Contents

1 Terminology

Definition 1 (Substructure)**.** *For a set U of elements of a τ-structure $\mathbb{A}$, the* substructure *of $\mathbb{A}$ induced by U is the τ-structure $\mathbb{A}'$ defined as follows and for each $R \in \tau$,*

- *the universe $\tilde{U}$ of $\mathbb{A}$ is $U \cup \{c^{\mathbb{A}} \mid c$ is a constant symbol$\}$,*
- *each constant symbol is interpreted as the same element as in $\mathbb{A}$,*
- *for each predicate $R \in \tau$, R is interpreted as $R^{\mathbb{A}} \cap \tilde{U}^{\mathsf{ar}(R)}$.*

Remark: Throughout the rest, we assume that the *original* τ consists of relational predicates only unless stated otherwise, i.e. it does not contain any constant symbol. The reason is because, when we consider r-neighborhood at $\vec{a}$ we consider the substructure as something induced by the r-ball at $\vec{a}$ with each entry of $\vec{a}$ defining a new constant, by expanding the vocabulary τ to $\tau \cup \{c_1, \ldots, c_\ell$, where $\ell = |\vec{a}|$. Carrying the original constants in each substructure is quite cumbersome especially we are introducing new constants along the way. But all presentation henceforth can be extended to the case when τ contains some constant symbols, in the manner of Definition 1.

Definition 2. *For a τ-structure $\mathbb{A}$, the* Gaifman graph *$G(\mathbb{A})$ of $\mathbb{A}$ is a graph whose vertex set is the universe of $\mathbb{A}$, and there is an edge (without any orientation) between a pair of distinct elements $u, v \in \mathbb{A}$ if and only if there is some $R \in \tau$ such that u, v is related in the relation $R^{\mathbb{A}}$. That is, there is a tuple $\vec{a} \in R^{\mathbb{A}}$ such that $u = a_i$ and $v = a_j$ for some $1 \leq i, j \leq \mathsf{ar}(R)$.*

The *distance* between two elements u, v of $\mathbb{A}$ in $\mathbb{A}$ is defined as their distance in the Gaifman graph of $\mathbb{A}$, denoted as $\mathsf{dist}_{\mathbb{A}}(u, v)$. For a tuple $\vec{u} = (u_1, \ldots, u_\ell$ of elements of $\mathbb{A}$, the distance $\mathsf{dist}_{\mathbb{A}}(\vec{u}, v)$ between $\vec{u}$ and v in $\mathbb{A}$ is $\min\{\mathsf{dist}_{\mathbb{A}}(u_i, v) \mid i \in [\ell]\}$.

Definition 3 (r-ball, r-neighborhood)**.** *For a τ-structure $\mathbb{A}$ and an element $a \in \mathbb{A}$, the r-*ball *$B_r^{\mathbb{A}}(a)$ at a in $\mathbb{A}$ is the set of elements of $\mathbb{A}$ whose distance to a is at most r in the Gaifman graph $G(\mathbb{A})$ of $\mathbb{A}$. For a* *tuple* $\vec{a} = (a_1, \ldots, a_\ell)$ *of* $\mathbb{A}$*, the* r*-ball* $B_r^{\mathbb{A}}(\vec{a})$ *at* $\vec{a}$ *in* $\mathbb{A}$ *is defined the same way:* $B_r^{\mathbb{A}}(\vec{a}) := \{v \in \mathbb{A} \mid \mathsf{dist}_{\mathbb{A}}(\vec{a}, v) \leq r\}$*. The* r-neighborhood of $\vec{a}$ in $\mathbb{A}$ *is a* τ_ℓ*-structure, where* $\tau_\ell = \tau \cup \{c_1, \ldots, c_\ell\}$*, which the substructure of* $\mathbb{A}$ *induced by the* r*-ball at* $\vec{a}$ *when you look at only the relational predicates, and each* c_i *is interpreted as* a_i*. It is denoted by* $N_r^{\mathbb{A}}(\vec{a})$

Definition 4. *(ℓ-queries) Given an integer* $\ell \geq 0$*, an* ℓ-query *on* τ*-structures is a map* Q *such that*

