Sparse-angle X-ray tomography

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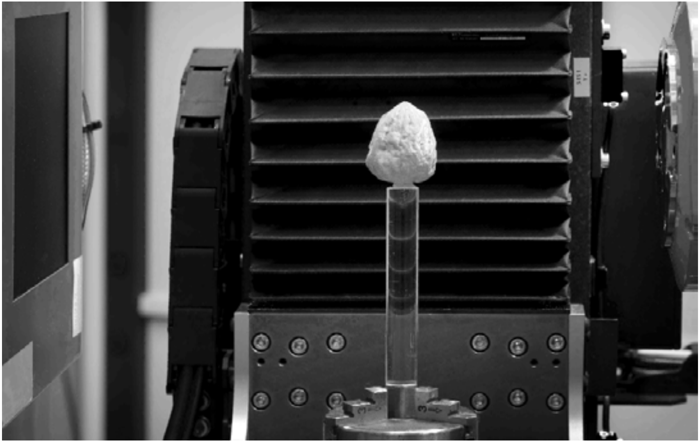
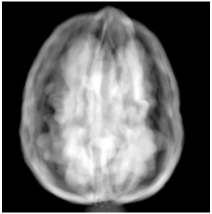
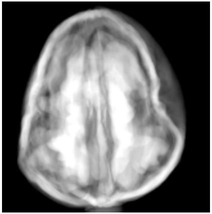
We collected X-ray projection data of a walnut from 1200 directions



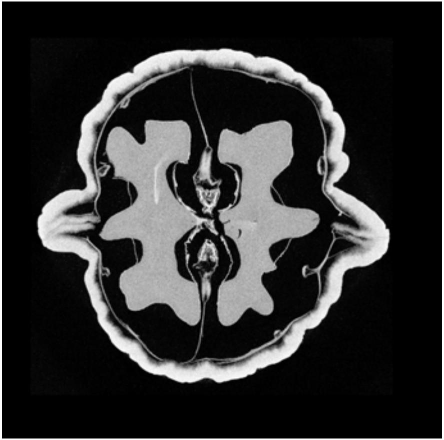




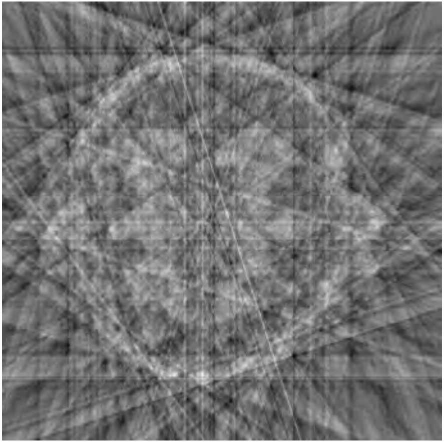
Data collection: thanks to Keijo Hämäläinen and Aki Kallonen, University of Helsinki.

The data is openly available at http://fips.fi/dataset.php, thanks to Esa Niemi and Antti Kujanpää

Reconstructions of a 2D slice through the walnut using filtered back-projection (FBP)

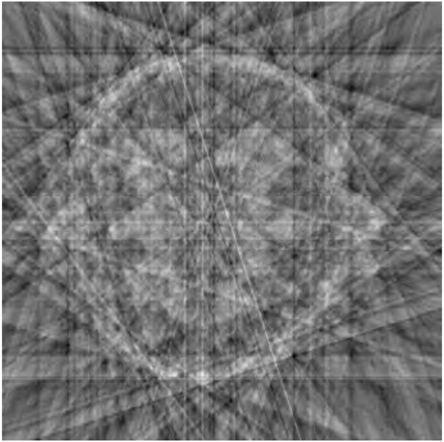


FBP with comprehensive data
(1200 projections)

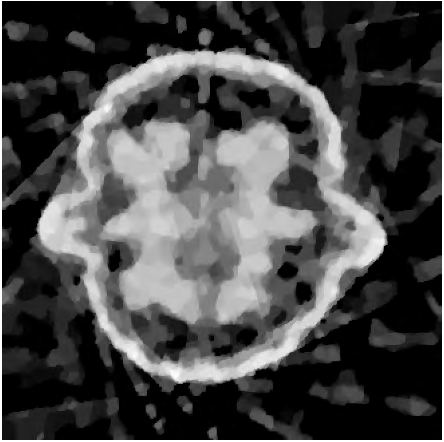


FBP with sparse data
(20 projections)

Sparse-data reconstruction of the walnut using non-negative total variation (TV) regularization

