Orders execution.

We are interested in markets with a single order book. Let $(A_i)_{i\in\mathbb{N}^*}$ be the set of agents. We call order a quadruplet of the form $o = (o_A, o_d, o_p, o_q)$ with $o_A \in (A_i)_{i\in\mathbb{N}^*},\ o_d \in \{\text{ask}, \text{bid}\},\ o_p > 0$ and $o_q > 0$. Ω denotes the set of orders and Ω_n denotes the set of parts $\mathcal{O}$ from Ω with n elements such that $\forall o \neq o' \in \mathcal{O},\ o_A \neq o'_A$. We also note w_i the wealth of agent i, $c_i \geqslant 0$ its initial cash and $n_i \geqslant 0$ its initial assets. It is also assumed that agents cannot have negative cash or assets.

If W is a welfare function taking as input the w_i, we can define $\tilde{W}$ taking as input the $(c_i)_{1\leqslant i\leqslant n}$, the $(n_i)_{1\leqslant i\leqslant n}$ and a sequence of orders $\mathcal{O} = (o_1, \ldots, o_n)$ and returning welfare $\tilde{W}((c_i), n_i)_{1\leqslant i\leqslant n}, \mathcal{O})$ after executing the $\mathcal{O}$ order sequence on the market initialised with an empty order book and whose agents are initialised with the initial conditions $CI = (c_i, n_i)_{1\leqslant i\leqslant n}$. Note $W_{CI} : \mathcal{O} \mapsto \tilde{W}(CI, \mathcal{O})$.

One wonders whether, with W fixed, there is a total order relationship $\preceq$ on the Ω set of possible orders such that for any finite subset $\mathcal{O} = \{o_1, \ldots, o_n\} \in \Omega_n$, for all initial conditions $CI \in \mathbb{N}^{2n}$, the sequence $(o_{\sigma(1)}, \ldots, o_{\sigma(n)})$ such that $o_{\sigma(1)} \preceq \ldots \preceq o_{\sigma(n)}$ maximises the welfare, i.e. :

$$W_{CI}(o_{\sigma(1)}, \ldots, o_{\sigma(n)}) = \max_{\tau\in\mathfrak{S}_n} W_{CI}(o_{\tau(1)}, \ldots, o_{\tau(n)})$$

Note: There is generally no single sequence maximising this welfare. We just require the one sorted by $\preceq$ to be one of them.

We note $\mathfrak{S}_{\mathcal{O}}$ the set of sequences whose elements are exactly the elements of $\mathcal{O}$ and W_u the utilitarian welfare, $W_{\min}$ the min welfare, $W_{\max}$ the max welfare and W_N the Nash welfare. We give ourselves $p_0 \geqslant 0$: this is the initial price given to the assets when no price has yet been set.

1 Two-order sequences

We will show the following (simple) result:

Property 1.1

If $\mathcal{O} \in \Omega_2$, there is a $s \in \mathfrak{S}_{\mathcal{O}}$ sequence maximising at once W_u, W_N, $W_{\min}$ and $W_{\max}$ regardless of the initial conditions.

Proof: If both orders have the same direction, or if one is an ask and the other is a bid with $p_{\text{ask}} > p_{\text{bid}}$, then no price is set and the result is trivial.
In the case where $\mathcal{O} = \{o_1 = (A_1, \text{ask}, p_a, q_a), o_2 = (A_2, \text{bid}, p_b, q_b)\}$ with $p_b \geqslant p_a$, let's note $q = \min(q_a, q_b)$.
Regardless of the sequence of execution, a price $p \in \{p_a, p_b\}$ will be set and a quantity q will be traded. So we will have $w_1 = (c_1 + qp) + (n_1 - q)p = c_1 + n_1 p$ and $w_2 = (c_2 - qp) + (n_2 + q)p = c_2 + n_2 p$, where $W_u = c_1 + c_2 + (n_1 + n_2)p$, $W_N = (c_1 + n_1 p)(c_2 + n_2 p)$, $W_{\min} = \min_i(c_i + n_i p)$ and $W_{\max} = \max_i(c_i + n_i p)$. All these quantities being increasing according to p, they are all maximised for the sequence (o_2, o_1) since the fixed price will be $p_b \geqslant p_a$. ■

By the way, we can show the following result:

Property 1.2

If $\mathcal{O}$ consists of an ask order o_a and a bid order o_b, then :

- If $p_a \geqslant p_b$, then the final welfare doesn't depend on the sequence of execution.
- If $p_a < p_b$, then $W_u(o_b, o_a) > W_u(o_a, o_b)$ and $W_N(o_b, o_a) > W_N(o_a, o_b)$
- If $p_a < p_b$, then $W_{\min}(o_b, o_a) \geqslant W_{\min}(o_a, o_b)$ with a tie when $n_b = 0$ and $c_b \leqslant c_a + n_a p_a$.
- If $p_a < p_b$, then $W_{\max}(o_b, o_a) \geqslant W_{\max}(o_a, o_b)$ with equal outcome when $n_b = 0$ and $c_b \geqslant c_a + n_a p_b$.

