CONTINUITY EQUATION

An equation which expresses the equality of incoming and outgoing charges in a system and follows the law of conservation of charge is known as equation of continuity.

The current density J and the charge density ρ are related at each point through a differential equation. This equation is based on the fact that electric charge can neither be created nor be destroyed and the rate of increase of the total charge inside any arbitrary volume must be equal to the net flow of charge into this volume.

The electric current through a closed surface S is

$$I = \oint \vec{J} \cdot d\vec{s}$$

$$= \int \vec{\nabla} \cdot \vec{J}\, d\vec{s} \quad \text{(Using Gauss divergence theorem)} \quad \text{— (1)}$$

Again considering charge leaving a volume V per second

= Rate of decrease of charge in volume V (current I)

$$\Rightarrow I = -\frac{\partial q}{\partial t} = -\frac{\partial}{\partial t}\left(\int \rho\, dV\right) = -\int_V \frac{\partial \rho}{\partial t}\, dV \quad \text{— (2)}$$

From (1) & (2)

$$\int \vec{\nabla} \cdot \vec{J}\, dV = -\int \frac{\partial \rho}{\partial t}\, dV$$

$$\Rightarrow \int \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

(∵ The integral is zero for any arbitrary volume)

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0} \quad \text{— (3)}$$

This is equation of continuity.

In steady state i.e. for steady current, the charge density at any point within the region is constant.

i.e. ρ = constant $\Rightarrow \frac{\partial \rho}{\partial t} = 0$.

Eqn (3) now

$$\boxed{\vec{\nabla} \cdot \vec{J} = 0} \quad \text{--- (4)}$$

This is eqn of continuity for steady state.

AMPERE'S CIRCUITAL LAW

The line integral of the magnetic field B along any closed loop C is proportional to the current I passing through the closed loop.

$[B = \mu H]$

$$\boxed{\oint B \cdot dl = \mu_0 I}$$

or $$\boxed{\oint H \cdot dl = I} \quad \text{—} \; (1)$$

Applying Gauss divergence theorem to L.H.S

$$\oint H \cdot dl = \int (\nabla \times H)\, ds = I. \quad \text{—} \; (2)$$

Also $$I = J \cdot ds \quad \text{—} \; (3)$$

Using (2) & (3) in (1)

$$\int (\nabla \times H)\, ds = \int J \cdot ds$$

$$\Rightarrow \boxed{\nabla \times H = J} \quad \text{—} \; (4)$$

This equation is differential form of Ampere's Law.

Taking divergence of eqn (4) from both sides

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}$$

$$0 = \vec{\nabla} \cdot \vec{J} \quad \text{—} \; (5)$$

($\because$ div of curl of any vector is always equal to zero)

This eqn (5) is satisfying equation of continuity for steady state only (which is $\nabla \cdot J = 0$).

This shows that Ampere's circuital law is valid only for steady states. When any circuit contains time varying electric field components, Ampere's law is not valid. Hence Maxwell modified Ampere's law and add one term for time varying electric fields.

The modified Ampere's law now

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d \quad \text{--- (6)}$$

Taking divergence from both sides

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J + \nabla \cdot J_d$$

$$\Rightarrow \quad 0 = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{J}_d \quad \text{--- (7)}$$

Comparing this equation of continuity, $\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$ --- (8)

we get

$$\nabla \cdot J_D = \frac{\partial \rho}{\partial t}$$

$$= \frac{\partial}{\partial t}(\nabla \cdot D)$$

(Gauss law in differential form $\nabla \cdot D = \rho$)

$$\Rightarrow \quad \nabla \cdot J_D = \nabla \cdot \left(\frac{\partial D}{\partial t}\right)$$

$$\Rightarrow \quad J_D = \frac{\partial D}{\partial t} \quad \text{--- (9)}$$

(Displacement current Density J_D)

Using (9) in (6)

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial D}{\partial t}} \quad \text{--- (10)}$$

This is modified Ampere's Law.