- $Q(\mathbb{A}) \subseteq A^\ell$ *for any* τ*-struture* $\mathbb{A}$*; and*
- *it is closed under isomorphism, that is, if two* τ*-structures* $\mathbb{A}, \mathbb{B}$ *are isomorphic by an isomorphism* $h : A \to B$*, then* $Q(\mathbb{B}) = h(Q(\mathbb{A}))$*. Here,*
$$h(Q(\mathbb{A})) = \{(h(a_1), \ldots, h(a_\ell)) : (a_1, \ldots, a_\ell) \in Q(\mathbb{A})\}.$$

In particular, we consider A^0 *as a singleton set so every* 0*-query is exactly a property on* σ*-structures.*

Definition 5. *(Definable ℓ-queries) Given an* ℓ*-query* Q *on* τ*-structures and a logic* $\mathcal{L}$*,* Q *is* definable in $\mathcal{L}$ *if there is a formula* $\varphi(x_1, \ldots, x_\ell)$ *of* $\mathcal{L}$ *in* τ *such that*
$$Q(\mathbb{A}) = \{(a_1, \ldots, a_\ell) \in A^\ell : \mathbb{A} \vDash \varphi(a_1, \ldots, a_\ell)\}$$
for every τ*-structure* $\mathbb{A}$*.*

For example, the set of vertex pairs of distance exactly two is a 2-query. It is also FO-definable using the formula $\varphi(x, y) := \exists z\ \mathsf{edge}(x, z) \wedge \mathsf{edge}(z, y) \wedge \neg(x, y)$.

The isomorphism between two structures over the same vocabulary is defined in the usual way.

Definition 6 (Isomorphism between two structures)**.** *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two* τ*-structures. A mapping* $\iota : \mathbb{A} \to \mathbb{B}$ *is an isomorphism between* $\mathbb{A}$ *and* $\mathbb{B}$ *if*

- ι *is a bijection,*
- *for every constant symbol* $c \in \tau$ *and for every* $i \leq \ell$*,* $\iota(c^{\mathbb{A}}) = c^{\mathbb{B}}$*,*
- *for every predicate* $R \in \tau$ *with* $\mathsf{ar}(R) = k$ *and for every* k*-tuple* $(a_1, \ldots, a_k) \in \mathbb{A}^k$*,* $R(a_1, \ldots, a_k)$ *if and only if* $R(\iota(a_1), \ldots, \iota(a_k))$*.*

We write $\mathbb{A} \cong \mathbb{B}$ *when there is* $\mathbb{A}$ *is isomorphic to* $\mathbb{B}$*.*

Note that a property $\mathcal{P}$ of τ-structures is (defined so that) closed under isomorphism. That is, if $\mathbb{A}$ has the property $\mathcal{P}$ and $\mathbb{B}$ is isomorphic to $\mathbb{A}$ then $\mathbb{B}$ also has the property $\mathcal{P}$. Note also that ℓ-query, a generalization of a property, defined so as to be closed under isomorphism.

Intuitively, an ℓ-query Q on τ-structures is Hanf local if the query is closed under the isomorphism of the r-neighborhood. Hanf locality is not guaranteed, and it is rather an anomaly. However, it turns out that an FO-definable ℓ-query is Hanf local.

Definition 7 (r-local isomorphism between two relational structures)**.** *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two* τ*-structures, where* τ *consists of relational predicates only (no constant symbols). We write* $\mathbb{A} \leftrightarrows_r \mathbb{B}$ *if there is a bijective mapping* $\iota : \mathbb{A} \to \mathbb{B}$ *(not necessarily isomorphism) such that for every* $a \in \mathbb{A}$*, it holds that* $N_r^{\mathbb{A}}(a) \cong \mathbb{N}_r^{\mathbb{B}}(\iota(a))$*.*

When we have $\mathbb{A} \leftrightarrows_r \mathbb{B}$, they have the same cardinality, they may not be isomorphic but 'locally' they are isomorphic everywhere. Note that in the r-local isomorphism, ι creates an element-to-element mapping designating 'which r-neighborhood to examine'. However, ι is not necessarily the isomorphism which witnesses $N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{B}}(\iota(a))$.