Proof: The first point is trivial and the second point is proven by noticing that $n_a > 0$.
The third point is longer to prove: inequality is obvious but the case of equality is not: we have to study when we have $\min(c_a + n_a p_a, c_b + n_b p_a) = \min(c_a + n_a p_b, c_b + n_b p_b)$. As $p_a < p_b$, having $n_b = 0$ is a necessary condition, which brings us back to study when $\min(c_a + n_a p_a, c_b) = \min(c_a + n_a p_b, c_b)$ which is true iff $c_b \leqslant c_a + n_a p_a$. It is then enough to check that the reciprocal (if $n_b = 0$ and $c_b \leqslant c_a + n_a p_a$ then we have equality) is true.
The fourth point can be treated as the third. ∎

We notice in the passage that we can't have both the tie case for $W_{\min}$ and the tie case for $W_{\max}$ when $p_a < p_b$.

2 Interlude: A useful lemma

We write down $\text{argsmax}_{x \in X} f(x)$ the set $\{x \in X, f(x) = \max_{y \in X} f(y)\}$.

Lemma 2.1

Let's call it $n \geqslant 2$. Let $\mathcal{O} = \{o_1, \ldots, o_n\} \in \Omega_n$. Let $\mathcal{O}'$ and $\mathcal{O}''$ be two subsets of $\mathcal{O}'$ of zero intersection. Let W be fixed. If there exists $CI \in \mathbb{N}^{2n}$ such that , by noting $S = \text{argsmax}_{s \in \mathfrak{S}_{\mathcal{O}}} W_{CI}(s)$, we have for all $s \in S$ the existence of $o_i \in \mathcal{O}'$ and $o_j \in \mathcal{O}''$ such that the order o_i appears before o_j in the sequence s, so if $\preceq$ exists, there's $o_i \in \mathcal{O}'$ and $o_j \in \mathcal{O}'$ such that $o_i \preceq o_j$.

Proof: Under the assumption that $\preceq$ exists, the sequence $s = (o_{\sigma(1)} \preceq \ldots \preceq o_{\sigma(n)})$ maximises W welfare for all initial conditions, so for CI in particular. So $s \in S$. Hypothetically, there is $o_i \in \mathcal{O}'$ and $o_j \in \mathcal{O}''$ such that o_i appears before o_j in s. So, by definition of s, $o_i \preceq o_j$ ∎

From this lemma and the property 1.2 we can deduce that :

Property 2.2

If $o_a, o_b \in \Omega^2$ with $o_{a,d} = \text{ask}$, $o_{b,d} = \text{bid}$ and $o_{a,p} < o_{b,p}$, then $o_b \preceq o_a$, if $W = W_u$, W_N, $W_{\min}$ or $W_{\max}$ and if $\preceq$ exists.

3 Non-existence of $\preceq$ for the usual welfare

We'll use these previous results to show that there is no $\preceq$ order on Ω verifying the desired property.

3.1 Minimum welfare and welfare of Nash

Theorem 3.1

Either $W = W_{\min}$ or W_N.
There's no such thing as a total order $\preceq$ in Ω such that for any set $\mathcal{O} = \{o_1, \ldots, o_n\} \in \Omega_n$, for all initial conditions $CI \in \mathbb{N}^{2n}$, the sequence $(o_{\sigma(1)}, \ldots, o_{\sigma(n)})$ such that $o_{\sigma(1)} \preceq \ldots \preceq o_{\sigma(n)}$ is one of the $\mathcal{O}$ sequences maximising welfare W.

Proof:

1st step : It exists $\mathcal{O} \in \Omega_3$ made up of two asks orders o_1 and o_2 and a bid order o_3 whose price is higher than those of the orders asks such that nor o_3, nor (o_3, o_2, o_1) nor (o_3, o_2, o_1) maximises welfare.
Let $\mathcal{O} = \{o_1 = (A_1, \text{ask}, p_1, q_1), o_2 = (A_2, \text{ask}, p_2, q_2), o_3 = (A_3, \text{bid}, p_3, q_3)\} \in \Omega_3$
and $CI = (c_i, n_i)_{1 \leqslant i \leqslant 3} \in \mathbb{N}^6$ such that :

1. $p_1 < p_3$
2. $p_2 < p_3$
3. $q_1 \leqslant n_1$
4. $q_2 \leqslant n_2$
5. $q_2 < q_3$
6. $q_3 - q_2 < q_1 < q_3$
7. $c_3 + n_3 p_3 < c_1 + n_1 p_3 < \min(c_2 + q_2 p_2 + n_2 p_3 - q_2 p_3, c_3 - q_2 p_2 + q_2 p_3 + n_3 p_3)$
8. $q_2(p_3 - p_2) < (c_2 + n_2 p_3) - (c_3 + n_3 p_3)$

The existence of such $\mathcal{O}$ and CI is not self-evident: an example is given in appendix A.1.
The following table shows the cash and the number of assets held by each agent after the execution of different sequences. The last fixed price is, in all cases, p_3. The details are given in the appendix A.2.

