COLUMBIA UNIVERSITY

Computation of European Call Option Greeks Using Pathwise Estimation and Likelihood Ratio Method

Final Project Report for MATH 5010 - Spring 2024

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1. Introduction

Black-Scholes (BS) option pricing allows us to compute and quantify risk in spot movements, time decay, volatility changes and higher order greeks. In theory, analytical formulas are useful for single options but in practice, Monte Carlo methods are used to price complex and exotic options. Finite-difference (FD) Monte Carlo in general has poor convergence rates and step-size dependent relative errors. Additionally, second-order greeks like Γ require 3 separate trajectories in order to compute a single iteration of the greek–a highly time and memory consuming process. Hence, in this report, we will show that Pathwise Estimator (PE) to compute Δ and ν and the Likelihood Ratio (LR) method to compute Γ all have better computational performance than the ‘bump-and-revalue’ finite-difference Monte Carlo method.

2. Pathwise Estimator Greeks as a function of spot

The Pathwise Estimator (PE) method provides a robust framework for analyzing the sensitivity of European call options to changes in the underlying spot price, often referred to as the "Greeks." The Pathwise approach, unlike finite-difference Monte Carlo methods, leverages the differentiation of the payoff function directly within the simulation paths (shown in Appendix), thereby enhancing computational efficiency and reducing errors associated with numerical approximation.

- Delta: As depicted in figure 1c), the Delta calculated via the Pathwise method (PE Delta) showcases a steady incline as the spot price increases, reflecting the increased likelihood of the option being in-the-money. The PE Delta closely tracks the theoretical values, indicating its reliability and precision in capturing the first-order sensitivity without the need for separate trajectory simulations.

- Vega: Figure 1d) highlights the Vega computed via the Pathwise method (PE Vega), which measures the sensitivity of the option's price to changes in the volatility of the underlying asset. Vega peaks near the at-the-money position, where the uncertainty about the option ending in-the-money is highest, and then diminishes as the option moves deeper into or out of the money. The slight deviation in the MC Vega at higher asset prices could be due to the inherent variance of the MC simulations, which follow GBM, which in turn follows log normal distribution – meaning the distribution is right skewed and bound by 0 at the lower end leading to higher concentration of paths between 0 and strike as compared to greater than strike, yet it still follows the expected pattern which aligns with the BS model.

- Gamma: Figure 1e) shows a similar situation as the Vega where due to the few GBM trajectories hitting the deep ITM region in spot price, the value of Gamma is noisier in that region. However, LR is a valid method for valuing BS Gamma.

- Theta: Figure 1b) shows a great deal of numerical instability associated with the finite difference approach to calculating MC theta. Perhaps a greater number of iterations are needed with a greater step size in maturity ('bump') to guarantee convergence to the BS result.

3. Convergence of Finite-Difference Monte Carlo Greeks with Iterations

The graph 1c) shows the comparison in the error of the estimation of Greeks, using Monte Carlo and Black-Scholes equation. As indicated on the x-axis, the number of iteration has to be big enough to minimize the errors, and by the definition of Monte Carlo, the more iteration is required, as we need more accuracy.

As observed, Monte-Carlo involves heavy-calculations that could vastly consume the time and energy of your computer. The Pathwise Estimator could reduce the loads for these calculations. Pathwise Estimator is more efficient than Monte Carlo when estimating Greeks, as long as the payoff function and its derivatives are well-behaved. This is because it integrates the computation of these derivatives into the simulation itself, avoiding the need for multiple sets of simulations required by the finite difference approach. This paper will discuss this in more detail with the comparison between Monte Carlo Estimator, and Pathwise Estimator.

4. Convergence of Pathwise Estimator and Likelihood ratio Greeks with Iterations

If we compare the convergence behavior between figure 1c) and 1d), we notice that the overall relative error of gamma computed using LR is lower than that computed using the 2nd order FD. The PE delta’s relative error is significantly more accurate than that calculated using FD. However, it seems that FD vega outperforms PE vega in terms of both accuracy and convergent rate. In general, the relative error of FD (1st and 2nd order) decreases with increasing number of iterations; that relationship is shared with PE greeks. However, the relative error of gamma computed with the LR method has a region where increasing the number of iterations increases the relative error, which suggests numerical instability could be an issue.

