Maxima and minima of a function of two variables ①

Let $f(x,y)$ be function of two independent variables x and y.

1) Determine critical or stationary points of $f(x,y)$ by solving $f_x = 0$ and $f_y = 0$.

2) Compute f_{xx}, f_{yy}, f_{xy} at each critical point (x,y).

(1) If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and f_{xx} or $f_{yy} > 0$ at a critical point (x,y) then that critical point is a minima.

(2) If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and f_{xx} or $f_{yy} < 0$ at a critical point (x,y) then that critical point is a maxima.

(3) If $f_{xx}f_{yy} - (f_{xy})^2 < 0$ then that critical point is a saddle point.

(4) If $f_{xx}f_{yy} - (f_{xy})^2 = 0$ then test is inconclusive.

Q Find minimum and maximum values of function

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$

→ First determine critical points.

$$f_x = 4x^3 + 0 - 4x + 4y - 0 = 4x^3 - 4x + 4y$$

$$f_y = 0 + 4y^3 - 0 + 4x - 4y = 4y^3 - 4y + 4x.$$

$f_x = 0$

$\Rightarrow 4x^3 - 4x + 4y = 0$

$\Rightarrow x^3 - x + y = 0$ — ①

$f_y = 0$

$4y^3 - 4y + 4x = 0$

$\Rightarrow y^3 - y + x = 0$ — ②

Add

$$x^3 \cancel{-x} \cancel{+y} + y^3 \cancel{-y} \cancel{+x} = 0$$

$$x^3 + y^3 = 0$$

$$y^3 = -x^3$$

$$\Rightarrow y = -x$$

Subtract ② from ①.

$$x^3 - x + y - y^3 + y - x = 0$$

$x^3 - y^3 - 2x + 2y = 0$

$(x-y)(x^2+xy+y^2) - 2(x-y) = 0$

$(x-y)(x^2+xy+y^2-2) = 0$

$\Rightarrow \quad x-y = 0$ or $x^2+xy+y^2-2=0$.

$x-y=0$ & $y=-x$ implies $x+x=0 \Rightarrow 2x=0 \Rightarrow x=0$.

Then, $y = -0 = 0$.

$\therefore$ Critical point from this is $(0,0)$.

$x^2+xy+y^2-2=0$ & $y=-x$ implies $\cancel{x^2} - \cancel{x^2} + x^2 - 2 = 0$

$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.

$x=\sqrt{2}$ implies $y=-\sqrt{2}$

$x=-\sqrt{2}$ implies $y=\sqrt{2}$

$\therefore$ Critical point from this is $(\sqrt{2},-\sqrt{2})$ & $(-\sqrt{2},\sqrt{2})$.

There are three critical points $(0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2})$

Now, compute f_{xx}, f_{yy}, f_{xy}.

$f_{xx} = 12x^2 - 4$

$f_{yy} = 12y^2 - 4$

$f_{xy} = 4$.

At critical point $(0,0)$,

$f_{xx}(0,0) = 0-4 = -4 < 0$

$f_{yy}(0,0) = 0-4 = -4 < 0$

$f_{xy}(0,0) = 4$

$f_{xx}f_{yy} - (f_{xy})^2 = (-4)(-4) - (4)^2 = 0$ $\therefore$ we cannot conclude anything at point $(0,0)$.

At critical point $(\sqrt{2},-\sqrt{2})$,

$f_{xx}(\sqrt{2},-\sqrt{2}) = 12(\sqrt{2})^2 - 4 = 24-4 = 20 > 0$

$f_{yy}(\sqrt{2},-\sqrt{2}) = 12(-\sqrt{2})^2 - 4 = 24-4 = 20 > 0$

$f_{xy}(\sqrt{2},-\sqrt{2}) = 4$

$f_{xx}f_{yy} - (f_{xy})^2 = (20)(20) - (4)^2 = 400 - 16 = 384 > 0$ $\therefore$ minima at $(\sqrt{2},-\sqrt{2})$.