2 Hanf locality: boolean query

Definition 8 (Hanf locality of boolean query)**.** *Let $\ell > 0$. A property $\mathcal{P}$ on τ-structures is* Hanf local *if there is some integer $r \geq 0$ such that whenever two τ-structures $\mathbb{A}$ and $\mathbb{B}$ satisfy $\mathbb{A} \leftrightarrows_r \mathbb{B}$, then $\mathbb{A} \in \mathcal{P}$ of and only if $\mathbb{B} \in \mathcal{P}$. The smallest such integer r is called the* Hanf locality rank*, $\mathsf{hlr}(\mathcal{P})$ in short.*

Hanf locality is a very useful tool when you want to prove inexpressibility in FO of a property on τ-structures. Here, the recipe for using Hanf locality for establishing inexpressibility of a property $\mathcal{P}$ is the following, for every $r \geq 0$.

1. Choose two τ-structures $\mathbb{A} \in \mathcal{P}$ and $\mathbb{B} \notin \mathcal{P}$.

2. Show that $\mathbb{A} \leftrightarrow_r \mathbb{B}$.

Beware we use the bijection ι from $\mathbb{A}$ to $\mathbb{B}$ (present in the definition of $\leftrightarrow_r$) to introduce the new constant $a \in \mathbb{A}$ and $\iota(a) \in \mathbb{B}$ whenever the r-neighborhood is constructed. So, in order to ensure that $N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{B}}(\iota(a))$ in the second stage of the recipe, the isomorphism between $N_r^{\mathbb{A}}(a)$ and $N_r^{\mathbb{B}}(\iota(a))$ is not necessarily the restriction of ι (in fact, it cannot), but the isomorphism maps a to $\iota(a)$ as they are the interpretations of the same constant symbol of τ_1, i.e. the expanded vocabulary.

Example 9 (The property CONNECTED.)**.** *Consider the vocabulary $\{\mathsf{edge}\}$. We want to show that the property* CONNECTED *is not Hanf local. Suppose that it is, with the Hanf locality rank r. The idea is to demonstrate two graphs G_1 and G_2 of the same size (vertex count), (i) which are indistinguishable when you look at any r-neighborhood of G_1 and the corresponding r-neighborhood, and (ii) one is connected whereas the other is not.*

Take G_1 as the disjoint union of two cycles, each of length $2r + 2$. Let G_2 be the cycle of length $4r + 4$. Let ι be an arbitrary bijection from G_1 to G_2. For any vertex v of G_1 or G_2, r-neighborhood at v in G_i is a path of length $2r$ whose two endpoints are non-adjacent. (We chose the length of each cycle of G_1 as the minimum integer so as to satisfy this property.) Therefore, $G_1 \leftrightarrows_r G_2$. However, G_1 is not connected and G_2 is connected. Therefore, CONNECTED *is not Hanf local!*

Theorem 10. *A* FO*-definable property is Hanf local.*

An immediate corollary of Theorem 10, together with the observation in Example 9 that CONNECTED is not Hanf local, means that CONNECTED is not FO-definable.

Inexpressibility in $\exists$ MSO: the example of ACYCLICITY. One can use Hanf locality to prove that some property $\mathcal{P}$ is not expressible in existential MSO. Let's consider the example of ACYCLICITY, the property of (undirected) graphs consisting of graphs without cycles. The vocabulary is over the usual one $\{\mathsf{edge}\}$.

First, let us express the complementary property, CYCLIC, consisting of graphs with some cycle(s). This property can be expressed in an existential MSO. (How?) Therefore, the negation of it yields a universal MSO-expression for ACYCLICITY. (Why?) Does there exist an existential MSO-expression for ACYCLIC-

ITY? It turns out not, and we can prove this using Hanf locality.

Suppose it is expressible in $\exists$MSO, with the sentence $\exists X_1 \cdots \exists X_\ell \varphi(X_1, \dots, X_\ell)$. Here φ is an FO-sentence over the expanded vocabulary

$$\tau' := \{\mathsf{edge}\} \cup \{X_1, \dots, X_\ell\},$$

where each X_i is a unary predicate. Therefore, thanks to Theorem 10, the property $\mathcal{P}$ of τ'-structures consisting of all *ℓ-colored* graphs (each unary predicate interpreted as a color class) satisfying φ. Let $d := \mathsf{hlr}(\mathcal{P})$. That is, if $(G_1, S_1, \dots, S_\ell) \leftrightarrows_d (G_2, T_1, \dots, T_\ell)$, then $(G_1, S_1, \dots, S_\ell) \models \varphi$ if and only if $(G_2, T_1, \dots, T_\ell) \models \varphi$. Especially, G_1 is acyclic if and only if G_2 is acyclic.