MAXWELL'S DISPLACEMENT CURRENT

Any circuit which contains a capacitor, the plates of the capacitor get charged with respect to time. During this charging process the plates of ~~the~~ capacitor gets charged, the electric field on the plates is changing with time.

Let at any instant the electric field on the plate is

$$\vec{E} = \frac{\sigma}{\epsilon_0} = \frac{q}{\epsilon_0 S}$$

where S = Area of plate, q = charge

σ = surface charge density $= \frac{q}{S}$.

$$\Rightarrow \quad \epsilon_0 \vec{E} = \frac{q}{S} \quad \Rightarrow \quad \vec{D} = \frac{q}{S}$$

Differentiating both sides by time t

$$\frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\frac{q}{S}\right) = \frac{1}{S}\frac{\partial q}{\partial t} = \frac{1}{S} I_D$$

$$\Rightarrow \boxed{I_D = S\frac{\partial D}{\partial t}}$$ I_D = Displacement current

$$\frac{I_D}{S} = \frac{\partial D}{\partial t} \Rightarrow \boxed{J_D = \frac{\partial D}{\partial t}}$$ J_D = Displacement current density.

Maxwell postulated that it is not only the current in a conductor produces magnetic field, but a changing electric field in vaccum or in dielectric also produces magnetic field. This changing magnetic field produces an emf i.e electric field. This changing electric field produces a current known as displacement current.

1. Displacement current is a current which flows when time-varying electric field is present.
2. It is not linked with the motion of charges.
3. Magnitude of displacement current is equal to rate of change of electric displacement - vector.
4. Displacement current serves the purpose to make the total current continuous across the discontinuity in the conduction current in a circuit.

Total current $J = J_C + J_D$

Modified Amperes Law in Integral form

We know $I_D = S\frac{\partial D}{\partial t} = S\frac{\partial}{\partial t}(\epsilon_0 E) = \epsilon_0\frac{\partial}{\partial t}(E \cdot S)$

$$= \epsilon_0 \frac{\partial \phi}{\partial t}.$$

$$\oint B \cdot dl = \mu_0(I + I_D) = \mu_0\left(I + \epsilon_0\frac{\partial \phi}{\partial t}\right)$$

OR $$\oint H \cdot dl = \int J\, ds + \epsilon_0\frac{\partial}{\partial t}\left(\int E \cdot ds\right) = \int J \cdot ds + \epsilon_0\frac{\partial \phi}{\partial t}$$

Distinction between displacement current and conduction current

Displacement current	Conduction Current
① It is not because of motion of charge carriers. It is a current which exist in vaccum or any medium when a time varying electric field is present.	① Conduction current is because of the actual flow of charge carriers of the conducting medium.
② It depends on the electric permittivity of the medium and the rate at which the electric field changes with time.	② It obeys Ohms law and depends on the resistance and potential difference of the conductor.

FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Induced emf in a conducting loop is equal to the negative rate of change of magnetic flux through the surface enclosed by the loop.

Induced emf $e = -\frac{\partial \phi_B}{\partial t}$ —— ①

But we know that induced emf $= e = \oint E \cdot dl$. ②

If S is surface enclosed by the loop, magnetic flux through the surface area S is $\phi_B = \int \vec{B} \cdot \vec{ds}$. —— ③

Using ② & ③ in ①

$$\boxed{\oint \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \int \vec{B} \cdot \vec{ds}} \quad \text{—— ④}$$

This is integral form of Faradays law.

Applying Stoke's theorem to L.H.S. of eqn ④

$$\oint \vec{E} \cdot \vec{dl} = \int (\vec{\nabla} \times \vec{E}) \cdot ds = -\int \frac{\partial B}{\partial t} ds$$

$$\Rightarrow \int (\vec{\nabla} \times \vec{E}) \cdot ds = -\int \frac{\partial B}{\partial t} \cdot ds .$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}}$$

This is Differential form of Faraday's Law.

GAUSS LAW FOR ELECTRIC FIELD

The total outward flux of an electric field vector over a closed surface is equal to $1/\epsilon_0$ times the total charge enclosed in a volume enclosed by the surface.

