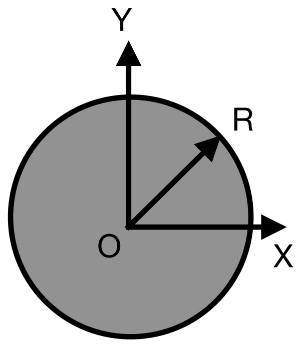
Test Case 1: Circular Cross Section PASS!

For the circular cross-section: $F(x,y) \; = \; x^2 \; + \; y^2 \; - \; R^2$

Therefore,

$$\phi_p \; = \; BF(x,y) \; = \; B(x^2 \; + \; y^2 \; - \; R^2)$$



Cross-Section

Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get $4B = -2G\theta$ or $B = -G\theta/2$.

Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Note: $J = \dfrac{\pi R^4}{2}$ for a circular shaft of radius R
$J \equiv Torsional\ Rigidity$

Owing to the above equations we get:

$$\sigma_{zy} \; = \; G\theta x; \Rightarrow \gamma_{zy} \; = \; \theta x$$

$$\sigma_{zx} \; = -G\theta y; \Rightarrow \gamma_{zx} \; = \; -\theta y$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \; \Rightarrow -\theta \; - \; (\theta) \; = -2\theta$$ (Satisfied!)

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} \; = \; 0 \; \Rightarrow \; 0 \; + \; 0 \; = \; 0$$ (Satisfied!)

Both the equations are trivially satisfied and thus no other constraint exists.

Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p \; = \; \frac{-G\theta}{2}(x^2 \; + \; y^2 \; - \; R^2)$$

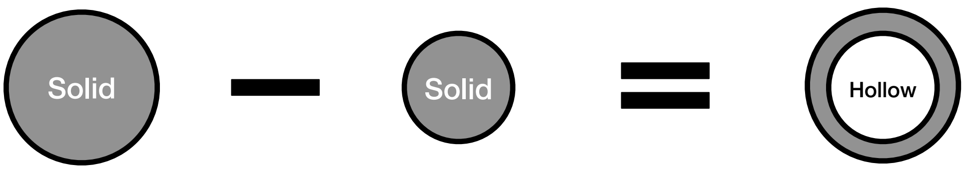
$$\sigma_{zx} \; = -G\theta y; \Rightarrow \gamma_{zx} \; = \; -\theta y$$

$$\sigma_{zy} \; = \; G\theta x; \Rightarrow \gamma_{zy} \; = \; \theta x$$

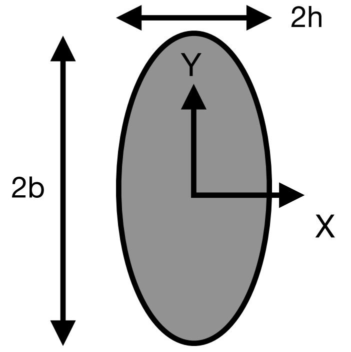
Remarks: Note that this will work for circular shafts as long as they are not composite.

Since the app has to be generalised for other arbitrary cross-sections as well, the integration of the composite problems in circular shafts would not be possible with other shafts in a single code. Hence, results can be calculated only for a single homogenous shaft made of a single material.

Also, results for hollow circular shafts can be obtained by superposing the results of the hollow portion and the solid portion. However, the calculation for the superposition of the respective outputs needs to be done manually as the program can only calculate the outputs for both the portions separately, owing to the generalisation that is required for all arbitrary shafts.



For the circular cross-section: $$F(x,y) = \frac{x^2}{h^2} + \frac{y^2}{b^2} - 1$$
Therefore,



$$\phi_p = BF(x,y) = B(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1)$$

Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get,

$$2B(\frac{1}{h^2} + \frac{1}{b^2}) = -2G\theta$$

Hence,

$$B = -\frac{h^2 b^2}{h^2 + b^2}(G\theta)$$

Note: $J = \dfrac{\pi b^3 h^3}{b^2 + h^2}$ for an elliptical shaft with dimensions as shown in figure

$J \equiv Torsional\ Rigidity$

Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$
$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to the above equations we get:

$$\sigma_{zx} = \frac{-2h^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zx} = \frac{-2h^2\theta y}{h^2 + b^2}$$
$$\sigma_{zy} = \frac{2b^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zy} = \frac{2b^2\theta y}{h^2 + b^2}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow -\frac{-2h^2\theta}{h^2 + b^2} - \frac{2b^2\theta}{h^2 + b^2} = -2\theta$$ (Satisfied!)