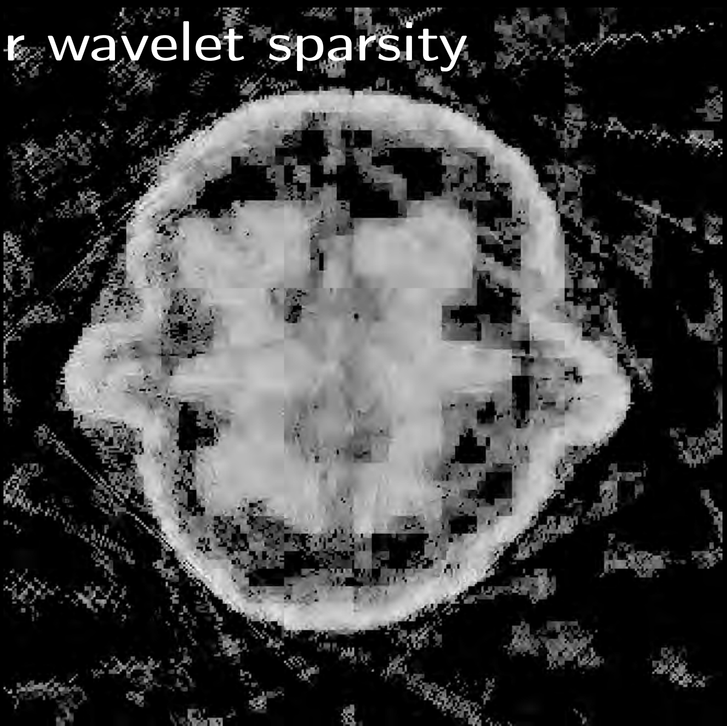


Filtered back-projection

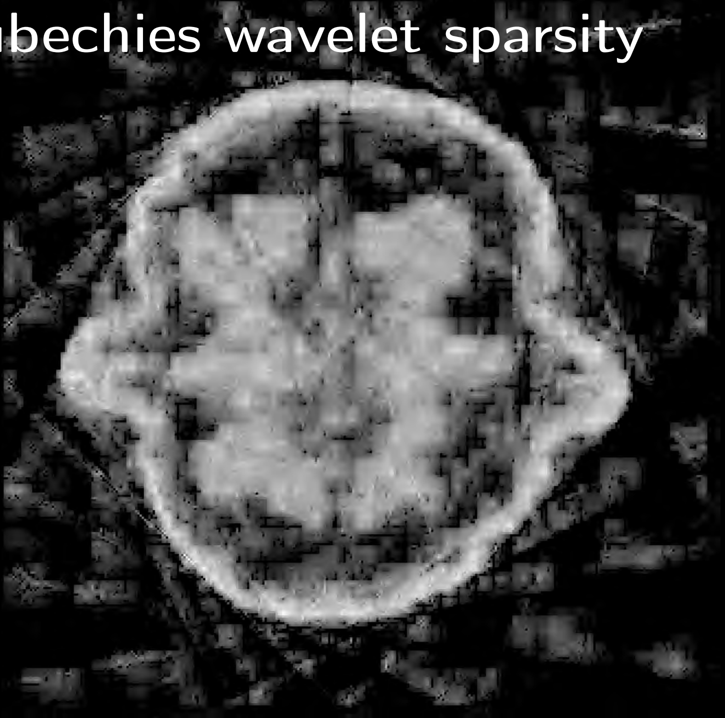


TV regularization: minimum of
$\|Ax - m\|_2^2 + \alpha(\|L_H x\|_1 + \|L_V x\|_1)$
over nonnegative images $x \in \mathbb{R}^{N \times N}$

