

FORMAL LANGUAGES AND AUTOMATA, 2024 FALL SEMESTER

Lec 09. Pushdown Automata

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PUSHDOWN AUTOMATA

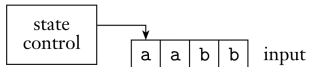


Figure 2.11, Sipser 2012

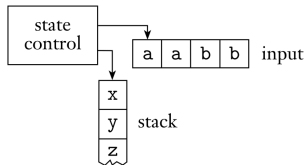


Figure 2.12, Sipser 2012

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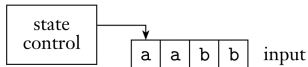


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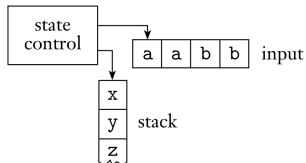


Figure 2.12, Sipser 2012

PDA: INFORMAL DESCRIPTION

A pushdown automata has three components: an input tape, a state (control), and a stack. At each step, PDA does the following

- **current state**: q .
- **read from input**: the first symbol a of the input tape; possibly ϵ .
- **pops off stack**: the first symbol x at (the top of) the stack; possibly ϵ .
- **transition**: depending on (q, a, x) , PDA
 - 1 updates the current state q to a new state q'
 - 2 pushes a symbol y to the stack; possibly ϵ .

PDA FOR $L = \{w \cdot w^R : w \in \{0, 1\}^*\}$

(INFORMAL) PDA RECOGNIZING A PALINDROME

- It has three states: q_{start} , q_{push} , $q_{pop\&match}$, q_{accept} .
- Starting at q_{start} , it pushes \$ to stack and updates to q_{push} .
- Stays in q_{push} while: it reads the symbol b from input and pushes b to stack.
- "Guess" the middle of the word; implemented as an update from q_{push} to $q_{pop\&match}$
- Stays in $q_{pop\&match}$ while: it reads the symbol b from input, pops the symbol b ; they match.
- Moves $q_{pop\&match}$ to q_{accept} if: there is nothing to read in the input when \$ is popped.
- In all other cases (e.g. "mismatch" between the input content and stack symbol): "dies"

FORMAL DEFINITION OF PDA

A PUSHDOWN AUTOMATA IS

a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where all entries except for q_0 is a finite set:

- Q is the set of states.
- Σ is the input alphabet.
- Γ is the stack alphabet.
- $\delta : Q \times \Sigma_{\epsilon} \times \Gamma_{\epsilon} \rightarrow \mathcal{P}(Q \times \Gamma_{\epsilon})$ is the transition function.
- $q_0 \in Q$ is the start state.
- $F \subseteq Q$ is the set of accept states.

LANGUAGE RECOGNIZED BY A PDA

PDA ACCEPTING A STRING

A pushdown automata $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts an input string $w \in \Sigma^*$ if

- I w can be written as $w_1 w_2 \cdots w_n$, where $w_i \in \Sigma_\epsilon$,
- II there is a sequence r_0, r_1, \dots, r_n of states, and
- III a sequence of strings s_0, s_1, \dots, s_n of strings over Γ^* ,

such that the following holds:

- 1 $r_0 = q_0$ and $s_0 = \epsilon$,
- 2 $s_i = xt$, $s_{i+1} = yt$ and $(r_{i+1}, y) \in \delta(r_i, w_{i+1}, x)$ hold for some $x, y \in \Gamma_\epsilon$ and $t \in \Gamma^*$
- 3 $r_n \in F$.

INSTANTANEOUS DESCRIPTION (A.K.A CONFIGURATION)

CONFIGURATION OF PDA

For a pushdown automata $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, an instantaneous description (ID), or equivalently a configuration of P is a triple (q, w, γ) . Essentially

- q is the current state.
- w is the remaining input string.
- γ is the string in the stack, read from top to bottom.

If $(q', y) \in \delta(q, a, x)$, then we write

$$(q, aw, x\beta) \vdash (q', w, y\beta).$$

This means that one can reach the configuration $(q', w, y\beta)$ from $(q, aw, x\beta)$ in a single step. The notation \vdash^* is used when one is reach from the other in $n \geq 0$ steps.

LANGUAGE RECOGNIZED BY A PDA

LANGUAGE RECOGNIZED BY PDA

For a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, the language recognized by PDA P is defined as

- the set of all strings in Σ^* accepted by P , or equivalently
- the set of all strings $w \in \Sigma^*$ such that $(q_0, w, \epsilon) \vdash^* (q, \epsilon, \gamma)$ for some $q \in F$ and $\gamma \in \Sigma^*$.

We write as $L(P)$ the language recognized by PDA P .

PDA FOR $L = \{0^n 1^n : n \geq 0\}$

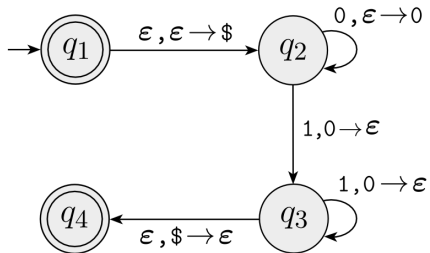


Figure 2.15, Sipser 2012

PDA FOR $L = \{0^n 1^n : n \geq 0\}$

Formal description of PDA recognizing L : it is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_1, F)$ as follows.

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{0, 1\}$
- $\Gamma = \{0, \$\}$
- the transition function δ is given as the transition table below.
- Start state is q_1
- $F = \{q_1, q_4\}$

Input:	0			1			ϵ		
Stack:	0	\$	ϵ	0	\$	ϵ	0	\$	ϵ
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$		$\{(q_3, \epsilon)\}$				
q_3					$\{(q_3, \epsilon)\}$			$\{(q_4, \epsilon)\}$	
q_4									

PDA FOR $L = \{w \cdot w^R : w \in \{0, 1\}^*\}$

PDA FOR $L = \{w \in \{a, b\}^* : |w|_a = |w|_b\}$

EQUIVALENCE OF PDA AND CFL

THEOREM

A language is context-free if and only if some pushdown automaton recognizes it.

Proof outline for (\Rightarrow).

- From a context-free grammar $G = (V, \Sigma, R, S)$, we aim to construct a PDA P such that $L(P) = L(G)$.

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- Key 1: we design PDA P which **accepts w if and only if w has a leftmost derivation** in G .
- Key 2: P simulates a leftmost derivation of $w \in L(G)$ by
 - 1 "matching" the input symbol and the stack symbol if the stack symbol is an element of Σ .
 - 2 "replacing" the stack symbol A by z if A is a variable of G and there is a rule $A \rightarrow z$.