EE23BTECH11054 - Sai Krishna Shanigarapu*

Gate MA 2022

14. The value of the integral

$$\int_C \frac{z^{100}}{z^{101}+1}\,dz$$

where C is the circle of radius 2 centred at the origin taken in the anti-clockwise direction is

(A) $-2\pi i$
(B) 2π
(C) 0
(D) $2\pi i$

(GATE 2022 MA)

Solution:

Cauchy's Theorem:
If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then

$$\int_C f(z) = 0 \tag{1}$$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{2}$$

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \tag{3}$$

Residue Theorem:

$$\int_C f(z)\,dz = 2\pi i \sum Res\, f(a) \tag{4}$$

where,

$$Res\, f(a) = \lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n\, f(z)\right]\right) \tag{5}$$

Solving the integral,

$$f(z) = \int_C \frac{z^{100}}{z^{101}+1}\,dz \tag{6}$$

$$z^{101}+1=0 \implies z=-1 \tag{7}$$

Since the pole $z=-1$ is inside the circle, Using eq (5)

$$Res\, f(-1) = \lim_{z\to -1}\left(\frac{z^{100}}{z^{101}+1}\right)\left(z^{101}+1\right) \tag{8}$$

$$= 1 \tag{9}$$

From eq (4), and eq (9)

$$\int f(z)\,dz = 2\pi i\,(1) \tag{10}$$

$$\implies \int_C \frac{z^{100}}{z^{101}+1}\,dz = 2\pi i \tag{11}$$

$\therefore$ option (D) is correct.

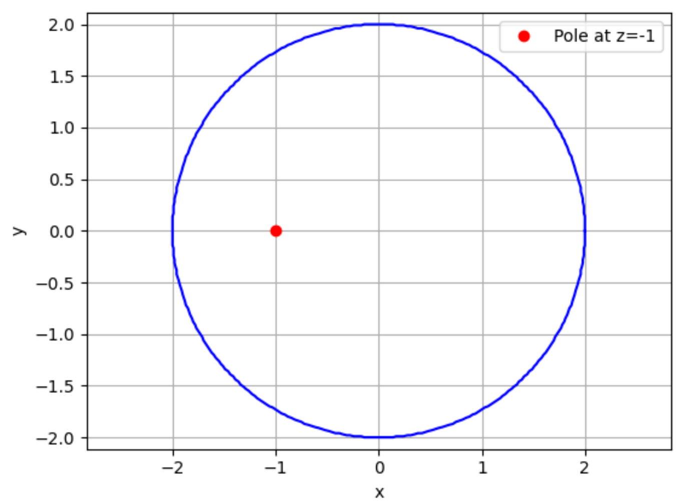

Fig. 1. plot of C with it's pole