CONTINUITY EQUATION

An equation which expresses the equality of incoming and outgoing charges in a system and follows the law of conservation of charge is known as equation of continuity.

The current density J and the charge density ρ are related at each point through a differential equation. This equation is based on the fact that electric charge can neither be created nor be destroyed and the rate of increase of the total charge inside any arbitrary volume must be equal to the net flow of charge into this volume.

The electric current through a closed surface S is

$$I = \oint \vec{J} \cdot d\vec{s}$$

$$= \int \vec{\nabla} \cdot \vec{J} \, dV \quad \text{(Using Gauss divergence theorem)} \quad \text{— (1)}$$

Again considering charge leaving a volume V per second = Rate of decrease of charge in volume V (= current I)

$$\Rightarrow I = -\frac{\partial q}{\partial t} = -\frac{\partial}{\partial t}\left(\int \rho \, dV\right) = -\int_V \frac{\partial \rho}{\partial t} dV \quad \text{— (2)}$$

From (1) & (2)

$$\int \vec{\nabla} \cdot \vec{J} \, dV = -\int \frac{\partial \rho}{\partial t} dV$$

$$\Rightarrow \int \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

(∵ The integral is zero for any arbitrary volume)

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0} \quad \text{— (3)}$$

This is equation of continuity.

In steady state i.e. for steady current, the charge density at any point within the region is constant.

i.e. ρ = constant $\Rightarrow \frac{\partial \rho}{\partial t} = 0$.

Eqn (3) now $\boxed{\vec{\nabla} \cdot \vec{J} = 0}$ — (4)

This is eqn of continuity for steady state.

AMPERE'S CIRCUITAL LAW

The line integral of the magnetic field B along any closed loop C is proportional to the current I passing through the closed loop. $[B = \mu H]$

$$\boxed{\oint B \cdot dl = \mu_0 I}$$

OR $\boxed{\oint H \cdot dl = I}$ — ①

Applying ~~Gauss~~ ~~divergence~~ Stokes theorem to L.H.S

$$\oint H \cdot dl = \int (\nabla \times H)\, ds = I. \quad — ②$$

Also $I = J \cdot ds$ — ③

Using ② & ③ in ①

$$\int (\nabla \times H)\, ds = \int J \cdot ds$$

$$\Rightarrow \boxed{\nabla \times H = J} \quad — ④$$

This equation is differential form of Ampere's Law.

Taking divergence of eqn ④ from both sides

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

$$0 = \vec{\nabla} \cdot \vec{J} \quad — ⑤$$

(∵ div of curl of any vector is always equal to zero)

This eqn (5) is satisfying equation of continuity for steady state only (which is $\nabla \cdot J = 0$)

This shows that Ampere's circuital law is valid only for steady states. When any circuit contains time varying electric field components, Ampere's law is not valid. Hence Maxwell modified Ampere's law and add one term for time varying electric fields.

The modified Ampere's law now

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d \quad \text{--- (6)}$$

Taking divergence from both sides

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J + \nabla \cdot J_d$$

$$\Rightarrow \quad 0 = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{J}_d \quad \text{--- (7)}$$

Comparing this equation of continuity, $\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$ --- (8)

we get

$$\nabla \cdot J_D = \frac{\partial \rho}{\partial t}$$

$$= \frac{\partial}{\partial t}(\nabla \cdot D) \quad \left(\text{Gauss law in differential form } \nabla \cdot D = \rho \right)$$

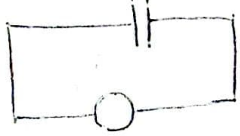
$$\Rightarrow \quad \nabla \cdot J_D = \nabla \cdot \left(\frac{\partial D}{\partial t}\right)$$

$$\Rightarrow \quad J_D = \frac{\partial D}{\partial t} \quad \text{--- (9)} \quad (\text{Displacement current Density } J_D)$$

Using (9) in (6)

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial D}{\partial t}} \quad \text{--- (10)}$$

This is modified Ampere's Law.



MAXWELL'S DISPLACEMENT CURRENT

Any circuit which contains a capacitor, the plates of the capacitor get charged with respect to time. During this charging process the plates of ~~the~~ capacitor gets charged, the electric field on the plates is changing with time.

Let at any instant the electric field on the plate is

$$\vec{E} = \frac{\sigma}{\epsilon_0} = \frac{q}{\epsilon_0 S}$$

where S = Area of plate, q = charge

σ = surface charge density $= \frac{q}{S}$.

$$\Rightarrow \epsilon_0 \vec{E} = \frac{q}{S} \Rightarrow \vec{D} = \frac{q}{S}$$

Differentiating both sides by time t

$$\frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\frac{q}{S}\right) = \frac{1}{S}\frac{\partial q}{\partial t} = \frac{1}{S} I_D$$

$$\Rightarrow \boxed{I_D = S\frac{\partial D}{\partial t}}$$ I_D = Displacement current

$$\frac{I_D}{S} = \frac{\partial D}{\partial t} \Rightarrow \boxed{J_D = \frac{\partial D}{\partial t}}$$ J_D = Displacement current density.