Sequence	A_1		A_2		A_3	
	cash	assets	cash	assets	cash	assets
(o_2, o_3, o_1)	$c_1 + (q_3 - q_2)p_3$	$n_1 - (q_3 - q_2)$	$c_2 + q_2p_2$	$n_2 - q_2$	$c_3 - q_2p_2$ $-(q_3 - q_2)p_3$	$n_3 + q_3$
(o_3, o_1, o_2)	$c_1 + q_1p_3$	$n_1 - q_1$	$c_2 + (q_3 - q_1)p_3$	$n_2 - (q_3 - q_1)$	$c_3 - q_3p_3$	$n_3 + q_3$
(o_3, o_2, o_1)	$c_1 + (q_3 - q_2)p_3$	$n_1 - (q_3 - q_2)$	$c_2 + q_2p_3$	$n_2 - q_2$	$c_3 - q_3p_3$	$n_3 + q_3$

So, it just comes that :

Sequence	w_1	w_2	w_3
(o_2, o_3, o_1)	$c_1 + n_1p_3$	$c_2 + q_2p_2 + (n_2 - q_2)p_3$	$c_3 + q_2(p_3 - p_2) + n_3p_3$
(o_3, o_1, o_2) (o_3, o_2, o_1)	$c_1 + n_1p_3$	$c_2 + n_2p_3$	$c_3 + n_3p_3$

Let $(o_3, \cdot, \cdot)$ be the sequences (o_3, o_1, o_2) and (o_3, o_2, o_1).
We conclude that $W_{\min,CI}(o_3, \cdot, \cdot) = \min(c_1 + n_1p_3,\ c_2 + n_2p_3,\ c_3 + n_3p_3) = c_3 + n_3p_3$ according to (7), (8) and (2).
Moreover, $W_{\min,CI}(o_2, o_3, o_1) = \min(c_1 + n_1p_3,\ c_2 + q_2p_2 + (n_2 - q_2)p_3,\ c_3 + q_2(p_3 - p_2) + n_3p_3) = c_1 + n_1p_3$ according to (7). Finally, (7) also gives that $\boldsymbol{W_{\min,CI}(o_2, o_3, o_1) = c_1 + n_1p_3 > c_3 + n_3p_3 = W_{\min,CI}(o_3, \cdot)}$.
We've made it clear what we want for $W_{\min,CI}$. Let's do the same for $W_{\mathbb{N},CI}$:

$W_{N,CI}(o_3, \cdot, \cdot) = (c_1 + n_1p_3)(c_2 + n_2p_3)(c_3 + n_3p_3)$ and
$W_{N,CI}(o_2, o_3, o_1) = (c_1 + n_1p_3)(c_2 + q_2p_2 + (n_2 - q_2)p_3)(c_3 + q_2(p_3 - p_2) + n_3p_3)$, hence

$$\begin{aligned}\frac{W_{N,CI}(o_2, o_3, o_1)}{W_{N,CI}(o_3, \cdot, \cdot)} &= \frac{(c_2 + q_2p_2 + (n_2 - q_2)p_3)(c_3 + q_2(p_3 - p_2) + n_3p_3)}{(c_2 + n_2p_3)(c_3 + n_3p_3)} \\ &= \dots \\ &= 1 + \frac{q_2(p_3 - p_2)[(c_2 + n_2p_3) - (c_3 + n_3p_3) - q_2(p_3 - p_2)]}{(c_2 + n_2p_3)(c_3 + n_3p_3)}\end{aligned}$$

with $q_2 \underbrace{(p_3 - p_2)}_{>0 \text{ as per } (2)} \underbrace{[(c_2 + n_2p_3) - (c_3 + n_3p_3) - q_2(p_3 - p_2)]}_{>0 \text{ as per } (8)} > 0$.
So $\boldsymbol{W_{N,CI}(o_2, o_3, o_1)} > \boldsymbol{W_{N,CI}(o_3, \cdot, \cdot)}$.

