Why the pole?

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Mohr circle remains an important tool in analysing the states of a second order tensor on different planes passing through a point. The second order tensors, very frequently used in geomechanics or geotechnical community, are stress, strain, and sometimes permeability. Oftentimes we require the components of these tensors on different planes to assess the pane of maximum criticality based on some criteria. Mohr circle appears to be most intutive graphical tool to use for achieving coordinate transformation of these tensors with ease of use.

From a pedagogical point of view, It appears to me that Mohr circle is not very clear to the students (not the entire technique but at-least a part of it). In specific, the following points are matter of concern and interest of this article

- The method of pole is very popular but why does it work is rarely shown in a class room.
- The use of a certain sign-convention and not the other i.e. it is often told to use compression and counterclockwise shear as positive or tension and clockwise shear as positive.

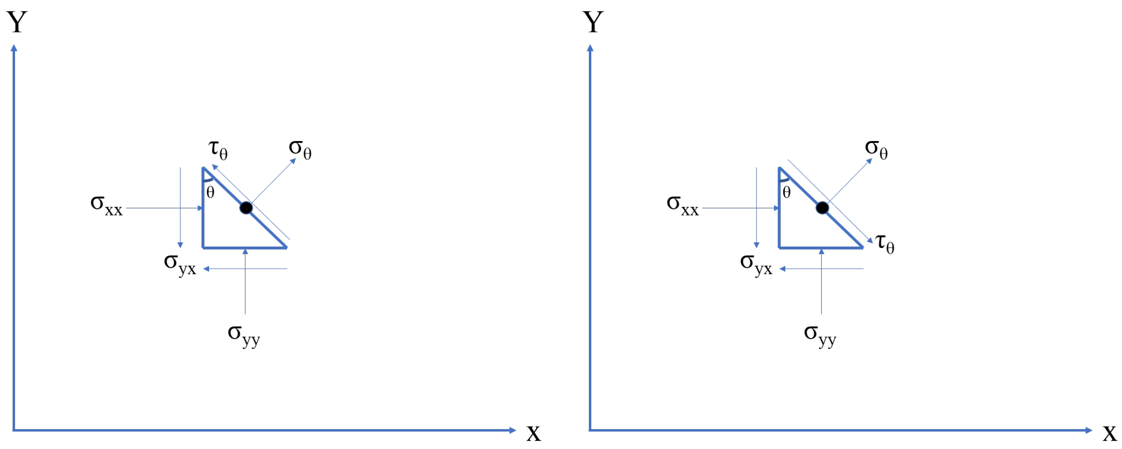
Warning: The use of any other combination of sign convention restricts the use of method of pole and corrupts the use of method of 2θ rotation from center.

1 Mohr circle sign convention

To clear things out, let us analyze the formulation of Mohr circle with different starting points.

1.1 First starting point

In this we analyze the stress state given in figure 1a. To keep confusion away, note that the sign convention are progeny of the assumptions that we make on variables (and not on the constant terms) while deriving the equations. For figure 1a, we assume that the variable normal and shear stress – on a plane inclined at θ counterclockwise from vertical – are considered to be in tension and acting in counterclockwise direction, respectively. For the configuration given in said figure, σ_θ and τ_θ can be estimated using force balance or tensor rotation (coordinate transformation) as



(a) Stress state 1

(b) Stress state 2

Figure 1: Stress states for Mohr circle

$$\sigma_\theta = -\frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{yy}-\sigma_{xx}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{1}$$

$$\tau_\theta = \frac{\sigma_{xx}-\sigma_{yy}}{2} + \sigma_{xy}\cos 2\theta \tag{2}$$

The above can be re-written as

$$\sigma_\theta = -\frac{\sigma_{xx}+\sigma_{yy}}{2} + R\cos\left(2\theta-\phi\right) \tag{3}$$

$$\tau_\theta = -R\sin\left(2\theta-\phi\right) \tag{4}$$

where

$$R = \sqrt{\left(\frac{\sigma_{yy}-\sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2}$$

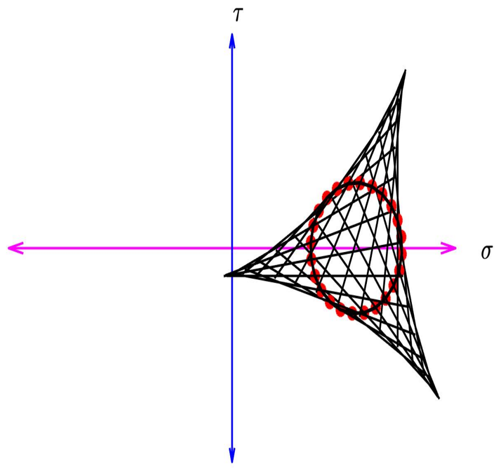
$$\tan\phi = \frac{2\sigma_{xy}}{\sigma_{yy}-\sigma_{xx}}$$

Let use bring this to the complex plane for further ease of handling,

$$\left[\sigma_\theta + \frac{\sigma_{xx}+\sigma_{yy}}{2}\right] + i\tau_\theta = Re^{i(\phi-2\theta)} \tag{5}$$

The stress states $(\sigma_\theta, \tau_\theta)$ from equation 5 lies on a circle whose radius is R and whose center lies at $\left(-\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$. Note that increasing θ results in counterclockwise rotation of θ plane (figure 1a) but results in clockwise rotation of radial vector in Mohr circle due to term $(\phi - 2\theta)$. This suggest that the 2θ rotation method does not work properly but in a reverse manner i.e. a clockwise rotation of plane should

Figure 2: Is there a pole here?



accompany counterclockwise rotation of radial vector. Now to look at where does pole come from amidst all this.

Now let us plot stress state on the Mohr circle for a plane inclined at counterclockwise θ from the vertical and also draw a line parallel to this plane passing through stress state point. Further, we vary the θ and observe what the result is!

The result of this exercise is given in figure.

1.2 Second starting point

$$\sigma_\theta = -\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{6}$$

$$\tau_\theta = -\frac{\sigma_{xx} - \sigma_{yy}}{2} - \sigma_{xy}\cos 2\theta \tag{7}$$

Following exactly same procedure as in section 1.1

$$\sigma_\theta = -\frac{\sigma_{xx} + \sigma_{yy}}{2} + R\cos\left(2\theta - \phi\right) \tag{8}$$

$$\tau_\theta = R\sin\left(2\theta - \phi\right) \tag{9}$$

where

$$R = \sqrt{\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\tan\phi = \frac{2\sigma_{xy}}{\sigma_{yy} - \sigma_{xx}}$$

Now bring this to the complex plane,

$$\left[\sigma_\theta + \frac{\sigma_{xx} + \sigma_{yy}}{2}\right] + i\tau_\theta = Re^{i(2\theta - \phi)}$$