Maxwell postulated that it is not only the current in a conductor produces magnetic field, but a changing electric field in vaccum or in dielectric also produces magnetic field. This changing magnetic field produces an emf i.e electric field. This changing electric field produces a current known as displacement current.

1. Displacement current is a current which flows when time-varying electric field is present.
2. It is not linked with the motion of charges.
3. Magnitude of displacement current is equal to rate of change of electric displacement - vector.
4. Displacement current serves the purpose to make the total current continuous across the discontinuity in the conduction current in a circuit.

Total current $J = J_c + J_D$

Modified Amperes Law in Integral form

We know $I_D = S\frac{\partial D}{\partial t} = S\frac{\partial}{\partial t}(\epsilon_0 E) = \epsilon_0 \frac{\partial}{\partial t}(E \cdot S)$

$$= \epsilon_0 \frac{\partial \phi}{\partial t}.$$

$$\oint B \cdot dl = \mu_0 (I + I_D) = \mu_0\left(I + \epsilon_0 \frac{\partial \phi}{\partial t}\right)$$

OR $\oint H \cdot dl = \int J\, dS + \epsilon_0 \frac{\partial}{\partial t}\left(\int E \cdot dS\right) = \int J \cdot dS + \epsilon_0 \frac{\partial \phi}{\partial t}$

Distinction between displacement current and conduction current

Displacement Current	Conduction Current
① It is not because of motion of charge carriers. It is a current which exist in vaccum or any medium when a time varying electric field is present.	① Conduction current is because of the actual flow of charge carriers of the conducting medium.
② It depends on the electric permittivity of the medium and the rate at which the electric field changes with time.	② It obeys Ohms law and depends on the resistance and potential difference of the conductor.

FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Induced emf in a conducting loop is equal to the negative rate of change of magnetic flux through the surface enclosed by the loop.

Induced emf $e = -\frac{\partial \phi_B}{\partial t}$ —— ①

But we know that induced emf $= e = \oint E \cdot dl$. ②

If S is surface enclosed by the loop, magnetic flux through the surface area S is $\phi_B = \int \vec{B} \cdot d\vec{S}$. —— ③

Using ② & ③ in ①

$$\boxed{\oint \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{S}} \quad \text{—— ④}$$

This is integral form of Faradays law.

Applying Stoke's theorem to L.H.S. of eqn ④

$$\oint \vec{E} \cdot \vec{dl} = \int (\vec{\nabla} \times \vec{E}) \cdot ds = -\int \frac{\partial B}{\partial t} ds$$

$$\Rightarrow \int (\vec{\nabla} \times \vec{E}) \cdot ds = -\int \frac{\partial B}{\partial t} \cdot ds .$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}}$$

This is Differential form of Faradays Law.

GAUSS LAW FOR ELECTRIC FIELD

The total outward flux of an electric field vector over a closed surface is equal to $1/\epsilon_0$ times the total charge enclosed in a volume enclosed by the surface.

Let q = charge enclosed, ds = surface area element.

$$\boxed{\int E \cdot ds = \frac{q}{\epsilon_0}} \text{ —— (1)}$$

or

$$\int E \cdot ds = \frac{1}{\epsilon_0} \int \rho \, dV \text{ —— (2)}$$

where ρ = charge density (volume) $\left(= \frac{dq}{dV} \Rightarrow q = \int dq = \int \rho dV\right)$

Applying Gauss divergence theorem to L.H.S of (2)

$$\int (\nabla \cdot E) \, dV = \frac{1}{\epsilon_0} \int \rho \, dV$$

$$\Rightarrow \boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}} \text{ or } \boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$ Differential form of Gauss Law

where $\vec{D} = \epsilon_0 \vec{E}$ = Electric displacement vector.

GAUSS LAW FOR MAGNETIC FIELD

The magnetic field lines due to a current carrying conductor are closed curves without any beginning or end.

Since magnetic field lines are continuous the magnetic flux entering any region is equal to the flux leaving it. So the net flux over a volume is zero, hence magnetic field is solenoidal.

$$\boxed{\nabla \cdot B = 0}$$ Differential form of Gauss Law in magnetostatics

Let ϕ_B = Magnetic flux, then $\phi_B = \oint B \cdot ds$.

Applying divergence theorem

$$\oint B \cdot ds = \int (\nabla \cdot B) dV = 0$$

$$\boxed{\oint B \cdot ds = 0}$$ Integral form of Gauss law for magnetic field.

