1. **Random numbers**
   The traditional random number generator (RNG) in computer science generates a sequence of the form

$$X_{n+1} = (aX_n + c) \mod k. \tag{1}$$

   (a) Verify the point made in the lecture, that such pseudo random numbers are highly correlated.

   (b) The performance of a bad RNG can be vastly improved by coupling two bad RNGs. We want to study this possibility. Run two differently seeded versions (try using different a, c, k also) of the above bad RNG. A random number from RNG one is only taken if RNG two produces a number which is a multiple of either $2, 3, 5, 7, 13$. Study the correlation of this new RNG.

2. **Monte Carlo (MC) Integration**
   We want to integrate the function

$$f(x,n) = \frac{n}{\sqrt{\pi}} e^{-x^2 n^2} , \tag{2}$$

   which parametrises the δ function in the limit of $n \to \infty$. First you have to estimate a good value for n under the premise that the integral over the interval $[-1, 1]$ reproduces the result within $\Delta = \pm 10^{-5}$. Your task then is to integrate the function $f(x, n)$ over the interval $[-1, 1]$ using MC integration. Using integration-by-rejection, integration-by-mean-value and integration-by-mean-value with a integral transformation. In order to be able to compare the results you should collect at least 1000 results by each method and plot the data using a histogram.

3. **Random Walk – Limp Darkening** In this exercise we will use a random walk method to model radiation in the photosphere of a star. We consider photons emitted from the inner boundary of the photosphere, which are scattered several times before they leave the photosphere through the inner or the outer boundary. We want to study the angular distribution of the photons that leave the star.

   - Instead of using the radius, it is more convenient to use the radial component of the optical depth $\tau^{(r)}$, which takes into account the changing opacity throughout the photosphere. The inner boundary is at the optical depth $\tau^{(r)} = \tau^{(r)}_{\max}$; the outer boundary is at $\tau^{(r)} = 0$.
   - Another useful quantity is $\mu = \cos\theta$ where θ is the angle between the radial axis and the direction of the photon.

- The optical path a photon travels before it is scattered is given by an exponential distribution
$$p(\tau)d\tau = e^{-\tau}d\tau. \tag{3}$$
Note that τ comprises the full optical path, not only the radial component.

- The scattering angle is isotropic.
$$p(\mu)d\mu = \begin{cases} \frac{1}{2}d\mu, & \text{for } \mu \in [-1,1] \\ 0, & \text{for } \mu \notin [-1,1] \end{cases} \tag{4}$$
(Can you explain why the distribution of μ is uniform instead of the distribution of the angle θ?)

- For the initial emission at the inner boundary, we assume the following angular distribution:
$$p(\mu)d\mu = \begin{cases} 2\mu d\mu, & \text{for } \mu \in [0,1] \\ 0, & \text{for } \mu \notin [0,1] \end{cases} \tag{5}$$

- The photosphere is a relatively thin region compared to the radius of the star. This justifies the plane-parallel approximation. That means the problem is essentially one-dimensional, i.e. we only have keep track of the radial component of the optical depth $\tau^{(r)}$.

- We work under the assumption of a so-called *grey atmosphere*, which means that the absorption and emission processes are independent of the energy of the photon and we can leave out energy in our model.

(a) Find a way to transform the uniform probability distribution that you get from the random number generator (RNG) into the distributions needed for this model. (In case you haven't implemented your own RNG yet, have a look at the selection offered by the GNU Scientific Library.)

(b) Given a random value for μ, how do you calculate the new value for τ.

(c) Implement the random walk of a single photon. Start at the optical depth $\tau_{\max}^{(r)} = 10$ and continue until you reach $\tau^{(r)} = 0$. Should the photon leave the photosphere through the inner boundary, simply restart the random walk with the emission of a new photon. Record the final value of μ.

(d) Modify your program so that it computes the random walk of a large number of photons (10^5 is a good trade-off between speed and accuracy). Acquire the distribution of the final values of μ. Instead of saving every single value, you can use bins, i.e. count the number of results in the intervals,
$$[0, \Delta\mu), [\Delta\mu, 2\Delta\mu), \cdots, [(n-1)\cdot\Delta\mu, 1), \tag{6}$$
where n is the number of bins.
Plot the results, normalised to the value at $\mu = 1$.

(e) So far, we have only calculated the distribution of the number of photons, not the radiation intensity. To get the intensity we can use the formula: $N(\mu)d\mu \propto I(\mu)\mu d\mu$. Plot $I(\mu)$, again normalised to the value of I at $\mu = 1$. Compare your result to the approximate expression:

$$I(\mu) \approx I(1)(0.4 + 0.6\mu) \tag{7}$$

This means that a star looks brighter at its centre than at its edge (also called *limb* in astrophysics). This effect is called *limb darkening*.