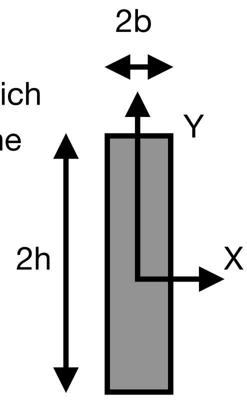
Test Case 6: Narrow Rectangular Cross-Section $(h >> b)$ PASS!



For the narrow rectangular cross-section:
We incorporate the Prandtl elastic-membrane analogy (analogue to a soap-film) which assumes that the lateral displacement $z(x, y)$ is mathematically equivalent to the Prandtl stress function $\phi_p(x, y)$ the relations for the various variables are substituted to get the equations in terms of the lateral displacement and then back-substitute it to get a valid Prandtl stress function.
We will not work the algebra out, since it requires more theoretical knowledge about the membrane analogy to get a better understanding.
However, after back-substituting we get the Prandtl-stress function as:

$$\phi_p = G\theta(b^2 - x^2)$$

Now remember that in case of the rectangular cross-section problem, we had:

$$\phi_p = B'(x^2y^2 - h^2x^2 - b^2y^2 + h^2b^2)$$

Thus, on comparing the two functions, we can say that we have removed the $'y'$ - dependence on from the earlier case and thus the Prandtl function in case of the narrow rectangular cross-section does not have any $'y'$ term present in its form.
Now, let's find the shear stresses and the corresponding shear strains:

$$\frac{-\partial\phi_p}{\partial x} = \sigma_{zy} \Rightarrow \gamma_{zy} = \frac{\sigma_{zy}}{G}$$

$$\frac{\partial\phi_p}{\partial y} = \sigma_{zx} \Rightarrow \gamma_{zx} = \frac{\sigma_{zx}}{G}$$

Owing to the above equations we get:

$$\sigma_{zy} = 2G\theta x \Rightarrow \gamma_{zy} = 2\theta x$$

$$\sigma_{zx} = 0 \Rightarrow \gamma_{zx} = 0 \; (No\ shear\ stresses\ or\ strains\ in\ the\ x\ direction\ on\ z\ plane!)$$

Now, we need to satisfy the compatibility equations for the strains and the equilibrium equations for the stresses (assuming no body forces are acting on the body):
Compatibility Equation:

$$\frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{zy}}{\partial x} = -2\theta \Rightarrow 0 - 2\theta = -2\theta$$ (Satisfied!)

Equilibrium Equation:

$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} = 0 \Rightarrow 0 - 0 = 0$$ (Satisfied!)

Remarks: Therefore in these cases, there is no shear stress in the direction of the dimension that is much larger than the other. This was also obvious from the Prandtl function itself which only depends on the shorter dimension and thus the other dimension has no contribution in the stress and the strain fields. Also note that in this case the torsion is computable. Hence,

$$T = 2\int_{-h}^{h}\int_{-b}^{b} \phi_p dx dy = \frac{1}{3}G\theta(2h)(2b)^3 = GJ\theta \implies J = \frac{1}{3}(2h)(2b)^3$$