<u>Linear Optimization</u>

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❖ Theorems Learnt in Course(Taken from class notes and IITB Professor Sundar's Lecture Notes)

Theorem 1: Suppose s1 = (v1, v2, v3….., vm) and s2 = (u1, u2,....., un) are the basis for a vector space V. Then, m=n, i.e. cardinality of both sets of basis is the same.

Theorem 2: A vector can be expressed exactly in 1 manner as a linear combination of basis.

Theorem 3: Given a set of equations(Ax = b). Suppose If Aj is replaced by c*Ai+Aj and bj is replaced by cbi+bj, then the solution set remains unchanged. It serves as the basis for doing row operations in Gaussian Elimination Process.

Theorem 4: In any vector space, there exists a basis where elements of basis are pairwise orthogonal to each other and magnitude of elements is 1. This is called an orthonormal basis. It can be noted that any vector can be divided by its magnitude to get a unit vector. It doesn't change the direction of that vector.

Theorem 5: In a matrix the number of linearly independent rows is equal to the number of linearly independent columns. The number of linearly independent columns is called the rank of the matrix. Inverse of a matrix exists if all rows and columns of a matrix are independent.

Theorem 6: Rank-Nullity Theorem: Rank(A) + Nullity(A) = n. A is the matrix and 'n' is the number of columns in A. Nullity is the number of vectors present in null space of A(Ax = 0).

Theorem 7: Let x0 be a vector satisfying (Ax = b). Then every solution to (Ax = b) can be written in the form (x0 + x′), where x′ is any vector satisfying (Ax′ = 0).

Theorem 8: A convex set is a set of points such that, given any two points A, B in that set, the line AB joining them lies entirely within that set. If S1 and S2 are two convex sets, then intersection of S1 and S2 is also a convex set.

Theorem 9: A point is a corner point(vertex) if the number of columns of a tight equation matrix is equal to rank of tight equation matrix.

Theorem 10: Given n vectors v1, . . . , vn, (i=1 to n) $\Sigma\lambda_i v_i$; $0 \le \lambda_i \le 1$; (i=1 to n) $\Sigma\lambda_i=1$, is called a convex combination of v1, …. , vn. Let p1,p2,...pt be extreme points of {x:Ax≤b}. Then every point in S = {x: Ax≤b} can be expressed as a convex combination of the points p1,p2,...pt.

Theorem 11: A linear objective function on S={x : Ax ≤ b} is maximized at an extreme point (corner point).

Theorem 12: Separating Hyperplane Theorem: Suppose we have a closed convex set of points. Suppose there is a point p which does not belong to this convex set. Then there is a hyperplane that separates the point p and our convex set.

❖ Simplex Algorithm

- **Given:** Constraints which are encoded as a matrix (Ax <= b). A linear objective function which can be encoded as Cx. C contains the coefficients of the linear objective function. A is m*n matrix, C is 1*n matrix, b is m*1 matrix.

- **Aim:** The aim of the algorithm is to maximise the objective function subject to the given constraints. If we are asked to minimise we can multiply objective function by -1 and then maximise the resulting objective function.

- **Degeneracy:** When number of tight rows at a point exceeds n(number of columns in A).

- **Finding the initial feasible point**

    - If all elements of the ‘b’ vector are greater then, (0,0,...0) n dimensional vector is our initial feasible point.
    - When one or more than one elements of b are negative. Let bk be the least value among b’s. One more variable (z >= 0) is added to the left hand side of given constraints(A matrix) with a coefficient -1(basically -1 is added at the end of all rows in A). And a new constraint corresponding to (z <= -bk) is added to the list of constraints to get a new matrix A’. Now we solve an auxiliary linear program where we maximize objective function (f = -z). (x1, x2, …, xn = 0 and z = -bk) is the initial feasible point

for this auxiliary program. If the optimal value of objective function, f >= 0 then our original linear program has a solution and optimal solution to the auxiliary program serves as the initial feasible point for it. Otherwise the original problem is infeasible. **We move ahead with the initial feasible point and original matrix A to the next step, where we loop till we get an optimal solution.**

- **Algorithm is as follows**

find an initial feasible point(corner point of the polyhedron.
loop:

1) move from one corner point to another.
2) exit the loop when moving away from the current point will decrease the objective function value. Now, we have achieved the optimal value.