2nd step : Conclusion
If by any chance there was an order relationship $\preceq_{\min}$ (resp. $\preceq_{\mathbb{N}}$) on Ω such that for any set $\mathcal{O}' = \{o'_1, \dots, o'_n\} \in \Omega_n$, for all initial conditions $CI' \in \mathbb{N}^{2n}$, the sequence $(o'_{\sigma(1)}, \dots, o'_{\sigma(n)})$ such that $o'_{\sigma(1)} \preceq \dots \preceq o'_{\sigma(n)}$ is one of the $\mathcal{O}'$ sequences maximising welfare $W_{\min,CI'}$, so:
As o_3 is a bid and as o_1, o_2 are asks such that $p_1 < p_3$ (1) and $p_2 < p_3$ (2), then from the property 2.2 comes the fact that $o_3 \preceq_{\min} o_1$, $o_3 \preceq_{\min} o_2$, $o_3 \preceq_N o_1$ and $o_3 \preceq_N o_2$. These inequalities are even strict because o_1, o_2 and o_3 are distinct. o_3 is therefore both the minimum of $\mathcal{O}$ for $\preceq_{\min}$ and for $\preceq_N$. The $\preceq_{\min}$ and $\preceq_N$ sequences of $\mathcal{O}$ ordered respectively according to $\preceq_{\min}$ and $\preceq_N$ begin both with o_3 i.e. are of the form $(o_3, \cdot)$. By hypothesis, $s_{\min}$ (resp. s_N) is thus a point at which $W_{\min,CI}$ (resp. $W_{N,CI}$) reaches its maximum, which contradicts the results of the first step. ■

3.2 Maximum welfare

Theorem 3.2

> There's no such thing as a total order of $\preceq$ on Ω such that for any set $\mathcal{O} = \{o_1, \dots, o_n\} \in \Omega_n$, for all initial conditions $CI \in \mathbb{N}^{2n}$, the sequence $(o_{\sigma(1)}, \dots, o_{\sigma(n)})$ such that $o_{\sigma(1)} \preceq \dots \preceq o_{\sigma(n)}$ is one of the $\mathcal{O}$ sequences maximising the welfare $W_{\max,CI}$.

Proof: The evidence is identical in all respects to the evidence for $W_{\min}$: the second step is identical, so we will just specify the first step.
1st step: There is $\boldsymbol{\mathcal{O} \in \Omega_3}$ made up of two orders asks $\boldsymbol{o_1}$ and $\boldsymbol{o_2}$ and a bid order $\boldsymbol{o_3}$ whose price is higher than those of the orders asks such that nor $\boldsymbol{o_3}$, nor $\boldsymbol{(o_3, o_2, o_1)}$ nor $\boldsymbol{(o_3, o_2, o_1)}$ maximises welfare.

Let $\mathcal{O} = \{o_1 = (A_1, \text{ask}, p_1, q_1), o_2 = (A_2, \text{ask}, p_2, q_2), o_3 = (A_3, \text{bid}, p_3, q_3)\} \in \Omega_3$ and let $CI = (c_i, n_i)_{1 \leqslant i \leqslant 3} \in \mathbb{N}^6$ such that :

1. $p_1 < p_3$
2. $p_2 < p_3$
3. $q_1 \leqslant n_1$
4. $q_2 \leqslant n_2$
5. $q_1 < q_3 < q_2$
6. $q_3 p_3 < c_3$
7. $\max(c_1 + q_1 p_1 + (n_1 - q_1) p_3,\ c_2 + n_2 p_3) < c_3 + q_1(p_3 - p_1) + n_3 p_3$
8. $\max(c_1 + n_1 p_3,\ c_2 + n_2 p_3) < c_3 + n_3 p_3$

An example is provided in Appendix 3. The following table shows the cash and the number of assets held by each agent after the execution of different sequences. The last fixed price is, in all cases, p_3. The details are given in the appendix B.2.

Sequence	A_1		A_2		A_3	
	cash	assets	cash	assets	cash	assets
(o_1, o_3, o_2)	$c_1 + q_1 p_1$	$n_1 - q_1$	$c_2 + (q_3 - q_1) p_3$	$n_2 - (q_3 - q_1)$	$c_1 - q_1 p_1$ $-(q_3 - q_1) p_3$	$n_3 + q_3$
(o_3, o_1, o_2)	$c_1 + q_1 p_3$	$n_1 - q_1$	$c_2 + (q_3 - q_1) p_3$	$n_2 - (q_3 - q_1)$	$c_3 - q_3 p_3$	$n_3 + q_3$
(o_3, o_2, o_1)	c_1	n_1	$c_2 + q_3 p_3$	$n_2 - q_3$	$c_3 - q_3 p_3$	$n_3 + q_3$

So, it turns out that :

