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Prony's method

Prony analysis (**Prony's method**) was developed by Gaspard Riche de Prony in 1795. However, practical use of the method awaited the digital computer.[1] Similar to the Fourier transform, Prony's method extracts valuable information from a uniformly sampled signal and builds a series of damped complex exponentials or sinusoids. This allows for the estimation of frequency, amplitude, phase and damping components of a signal.

Contents

The method

Let $f(t)$ be a signal consisting of N evenly spaced samples. Prony's method fits a function

$$\hat{f}(t) = \sum_{i=1}^{M} A_i e^{\sigma_i t} \cos(2\pi f_i t + \phi_i)$$

to the observed $f(t)$. After some manipulation utilizing Euler's formula, the following result is obtained. This allows more direct computation of terms.

$$\begin{aligned}\hat{f}(t) &= \sum_{i=1}^{M} A_i e^{\sigma_i t} \cos(2\pi f_i t + \phi_i) \\ &= \sum_{i=1}^{M} \frac{1}{2} A_i e^{\pm j\phi_i} e^{\lambda_i t}\end{aligned}$$

where:

- $\lambda_i = \sigma_i \pm j\omega_i$ are the eigenvalues of the system,
- $\sigma_i = -\omega_{0,i}\xi_i$ are the damping components,
- $\omega_i = \omega_{0,i}\sqrt{1-\xi_i^2}$ are the angular frequency components
- ϕ_i are the phase components,
- $f_i = \frac{\omega_i}{2\pi}$ are the frequency components,
- A_i are the amplitude components of the series, and
- j is the imaginary unit ($j^2 = -1$).

Representations

Prony's method is essentially a decomposition of a signal with M complex exponentials via the following process:

Regularly sample $\hat{f}(t)$ so that the n-th of N samples may be written as

$$F_n = \hat{f}(\Delta_t n) = \sum_{m=1}^{M} \mathrm{B}_m e^{\lambda_m \Delta_t n}, \quad n = 0, \dots, N-1.$$

If $\hat{f}(t)$ happens to consist of damped sinusoids, then there will be pairs of complex exponentials such that

$$\begin{aligned}
\mathrm{B}_a &= \frac{1}{2} A_i e^{\phi_i j}, \\
\mathrm{B}_b &= \frac{1}{2} A_i e^{-\phi_i j}, \\
\lambda_a &= \sigma_i + j\omega_i, \\
\lambda_b &= \sigma_i - j\omega_i,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{B}_a e^{\lambda_a t} + \mathrm{B}_b e^{\lambda_b t} &= \frac{1}{2} A_i e^{\phi_i j} e^{(\sigma_i + j\omega_i)t} + \frac{1}{2} A_i e^{-\phi_i j} e^{(\sigma_i - j\omega_i)t} \\
&= A_i e^{\sigma_i t} \cos(\omega_i t + \phi_i).
\end{aligned}$$

Because the summation of complex exponentials is the homogeneous solution to a linear difference equation, the following difference equation will exist:

$$\hat{f}(\Delta_t n) = -\sum_{m=1}^{M} \hat{f}[\Delta_t (n-m)] P_m, \quad n = M, \dots, N-1.$$

The key to Prony's Method is that the coefficients in the difference equation are related to the following polynomial:

$$z^M - P_1 z^{M-1} - \dots - P_M = \prod_{m=1}^{M} \left(z - e^{\lambda_m}\right).$$

These facts lead to the following three steps to Prony's Method:

1) Construct and solve the matrix equation for the P_m values:

$$\begin{bmatrix} -F_M \\ \vdots \\ -F_{N-1} \end{bmatrix} = \begin{bmatrix} F_{M-1} & \dots & F_0 \\ \vdots & \ddots & \vdots \\ F_{N-2} & \dots & F_{N-M-1} \end{bmatrix} \begin{bmatrix} P_1 \\ \vdots \\ P_M \end{bmatrix}.$$

Note that if $N \neq 2M$, a generalized matrix inverse may be needed to find the values P_m.

2) After finding the P_m values find the roots (numerically if necessary) of the polynomial

$$z^M - P_1 z^{M-1} - \dots - P_M,$$

The m-th root of this polynomial will be equal to e^{λ_m}.

3) With the e^{λ_m} values the F_n values are part of a system of linear equations that may be used to solve for the B_m values:

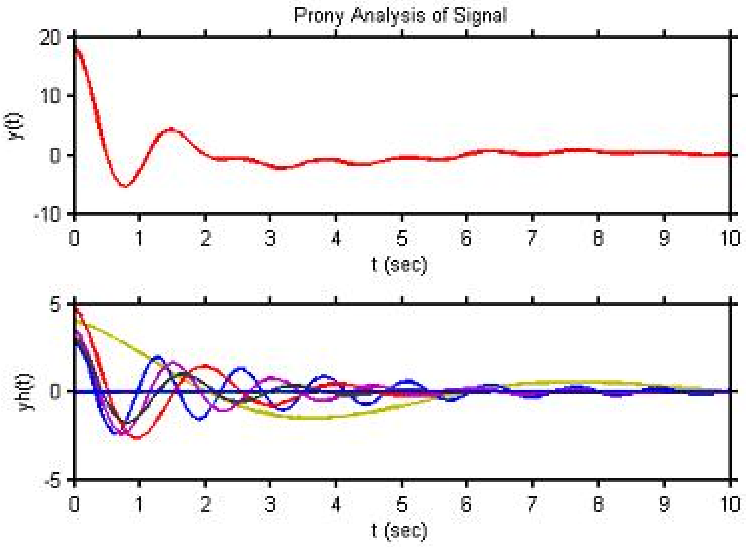
$$\begin{bmatrix} F_{k_1} \\ \vdots \\ F_{k_M} \end{bmatrix} = \begin{bmatrix} (e^{\lambda_1})^{k_1} & \dots & (e^{\lambda_M})^{k_1} \\ \vdots & \ddots & \vdots \\ (e^{\lambda_1})^{k_M} & \dots & (e^{\lambda_M})^{k_M} \end{bmatrix} \begin{bmatrix} \mathrm{B}_1 \\ \vdots \\ \mathrm{B}_M \end{bmatrix},$$

where M unique values k_i are used. It is possible to use a generalized matrix inverse if more than M samples are used.

Note that solving for λ_m will yield ambiguities, since only e^{λ_m} was solved for, and $e^{\lambda_m} = e^{\lambda_m + q2\pi j}$ for an integer q. This leads to the same Nyquist sampling criteria that discrete Fourier transforms are subject to:

$$|\mathrm{Im}(\lambda_m)| = |\omega_m| < \frac{\pi}{\Delta_t}.$$

Example



Notes

1. Hauer, J.F.; Demeure, C.J.; Scharf, L.L. (1990). "Initial results in Prony analysis of power system response signals". *IEEE Transactions on Power Systems*. **5**: 80–89. doi:10.1109/59.49090 (https://doi.org/10.1109%2F59.49090).

References

- Carriere, R.; Moses, R.L. (1992). "High resolution radar target modeling using a modified Prony estimator". *IEEE Transactions on Antennas and Propagation*. **40**: 13–18. doi:10.1109/8.123348 (https://doi.org/10.1109%2F8.123348).