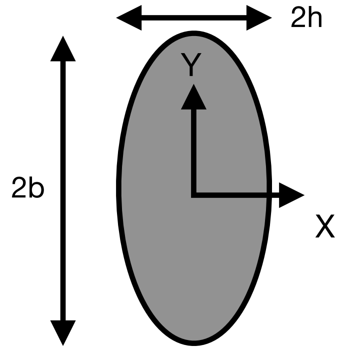
Test Case 2: Elliptical Cross-Section PASS!

For the circular cross-section: $F(x, y) = \frac{x^2}{h^2} + \frac{y^2}{b^2} - 1$
Therefore,

$$\phi_p = BF(x, y) = B(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1)$$



Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get,

$$2B(\frac{1}{h^2} + \frac{1}{b^2}) = -2G\theta$$

Hence,

$$B = -\frac{h^2 b^2}{h^2 + b^2}(G\theta)$$

Note: $J = \frac{\pi b^3 h^3}{b^2 + h^2}$ for an elliptical shaft with dimensions as shown in figure

$J \equiv Torsional\ Rigidity$

Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to the above equations we get:

$$\sigma_{zx} = \frac{-2h^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zx} = \frac{-2h^2 \theta y}{h^2 + b^2}$$

$$\sigma_{zy} = \frac{2b^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zy} = \frac{2b^2 \theta y}{h^2 + b^2}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):
Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow -\frac{-2h^2\theta}{h^2 + b^2} - \frac{2b^2\theta}{h^2 + b^2} = -2\theta$$

(Satisfied!)

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow 0 + 0 = 0$$

(Satisfied!)

Both the equations are trivially satisfied and thus no other constraint exists.
Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p = -\frac{h^2 b^2 (G\theta)}{h^2 + b^2}(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1)$$

$$\sigma_{zy} = \frac{2b^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zy} = \frac{2b^2 \theta y}{h^2 + b^2}$$

$$\sigma_{zx} = \frac{-2h^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zx} = \frac{-2h^2 \theta y}{h^2 + b^2}$$

Remarks: Note that we can also calculate for a hollow elliptical shaft just like a hollow circular shaft discussed above, by considering negative mass of the hollow part and superposing the results with the solid shaft. Since, it's one of the standard shapes, its torsional rigidity J is known and hence the torque can be calculated. Therefore, tha app is able to give the value of torque for this case.