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow 0 + 0 = 0$$ (Satisfied!)

Both the equations are trivially satisfied and thus no other constraint exists.

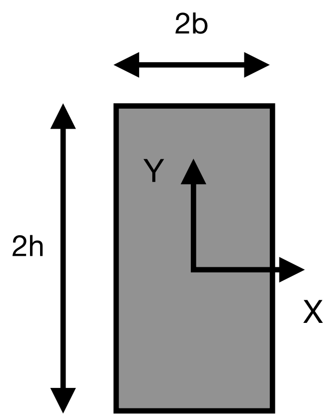
Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p = -\frac{h^2 b^2 (G\theta)}{h^2 + b^2}(\frac{x^2}{h^2} + \frac{y^2}{b^2} - 1)$$
$$\sigma_{zy} = \frac{2b^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zy} = \frac{2b^2\theta y}{h^2 + b^2}$$
$$\sigma_{zx} = \frac{-2h^2 G\theta y}{h^2 + b^2} \Rightarrow \gamma_{zx} = \frac{-2h^2\theta y}{h^2 + b^2}$$

Remarks: Note that we can also calculate for a hollow elliptical shaft just like a hollow circular shaft discussed above, by considering negative mass of the hollow part and superposing the results with the solid shaft. Since, it's one of the standard shapes, its torsional rigidity J is known and hence the torque can be calculated. Therefore, tha app is able to give the value of torque for this case.

For the rectangular cross-section:

$$F(x, y) = (x - b)(x + b)(y - h)(y + h) = x^2y^2 - h^2x^2 - b^2y^2 + h^2b^2$$



Cross-section

Therefore,

$$\phi_p = BF(x, y) = B(x^2y^2 - h^2x^2 - b^2y^2 - h^2b^2)$$

Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2\phi_p}{\partial y^2} + \frac{\partial^2\phi_p}{\partial x^2} = -2G\theta$$

We get,

$$2B(x^2 - b^2) + 2B(y^2 - h^2) = -2G\theta$$

Hence,

$$B = \frac{-G\theta}{x^2 + y^2 - h^2 - b^2}$$

Since B that is expected to be a constant has a dependance on both x and y coordinate, the Laplacian cannot be solved in this case like it was solved in the above cases.

Remarks: The term x^2y^2 is producing the terms x^2 and y^2 after the Laplacian acts on it and thus it creates a dependance on which the constant B has to rely.

Note that to remove the dependance B has on x and y, the term $x^2 + y^2$ has to be constant, but that would mean that the points on the cross section- lie on a circle, which contradicts the fact that the cross-section we chose is rectangular.

Hence, a rectangular cross-section is **a clear fail case for the app and the method of solving as well**.

The app identifies any function that does not satisfy the Laplacian (the Poisson equation) for the Prandtl stress function as a “wrong input” and hence the user gets to know that the input function is not a valid Prandtl stress function.

Note that this does not mean that the solution does not exist for the Laplacian, or the problem in general.