Sequence	w_1	w_2	w_3
(o_1, o_3, o_2)	$c_1 + q_1 p_1 + (n_1 - q_1) p_3$	$c_2 + n_2 p_3$	$c_3 + q_1(p_3 - p_1) + n_3 p_3$
(o_3, o_1, o_2) (o_3, o_2, o_1)	$c_1 + n_1 p_3$	$c_2 + n_2 p_3$	$c_3 + n_3 p_3$

Denote as $(o_3, \cdot, \cdot)$ the sequences (o_3, o_1, o_2) and (o_3, o_2, o_1).
We have $W_{\max,CI}(o_3, \cdot, \cdot) = \max(c_1 + n_1 p_3,\ c_2 + n_2 p_3,\ c_3 + n_3 p_3) = c_3 + n_3 p_3$ according to (8).
And $W_{\max,CI}(o_1, o_3, o_2) = \max(c_1 + q_1 p_1 + (n_1 - q_1) p_3,\ c_2 + n_2 p_3,\ c_3 + q_1(p_3 - p_1) + n_3 p_3) = c_3 + q_1(p_3 - p_1) + n_3 p_3$ according to (7),
hence $\boldsymbol{W_{\max,CI}(o_3, \cdot, \cdot) = c_3 + n_3 p_3 < c_3 + q_1(p_3 - p_1) + n_3 p_3 = W_{\max,CI}(o_1, o_3, o_2)}$ according to (1). ■

3.3 Utility welfare

Note in this part the initial conditions $CI = (c_i(0), n_i(0))_{1 \leqslant i \leqslant n}$. We call $t + 1$ the time instant at which the first order is placed or the first price is set from instant t.

Lemma 3.3

Let CI be fixed.
There is $C \geqslant 0$ and $N \geqslant 0$ such that $\forall \mathcal{O} = \{o_1, \dots, o_n\} \in \Omega_n, \forall m \leqslant n$, $W_{u,CI}(o_1), \dots, o_m) = C + N p_m$ where p_m is the last order set after execution of the sequence $(o_1, \dots, o_m)$.

Proof: Let $t \geqslant 0$.
If the time $t+1$ was fixed by a placed order, we trivially have $\sum_{1 \leqslant i \leqslant n} c_i(t+1) = \sum_{1 \leqslant i \leqslant n} c_i(t)$ and $\sum_{1 \leqslant i \leqslant n} n_i(t+1) = \sum_{1 \leqslant i \leqslant n} n_i(t)$.
If $t + 1$ was fixed by a price that was fixed, note A_i the agent from which the ask order originated, A_j the agent from which the bid originated, q the quantity traded and p the fixed price. We have $\forall k \notin \{i, j\}, c_k(t+1) = c_k(t)$ and $n_k(t+1) = n_k(t)$ and :
$c_i(t+1) = c_i(t) + qp$, $n_i(t+1) = n_i(t) - q$, $c_j(t+1) = c_j(t) - qp$ and $n_j(t+1) = n_j(t) + q$.
So $\sum_{1 \leqslant i \leqslant n} c_i(t+1) = \sum_{1 \leqslant i \leqslant n} c_i(t)$ and $\sum_{1 \leqslant i \leqslant n} n_i(t+1) = \sum_{1 \leqslant i \leqslant n} n_i(t)$ in all cases: $\sum_i c_i$ and $\sum_i n_i$ are constant.
They are noted as C and N respectively. Then $W_{u,CI}(t) = \sum_i w_i(t) = \sum_i c_i(t) + n_i(t) p = C + Np$ with p the last fixed price. ■

In fact, it can be shown that there are no counter-examples with 3 orders, but there are counter-examples with 4 orders.

4 Leximin welfare

4.1 Formalism

Definition 4.1 (*Pre-order total leximin*)

> Be $n \in \mathbb{N}^*$ and $x, y \in \mathbb{R}^n$. We define the total pre-order leximin $\preceq_{\mathrm{Lm}}$ by $x \preceq_{\mathrm{Lm}} y$ iff, noting $\sigma, \tau \in \mathfrak{S}_n$ of the permutations such that $x_{\sigma(1)} \leqslant \ldots \leqslant x_{\sigma(n)}$ and $y_{\tau(1)} \leqslant \ldots \leqslant y_{\tau(n)}$, there is $k \in [\![1, n+1]\!]$ such that $\forall i < k, x_{\sigma(i)} = y_{\tau(i)}$ and, if $k \leqslant n$, $x_{\sigma(k)} < y_{\tau(k)}$.
> Informally, x is smaller than y when the sequence of x coordinates sorted in ascending order is lexicographically smaller than the sequence of y coordinates sorted the same way in ascending order.

