Contents

1 Approximation for Cut problems via simple randomized rounding

1.1 $(2 - \frac{2}{k})$-approximation for EDGE MULTIWAY CUT

Let $T = \{t_1, \dots, t_k\}$ be the set of terminals and for $1 \leq i < j \leq k$, let $\mathcal{P}_{i,j}$ be the collection of all (t_i, t_j)-path in G. The linear program for EDGE MULTIWAY CUT is formulated as follows.

$$\begin{aligned} \min \sum_{e \in E(G)} \omega_e \cdot x_e & & \\ \sum_{e \in P} x_e \geq 1 & \qquad \forall P \in \mathcal{P}_{i,j} \quad \text{for all } 1 \leq i < j \leq k \\ x_e \geq 0 & \qquad \forall e \in E(G) \end{aligned}$$

Algorithm 1 Randomized Rounding for EDGE MULTIWAY CUT

procedure θ-**mwc**(G, ω, T)
- Solve LP for MINIMUM (s,t)-CUT to obtain an optimal fractional solution x^*.
- For each $i \in [k]$ and $v \in V$, compute $\texttt{dist}_G(t_i, v)$ with x_e^* interpreted as the length of edge e.
- Choose $\theta \in (0, \frac{1}{2})$ uniformly at random.
- For each $i \in [k]$, let $S_i := B(t_i, \theta) \subseteq V$ be the set of vertices v with $\texttt{dist}_G(t_i, v) \leq \theta$.
- **return** $\bigcup_{i=1}^k \delta(S_i)$.

Lemma 1. *The output set of the procedure θ-**mwc**(G, ω, T) is a multiway cut.*

Proof: Clearly, $t_i \in S_i$ for each $i \in [k]$. Moreover, S_i does not contain any terminal t_j for $j \neq i$ because $\texttt{dist}_G(t_i, t_j) \geq 1 > \theta$. Therefore $\delta(S_i)$ is an isolating cut for t_i, i.e. an $(t_i, T \setminus t_i)$-cut of G. It follows that $\bigcup_{i=1}^k \delta(S_i)$ pairwise separates all terminals of T. □

Lemma 2. *The expected weight of an output set is at most $2 \cdot \sum_{e \in E} \omega_e \cdot x_e^*$.*

Proof: The expected weight of an output set is at most

$$\sum_{e = xy \in E} \omega_e \cdot \texttt{prob}[\texttt{dist}_G(t, x) \leq \theta \text{ and } \texttt{dist}_G(t, y) > \theta \text{ for some } t \in T],$$

where x is an endpoint of e which not farther from t_i than the other endpoint y is, and this choice can be different for a fixed edge e depending on the terminal t_i.

To analyze the probability of the event $[\texttt{dist}_G(t, x) \leq \theta$ and $\texttt{dist}_G(t, y) > \theta$ for some $t \in T]$, we define $B_i^{0.5}$ as the set of vertices v with $\texttt{dist}_G(t_i, v) < 0.5$.

There are a few cases to consider for the type of an edge $e = xy$. The cases are exclusive, and we assume that none of the preceding cases are applicable for the later case.

1. an endpoint of e, say x, belongs to at least two balls $B_i^{0.5}$ and $B_j^{0.5}$.
2. (1 is not applicable, and) $x \in B_i^{0.5}$ and $y \in B_j^{0.5}$ for $i \neq j$.
3. (1-2 are not applicable, and) there exists a unique ball $B_i^{0.5}$ such that $B_i^{0.5} \cap \{x, y\} \neq \emptyset$.
4. (1-3 are not applicable, and) there is no ball $B_i^{0.5}$ such that $B_i^{0.5} \cap \{x, y\} \neq \emptyset$.