MAXWELL'S EQUATIONS →

① $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ or $\vec{\nabla}\cdot\vec{D} = \rho$ ① $\int E\cdot ds = \frac{1}{\epsilon_0}\int \rho\, dV$

② $\nabla\cdot B = 0$ ② $\int B\cdot ds = 0$

③ $\nabla\times E = -\frac{\partial B}{\partial t}$ ③ $\oint E\cdot dl = -\frac{\partial}{\partial t}\int_s B\cdot ds$

④ $\nabla\times H = J + \frac{\partial D}{\partial t}$ ④ $\oint H\cdot dl = \int_s \left(J + \frac{\partial D}{\partial t}\right)\cdot ds$.

① GAUSS LAW → It states the relation between electric field and the charges that produced it. In static conditions, it is equivalent to Coulomb's law and relates the electric flux through a closed surface to the charge enclosed. This law is actually a generalised form of Coulomb's law. Although Coulomb's law is valid for static charge only, Gauss's law holds even if the charges are in arbitrary motion i.e if the electric field varies with time.

② GAUSS LAW FOR MAGNETIC FIELD →

It states that there are no magnetic charges or magnetic monopoles, which would generate a magnetic field in the same way as electric charges ct create an electric field. As monopoles are not existing, $\nabla\cdot B = 0$.

$\oint B\cdot ds = 0$ says that the net magnetic flux ~~over~~ out of any closed surface is zero. This is because, the magnetic flux directed towards (inwards) the south pole of a magnetic dipole kept in any closed surface is equal to the flux towards north pole. Therefore net flux is zero for dipole sources.

③ FARADAY'S LAW → It shows the relation between the induced electric field generated by a changing magnetic flux. It shows that a varying magnetic field acts as one of the possible sources of an electric field. This -ve sign shows the induced electric field would give rise to an induced current that opposes the change in magnetic flux.

④ MODIFIED AMPERE'S LAW ⇒ It states the relation between a magnetic field and the current that gives rise to the field. It shows that the magnetic field is produced by an electric current or by changing electric flux or field. The second term representing the rate of change of electric field flux is known as displacement current ~~displacement~~, distribution. Thus we see that the conduction currents and displacement currents are two possible sources of magnetic field.

As a changing magnetic field produces an electric field and a changing electric field produces an magnetic field. This in turn produces electromagnetic waves

POYNTING THEOREM

This theorem is the conservation of electromagnetic energy.
Poynting theorem states that the rate of decrease of electromagnetic energy in a certain volume V is equal to the work done by the field forces per unit time plus the power transferred into the electromagnetic field.

$$\boxed{-\frac{\partial u}{\partial t} = \nabla \cdot S + E \cdot J}$$

$D = \epsilon E$
$B = \mu H$

Maxwell's eqns are $\nabla \cdot E = \rho/\epsilon_0$, $\nabla \cdot B = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$ — ③ $\nabla \times H = J + \frac{\partial D}{\partial t}$ — ④

Taking dot product of H with eqn ③ & E with eqn ④

$$H \cdot (\nabla \times E) = -H \cdot \frac{\partial B}{\partial t} \quad — ⑤$$

$$E \cdot (\nabla \times H) = E \cdot J + E \cdot \frac{\partial D}{\partial t} \quad — ⑥$$

Using vector identity $\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H)$ — ⑦

Using ⑤ & ⑥ in ⑦

$$\nabla \cdot (E \times H) = -H \cdot \frac{\partial B}{\partial t} - E \cdot \frac{\partial D}{\partial t} - E \cdot J.$$

$$E \cdot \frac{\partial D}{\partial t} = E \cdot \frac{\partial}{\partial t}(\epsilon E) = \frac{1}{2}\epsilon \frac{\partial (E^2)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E \cdot D)$$

$$H \cdot \frac{\partial B}{\partial t} = H \cdot \frac{\partial}{\partial t}(\mu H) = \frac{1}{2}\mu \frac{\partial (H^2)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(H \cdot B)$$

$$\nabla \cdot (E \times H) = -\frac{\partial}{\partial t}(E \cdot D) - \frac{\partial}{\partial t}(H \cdot B) - E \cdot J.$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}(E \cdot D + H \cdot B) - E \cdot J.$$

Integrating this eqn over a volume bounded by a surface

$$\int \nabla \cdot (E \times H)\, dV = -\int \frac{\partial}{\partial t}\left(\frac{1}{2}(E \cdot D + H \cdot B)\right) dV - \int E \cdot J\, dV$$

Using Gauss div theorem

$$\int (E \times H)\, ds = -\frac{\partial}{\partial t}\int \frac{1}{2}(E \cdot D + H \cdot B)\, dV - \int E \cdot J\, dV$$

$$\boxed{-\frac{\partial}{\partial t}\int \frac{1}{2}(E \cdot D + H \cdot B)\, dV = \int (E \times H)\, ds + \int (E \cdot J)\, dV}$$

Significance of $\int (E \cdot J)\, dV$

Let a charged particle q is moving (in an isotropic homogeneous medium (ϵ, μ, σ) in an electromagnetic field with a velocity v.
The force acting on the charge

$$F = q(E + v \times B)$$

Significance of $\int (E \cdot J) dV$

Let a charged particle q is moving in an isotropic homogeneous medium in an electromagnetic field with a velocity v.

