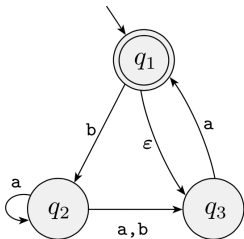# Lec 03. Equivalence of DFA and NFA & Closure under Regular Operations

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FORMAL DEFINITION OF NFA

NONDETERMINISTIC FA IS A 5-TUPLE $(Q, \Sigma, \delta, q_0, F)$

- Q a finite set called the states,

- Σ a finite set called the alphabet,

- δ a function from $Q \times \Sigma_\epsilon$ to 2^Q called the transition function,

- $q_0 \in Q$ the start state,
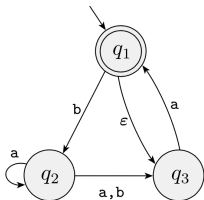
- $F \subseteq Q$ the set of accept states.

Write a formal description of this NFA

LANGUAGE RECOGNIZED BY NFA

NFA N ACCEPTS w IF

1 w can be written as $y_1, \ldots, y_m$ with $y_i \in \Sigma_\epsilon = \Sigma \cup \{\epsilon\}$,

2 there exists a sequence of states $r_0, \ldots, r_m$ s.t.

- $r_0 = q_0$,
- r_{i+1} (??) $\delta(q_i, y_i)$,
- $r_m \in F$.

Write a computation tree for $w = $ *baabaaa*. How many accepting paths?

CLOSURE UNDER REGULAR OPERATION

UNION OPERATION

Let A_1 and A_2 be two languages recognized by NFA N_1 and N_2 respectively. Then $A_1 \cup A_2$ is recognized by some NFA.

CLOSURE UNDER REGULAR OPERATION

CONCATENATION OPERATION

Let A_1 and A_2 be two languages recognized by NFA N_1 and N_2 respectively. Then $A_1 \circ A_2$ is recognized by some NFA.

CLOSURE UNDER REGULAR OPERATION

Let A a languages recognized by NFA N. Then A^* is recognized by some NFA.

CLOSURE UNDER COMPLEMENTATION

COMPLEMENTATION OPERATION

Let A a languages recognized by NFA N. Then $\bar{A}$, that is, $\Sigma^* - A$ is recognized by some NFA.

- For a regular language L, we can obtain a DFA recognizing the complement of L.
- ...using the trick...
- Can we use the same trick for NFA in general?

CLOSURE UNDER INTERSECTION

INTERSECTION OPERATION

Let A_1 and A_2 be two languages recognized by NFA N_1 and N_2 respectively. Then $A_1 \cap A_2$ is recognized by some NFA.

- Use the expression that $A_1 \cap A_2 =$??????.
- Combine the above (which ones?) operations on NFAs...
- Direct way with two DFAs M_1 and M_2 by <u>simulating</u> both automata <u>simultaneously</u>.

CLOSURE UNDER INTERSECTION

- Direct way with two DFAs M_1 and M_2 by <u>simulating</u> both automata <u>simultaneously</u>.
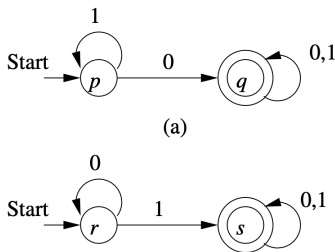

(a)

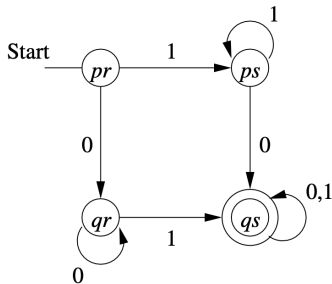Figure 4.4 (a)-(b), Hopcroft et al. 2014.

Figure 4.4 (c), Hopcroft et al. 2014.

EQUIVALENCE OF NFA AND DFA

NFA AND DFA OWN THE SAME COMPUTATIONAL POWER

For every NFA, there exists a deterministic finite automata which recognizes the same language (a.k.a. equivalent DFA).

Proof outline.

- Let $N = (Q, \Sigma_\epsilon, \delta, q_0, F)$ be an NFA.
- We want to construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ such that $L(N) = L(M)$.
- Define $Q' := 2^Q$, i.e. the collection of all subsets of Q.
- Let us define δ', $q'_0 \in 2^Q$ and $F' \subseteq 2^Q$,

PROOF: CONSTRUCTING DFA M, WITHOUT ϵ-TRANSITION

- $q_0' = q_0$.

- transition function δ' from $2^Q \times \Sigma$ to 2^Q:
  for every $R \in 2^Q$ (R is a subset of Q) and every symbol $a \in \Sigma$,

$$\delta'(R, a) := \bigcup_{r \in R} \delta(r, a)$$

- Define $F' \subseteq 2^Q$ as the collection of all subsets of Q containing at least one accept state of N.

