Jacobians

Let $u = u(x,y)$ and $v = v(x,y)$ be two continuous functions of independent variables such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continous in x and y.

Jacobian of u and v with respect to x and y is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$ and is given by

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

- If u,v,w are function of x,y & z then Jacobian of u,v,w with respect x,y,z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

- <u>Chain Rule for Jacobians</u>

If u,v are functions of x,y and x,y are further functions of variables r & s then

$$J\left(\frac{u,v}{x,y}\right) J\left(\frac{x,y}{r,s}\right) = J\left(\frac{u,v}{r,s}\right).$$

- Put $r = u$ & $s = v$ in above result.

$$J\left(\frac{u,v}{x,y}\right) J\left(\frac{x,y}{u,v}\right) = J\left(\frac{u,v}{u,v}\right)$$
$$= 1$$

$$\Rightarrow \quad J\left(\frac{x,y}{u,v}\right) = \frac{1}{J\left(\frac{u,v}{x,y}\right)}.$$

Jacobian of Implicit Functions

- If u & v are implicit functions of variables x and y connected by relations $f_1(u,v,x,y) = 0$ and $f_2(u,v,x,y) = 0$.

Then, $$J\left(\frac{u,v}{x,y}\right) = (-1)^2 \frac{J\left(\frac{f_1,f_2}{x,y}\right)}{J\left(\frac{f_1,f_2}{u,v}\right)}.$$

- If u,v & w are implicit functions of x,y & z related by relations $f_1(u,v,w,x,y,z) = 0$, $f_2(u,v,w,x,y,z) = 0$ & $f_3(u,v,w,x,y,z) = 0$.

Then $$J\left(\frac{u,v,w}{x,y,z}\right) = (-1)^3 \frac{J\left(\frac{f_1,f_2,f_3}{x,y,z}\right)}{J\left(\frac{f_1,f_2,f_3}{u,v,w}\right)}.$$

Q If $u = x^2 - y^2$, $v = 2xy$, $x = r\cos\theta$, $y = r\sin\theta$, compute Jacobian $J\left(\frac{u,v}{r,\theta}\right)$.

$$J\left(\frac{u,v}{r,\theta}\right) = J\left(\frac{u,v}{x,y}\right) J\left(\frac{x,y}{r,\theta}\right).$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (4x^2 - (-4y^2))(r\cos^2\theta - (-r\sin^2\theta))$$

$$= (4x^2 + 4y^2) r(\cos^2\theta + \sin^2\theta)$$

$$= 4(x^2+y^2)\, r(1)$$

$$= 4r^2 r = 4r^3$$

Q If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$.
Find $J\left(\frac{x,y,z}{u,v,w}\right)$.

$\rightarrow$ $J\left(\frac{x,y,z}{u,v,w}\right) = \dfrac{1}{J\left(\frac{u,v,w}{x,y,z}\right)}$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cancel{x}yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$

$$= \begin{vmatrix} yz & z(x-y) & y(x-z) \\ 2x & 2(y-x) & 2(z-x) \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 1\left[z(x-y)2(z-x) - 2(y-x)y(x-z)\right]$$

$$= 2(x-y)(z-x)\left(z - y(-1)(-1)\right)$$

$$= 2(x-y)(z-x)(z-y)$$

$$\therefore J\left(\frac{x,y,z}{u,v,w}\right) = \frac{1}{2(x-y)(z-x)(z-y)}$$

Q If $u^2+v+w = x+y^2+z^2$, $u+v^3+w = x^2+y+z^2$ (4)
and $u+v+w^3 = x^2+y^2+z$, compute $J\left(\frac{u,v,w}{x,y,z}\right)$.

→ Given relations can be written as:

$f_1(u,v,w,x,y,z) = u^2+v+w-x-y^2-z^2=0$.

$f_2(u,v,w,x,y,z) = u+v^3+w-x^2-y-z^2=0$.

$f_3(u,v,w,x,y,z) = u+v+w^3-x^2-y^2-z=0$.

$$J\left(\frac{u,v,w}{x,y,z}\right) = (-1)^3 \frac{J\left(\frac{f_1,f_2,f_3}{x,y,z}\right)}{J\left(\frac{f_1,f_2,f_3}{u,v,w}\right)}.$$

$$J\left(\frac{f_1,f_2,f_3}{x,y,z}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$$

$$= -1\left[(-1)(-1)-(-2y)(-2z)\right] - (-2y)\left[(-2x)(-1) - (-2x)(-2z)\right]$$

$$-2z\left[(-2x)(-2y)-(-2x)(-1)\right]$$

$$= -1\left[1-4yz\right) + 2y(2x-4xz) - 2z(4xy-2x)$$

$$= -1+4yz+4yx-8xyz-8xyz+4xz$$

$$= -1+4(yz+yx+xz)-16xyz$$

$$J\left(\frac{f_1, f_2, f_3}{u, v, w}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$

$$= 3u^2(9v^2w^2 - 1) - 1(3w^2 - 1) + 1(1 - 3v^2)$$

$$= 27u^2v^2w^2 - 3u^2 - 3w^2 + 1 + 1 - 3v^2$$

$$= 27u^2v^2w^2 - 3u^2 - 3w^2 - 3v^2 + 2$$

$$\therefore J\left(\frac{u, v, w}{x, y, z}\right) = -1 \times \frac{-1 + 4(yz + yx + xz) - 16xyz}{27u^2v^2w^2 - 3u^2 - 3w^2 - 3v^2 + 2}.$$

$$= \frac{1 - 4(yz + yx + xz) + 16xyz}{27u^2v^2w^2 - 3(u^2 + v^2 + w^2) + 2}.$$

Partial derivative from Implicit functions using Jacobians

Let u, v be implicit functions of independent variables x, y related by function $f_1(u, v, x, y) = 0$, $f_2(u, v, x, y) = 0$.

