KAIST, School of Computing, Spring 2024 **Scribed By: Eunjung KIM**
Algorithms for NP-hard problems (CS492) **Lecture #23**
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Contents

1 Combinatorial Approximation for Cut Problems

1.1 Greedy $(2-\frac{2}{k})$-approximation for Edge Multiway Cut

Given an undirected graph G and a set of terminals $T \subseteq V(G)$, an *edge multiway cut* is a set of edges which pairwise separates T. In other words, in the graph $G - X$, there is no (x, y)-path for any pair of terminals $x, y \in T$.

Edge Multiway Cut
Instance: an undirected graph $G = (V, E)$ and a set $T \subseteq V$ of k terminals, an edge weight $\omega : E \to Q^+$.
Goal: Find a minimum weight edge set $X \subseteq E$ such that any two vertices of T are in distinct connected components of $G - X$.

When $|T| = 2$, the problem is precisely finding a minimum weight (s, t)-cut. For $|T| \geq 3$, the (decision version of) Edge Multiway Cut is NP-complete. While finding a small multiway cut is hard, finding a small weighted cut between a fixed terminal t and the rest of the terminals $T \setminus t$ can be done efficiently using any of the polynomial-time algorithm for Minimum (s, t)-cut. Let us call such a cut separating $t \in T$ and $T \setminus t$ an *isolating cut for* t. Clearly, the union of isolating cuts for all terminals is a multiway cut. In fact, you can skip an isolating cut for some arbitrary $t^o \in T$ in the union and the resulting set is still a multiway cut. It turns out that this greedy approach gives $2 - 2/k$-approximation algorithm for Edge Multiway Cut already.

Algorithm 1 Algorithm for Edge Multiway Cut

1: **procedure edgeMWC**(G, T, ω)
2: **for** all $t \in T$ **do**
3: $W_t \leftarrow$minimum weight $(t, T \setminus t)$-cut of G. ▷ W_t can be found in polynomial time.
4: Let $t^o \in T$ be a terminal such that W_{t^o} is maximum.
5: **return** $\bigcup_{t \in T \setminus t^o} W_t$.

Analysis of the procedure edgeMWC(G, T, ω). Let $T = \{t_1, \ldots, t_k\}$. Let $S^* \subseteq E$ be an optimal multiway cut and let $S_i^* \subseteq S^*$ be the minimal $(t_i, T \setminus t_i)$-cut contained in S^*. Note

$$\sum_{i=1}^{k} \omega(S_i^*) = 2 \cdot \omega(S^*)$$

because each edge e of S^* appears in precise two isolating cuts S_i^* and S_j^*, $i \neq j$. This is because $G - S^*$ consists of k connected components $V_1, \ldots, V_k$ with $t_i \in V_i$ for each $i \in [k]$ and an edge $e = uv$ is contained in S^* if and only if $u \in V_i$ and $v \in V_j$ for some $i \neq j$. Note that e appears in S_i^* and S_j^*, and in no other isolating cut.

Let W_i be the isolating cut for t_i found in line 3. Observe

$$\omega(\bigcup_{i=1}^{k} W_i) \leq \sum_{i=1}^{k} \omega(W_i) \leq \sum_{i=1}^{k} \omega(S_i^*) = 2 \cdot \omega(S^*).$$

As we omitted the heaviest isolating set, say W_k, in the output solution and $\omega(W_k) \geq \frac{1}{k} \cdot \left(\sum_{i=1}^{k-1} \omega(W_i)\right)$,

$$\begin{aligned}
\omega(\bigcup_{i=1}^{k-1} W_i) &\leq \sum_{i=1}^{k-1} \omega(W_i) \\
&\leq (1 - \frac{1}{k}) \cdot \sum_{i=1}^{k-1} \omega(W_i) \\
&\leq (1 - \frac{1}{k}) \cdot \sum_{i=1}^{k-1} \omega(S_i^*) \\
&\leq 2(1 - \frac{1}{k}) \cdot \omega(S^*).
\end{aligned}$$

1.2 Gomory-Hu Tree

Alternative views on cuts and submodularity. For vertex sets $X, Y \subseteq V$, the set of edges with one endpoint in X and another endpoint in Y is denoted by $E(X, Y)$. Note that $E(X, Y)$ also contains all edges whose both endpoints are in $X \cap Y$. When $Y = \bar{X} := V \setminus X$, we write $E(X, \bar{X})$ as $\delta(X)$. Note that $\delta(X) = \delta(\bar{X})$.