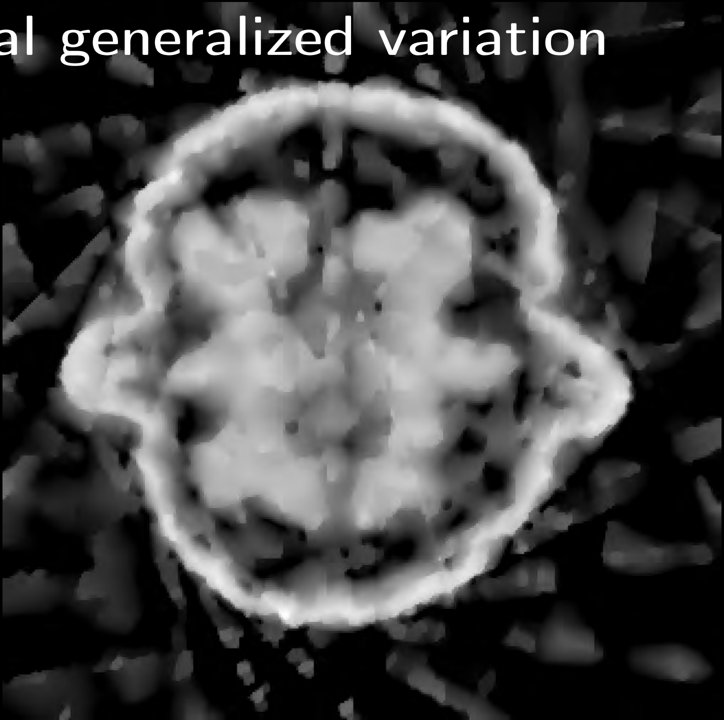
Haar wavelet sparsity



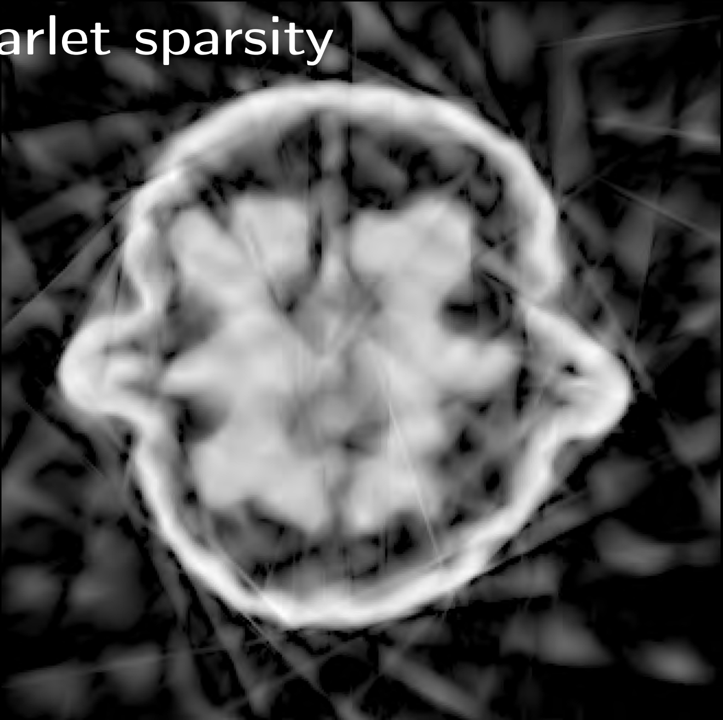
Daubechies wavelet sparsity



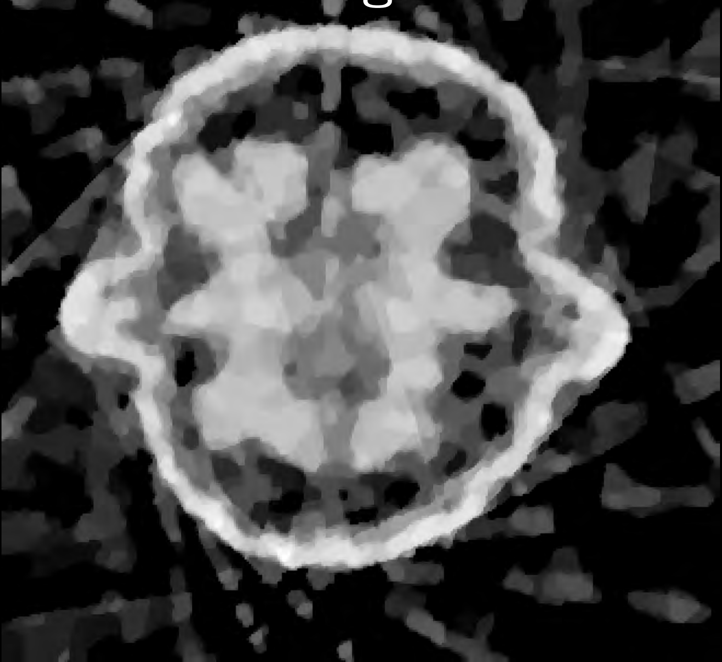
Total generalized variation



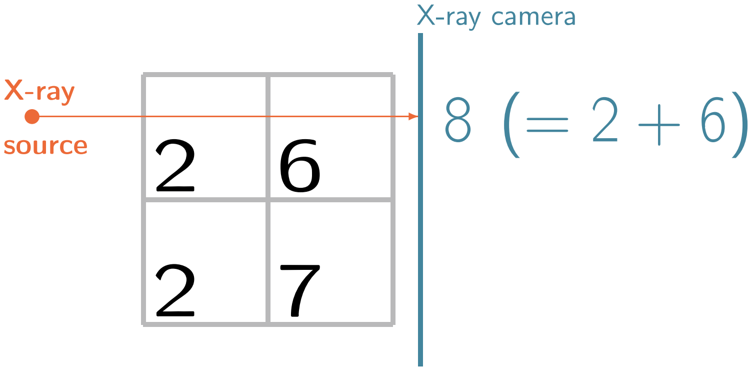
Shearlet sparsity



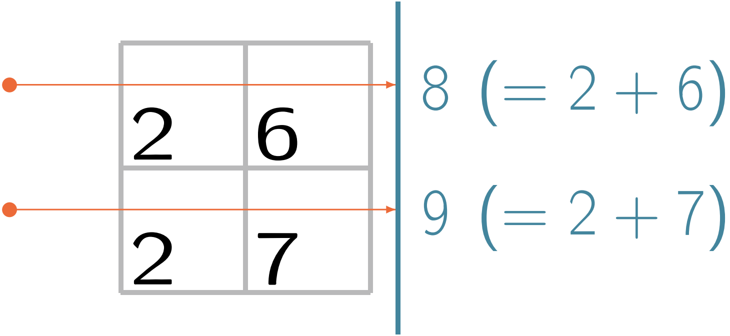
Total variation regularization



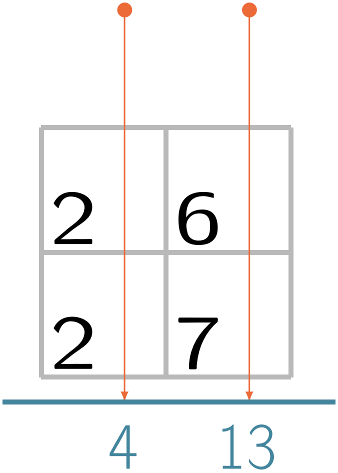
Consider a simple example of a 2D slice with internal structures consisting of small squares



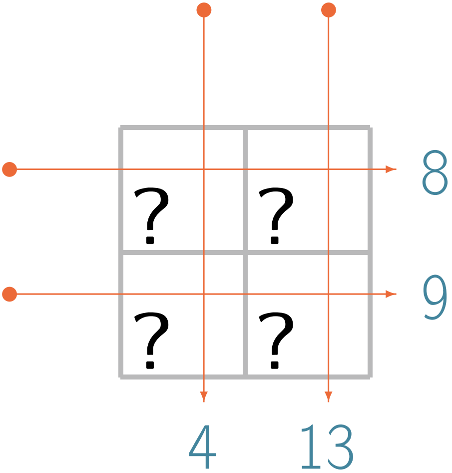
Two horizontal X-rays give us two numbers: row sums of the 2×2 array of attenuations



Tomographic imaging requires collecting X-ray data along another direction as well



“Inverse problem” in this example is to recover the interior numbers from the measurements



With such a limited amount of data, the inverse problem has multiple solutions!

2	6	8
2	7	9
4	13	

-1	9	8
5	4	9
4	13	

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the inverse problem has multiple solutions!

2	6	8
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-1	9	8
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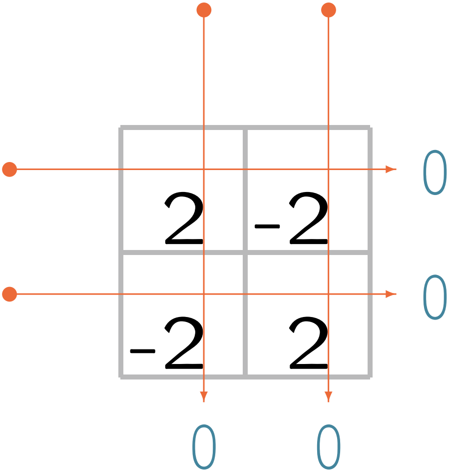
With such a limited amount of data,
the inverse problem has multiple solutions!

2	6	8
2	7	9
4	13	

-1	9	8
5	4	9
4	13	

4	4	8
0	9	9

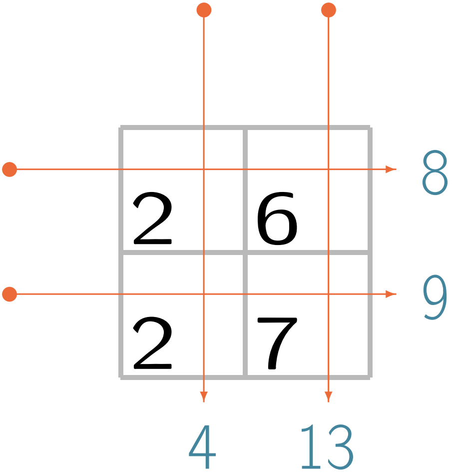
So-called "ghosts," or targets with zero data, are the source of multiple solutions

