

Lec 22. More on NP-completeness

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TRANSMITTING HARDNESS

THE FOLLOWING PROBLEMS ARE NP-COMPLETE

- 3-CNF
- INDEPENDENT SET
- 3-COLORING
- DIRECTED HAMILTONIAN PATH

CONJUNCTIVE NORMAL FORM (CNF)

- CNF (Conjunctive Normal Form) formula: \wedge ('AND') of \vee 's ('OR) of variables and their negations.

$$(x_1 \vee \bar{x}_3) \wedge (x_2 \vee x_4) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_5) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4 \vee x_5).$$

- Literals are variables and its negations.
- A clause is a OR's over literals.
- CNF formula is a conjunction ('AND') of clauses.
- A formula φ is k -CNF if each clause contains at most k literals.

PROBLEM DEFINITION: CNF-SAT

INPUT: a CNF formula φ

QUESTION: is there a satisfying assignment to the variables of φ ?

REDUCTION: CNF-SAT TO 3CNF-SAT

PROBLEM 3CNF-SAT

INPUT a 3-CNF formula φ

QUESTION is there a satisfying assignment to the variables of φ ?

KEY STEP IN THE REDUCTION

$$C = (x_1 \wedge (\bar{x}_2 \vee x_3) \vee \bar{x}_4) \rightsquigarrow C_1 = (\textcolor{brown}{x_1} \vee \bar{x}_2 \vee z), C_2 = (\textcolor{blue}{x_3} \vee \bar{x}_4 \vee \bar{z})$$

REDUCTION: CNF-SAT TO 3CNF-SAT

PROBLEM 3CNF-SAT

INPUT a 3-CNF formula φ

QUESTION is there a satisfying assignment to the variables of φ ?

KEY STEP IN THE REDUCTION

$$C = (x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4) \rightsquigarrow C_1 = (x_1 \vee \bar{x}_2 \vee z), C_2 = (x_3 \vee \bar{x}_4 \vee \bar{z})$$

REDUCTION: CNF-SAT TO 3CNF-SAT

- Replace each clause C with $\ell \geq 4$ literals by two clauses C_1, C_2 .
- C_1, C_2 are (simultaneously satisfied) if and only if C is satisfied .
- How to: divide the literals of C roughly into two parts so that one part has two literals (the other $\ell - 2$), and add a new variable y to toggle

KEY STEP IN THE REDUCTION

$$C = (z_1 \vee z_2 \vee z_3 \vee \dots \vee z_\ell) \rightsquigarrow C_1 = (z_1 \vee z_2 \vee y), C_2 = (z_3 \vee \dots \vee z_\ell \vee \bar{y})$$

REDUCTION: CNF-SAT TO 3CNF-SAT

Proof. Let φ be an arbitrary formula in CNF.

- 1 Construction: for each clause C with at least $\ell \geq 4$ literals, we replace C by the two clauses C_1 and C_2 as described in the previous slide.
- 2 Polynomial-time: By each replacement the following value

$$\sum_{C \in \varphi : |C| \geq 4} |C|$$

strictly decrease. Therefore after $||\varphi||$ rounds of replacement, we obtain a 3-CNF formula φ' .

- 3 (This implies that the entire construction can be completed in polynomial-time in $||\varphi||$. Why?)
- 4 Equivalence: That φ is satisfiable if and only if φ' follows from that C is satisfiable if and only if C_1 and C_2 above.

REDUCTION: 3CNF-SAT TO IND

An independent set I of a graph is a vertex subset I of G s.t. any two vertices $u, v \in I$ is non-adjacent.

PROBLEM INDEPENDENT SET

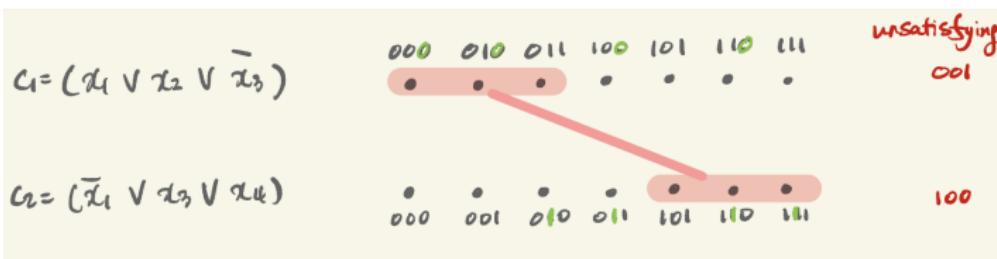
INPUT a graph G and an integer k

QUESTION does G have an independent set of size (at least) k ?