A pair of vertex sets (X, Y) is a *cut* of G if $X \cup Y = V$ and $X \cap Y = \emptyset$. The *order* of a cut $(X, \bar{X})$ is the cardinality of $\delta(X)$. When the graph G at hand comes with an edge-weight $\omega : E \to \mathbb{R}$, the *weight* of the cut $(X, \bar{X})$ is defined as the value of $\omega(\delta(X))$. We say that a cut $(X, \bar{X})$ is an (x, y)-cut if $|X \cap \{x, y\}| = 1$, or equivalently the edge set $\delta(X)$ is an (x, y)-cut of G.

We say that two cuts $(X, \bar{X})$ and $(A, \bar{A})$ are *crossing* if all four sets $X \cap A$, $X \cap \bar{A}$, $\bar{X} \cap A$ and $\bar{X} \cap \bar{A}$ are non-empty. Two cuts are *non-crossing* if they are not crossing. The following observation is immediate from definition.

Observation 1. *If two cuts (A_1, A_2) and (B_1, B_2) are non-crossing, for some $i, j \in [2]$ it holds $A_i \subseteq B_j$.*

Note that a cut $(X, \bar{X})$ as a vertex bipartition can be uniquely determined by considering the vertex set X or $V \setminus X$. From this perspective, an essential property of cuts called *submodularity of cut functions* can be observed.

A set function $f : 2^E \to \mathbb{R}$ is *submodular* if for every sets $A, B \subseteq E$

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

holds. For an edge-weighted graph $(G = (V, E), \omega)$, the function $\delta^\omega : 2^V \to \mathbb{R}$ defined as

$$\delta^\omega(S) := \omega(\delta(S)) \qquad \text{for all } S \subseteq V,$$

often known as a *cut function*, is submodular.

Gomory-Hu tree.

Let $\alpha : \binom{V}{2} \to Q^+$ be the function mapping every pair (x, y) of vertices of V to the the weight of a minimum (x, y)-cut. That is, $\alpha(x, y)$ is the weight of a min (x, y)-cut.

Definition 1. *Let $(G = (V, E), \omega)$ be an edge-weighted graph. Consider a tree $(T, \alpha|_{E(T)})$ on the vertex set $V(T) = V(G)$ with an edge-weight inheriting the weight function α restricted on the vertex pairs corresponding to $E(T)$. We say that (T, α) is a* Gomory-Hu Tree *of (G, ω) if for every edge xy of T,*

$$\omega(V_x, V_y) = \alpha(xy),$$

where V_x and V_y are the vertex sets of the two connected components of $T - xy$.

For a tree edge $xy \in T$, there is a corresponding cut (V_x, V_y) of G *displayed by the tree edge xy*, where V_x and V_y are the vertex sets of the two connected components of $T - xy$. Definition 1 says that for every edge xy of a Gomory-Hu tree, each cut displayed by a xy is a minimum (x, y)-cut of G. Especially, the $(n-1)$ mininimum cuts displayed by the tree edges pairwise non-crossing. Moreover, Theorem 1 says that these pairwise non-crossing $n-1$ cuts (as vertex bipartition) of Gomory-Hu tree describes minimum (x, y)-cuts for all pairs $(x, y) \in \binom{V}{2}$.

To summarize, there are two surprising facts about Gomory-Hu cut. First, if a Gomory-Hu tree exists, you know not only minimum cuts for $n-1$ vertex pairs, but also all possible $\binom{n}{2}$ min cuts of G. This is stated in Theorem 1. Second, Gomory-Hu tree actually exists, the result of Theorem 2.

Theorem 1. *Let (T, α) be a Gomory-Hu tree of (G, ω). Then for every vertex pair $(u, v) \in \binom{V}{2}$,*

$$\alpha(u, v) = \text{the weight of a min } (u, v)\text{-cut of } (T, \alpha)$$

Proof: We first observe the following.

