TUTORIAL - 1

① If $\phi = x^{3/2} + y^{3/2} + z^{3/2}$, find $\vec{\nabla}\phi$

② Given $\vec{A} = x^2 y \hat{i} + (x-y)\hat{k}$. Find $\vec{\nabla}\cdot\vec{A}$ and $\vec{\nabla}\times\vec{A}$

③ The vector $\vec{A} = (x+3y)\hat{i} + (2y+3z)\hat{j} + (x+az)\hat{k}$ is a solenoidal vector. Find the value of a [-3]

④ A vector field is given by $\vec{A} = \hat{i}y^2 + 2xy\hat{j} - z^2\hat{k}$. Determine divergence and curl of $\vec{A}$ at point (1, 2, 1). Also determine the vector field $\vec{A}$ is solenoidal or irrotational.

⑤ Find constants a, b, c so that vector $\vec{A}$ is irrotational.
where $\vec{A} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + y + 2z)\hat{k}$.

⑥ For position vector $\vec{r}$ show that $\vec{\nabla} r^n = n\, r^{n-1}\vec{r}$.

⑦ Find $\vec{\nabla}\phi$, where $\phi = \frac{1}{r}$, where $\vec{r}$ is position vector.

Solution Tutorial 1

(1) $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^{3/2} + y^{3/2} + z^{3/2}\right)$

$= \frac{3}{2}\left(\hat{i}\,x^{1/2} + \hat{j}\,y^{1/2} + \hat{k}\,z^{1/2}\right)$

(2) $\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(x^2y\,\hat{i} + (x-y)\hat{k}\right)$

$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial z}(x-y) = 2xy.$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ x^2y & 0 & x-y \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(0)\right) + \hat{j}\left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(x-y)\right] + \hat{k}\left(0 - \frac{\partial}{\partial y}(x^2y)\right)$$

$= i(-1) + \hat{j}(-1) + k(-x^2)$

$\vec{\nabla}\times\vec{A} = -\hat{i} - \hat{j} - \hat{k}x^2.$

(3) $\vec{\nabla}\cdot\vec{A} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left((x+3y)\hat{i} + (2y+3z)\hat{j} + (x+az)\hat{k}\right)$

$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(2y+3z) + \frac{\partial}{\partial z}(x+az)$

$= 1 + 2 + a = 3 + a.$

$\vec{\nabla}\cdot\vec{A} = 0$ (for solenoidal vector $\vec{A}$)

$\Rightarrow 3 + a = 0 \Rightarrow \boxed{a = -3}$

(4) $\vec{\nabla}\cdot\vec{A} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}y^2 + 2xy\,\hat{j} - z^2\hat{k}\right)$

$= \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial z}(2xy) + \frac{\partial}{\partial z}(-z^2\hat{k})$

$= -2z.$

At $(1,2,1)$, $\vec{\nabla}\cdot\vec{A} = -2\times 1 = -2$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} i & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & -z^2 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(2xy)\right) + \hat{j}\left(\frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(-z^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(y^2)\right)$$

$= \hat{k}(2y - 2y) = 0$

$\Rightarrow \vec{A}$ vector is irrotational.

Also $\vec{A}$ is not solenoidal as $(\vec{\nabla}\cdot\vec{A} \neq 0)$

⑥ $\vec{\nabla} r^n = n r^{n-1} \vec{r}$

Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$|r| = (x^2 + y^2 + z^2)^{1/2}$

$$\vec{\nabla} r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^2 + y^2 + z^2)^{n/2}$$

$$= \frac{n}{2}(\hat{i}2x + \hat{j}2y + \hat{k}2z)\left((x^2+y^2+z^2)^{1/2}\right)^{n-1}$$

$$= n(\hat{i}x + \hat{j}y + kz)\left((x^2+y^2+z^2)^{1/2}\right)^{n-1}$$

$$= n\,\vec{r}\,|r|^{n-1}$$

$$\Rightarrow \vec{\nabla} r^n = n\,|\vec{r}|^{n-1}\,\vec{r}.$$

⑦ $\phi = \frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$

$$\vec{\nabla}\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(\hat{i}2x + \hat{j}2y + k2z)(x^2+y^2+z^2)^{-3/2}$$

$$= -\vec{r}\,|r|^{-3}. = -\frac{\vec{r}}{r^3}.$$

⑤ $$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+y+2z) \end{vmatrix} = 0$$