EE23BTECH11054 - Sai Krishna Shanigarapu*

Gate MA 2022

14. The value of the integral

$$\int_C \frac{z^{100}}{z^{101}+1}\,dz$$

where C is the circle of radius 2 centred at the origin taken in the anti-clockwise direction is

(A) $-2\pi i$
(B) 2π
(C) 0
(D) $2\pi i$

(GATE 2022 MA)

Solution:

Cauchy's Theorem:
From Figure 1

$$\int_C f(z)\,dz = \int_{Cr} f(z)\,dz \tag{1}$$

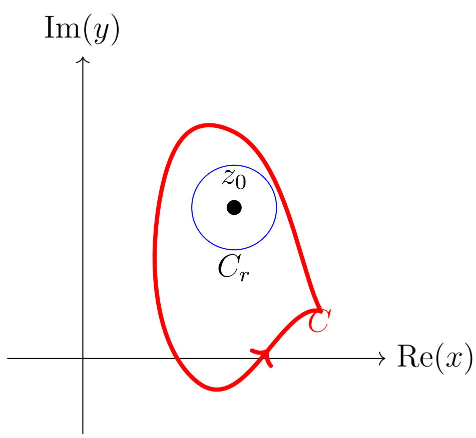

Fig. 1. Figure1

Since $g(z)$ is continuous we know that $|g(z)|$ is bounded inside C_r. Say, $|g(z)| < M$. The corollary to the triangle inequality says that

$$\left|\int_{C_r} f(z)\ dz\right| \leq M2\pi r. \tag{2}$$

Since r can be as small as we want, this implies that

$$\int_{Cr} f(z)\,dz = 0 \tag{3}$$

let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \tag{4}$$

$$\lim_{z\to z_0} g(z) = f'(z_0) \tag{5}$$

$$\int_C g(z)\ dz = 0, \implies \int_C \frac{f(z)-f(z_0)}{z-z_0}\ dz = 0 \tag{6}$$

Thus,

$$\int_C \frac{f(z)}{z-z_0}\ dz = \int_C \frac{f(z_0)}{z-z_0}\ dz = 2\pi i f(z_0) \tag{7}$$

Using Cauchy's Theorem,

$$\int_C f(z) = 0 \tag{8}$$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{9}$$

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \tag{10}$$

Residue Theorem:
From eq (7)

$$\int_C f(z)\ dz = 2\pi i \sum Res\, f(a) \tag{11}$$

where, for n repeated poles,

$$Res\, f(a) = \lim_{z\to a}\frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n\ f(z)\right]\right) \tag{12}$$

Solving the integral,

$$f(z) = \int_C \frac{z^{100}}{z^{101}+1}\,dz \tag{13}$$

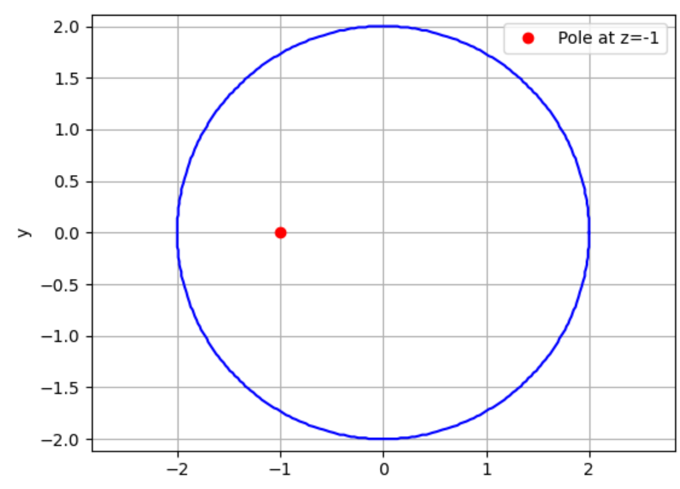

Fig. 2. plot of C with it's pole

Since the pole $z = -1$ is inside the circle, Using eq (12)

$$Res\, f(-1) = \lim_{z\to -1}\left(\frac{z^{100}}{z^{101}+1}\right)\left(z^{101}+1\right) \tag{14}$$

$$= 1 \tag{15}$$

From eq (11), and eq (15)

$$\int f(z)\, dz = 2\pi i\,(1) \tag{16}$$

$$\implies \int_C \frac{z^{100}}{z^{101}+1}\, dz = 2\pi i \tag{17}$$

$\therefore$ option (D) is correct.