Contents

1 Terminology

Definition 1 (Substructure)**.** *For a set U of elements of a τ-structure $\mathbb{A}$, the* substructure *of $\mathbb{A}$ induced by U is the τ-structure $\mathbb{A}'$ defined as follows and for each $R \in \tau$,*

- *the universe of $\mathbb{A}$ is $U \cup \{c^{\mathbb{A}} \mid c$ is a constant symbol$\}$,*
- *each constant symbol is interpreted as the same element as in $\mathbb{A}$,*
- *for each predicate $R \in \tau$, R is interpreted as $R^{\mathbb{A}} \cap U^{\mathsf{ar}(R)}$.*

Definition 2. *For a τ-structure $\mathbb{A}$, the* Gaifman graph $G(\mathbb{A})$ *of $\mathbb{A}$ is a graph whose vertex set is the universe of $\mathbb{A}$, and there is an edge (without any orientation) between a pair of distinct elements $u, v \in \mathbb{A}$ if and only if there is some $R \in \tau$ such that u, v is related in the relation $R^{\mathbb{A}}$. That is, there is a tuple $\vec{a} \in R^{\mathbb{A}}$ such that $u = a_i$ and $v = a_j$ for some $1 \le i, j \le \mathsf{ar}(R)$.*

The *distance* between two elements u, v of $\mathbb{A}$ in $\mathbb{A}$ is defined as their distance in the Gaifman graph of $\mathbb{A}$, denoted as $\mathsf{dist}_{\mathbb{A}}(u, v)$. For a tuple $\vec{u} = (u_1, \ldots, u_\ell$ of elements of $\mathbb{A}$, the distance $\mathsf{dist}_{\mathbb{A}}(\vec{u}, v)$ between $\vec{u}$ and v in $\mathbb{A}$ is $\min\{\mathsf{dist}_{\mathbb{A}}(u_i, v) \mid i \in [\ell]\}$.

Definition 3 (r-ball, r-neighborhood)**.** *For a τ-structure $\mathbb{A}$ and an element $a \in \mathbb{A}$, the* r-ball $B_r^{\mathbb{A}}(a)$ *at a in $\mathbb{A}$ is the set of elements of $\mathbb{A}$ whose distance to a is at most r in the Gaifman graph $G(\mathbb{A})$ of $\mathbb{A}$. For a tuple $\vec{a} = (a_1, \ldots, a_\ell)$ of $\mathbb{A}$, the r-ball $B_r^{\mathbb{A}}(\vec{a})$ at $\vec{a}$ in $\mathbb{A}$ is defined the same way: $B_r^{\mathbb{A}}(\vec{a}) := \{v \in \mathbb{A} \mid \mathsf{dist}_{\mathbb{A}}(\vec{a}, v) \le r\}$. The* r-neighborhood of $\vec{a}$ in $\mathbb{A}$ *is the substructure of $\mathbb{A}$ induced by the r-ball at $\vec{a}$, and it is denoted by $N_r^{\mathbb{A}}(\vec{a})$*

Definition 4. *(ℓ-queries) Given an integer $\ell \ge 0$, an ℓ-*query *on τ-structures is a map Q such that*

- *$Q(\mathbb{A}) \subseteq A^\ell$ for any τ-struture $\mathbb{A}$; and*
- *it is closed under isomorphism, that is, if two τ-structures $\mathbb{A}, \mathbb{B}$ are isomorphic by an isomorphism $h : A \to B$, then $Q(\mathbb{B}) = h(Q(\mathbb{A}))$. Here,*

$$h(Q(\mathbb{A})) = \{(h(a_1), \ldots, h(a_\ell)) : (a_1, \ldots, a_\ell) \in Q(\mathbb{A})\}.$$

In particular, we consider A^0 as a singleton set so every 0-query is exactly a property on σ-structures.

Definition 5. *(Definable ℓ-queries) Given an ℓ-query Q on τ-structures and a logic $\mathcal{L}$, Q is* definable in $\mathcal{L}$ *if there is a formula $\varphi(x_1, \ldots, x_\ell)$ of $\mathcal{L}$ in τ such that*

$$Q(\mathbb{A}) = \{(a_1, \ldots, a_\ell) \in A^\ell : \mathbb{A} \vDash \varphi(a_1, \ldots, a_\ell)\}$$

for every τ-structure $\mathbb{A}$.

For example, the set of vertex pairs of distance exactly two is a 2-query. It is also FO-definable using the formula $\varphi(x, y) := \exists z \, \mathsf{edge}(x, z) \wedge \mathsf{edge}(z, y) \wedge \neg(x, y)$.

