Lec 18. Reduction and undecidable languages II

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E_{LBA} IS UNDECIDABLE

$E_{LBA} = \{\langle M \rangle : M \text{ is LBA and } L(M) = \emptyset\}$.

UNDECIDABILITY OF E_{LBA}

E_{LBA} is undecidable.

- Reduce from A_{TM} to E_{LBA}.
- Can we use the same reduction from A_{TM} to E_{TM}?

E_{LBA} IS UNDECIDABLE

$E_{LBA} = \{\langle M \rangle : M \text{ is LBA and } L(M) = \emptyset\}$.

- D upon an input $\langle M, w \rangle$ does the following:
  1. Compute & write an encoding $\langle B^{M,w} \rangle$ of LBA $B^{M,w}$ s.t.

$$L(B^{M,w}) = \begin{cases} \{\text{the accepting computation history of } M \text{ on } w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ does not accept } w \end{cases}$$

  2. Run E on $\langle B^{M,w} \rangle$.
  3. D outputs

$$\begin{cases} \text{NO} & \text{if } E \text{ outputs YES} \\ \text{YES} & \text{if } E \text{ outputs NO} \end{cases}$$

E_{LBA} IS UNDECIDABLE

How does the LBA $B^{M,w}$ work internally?

$$L(B^{M,w}) = \begin{cases} \{\text{the accepting computation history of } M \text{ on } w\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } M \text{ accepts } w \\ \emptyset \quad \text{if } M \text{ does not accept } w \end{cases}$$

Upon an input string $x \in \Sigma^*$, we want:

- $B^{M,w}$ rejects x if it is not in the form

$$\#C_1\#C_2\#\cdots\#C_\ell\#$$

  for some ℓ where
  - each C_i is a configuration of M,
  - C_1 is a starting configuration of M on w, i.e. $q_{init}\ w$,
  - C_ℓ is an accepting configuration of M, i.e. $y\ q_{accept}\ z$ for some $y, z \in \Gamma^*$.
- $B^{M,w}$ zig-zags between C_i and C_{i+1} and check $C_i \vdash_M C_{i+1}$. Reject if not.
- Accept the input x if nothing went wrong for all $i \leq \ell - 1$.

TM computing a function

Let's use the writing power of TM to have more than 'yes'-'no' answers.

TM computing a function in general

Consider a single-tape TM $M = (Q, \Sigma, \delta, q_0, q_{final})$:

- the contents of the tape when M reaches q_{final} (halting/final state, and terminate immediately) is said to be the output of M on w, written as $M(w)$.

We say that M computes a function $f : \Sigma^* \to \Sigma^*$ if for every input $w \in \Sigma^*$,

$$M(w) = f(w).$$

Especially, TM computing a function must halt on every input w.

A function f is (Turing)-computable if there exists TM that computes f.

Instead of a single-tape TM and $f(w)$ is the content of the tape in the halting state, we can consider a multitape TM and designate a specific tape so that $f(w)$ is the content of the said tape.

TM computing a partial function

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Consider a TM $M = (Q, \Sigma, \delta, q_0, q_{final})$ as before.

We say that M computes a partial function $f : \Sigma^* \to \Sigma^*$ if for every input $w \in \Sigma^*$,

$$M(w) = f(w)$$

whenever $f(w)$ is defined and M does not halt if $f(w)$ is not defined.

Mapping-reducibility

Mapping-reducibility: definition

Let $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^*$ be two languages.

We say that A is mapping-reducible (or many-one reducible) to B, written as $A \leq_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that for every input $w \in \Sigma^*$,

$$w \in A \text{ if and only if } f(w) \in B.$$

$A \leq_m B$ means B is as hard as A

Decidability propagates backwards

If $A \leq_m B$ and B is decidable, then A is decidable.

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If $A \leq_m B$ and B is decidable, then A is decidable.

Proof: build a TM M_A which decides A, using the decider M_B for B and the TM R for reduction; R halts on every input $x \in \Sigma^*$ and $R(x) \in B$ if and only if $x \in A$.

M_A upon an input string $w \in \Sigma^*$ does the following.

1. Run R on w and output $f(w)$.
2. Run M_B on $f(w)$: if M_B accepts $f(w)$, then M_A accepts. Otherwise, M_A rejects.

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Undecidability propagates forwards

If $A \leq_m B$ and A is undecidable, then B is undecidable.