5. Conclusions

We have surveyed both the pathwise estimator and likelihood ratio method to calculate first and second order greeks for a European Call option. We have found an overall better convergence for both methods than the brute force finite difference monte carlo method. Both methods show great agreement with analytical Black-Scholes Greeks. This is important in valuing complex exotic options and derivatives with path-dependent payoffs. An interesting extension to this project is to value and calculate Greeks for Asian Options or Digital Options.

References:

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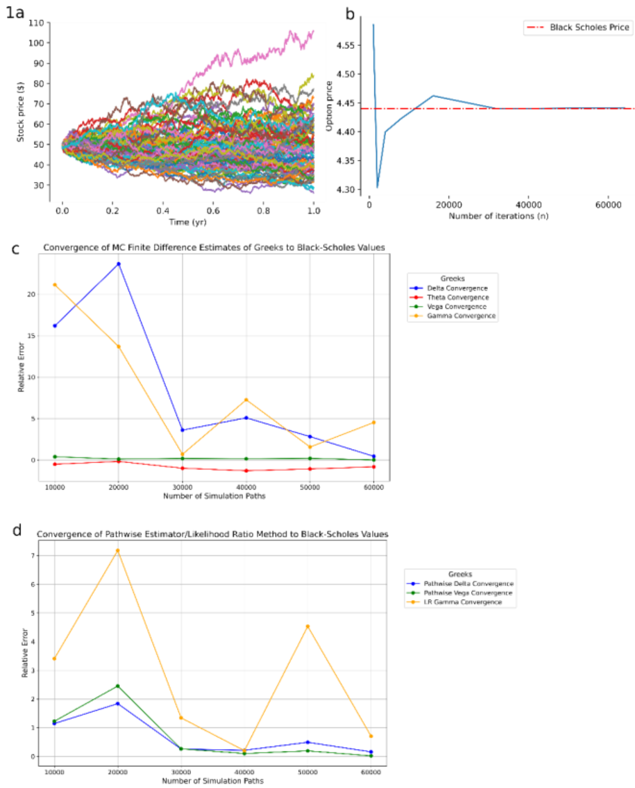
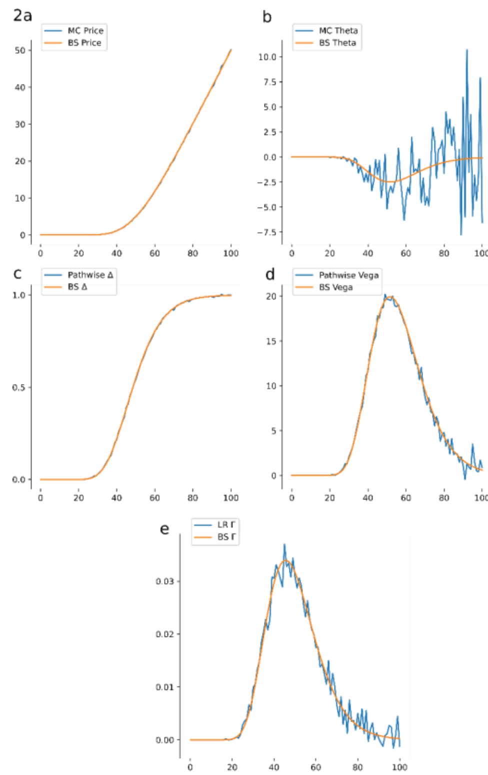
- Haugh, Martin. “IEOR E4703: Monte-Carlo Simulation.” Https://Martin-Haugh.Github.Io, Columbia University, martin-

haugh.github.io/files/MonteCarlo/MCS_Greeks_MasterSlides.pdf. Accessed 4 Nov. 2023.

Figures


1a
b
Stock price ($)
Time (yr)
Option price
Black Scholes Price
Number of iterations (n)
c
Convergence of MC Finite Difference Estimates of Greeks to Black-Scholes Values
Relative Error
Number of Simulation Paths
Greeks
Delta Convergence
Theta Convergence
Vega Convergence
Gamma Convergence
d
Convergence of Pathwise Estimator/Likelihood Ratio Method to Black-Scholes Values
Relative Error
Number of Simulation Paths
Greeks
Pathwise Delta Convergence
Pathwise Vega Convergence
LR Gamma Convergence


2a
MC Price
BS Price
b
MC Theta
BS Theta
c
Pathwise Δ
BS Δ
d
Pathwise Vega
BS Vega
e
LR Γ
BS Γ

Appendix

Derivation of Finite-Difference Monte Carlo greeks:

To calculate a European call option with a payoff:
$f(S_T) = \max(S_T - K, 0)$ at t=T.
We can calculate the following quantity:
$E^Q[max(S_T - K, 0)]$, where $E^Q[\cdot]$ is taken with respect to the risk neutral density--the probability density under which the expectation of the discounted payoff is a martingale.The theoretical pricing formula is done with the Black-Scholes Model:
$$C(S,t) = S_t N(d_1) - KN(d_2)e^{-rT}$$
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)T\right]$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Let V be the value of the derivative and S_0 be the current spot price:
$$\Delta \approx \frac{V(S+\delta S) - V(S)}{\delta S}.$$
For Γ, we require a second-order centered finite difference:
$$\Gamma \approx \frac{V(S+\delta S) - 2V(S) + V(S-\delta S)}{(\delta S)^2}.$$

Derivation of Pathwise Estimator Greeks:

To simulate the stock paths, we can discretize the SDE $\frac{dS_t}{S_t} = rdt + \sigma dW_t$ using Euler discretization:$S_{n+1} = S_n(1 + r\Delta t) + \sigma S_n \sqrt{\Delta t} Z_n$, where $Z_n \sim N(0,1)$.

Delta:
Since we know the call price is:$C(S_0, 0) = e^{-rT}E^Q[\max(S_T - K, 0)]$To get the delta, we need to compute $\frac{\partial C(S_0,0)}{\partial S_0}$:
$$\frac{\partial C(S_0,0)}{\partial S_0} = e^{-rT}E^Q[\frac{\partial \max(S_T - K, 0)}{\partial S_T}\frac{\partial S_T}{\partial S_0}] = e^{-rT}E^Q[1_{S_T>K}\frac{S_T}{S_0}] \approx e^{-rT}\frac{1}{n}\sum_{i=0}^{n} 1_{S_T>K}\frac{S_T}{S_0}$$

In other words, we only count the fraction $\frac{S_T}{S_0}$ if $S_T > K$ and take an average to compute the Δ

To compute vega in a pathwise estimator fashion, we likewise take a derivative of the payoff function with respect to implied volatility:

$$\nu = E^Q\left[\frac{\partial\left(e^{-rT}\max(S_T-K,0)\right)}{\partial\sigma}\right]$$

$$\nu = E^Q\left[e^{-rT}\left(\frac{\ln S_T/S_0-(r+\sigma^2/2)T}{\sigma}\right)S_T I_{S_T>K}\right]$$

To fit gamma, we cannot differentiate the above expression another time, since it will give us a Dirac Delta function and makes the average ill-defined. We can approach it from the likelihood ratio method.

$$C(S_0,0)=e^{-rT}E^Q[f(S_T)]=e^{-rT}\int f(S_T)p_{S_0}(S_T)dS_T$$

We can differentiate the above expression again with respect to S_0 twice to get the Γ:

$$\Gamma=\frac{\partial^2 C(S_0,0)}{\partial S_0^2}=e^{-rT}\int\max(S_T-K,0)\frac{\partial^2 p_{S_0}(S_T)}{\partial S_0^2}dS_T=e^{-rT}\int\left[\max(S_T-K,0)\frac{\partial^2 p_{S_0}(S_T)}{\partial S_0^2}\frac{1}{p_{S_0}(S_T)}\right]p_{S_0}(S_T)dS_T$$

$$\Gamma\approx e^{-rT}\frac{1}{n}\sum_{i=0}^{n}\max(S_T-K,0)\frac{\partial^2 p_{S_0}(S_T)}{\partial S_0^2}\frac{1}{p_{S_0}(S_T)}$$ Since

$$p(S_T)=\frac{1}{\sqrt{2\pi\sigma^2 T}S_T}\exp\left(-\frac{(\ln(S_T/S_0)-(r-\frac{1}{2}\sigma^2T))^2}{(2\sigma^2T)}\right)$$

$$\frac{\partial\ln p(S_T)}{\partial S_0}=\frac{\ln(\frac{S_T}{S_0})-(r-0.5\sigma^2)T}{S_0\sigma^2T}\frac{1}{p(S_T)}\frac{\partial^2 p(S_T)}{\partial S_0^2}=\frac{4\ln^2\left(\frac{S_T}{S_0}\right)-\sigma^2T\left(\sigma^2T+4\right)}{4S_0^2\sigma^4T^2}$$