Lec 19. Reduction and undecidable languages III

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Post Correspondence Problem (PCP)

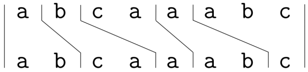
$$\left\{ \left[\frac{\texttt{b}}{\texttt{ca}}\right], \left[\frac{\texttt{a}}{\texttt{ab}}\right], \left[\frac{\texttt{ca}}{\texttt{a}}\right], \left[\frac{\texttt{abc}}{\texttt{c}}\right] \right\}$$ Chapter 5.2, Sipser 2012.

Emil Post's Correspondence Problem

Input: a (finite) set $P = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots (\alpha_k, \beta_k)\}$ of ordered pairs (called dominoes) of strings over Σ.

Question: Is there a match, i.e. a sequence $i_1, \ldots, i_m \in [k]$ such that $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$?

$$\left[\frac{\texttt{a}}{\texttt{ab}}\right]\left[\frac{\texttt{b}}{\texttt{ca}}\right]\left[\frac{\texttt{ca}}{\texttt{a}}\right]\left[\frac{\texttt{a}}{\texttt{ab}}\right]\left[\frac{\texttt{abc}}{\texttt{c}}\right]$$



Post Correspondence Problem

Key idea: many-one reduction (mapping-reduction).

- Many-one reduce from $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM accepting } w\}$ to PCP.
- As an intermediary problem we introduce a decision problem <u>Modified PCP</u> (MPCP), in which an instance of PCP is a YES-instance iff there is a match which begins with the first domino (α_1, β_1).
- Combine two (many-one) reductions: from A_{TM} to MPCP, and one from MPCP to PCP.

Post Correspondence Problem

Set-up

1. We assume that the TM M of instance $\langle M, w \rangle$ satisfies:
   - it is deterministic, with left/right move only.
   - M never attempts to move the header to the left when it is in the left-most cell of the tape.
   - if $w = \epsilon$, the string w is encoded as B, where B is a symbol in the alphabet.
2. Reduction from MPCP to PCP is simple:

$$\left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \left[\frac{t_3}{b_3}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$

$$\left\{ \left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_1}{b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \left[\frac{\star t_3}{b_3 \star}\right], \ldots, \left[\frac{\star t_k}{b_k \star}\right], \left[\frac{* \diamond}{\diamond}\right] \right\}$$

Chapter 5.2, Sipser 2012.

Post Correspondence Problem

Key idea for many-one reduction from A_{TM} to MPCP:

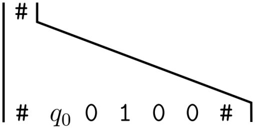
From an instance $\langle M, w\rangle$ to A_{TM}, create an instance (i.e. the set of dominoes) to MPCP so that there is a match if and only if there is an accepting computation history of M on w.

Implementing the idea:

- In a match, the string is an accepting computation history of the form

$$\#C_1\#C_2\#\cdots\#C_\ell\#$$

- The first domino is $(\#, \#q_0\ w\#)$, so the match begins in a form

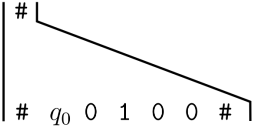


Chapter 5.2, Sipser 2012.

Post Correspondence Problem

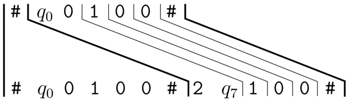
Implementing the idea: In a match, the dominoes are grouped into blocks (contiguous dominoes), where each group is one of the following forms:

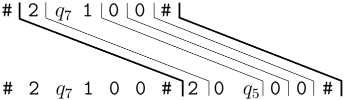
1. Stage 1: expresses a starting configuration. The first domino forms a single group and falls into this stage.



Chapter 5.2, Sipser 2012.

2. Stage 2: expresses a transition from the config C_i to C_{i+1}.



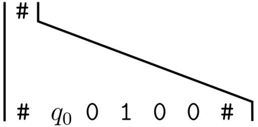


3. Stage 3: once the bottom string reaches an accept state, the dominoes let the upper string to catch up with the bottom string. (Details later.)

Post Correspondence Problem

Implementing details using "gadgets": given the instance $\langle M, w \rangle$ to A_{TM}, we progressively construct the instance P to MPCP by adding the following dominoes.

1. Gadget for Stage 1: we (the algorithm / TM) adds the domino of the form $(\#, \#q_0\ w\#)$



Chapter 5.2, Sipser 2012.

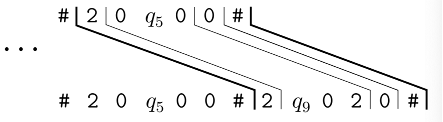
Post Correspondence Problem

1. Gadgets for Stage 2: add dominoes for expressing the transitions as well as the tape content.
   1. Right move: For each $a, b \in \Gamma$ and each $q, r \in Q$ where $q \neq q_{reject}$, add *the domino* (qa, br) *if* $\delta(q, a) = (r, b, R)$
   2. Left move: For each $a, b, c \in \Gamma$ and each $q, r \in Q$ where $q \neq q_{reject}$, add *the domino* (cqa, rcb) *if* $\delta(q, a) = (r, b, L)$
   3. Symbol domino: for each $a \in \Gamma$, add the domino (a, a)
   4. Expressing the end of the tape content / the unused cell on the right: add the dominoes $(\#, \#)$ and $(\#, B\#)$.