Then,
$$\frac{\partial u}{\partial x} = -\frac{J\left(\frac{f_1, f_2}{x, v}\right)}{J\left(\frac{f_1, f_2}{u, v}\right)}, \quad \frac{\partial v}{\partial x} = -\frac{J\left(\frac{f_1, f_2}{u, x}\right)}{J\left(\frac{f_1, f_2}{u, v}\right)},$$

$$\frac{\partial u}{\partial y} = -\frac{J\left(\frac{f_1, f_2}{y, v}\right)}{J\left(\frac{f_1, f_2}{u, v}\right)}, \quad \frac{\partial v}{\partial y} = -\frac{J\left(\frac{f_1, f_2}{u, y}\right)}{J\left(\frac{f_1, f_2}{u, v}\right)}.$$

Q Find $\frac{\partial u}{\partial x}$ using Jacobian if $u^2+xy^2-xy=0$ & (6)

$u^2+uvx+v^2=0$.

$$\left(\begin{aligned} xy^2-xy &= -u^2 \\ x(y^2-y) &= -u^2 \\ y^2-y &= \frac{-u^2}{x} \end{aligned} \right)$$

$\longrightarrow$ $f_1 = u^2+xy^2-xy = 0$

$f_2 = u^2+uvx+v^2 = 0$

$$\frac{\partial u}{\partial x} = -\frac{J\left(\frac{f_1, f_2}{x, v}\right)}{J\left(\frac{f_1, f_2}{u, v}\right)}$$

$$J\left(\frac{f_1, f_2}{x, v}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} y^2-y & 0 \\ uv & ux+2v \end{vmatrix}$$

$$= (y^2-y)(ux+2v).$$

$$= -\frac{u^2}{x}(ux+2v).$$

$$J\left(\frac{f_1, f_2}{u, v}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 2u+vx & ux+2v \end{vmatrix}$$

$$= 2u(ux+2v).$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{\frac{-u^2}{x}\cancel{(ux+2v)}}{2u\cancel{(ux+2v)}}$$

$$= \frac{u^2}{x} \times \frac{1}{2u} = \frac{u}{2x}.$$

Functional dependence of Two functions using Jacobian

Two function $f_1(x,y)$ and $f_2(x,y)$ are functionally dependent if they are functions of each other.

If $J\left(\frac{f_1,f_2}{x,y}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = 0$ then

functions f_1 & f_2 are dependent otherwise not.

Q Examine functional dependence of $u = \frac{x-y}{1+xy}$ and $v = \tan^{-1}x - \tan^{-1}y$. If dependent, find relation.

→ Compute $J\left(\frac{u,v}{x,y}\right)$.

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1+xy)^2} & \frac{-(1+x^2)}{(1+xy)^2} \\ \frac{1}{1+x^2} & \frac{-1}{1+y^2} \end{vmatrix}$$

$$= \frac{-1}{(1+xy)^2} + \frac{1}{(1+xy)^2} = 0$$

∴ u & v are functional dependent.

As $v = \tan^{-1}x - \tan^{-1}y$,

$$\tan v = \tan\left(\underbrace{\tan^{-1}x}_{a} - \underbrace{\tan^{-1}y}_{b}\right) = \frac{\tan(\tan^{-1}x) - \tan(\tan^{-1}y)}{1+\tan(\tan^{-1}x)\tan(\tan^{-1}y)}$$

$$= \frac{x-y}{1+xy} = u.$$

∴ $\boxed{u = \tan v}$

Q Show functions $u = x+y+z$, $v = x^3+y^3+z^3-3xyz$, $w = x^2+y^2+z^2-xy-yz-zx$ are dependent. (8)

$$\rightarrow J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \\ 2x-y-z & 2y-x-z & 2z-y-x \end{vmatrix}$$

$\cancel{C_2 \to C_2 - C_1,\ C_3 \to}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \\ 2x-y-z & 2y-x-z & 2z-y-x \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 1 & 1 \\ x^2-yz & y^2-xz & z^2-xy \\ 2x-y-z & 2y-x-z & 2z-y-x \end{vmatrix}$$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2-yz & y^2-x^2-z(x-y) & z^2-x^2-y(x-z) \\ 2x-y-z & 3(y-x) & 3(z-x) \end{vmatrix}$$

$= 0.$

$\therefore$ Dependent.

Also, $\boxed{V = UW}$ by identity.