PASS!

A) Second-Degree Polynomials:

$$\phi_p = ax^2 + by^2 + cxy + dx + ey + f$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get the equation $a + b = -G\theta$ (the same as in homogenous case!) putting a constraint on the coefficients a and b. The constants c, d, e, f have no constraints as of now.

Now, we find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to these equations we get:

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

$$\sigma_{zx} = 2by + cx \Rightarrow \gamma_{zx} = \frac{2by + cx}{G}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow \frac{2b}{G} - \frac{(-2a)}{G} = \frac{2(a+b)}{G} = \frac{2(-G\theta)}{G} = -2\theta \quad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow c - c = 0 \quad \text{(Satisfied!)}$$

The equations are trivially satisfied on their own. Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p = ax^2 - (a + G\theta)y^2 + cxy + dx + ey + f$$

$$\sigma_{zx} = -2(a + G\theta)y + cx \Rightarrow \gamma_{zx} = \frac{-2(a + G\theta)y + cx}{G}$$

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

Remarks: Notice that the addition of the first-order terms have made no difference to the field descriptions. However, they surely have increased the number of possible functions that can satisfy the Poisson equation by making the Prandtl stress function non-homogenous and therefore, even though they don't change the outputs, they can surely change the input in a way that it also produces a physically possible case of a boundary equation along with satisfying the Laplacian.

B) Third-Degree Polynomials:

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$$\phi_p = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get:

$$6dy + 2cx + 2g + 6ax + 2by + 2e = -2G\theta$$

In order to satisfy this equation for all x and all y, following relations must be established:

$$b = -3d \mid c = -3a \mid e + g = -G\theta$$

Now, we find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to these equations and eliminating c, d, g using the relations obtained above, we get:

$$\sigma_{zx} = b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i \Rightarrow \gamma_{zx} = \frac{b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i}{G}$$

$$\sigma_{zy} = -(3a(x^2 - y^2) + 2bxy + 2ex + fy + h) \Rightarrow \gamma_{zy} = \frac{-(3a(x^2 - y^2) + 2bxy + 2ex + fy + h)}{G}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow \frac{-2by}{G} + \frac{(-6ax)}{G} + \frac{-2(e + G\theta)}{G} - \left\{ \frac{-6ax}{G} + \frac{-2by}{G} + \frac{-2e}{G} \right\} = -2\theta \quad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow 2bx - 6ay + f + (6ay - 2bx - f) = 0 \quad \text{(Satisfied!)}$$

The equations are trivially satisfied on their own. Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p = ax^3 + bx^2y - 3axy^2 - (b/3)y^3 + ex^2 + fxy - (e + G\theta)y^2 + hx + iy + j$$

$$\sigma_{zx} = b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i \Rightarrow \gamma_{zx} = \frac{b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i}{G}$$

$$\sigma_{zy} = -(3a(x^2 - y^2) + 2bxy + 2ex + fy + h) \Rightarrow \gamma_{zy} = \frac{-(3a(x^2 - y^2) + 2bxy + 2ex + fy + h)}{G}$$

Remarks: Again, just like the first order terms, terms of degree three remain in the Prandtl stress function with the coefficients obeying some particular relations. Again, this opens a lot of other functions that might be analysed. Similar arguments can be continued for polynomials of higher degree; the only difference is that the number of relations between the coefficients will increase and the algebra will become a bit clumsy and messy. However, the fact that the second order polynomials are required for the function to prove to be a valid Prandtl function does not change. **On the other hand, if any function does not adhere to the relations of the coefficients determined above, it will result into a fail case.**