Example: $\delta(q_5, 0) = (q_9, 2, L)$

$$\left[\frac{0q_5 0}{q_9 02}\right], \left[\frac{1q_5 0}{q_9 12}\right], \left[\frac{2q_5 0}{q_9 22}\right], \text{ and } \left[\frac{\sqcup q_5 0}{q_9 \sqcup 2}\right]$$

Chapter 5.2, Sipser 2012.



Post Correspondence Problem

1. Gadgets for Stage 3: add dominoes so that the upper string catches up with the bottom string once (the bottom) reaches the accept state.
   1. "Eat-up" the leftover tape content: for each $a \in \Gamma$, add the domino

$$\left[\frac{a\, q_{\text{accept}}}{q_{\text{accept}}}\right] \text{ and } \left[\frac{q_{\text{accept}}\ a}{q_{\text{accept}}}\right]$$

   2. Finish the match: add the domino

$$\left[\frac{q_{\text{accept}}\#\#}{\#}\right]$$

Post Correspondence Problem

Finishing the reduction: to show that there is a (many-one) reduction from A_{TM} to PCP consists of two parts.

1. Construct a reduction. That is, we show an algorithm which maps an arbitrary instance $\langle M, w\rangle$ to A_{TM} to a suitable instance P to MPCP.
2. Establish the equivalence. we need to show that $\langle M, w\rangle \in A_{TM}$ if and only if $P \in MPCP$. That is, $\langle M, w\rangle$ is a YES-instance to A_{TM} if and only if the constructed instance P is a YES-instance to MPCP.
3. So far, we constructed a reduction (= designed an algorithm outputting an equivalent instance of MPCP from an instance of A_{TM}).

POST CORRESPONDENCE PROBLEM

Finishing the reduction: Establishing the equivalence consists of two directions.

1. From A_{TM} to MPCP. One shows that if $\langle M, w \rangle \in A_{TM}$ (equivalently, it is a YES-instance / M accepts w), then the constructed instance P is a YES-instance to MPCP, i.e. there is a match in P beginning with the first domino.
2. From MPCP to A_{TM} One shows that if the constructed instance P allows a match starting with the first domino, then $\langle M, w \rangle$ is a YES-instance to A_{TM}.

Post Correspondence Problem

If $\langle M, w\rangle \in A_{TM}$, then P allows a match starting with the first domino.

Let $C_1, C_2, \cdots C_\ell$ be the computation history of M on w. We construct a solution of the instance P as follows.

1. Place the first domino; $s_1 = \#$, $s_2 = \#C_1\#$.
2. Use a sequence of 'symbol domino', 'transition domino', 'end-of-string/move-off-to-right domino' as needed to get

$$s_1 = \#C_1\# \qquad \text{and } s_2 = \#C_1\#C_2\#.$$

3. Continue the above 2 until you get

$$s_1 = \#C_1\#C_2\#\cdots\#C_{\ell-1}\# \qquad \text{and } s_2 = \#C_1\#C_2\#\cdots\#C_{\ell-1}\#C_\ell\#,$$

where C_ℓ is the accepting configuration.

Post Correspondence Problem

If $\langle M, w\rangle \in A_{TM}$, then P allows a match starting with the first domino.

We finish the partial match forming $s_1 = ...\#$ and $s_2 = ...\#C_\ell\#$ as follows.

4. Use the symbol dominoes and 'eat-up' dominoes for Stage 3 of the form

$$\left[\frac{a\, q_{\text{accept}}}{q_{\text{accept}}}\right] \text{ and } \left[\frac{q_{\text{accept}}\ a}{q_{\text{accept}}}\right]$$

   so that

$$s_1 = ...\#C_\ell\# \qquad \text{and } s_2 = ...\#C_\ell\#C_\ell^1.$$

   where C_ℓ^1 is a string having precisely one less symbol than C_ℓ around q_{accept}.

5. Repeat the above 4 to extend s_1 and s_2 into the forms

$$s_1 = ...\#C_\ell\#C_\ell^1\#\cdots\#C_\ell^{j-1}\# \qquad \text{and } s_2 = ...\#C_\ell\#C_\ell^1\#\cdots\#C_\ell^j\#$$

   until $C_\ell^j = q_{accept}$.

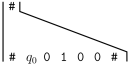
6. Use the 'finish-the-match' domino and finish the match.

$$\left[\frac{q_{\text{accept}}\#\#}{\#}\right]$$

Post Correspondence Problem

If P allows a match starting with the first domino, then $\langle M, w\rangle \in A_{TM}$ holds.
Suppose that $s_1 = s_2$ is the string formed by a match $i_1, \ldots, i_m$.

