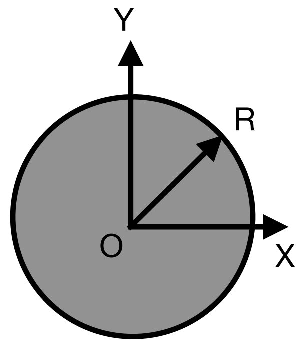
Test Case 1: Circular Cross Section PASS!



Cross-Section

For the circular cross-section: $F(x,y) \; = \; x^2 \; + \; y^2 \; - \; R^2$

Therefore, $\phi_p \; = \; BF(x,y) \; = \; B(x^2 \; + \; y^2 \; - \; R^2)$

Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get $4B = -2G\theta$ or $B = -G\theta/2$.

Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to the above equations we get:

$$\sigma_{zy} \; = \; G\theta x; \Rightarrow \gamma_{zy} \; = \; \theta x$$

$$\sigma_{zx} \; = -G\theta y; \Rightarrow \gamma_{zx} \; = \; -\theta y$$

Note: $J = \dfrac{\pi R^4}{2}$ for a circular shaft of radius R

$J \equiv Torsional \; Rigidity$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \; \Rightarrow -\theta \; - \; (\theta) \; = -2\theta$$

(Satisfied!)

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} \; = \; 0 \; \Rightarrow \; 0 \; + \; 0 \; = \; 0$$

(Satisfied!)

Both the equations are trivially satisfied and thus no other constraint exists.

Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p \; = \; \frac{-G\theta}{2}(x^2 \; + \; y^2 \; - \; R^2)$$

$$\sigma_{zx} \; = -G\theta y; \Rightarrow \gamma_{zx} \; = \; -\theta y$$

$$\sigma_{zy} \; = \; G\theta x; \Rightarrow \gamma_{zy} \; = \; \theta x$$

Remarks: Note that this will work for circular shafts as long as they are not composite.

Since the app has to be generalised for other arbitrary cross-sections as well, the integration of the composite problems in circular shafts would not be possible with other shafts in a single code. Hence, results can be calculated only for a single homogenous shaft made of a single material.

Also, results for hollow circular shafts can be obtained by superposing the results of the hollow portion and the solid portion. However, the calculation for the superposition of the respective outputs needs to be done manually as the program can only calculate the outputs for both the portions separately, owing to the generalisation that is required for all arbitrary shafts.