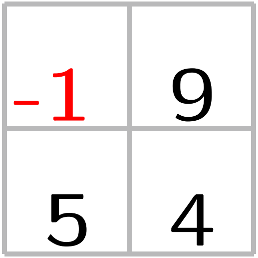


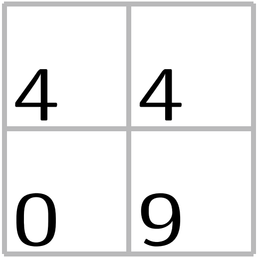
Adding a ghost does not change the data

$$\begin{array}{cc|c} 2 & 6 & 8 \\ 2 & 7 & 9 \\ \hline 4 & 13 & \end{array} + \begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \\ \hline 0 & 0 & \end{array} = \begin{array}{cc|c} 4 & 4 & 8 \\ 0 & 9 & 9 \\ \hline 4 & 13 & \end{array}$$

How can a reconstruction method pick out the correct image among all that match the data?







Consider these three candidates for reconstruction

True target

2	6
2	7

Wrong data,
good “tissue type”

3	3
3	3

Right data,
bad “tissue type”

4	4
0	9

Penalty calculation for candidate 1 (true target). First the penalty from (mis)matching X-ray data

2	6	$(8-8)^2$
2	7	$(9-9)^2$
$(4-4)^2$	$(13-13)^2$	

Data penalty: $(8-8)^2 + (9-9)^2 + (4-4)^2 + (13-13)^2 = 0.$

Penalty calculation for candidate 1.
Then the penalty from prior information

2	6
2	7

Data penalty: $(8-8)^2 + (9-9)^2 + (4-4)^2 + (13-13)^2 = 0.$

Prior penalty: $|2-6|$

Penalty calculation for candidate 1.
Then the penalty from prior information

2	6
2	7

Data penalty: $(8-8)^2 + (9-9)^2 + (4-4)^2 + (13-13)^2 = 0.$

Prior penalty: $|2-6| + |2-7|$

Penalty calculation for candidate 1.
Then the penalty from prior information

2	6
2	7

Data penalty: $(8-8)^2+(9-9)^2+(4-4)^2+(13-13)^2=0.$

Prior penalty: $|2-6|+|2-7|+|2-2|$

Penalty calculation for candidate 1.
Then the penalty from prior information

2	6
2	7

Data penalty: $(8-8)^2 + (9-9)^2 + (4-4)^2 + (13-13)^2 = 0.$

Prior penalty: $|2-6| + |2-7| + |2-2| + |6-7| = 4 + 5 + 0 + 1 = 10.$

Penalty calculation for candidate 1. Total penalty is the sum of data and prior penalties

2	6
2	7

	data penalty	0
+	prior penalty	10
=	total penalty	€10

Penalty calculation for candidate 2.
First the penalty from (mis)matching X-ray data

3	3	$(6-8)^2$
3	3	$(6-9)^2$
$(6-4)^2$	$(6-13)^2$	

Data penalty: $2^2 + 3^2 + 2^2 + 7^2 = 4 + 9 + 4 + 49 = 66$.

Penalty calculation for candidate 2.
Then the penalty from prior information

3	3
3	3

Data penalty: $2^2 + 3^2 + 2^2 + 7^2 = 4 + 9 + 4 + 49 = 66.$

Prior penalty: $|3-3| + |3-3| + |3-3| + |3-3| = 0.$

Penalty calculation for candidate 2. Total penalty is the sum of data and prior penalties

3	3
3	3

data penalty	66
+ prior penalty	0
= total penalty	€66

Penalty calculation for candidate 3.
First the penalty from (mis)matching X-ray data

$$\begin{array}{r|c|c|}
\cline{2-3}
(8-8)^2 & 4 & 4 \\
\cline{2-3}
(9-9)^2 & 0 & 9 \\
\cline{2-3}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{(4-4)^2} & \multicolumn{1}{c}{(13-13)^2}
\end{array}$$

Data penalty: $(8-8)^2 + (9-9)^2 + (4-4)^2 + (13-13)^2 = 0$.

Penalty calculation for candidate 3.
Then the penalty from prior information

4	4
0	9

Data penalty: $(8-8)^2 + (9-9)^2 + (4-4)^2 + (13-13)^2 = 0.$

Prior penalty: $|4-4| + |0-9| + |4-0| + |4-9| = 0 + 9 + 4 + 5 = 18.$

Penalty calculation for candidate 3. Total penalty is the sum of data and prior penalties

4	4
0	9

data penalty	0
+ prior penalty	18
= total penalty	€18

Which of candidates has smallest total penalty?

True target

2	6
2	7

data penalty	0
+ prior penalty	10
= total penalty	€10

Wrong data,
good “tissue type”

3	3
3	3

data penalty	66
+ prior penalty	0
= total penalty	€66

Right data,
bad “tissue type”

4	4
0	9

data penalty	0
+ prior penalty	18
= total penalty	€18

The problem can be solved in general using optimization

General target

x_1	x_3	8
x_2	x_4	9
4	13	

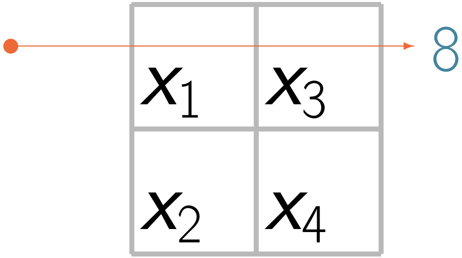
Find numbers $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and $x_4 \geq 0$ such that the sum of these two penalties is as small as possible:

Data penalty: $$(x_1+x_3-8)^2+(x_2+x_4-9)^2 +(x_1+x_2-4)^2+(x_3+x_4-13)^2$$

Prior penalty: $$|x_1-x_3|+|x_2-x_4| +|x_1-x_2|+|x_3-x_4|$$

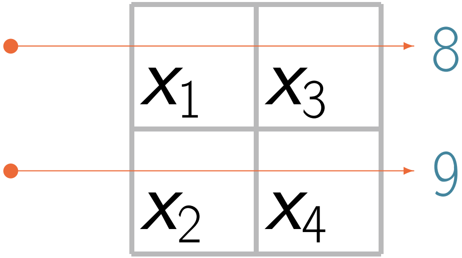
This method is called **total variation regularization.**