1. The match must begin with the block of the form



Chapter 5.2, Sipser 2012.

2. One should replicate C_1 in the upper string. The only dominoes containing some state are the transition , 'eat-up' dmino, or the finish-the-match domino. Assuming $w \neq \epsilon$. there is a prefix of the form $s_1 = \#C_1\#C_2\#\cdots\#C_{\ell-1}\#$ and $s_2 = \#C_1\#\cdots\#C_2\#\cdots\#C_{\ell-1}\#C_\ell\#$, where C_i is a legal move of M from C_{i-1}.
3. Similarly, you can argue that the string from a match should be of the form $s = \#C_1\#\cdots\#C_\ell\#C_\ell^1\#\cdots\#C_\ell^j\#\#,$ where C_ℓ^i omits one symbol around q_{accept} from C_ℓ^{i-1} for all $i \leq j$, and $C_\ell^j = q_{accept}$.
4. We conclude that there is a sequence of legal moves (i.e. computation history) of M on w.

Revisiting many-one reduction

Many-one reducibility: definition

Let $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$ be two languages.

We say that A is mapping-reducible (or many-one reducible) to B, written as $A \leq_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that for every input $w \in \Sigma^*$,

$$w \in A \text{ if and only if } f(w) \in B.$$

Revisiting many-one reduction

Many-one reducibility: rephrase

Let $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$ be two languages. A (many-one / mapping) reduction from A to B is an algorithm which

INPUT given an instance x to A as input

OUTPUT computes an equivalent instance y to B. That is, x is a YES-instance to A if and only if y is a YES-instance to B (i.e. $x \in A$ if and only if $y \in B$).

- When $A = A_{TM}$ and $B =$MPCP, we designed an algorithm which gets $x = \langle M, w \rangle$ as input and output an instance P to MPCP..

Revisiting many-one reduction

- When $A = A_{TM}$ and $B =$MPCP, we designed an algorithm which gets $x = \langle M, w \rangle$ as input and output an instance P to MPCP..
- What if the input x is not of the form $\langle M, w \rangle$?
- The reduction (algorithm) can always check if x conforms the format of an instance to A, and output a trivial No-instance to B if it is not the case.
- So we can assume that x is in the form $\langle M, w \rangle$.
- This step of reduction for checking if the input is "well-formed" is often skipped (as this can be easily done).

Revisit: Turing-reduction from A_{TM} to REG_{TM}

$REG_{TM} = \{M : M \text{ is TM and } L(M) \text{ is a regular language}\}$.

- D, upon an input $\langle M, w\rangle$ to A_{TM}, does the following:
  1. Compute & write an encoding $\langle M_o^{M,w}\rangle$ of TM $M_o^{M,w}$ such that

$$L(M_o^{M,w}) = \begin{cases} \{0^n 1^n \mid n \geq 0\} & \text{if } M \text{ does not accept } w \\ \Sigma^* & \text{if } M \text{ accepts } w \end{cases}$$

  2. Run R on $\langle M_o^{M,w}\rangle$.
  3. D outputs

$$\begin{cases} \text{YES} & \text{if } R \text{ outputs YES} \\ \text{NO} & \text{if } R \text{ outputs NO} \end{cases}$$

- Can we turn the above Turing-reduction into a many-one reduction?

REVISIT: TURING-REDUCTION FROM A_{TM} TO E_{TM}

Emptiness problem: $E_{TM} = \{M : M \text{ is TM and } L(M) = \emptyset\}$.

- D, upon an input $\langle M, w\rangle$ to A_{TM}, does the following:
  1. Compute an encoding of a TM M_o^w such that

$$L(M_o^w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ does not accept } w \end{cases}$$

  2. Run R on $\langle M_o^w\rangle$.
  3. D outputs

$$\begin{cases} \text{NO} & \text{if } E(\langle M_o^w\rangle) = \text{YES} \\ \text{YES} & \text{if } E(\langle M_o^w\rangle) = \text{NO} \end{cases}$$

- Can we turn the above Turing-reduction into a many-one reduction?

Turing-reduction vs mapping-reduction

Emptiness problem: $E_{TM} = \{M \text{ is TM and } L(M) = \emptyset\}$.
Non-emptiness problem: $SOME_{TM} = \{M : M \text{ is TM and } L(M) \neq \emptyset\}$.

A_{TM} is Turing-reducible to E_{TM} (last week).

A_{TM} is not many-one reducible to E_{TM}.

- Complement of E_{TM}, i.e. $SOME_{TM}$ is Turing-recognizable (how so?).
- We know $\neg A_{TM}$ is not Turing-recognizable (why?)
- If $A_{TM} \leq_m E_{TM}$, then $\neg A_{TM} \leq_m SOME_{TM}$, contradiction.