Claim 1. *For any vertices $x, y, z \in V$, $\alpha(x, z) \geq \min\{\alpha(x, y), \alpha(y, z)\}$.*

PROOF OF THE CLAIM: Consider a min (x, z)-cut (X, Z), where $x \in X$ and $z \in Z$. If $y \in X$, then (X, Z) is a (y, z)-cut and thus $\alpha(x, z) \geq \alpha(y, z)$ holds. Similarly if $y \in Z$, it holds $\alpha(x, z) \geq \alpha(x, y)$. The claimed inequality follows. $\diamondsuit$

If uv is an edge of T, the statement holds by definition of Gomory-Hu tree. Suppose that $u = u_0$ and $v = u_\ell$ are non-adjacent in T and let $u_0, u_1 \ldots, u_\ell$ be the (u, v)-path of T. By Claim 1,

$$\alpha(u, v) \geq \min\{\alpha(u_i, u_{i+1}) : i = 0, \ldots, \ell - 1\}.$$

On the other hand, any cut $(U_i, \bar{U}_i)$ is a (u, v)-cut by definition of Gomory-Hu cut, where $(U_i, \bar{U}_i)$ the cut displayed by the tree edge $u_i u_{i+1}$. This in particular means

$$\alpha(u, v) \leq \min\{\alpha(u_i, u_{i+1}) : i = 0, \ldots, \ell - 1\},$$

thus settling the claimed equation. $\square$

The next is an immediate consequence of Theorem 1.

Corollary 1. *Let (T, α) be a Gomory-Hu tree of (G, ω). Then for every vertex pair $(u, v) \in \binom{V}{2}$, $E(V_x, V_y)$ is a min (u, v)-cut, where xy is the lightest edge on the (u, v)-path of T.*

Gomory-Hu tree exists.

Theorem 2. *For any edge-weighted graph (G, ω), a Gomory-Hu tree (T, α) exists. Moreover, a Gomory-Hu tree can be constructed in polynomial time.*

We present a couple of technical lemmas that are essential for proving Theorem 2.

Lemma 1. *Let $s, t \in V$ be two arbitrary vertices of G and $(S, \bar{S})$ be a minimum (s,t)-cut of G. Then for any $u, w \in S$, there exists a minimum (u, w)-cut $(U, \bar{U})$ which is non-crossing with $(S, \bar{S})$*

Proof: Let $(U, \bar{U})$ be a min (u, w)-cut of G and without loss of generality we assume $u \in U$,$w \in \bar{U}$, $s \in S$ and $t \in \bar{S}$. If $(U, \bar{U})$ and $(S, \bar{S})$ are non-crossing, the statement vacuously holds. So, suppose they are crossing. Note that s either belongs to U or $\bar{U}$, and without loss of generality we may assume that $s \in U$ (otherwise we exchange the roles of u and w in the subsequent proof). To summarize, the setup is as follows.

- $u \in S \cap U$, $w \in S \setminus U$, and
- $s \in S \cap U$, $t \in \bar{S}$.

There are two possibilities depending on whether t belongs to U or not. Let $\delta^\omega : V \to Q^+$ be defined by $\delta^\omega(W) := \omega(\delta(W))$ and observe that δ^ω is submodular.

Case when $t \notin U$. By submodularity of δ^ω,

$$\delta^\omega(S) + \delta^\omega(U) \geq \delta^\omega(S \cap U) + \delta^\omega(S \cup U).$$

Because $\delta(S \cap U)$ and $\delta(S \cap U)$ are a (u, v)-cut and an (s, t)-cut of G respectively, we have

$$\delta^\omega(S \cap U) \geq \delta^\omega(U)$$

and

$$\delta^\omega(S \cup U) \geq \delta^\omega(S).$$

This implies that $\delta^\omega(S \cap U) = \delta^\omega(U)$ (and $\delta^\omega(S \cup U) = \delta^\omega(S)$ as well), or equivalently, the cut $(S \cap U, S \bar{\cap} U)$ is a minimum (u, v)-cut of G. It remains to observe that $(S \cap U, S \bar{\cap} U)$ and $(S, \bar{S})$ are non-crossing.