Writing the data penalty in matrix form



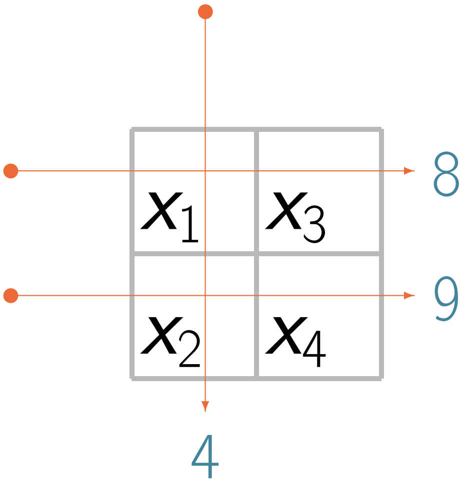
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \color{teal}{8} \\ \\ \\ \end{bmatrix}$$

Writing the data penalty in matrix form



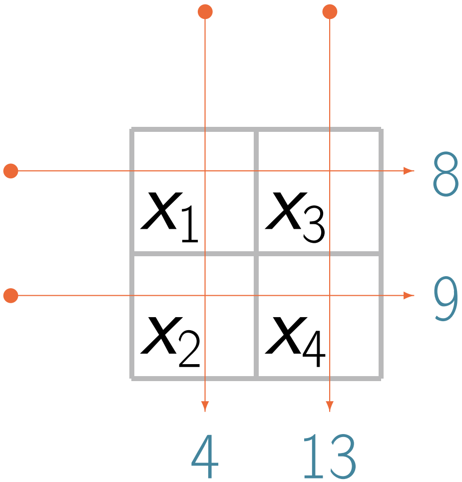
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \\ \end{bmatrix}$$

Writing the data penalty in matrix form



$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 4 \\ \ \end{bmatrix}$$

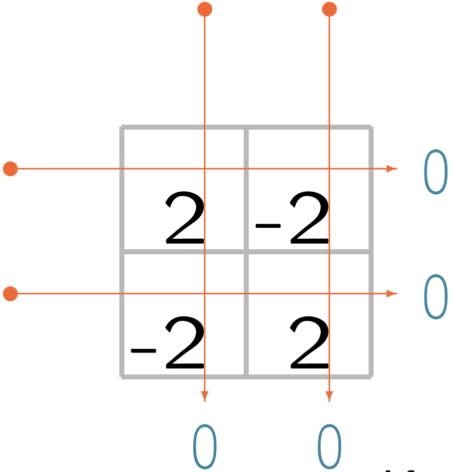
Writing the data penalty in matrix form



$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 4 \\ 13 \end{bmatrix}$$

$$Ax = m$$

Kernel of matrix A is the origin of ghosts



$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Ker}(A) = \left\{ \begin{bmatrix} t \\ -t \\ t \\ -t \end{bmatrix} \ : \ t \in \mathbb{R} \right\}$$

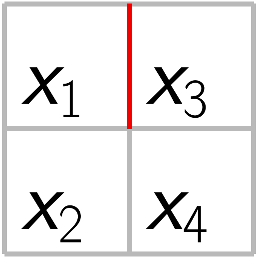
Writing the data penalty in matrix form

Recall that for a vector $y \in \mathbb{R}^n$ we have $\|y\|_2^2 = y_1^2 + y_2^2 + \cdots + y_n^2$.

Therefore, the data penalty can be written as

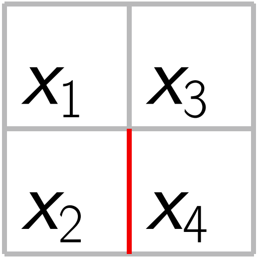
$$\begin{aligned}\|Ax - m\|_2^2 \;=\; & (x_1 + x_3 - 8)^2 + (x_2 + x_4 - 9)^2 + \\ & (x_1 + x_2 - 4)^2 + (x_3 + x_4 - 13)^2.\end{aligned}$$

Writing the prior penalty in matrix form: construction of the horizontal difference matrix L_H



$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ \\ \end{bmatrix}$$

Writing the prior penalty in matrix form: construction of the horizontal difference matrix L_H



$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_4 \end{bmatrix}$$

Writing the prior penalty in matrix form: construction of the horizontal difference matrix L_H

x_1	x_3
x_2	x_4

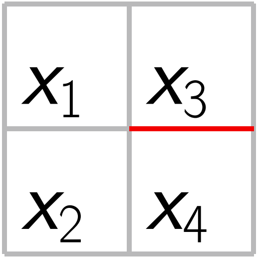
$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}}_{L_H} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_4 \end{bmatrix}$$

Writing the prior penalty in matrix form: construction of the vertical difference matrix L_V

x_1	x_3
x_2	x_4

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ \\ \end{bmatrix}$$

Writing the prior penalty in matrix form: construction of the vertical difference matrix L_V



$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_3 - x_4 \end{bmatrix}$$

Writing the prior penalty in matrix form: construction of the vertical difference matrix L_V

x_1	x_3
x_2	x_4

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{L_V} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_3 - x_4 \end{bmatrix}$$

Writing the prior penalty in matrix form

Recall that for a vector $y \in \mathbb{R}^n$ we have $\|y\|_1 = |y_1| + |y_2| + \cdots + |y_n|$.

Therefore, the prior penalty can be written as

$$\begin{aligned} \|L_H x\|_1 + \|L_V x\|_1 &= |x_1 - x_3| + |x_2 - x_4| \\ &\quad + |x_1 - x_2| + |x_3 - x_4|. \end{aligned}$$

Quadratic programming for TV regularization

We want to minimize the non-quadratic functional

$$\|Ax - m\|_2^2 + \|L_H x\|_1 + \|L_V x\|_1$$

over non-negative image vectors $x \in \mathbb{R}^4$. This task can be converted into minimizing the quadratic functional

$$\frac{1}{2} z^T Q z + c^T z$$

over non-negative $z \in \mathbb{R}^{12}$ with equality constraints $Ez = b$.