The force acting on the charge is

$$F = q(E + v \times B)$$

The rate of work done by electromagnetic fields on the charge q is

$$\frac{dW}{dt} = \frac{d}{dt}(F \cdot x) = F \cdot v = q(E + v \times B) \cdot v$$
$$= qEv.$$

If there is a continuous distribution of charge, the total rate of work done by electromagnetic field is

$$\Sigma q E v \quad \ni \left(\Sigma q = (q_1 + q_2 \cdots q_n) \text{ all charges} \right)$$
$$\Sigma q = \int dq = \int \rho dV.$$

$$\Sigma q E v = \int E v \rho dV = \int E \cdot J dV$$

where $J = \Sigma n q v$ (n = no of charge / Volume)
$= N/V$

$$= \frac{Nq}{V} v = \rho v \qquad \left(\begin{array}{l} Nq = \text{Total no of charges} \\ V = \text{Total volume} \end{array} \right)$$

$$\Rightarrow J = \rho v$$

II Term . $-\frac{\partial}{\partial t} \int \frac{1}{2} (E \cdot D + H \cdot B) dV.$

$\int \frac{1}{2} E \cdot D \, dV = U_e$ = Electrostatic potential energy in volume V

$\int \frac{1}{2} H \cdot B \, dV = U_m$ = Magnetic energy in volume V.

$U = U_e + U_m$ = Electromagnetic field energy in volume V.

$-\frac{\partial U}{\partial t}$ = Rate of decrease of electromagnetic energy in a volume V.

III Term $\quad S = E \times H.$

$\vec{S} = \vec{E} \times \vec{H}$ = Poynting vector

= Power density associated with Electromagnetic field at a point.

= Amount of electromagnetic energy crossing unit area per unit time.

$\int (E \times H)\, ds$ is the amount of electromagnetic energy crossing the closed surface per second.

Also known as power flux.

Unit of S

$$S = E \times H = \frac{\text{Energy}}{\text{Area Time}} = \frac{\text{Power}}{\text{Area}} = \frac{\text{Watt}}{m^2}$$

$$= \frac{\text{Power}}{\text{Area}} = \text{Power Density}.$$

PROPAGATION OF ELECTROMAGNETIC WAVE IN FREE SPACE

In this section we will discuss how the electromagnetic fields travel through free space in the form of waves.

We know the Maxwells equations are

① $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ or $\nabla \cdot D = \rho$

② $\vec{\nabla} \cdot \vec{B} = 0$

③ $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$

④ $\vec{\nabla} \times \vec{H} = J + \frac{\partial D}{\partial t}$

We know that $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{J} = \sigma \vec{E}$

σ = conductivity of the medium

For free space $\epsilon = \epsilon_0$, $\mu = \mu_0$
$\sigma = 0$, $\rho = 0$

Now Maxwells eqns for free space can be written as

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{——— ①}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{——— ②}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial H}{\partial t} \quad \text{——— ③}$$

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial E}{\partial t} \quad \text{——— ④}$$

Taking curl of eqn ③ from both sides

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \left(\mu_0 \frac{\partial H}{\partial t}\right) \quad \text{——— ⑤}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \nabla(\nabla \cdot E) - \nabla^2 E \quad \text{(This is an identity)}$$

$$= \nabla(0) - \nabla^2 E \quad (\text{From eqn ① } \nabla \cdot E = 0)$$

$$= -\nabla^2 E \quad \text{——— ⑥}$$

Using ⑥ in ⑤

$$-\nabla^2 E = -\mu_0 \vec{\nabla} \times \left(\frac{\partial H}{\partial t}\right) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times H)$$

$$= -\mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial E}{\partial t}\right) \quad \left(\text{Using eqn ④ } \vec{\nabla} \times H = \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}} \quad \text{——} (7)$$

Taking curl of eqn (4) we have

$$\nabla \times \nabla \times H = \nabla \times \frac{\partial D}{\partial t} = \nabla \times \epsilon_0 \frac{\partial E}{\partial t}$$

$$\Rightarrow \nabla(\nabla \cdot H) - \nabla^2 H = \epsilon_0 \frac{\partial}{\partial t}(\nabla \times E)$$

$$0 - \nabla^2 H = \epsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial H}{\partial t}\right) \quad (\text{using eqn } (3))$$

$$\Rightarrow \boxed{\nabla^2 H = \mu_0 \epsilon_0 \frac{\partial^2 H}{\partial t^2}} \quad \text{——} (8)$$

Comparing eqn (7) and (8) with general wave equation

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \text{——} (9)$$

we get $\frac{1}{v^2} = \mu_0 \epsilon_0$.

where v = velocity of wave

ψ = wave function.