Basics about mapping-reducibility

Recognizability propagates backwards

If $A \leq_m B$ and B is Turing-recognizable, then A is recognizable.

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If $A \leq_m B$ and B is Turing-recognizable, then A is recognizable.

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Basics about mapping-reducibility

Transitivity

If $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$.

Mapping-reducibility for complements

If $A \leq_m B$, then $\neg A \leq_m \neg B$.

Halting problem is undecidable

Halting problem: $HALT_{TM} = \{(M, w) : M \text{ is TM and } M \text{ halts on } w\}$.

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Proof: We build TM T which converts an input (M, w) to A_{TM} to an equivalent input (M', w') to $HALT_{TM}$.

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Proof: We build TM T which converts an input (M, w) to A_{TM} to an equivalent input (M', w') to $HALT_{TM}$.

T works as follows on input (M, w):

1. T internally builds a new (description of) TM M' which, on input string x,
   - simulates M on x,
   - if $M(x) = 1$, then $M'(x) = 1$,
   - if $M(x) = 0$, then M' loops.
2. T outputs (M', w).

Turing-reduction vs mapping-reduction

Emptiness problem: $E_{TM} = \{M \text{ is TM and } L(M) = \emptyset\}$.
Non-emptiness problem: $SOME_{TM} = \{M : M \text{ is TM and } L(M) \neq \emptyset\}$.

A_{TM} is Turing-reducible to E_{TM}.

A_{TM} is not mapping-reducible to E_{TM}.

- Complement of E_{TM}, i.e. $SOME_{TM}$ is Turing-recognizable (how so?).
- We know $\neg A_{TM}$ is not Turing-recognizable.
- If $A_{TM} \leq_m E_{TM}$, then $\neg A_{TM} \leq_m SOME_{TM}$, contradiction.

Post Correspondence Problem (PCP)

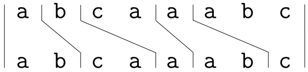
$$\left\{ \left[\frac{\mathtt{b}}{\mathtt{ca}}\right], \left[\frac{\mathtt{a}}{\mathtt{ab}}\right], \left[\frac{\mathtt{ca}}{\mathtt{a}}\right], \left[\frac{\mathtt{abc}}{\mathtt{c}}\right] \right\}$$ Chapter 5.2, Sipser 2012.

Emil Post's Correspondence Problem

Input: a (finite) set $P = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots (\alpha_k, \beta_k)\}$ of ordered pairs (called dominoes) of strings over Σ.

Question: Is there a match, i.e. a sequence $i_1, \ldots, i_m \in [k]$ such that $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$?

$$\left[\frac{\mathtt{a}}{\mathtt{ab}}\right]\left[\frac{\mathtt{b}}{\mathtt{ca}}\right]\left[\frac{\mathtt{ca}}{\mathtt{a}}\right]\left[\frac{\mathtt{a}}{\mathtt{ab}}\right]\left[\frac{\mathtt{abc}}{\mathtt{c}}\right]$$



Post Correspondence Problem

Key idea: many-one reduction (mapping-reduction).

- Many-one reduce from $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM accepting } w\}$ to PCP.
- As an intermediary problem we introduce a decision problem Modified PCP (MPCP), in which an instance of PCP is a YES-instance iff there is a match which begins with the first domino (α_1, β_1).
- Combine two (many-one) reductions: from A_{TM} to MPCP, and one from MPCP to PCP.

Post Correspondence Problem

Set-up

1. We assume that the TM M of instance $\langle M, w\rangle$ satisfies:
   - it is deterministic, with left/right move only.
   - M never attempts to move the header to the left when it is in the left-most cell of the tape.
   - if $w = \epsilon$, the string w is encoded as B, where B is a symbol in the alphabet.
2. Reduction from MPCP to PCP is simple:

$$\left\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \left[\frac{t_3}{b_3}\right], \ldots, \left[\frac{t_k}{b_k}\right]\right\}$$

$$\left\{\left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_1}{b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \left[\frac{\star t_3}{b_3 \star}\right], \ldots, \left[\frac{\star t_k}{b_k \star}\right], \left[\frac{* \diamond}{\diamond}\right]\right\}$$

Chapter 5.2, Sipser 2012.

Post Correspondence Problem

Key idea for many-one reduction from A_{TM} to MPCP:

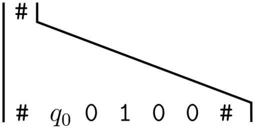
From an instance $\langle M, w\rangle$ to A_{TM}, create an instance (i.e. the set of dominoes) to MPCP so that there is a match if and only if there is an accepting computation history of M on w.