Rewriting the TV regularization

Write the horizontal and vertical differences in the form

$$L_{\mathsf{H}}x = u_{\mathsf{H}}^{+} - u_{\mathsf{H}}^{-} \quad \text{and} \quad L_{\mathsf{V}}x = u_{\mathsf{V}}^{+} - u_{\mathsf{V}}^{-},$$

using **non-negative** vectors $u_{\mathsf{H}}^{\pm}, u_{\mathsf{V}}^{\pm} \in \mathbb{R}^2$.

Then TV regularization is equivalent to minimizing

$$x^T A^T A x - 2x^T A^T m + \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T (u_{\mathsf{H}}^{+} + u_{\mathsf{H}}^{-} + u_{\mathsf{V}}^{+} + u_{\mathsf{V}}^{-}),$$

over non-negative vectors $x \in \mathbb{R}^4$.

Reduction of TV regularization to the quadratic problem $\arg\min_{z \in \mathbb{R}^{12}_+} \left\{ \frac{1}{2} z^T Q z + c^T z \right\}$ with $Ez = b$

We now minimize over non-negative vectors $z \in \mathbb{R}^{12}$ connected to TV regularization by this formula:

$$z = \begin{bmatrix} x \\ u_{\mathrm{H}}^{+} \\ u_{\mathrm{H}}^{-} \\ u_{\mathrm{V}}^{+} \\ u_{\mathrm{V}}^{-} \end{bmatrix}.$$

The 12×12 matrix Q and the vector $c \in \mathbb{R}^{12}$ are these:

$$Q = \begin{bmatrix} 2A^T A & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & 0 \end{bmatrix}, \quad c = \begin{bmatrix} -2A^T m \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Reduction of TV regularization to the quadratic problem $\arg\min_{z\in\mathbb{R}^{12}_+} \left\{\frac{1}{2}z^TQz + c^Tz\right\}$ with $Ez = b$

The equality constraint $Ez = b$ is needed for enforcing the identities $L_{\mathsf{H}}x - u_{\mathsf{H}}^+ + u_{\mathsf{H}}^- = 0$ and $L_{\mathsf{V}}x - u_{\mathsf{V}}^+ + u_{\mathsf{V}}^- = 0$. Since

$$z = \begin{bmatrix} x \\ u_{\mathsf{H}}^+ \\ u_{\mathsf{H}}^- \\ u_{\mathsf{V}}^+ \\ u_{\mathsf{V}}^- \end{bmatrix} \in \mathbb{R}^{12},$$

we have

$$\begin{bmatrix} L_H & -I & I & 0 & 0 \\ L_V & 0 & 0 & -I & I \end{bmatrix} z = 0.$$

Reduction of TV regularization to the quadratic problem $\arg\min_{z\in\mathbb{R}^{12}_+}\left\{\frac{1}{2}z^TQz + c^Tz\right\}$ with $Ez = b$

Finally we get

$$E = \left[\begin{array}{cccc|cc|cc|cc|cc} 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

and

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Total variation regularization for the $2{\times}2$ problem

True target		Total variation regularization	
2	6	$2\frac{1}{4}$	$6\frac{1}{4}$
2	7	$2\frac{1}{4}$	$6\frac{1}{4}$

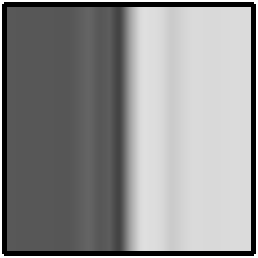
	True target		Total variation regularization
data penalty	0	data penalty	1
+ prior penalty	10	+ prior penalty	8
= total penalty	€10	= total penalty	€9

Total variation regularization for the 2×2 problem

True target

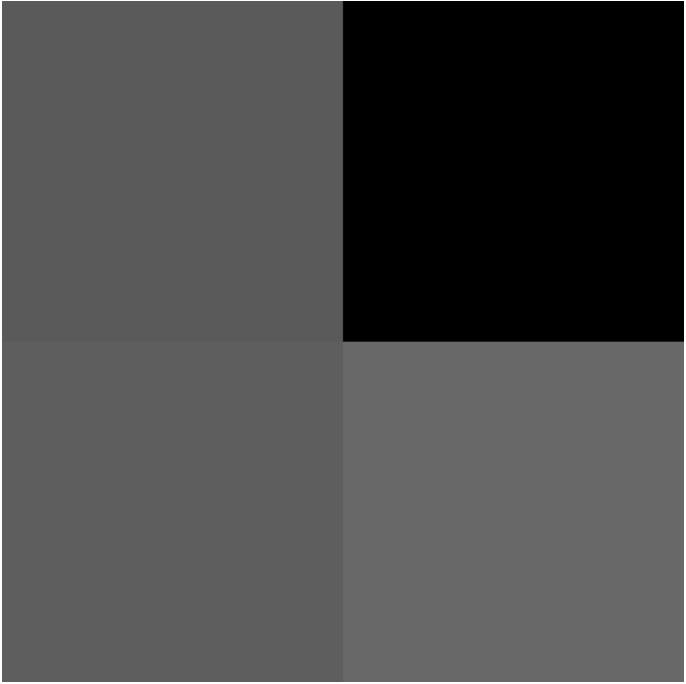
data penalty	0
+ prior penalty	10
= total penalty	€10