At critical point $(-\sqrt{2},\sqrt{2})$,

$f_{xx}(-\sqrt{2},\sqrt{2}) = 20 > 0$

$f_{yy}(-\sqrt{2},\sqrt{2}) = 20 > 0$

$f_{xy}(-\sqrt{2},\sqrt{2}) = 4$

$f_{xx}f_{yy}-(f_{xy})^2 = (20)(20)-(4)^2 = 384 > 0$ ∴ minima at $(-\sqrt{2},\sqrt{2})$.

Critical points $(\sqrt{2},-\sqrt{2})$ & $(-\sqrt{2},\sqrt{2})$ are minima.

Minimum value is

$f(\sqrt{2},-\sqrt{2}) = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$

$= \cancel{4}+\cancel{4}-\cancel{4}-8-\cancel{4}$

$= -8.$

Q Find dimensions of rectangular box (without top) with given volume so that material used is minimum.

→ let length − x

breadth − y

height − z.

Volume $V = xyz$ (fixed). $\Rightarrow \frac{V}{xy} = z$.

Material used is equal to surface area of box and is given by $S = xy + 2yz + 2xz$.

$$S = xy + (2y+2x)\left(\frac{V}{xy}\right) \qquad \left(\text{Put } z=\frac{V}{xy}\right)$$

$$S = xy + \frac{2V}{x} + \frac{2V}{y}.$$

$$S_x = y - \frac{2V}{x^2} + 0 = y - \frac{2V}{x^2}.$$

$$S_y = x + 0 - \frac{2V}{y^2} = x - \frac{2V}{y^2}.$$

$$S_x = 0 \Rightarrow y - \frac{2V}{x^2} = 0 \Rightarrow y = \frac{2V}{x^2}$$

$$\Rightarrow x^2y = 2V.$$

$$S_y = 0$$

$$x - \frac{2V}{y^2} = 0$$

$$x = \frac{2V}{y^2}$$

$$xy^2 = 2V.$$

$x^2y = 2V$ and $xy^2 = 2V$ implies $x^2y = xy^2$.

$$x^2y = xy^2$$

$$\Rightarrow x^2y - xy^2 = 0$$

$$\Rightarrow xy(x-y) = 0$$

$$x=0,\ y=0,\ x-y=0$$

$$\Rightarrow x=0,\ y=0,\ x=y.$$

$x=y$ implies $y^3 = 2V \Rightarrow y = (2V)^{1/3}$.

$$x = (2V)^{1/3}$$

$\therefore x = y = (2V)^{1/3}$

Critical points ~~are (0,0)~~ is $((2V)^{1/3}, (2V)^{1/3})$

$$S_{xx} = 0 + \frac{4V}{x^3} = \frac{4V}{x^3}$$

$$S_{yy} = 0 + \frac{4V}{y^3} = \frac{4V}{y^3}$$

$$S_{xy} = 1 - 0 = 1.$$

At critical point $((2V)^{1/3}, (2V)^{1/3})$,

$$S_{xx} = \frac{4V}{((2V)^{1/3})^3} = \frac{4V}{2V} = 2 > 0$$

$$S_{yy} = \frac{4V}{((2V)^{1/3})^3} = \frac{4V}{2V} = 2 > 0$$

$$S_{xy} = 1$$

$S_{xx}S_{yy} - (S_{xy})^2 = (2)(2) - (1)^2 = 4 - 1 = 3 > 0$ $\therefore$ minimum at $((2V)^{1/3}, (2V)^{1/3})$.

$\therefore$ length $= x = (2V)^{1/3}$

Breadth $= y = (2V)^{1/3}$.

Height $= z = \dfrac{V}{xy} = \dfrac{V}{(2V)^{1/3}(2V)^{1/3}}$

$$= \frac{(V^3)^{1/3}}{(4V^2)^{1/3}} = \left(\frac{V^3}{4V^2}\right)^{1/3}$$

$$= \left(\frac{V}{4}\right)^{1/3}.$$

$$= \left(\frac{2V}{8}\right)^{1/3}$$

$$= \frac{(2V)^{1/3}}{2} = \frac{x}{2}.$$

Box should be such that its length & breadth are equal and height is half the length.