The isomorphism between two structures over the same vocabulary is defined in the usual way.

Definition 6 (Isomorphism between two structures)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be two τ-structures. A mapping $\iota : \mathbb{A} \to \mathbb{B}$ is an isomorphism between $\mathbb{A}$ and $\mathbb{B}$ if*

- *ι is a bijection,*
- *for every constant symbol $c \in \tau$ and for every $i \leq \ell$, $\iota(c^{\mathbb{A}}) = c^{\mathbb{B}}$,*
- *for every predicate $R \in \tau$ with $\mathsf{ar}(R) = k$ and for every k-tuple $(a_1, \ldots, a_k) \in \mathbb{A}^k$, $R(a_1, \ldots, a_k)$ if and only if $R(\iota(a_1), \ldots, \iota(a_k))$.*

We write $\mathbb{A} \cong \mathbb{B}$ when there is $\mathbb{A}$ is isomorphic to $\mathbb{B}$.

Note that a property $\mathcal{P}$ of τ-structures is (defined so that) closed under isomorphism. That is, if $\mathbb{A}$ has the property $\mathcal{P}$ and $\mathbb{B}$ is isomorphic to $\mathbb{A}$ then $\mathbb{B}$ also has the property $\mathcal{P}$. Note also that ℓ-query, a generalization of a property, defined so as to be closed under isomorphism.

Intuitively, an ℓ-query Q on τ-structures is Hanf local if the query is closed under the isomorphism of the r-neighborhood. Hanf locality is not guaranteed, and it is rather an anomaly. However, it turns out that an FO-definable ℓ-query is Hanf local.

Definition 7 (r-local isomorphism between two structures)**.** *Let $\mathbb{A}$ and $\mathbb{B}$ be two τ-structures. We write $\mathbb{A} \leftrightarrows_r \mathbb{B}$ if there is a bijective mapping $\iota : \mathbb{A} \to \mathbb{B}$ (not necessarily isomorphism) such that for every $a \in \mathbb{A}$, it holds that $N_r^{\mathbb{A}}(a) \cong \mathbb{N}_r^{\mathbb{B}}(\iota(a))$.*

When we have $\mathbb{A} \leftrightarrows_r \mathbb{B}$, they have the same cardinality, they may not be isomorphic but 'locally' they are isomorphic everywhere. Note that in the r-local isomorphism, ι creates an element-to-element mapping designating 'which r-neighborhood to examine'. However, ι is not necessarily the isomorphism which witnesses $N_r^{\mathbb{A}}(a) \cong \mathbb{N}_r^{\mathbb{B}}(\iota(a))$.

2 Hanf locality

Definition 8 (Hanf locality of boolean query)**.** *Let $\ell > 0$. A property $\mathcal{P}$ on τ-structures is Hanf local if there is some integer $r > 0$ such that whenever two τ-structures $\mathbb{A}$ and $\mathbb{B}$ satisfy $\mathbb{A} \leftrightarrows_r \mathbb{B}$, then $\mathbb{A} \in \mathcal{P}$ of and only if $\mathbb{B} \in \mathcal{P}$. The smallest such integer r is called the* Hanf locality rank*, $\mathsf{hlr}(\mathcal{P})$ in short.*

Example 9. *Consider the vocabulary $\{\mathsf{edge}\}$. We want to show that the property* CONNECTED *is not Hanf local. Suppose that it is, with the Hanf locality rank r. The idea is to demonstrate two graphs G_1 and G_2*

of the same size (vertex count), (i) which are indistinguishable when you look at any r-neighborhood of G_1 and the corresponding r-neighborhood, and (ii) one is connected whereas the other is not.

Take G_1 as the disjoint union of two cycles, each of length $2r + 2$. Let G_2 be the cycle of length $4r + 4$. Let ι be an arbitrary bijection from G_1 to G_2. For any vertex v of G_1 or G_2, r-neighborhood at v in G_i is a path of length $2r$ whose two endpoints are non-adjacent. (We chose the length of each cycle of G_1 as the minimum integer so as to satisfy this property.) Therefore, $G_1 \leftrightarrows_r G_2$. However, G_1 is not connected and G_2 is connected. Therefore, CONNECTED *is not Hanf local!*

Theorem 10. *A* FO*-definable property is Hanf local.*

An immediate corollary of Theorem 10, together with the observation in Example 9 that CONNECTED is not Hanf local, means that CONNECTED is not FO-definable.

References