The Laplacian is solved by taking a particular solution superposed with a homogenous solution. The homogenous part is solved by separation of variables method. Thus, solution for the homogenous part will give a generalised solution to the Laplacian. It is seen that the homogenous part eventually becomes a problem of Fourier series and thus has to be solved with Fourier analysis, making the homogenous part a series of sinusoidal functions. This make the computation of the problem of rectangular shafts very complex and beyond the scope of the fundamental study on “Torsion in Arbitrary Shafts". A glimpse of the mathematical approach in the generalised solution is shown:

$$V(x, y) = f(x)g(y)\ (assumed)$$

$$where\ V = \{0\ for\ x = \pm b\ and\ G\theta(h^2 = x^2)\ for\ y = \pm h\}$$

$$\phi_p = G\theta(h^2 - x^2) + V(x, y)$$

Then the Laplacian for V gives:

$$g''f + f''g = 0 \implies \frac{f''}{f} = -\frac{g''}{g} = \lambda^2 \implies \begin{aligned} f(x) &= Acos(\lambda x) + Bsin(\lambda x) \\ g(y) &= Ccos(\lambda y) + Dsin(\lambda y) \end{aligned}$$

Ultimately, $B = D = 0\ and\ \lambda = \frac{n\pi b}{2h}\ (n = 1,2,3...)$ is what we get after the condition that V must be and even function of the coordinates.

$$\implies V = \sum_{i=1}^{n} A_n cos\left(\frac{n\pi x}{2h}\right) cos\left(\frac{n\pi y}{2h}\right)$$

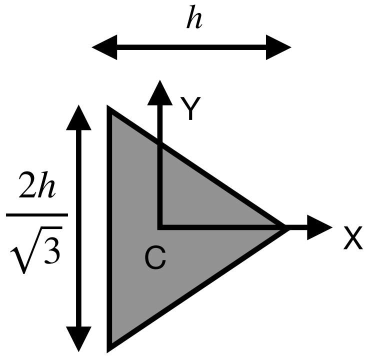
(The constant A_n further needed to be found by boundary equations which will involve evaluating Fourier series and beyond the fundamental problems that are considered)

For the given cross-section:

$$F(x,y) \ = \ (x+\frac{h}{3})(x-\sqrt{3}y-\frac{2h}{3})(x+\sqrt{3}y-\frac{2h}{3})$$
$$\implies F(x,y) = x^3 - hx^2 - 3xy^2 - hy^2 + 4h^3/27$$

Therefore,

$$\phi_p \ = \ BF(x,y) \ = \ B(x^3 - hx^2 - 3xy^2 - hy^2 + 4h^3/27)$$



Cross-section
Note: Axes with reference to centroid (C)

Solving the Laplacian for above Prandtl stress function,

$$\frac{\partial^2\phi_p}{\partial y^2} + \frac{\partial^2\phi_p}{\partial x^2} \ = -2G\theta$$

We get,

$$B(6x-2h-6x-2h) = -2G\theta$$

Hence,

$$B = \frac{G\theta}{2h}$$

Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{-\partial\phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

$$\frac{\partial\phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

Note: $J = \frac{h^3}{15\sqrt{3}}$ for the given cross-section
$J \equiv Torsional \ Rigidity$

Owing to the above equations we get:

$$\sigma_{zy} \ -\frac{G\theta}{2h}(3x^2-3y^2-2hx) \Rightarrow \gamma_{zy} \ = -\frac{\theta}{2h}(3x^2-3y^2-2hx)$$
$$\sigma_{zx} \ = -\frac{G\theta}{2h}(3x+h) \Rightarrow \gamma_{zx} \ = -\frac{\theta y}{2h}(3x+h)$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{zy}}{\partial x} \ = -2\theta \ \Rightarrow \frac{-3\theta x}{h} - \theta - \left(\frac{-3\theta x}{h}+\theta\right) = -2\theta \qquad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} \ = \ 0 \ \Rightarrow \frac{3G\theta y}{h} - \frac{3G\theta y}{h} = 0 \qquad \text{(Satisfied!)}$$

The equations are trivially satisfied and no other constraint exists. Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\sigma_{zy} \ = \ \frac{2b^2G\theta y}{h^2+b^2} \Rightarrow \gamma_{zy} \ = \ \frac{2b^2\theta y}{h^2+b^2}$$

$$\phi_p \ = -\frac{h^2b^2(G\theta)}{h^2+b^2}(\frac{x^2}{h^2}+\frac{y^2}{b^2}-1)$$

$$\sigma_{zx} \ = \ \frac{-2h^2G\theta y}{h^2+b^2} \Rightarrow \gamma_{zx} \ = \ \frac{-2h^2\theta y}{h^2+b^2}$$

Remarks: The above problem has been solved by taking the centroid of the origin as the origin and the function $F(x,y)$ has been formulated by multiplying the equations of the line that make the boundary of the triangular cross section. The vertices are points of discontinuity since there are two lines that pass through a vertex (two tangents at one point!). However, the domain of the resulting Prandtl function from these three is taken to be continuous over the entire plane of cross-section.