We then divide $\mathbb{R}^n$ by the equivalence relation $\sim$ defined by $x \sim y$ when $x \preceq_{\mathrm{Lm}} y$ and $y \preceq_{\mathrm{Lm}} x$ (i.e. when the coordinates of x are a permutation of the coordinates of y). The relation $\precsim_{\mathrm{Lm}}$ on $\mathbb{R}^n / \sim$ defined by $X \precsim_{\mathrm{Lm}} Y$ when $\forall x \in X, \forall y \in Y, x \preceq_{\mathrm{Lm}} y$ is then a relation of total order [1] on $\mathbb{R}^n / \sim$. For a finite part P of $\mathbb{R}^n$, we then define $\max_{\precsim_{\mathrm{Lm}}} P$ as being equal to the intersection between P and the maximum of $\{\bar{x}, x \in P\}$ for $\precsim_{\mathrm{Lm}}$, where $\bar{x}$ designates the equivalence class of x by $\sim$.

One wonders whether, by calling welfare leximin W the identity function defined on the union of R^n with $n \in \mathbb{N}^*$, there is a total order $\preceq$ on Ω such that for any n, for any $\mathcal{O} = \{o_1, \ldots, o_n\} \in \Omega_n$, for all initial conditions $CI \in \mathbb{N}^{2n}$, we have $W(o_{\sigma(1)}, \ldots, o_{\sigma(n)}) \in \max_{\precsim_{\mathrm{Lm}}, \tau \mathfrak{S}_n} W(o_{\tau(1)}, \ldots, o_{\tau(n)})$ with $\sigma \in \mathfrak{S}_n$ such that $o_{\sigma(1)} \preceq \ldots \preceq o_{\sigma(n)}$.

The answer is no: just take the counter-example of $W_{\min}$ (and the property 1.2 remains true).

[1] Bourbaki, Éléments de mathématique : Théorie des ensembles, Paris, Masson, 1998, ch III, §1, n$^o 2, p3$

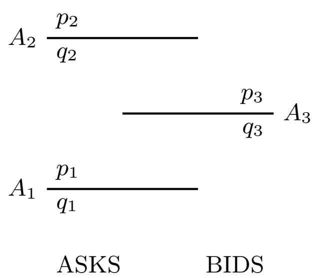
A Details of the proof of the theorem 3.1

A.1 Example of $\mathcal{O}$ and CI sets testing assumptions

i	p_i	q_i	c_i	n_i
1	1463	3	20932	4
2	1248	4	45856	24
3	5528	6	12339	5

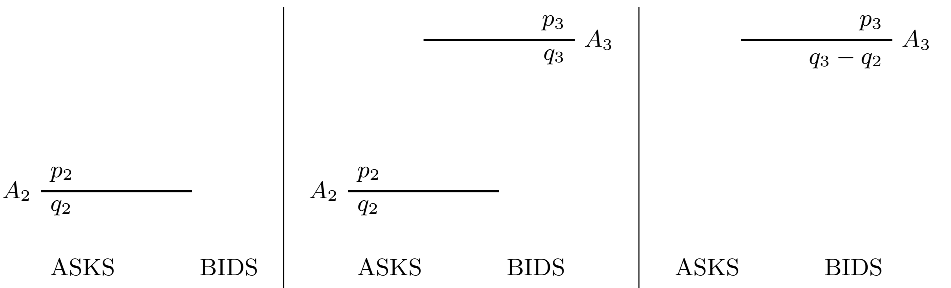
A.2 Proof of the values obtained for cash and assets after the execution of different sequences

An order book is represented as follows:

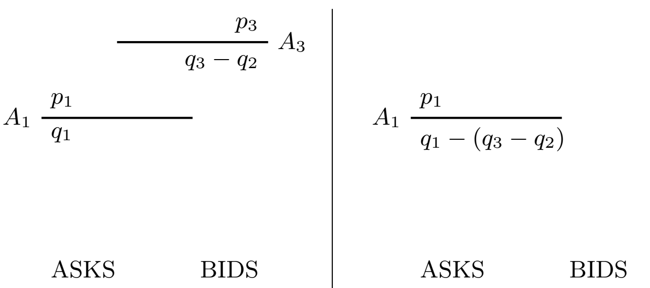


where, in this example, A_i is an agent having placed an order at price p_i for a quantity q_i. Orders 1 and 2 are asks, order 3 is a bid, and the vertical scale is the price scale: $p_1 < p_3 < p_2$.