This shows that the field vectors $\vec{E}$ and $\vec{H}$ propagate as waves in free space.

The speed of propagation is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Henery/m & $\epsilon_0 = 8.85 \times 10^{-12}$ Farad/m

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s} = c$$

= speed of light.

So the electric field vector and magnetic field vector travel like waves in free space with speed of light c.

We can write the wave eqns for electric field vector ($\vec{E}$) and magnetic field vector $\vec{H}$ as

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \quad \text{——— (10)}$$

$$\nabla^2 H - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = 0 \quad (11)$$

Let us write the plane wave solution of above equation as

$$E(r,t) = E_0 e^{i(K\cdot R - \omega t)} \quad \text{— (12)}$$

$$H(r,t) = H_0 e^{i(K\cdot R - \omega t)} \quad \text{— (13)}$$

where E_0 & H_0 are amplitudes of which are constant in space and time.

K = wave propagation vector.

$$\vec{K} = |K|\,\hat{n} = \frac{2\pi}{\lambda}\hat{n}.$$

$$|K| = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{\lambda\nu} = \frac{\omega}{v} = \frac{\omega}{c} \quad \text{— (14)}$$

$$\vec{K} = \hat{i}K_x + \hat{j}K_y + \hat{k}K_z. \quad \text{——— (15)}$$

$$\vec{K}\cdot\vec{r} = \hat{i}(K_x + \hat{j}K_y + \hat{k}K_z)\cdot(\hat{i}x + \hat{j}y + \hat{k}z) \quad \text{——— (16)}$$

$$= K_x x + K_y y + K_z z$$

$$\nabla \cdot E = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[E_0\, e^{i k \cdot r - i\omega t}\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z) - i\omega t}\right]$$

where $E_0 = E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}$.

$$\vec{\nabla}\cdot\vec{E} = \hat{i}\frac{\partial}{\partial x}\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$+ \hat{j}\frac{\partial}{\partial y}\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$+ \hat{k}\frac{\partial}{\partial z}\left[(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$= i\left[E_{0x}k_x\right] e^{i(k_x x + k_y y + k_z z - \omega t)} + i(E_{0z}k_z)\, e^{i(k_z x + k_y y + k_z z - \omega t)}$$

$$+ i(E_{0y}k_y)\, e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= i(E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)\, e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= i(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\cdot(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z})\, e^{i(k\cdot r - \omega t)}$$

$$= i\vec{K}\cdot E_0\, e^{i(k\cdot r - \omega t)}$$

$$= i\vec{K}\cdot\vec{E}.$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = i\vec{K}\cdot\vec{E}} \quad \text{— (17)}$$

Similarly $\nabla \cdot H = iK\cdot H$. — (18)

We get $\nabla \cdot E = iK \cdot E$ and $\nabla \cdot H = iK \cdot H$.

From eqn ① we know that $\nabla \cdot E = 0$

$\Rightarrow iK \cdot E = 0$

$\Rightarrow K \perp E$. —— (16)

Similarly from eqn ② we know that

$\nabla \cdot H = 0$

$\Rightarrow iK \cdot H = 0$

$\Rightarrow K \perp H$. —— (17)

This shows that electromagnetic field vectors $\vec{E}$ and $\vec{H}$ are both perpendicular to the wave propagation vector $\vec{K}$.

Maxwell's third eqn (3) shows that

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial H}{\partial t}.$$

Using (12) & (13) in this eqn ③ we get

$$iK \times E = +i\mu_0 \omega H. \quad \text{—— (18)}$$

$$\because \vec{\nabla} \times E = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times E_0 e^{i(K\cdot r - \omega t)}$$

$$= i\vec{K} \times \vec{E} \quad \text{—— (19)}$$

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\left(H_0 e^{i(K\cdot r - \omega t)}\right) = -i\omega H_0 e^{i(K\cdot r - \omega t)}$$

$$= -i\omega H \quad \text{—— (20)}$$

$$\Rightarrow K \times E = +\mu_0 \omega H$$

$$\Rightarrow H = \frac{1}{\mu_0 \omega}(K \times E)$$

$$= \frac{|K|}{\mu_0 \omega}(\hat{n} \times \vec{E}) = \frac{1}{\mu_0\, \omega/|K|}(\hat{n} \times \vec{E})$$

$$H = \frac{1}{\mu_0 c}(\hat{n} \times \vec{E}) \quad \text{—— (21)}$$

(Using eqn (14) $K = \omega/c \Rightarrow \frac{\omega}{K} = c$)

This eqn shows that

This eqn (21) shows that field vector $\vec{H}$ is perpendicular to the plane containing K and E.
Eqn (16) & (17) shows that $K \perp E$ & $K \perp H$.
This imply that E & H are mutually perpendicular and also perpendicular to the wave propagation vector.
Hence ~~(E H~~ (E, H, k) form a set of orthogonal vectors.
Hence electromagnetic wave is transverse in nature.