Test Case 5: Triangular (Right-Angled) Cross-Section FAIL!

For the given cross-section:

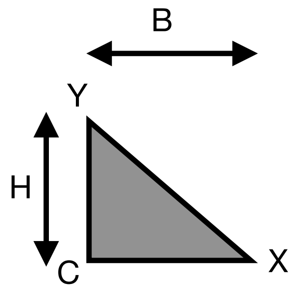
$$F(x,y) = (x)(y)(By + Hx - HB)$$

$$\Rightarrow F(x,y) = Bxy^2 + Hx^2y - HBxy$$

Therefore,

$$\phi_p = B'F(x,y) = B'(Bxy^2 + Hx^2y - HBxy)$$

$$Note\ that\ B' \neq B!\ B \equiv base,\ B' \equiv constant$$



Cross-Section

Note: The axes are taken along the sides making the right angle.

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get,

$$2B'(Bx + Hy) = -2G\theta$$

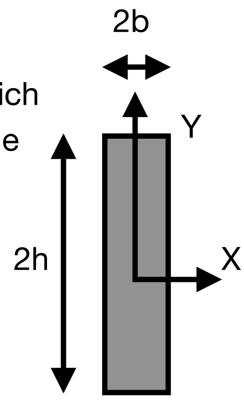
Hence,

$$B' = \frac{-G\theta}{Bx + Hy}$$

Here also, B' which should have been a constant depends on the coordinates of the cross-section and hence, this case is **also a fail case in theory as well as in the app**.

Remarks: The right angled triangle has a rather simpler formulation than the equilateral triangle due to two of its sides being perpendicular and providing us an easy choice for the reference axes. Even though simpler in formulation, the Prandtl stress function could not be defined for the given cross-section again due to the dependency of the constant B' that is multiplied to $F(x,y)$ and this leads to another discussion similar to that in the rectangular cross-sectional case that was analysed earlier, where there might be a solution possible with a higher-level mathematics involved. However, given the method we're generalising in the app and worked out in this manual, the above case is a definite fail case for the app and the method of solving.

Test Case 6: Narrow Rectangular Cross-Section $(h >> b)$ PASS!



For the narrow rectangular cross-section:
We incorporate the Prandtl elastic-membrane analogy (analogue to a soap-film) which assumes that the lateral displacement $z(x, y)$ is mathematically equivalent to the Prandtl stress function $\phi_p(x, y)$ the relations for the various variables are substituted to get the equations in terms of the lateral displacement and then back-substitute it to get a valid Prandtl stress function.
We will not work the algebra out, since it requires more theoretical knowledge about the membrane analogy to get a better understanding.
However, after back-substituting we get the Prandtl-stress function as:

$$\phi_p = G\theta(b^2 - x^2)$$

Now remember that in case of the rectangular cross-section problem, we had:

$$\phi_p = B'(x^2y^2 - h^2x^2 - b^2y^2 + h^2b^2)$$

Thus, on comparing the two functions, we can say that we have removed the $'y'$ - dependence on from the earlier case and thus the Prandtl function in case of the narrow rectangular cross-section does not have any $'y'$ term present in its form.
Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{-\partial\phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

$$\frac{\partial\phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

Owing to the above equations we get:

$$\sigma_{zy} \ = \ 2G\theta x \Rightarrow \gamma_{zy} \ = \ 2\theta x$$

$$\sigma_{zx} \ = \ 0 \Rightarrow \gamma_{zx} \ = \ 0 \ (No\ shear\ stresses\ or\ strains\ in\ the\ x\ direction\ on\ z\ plane!)$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):
Compatibility Equation:

$$\frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{zy}}{\partial x} = -2\theta \Rightarrow 0 - 2\theta = -2\theta$$ (Satisfied!)