We want to design a pair of ℓ-colored graphs which is equivalent under $\cong_d$ whereas one is cyclic and the other is not. This shows that the initial assumption is wrong, that is ACYCLICITY is not expressible in existential MSO.

How do we build such a pair of ℓ-colored graphs? We use a similar construction as in case of TREES (inexpressibility in FO using G_1 as a long path, and G_2 as a disjoint union of a path and a cycle). But in order to establish the bijection between two ℓ-colored graphs with $\leftrightarrows_d$ one needs to be extra careful (and increase the length of paths and cycles quite long). Consider G_1 is an ℓ-colored path of length sufficiently long. The key observation is that there are bounded number of *r-neighborhood isomorphism types* in a ℓ-colored graphs of bounded degree. Therefore, one can find two vertices u, v on the path G_1 such that

- u is before v (fix the starting and endpoint s and t of the path G_1),
- the distance between u, v on G is at least $2r + 2$,
- the distance between s, u as well as v, t is at least r,
- and mostly importantly, the *r-neighborhood isomorphism type* around u and v are identical, even after considering the orientation from left-to-right.

One can prove that as there are bounded (by a function of r and ℓ) number of r-isomorphism types, there must exist such two vertices, sufficiently far from each other. Now, we create a new graph G_2 by

- disconnecting / connecting the predecessor of u from u / with v , to make the former as the predecessor of v, and
- disconnecting / connecting the predecessor of v from v / with u, creating a cycle.

The bijection from G_1 to G_2 is canonical; one can easily see that this certifies that $(G_1, S_1, \dots, S_\ell) \leftrightarrows_d (G_2, T_1, \dots, T_\ell)$. Therefore, by Theorem 10 the ℓ-colored graph G_1 is in $\mathcal{P}$ and only if G_2 as the ℓ-colored graph does. That is, the original existential MSO-sentence $\exists X_1 \cdots \exists X_\ell \varphi(X_1, \dots, X_\ell)$ is satisfied by (uncolored) G_1 if and only if it is satisfied by G_2. However, one is acyclic while the other is not, and the sentence does not express ACYCLICITY.

3 Gaifman Locality: non-booean query

We can extend the notion for a property $\mathcal{P}$ being r-local in the sense of Definition 8 to ℓ-queries. Again, we assume that the initial vocabulary τ consists of relational predicates only.

Intuitively, a query Q is *local* if one can decide if an element $v \in \mathbb{A}$ is in the query $Q(\mathbb{A})$, i.e. whether the element satisfies the property in $\mathbb{A}$, only by looking at the neighborhood of v. Let us see some examples of queries. If Q is an ℓ-query in general, we look at the neighborhood of a ℓ-tuple $\vec{v}$ of vertices. Which one seems local and which ones are not?

- 1-query Q consisting of all vertices of a graph contained in some cycle.
- 1-query Q consisting of all vertices of a graph contained in a triangle.
- 1-query Q consisting of all vertices of a graph contained in a cycle of length at most $2d$ for some fixed d.
- 2-query Q consisting of all vertex pairs with at least three common neighbors
- 2-query Q consisting of all vertex pairs contained in a cycle of length at most $2d$
- 2-query Q consisting of all vertex pairs contained in a cycle of length at least $2d$
- 2-query Q consisting of all vertex pairs (u, v) such that there are two vertex disjoint triangles, one containing u the other containing v.

Definition 11 (Gaifman locality of ℓ-query)**.** *Let $\ell > 0$. An ℓ-query Q on τ-structures is* Gaifman local *if there is some integer $r \geq 0$ such that the following holds:*

$$\text{for any } \tau\text{-structures } \mathbb{A} \text{ and two } \ell\text{-tuples } \vec{a}, \vec{b} \in \mathbb{A}^{\ell},$$
$$\text{if } N_r^{\mathbb{A}}(\vec{a}) \cong N_r^{\mathbb{A}}(\vec{b}) \text{ then } \vec{a} \in Q(\mathbb{A}) \text{ if and only if } \vec{b} \in Q(\mathbb{A}).$$

The smallest such integer r is called the (Gaifman) locality rank*, or simply locality rank,* $\mathsf{lr}(Q)$ *in short.*

Example 12. Transitive Closure. *Consider the 2-query* TC *on directed graphs, a digraph seen as a relational structure over the vocabulary* $\tau = \{\mathsf{edge}\}$ *interpreted as the arcs of the directed graph.*

$$\mathsf{TC}(G) = \{(u, v) \mid \text{there is a directed } (u, v)\text{-path in } G\}$$