Implementing the idea:

- In a match, the string is an accepting computation history of the form

$$\#C_1\#C_2\#\cdots\#C_\ell\#$$

- The first domino is $(\#, \#q_0\ w\#)$, so the match begins in a form

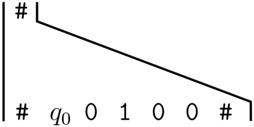


Chapter 5.2, Sipser 2012.

Post Correspondence Problem

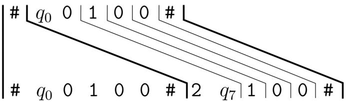
Implementing the idea: In a match, the dominoes are grouped into blocks (contiguous dominoes), where each group is one of the following forms:

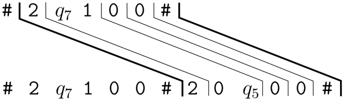
1. Stage 1: expresses a starting configuration. The first domino forms a single group and falls into this stage.



Chapter 5.2, Sipser 2012.

2. Stage 2: expresses a transition from the config C_i to C_{i+1}.



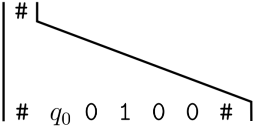


3. Stage 3: once the bottom string reaches an accept state, the dominoes let the upper string to catch up with the bottom string. (Details later.)

Post Correspondence Problem

Implementing details using "gadgets": given the instance $\langle M, w \rangle$ to A_{TM}, we progressively construct the instance P to MPCP by adding the following dominoes.

1. Gadget for Stage 1: we (the algorithm / TM) adds the domino of the form $(\#, \#q_0\, w\#)$



Chapter 5.2, Sipser 2012.

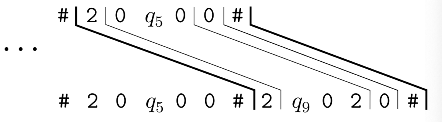
Post Correspondence Problem

1. Gadgets for Stage 2: add dominoes for expressing the transitions as well as the tape content.
   1. Right move: For each $a, b \in \Gamma$ and each $q, r \in Q$ where $q \neq q_{reject}$, add *the domino* (qa, br) *if* $\delta(q, a) = (r, b, R)$
   2. Left move: For each $a, b, c \in \Gamma$ and each $q, r \in Q$ where $q \neq q_{reject}$, add *the domino* (cqa, rcb) *if* $\delta(q, a) = (r, b, L)$
   3. Writing a string: for each $a \in \Gamma$, add the domino (a, a)
   4. Expressing the end of the tape content / the unused cell on the right: add the dominoes $(\#, \#)$ and $(\#, B\#)$.

Example: $\delta(q_5, 0) = (q_9, 2, L)$

$$\left[\frac{0q_5 0}{q_9 02}\right], \left[\frac{1q_5 0}{q_9 12}\right], \left[\frac{2q_5 0}{q_9 22}\right], \text{ and } \left[\frac{\sqcup q_5 0}{q_9 \sqcup 2}\right]$$

Chapter 5.2, Sipser 2012.



Post Correspondence Problem

1. Gadgets for Stage 3: add dominoes so that the upper string catches up with the bottom string once (the bottom) reaches the accept state.
   1. "Eat-up" the leftover tape content: for each $a \in \Gamma$, add the domino
      $$\left[\frac{a\, q_{\text{accept}}}{q_{\text{accept}}}\right] \text{ and } \left[\frac{q_{\text{accept}}\ a}{q_{\text{accept}}}\right]$$
   2. Finish the match: add the domino
      $$\left[\frac{q_{\text{accept}}\#\#}{\#}\right]$$

POST CORRESPONDENCE PROBLEM

Finishing the reduction: to show that there is a (many-one) reduction from A_{TM} to PCP consists of two parts.

1. Construct a reduction. That is, we show an algorithm which maps an arbitrary instance $\langle M, w\rangle$ to A_{TM} to a suitable instance P to MPCP.
2. Establish the equivalence. we need to show that $\langle M, w\rangle \in A_{TM}$ if and only if $P \in MPCP$. That is, $\langle M, w\rangle$ is a YES-instance to A_{TM} if and only if the constructed instance P is a YES-instance to MPCP.
3. So far, we have constructed a reduction.