- **Steps that are executed in a loop to reach an optimal point**

  - Start with feasible point = initial feasible point = x0
  - Below mentioned bullet points are an expansion of line number 2 mentioned in the just above mentioned algorithm section.
  - Find n tight independent tight rows from A corresponding to the feasible point. Name the matrix **T**. The rows that don't appear in T, club them together in **P**.
  - (-1)*inverse(T) = (-1)*T' = direction vectors of neighbours.
  - Find a column in -T' whose dot product with C is positive. In case no such column exists we have reached the **OPTIMAL POINT(report and exit in this case)**. Let the found column be **V**.
  - If the product of V with P gives all non-positive values then the problem is **UNBOUNDED**. We report it and exit.
  - Otherwise, find

$$t_k = min_s \frac{b_s - A_s x_0}{A_s v_i}$$

s, corresponds to rows which are not in T. vi is the same as V(found earlier). $A_s v_i > 0$. Let t = t_k.
This is the jump we need to make to reach the next corner vertex.
x0 = x0 + t*V (updated corner point).

  - Go to the line number 1 of the algorithm mentioned in the above section(next iteration of the loop).

- **How Degeneracy is solved**

- Add a random small positive random noise to the rhs of the constraints(vector b). Start the normal simplex algo.
- Run simplex for a fixed number of iterations(to catch infinite loop). If it fails to terminate, repeat bullet point 1 with a new set of random noises.
- If it successfully terminates. Then we need to recover the original optimal point and solution from the solution modified system. We find the tight constraints of the optimal solution wrt to modified constraints. Then the intersection of those tight constraints with respect to the original system is computed. This will be the optimal solution.

❖ Primal-Dual Theory

- Every LP problem(primal) has a corresponding another LP problem associated called dual. Below is an example given.
- Primal:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0. \end{array}$$

- Dual:

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c \\ & y \geq 0. \end{array}$$

- RHS in original constraints becomes the 'Coefficient vector' of the objective in dual. The constraint matrix in dual is the transpose of the constraint matrix of primal. Coefficients of objective in primal become RHS of constraint matrix in dual. We minimize the objective in dual. And inequality in constraints changes to '>=' inplace of '<=' in dual.

- **Theorem:** Dual of dual is primal.

Suppose the primal is

$$\text{Maximize} \quad Z = c^T x$$

such that

$$Ax \leq b, \qquad x \geq 0.$$

Its dual is

$$\text{Minimize} \quad W = b^T y$$

such that

$$A^T y \geq c, \qquad y \geq 0,$$

which can be written in standard form

$$\text{Maximize} \quad -W = (-b)^T y$$

such that

$$(-A)^T y \leq -c, \qquad y \geq 0.$$

The dual of the dual is therefore

$$\text{Minimize} \quad \bar{Z} = (-c)^T x$$

such that

$$\left((-A)^T\right)^T x \geq -b, \qquad x \geq 0,$$

But this equivalent to

$$\text{Maximize} \quad Z = c^T x$$

such that

$$Ax \leq b, \qquad x \geq 0.$$

Hence, Proved.

- **Weak Duality Theorem:** Let x0 be a feasible solution of primal and y0 be a feasible solution for dual. Then dot(c, x0) <= dot(b, y0). Dot means dot product.

- **Theorem(Certificate of Optimality)**: Let x0 be a feasible solution of primal and y0 be a feasible solution for dual. Then, dot(c, x0) = dot(b, y0) --> x0, y0 are optimal solutions to their corresponding linear problems. This theorem has many important implications:
    - If primal is feasible and bounded so is the dual and vice-versa. The optimal value of both primal and dual linear programs is equal.
    - If primal is unbounded then dual is infeasible. If dual is unbounded then primal is infeasible.

❖ References

- http://www.dam.brown.edu/people/huiwang/classes/am121/Archive/dual_121.pdf
- https://web.stanford.edu/~ashishg/msande111/notes/chapter4.pdf
- https://www.cse.iitb.ac.in/~sundar/cs435/