Total variation regularization

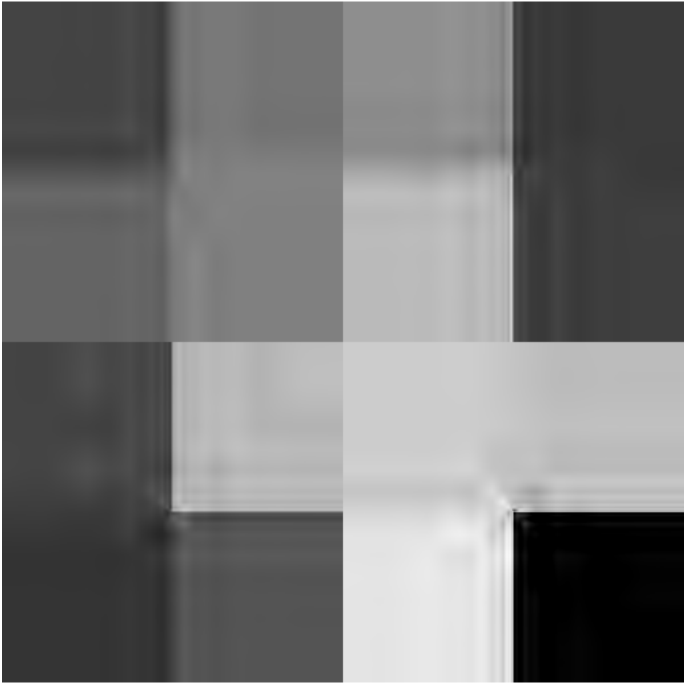


data penalty	1
+ prior penalty	8
= total penalty	€9

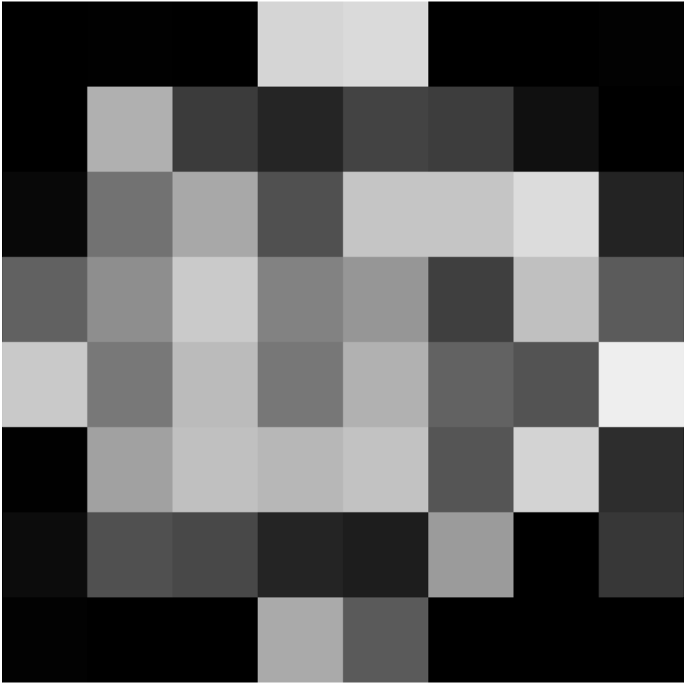
Resolution
2×2



Resolution
4×4

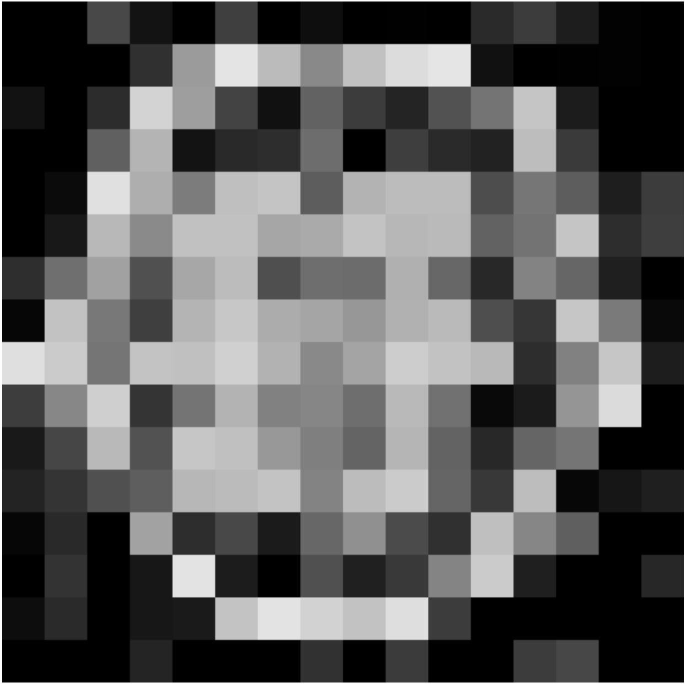


Resolution
8×8



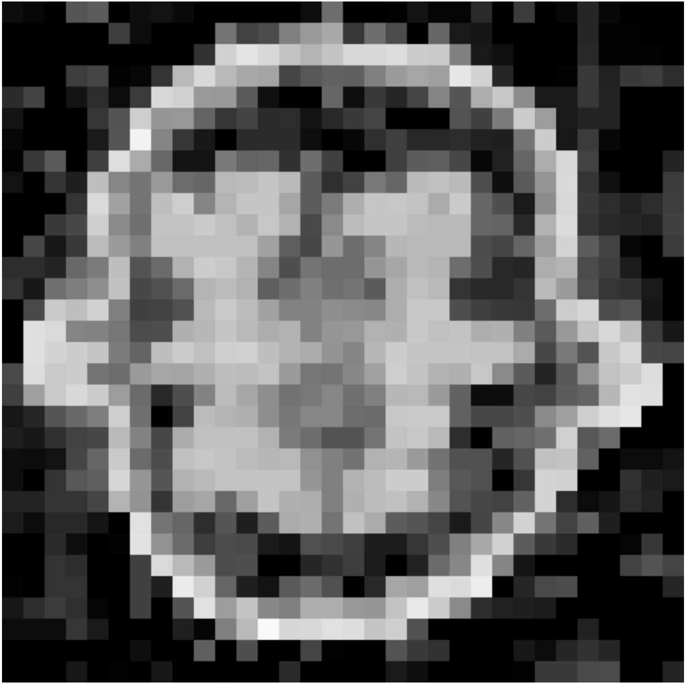
Resolution

16×16

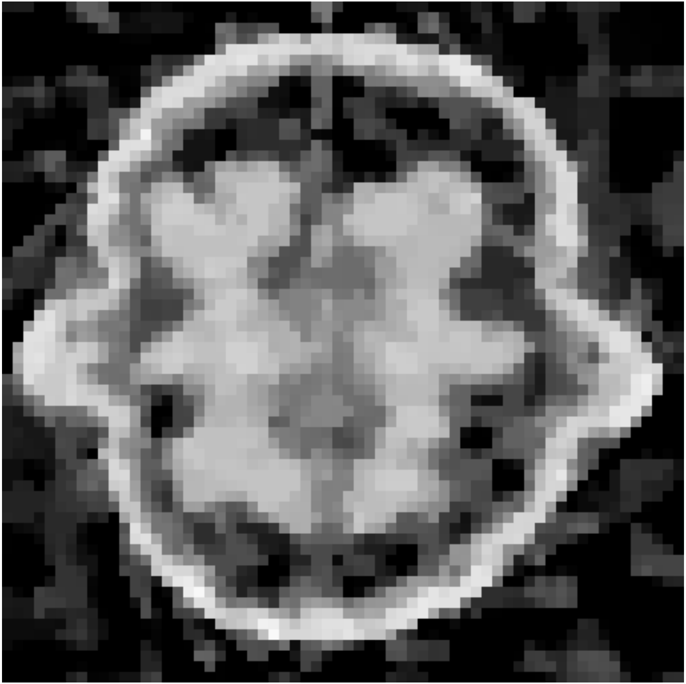


Resolution

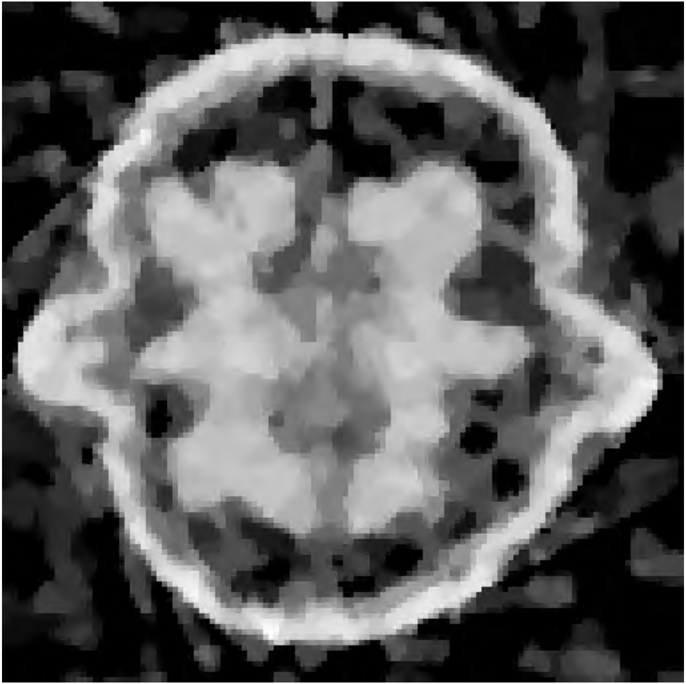
32×32



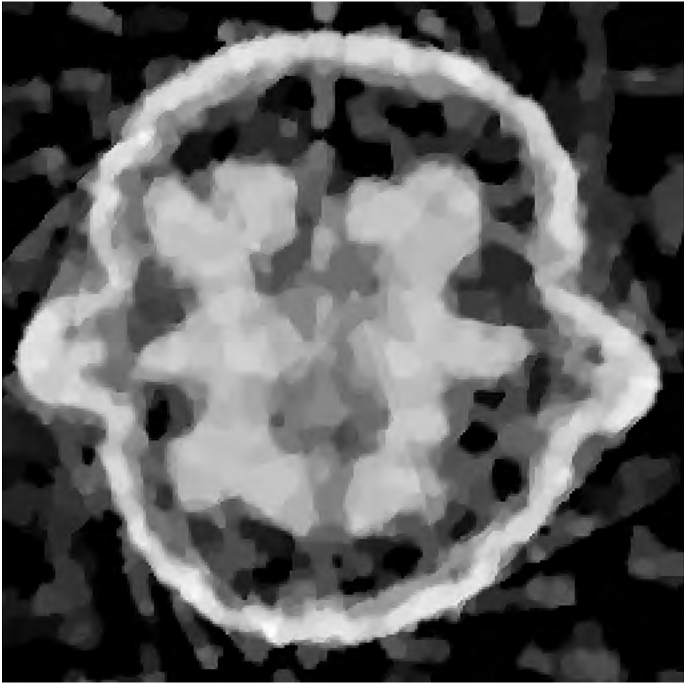
Resolution
64×64