Case when $t \in U$. In this case, we apply the submodularity to the sets S and $\bar{U}$. That is,

$$\delta^\omega(S) + \delta^\omega(\bar{U}) \geq \delta^\omega(S \cap \bar{U}) + \delta^\omega(S \cup \bar{U}).$$

Now that $\delta(S \cap \bar{U})$ and $\delta(S \cup \bar{U})$ are a (u, v)-cut and an (s, t)-cut of G respectively, and consequently $\delta^\omega(S \cap \bar{U}) = \delta^\omega(U)$. That is $(S \setminus U, S \bar{\setminus} U)$ is a minimum (u, v)-cut, and it is non-crossing with $(S, \bar{S})$. $\square$

Proof of Theorem 2:

To demonstrate the existence of a Gomory-Hu tree, we shall construct a sequence of trees T_i on a set $R_i \subseteq V$ of i nodes for $i = 1$ up to n while maintaining the following invariant:

($\dagger$) there is a surjective mapping $\varphi_i : V \to R_i$ from the vertices of G onto R_i such that $\varphi(r) = r$ for every $r \in R_i$,

($\star$) for each tree edge $e = xy$ of T_i, the cut $(U, \bar{U})$ of G displayed by e is a minimum (x, y)-cut of G. Here, $U = \bigcup_{r \in K} \varphi^{-1}$ and $K \subseteq R_i$ is a connected component of $T_i - e$.

Clearly, $(T_n, \alpha|_{E(T)})$ is a Gomory-Hu tree of (G, ω).

Let $R_1 = \{s\}$ where s is an arbitrary vertex of G. The trivial tree T_1 on $\{s\}$ with $\varphi_1(V) = \{s\}$ satisfies the invariants (†) and ($\star$) trivially. Assume that T_i is a tree on $R_i \in V$ satisfying (†) and ($\star$) with a mapping φ_i for $i < n$. Choose a vertex $x \in R_i$ such that $\varphi^{-1}(x)$ contains at least two vertices of G and let y be an arbitrary vertex of $\varphi_i^{-1}(x) \setminus x$. Let $(U, \bar{U})$ be a minimum (x, y)-cut of G.

We claim that there is a minimum (x, y)-cut $(U^*, \bar{U}^*)$ which is non-crossing with each cut of G displayed by a tree edge in T_i incident with x. Once such an (x, y)-cut $(U^*, \bar{U}^*)$ is found, we refine T_i into T_{i+1} as follows. Without loss of generality, we assume $x \in U^*$.

1. Let $w_1, \ldots, w_\ell \in R_i$ be the neighbors of x in T_i.
2. Let $W_j \subseteq V$ be the vertex set $\bigcup_{r \in K_j} \varphi_i^{-1}(r)$, where K_j is the connected component of $T_i - x$ containing w_j. If we see T_i as a tree rooted at x and thus w_j's as the children of x in T_i, W_j is the set of all vertices of V which are assigned to the subtree rooted at w_j.
3. Classify each vertex w_j into one of the two parts depending on whether the corresponding set W_j is fully contained in U^* or in $\bar{U}^*$. Recall that one of the two situations should occur, see Observation 1.
4. Let $J \subseteq [\ell]$ be the set of children w_j such that $W_j \in U^*$.
5. Let $R_{i+1} = R_i \cup \{y\}$.
6. T_{i+1} is obtained from T_i by substituting the node x by the edge xy and
   - each subtree rooted at w_j with $j \in J$ is a child of x, (recall $x \in U^*$ and $y \in \bar{U}^*$)
   - each subtree rooted at w_j with $j \in [\ell] \setminus J$ is a child of y.
7. Finally, $\varphi_{i+1}(v) = x$ for all $v \in \varphi_i^{-1}(x) \cap U^*$ and $\varphi_{i+1}(v) = y$ for all $v \in \varphi_i^{-1}(x) \setminus U^*$, and for all other vertices of G assigned to a tree node other than x by φ_i, we keep the same assignment in the new φ_{i+1}.

It is tedious to verify that the newly constructed structure $(R_{i+1}, T_{i+1}, \varphi_{i+1})$ meets the invariants (†) and ($\star$). Moreover, such a cut $(U^*, \bar{U}^*)$ can be computed by a single application of minimum cut algorithm (How?). This complete the proof of the theorem.