Type 1 edges do not exist because if $x \in B_i^{0.5} \cap B_j^{0.5}$, then $\mathtt{dist}_G(t_i, t_j) < 1$ and x^* is infeasible. Consider a Type 2 edge $e = xy$ and note that $\mathtt{dist}_G(t_i, y) > 0.5$ and $\mathtt{dist}_G(t_j, x) > 0.5$. Therefore,

$$
\begin{aligned}
&\mathtt{prob}[\mathtt{dist}_G(t, x) \le \theta \text{ and } \mathtt{dist}_G(t, y) > \theta \text{ for some } t \in T] \\
&\le \max\{\mathtt{prob}[\mathtt{dist}_G(t_i, x) \le \theta], \mathtt{prob}[\mathtt{dist}_G(t_j, y) \le \theta]\} \\
&\le \frac{\max\{0.5 - \mathtt{dist}_G(t_i, x), 0.5 - \mathtt{dist}_G(t_j, y)\}}{\text{the length of the interval that } \theta \text{ is chosen from}}\} \\
&\le 2 \cdot (1 - \mathtt{dist}_G(t_i, x) - \mathtt{dist}_G(t_j, y)) \\
&\le 2 \cdot x_e^*
\end{aligned}
$$

where the last inequality comes from $\mathtt{dist}_G(t_i, x) + x_e^* + \mathtt{dist}_G(t_j, y) \ge \mathtt{dist}_G(t_i, t_j) \ge 1$.

For Type 3 edge e, the same analysis for the randomized rounding for MINIMUM (s, t)-CUT easily yields the probability bound at most $2x_e^*$. Notice that both endpoints or exactly one endpoint of e may lie in $B_i^{0.5}$. For Type 4 edge e, the probability is simply zero.

To sum up,

$$
\sum_{e=xy \in E} \omega_e \cdot \mathtt{prob}[\mathtt{dist}_G(t, x) \le \theta \text{ and } \mathtt{dist}_G(t, y) > \theta \text{ for some } t \in T] \le 2 \cdot \sum_{e=xy \in E} \omega_e \cdot x_e^*,
$$

which completes the proof. □

1.2 2-Approximation for EDGE MULTICUT ON TREES

EDGE MULTICUT ON TREES
Instance: a tree $T = (V, E)$, a set k terminal pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\}$ and edge weight $\omega : E \to Q^+$.
Goal: Find a minimum weight edge set $X \subseteq E$ such that no terminal pair (s_i, t_i) has a path in $T - X$.

EDGE MULTICUT ON TREES is NP-complete even on a tree of height 1 (star) and uniform weight on the edges. In this section, we present a 2-approximation algorithm for EDGE MULTICUT ON TREES based on a simple randomized rounding of LP solution.

Note that any pair of vertices on a tree is connected by a unique path. Let P_i be the unique (s_i, t_i)-path on T. The linear program for EDGE MULTICUT ON TREES is formulated as follows.

$$
\begin{aligned}
\min \sum_{e \in E(G)} \omega_e \cdot x_e & \\
\sum_{e \in P_i} x_e \ge 1 & \qquad \forall i \in [k] \\
x_e \ge 0 & \qquad \forall e \in E(G)
\end{aligned}
$$

Let x^* be an optimal fractional multicut, that is, an optimal solution to the above LP. Again, we interpret x_e^* as the edge length on e. We construe T as a rooted tree, and let r be the (arbitrarily chosen) root of T. With

$x^* : E \to Q^+$ interpreted as the edge length function[1] we define the *distance* of a vertex $v \in V$ from the root in the usual way:

$$\mathtt{dist}_{x^*}(u, v) := \sum_{e \in E(P_{u,v})} x^*_e,$$

where $P_{u,v}$ is the unique (u, v)-path on T.

Choose $\theta \in (0, \frac{1}{2})$ uniformly at random. Let $B(r, d)$ be the set of all vertices v such that $\mathtt{dist}_{x^*}(r, v) \leq d$. We *cut* all the edges in $\delta(B(r, \theta))$, $\delta(B(r, \theta + \frac{1}{2}))$, $\delta(B(r, \theta + 1))$, $\delta(B(r, \theta + \frac{3}{2}))$. We argue that the set of deleted edges, denoted as D, is a multicut within $2 \cdot \omega(\mathtt{lp\ opt})$.