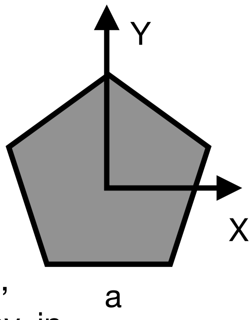
Equilibrium Equation:

$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} = 0 \Rightarrow 0 - 0 = 0$$ (Satisfied!)

Remarks: Therefore in these cases, there is no shear stress in the direction of the dimension that is much larger than the other. This was also obvious from the Prandtl function itself which only depends on the shorter dimension and thus the other dimension has no contribution in the stress and the strain fields. Also note that in this case the torsion is computable. Hence,

$$T = 2\int_{-h}^{h}\int_{-b}^{b}\phi_p dx dy = \frac{1}{3}G\theta(2h)(2b)^3 = GJ\theta \implies J = \frac{1}{3}(2h)(2b)^3$$

Discussion/Remarks:



In case of a regular polygon (pentagon taken just for reference), the app will definitely be unable to find the required outputs. However, even if we analyse it theoretically just like the rest of the cases, the five sides will produces five different equations and hence there will be a lot of terms to with maximum degree being five, for a pentagon. Now, this is not only complex but also tedious and very lengthy in calculation. Also, since it is a regular polygon and all sides have the same dimension, it is very probable that just like the rectangular cross-section case, this will lead to a solution that will require higher mathematics to be solved or maybe unsolvable depending on the function it yields. Also, while calculating the torsion of such a complex problem, there will be a double integral will limits required for both the spatial coordinates. Since there are 5 different lines enclosing the figure, a universal relation for the limit won't exist. One would have to break the entire geometry into parts that can be analysed with relatively simpler and calculable limits.

Hence, **all polygonal cross-sections with more than four sides are fail cases for the app due to mathematical limitations in this complex problem.**

A) Second-Order Polynomials: PASS!

$$\phi_p = ax^2 + by^2 + cxy + d$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial^2 y} + \frac{\partial^2 \phi_p}{\partial^2 x} = -2G\theta$$

We get the equation $a + b = -G\theta$ putting a constraint on the coefficients a and b. The constants c and d have no constraints on them (as for now).

Now, we find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to these equations we get:

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

$$\sigma_{zx} = 2by + cx \Rightarrow \gamma_{zx} = \frac{2by + cx}{G}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow \frac{2b}{G} - \frac{(-2a)}{G} = \frac{2(a+b)}{G} = \frac{2(-G\theta)}{G} = -2\theta \quad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow c - c = 0 \quad \text{(Satisfied!)}$$

The equations are satisfied on their own. Thus, the Prandtl stress function the shear stresses and shear strains take the form:

$$\phi_p = ax^2 - (a + G\theta)y^2 + cxy + d$$

$$\sigma_{zx} = -2(a + G\theta)y + cx \Rightarrow \gamma_{zx} = \frac{-2(a + G\theta)y + cx}{G}$$

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

Here, the coefficients a, c and d are arbitrary and can take any real value. Only one relation was established and therefore only one variable has been reduced. The rest all variables are free to take any values without affecting the compatibility or the equilibrium.

Remarks: Note that standard cases such as circle and ellipse will be sub-cases of the above set of polynomials on choosing the appropriate values of the coefficients a, c and d. However, also note that some of the valid functions of this form might not have any physical meaning in the form of a boundary equation.

B) Third-Order Polynomials:

FAIL!

$$\phi_p = ax^3 + bx^2y + cxy^2 + dy^3 + e$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get the equation: $(6a + 2c)x + (2b + 6d)y = -2G\theta$

Now, in order to be independent of the coordinates, the coefficients have to follow the relations: $c = -3a$ & $b = -3d$.