Taking magnitude of eqn (21)

$$|H| = \left| \frac{1}{\mu_0 c} (\hat{n} \times \vec{E}) \right|$$

$$\Rightarrow H = \frac{1}{\mu_0 c} E.$$

$$\Rightarrow \frac{E}{H} = \mu_0 c = \frac{\mu_0}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$= 377 \text{ ohms}$$

$$\left|\frac{E}{H}\right| = Z_0 = 377 \text{ ohms}$$

= Wave Impedance in free space.

The unit of $Z_0 = \frac{\text{Unit of E}}{\text{Unit of H}} = \frac{\text{Volt/m}}{\text{Amp/m}} = \frac{\text{Volt}}{\text{Amp}}$

$= \frac{V}{I} = R =$ Resistance Unit

= Ohm.

The ratio $\frac{E}{H} = 377$ ohm, which is real and positive
This implies that the field vectors E and H are in same phase.

The Poynting vector i.e the energy flow per unit area per unit time for a plane electromagnetic wave is given by

$$S = E \times H$$

$$= E \times \frac{\hat{n} \times E}{\mu_0 c}$$

$$= \frac{1}{\mu_0 c} (E \times n \times E)$$

$$= \frac{1}{\mu_0 c} (\hat{n} (E \cdot E) - E (E \cdot n))$$

$$= \frac{1}{\mu_0 c} (\hat{n} E^2 - 0) = \frac{E^2}{\mu_0 c} \hat{n}$$

$$S = \frac{E^2}{Z_0} \hat{n}$$

For a plane electromagnetic wave of angular frequency ω the average value of S over a complete cycle is given by

$$\langle S \rangle = \frac{1}{Z_0} \langle E^2 \rangle \hat{n} = \frac{1}{Z_0} \langle (E_0 e^{ik \cdot r - i\omega t})^2 \rangle_{Real}$$

$$= \frac{1}{Z_0} \langle (E_0^2 (\cos^2 (k \cdot r - \omega t) + i \sin (k \cdot r - \omega t)))^2 \rangle_{Real} \hat{n}$$

$$= \frac{1}{Z_0} E_0^2 \langle \cos^2 (k \cdot r - \omega t) \rangle \hat{n}$$

$$= \frac{1}{Z_0} E_0^2 \times \frac{1}{2} = \frac{1}{Z_0} \left(\frac{E_0}{\sqrt{2}} \right)^2$$

$$= \frac{1}{Z_0} E_{RMS}^2 .$$

This shows that the direction of Poynting vector is along the direction of propagation of electromagnetic wave. This means that the flow of energy in a plane electromagnetic wave in free space is along the direction of wave.

We know that electrostatic energy density U_e & magnetic energy density (U_m) are

$$U_e = \cancel{U_m}\ \frac{1}{2}\epsilon_0 E^2 \quad , \quad U_m = \frac{1}{2}\mu_0 H^2$$

$$\frac{U_e}{U_m} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\frac{E^2}{H^2} = \frac{\epsilon_0}{\mu_0}\left(\sqrt{\frac{\mu_0}{\epsilon_0}}\right)^2 = 1$$

$$\Rightarrow \quad U_e = U_m.$$

Total electromagnetic energy densities are

$$U = U_e + U_m = 2U_e = \left(\frac{1}{2}\epsilon_0 E^2\right)\times 2 = \epsilon_0 E^2.$$

$$U = \epsilon_0 E^2.$$

Time average of energy density

$$\langle U \rangle = \langle \epsilon_0 E^2 \rangle = \epsilon_0 \left\langle \left(E_0 e^{i(k\cdot r - \omega t)}\right)^2 \right\rangle$$

$$= \epsilon_0 \left\langle E_0^2 \left(\cos(k\cdot r - \omega t) + i \sin(k\cdot r - \omega t)\right)^2 \right\rangle_{Real}$$

$$= \epsilon_0 E_0^2 \langle \cos^2(k\cdot r - \omega t) \rangle$$

$$= \frac{1}{2}\epsilon_0 E_0^2 = \cancel{\frac{1}{2}}\ \epsilon_0 \left(\frac{E_0}{\sqrt{2}}\right)^2$$