REDUCTION: 3CNF-SAT TO IND

KEY STEP IN THE REDUCTION

- For each variable x_i , create two vertices v_i and v'_i (correspond to positive and negative literals).
- For a clause $C_i = (x \vee y \vee z)$ in the 3-CNF formula ϕ , create 7 vertices.
- Each of 7 vertices corresponds to a **satisfying partial assignment**. That is, an assignment on $V(C_i)$ satisfying C_i .
- Incompatible assignments** between two clauses is expressed as an edge.



REDUCTION: 3CNF-SAT TO IND

- Construction: set $k = |\varphi|$, i.e. the number of clauses in φ . The 7 vertices from $C \in \varphi$ form a clique.
- Easy to see that the construction can be done in polynomial time
- (\Rightarrow) If γ is a satisfying assignment to φ , then for each 3-clause C one can choose a vertex which corresponds to the assignment γ on the variables of C .
- The set of $|\varphi|$ -many chosen vertices are independent.
- (\Leftarrow) If there is an independent set I of size $k = |\varphi|$, I has precisely one vertex out of the 7 vtxs.
- The set I encodes a consistent assignment (no edge), and they satisfy each $C \in \varphi$.

REDUCTION: 3CNF-SAT TO 3-COL

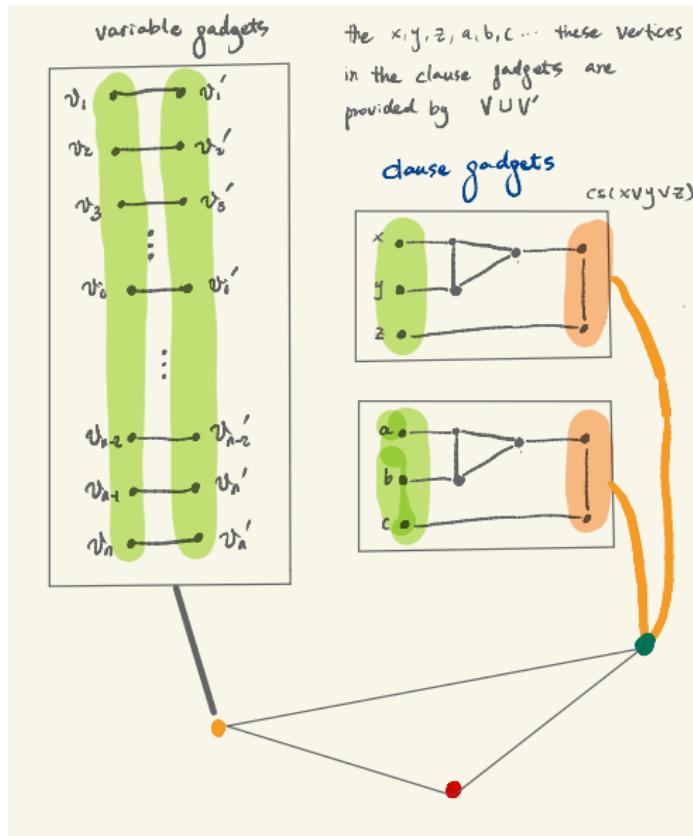
A 3-Coloring of a graph $G = (V, E)$ is a mapping from V to $\{\text{red}, \text{yellow}, \text{green}\}$ such that each color class is an independent set.

PROBLEM 3-COLORING

INPUT a graph G .

QUESTION does G have 3-Coloring?

