Why the pole?

Saurabh

May 25, 2019

Mohr circle remains an important tool in analyzing the states of a second order tensor on different planes passing through a point. The second order tensors, very frequently used in geomechanics or geotechnical community, are stress, strain, and sometimes permeability. Oftentimes we require the components of these tensors on planes different from standard to assess the plane of maximum criticality based on some criteria. Mohr circle appears to be most intuitive graphical tool to use for achieving such coordinate transformation of these tensors with ease.

From a pedagogical point of view, It appears to me that Mohr circle is not very clear to the students (not the entire technique but at-least a part of it). In specific, the following points are matter of concern and interest of this article

- The method of pole is very popular but why does it work is rarely shown in a class room.
- The use of a certain sign-convention and not the other i.e. it is often told to use compression and counterclockwise shear as positive or tension and clockwise shear as positive. Why not other combinations?

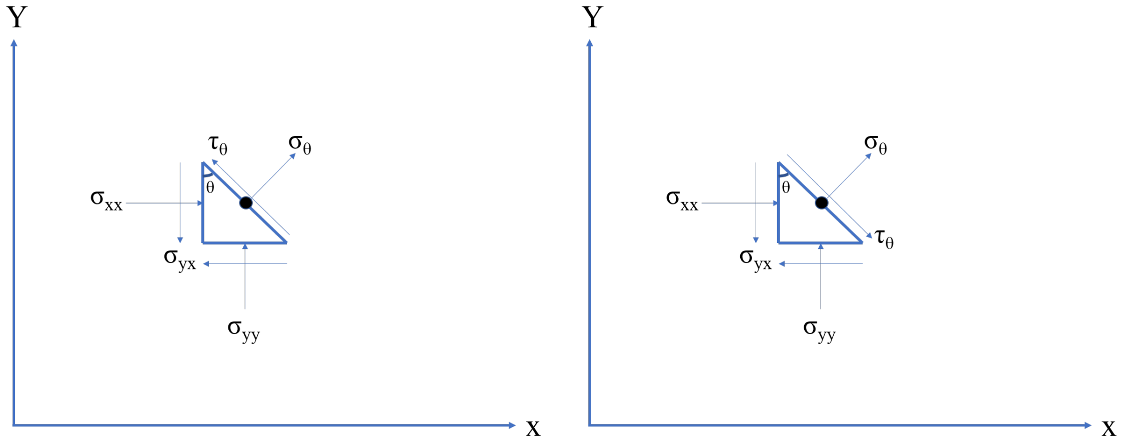
Warning: The use of any other combination of sign convention restricts the use of method of pole and corrupts the use of method of 2θ rotation from center.

1 Mohr circle sign convention

To clear things out, let us analyze the formulation of Mohr circle with two different starting points (sign conventions).

1.1 First starting point

In this section, we analyze the stress state given in figure 1a. To keep confusion away, note that the sign conventions are progeny of the assumptions that we make on variables (and not on the constant terms) while deriving the equations. For figure 1a, we assume that the variable normal and shear stress – on a plane inclined at θ counterclockwise from vertical – are considered to be in tension and acting in counterclockwise direction, respectively i.e. tension and counterclockwise shear is considered



(a) Stress state 1 (b) Stress state 2

Figure 1: Stress states for Mohr circle

positive. For the configuration given in said figure, σ_θ and τ_θ can be estimated using force balance or tensor rotation (coordinate transformation) as

$$\sigma_\theta = -\frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{yy}-\sigma_{xx}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{1}$$

$$\tau_\theta = \frac{\sigma_{xx}-\sigma_{yy}}{2} + \sigma_{xy}\cos 2\theta \tag{2}$$

The above can be re-written as

$$\sigma_\theta = -\frac{\sigma_{xx}+\sigma_{yy}}{2} + R\cos(2\theta-\phi) \tag{3}$$

$$\tau_\theta = -R\sin(2\theta-\phi) \tag{4}$$

where

$$R = \sqrt{\left(\frac{\sigma_{yy}-\sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\tan\phi = \frac{2\sigma_{xy}}{\sigma_{yy}-\sigma_{xx}}$$

Let us bring this to the complex plane for further ease of handling,

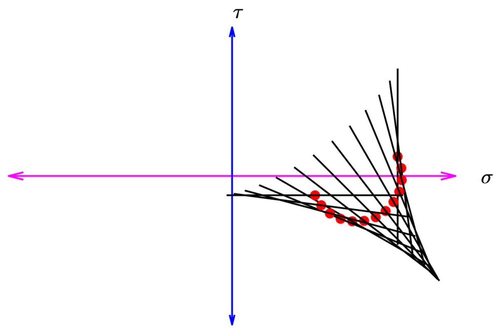
$$\left[\sigma_\theta + \frac{\sigma_{xx}+\sigma_{yy}}{2}\right] + i\tau_\theta = Re^{i(\phi-2\theta)} \tag{5}$$

The stress states $(\sigma_\theta, \tau_\theta)$ from equation 5 lies on a circle whose radius is R and whose center lies at $\left(-\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$. Note that increasing θ results in counterclockwise rotation of θ plane (figure 1a) but results in clockwise rotation of radial vector in

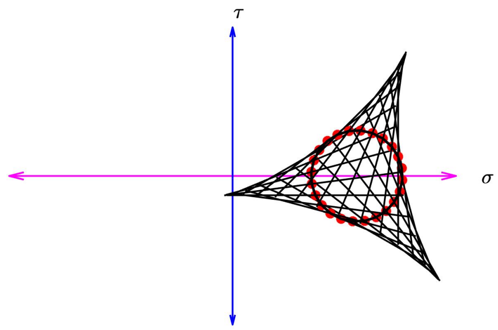
Mohr circle due to term $(\phi - 2\theta)$. This suggest that the 2θ rotation method does not work properly but works in a reverse manner i.e. a clockwise rotation of plane should accompany counterclockwise rotation of radial vector in Mohr circle. Now, we look at where does pole come from amidst all this.