A.2.1 Sequence (o_2, o_3, o_1)

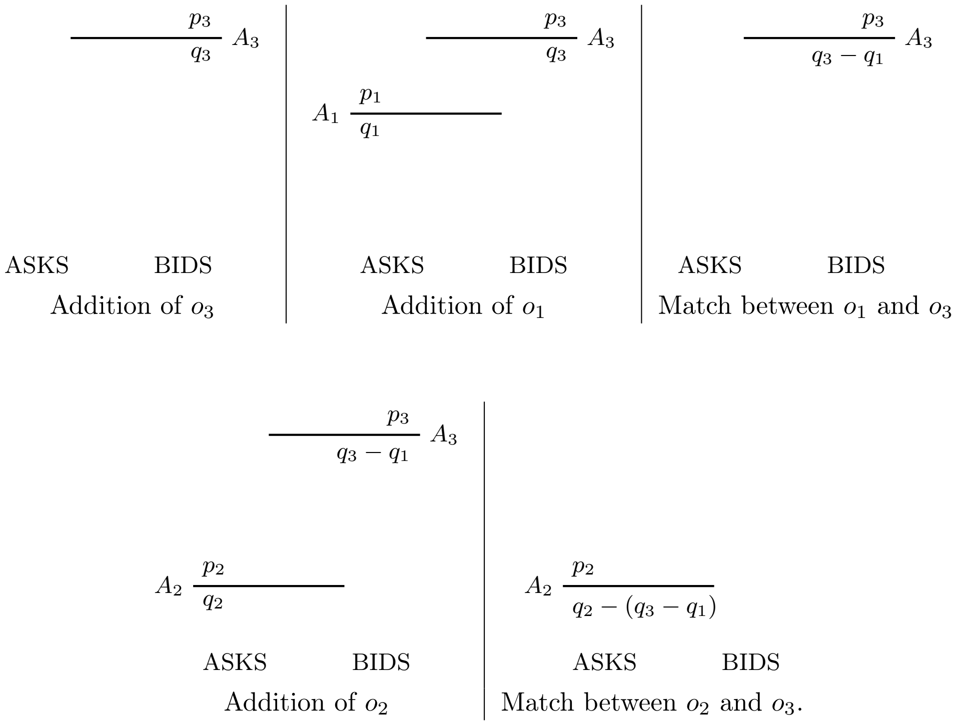


Addition of o_2 | Addition of o_3 | Match between o_2 and o_3

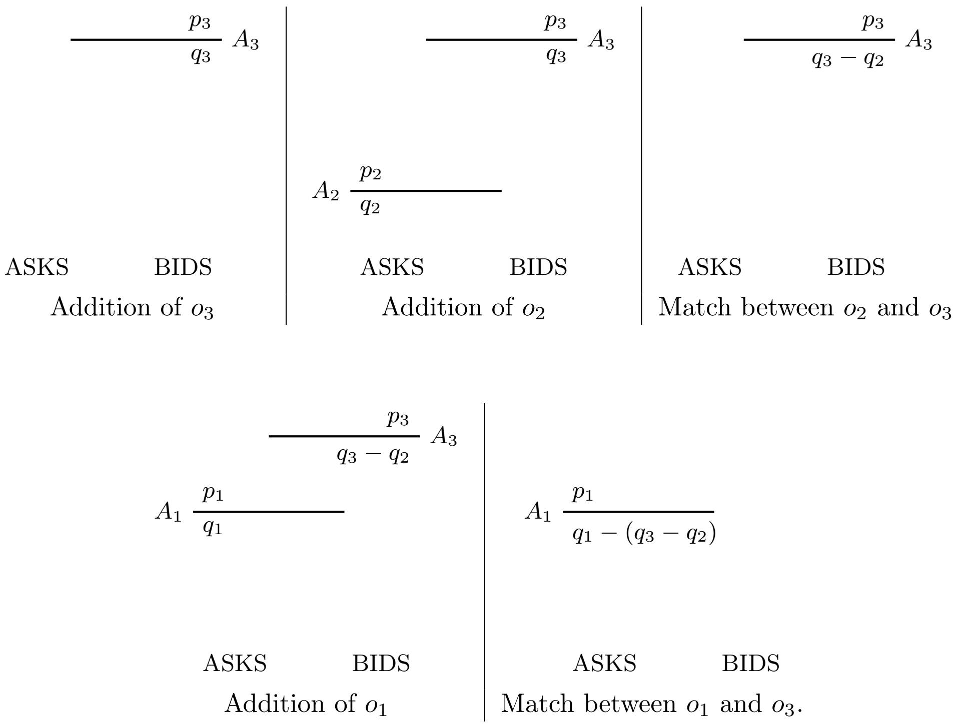


Addition of o_1 | Match between o_1 and o_3.