We want to argue that TC *is not Gaifman local. Suppose it is, with* $\mathsf{lr}(\mathsf{TC}) = r$*. Consider a directed graph G, sufficiently long ($\geq 4r + 2$ suffices). Then consider two vertices u and v, u precedes v on the path with* $\mathsf{dist}_G(u, v) \geq 2r + 2$, $\mathsf{dist}_G(\mathit{first}, u) \geq r$ *and* $\mathsf{dist}_G(v, \mathit{last}) \geq r$*. Then*

- *Each of $N_r(u, v)$ and $N_r(v, u)$ is the disjoint union of two directed paths, one centered at u the other centered at v,*
- *therefore, $N_r(u, v) \cong N_r(v, u)$,*
- *by definition of Gaifman locality, we have $(u, v) \in \mathsf{TC}$ if and only if $(v, u) \in \mathsf{TC}$,*
- *but we know that this is not the case,*
- *Conclusion:* TC *is not Gaifman local.*

In the example above, you proved that TC is not FO-definable by demonstrating *some* structure, and *some* pair of 2-tuples which are indistinguishable locally in the structure, but one is in the query while the other is not.

Theorem 13. *Any* FO*-definable query Q is Gaifman local. Moreover, if Q is defined by an* FO*-formula of quantifier rank at most q, then the Gaifman locality rank of Q is at most $\frac{3^{q+1}-1}{2}$.*

Example 14 (Gaifman locality is not complete for FO.)**.** *Is Gaifman locality is complete for* FO*-logic? Theorem 13 says that an* FO*-definable query is Gaifman local. Is the converse also true? That is, if a query is Gaifman local, then is it* FO*-definable?*

Consider the 1-query CYCLE-IN-NEIGHBORHOOD *on graphs which consists of all vertices v such that the subgraph induced by $N_G(v)$ contains a cycle. The query* CYCLE-IN-NEIGHBORHOOD *is certainly Gaifman local, with* $\mathsf{hlr}(\text{CYCLE-IN-NEIGHBORHOOD}) = 1$. *Indeed, if the 1-neighborhood centered at a vertex v in G is isomorphic to the 1-neighborhood centered at a vertex v' in G', and if the former contains a cycle in $G[N(v)]$ then the latter contains a cycle in $G'[N(v')]$ and vice versa. But it seems highly unlikely that this 1-query is* FO*-definable.*

In fact one can use Hanf locality for boolean query in a different way to show that that CYCLE-IN-NEIGHBORHOOD *is not* FO*-definable. It works in the same principle as proving that a certain property is not* FO*-expressible. Consider the vocabulary $\tau_1 := \{\mathsf{edge}\} \cup \{c\}$. Suppose* CYCLE-IN-NEIGHBORHOOD *is* FO*-definable with $\varphi(x)$. Now we consider an alternative (boolean) query, namely the property $\mathcal{P}$ on τ_1-structures which consists of all graphs with a distinguished vertex c^G, as an interpretation of the constant symbol c, such that c^G is a universal vertex (i.e. adjacent all other vertices) and there is a cycle which do not traverse the distinguished vertex. Note that $\varphi(x)$ can be seen (or converted trivially) as a sentence φ' over τ_1 where x plays the role of the constant symbol c. Note that $(G, v) \models \varphi'$ if and only if $G \models \varphi(v)$. This implies that $\mathcal{P}$ is defined φ', Let r be the Gaifman locality rank of $\mathcal{P}$; such r exists by Theorem 13.*

Consider two τ_1-structures (G_1, v_1) and (G_2, v_2) obtained from the graphs of Example 9 by adding v_1 and v_2 as the universal vertex respectively. With the universal vertices added, the two graphs are still r-locally indistinguishable. Specifically, $(G_1, v_1) \leftrightarrows_r (G_2, v_2)$ and $(G_1, v_1) \models \varphi'$ if and only if $(G_2, v_2) \models \varphi'$. As observed above, this is equivalent to $G_1 \models \varphi(v_1)$ if and only if $G_2 \models \varphi(v_2)$. However, G_1 contains a cycle in the open neighborhood of v_1 while v_2 is not, a contradiction.