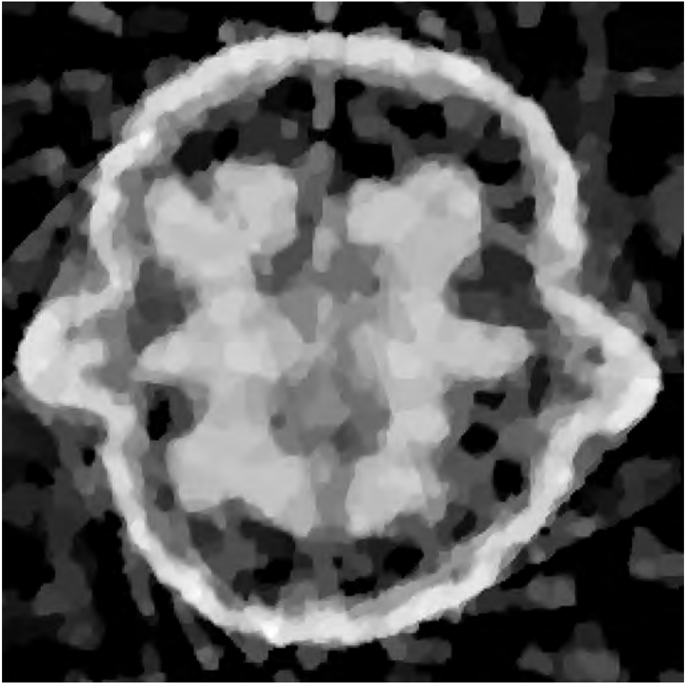
Resolution
128×128



Resolution
256×256



Resolution
512×512



Computational resources

You can find Matlab code for total variation regularized sparse tomography at
https://blog.fips.fi/tomography/x-ray/total-variation-regularization-for-x-ray-tomography/

Another computational method for the same problem is here:
https://blog.fips.fi/tomography/x-ray/total-variation-regularization-for-x-ray-tomography-experimental-data/

fips Finnish Inverse Problems Society