Note that the proof is constructive and can be turned into an efficient algorithm, provided that a desired cut $(U^*, \bar{U}^*)$ above can be found in polynomial time. This is left as an exercise for the readers. □

1.3 Greedy $(2 - \frac{2}{k})$-approximation for k-Cut

k-Cut
Instance: an undirected graph $G = (V, E)$, an edge weight $\omega : E \to Q^+$ and a positive integer k.
Goal: Find a minimum weight edge set $X \subseteq E$ such that $G - X$ consists of k connected components.

Using the result of the previous subsection, we present a greedy approximation algorithm for k-Cut with approximation ratio $(2 - 2/k)$.

Analysis of the procedure kCUT(G, T, ω). Let $S^* \subseteq E$ be an optimal k-cut of G and

Algorithm 2 Algorithm for k-CUT

1: **procedure** k**CUT**(G, T, ω)
2: Compute a Gomory-Hu Tree T of G.
3: Let $f_1, \dots, f_{k-1}$ be a set of $k-1$ lightest edge in T.
4: Let $W_1, \dots, W_{k-1}$ be the edge sets of G displayed by $f_1, \dots, f_{k-1}$.
5: **return** $\bigcup_{i=1}^{k-1} W_i$.

- let V_i for $1 \le i \le k$ be the connected components of $G - S^*$,
- let $S_i^* := E(V_i, V \setminus V_i)$ for $1 \le i \le k$.

Without loss of generality, we assume that $\omega(S_i^*) \le \omega(S_k^*)$ for every $i \in [k]$. Clearly, it holds that $\omega(S_k^*) \ge (1/k) \cdot \sum_{i=1}^{k} \omega(S^* i)$. For now, let us assume that G is connected. The case when G is not connected will be discussed at the end of the analysis.

Consider a Gomory-Hu tree (T, α) of (G, ω). Because T spans the vertex set of G, one can greedily choose a subset $F \subseteq E(T)$ of $k-1$ tree edges so that the following holds:

- for every $i \in [k-1]$, F has exactly one edge $e_i = (v_i, v_i')$ such that $v_i \in V_i$ and $v_i' \in V \setminus V_i$, and
- the graph H obtained from (V, F) by identifying each V_i, $i \in [k]$, into a single vertex is a tree.

Now, the key fact here is that for each $i \in [k-1]$, S_i^* separates V_i and $V \setminus V_i$. This in particular means that S_i^* is a (v_i, v_i')-cut of G, and consequently, $\alpha(v_i, v_i') \le \omega(S_i^*)$. As the edges e_i are all distinct edges of T, and $W_1, \dots, W_{k-1}$ chosen during the the procedure k**CUT**(G, T, ω) are the $k-1$ lightest edges of T, we have the lower bound

$$\sum_{i=1}^{k-1} \omega(W_i) = \sum_{i=1}^{k-1} \alpha_T(f_i) \le \sum_{i=1}^{k-1} \alpha_T(e_i) \le \sum_{i=1}^{k-1} \omega(S_i^*).$$

From $\sum_{i=1}^{k} \omega(S_i^*) = 2 \cdot \omega(S^*)$ and $\omega(S_k^*) \ge (1/k) \cdot \sum_{i=1}^{k} \omega(S_i^*)$, we conclude

$$\omega(\bigcup_{i=1}^{k-1} W_i) \le \sum_{i=1}^{k} \omega(W_i) \le 2(1 - \frac{1}{k}) \cdot \omega(S^*).$$

Notice that the first equation holds by definition of Gomory-Hu tree.

It remains to see that $\bigcup_{i=1}^{k-1} W_i$ is indeed a k-cut of G. By definition of Gomory-Hu Tree, with each addition of W_i to the output solution we separate a new vertex pair, thus strictly increasing the number of connected components. Therefore, $G - \bigcup_{i=1}^{k-1} W_i$ has at least k connected components.

Remark on the case when G is not connected. Note that Gomory-Hu tree can be constructed regardless of whether G is connected or not, and distinct connected components are displayed by a tree edge with zero weight. The same analysis goes through in the presence of multiple connected components in G.