A.2.2 Sequence (o_3, o_1, o_2)



A.2.3 Sequence (o_3, o_2, o_1)



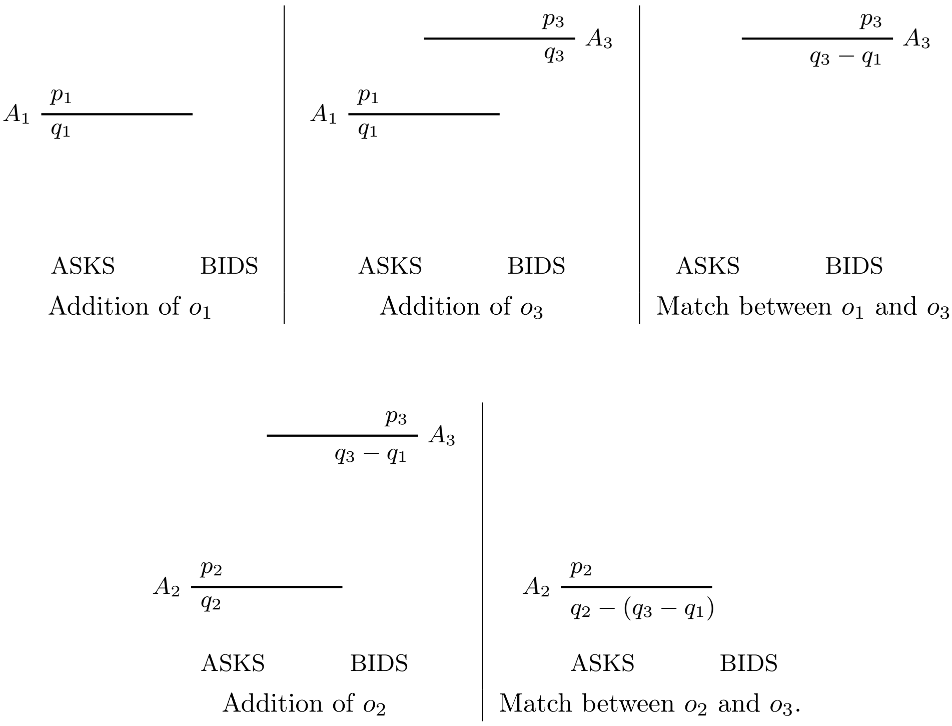
B Details of the proof of the theorem 3.2

B.1 Example of $\mathcal{O}$ and CI verifying the hypotheses

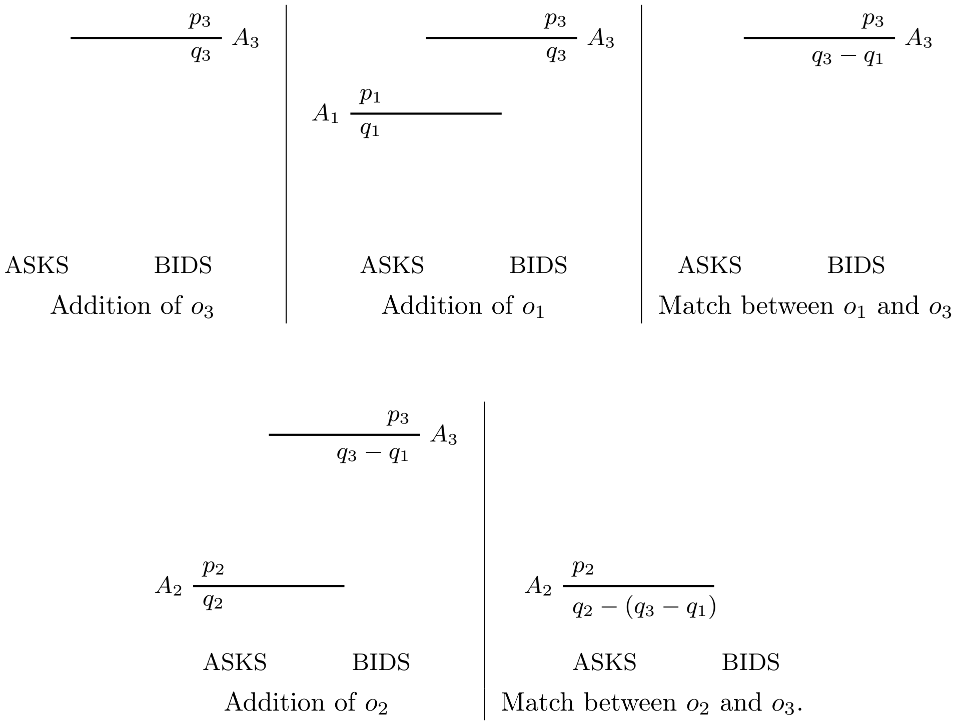
i	p_i	q_i	c_i	n_i
1	1166	1	23500	11
2	1002	13	14969	15
3	2048	11	32763	24

B.2 Proof of the values obtained for cash and assets after the execution of different sequences

B.2.1 Sequence (o_1, o_3, o_2)



B.2.2 Sequence (o_3, o_1, o_2)



B.2.3 Sequence (o_3, o_2, o_1)