REDUCTION: 3CNF-SAT TO 3-COL



REDUCTION: 3CNF-SAT TO 3-COL

- The construction can be clearly done in polynomial time.
- (\Rightarrow) Let γ be a satisfying assignment to φ .
- Consider the following (partial) coloring c on V

$$c(v_i) = \begin{cases} \text{green} & \text{if } \gamma(x_i) = T \\ \text{red} & \text{if } \gamma(x_i) = F \end{cases}$$

- Verify that c can be extended into a full 3-coloring.
- (\Leftarrow) Let $c : V(G_\varphi) \rightarrow \{\text{green}, \text{red}, \text{yellow}\}$.
- Key obs: in each clause gadget, at least one literal vertex should be colored green.
- Consider the assignment $\alpha : V(\varphi) \rightarrow \{T, F\}$ s.t. $\alpha(x_i) = T$ if and only if $c(v_i) = \text{green}$.
- Due to *Key Obs*, this is a satisfying assignment.

REDUCTION: 3CNF-SAT TO DIRECTED HAMILTONIAN PATH

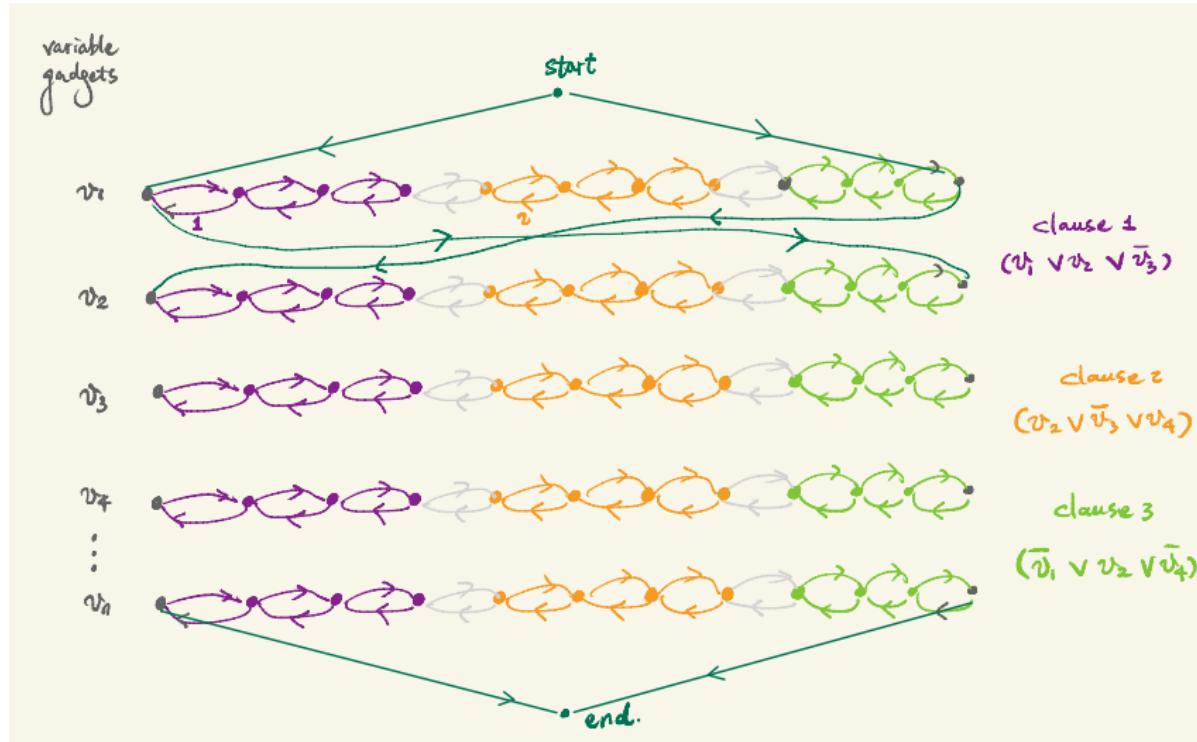
A Hamiltonian path of a directed graph $G = (V, E)$ is a directed path which visits every vertex of G precisely once.

PROBLEM DIRECTED HAMILTONIAN PATH

INPUT a graph G .

QUESTION does G have a Hamiltonian path?

REDUCTION: 3CNF-SAT TO DIRECTED HAMILTONIAN PATH



REDUCTION: 3CNF-SAT TO DIRECTED HAMILTONIAN PATH