4 Gaifman's Locality Theorem

Note that whether $u \in \mathbb{A}$ is in the r-ball around $\vec{a} \in \mathbb{A}^\ell$ can be verified with an FO-formula. First, the 2-query Q which consists of pairs (u, v) of elements of any τ-structure $\mathbb{A}$ whose distance in the Gaifman graph of $\mathbb{A}$ is exactly 1 can be FO-expressed easily:

$$\begin{aligned}\mathsf{dist}_1(x,y) := &\bigvee_{\substack{R\in\tau;\\ k:=\mathsf{ar}(R)\geq 2}} \exists z_1 \cdots \exists z_k \bigvee_{1\leq i<j\leq \mathsf{ar}(R)} R(z_1,\ldots,z_{i-1},x,z_{i+1},\ldots,z_{j-1},y,z_{j+1},\ldots,z_k)\\ &\vee \bigvee_{\substack{R\in\tau;\\ k:=\mathsf{ar}(R)\geq 2}} \exists z_1 \cdots \exists z_k \bigvee_{1\leq i<j\leq \mathsf{ar}(R)} R(z_1,\ldots,z_{i-1},y,z_{i+1},\ldots,z_{j-1},x,z_{j+1},\ldots,z_k).\end{aligned}$$

One can FO-define the 2-query which consists of two elements of distance exactly k in the Gaifman graph.

$$\mathsf{dist}_k(x,y) := \exists z_1 \cdots \exists z_{k-1}\, \mathsf{dist}_1(x,z_1) \wedge \mathsf{dist}_1(z_1,z_2) \wedge \cdots \wedge \mathsf{dist}_1(z_{k-2},z_{k-1}) \wedge \mathsf{dist}_1(z_{k-1},y)$$

Using the above formula checking if two elements have distance k in the Gaifman graph, one can check if an element is in the r-ball around some ℓ-tuple $\vec{a}$ in $\mathbb{A}$.

$$\mathsf{dist}_{\le r}(y,\vec{x}) := \bigvee_{i=1}^{\ell} \big(y = x_i \vee \bigvee_{k=1}^{r} \mathsf{dist}_k(y,x_i)\big).$$

We write $y \in B_r(\vec{x})$ as a shorthand for the formula $\mathsf{dist}_{\le r}(y,\vec{x})$. Intuitively, we say that an FO-formula is local (r-local) if we restrict the interpretation of all quantified variables in the r-ball around (the interpretation of) the free variables.

Definition 15 (Locality of a formula). *Let* $\mathsf{dist}_{>r}(x,y) := \neg\mathsf{dist}_{\le r}(x,y)$.

- *An* FO-*formula* $\varphi(\vec{x})$ *is said to be* r-local *around* $\vec{x}$ *if every quantifier in* φ *is in the form* $\exists y \in B_r(\vec{x})$.
- *An* FO-*sentence* Φ *is said to be* basic r-local *if it is in the form*

$$\Phi = \exists x_1 \cdots \exists x_\ell \ \big(\bigwedge_{i=1}^{\ell} \alpha^{(r)}(x_i) \ \wedge \ \bigwedge_{i \ne j} \mathsf{dist}_{>2r}(x_i,x_j) \ \big).$$

Here $\alpha^{(r)}(x)$ *is a* r*-local formula around the single variable* x.

Note that $\alpha^{(r)}(x)$ above can be replaced by $\alpha^{(r')}(x)$ for any $r' \le r$ because one can add $\mathsf{dist}_{\le r'}(y)$ for any quantified variable y occurring in $\alpha^{(r)}(x)$.

Let us contemplate on Definition 15. When a sentence if basic r-local, it asks if there exists a set of elements ($\exists x_1 \cdots \exists x_\ell$) which is pairwise faraway ($\bigwedge_{i \ne j} \mathsf{dist}_{>2r}(x_i,x_j)$), and each element satisfies some r-local formula ($\alpha^{(r)}(x)$). Especially, the last part checks if the r-neighborhood around the element has certain property, expressible in FO.

Theorem 16 (Gaifman Locality Theorem, 1982). *[1] Every* FO-*sentence is equivalent to a boolean combination of basic local sentences. Moreover, given any* FO-*sentence* ψ

- *its Gaifman Normal Form is computable*[1]*, and*
- *the quantifier rank of each basic local sentence is at most* $\mathsf{qr}(\psi)$*, and* $r \le 7^{\mathsf{qr}(\psi)}$.