Algorithm 2 Randomized Rounding for EDGE MULTICUT ON TREES

```
procedure θ-mwctree(G, ω, T)
    Solve LP for EDGE MULTICUT ON TREES to obtain an optimal fractional solution x*.
    Compute the distance dist_{x*}(r, v) for each v ∈ V.
    Choose θ ∈ (0, 1/2) uniformly at random.
    For i ∈ ℕ, let S_i := B(r, θ + i · 1/2) ⊆ V be the set of vertices v with dist_{x*}(r, v) ≤ θ + 1/2 · i.
    return ⋃_{i=1}^∞ δ(S_i).
```

The next observation is immediate from $\mathtt{dist}_{x^*}(s_i, \ell_i) + \mathtt{dist}_{x^*}(t_i, \ell_i) = \mathtt{dist}_{x^*}(s_i, t_i) \geq 1$ and the feasibility of x^*.

Observation 1. *Let ℓ_i be the least common ancestor of s_i and t_i on T. It holds that*

$$\max\{\mathtt{dist}_{x^*}(s_i, \ell_i), \mathtt{dist}_{x^*}(t_i, \ell_i)\} \geq 0.5.$$

Lemma 3. *The edge set B is a multicut.*

Proof: It suffices to argue that for each $i \in [k]$, at least one of the (s_i, ℓ_i)-path and (t_i, ℓ_i)-path is cut by D. By Observation 1, we may assume that $\mathtt{dist}_{x^*}(s_i, \ell_i) \geq 0.5$ without loss of generality. Let p be the least integer such that $\ell_i \in B(r, \theta + p \cdot \frac{1}{2})$.

If $p = 0$, then it holds that $\mathtt{dist}_{x^*}(r, s_i) \geq \mathtt{dist}_{x^*}(\ell_i, s_i) \geq 0.5 > \theta \geq \mathtt{dist}_{x^*}(r, \ell_i)$, implying that $s_i \notin B(r, \theta)$ and the (ℓ_i, s_i)-path is cut by D.

If $p \geq 1$, observe

$$\mathtt{dist}_{x^*}(r, s_i) = \mathtt{dist}_{x^*}(r, \ell_i) + \mathtt{dist}_{x^*}(\ell_i, s_i) > \big(\theta + (p-1) \cdot \frac{1}{2}\big) + \frac{1}{2} = \theta + p \cdot \frac{1}{2}.$$

This again implies that $s_i \notin B(r, \theta + p \cdot \frac{1}{2})$ and the (ℓ_i, s_i)-path is cut by D. $\square$

Lemma 4. *The expected weighted of the chosen edge set is at most* $2 \cdot \sum_{e \in E} \omega_e \cdot x^*_e$.

Proof: The probability that each edge is chosen for D is at most $2 \cdot x^* e$ and the statement follows. $\square$

[1]When all coefficients are rational, there is a rational LP optimal solution and the usual polynomial-time algorithm finds such a solution.

2 Half-integrality of VERTEX MULTIWAY CUT via Complementary Slackness

VERTEX MULTIWAY CUT
Instance: an undirected graph $G = (V, E)$, a set $T \subseteq V$ of k terminals $t_1, \ldots, t_k$, a vertex weight $\omega : V \setminus T \to Q^+$.
Goal: Find a minimum weight vertex set $X \subseteq V \setminus T$ such that any two vertices of T are in distinct connected components of $G - X$.

Let $T = \{s_1, \ldots, s_k\}$ be the set of terminals and for $1 \leq i < j \leq k$, let $\mathcal{P}_{i,j}$ be the collection of all (t_i, t_j)-path in G. The collection $\mathcal{P}$ is the union of $\mathcal{P}_{i,j}$ for all $1 \leq i < j \leq k$. The linear program for VERTEX MULTIWAY CUT is formulated as follows.

$$\begin{aligned}
\min \sum_{v \in V \setminus T} & \omega_v \cdot x_v & \\
\sum_{v \in V(P) \setminus T} & x_v \geq 1 & \forall P \in \mathcal{P} \\
& x_v \geq 0 & \forall v \in E(G)
\end{aligned}$$

The dual LP for VERTEX MULTIWAY CUT is:

$$\begin{aligned}
\max \sum_{P \in \mathcal{P}} & f_P & \\
\sum_{\substack{P \in \mathcal{P}: \\ P \text{ traverses } v}} & f_P \leq \omega_v & \forall v \in V \setminus T \\
& f_P \geq 0 & \forall P \in \mathcal{P}
\end{aligned}$$

We prove that the LP for VERTEX MULTIWAY CUT always has a half-integral solution, that is, an optimal solution to the primal LP such that each variable is assigned with a value in $\{0, 0.5, 1\}$.

