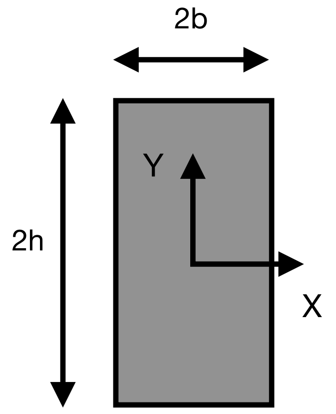
Test Case 3: Rectangular Cross-Section ($h \approx b$) FAIL!

For the rectangular cross-section:



Cross-section

$$F(x, y) \;=\; (x-b)(x+b)(y-h)(y+h) \;=\; x^2y^2 - h^2x^2 - b^2y^2 + h^2b^2$$

Therefore,

$$\phi_p \;=\; BF(x, y) \;=\; B(x^2y^2 - h^2x^2 - b^2y^2 - h^2b^2)$$

Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get,

$$2B(x^2 - b^2) + 2B(y^2 - h^2) = -2G\theta$$

Hence,

$$B = \frac{-G\theta}{x^2 + y^2 - h^2 - b^2}$$

Since B that is expected to be a constant has a dependance on both x and y coordinate, the Laplacian cannot be solved in this case like it was solved in the above cases.

Remarks: The term x^2y^2 is producing the terms x^2 and y^2 after the Laplacian acts on it and thus it creates a dependance on which the constant B has to rely.

Note that to remove the dependance B has on x and y, the term $x^2 + y^2$ has to be constant, but that would mean that the points on the cross section- lie on a circle, which contradicts the fact that the cross-section we chose is rectangular.

Hence, a rectangular cross-section is **a clear fail case for the app and the method of solving as well**.

The app identifies any function that does not satisfy the Laplacian (the Poisson equation) for the Prandtl stress function as a "wrong input" and hence the user gets to know that the input function is not a valid Prandtl stress function.

Note that this does not mean that the solution does not exist for the Laplacian, or the problem in general.

The Laplacian is solved by taking a particular solution superposed with a homogenous solution. The homogenous part is solved by separation of variables method. Thus, solution for the homogenous part will give a generalised solution to the Laplacian. It is seen that the homogenous part eventually becomes a problem of Fourier series and thus has to be solved with Fourier analysis, making the homogenous part a series of sinusoidal functions. This make the computation of the problem of rectangular shafts very complex and beyond the scope of the fundamental study on "Torsion in Arbitrary Shafts". A glimpse of the mathematical approach in the generalised solution is shown:

$$V(x, y) = f(x)g(y)\;(assumed)$$

$$where\; V = \{0\; for\; x = \pm b\; and\; G\theta(h^2 = x^2)\; for\; y = \pm h\}$$

$$\phi_p = G\theta(h^2 - x^2) + V(x, y)$$

Then the Laplacian for V gives:

$$g''f + f''g = 0 \implies \frac{f''}{f} = -\frac{g''}{g} = \lambda^2 \implies \begin{matrix} f(x) = Acos(\lambda x) + Bsin(\lambda x) \\ g(y) = Ccos(\lambda y) + Dsin(\lambda y) \end{matrix}$$

Ultimately, $B = D = 0\; and\; \lambda = \dfrac{n\pi b}{2h}$ $(n = 1,2,3...)$ is what we get after the condition that V must be and even function of the coordinates.

$$\implies V = \sum_{i=1}^{n} A_n cos\left(\frac{n\pi x}{2h}\right) cos\left(\frac{n\pi y}{2h}\right)$$

(The constant A_n further needed to be found by boundary equations which will involve evaluating Fourier series and beyond the fundamental problems that are considered)