Consider a deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ and let $L = L(M)$. Fix two (distinct) states $q_1, q_2 \in Q$ and let

$$P_{s_1,s_2} = \{w \in L \mid \text{the computation history of } M \text{ on } w \text{ visits each of } s_1 \text{ and } s_2 \text{ exactly once}\}.$$

We want to show that P_{s_1,s_2} is regular.

Recall that $\equiv_{P_{s_1,s_2}}$ is the equivalence relation on Σ^* such that

$$x \equiv_{P_{s_1,s_2}} y \text{ if and only if for every } z \in \Sigma^*, x \circ z \in P_{s_1,s_2} \text{ if and only if } y \circ z \in P_{s_1,s_2}.$$

By Myhill-Nerode Theorem, P_{s_1,s_2} is regular if and only if there are finitely many equivalence classes of $\equiv_{P_{s_1,s_2}}$. Therefore, in order to show that P_{s_1,s_2} is regular, we shall establish the latter statement for $\equiv_{P_{s_1,s_2}}$.

For proving that $\equiv_{P_{s_1,s_2}}$ has finitely many equivalence classes, it suffices to

1. define a partition $\mathcal{P}$ on Σ^*,
2. show that any two strings $x, y \in P$ for any $P \in \mathcal{P}$ are indistinguishable by $\equiv_{P_{s_1,s_2}}$, and
3. show that $\mathcal{P}$ has finitely many parts.

The first and second steps establishes that $\mathcal{P}$ is a refinement of $\Sigma^*/\equiv_{P_{s_1,s_2}}$ (the equivalence classes of $\equiv_{P_{s_1,s_2}}$). This implies that the number of parts in $\Sigma^*/\equiv_{P_{s_1,s_2}}$ is at most the number of parts in $\mathcal{P}$. This, together with the third step, proves that there are finitely many equivalence classes of $\equiv_{P_{s_1,s_2}}$.

Define a partition $\mathcal{P}$ on Σ^*. For each i, j with $0 \leq i, j \leq 2$ and for each $q \in Q$, let $L^q_{i,j}$ be the set of all strings $w \in \Sigma^*$ such that the computation history of M on w visits

- s_1 exactly i times if $i \leq 1$, or at least i times when $i = 2$, and
- s_2 exactly j times if $j \leq 1$, or at least j times when $j = 2$

Notice that "The computation history of M on w visits a state q'" is a loose way of saying that there is a prefix w' of w such that $\hat{\delta}(q_0, w') = q'$. It is clear that $\mathcal{P} := \{L^q_{i,j} \mid q \in Q \text{ and } (i, j) \in \{0, 1, 2\}^2\}$ partitions Σ^*.

Any two strings $x, y \in L^q_{i,j}$ are indistinguishable by $\equiv_{P_{s_1,s_2}}$ for any $q \in Q$ and $(i, j) \in \{0, 1, 2\}^2$. We consider every possible extension $z \in \Sigma^*$, and argue that $x \circ z \in P_{s_1,s_2}$ if and only if $y \circ z \in P_{s_1,s_2}$. For this, we make a case distinction on z. Note that the case distinction on z does not need to be exclusive; it only needs to be exhaustive (we do not miss out any possible form of z).

For each $q \in Q$, note that z falls into one of the following cases where (i, j) is taken over $\{0, 1, 2\}^2$.

1. T^q_{non} be the set of all strings z such that $\hat{\delta}(q, z) \notin F$.
2. $T^q_{i,j}$ be the set of all strings z such that $\hat{\delta}(q, z) \in F$ and the computation history of M on z starting with the configuration (q, z) visits

- s_1 exactly i times if $i \leq 1$, or at least i times when $i = 2$, and
- s_2 exactly j times if $j \leq 1$, or at least j times when $j = 2$

Notice that the purpose of defining these sets is *NOT* to partition the extensions z but to consider all possible cases of z. In particular, the case distinction does not need to be exclusive, but it only needs to exhaustive.

Finally, to see that any two strings $x, y \in L^q_{i,j}$ are indistinguishable by $\equiv_{P_{s_1,s_2}}$, observe that $x \cdot z$ is in P_{s_1,s_2} if and only if $x \cdot z \in L$ and $\hat{\delta}(x \cdot z)$ visits s_1 and s_2 precisely once. The latter condition holds if and only if the computation history of M starting with the configuration (q_0, x) visits s_1 i times and the computation history of M starting with the configuration (q, z) visits s_1 i' times with $i + i' = 1$, likewise for s_2. For any $x \in L^q_{i,j}$, it holds that $x \cdot z \notin L$ thus $x \cdot z \notin P_{s_1,s_2}$ for any $z \in T^q_{non}$, and $x \cdot zi \in P_{s_1,s_2}$ if and only if $z \in T^q_{i',j'}$ with $i + i' = 1$ and $j + j' = 1$. (This step fills "the table" we saw during the class.) In particular, this implies that any two strings of $L^q_{i,j}$ are indistinguishable.

$\mathcal{P}$ **has finitely many parts.** Observe that $|\mathcal{P}| = 9 \cdot |Q|$.