$$= \cancel{\frac{1}{2}}\ \epsilon_0 E_{RMS}^2 .$$

$$\frac{\langle S \rangle}{\langle U \rangle} = \frac{\frac{1}{Z_0} E_{RMS}^2 \hat{n}}{\epsilon_0 E_{RMS}^2} = \frac{1}{\epsilon_0 Z_0} = \frac{1}{\epsilon_0 \sqrt{\frac{\mu_0}{\epsilon_0}}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c\,\hat{n}$$

$$\Rightarrow \quad \langle S \rangle = \langle U \rangle c\,\hat{n}$$

$\Rightarrow$ Energy flux = Energy density $\times$ velocity of light in vaccum.

Hence energy density associated with electromagnetic wave ~~wave~~ in free space propagates with speed of light with which the field vectors do.

PROPAGATION OF ELECTROMAGNETIC WAVE IN DIELECTRIC MEDIUM.

Let us consider electromagnetic wave is propagating in a dielectric medium of permittivity ϵ & permiability μ.

The Maxwells equations are

$$\boxed{\begin{aligned} &① \ \vec{\nabla}\cdot\vec{D} = \rho \\ &② \ \vec{\nabla}\cdot\vec{B} = 0 \\ &③ \ \vec{\nabla}\times\vec{E} = -\partial B/\partial t \\ &④ \ \vec{\nabla}\times\vec{H} = J + \frac{\partial D}{\partial t} \end{aligned}}$$

We know that $D = \epsilon E$, $B = \mu H$, $J = \sigma E$

For dielectric medium $\sigma = 0$, $J = 0$. Also $\rho = 0$.

Hence Maxwells eqns can be written as

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 0 && — ① \\ &\vec{\nabla}\cdot\vec{B} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{H} = 0 && — ② \\ &\vec{\nabla}\times\vec{E} = -\mu\frac{\partial H}{\partial t} && — ③ \\ &\vec{\nabla}\times\vec{H} = \epsilon\frac{\partial E}{\partial t} && — ④ \end{aligned}}$$

Taking curl of eqn ③

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu\left(\nabla\times\frac{\partial H}{\partial t}\right) \quad — ⑤$$

$$= -\mu\frac{\partial}{\partial t}(\nabla\times H)$$

But $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})} - \cancel{\nabla^2 E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E$

$$= 0 - \nabla^2 E. \quad — ⑥.$$

Using ⑥ & ④ in eqn ⑤

$$-\nabla^2 E = -\mu\frac{\partial}{\partial t}\left(\epsilon\frac{\partial E}{\partial t}\right) = -\mu\epsilon\frac{\partial^2 E}{\partial t^2}.$$

$$\Rightarrow \boxed{\nabla^2 E = \mu\epsilon\frac{\partial^2 E}{\partial t^2}} \quad — ⑦$$

Similarly taking curl of eqn ④

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\mu\left(\nabla\times\frac{\partial E}{\partial t}\right) = -\mu\frac{\partial}{\partial t}(\nabla\times E)$$

$$\Rightarrow -\nabla^2 \vec{H} = -\mu\epsilon\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 H = \mu\epsilon\frac{\partial^2 H}{\partial t^2}} \quad — ⑧$$

Comparing eqn (7) & (8) with the general wave eqn

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \text{— (9)}$$

where v = velocity of wave

$$v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{1}{\sqrt{\mu_0 \epsilon_0 \mu_r \epsilon_r}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

$\because$ $\mu_r > 1$ & $\epsilon_r > 1$ we can say

$v < c$.

Hence velocity of electromagnetic wave in dielectric medium is less than that in free space.

Replacing by $\frac{1}{v^2}$ eqn (7) & (8) can be written as

$$\boxed{\nabla^2 E = \frac{1}{v^2} \frac{\partial^2 E}{\partial t^2}} \quad \text{— (10)}$$

$$\boxed{\nabla^2 H = \frac{1}{v^2} \frac{\partial^2 H}{\partial t^2}} \quad \text{— (11)}$$

Let us write the plane of wave solution of eqn (10) & (11) can be written as

$$E(r, t) = E_0 e^{i(k \cdot r - \omega t)} \quad \text{— (12)}$$

$$H(r, t) = H_0 e^{i(k \cdot r - \omega t)} \quad \text{— (13)}$$

where E_0 & H_0 are complex amplitudes which are constant in space and time.

k = wave propagation vector, $\vec{k} = |\vec{k}| \hat{n} = \frac{2\pi}{\lambda} \hat{n} = \frac{\omega}{v} \hat{n}$

We can get

$$\boxed{\nabla \cdot E = i k \cdot E = 0} \Rightarrow \boxed{k \perp E} \quad \text{— (14)}$$

$$\boxed{\nabla \cdot H = i k \cdot H = 0} \Rightarrow \boxed{k \perp H} \quad \text{— (15)}$$

This shows that electric field vector E & magnetic field vector H both are $\perp r$ to the direction of propagation vector k.

