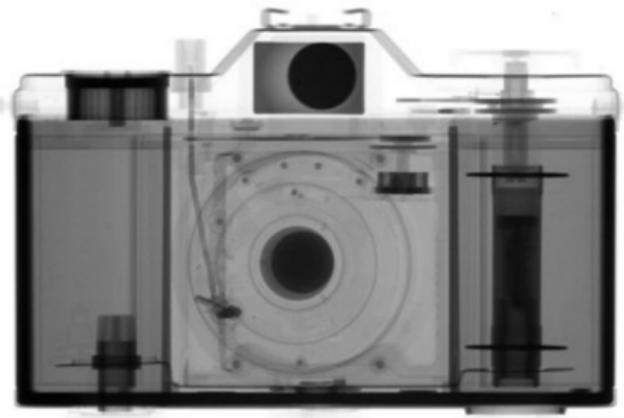
Electrons

Neutrons

Neutron beams deliver projection images different from X-ray radiographs



X-ray image



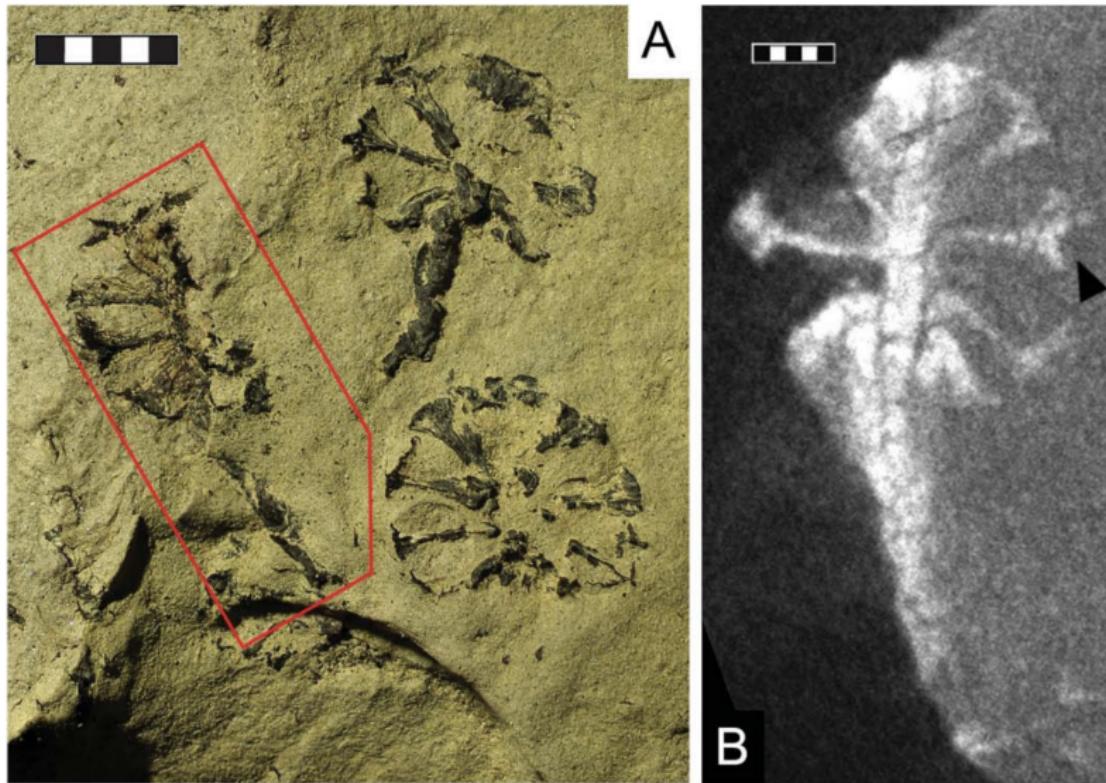
Neutron image

**Imaging with neutrons opens up new possibilities
as water attenuates but metal is transparent**



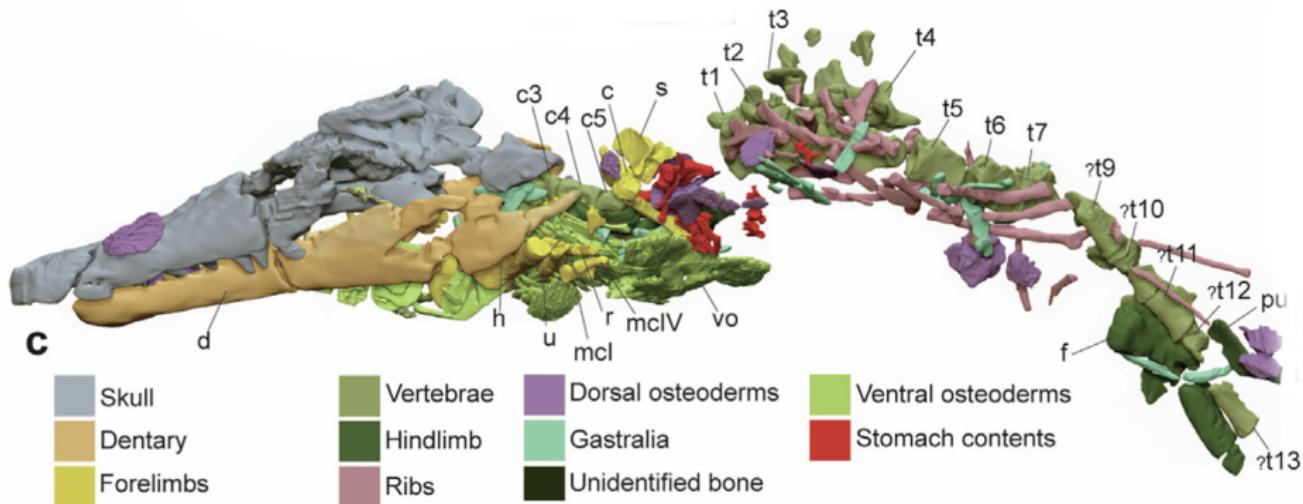
Video:
Anders Kaestner
Neutron Imaging and
Activation Group,
Paul Scherrer Institute

Neutron tomography for fossilized cones



Mays, Cantrill, Stilwell & Bevitt 2018

Neutron tomography revealed the lunch of a fossil crocodile *Confractosuchus sauroktonos*



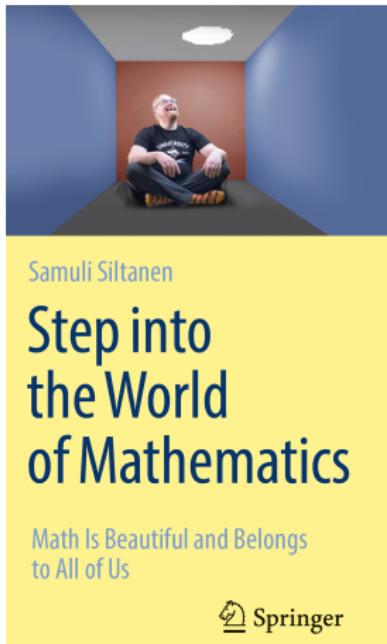
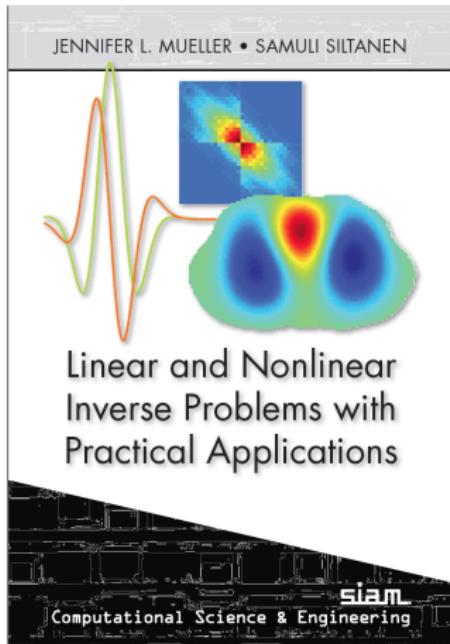
White, Bell, Campione, Sansalone, Brougham, Bevitt, Molnar, Cook,
Wroe & Elliott 2022

Bon appetit, *Confractosuchus sauroktonos*!



Image credit: Dr. Matt White / Australian age of dinosaurs

In case you liked this material, here are a couple of pointers to further cool stuff



<https://blog.fips.fi/>

More books recommended for learning the mathematics of practical X-ray tomography

2008 Buzug: Computed Tomography: From Photon Statistics to Modern Cone-Beam CT

2008 Epstein: Introduction to the mathematics of medical imaging

2010 Hansen: Discrete inverse problems

2012 Mueller & S: Linear and Nonlinear Inverse Problems with Practical Applications

2014 Kuchment: The Radon Transform and Medical Imaging

2021 Hansen, Jørgensen, Lionheart (eds.): Computed Tomography: Algorithms, Insight, and Just Enough Theory

Thank you for your attention!



← Slime mold called *Lycogala conicum*