Some examples. k-INDEPENDENT SET, the property of graph which has an independent set of size k, can be expressed with a basic r-local sentence for some fixed r.

$$k\mathsf{IND} := \exists x_1 \cdots \exists x_k \bigwedge_{i \ne j} \mathsf{dist}_{>1}(x_i,x_j).$$

How about the property CONTAINMENT OF C_{2d+1} of graphs having a cycle of length exactly $2d+1$? Consider the 1-query which says that a vertex contained in a cycle of length $2d+1$. This can be expressed in FO, and in particular with a d-local formula. Indeed, for a vertex v on a cycle of length $2d+1$, all vertices of the cycle is in the d-ball around v, and one can check if there is a cycle of length $2d+1$ in the d-ball around v:

$$\alpha(x) := \exists z_1 \in B_d(x) \cdots \exists z_{2d} \in B_d(x) \ \big(\ \mathsf{edge}(x,z_1) \wedge \mathsf{edge}(z_1,z_2) \wedge \cdots \wedge \mathsf{edge}(z_{2d},x) \ \big)$$

[1] The length of the obtained sentence can be very large; there is a nonelementary lower bound known [2]. This also implies that the running time of converting a given FO-formula ψ into Gaifman Normal Form can take $f(|\psi|)$ steps with non-element function f.

Certainly, $\alpha(x)$ is a d-local formula around x. Now $\Phi := \exists x\ \alpha(x)$ defines the graph property of having a cycle of length $2d+1$ with a basic d-local sentence.

Let TWO DISJOINT C_{2d+1} be the graph property of having two disjoint cycles of length exactly $2d+1$ each. Is this FO-definable? Let us express this property as a boolean combination of basic d-local sentences. There are two possibilities, if there exists two disjoint copies C_1, C_2 of length $2d+1$ in a graph G. One case is when there are two cycles whose distance is at least $4d+1$. Then one can certify the existence of two such cycles with a basic $2d$-local sentence

$$\Phi_1 := \exists x_1 \exists x_2 \big(\ \alpha(x_1) \wedge \alpha(x_2) \wedge \mathsf{dist}_{>4d}(x_1, x_2)\ \big),$$

where $\alpha(x)$ is defined as above. (Recall that d-local formula is $2d$-local formula as well.) Notice that if $\alpha(v_1)$ and $\alpha(v_2)$ are satisfied, then the witnessing cycles (an interpretation of variables in $\alpha(v_i)$) must be disjoint as each cycle is contained in the d-neighborhood of v_i whereas the distance between v_1 and v_2 is sufficiently far apart.

The other case when two copies are of distance at most $4d$ can be expressed with a $2d$-local formula which verifies if there is a connected subgraph, within $2d$-neighborhood of some vertex, consisting of two disjoint cycles connected by a path of length at most $4d$. Whenever such two copies exist, a central vertex in the connecting path, say v, sees all vertices in the path as well as the two cycles in $N_{2d}(v)$. One can write (tediously long, but trivial) a $2d$-local FO-formula $\beta(x)$ which defines the 1-query that there is such a connected graph in $2d$-neighborhood of v. Now,

$$\Phi := \Phi_1 \vee \exists x\ \beta(x)$$

defines the property TWO DISJOINT C_{2d+1}.

How about k-DOMINATING SET? It is the property of graphs which has a dominating set[2] of size k, can be expressed by an FO-sentence.

$$\Phi_1 := \exists x_1 \cdots \exists x_k\ \forall y\ \bigwedge_{i=1}^{k} \big(x_i = y \vee \mathsf{edge}(x_i, y)\big).$$

Directly obtaining an equivalent FO-sentence as a boolean combination of basic local sentences is nontrivial, even though such a sentence exists due to Gaifman Locality Theorem. Let us consider the case of $k = 1$. The foremost challenge is to get rid of the universal quantifier. We first want to exclude the obvious case when at least two vertices are necessary for dominating the entire vertex set; that is, when you have a vertex pair whose distance is at least three. This can be expressed by a (negation of) basic 1-local sentence $\Phi_2 := \neg \exists x_1 \exists x_2\ \mathsf{dist}_{>2}(x_1, x_2)$. Notice that omitting a local formula of the form $\alpha^{(1)}(x_i)$ is equivalent to placing a trivially satisfiable 1-local formula.

