Lec 10. Pushdown Automata and CFG

Eunjung Kim

PDA for $L = \{w \cdot w^R : w \in \{0,1\}^*\}$

PDA FOR $L = \{w \in \{a,b\}^* : |w|_a = |w|_b\}$

Equivalence of CFG and PDA

Theorem

A language is context-free if and only if some pushdown automaton recognizes it.

- ($\Rightarrow$): converting a CFG to a PDA.
- ($\Leftarrow$): converting a PDA to a CFG.

Converting CFG to PDA

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- Key idea: we design PDA P which simulates a leftmost derivation of w.
  1. It "matches" the input symbol and the stack symbol if the stack symbol is an element of Σ.
  2. It "replaces" the stack symbol A by γ if A is a variable of G and there is a rule $A \to \gamma$.

Converting CFG to PDA: example

$L = \{0^n 1^n : n \geq 0\}$ is the language of the grammar $S \rightarrow 0S1 \quad | \quad \epsilon$.

Converting CFG to PDA

Construct a PDA P as follows.

1. There are three states q_{start}, q, q_{accept}.
2. The stack alphabet is $V \cup \Sigma \cup \{\$\}$.
3. Initially, P places the marker \$ onto the (empty) stack, then the start symbol S on the CFG G.
4. It loops at the state q and executes the following unless the stack symbol is \$
   - If the stack symbol is $A \in V$, then P nondeterministically chooses a rule of the form $A \to \gamma$ and pushes γ onto stack so that the first symbol of γ is at the top.
   - If the stack symbol is $a \in \Sigma$, then P reads the symbol $a \in \Sigma$ in the input and stays in the current state. If a cannot be read ("does not match"), no move is defined and the current computation branch dies out.
5. If the stack symbol is \$, then it goes to q_{accept}. The input string is accepted if the string has been read fully. If not, the current branch dies out.

Converting CFG to PDA

Construct a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ from $G = (V, \Sigma, R, S)$:

1. $Q = \{q_{start}, q, q_{accept}\}$. $\Gamma = V \cup \Sigma \cup \{\$\}$.
2. $\delta(q_{start}, \epsilon, \epsilon) = \{(q, S\$)\}$.
3. For each stack symbol in $V \cup \Sigma \cup \{\$\}$
   - for every $A \in V$: $\delta(q, \epsilon, A) = \{(q, \gamma) : \text{ for all rules } A \rightarrow \gamma \text{ in } G\}$
   - for every $a \in \Sigma$: $\delta(q, a, a) = \{(q, \epsilon)\}$
   - $\delta(q, \epsilon, \$) = \{q_{accept}, \epsilon\}$.

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3. For each stack symbol in $V \cup \Sigma \cup \{\$\}$
    - for every $A \in V$: $\delta(q, \epsilon, A) = \{(q, \gamma) :$ for all rules $A \to \gamma$ in $G\}$
    - for every $a \in \Sigma$: $\delta(q, a, a) = \{(q, \epsilon)\}$
    - $\delta(q, \epsilon, \$) = \{q_{accept}, \epsilon\}$.

How to implement a transition such as $\{(q, \gamma) \in \delta(q, \epsilon, A)$ when γ is a string, not necessarily a symbol in Γ_ϵ?

Converting PDA to CFG

Step A. Streamlining the PDA.

1. It has a single accept state q_{accept}.
2. It empties its stack before accepting.
3. Each transition move either pushes a symbol onto the stack (push move) or pops a symbol off the stack (pop move), but does not do both at the same time.

Converting PDA to CFG

Step B. Variables A_{pq} for all $p, q \in Q$.

1. Meaning of A_{pq}: we intend to design CFG G so that

$$L(A_{pq}) := \{w : A \Rightarrow_G^* w\}$$

coincides with

$$\{w : (p, w, \epsilon) \vdash_G^* (q, \epsilon, \epsilon)\}$$

2. Take A_{st} as the start variable of CFG G, where $s = q_0$ and $t = q_{accept}$.
3. Then $L(G)(= L(A_{st}))$ coincides with

$$\{w : (q_0, w, \epsilon) \vdash_G^* (q_{accept}, \epsilon, \epsilon)\},$$

which is precisely $L(P)$.

CONVERTING PDA TO CFG

Step C. Designing a production rule for the variable A_{pq}.

1. For a string w in

$$\{w : (p, w, \epsilon) \vdash_G^* (q, \epsilon, \epsilon)\},$$

two situation can occur when P runs on w.

   A. the stack gets empty while running
   B. the symbol pushed at the beginning is never popped till the last moment.

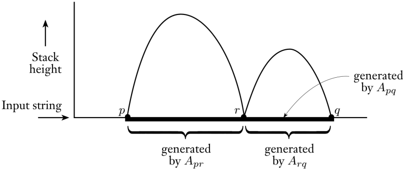

Figure 2.28, Sipser 2012

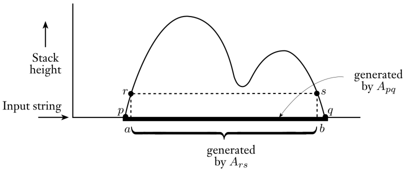

Figure 2.29, Sipser 2012

Converting PDA to CFG

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1. For a string w in

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two situation can occur when P runs on w.

A. the stack gets empty while running:
i.e. $w \in L(A_{pr}) \cdot L(A_{rq})$.

B. the symbol pushed at the beginning is never popped till the last moment.
i.e. $w \in aL(rs)b$ for all $a, b \in \Sigma$ whenever $\delta(p, a, \epsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ϵ) for some $u \in \Gamma$.

2. The trivial case $(p, \epsilon, \epsilon) \vdash_P^* (p, \epsilon, \epsilon)$.

Each case is simulated by the next rules.

- case A: $A_{pq} \to A_{pr}A_{rq}$ for all $p, q, r \in Q$
- case B: $A_{pq} \to aA_{rs}b$ for all $p, q, r, s \in Q$ and $a, b \in \Sigma_\epsilon$, and $u \in \Gamma$ such that $\delta(p, a, \epsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ϵ).
- case C: $A_{pp} \to \epsilon$.

The new CFG G contains all the above rules.

CONVERTING PDA TO CFG

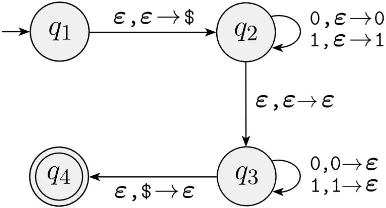

Figure 2.19, Sipser 2012