This shows that em waves are transverse in character.

Using (14) & (15) in eqn (3)

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\Rightarrow \quad i K \times E = -\mu(-i\omega H) = i\mu\omega H$$

$$\Rightarrow \quad K \times E = \mu\omega H$$

$$\Rightarrow \quad \boxed{H = \frac{1}{\mu\omega}(\vec{K} \times \vec{E})} \quad \text{—— (16)}$$

$\Rightarrow$ $\vec{H}$ is $\perp$r to $\vec{K}$ and $\vec{E}$

Using Maxwells 4^{th} eqn

$$\nabla \times H = \epsilon_0 \frac{\partial E}{\partial t}.$$

$$\Rightarrow \quad i k \times H = \epsilon_0(-i\omega E) \Rightarrow K \times H = -\omega\epsilon_0 E.$$

$$\Rightarrow \quad \boxed{\vec{E} = -\frac{1}{\epsilon_0\omega}(\vec{K} \times \vec{H})} \quad \text{—— (17)}$$

Eqns (14) (15), (16), (17) shows that

· E, H & k are $\perp$r to each other.

So (E, H, k) form a set of orthogonal vectors which forms a right handed co-ordinate system.

From eqn (16) we can write

$$|H| = \left|\frac{1}{\mu\omega}(\vec{K} \times \vec{E})\right| = \frac{k}{\mu\omega}|E| = \frac{\omega/v}{\mu\omega}E = \frac{|E|}{\mu v}$$

$$\frac{|E|}{|H|} = Z = \mu v = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}}$$

$$\boxed{Z = \frac{|E|}{|H|} = \text{Real quantity.}}$$

Hence E and H are in the same phase.

i.e they have same relative magnitudes at all points at all time.

Unit of Z is, $Z = \frac{|E|}{|H|} = \frac{Volt/m}{Amp/m} = \frac{Volt}{Amp} = Ohm.$

Poynting vector i.e. energy flow per unit area per unit time for a plane em wave is

$$S = E \times H = E \times \left(\frac{n \times E}{\mu_0 c}\right) = \frac{E^2}{Z_0}\hat{n}.$$

For a plane em wave of ~~am~~ $\left[\begin{aligned} E \times n \times E &= \hat{n}(E \cdot E) - E(E \cdot n) \\ &= E^2 \hat{n} \end{aligned}\right]$

angular frequency ω, average value of S over a complete cycle is

$$\langle S \rangle = \frac{1}{Z_0}\langle E^2 \rangle \hat{n} = \frac{1}{Z_0}\left\langle \left(E_0 e^{ik \cdot r - i\omega t}\right)^2 \right\rangle_{Real} \hat{n}$$

$$= \frac{1}{Z_0} E_0^2 \langle \cos^2(k \cdot r - \omega t) \rangle \hat{n} = \frac{1}{Z_0} E_0^2 \frac{1}{2} \hat{n}$$

$$= \frac{1}{Z_0} E_{rms}^2 \hat{n}$$

So direction of Poynting vector is along the direction of em wave

Ratio of electrostatic and magnetic energy densities is

$$\frac{U_e}{U_m} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\left(\frac{E}{H}\right)^2 = \frac{\epsilon_0}{\mu_0}\left(\sqrt{\frac{\mu_0}{\epsilon_0}}\right)^2 = 1$$

$\Rightarrow \boxed{U_e = U_m.} \Rightarrow$ Electrostatic energy density = Magnetic energy density.

Total electromagnetic energy density

$$\boxed{U = U_e + U_m = \epsilon_0 E^2.}$$

Avg value of $U = \langle U \rangle = \langle \epsilon_0 E^2 \rangle = \epsilon_0 \left\langle \left(E_0 e^{ik \cdot r - i\omega t}\right)^2 \right\rangle_{Real}$

$$= \epsilon_0 E_0^2 \langle \cos^2(k \cdot r - \omega t) \rangle = \epsilon_0 \frac{E_0^2}{2} = \epsilon_0 E_{RMS}^2.$$

$$\frac{\langle S \rangle}{\langle U \rangle} = \frac{\frac{1}{Z} E_{RMS}^2 \hat{n}}{\epsilon E_{RMS}^2} = \frac{1}{\epsilon_0 Z_0} = \frac{\hat{n}}{\epsilon Z.} = ~~v\hat{n}~~.\ v\hat{n}$$

$$\langle S \rangle = \langle U \rangle v \hat{n}.$$

Energy flux = Energy density $\times$ Velocity of light in vaccum.