Unit-1

Partial differentiation

- Let $z = f(x,y)$ be function of two independent variables x and y.

In order to obtain derivatives of z, we can differentiate either with respect to x or with respect to y.

To obtain derivative of z with respect to x, we diffetiate all the terms with respect to x keeping y as constant. This is known as partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$.

Partial derivative of z with respect to y is denoted by $\frac{\partial z}{\partial y}$ and is obtained by differentiating $z=f(x,y)$ with respect to y treating x as constant.

Similarly, higher order derivatives can be obtained.

Consider, $z = x^2y^2 + x^3 + y^4$.

$$\frac{\partial z}{\partial x} = 2xy^2 + 3x^2 + 0 = 2xy^2 + 3x^2.$$

$$\frac{\partial z}{\partial y} = 2x^2y + 0 + 4y^3 = 2x^2y + 4y^3$$

$$\frac{\partial^2 z}{\partial x^2} = 2y^2 + 6x$$

$$\frac{\partial^2 z}{\partial y^2} = 2x^2 + 12y^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = 4xy + 0 = 4xy.$$

Q $u = e^{xyz}$. Find $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

$$\frac{\partial u}{\partial z} = xy\, e^{xyz}$$

$$\frac{\partial^2 u}{\partial y \partial z} = x e^{xyz} + x^2 y z\, e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 1\, e^{xyz} + xyz e^{xyz} + 2xyz e^{xyz} + x^2y^2z^2 e^{xyz}$$

$$= e^{xyz} + 3xyz e^{xyz} + x^2y^2z^2 e^{xyz}$$

$$= (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

Q Show that, at a point on the surface $x^x y^y z^z = c$ where $x = y = z$, we have $\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x \log(ex)}$.

$\longrightarrow$ $x^x y^y z^z = c$

Take logarithm of both sides

$$\log(x^x y^y z^z) = \log c$$

$$\log x^x + \log y^y + \log z^z = \log c$$

$$x \log x + y \log y + z \log z = \log c \quad \text{—— } (*)$$

Partially differentiate both sides wrt y.

$$0 + 1 \log y + y \cdot \frac{1}{y} + 1 \frac{\partial z}{\partial y} \log z + z\left(\frac{1}{z} \frac{\partial z}{\partial y}\right) = 0$$

$$\log y + 1 + (\log z + 1) \frac{\partial z}{\partial y} = 0$$

$$(\log z+1)\,\frac{\partial z}{\partial y} = -(1+\log y)$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z} \quad \text{——} ①$$

Similarly, if we partially differentiate ⊛ with respect to x, we get $\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{1+\log z}$.

Differentiate ⊛ wrt to x.

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{-(1+\log y)}{1+\log z}\right)$$

$$= -(1+\log y)\,\frac{\partial}{\partial x}\left(\frac{1}{1+\log z}\right)$$

$$= -(1+\log y)\,\frac{\partial}{\partial x}(1+\log z)^{-1}$$

$$= -(1+\log y)\;(-1)(1+\log z)^{-2}\left(0+\frac{1}{z}\frac{\partial z}{\partial x}\right)$$

$$= +(1+\log y)\,\frac{1}{(1+\log z)^2}\left(\frac{1}{z}\times\frac{-(1+\log x)}{(1+\log z)}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y} = -\frac{(1+\log y)(1+\log x)}{z\,(1+\log z)^3}$$

At $x=y=z$,

$$\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{x=y=z} = \frac{-(1+\log x)(1+\log x)}{x\,(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

$$= \frac{-1}{x(\log e+\log x)}$$

$$= \frac{-1}{x\log ex}$$

① $u = \log(x^3+y^3+z^3-3xyz)$.

To prove: $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

consider,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \quad (*)$$

$u = \log(x^3+y^3+z^3-3xyz)$.

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz},$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xz-yz-xy)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3(x^2+y^2+z^2-xz-yz-xy)}{(x+y+z)(x^2+y^2+z^2-xz-yz-xy)}$$

$$= \frac{3}{x+y+z}.$$

$(*)$ becomes

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}.$$

Hence Proved.