Lec 18. Reduction and undecidable languages I

Eunjung Kim

How to show that a language is undecidable

1. Diagonal argument: used to show that A_{TM} is undecidable.

How to show that a language is undecidable

1. Diagonal argument: used to show that A_{TM} is undecidable.
2. Reduction, a.k.a. Turing-reduction:

A Turing-reducible to B

We say that A is Turing-reducible to B if there is a TM deciding A which uses a (hypothetical) TM deciding B.

Observe: Undecidability of A implies the undecidability of B. ("B is as hard as A")

How to show that a language is undecidable

1. Diagonal argument: used to show that A_{TM} is undecidable.
2. Reduction, a.k.a. Turing-reduction:

A Turing-reducible to B

We say that A is Turing-reducible to B if there is a TM deciding A which uses a (hypothetical) TM deciding B.

Observe: Undecidability of A implies the undecidability of B. ("B is as hard as A")

3. We shall later see a third technique called mapping-reduction (a.k.a many-one reduction).

SOME TERMINOLOGY IN CLASS / TEXT

We say a TM D

- runs a TM M on w if M is a subroutine of D; it is hardwired in D.
- simulates a TM M on w if $\langle M \rangle$ is written on the tape of D and D has a universal TM as hardwired subroutine. In this case D is running the universal TM on $\langle M \rangle$ and $\langle w \rangle$.

Halting problem is undecidable

Halting problem: $HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is TM and } M \text{ halts on } w\}$.

From undecidability of A_{TM}, we derive

$HALT_{TM}$ is undecidable.

Halting problem is undecidable

Halting problem: $HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is TM and } M \text{ halts on } w\}$.

From undecidability of A_{TM}, we derive

$HALT_{TM}$ is undecidable.

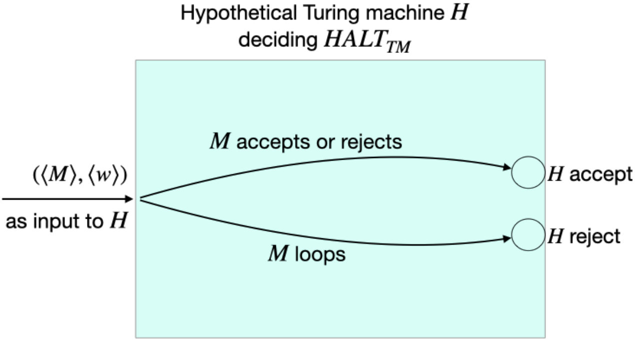
Proof: we crucially use that $A_{TM} = \{\langle M, w \rangle \mid M \text{ is TM and } M \text{ accepts } w\}$ is undecidable.

Halting problem is undecidable

Suppose the contrary; let H be a decider TM for $HALT_{TM}$.

Halting problem is undecidable

Suppose the contrary; let H be a decider TM for $HALT_{TM}$.



Halting problem is undecidable

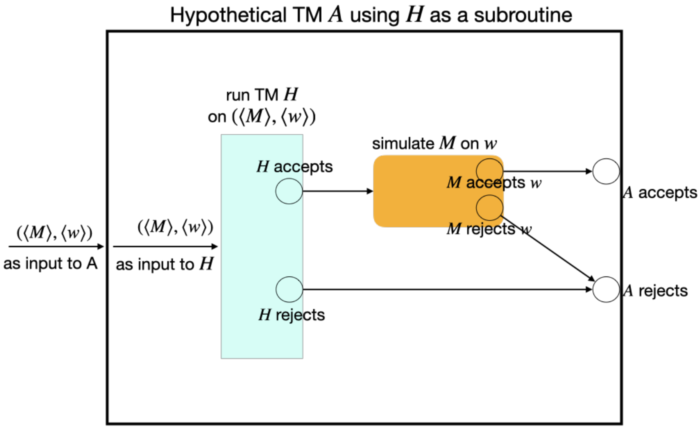
Suppose the contrary; let H be a decider TM for $HALT_{TM}$.

Using the hypothetical decider H for $HALT_{TM}$ as a subroutine, we can build a decider A for A_{TM} as follows (Turing-reduction):

Halting problem is undecidable

Suppose the contrary; let H be a decider TM for $HALT_{TM}$.