Let q = charge enclosed, ds = surface ~~endo~~ element

$$\boxed{\int E \cdot ds = \frac{q}{\epsilon_0}} \quad \text{--- (1)}$$

or

$$\int E \cdot ds = \frac{1}{\epsilon_0} \int \rho \, dV \quad \text{--- (2)}$$

where ρ = charge density (volume) $\left(= \frac{dq}{dV} \Rightarrow q = \int dq = \int \rho dV \right)$

Applying Gauss divergence theorem to L.H.S of (2)

$$\int (\nabla \cdot E) \, dV = \frac{1}{\epsilon_0} \int \rho \, dV$$

$$\Rightarrow \boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}} \quad \text{or} \quad \boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$ Differential form of Gauss Law

where $\vec{D} = \epsilon_0 \vec{E}$ = Electric displacement vector.

GAUSS LAW FOR MAGNETIC FIELD

The magnetic field lines due to a current carrying conductor are closed curves without any beginning or end.

Since magnetic field lines are continuous the magnetic flux entering any region is equal to the flux leaving it. So the net flux over a volume is zero, hence magnetic field is solenoidal.

$$\boxed{\nabla \cdot B = 0}$$ Differential form of Gauss Law in magnetostatics

Let ϕ_B = Magnetic flux, then $\phi_B = \oint B \cdot ds$.

Applying divergence theorem

$$\oint B \cdot ds = \int (\nabla \cdot B) dV = 0$$

$$\boxed{\oint B \cdot ds = 0}$$ Integral form of Gauss law for magnetic field.

MAXWELL'S EQUATIONS →

① $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ OR $\vec{\nabla} \cdot \vec{D} = \rho$ ① $\int E \cdot ds = \frac{1}{\epsilon_0} \int \rho \, dV$

② $\nabla \cdot B = 0$ ② $\int B \cdot ds = 0$ $\int B \cdot ds = Q_m = 0$

③ $\nabla \times E = -\frac{\partial B}{\partial t}$ ③ $\oint E \cdot dl = -\frac{\partial}{\partial t} \int_s B \cdot ds$

④ $\nabla \times H = J + \frac{\partial D}{\partial t}$. ④ $\oint H \cdot dl = \int_s \left(J + \frac{\partial D}{\partial t}\right) \cdot ds$.

① GAUSS LAW → It states the relation between electric field and the charges that produced it. In static conditions, it is equivalent to Coulomb's law and relates the electric flux through a closed surface to the charge enclosed. This law is actually a generalised form of Coulomb's law. Although Coulomb's law is valid for static charge only, Gauss's law holds even if the charges are in arbitrary motion i.e if the electric field varies with time.

② GAUSS LAW FOR MAGNETIC FIELD →

It states that there are no magnetic charges or magnetic monopoles, which would generate a magnetic field in the same way as electric charges et create an electric field. As monopoles are not existing, $\nabla \cdot B = 0$.
$\oint B \cdot ds = 0$ says that the net magnetic flux ~~over~~ out of any closed surface is zero. This is because, the magnetic flux directed towards (inwards) the south pole, of a magnetic dipole kept in any closed surface is equal to the flux towards north pole. Therefore net flux is zero for dipole sources.

③ FARADAY'S LAW → It shows the relation between the induced electric field generated by a changing magnetic flux. It shows that a varying magnetic field acts as one of the possible sources of an electric field. This -ve sign shows the induced electric field would give rise to an induced current that opposes the change in magnetic flux.

④ MODIFIED AMPERE'S LAW ⇒ It states the relation between a magnetic field and the current that gives rise to the field. It shows that the magnetic field is produced by an electric current or by changing electric flux or field. The second term representing the rate of change of electric field flux is known as displacement current ~~displacement~~, distribution. Thus we see that the conduction currents and displacement currents are two possible sources of magnetic field.

As a changing magnetic field produces an electric field and a changing electric field produces an magnetic field. This in turn produces electromagnetic waves