However, note that in doing so the left-hand side of the equation goes to zero and hence the equation besoms redundant.

Thus, no set of coefficients will ever be able to satisfy the Laplacian. Hence,

All homogenous third-order polynomials will be 'Fail Cases' both in theory as well as in the app.

Remarks: Notice that the x and y dependencies were removed by the above mentioned relations between the coefficients. The only term needed was a constant that can be set equal to the term on the right-hand side. A term that comes out as a constant from a Laplacian can only be of the form ax^2 or by^2 where a and b are arbitrary constants. Thus, only a second-order term only in x or in y can satisfy the right-hand side of the equation. However, since we are talking about homogenous polynomials, a third-order polynomial will not have any second-order terms and hence the case fails. Now, this logic can be extended to all the polynomials with order greater than two. Let us follow this discussion up with the generalised case below.

C) Nth-Order Polynomials:

FAIL!

$$\phi_p = \sum_{n=0}^{N} a_n y^n x^{N-n} = a_0 x^N + a_1 x^{N-1} y + a_2 x^{N-2} y^2 + \dots$$

Here again, the order is greater than two and being homogenous, the polynomial will not contain any term that is of order other than N. Therefore, the Laplacian in this case will yield the terms that will have at least one power of either x or y or both but no constant term that can help satisfy the governing equation.

Remarks: Note that we did not take first-order homogenous polynomials since they will simply yield a redundant equation when it comes to the Laplacian equation. Since the Laplacian has partial derivatives of order two, the first-order terms will just vanish on the left-hand side leaving unequal left and right-and sides. Therefore, from the cases A, B and C that we discussed under homogenous polynomial, we can say that only second-order homogenous polynomials will have the potential to satisfy the Laplacian or the Poisson equation for the Prandtl stress function and create a situation where there is a possibility of a valid solution emerging from the governing equations.

A) Second-Degree Polynomials: PASS!

$$\phi_p = ax^2 + by^2 + cxy + dx + ey + f$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get the equation $a + b = -G\theta$ (the same as in homogenous case!) putting a constraint on the coefficients a and b. The constants c, d, e, f have no constraints as of now.

Now, we find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to these equations we get:

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

$$\sigma_{zx} = 2by + cx \Rightarrow \gamma_{zx} = \frac{2by + cx}{G}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow \frac{2b}{G} - \frac{(-2a)}{G} = \frac{2(a+b)}{G} = \frac{2(-G\theta)}{G} = -2\theta \quad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow c - c = 0 \quad \text{(Satisfied!)}$$

The equations are trivially satisfied on their own. Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p = ax^2 - (a + G\theta)y^2 + cxy + dx + ey + f$$

$$\sigma_{zx} = -2(a + G\theta)y + cx \Rightarrow \gamma_{zx} = \frac{-2(a + G\theta)y + cx}{G}$$

$$\sigma_{zy} = -2ax - cy \Rightarrow \gamma_{zy} = \frac{-2ax - cy}{G}$$

Remarks: Notice that the addition of the first-order terms have made no difference to the field descriptions. However, they surely have increased the number of possible functions that can satisfy the Poisson equation by making the Prandtl stress function non-homogenous and therefore, even though they don't change the outputs, they can surely change the input in a way that it also produces a physically possible case of a boundary equation along with satisfying the Laplacian.

B) Third-Degree Polynomials:

PASS!

$$\phi_p = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

Solving the Laplacian for above Prandtl stress function:

$$\frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial x^2} = -2G\theta$$

We get:

$$6dy + 2cx + 2g + 6ax + 2by + 2e = -2G\theta$$

In order to satisfy this equation for all x and all y, following relations must be established:

$$b = -3d \mid c = -3a \mid e + g = -G\theta$$

Now, we find the shear stresses and the corresponding shear strains:

$$\frac{\partial \phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

$$\frac{-\partial \phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

Owing to these equations and eliminating c, d, g using the relations obtained above, we get:

$$\sigma_{zx} = b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i \Rightarrow \gamma_{zx} = \frac{b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i}{G}$$

$$\sigma_{zy} = -(3a(x^2 - y^2) + 2bxy + 2ex + fy + h) \Rightarrow \gamma_{zy} = \frac{-(3a(x^2 - y^2) + 2bxy + 2ex + fy + h)}{G}$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):