Using the hypothetical decider H for $HALT_{TM}$ as a subroutine, we can build a decider A for A_{TM} as follows (Turing-reduction):



HALTING PROBLEM IS UNDECIDABLE

Using the hypothetical decider H for $HALT_{TM}$, we can build a decider A for A_{TM} as follows:

DECIDER A FOR A_{TM} USING H AS A SUBROUTINE

Decider A upon input $\langle M, w \rangle$

1. Run H on the string $\langle M, w \rangle$.
2. if H accepts $\langle M, w \rangle$ (i.e. M halts on w), then A simulates M on w and outputs

$$A(\langle M, w \rangle) = \begin{cases} \text{YES} & \text{if } M(w) = \text{YES} \\ \text{NO} & \text{if } M(w) = \text{NO} \end{cases}$$

3. if H rejects $\langle M, w \rangle$ (i.e. M loops on w), then A rejects $\langle M, w \rangle$.

Halting problem is undecidable

A is a decider.

1. case 1. when M accepts w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs Yes.

HALTING PROBLEM IS UNDECIDABLE

A is a decider.

1. case 1. when M accepts w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs YES.
2. case 2. when M rejects w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs NO.

HALTING PROBLEM IS UNDECIDABLE

A is a decider.

1. case 1. when M accepts w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs YES.
2. case 2. when M rejects w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs NO.
3. case 3. when M loops on w: then H rejects (M, w). By design of A, A outputs NO.

Halting problem is undecidable

A is a decider.

1. case 1. when M accepts w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs Yes.
2. case 2. when M rejects w: then H accepts (M, w), meaning that M halts on w. Therefore, when A simulates M on w, A also halts. By design of A, A outputs No.
3. case 3. when M loops on w: then H rejects (M, w). By design of A, A outputs No.

$L(A) = \{(M, w) \mid M \text{ accepts } w\} = A_{TM}$.

This contradicts that A_{TM} is undecidable.

Emptiness problem is undecidable

Emptiness problem: $E_{TM} = \{M : M \text{ is TM and } L(M) = \emptyset\}$.

Undecidability of E_{TM}

E_{TM} is undecidable.

Emptiness problem is undecidable

Emptiness problem: $E_{TM} = \{M : M \text{ is TM and } L(M) = \emptyset\}$.

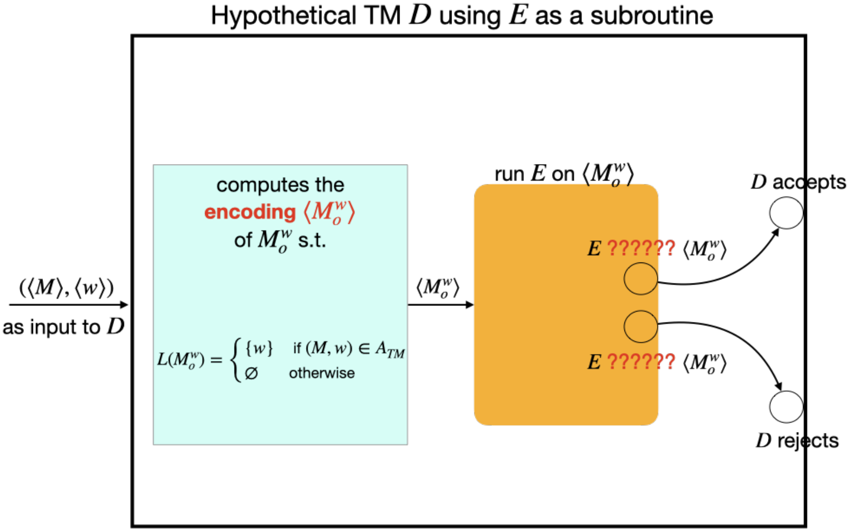
Undecidability of E_{TM}

E_{TM} is undecidable.

Proof outline: Turing-reduction from A_{TM} to E_{TM}.

- Prove that if E_{TM} has a decider E, then this can be used (as a subroutine) to build a decider D for $A_{TM} = \{(M, w) : M \text{ is TM and } M \text{ accepts } w\}$.
- This contradicts the undecidability of A_{TM}.

EMPTINESS PROBLEM IS UNDECIDABLE



Emptiness problem is undecidable

Emptiness problem: $E_{TM} = \{M : M \text{ is TM and } L(M) = \emptyset\}$.

Undecidability of E_{TM}

E_{TM} is undecidable.

