A) Second-Order Polynomials:

PASS!

$$\phi_p = ax^2 + by^2 + cxy + d$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial^2 y} + \frac{\partial^2 \phi_p}{\partial^2 x} = -2G\theta$$

We get the equation $a + b = -G\theta$ putting a constraint on the coefficients a and b. The constants c and d have no constraints on them (as for now).
Now, we find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to these equations we get:

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

$$\sigma_{zx} = 2by + cx \Rightarrow \gamma_{zx} = \frac{2by + cx}{G}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):
Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow \frac{2b}{G} - \frac{(-2a)}{G} = \frac{2(a+b)}{G} = \frac{2(-G\theta)}{G} = -2\theta \quad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow c - c = 0 \quad \text{(Satisfied!)}$$

The equations are satisfied on their own. Thus, the Prandtl stress function the shear stresses and shear strains take the form:

$$\phi_p = ax^2 - (a + G\theta)y^2 + cxy + d$$

$$\sigma_{zx} = -2(a + G\theta)y + cx \Rightarrow \gamma_{zx} = \frac{-2(a + G\theta)y + cx}{G}$$

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

Here, the coefficients a, c and d are arbitrary and can take any real value. Only one relation was established and therefore only one variable has been reduced. The rest all variables are free to take any values without affecting the compatibility or the equilibrium.

Remarks: Note that standard cases such as circle and ellipse will be sub-cases of the above set of polynomials on choosing the appropriate values of the coefficients a, c and d. However, also note that some of the valid functions of this form might not have any physical meaning in the form of a boundary equation.

FAIL!

B) Third-Order Polynomials:

$$\phi_p = ax^3 + bx^2y + cxy^2 + dy^3 + e$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get the equation: $(6a + 2c)x + (2b + 6d)y = -2G\theta$

Now, in order to be independent of the coordinates, the coefficients have to follow the relations: $c = -3a$ & $b = -3d$.

However, note that in doing so the left-hand side of the equation goes to zero and hence the equation besoms redundant.

Thus, no set of coefficients will ever be able to satisfy the Laplacian. Hence,

All homogenous third-order polynomials will be 'Fail Cases' both in theory as well as in the app.

Remarks: Notice that the x and y dependencies were removed by the above mentioned relations between the coefficients. The only term needed was a constant that can be set equal to the term on the right-hand side. A term that comes out as a constant from a Laplacian can only be of the form ax^2 or by^2 where a and b are arbitrary constants. Thus, only a second-order term only in x or in y can satisfy the right-hand side of the equation. However, since we are talking about homogenous polynomials, a third-order polynomial will not have any second-order terms and hence the case fails. Now, this logic can be extended to all the polynomials with order greater than two. Let us follow this discussion up with the generalised case below.

FAIL!

C) N^th^-Order Polynomials:

$$\phi_p = \sum_{n=0}^{N} a_n y^n x^{N-n} = a_0 x^N + a_1 x^{N-1} y + a_2 x^{N-2} y^2 + \dots$$

Here again, the order is greater than two and being homogenous, the polynomial will not contain any term that is of order other than N. Therefore, the Laplacian in this case will yield the terms that will have at least one power of either x or y or both but no constant term that can help satisfy the governing equation.

Remarks: Note that we did not take first-order homogenous polynomials since they will simply yield a redundant equation when it comes to the Laplacian equation. Since the Laplacian has partial derivatives of order two, the first-order terms will just vanish on the left-hand side leaving unequal left and right-and sides. Therefore, from the cases A, B and C that we discussed under homogenous polynomial, we can say that only second-order homogenous polynomials will have the potential to satisfy the Laplacian or the Poisson equation for the Prandtl stress function and create a situation where there is a possibility of a valid solution emerging from the governing equations.