Once a graph satisfies Φ_1, every pair of vertices have distance at most two. Here, we can use Φ_1 for $k = 1$ where every quantified variable z is follows by the extra locality checking formula $B_r(z)$ for $r = 2$! To summarize,

$$\mathsf{1DOM} := \neg \exists x_1 \exists x_2\ \mathsf{dist}_{>2}(x_1, x_2)\ \wedge\ \exists x \big(\Phi_1(2)(x)\big)$$

where $\Phi_1(2)(x)$ is obtained from Φ_1 by substituting $\exists x_i$ with $\exists x_i \in B_2(x)$ and $\forall y\ \varphi$ with $\forall y\ \big(y \in B_2(x) \to \varphi\big)$.

[2] A set D of vertices in G is a dominating set if every vertex of G is either in D or has a neighbor in D.

5 FO-model checking on graphs of bounded degree

Using Theorem 16, Seese proved in [3] that an FO-sentence ϕ can be evaluated on an arbitrary structure in time $f(\phi, d) \cdot n$ when the structure has degree (the degree in the Gaifman graph) is bounded by d. Here n is the size of the universe. We prove this result on graphs; the generalization to an arbitrary (finite) vocabulary is immediate.

Theorem 17 (FO-model checking, Seese 1996)**.** *[3] Let $\mathcal{C}$ be a class of graphs, all of which have maximum degrees at most d. Then there is a function $f_d : \mathbb{N} \to \mathbb{N}$ such that for any* FO*-sentence ϕ on graphs, one can decide whether $G \models \phi$ in $\mathcal{C}$ in time $f_d(|\phi|) \cdot |V(G)|$.*

By Theorem 16, it suffices to present an algorithm which works for a basic local sentence. Consider a basic r-local sentence

$$\Psi = \exists x_1 \cdots \exists x_k \ \big(\ \bigwedge_{i=1}^{k} \varphi^{(r)}(x_i) \ \wedge \ \bigwedge_{i \neq j} \mathsf{dist}_{>2r}(x_i, x_j) \ \big).$$

Verifying if G satisfies Ψ asks you to do two very different tasks:

1. check if there are pairwise faraway vertices,
2. check if locally a desired property holds.

It turns out that these two tasks are all we need. The FO-model checking algorithm for Ψ does the following.

1. On each vertex $v \in V(G)$, decide if v satisfies the 1-query defined by $\varphi^{(r)}(x)$. Let S_φ be the set of vertices satisfying $\varphi^{(r)}(x)$.
2. Decide if there are k vertices in S_φ that are $2r$-*scattered*, i.e. k vertices whose pairwise distance is strictly larger than $2r$.

The crucial property we use for degree-bounded graphs is the following.

Observation 18. *On a graph G of maximum degree at most d, the r-ball of any vertex has at most $\sum_{i=0}^{r} d^i \leq d^{r+1}$ vertices.*

For evaluating $\varphi^{(r)}(x)$ with $x = v$, it suffices to evaluate φ on the r-neighborhood of v. By Observation 18 there are at most d^{r+1} possible vertices you can assign to a variable of φ. So one can enumerate all possibilities of assignment, there are at most $d^{(r+1)\cdot|\varphi|}$ many of them, and decide the truth value of each assignment. So, one can compute the set S_φ in time $O(d^{(r+1)\cdot|\varphi|} \cdot n)$.

One can decide if there is an r-scattered set of size k contained in $S \subseteq V(G)$ in linear time as well. Take a maximal r-scattered set S' contained in S greedily. That is, starting from $S' = \emptyset$, add to S' a vertex v from $S \setminus S'$ if v has distance at least $r+1$ from all vertices currently in S'. If we can do this for at least k iterations, we have found an r-scattered set inside S with k vertices. If we end up in S' with at most $k-1$ vertices, then it holds that $|V(G)| \leq (k-1) \cdot d^{r+1}$. The key observation here is that, the r-ball $B_r(S') = \bigcup_{v \in S'} B_r(v)$ around S' contains *all* vertices of G. Indeed, if there is a vertex which is still left out, it has distance strictly larger than r from all current vertices of S' and we should have proceeded with the next iteration. Now that the number of vertices of G is bounded by a function of k, we can try brute-force to find a r-scattered set in S of size k or correctly decide if there is no such a set. This take $c \cdot \big((k-1) \cdot d^{r+1}\big)^k$ steps for some constant c - the constant is the additional factor incurred for checking if a set is indeed r-scattered in G.

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