Compatibility Equation:

$$\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{zy}}{\partial x} = -2\theta \Rightarrow \frac{-2by}{G} + \frac{(-6ax)}{G} + \frac{-2(e + G\theta)}{G} - \left\{ \frac{-6ax}{G} + \frac{-2by}{G} + \frac{-2e}{G} \right\} = -2\theta \quad \text{(Satisfied!)}$$

Equilibrium Equation:

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0 \Rightarrow 2bx - 6ay + f + (6ay - 2bx - f) = 0 \quad \text{(Satisfied!)}$$

The equations are trivially satisfied on their own. Thus, the Prandtl stress function, the shear stresses and shear strains take the form:

$$\phi_p = ax^3 + bx^2y - 3axy^2 - (b/3)y^3 + ex^2 + fxy - (e + G\theta)y^2 + hx + iy + j$$

$$\sigma_{zx} = b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i \Rightarrow \gamma_{zx} = \frac{b(x^2 - y^2) - 6axy + fx - 2(e + G\theta)y + i}{G}$$

$$\sigma_{zy} = -(3a(x^2 - y^2) + 2bxy + 2ex + fy + h) \Rightarrow \gamma_{zy} = \frac{-(3a(x^2 - y^2) + 2bxy + 2ex + fy + h)}{G}$$

Remarks: Again, just like the first order terms, terms of degree three remain in the Prandtl stress function with the coefficients obeying some particular relations. Again, this opens a lot of other functions that might be analysed. Similar arguments can be continued for polynomials of higher degree; the only difference is that the number of relations between the coefficients will increase and the algebra will become a bit clumsy and messy. However, the fact that the second order polynomials are required for the function to prove to be a valid Prandtl function does not change. **On the other hand, if any function does not adhere to the relations of the coefficients determined above, it will result into a fail case.**

IMPORTANT ASSUMPTIONS, NOTABLE DISCUSSIONS AND A SUMMARY OF THE LIMITATIONS

DISCLAIMER: The overall focus of the app is on the field descriptions and the warping function. In this fail case manual, there is a generalised theoretical approach to mainly two class of problems: one is a set of standard or commonly known shapes, and the other is a functional approach towards guessing the possible sets of Prandtl stress functions. The algebra has been explicitly worked out in the problems till the point of obtaining the stress and the strain field descriptions, simply because in the process of integration of the strain descriptions to get the displacements produce a lot of unnecessary constants in the overall generalised expression. Since we have taken generalised expressions for almost all the cases and have also kept all other parameters (more or less) the same to generalise the approach and the process, the algebra has been worked out till the field descriptions only. However, since the app code has been designed once and for all, the app has the ability to calculate the displacement fields for any arbitrary case and also display surface plots on the app interface.

Following are a set of points one needs to keep in mind for getting a deeper and a finer sense of the topic and an insight into the working of the problems:

1. The first and foremost assumption (a note) that one must consider is: The shaft member is made of an isotropic material that is in the linear elastic domain (range). Hence, the behaviour of the material depends only on two properties viz. Young's Modulus E and Poisson's Ratio ν. These might not be obvious while giving the inputs since one uses the Modulus of Rigidity G for the class of problems under torsion in arbitrary shafts. However, the relation $G = E/2(1 + \nu)$ shows that both E and ν are indeed involved in the calculations. That also means that the topics of failure criteria, yielding and plastic torsion are not to be discussed in this context.
2. Another important nuance about the functioning of the formulation of the problems is that St. Venant's principle is taken into account. That is, the cross-section that we're analysing is not a cut-section near the ends; it is a cut-section far enough from the ends to make the loading uniform and to ignore the end effects.
3. The app will not be able to distinguish between a function that represents a closed boundary i.e. a closed cross-section of the shaft and any other function (be it of polynomial form, sinusoidal or exponential!). This means, as far as the Laplacian is satisfied, the app will continue to assume that it represents an actual closed boundary and will produce results corresponding to that form of the function. Thus, there will be no physical meaning of the outputs shown in such a condition. However, for an invalid function (that does not satisfy the Laplacian), the app does give out a message saying "wrong input" in a check box.
4. The app takes the inputs and gives the outputs in SI units (e.g. the modulus of rigidity is to be filled in Giga-Pascals and the displacements calculated are in meters). However, in many questions there might be other units (which are more common in real life situations) that are used. Hence the app doesn't have the flexibility of unit conversion while solving the problem. The user will then have to manually convert the results (or the inputs) into SI units accordingly.
5. Since solving for a single arbitrary shaft is a complex problem in itself, the app has limitations regarding composite shafts with different arbitrary cross-sections (e.g. two slender shafts of different cross-sections joined to form a composite shaft).

6. The app is also made for solving problems on shafts of arbitrary cross-sections made of a single material and no variation in terms of material composition is accounted for, since that would make the problem extremely difficult to solve (even manually!)
7. For the fail case of a rectangular shaft, the Laplacian can be solved by breaking the equation into a particular and a general solution and the method ends up with a series solution to the Poisson equation. This is not shown explicitly as it is beyond the scope of the fundamental cases in torsion in arbitrary shafts. Instead, one can take narrow rectangular cross-sections as one of the cases that is seen in many of the engineering applications which is a passed case and hence can be exploited.
8. Problems on hollow elliptical (or circular) shafts can not entirely be solved by the app in one go. For a hollow member, the outputs will be given for the solid and the hollow sections, but the computation of the entire shaft needs to be done by the user manually (by subtracting the values associated with the hollow section from those associated with the solid section) to get the final values.
9. The app has a limitation on the calculation of torque on the shaft from the Prandtl function. Since the torque is related to the Prandtl function by the relation $T = 2\int \phi_p dA = 2\int \phi_p dx dy$ it has to be integrated twice and that becomes difficult when the function becomes implicit. Implicit functions will have a variation of their integration limits dependent on the coordinates. An examples would be $x : 0 \rightarrow 1$, $y : 0 \rightarrow 1 - x$
   In such scenarios, the integration limits need to be worked out for the specific problem at hand. That would require the app to recognise every possible scenario and the generalisation of the problem will become nearly an impossible task. Therefore, in general, for the arbitrary cross-sections, the torque is not considered in the parameters that are to be found.
10. For the equilateral triangle cross-section, although it is a textbook case, where even the torsional rigidity is also known in terms of the material properties and the dimensions of the cross-section, the app does not include it under the “Standard Shapes” tab since the Prandtl function will depend on the choice of the axes that the user defines i.e. the orientation of the triangle w.r.t the user-defined axes. The origin in the theoretical case has been taken at the centroid of the triangle. However it, may also be taken as one of the vertices or at any point along the edges of the triangle by the user and thus becomes very specific and adds complexity to the problem (such as the angle between the user’s axis and the symmetry axis of the triangle and the origin in both the reference frames). Therefore, the function corresponding to the axes chosen should be put in as an arbitrary function in the app.
11. One of the key assumptions while solving engineering problems and also from a design perspective is the assumption of a slender member - where the length of the member is *at least* 5 times greater than the other two cross-sectional dimensions (e.g. $\frac{L}{D} > 5$ where $D = 2R$ for a circle of radius R and length L and $\frac{L}{max\{2a,2b\}}$ for an ellipse of semi-major axis a, semi-minor axes b and length L). Since the length of the problem does not get involved into the process of calculating the field descriptions, the verification of the shaft member being a slender member is not possible with the given input options in the app. As a result, the app may give results for those cross-sections with dimensions that are comparable in magnitude with the length of the shaft member.