- D upon an input $\langle M, w\rangle$ does the following:
  1. Compute an encoding of a TM M_o^w such that

$$L(M_o^w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ does not accept } w \end{cases}$$

  2. Run E on $\langle M_o^w\rangle$.
  3. D outputs

$$\begin{cases} ?????????? & \text{if } E(\langle M_o^w\rangle) = \text{YES} \\ ?????????? & \text{if } E(\langle M_o^w\rangle) = \text{NO} \end{cases}$$

- How to design (compute) such TM M_o^w?

Emptiness problem is undecidable

We want D to compute (i.e. write the encoding of) another TM M_o^w such that

$$L(M_o^w) = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ does not accept } w \end{cases}$$

How does a TM M_o^w work internally? And how can a TM output its encoding?

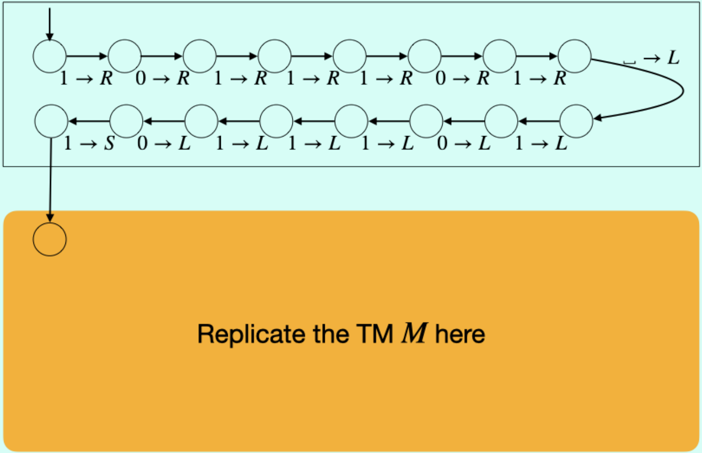
TM M_o^w does the following on input string x:

$$M_o^w(x) = \begin{cases} \text{No} & \text{if } x \neq w \\ \text{Yes} & \text{if } x = w \text{ and } M \text{ accepts } w \end{cases}$$

M_o^w may loop on w (this happens when M loops on w). But it's fine for the purpose of obtaining a desired decider D as D does not simulate M_o^w but only computes its encoding $\langle M_o^w \rangle$.

Emptiness problem is undecidable

TM M_o^w with a hardwired string $w = 1011101$



Non-emptiness is undecidable

$SOME_{TM} = \{M : M \text{ is TM and } L(M) \neq \emptyset\}$.

Undecidability of $SOME_{TM}$ from that of E_{TM}

$SOME_{TM}$ is undecidable.

- Key idea: (Turing-)reduce from E_{TM} to $SOME_{TM}$ by designing a decider D of E_{TM} using the hypothetical TM S deciding $SOME_{TM}$.
- D upon an input $\langle M \rangle$ does the following:
  1. Check if $\langle M \rangle$ is a legit encoding of a TM.
  2. Run S on $\langle M \rangle$.
  3. D outputs

$$D(\langle M \rangle) = \begin{cases} ??????? & \text{if } S(\langle M \rangle) = \text{Yes} \\ ??????? & \text{if } S(\langle M \rangle) = \text{No} \end{cases}$$

REG_{TM} IS UNDECIDABLE

$REG_{TM} = \{M : M \text{ is TM and } L(M) \text{ is a regular language}\}$.

UNDECIDABILITY OF REG_{TM}

REG_{TM} is undecidable.