Theorem 1. *There is an optimal solution x^{**} to LP for* VERTEX MULTIWAY CUT *such that $x_v^{**} \in \{0, 0.5, 1\}$ for every $v \in V \setminus T$. Moreover, such an optimal solution can be constructed in polynomial time.*

Corollary 1. *In polynomial time, one can find a vertex multiway cut S to an input (G, T, ω) such that $\omega(S) \leq 2 \cdot \sum_{v \in V \setminus T} \omega_v \cdot x_v^*$, where x^* is an optimal solution to LP of* VERTEX MULTIWAY CUT.

Proof: We apply Theorem 1 and find a half-integral optimal LP solution x^*. Let $X = \{v \in V \setminus T : x_v^* > 0\}$. It is clear that X is a multiway cut for (G, T). Observe that $\sum_{v \in X} \omega_v \leq \sum_{v \in X} \omega_v \cdot 2x_v^* = 2 \sum_{v \in V \setminus T} \omega_v \cdot x_v^*$, as claimed. □

Henceforth, we prove the half-integrality of VERTEX MULTIWAY CUT.

Let x^* and f^* be optimal solutions to the primal and dual LP of VERTEX MULTIWAY CUT, and beware that x^* and f^* are not necessarily half-integral. As in the case of MINIMUM (s, t)-CUT and EDGE MULTIWAY CUT we interpret the assigned value x_v^* on v by the primal optimal fractional solution x^* as the length (or cost) of the vertex v. For $v \in V$ and a terminal $s \in T$, the distance of v from t is

$$\mathtt{dist}_{x^*}(t,v) := \begin{cases} 0 & \text{for } v = t \\ \min_{P \in \mathcal{P}_{t,v}} \sum_{u \in V(P) \setminus T} x^*_u & \text{for } v \in T \setminus t \end{cases}$$

where $\mathcal{P}_{t,v}$ is the collection of all (s,v)-path of G. Because x^* is a feasible solution to the primal LP and $\mathcal{P}_{t,t'} \subseteq \mathcal{P}$ for every $t, t' \in T$, we have $\mathtt{dist}_{x^*}(t,t') \geq 1$ for every $t, t' \in T$.

Region, Boundary, Communal and private boundary. For $i \in [k]$, let $T_i \subseteq V$ be the set $\{v \in V : \mathtt{dist}_{x^*}(t_i, v) = 0\}$, called the *region of terminal* t_i, and $B_i := N_G(T_i)$, i.e. the vertices of $V \setminus T_i$ which has a neighbor in T_i. Let us call B_i the *boundary of the region* T_i and B be the union of all boundaries, i.e. $\bigcup_{i=1} B_i$. Note that

$$x^*_v > 0 \qquad \forall v \in B \tag{1}$$

since otherwise (i.e. $x^*_v = 0$) v would have been included in T_i. Moreover, all vertices of T_i for any $i \in [k]$ is assigned 0 by x^*_v whereas $x^*_v > 0$ for all vertices $v \in B_i$ for any $i \in [k]$, implying

$$B \cap T_i = \emptyset \qquad \text{for all } i \in [k].$$

A vertex v of $B = \bigcup_{i=1}^k B_i$ falls into one of the two types:

- there are (at least) two indices $i, j \in [k]$ such that $v \in B_i \cap B_j$; let B^{com} denote the set of these vertices.
- there is a unique index $i \in [k]$ such that $v \in B_i$; let B^{prv} denote the set of such vertices.

Lemma 5. $x^*_v = 1$ *for every* $v \in B^{com}$.

Complementary Slackness Condition.