Let us plot stress state on the Mohr circle for a plane inclined at counterclockwise θ from the vertical and also draw a line parallel to this plane passing through stress state point in the Mohr circle. Further, we vary the θ and follow the said exercise. The result is shown in figure 2.

This sign convention does not give us one pole but three poles. I am not sure if these poles are of any use to us. But clearly method of poles does not work for current sign convention i.e. tension and counterclockwise shear – positive.



(a) Constructing the pole?



(b) Three Poles!

Figure 2: Finding pole

1.2 Second starting point

This section correspond to stress state shown in figure 1b which assumes normal component as tension and shear stress in clockwise sense i.e. tension and clockwise shear is considered positive. Now following the procedure detailed in section 1.1, we discuss the outcome of this sign convention. The stresses on θ plane with this sign convention are

$$\sigma_\theta = -\frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta \tag{6}$$

$$\tau_\theta = -\frac{\sigma_{xx} - \sigma_{yy}}{2} - \sigma_{xy} \cos 2\theta \tag{7}$$

Furher,following section 1.1

$$\sigma_\theta = -\frac{\sigma_{xx} + \sigma_{yy}}{2} + R \cos(2\theta - \phi) \tag{8}$$

$$\tau_\theta = R \sin(2\theta - \phi) \tag{9}$$

where

$$R = \sqrt{\left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)^2 + \sigma_{xy}^2}$$

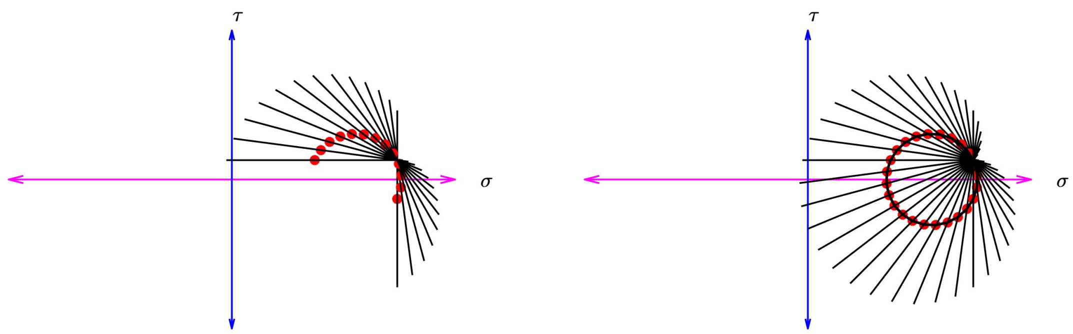
$$\tan\phi = \frac{2\sigma_{xy}}{\sigma_{yy} - \sigma_{xx}}$$

Now bring things to the complex plane,

$$\left[\sigma_\theta + \frac{\sigma_{xx} + \sigma_{yy}}{2}\right] + i\tau_\theta = Re^{i(2\theta - \phi)}$$

Observe that as we increase θ by α, variable plane in the element moves along counterclockwise by α angle and radial vector in the Mohr circle also moves in counterclockwise sense by a 2α angle. This suggest that, method of 2θ rotation works for this convention. Now let us look for pole.

Using a similar procedure as in section 1.1, we plot the Mohr circle with stress states as points and corresponding planes as lines. The figure 3 suggest the existence of pole.



(a) Constructing the pole

(b) Got the pole

Figure 3: Getting pole

It can be shown, by following exact same procedure, that method of pole and 2θ rotation will also hold for compression and counterclockwise shear as positive and becomes erroneous when compression and clockwise shear is considered positive.

2 Remarks

The method of pole and 2θ rotation appears to be quite handy tool in geomechanics. The article suggests and clears why these two techniques work correctly if sign convention is assumed either of the following

1. Compression and counterclockwise shear as positive

2. Tension and clockwise shear as positive

Further, convention 1 is popular in geoengineering community whereas convention 2 is used by everybody else.

Note that the Mohr circle considered here is for a triaxial state of stress where two principal stresses are equal. For a general stress state, we need to consider a three dimensional Mohr circle or most critical in one of the $3-D$ Mohr circle.

3 References

Most of the material used here is taught in a typical geomechanics class. Material in this article is either from class room or from writer's exploration on why pole works in Mohr circle. Follow the link https://github.com/ssingh09299/Notes-on-Geomechanics/tree/master/Pole related documents. For any comment and criticism, please write to saurabhsingh@iisc.ac.in

4 Further reading

For further exploration of Mohr circle refer to A detailed article on Mohr circle by Prof. Rebecca Brannon. Refer to the book by Prof Richard H. G. Parry for geotechnical specific use of Mohr circle.