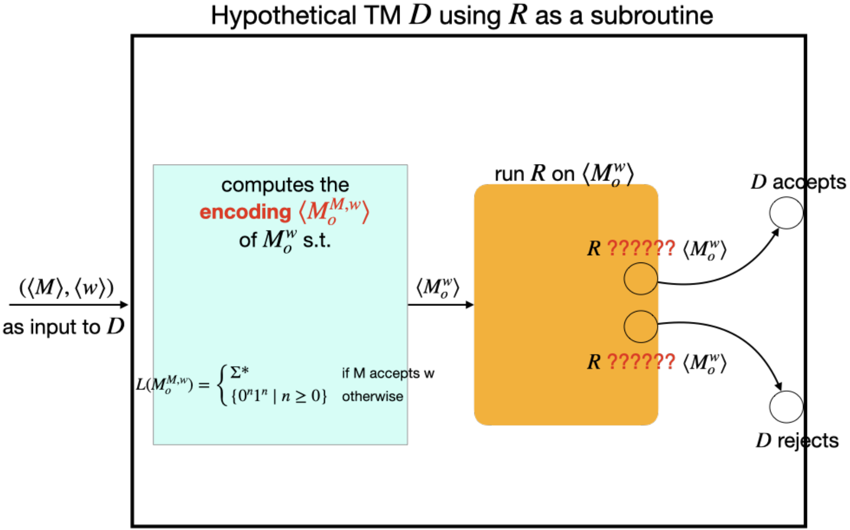
- Key idea: reduction from A_{TM} to REG_{TM}. Suppose that R is a decider for REG_{TM}.
- D, upon an input $\langle M, w\rangle$, does the following:
  1. Compute & write an encoding $\langle M_o^{M,w}\rangle$ of TM $M_o^{M,w}$ such that

$$L(M_o^{M,w}) = \begin{cases} \{0^n1^n \mid n \geq 0\} & \text{if } M \text{ does not accept } w \\ \Sigma^* & \text{if } M \text{ accepts } w \end{cases}$$

  2. Run R on $\langle M_o^{M,w}\rangle$.
  3. D outputs

$$\begin{cases} \text{YES} & \text{if } R \text{ outputs YES} \\ \text{NO} & \text{if } R \text{ outputs NO} \end{cases}$$

Emptiness problem is undecidable



REG_{TM} is undecidable

We want D to compute the encoding of $M_o^{M,w}$ such that

$$L(M_o^{M,w}) = \begin{cases} \{0^n 1^n \mid n \geq 0\} & \text{if } M \text{ does not accept } w \\ \Sigma^* & \text{if } M \text{ accepts } w \end{cases}$$

How does a TM $M_o^{M,w}$ work internally?

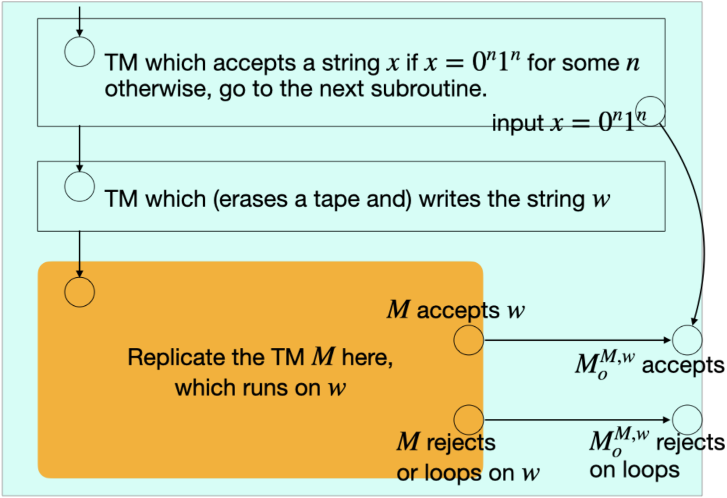
TM $M_o^{M,w}$ does the following on input string x:

$$M_o^{M,w}(x) = \begin{cases} \text{YES} & \text{if } x \text{ is of the form } 0^n 1^n \text{ for some } n \geq 0 \\ \text{YES} & \text{if } M \text{ accepts } w \end{cases}$$

$M_o^{M,w}$ may loop on w. But it's fine to obtain a decider D as D does not simulate $M_o^{M,w}$ but only computes its encoding.

Emptiness problem is undecidable

TM $M_o^{M,w}$ with a hardwired string $w = 1011101$
and using M as a subroutine



Linear Bounded Automaton

Linear bounded Automaton

A linear bounded automaton is a Turing machine with the following restriction: its header is not allowed to move off the portion of the (single) tape containing the input. When the TM instructs the header to move to the right of the right-end of the input, then it stays where it is.

Key observation: For an input of length n, a linear bounded automaton on w can go through at most

$$|Q| \cdot n \cdot |\Gamma|^n$$

distinct configurations. This means

Halting problem for LBA

A linear bounded automaton M halts on an input w of length n if it halts in the first $|Q| \cdot n \cdot |\Gamma|^n$ steps. Does the converse hold?