Lemma 6. *Let* $P \in \mathcal{P}$ *be a* (t_i, t_j)*-path with* $f^*_P > 0$*. Then either (i)* $V(P) \cap B = \{v\}$ *for some* $v \in B^{com}$*, or (ii)* $V(P) \cap B = \{u, v\}$ *for some* $u \in B_i \cap B^{prv}$ *and* $v \in B_j \cap B^{prv}$.

Proof: Note that Dual complementary slackness condition says:

$$\text{for every } P \in \mathcal{P} \text{ with } f^*_P > 0, \text{ we have } \textstyle\sum_{v \in V(P) \setminus T} x^*_v = 1.$$

Together with Lemma 5, we know that if such P intersects with B^{com}, say at v, P cannot contain any other vertex with positive value at x^*. Especially, P does not contain any other vertex of B other than v due to Equation 1.

Suppose that $V(P) \cap B^{com} = \emptyset$. As P connects two terminals t^1 and t^2 and P can run between distinct terminals only via some boundary vertex of B_1 and B_2, we have P traverses at least two vertices $u \in B_i \cap B^{prv}$ and $v \in B_j \cap B^{prv}$. Suppose that P traverses another boundary vertex $z \in B_\ell \cap B^{prv}$ (possibly $\ell = i$ or $\ell = j$). As $i \neq j$, we may assume that ℓ is different from at least one of i and j. Without loss of generality, assume $\ell \neq i$. Then the (t_i, t_ℓ)-path Q can be obtained from P by taking the (t_i, z)-subpath of P and appending it with the edge zt_ℓ. On the other hand, Q does not contain v while $x^*_v > 0$, which implies

$$\sum_{w \in V(Q) \setminus T} x^*_w \leq \Big(\sum_{w \in V(P) \setminus T} x^*_w \Big) - x^*_v = 1 - x^*_v < 1$$

where the second equality holds due to the dual complementary slackness condition. This means that x^* violates the primal constraint corresponding to Q, a contradiction. Therefore, we conclude that if $V(P) \cap B^{com} = \emptyset$, then the case (ii) in the statement holds. $\square$

Now we define a half-integral solution x' to LP of VERTEX MULTIWAY CUT and prove that x' is an optimal LP solution.

$$x'_v = \begin{cases} 1 & \text{if } v \in B^{com} \\ \frac{1}{2} & \text{if } v \in B^{prv} \\ 0 & \text{otherwise} \end{cases}$$

It is trivial to verify that x' is a feasible solution to the primal LP. Recall that due to Weak LP duality and the optimality of f^*, the following holds for x' and f^*.

$$\sum_{P \in \mathcal{P}} f_P^* \leq \sum_{P \in \mathcal{P}} f_P^* \cdot \Big(\sum_{v \in V(P) \setminus T} x_v^* \Big) = \sum_{v \in V(P) \setminus T} x_v^* \cdot \Big(\sum_{\substack{P \in \mathcal{P}: \\ P \text{ traverses } v}} f_P^* \Big) \leq \sum_{v \in V \setminus T} \omega_v \cdot x_v^*,$$

and the Complementary Slackness Condition for primal and dual optimal solutions says that x' is an optimal solution to the primal LP if and only if the following holds (f^* is a dual optimal already).

Primal complementary slackness for every v with $x'_v > 0$, we have $\sum_{\substack{P \in \mathcal{P}: \\ P \text{ traverses } v}} f_P^* = \omega_v$.

Dual complementary slackness for every $P \in \mathcal{P}$ with $f_P^* > 0$, we have $\sum_{v \in V(P) \setminus T} x'_v = 1$.

Due to the optimality of x^*, for every $u \in \{v \in V \setminus T : x_v^* > 0\}$ it holds that $\sum_{\substack{P \in \mathcal{P}: \\ P \text{ traverses } u}} f_P^* = \omega_u$. From $x_v^* > 0$ for every $v \in B$ (see Equation 1) and

$$\{v \in V \setminus T : x'_v > 0\} = B \subseteq \{v \in V \setminus T : x_v^* > 0\},$$

we know that the primal complementary slackness condition holds for x' and f^*. The dual complementary slackness holds due to Lemma 6 and by the construction of x'.