PROOF: CONSTRUCTING DFA *M*

- How to define the initial state for DFA: from a state $q \in Q$ of NFA N, any other state q' that can be reached by reading a string ϵ, can be aggregated with q to form a single state in DFA.

- Define $ext(q) \subseteq Q$ as the set of all states q' of N such that there is a directed path from q to q' in (the state diagram of) N each of whose arcs carries the label ϵ. Extend the definition $ext(X) := \bigcup_{q \in X} ext(q)$.

- transition function δ' from $2^Q \times \Sigma$ to 2^Q:
  for every $R \in 2^Q$ (R is a subset of Q) and every symbol $a \in \Sigma$,

$$\delta'(R, a) := ext(\bigcup_{r \in R} \delta(r, a))$$

- Define $q_0' := ext(q_0) \in 2^Q$. Note that q_0' corresponds to a subset of Q.

- Define $F' \subseteq 2^Q$ as the family of all subsets of Q containing at least one accept state of N.

PROOF: $L(N) \subseteq L(M)$

- Let $\pi = (q_0, w = w_0), \ldots, (q_i, w_i), \ldots, (q_s, w_s = \epsilon)$ be an accepting computation history of N for w, that is

$$q_i \in \delta(q_{i-1}, y_i) \text{ for every } i \in [s] \text{ and } q_s \in F,$$

where $w_i = y_i \circ w_{i-1}$ with $y_i \in \Sigma \cup \{\epsilon\}$ for every $i \in [s]$.

- Let $0 \leq i_0 < \cdots < i_{t-1} < s$ be the indices in π such that the leading symbol of w_i in (q_i, w_i) is in Σ (not ϵ). Let $i_t = s$.

- Let $Q_0 = q_0'$. Inductively for each $i \in [t]$, let

$$Q_i = \delta'(Q_{i-1}, y_i).$$

- Now we have a computation history for $w = y_1 \cdots y_t$ as follows

$$\pi' = (Q_0, w), (Q_1, y_2 \cdots y_t), \cdots (Q_t, \epsilon).$$

PROOF: $L(N) \subseteq L(M)$

- It remains to see Q_t is an accept state of M, i.e. the subset $Q_t \subseteq Q$ contains at least one accept state of N.

- It suffices to prove $q_{i_j} \in Q_j$ for each $0 \leq j \leq t$ (this implies the above).

- By induction. Base case holds because $Q_0 = q'_0 = ext(q_0)$ and q_{i_0} is reachable from q_0 by reading a number of ϵ's.

- From the partition using $i_0, \ldots, i_t$, q_{i_j} is reachable from $q_{i_{j-1}}$ by reading a single symbol $a_{i_{j-1}+1} \in \Sigma$ and additional ϵ's. Hence
$q_{i_j} \in ext(\delta(q_{i_{j-1}}, a_{i_{j-1}+1}) \subseteq ext(\delta(Q_{j-1}, a_{i_{j-1}+1}) = \delta'(Q_{j-1}, a_{i_{j-1}+1}) = Q_j$.

PROOF: $L(M) \subseteq L(N)$

- Let $\pi' = (Q_0, w = w_0), \ldots, (Q_i, w_i), \ldots, (Q_t, w_t = \epsilon)$ be an accepting computation history of M for w. By definition of computation history $Q_i = \delta'(Q_{i-1}, x_i)$, where $x_i \in \Sigma$ is the leading symbol of w_{i-1}.

- We construct an accepting computation history of N by following the sequence π' backwardly.

- Let $q_f \in Q_t$ be an accept state of N.

- Observe: for each $q \in Q_i \subseteq Q$, there exists a state $q' \in Q_{i-1}$ such that from q' to q there is a computation history consisting of readying a string consisting of x_i followed by ϵ's.

- Now starting from q_f, we concatenate computation histories witnessed by the previous observation.

PROOF: CONSTRUCTING DFA *M* WITH ϵ-TRANSITION

- Define $ext(q) \subseteq Q$ as the set of all states q' of NFA N such that there is a directed path from q to q' in (the state diagram of) N each of whose arcs carries the label ϵ. Let $ext(X) := \bigcup_{q \in X} ext(q)$.

- transition function δ' from $2^Q \times \Sigma_\epsilon$ to 2^Q:
  for every $R \in 2^Q$ (R is a subset of Q) and every symbol $a \in \Sigma$,

$$\delta'(R, a) := ext(\bigcup_{r \in R} \delta(r, a))$$

- Define $q_0' := ext(q_0) \in 2^Q$. Note that q_0' corresponds to a subset of Q.

- Define $F' \subseteq 2^Q$ as the family of all subsets of Q containing at least one accept state of N.