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Chapter 1

Motion of the Charged Particles in Electromagnetic Fields

1.1 Introduction

Various devices such as radio, television, electric motor, generator, computer, particle accelerator and so on are based on the laws of electricity and magnetism. Man was aware of electric and magnetic phenomena as early as 700 B.C. Understanding and mathematical formulation of these phenomena was possible by the work of Coulomb, Faraday, Maxwell and many other physicists.

1.2 Electric Field

Coulomb's law describes the force between electric charges, but it does not explain or explore the mechanism for these forces. It was known for many years that iron filing trace out a pattern in the vicinity of a magnet. Fine particles like saw dust exhibits a similar pattern in the vicinity of electric charges. Search for the mechanism for Coulomb's law and the pattern around electric charges resulted in the concept of electric field. It was proposed by Faraday.

The electric field at some point in space is defined as the electric force **F** acting on positive test charge q_0 placed at that point divided by the magnitude of the test charge q_0 i.e.,

$$\mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0} = \frac{\mathbf{F}}{q_0} \qquad ...(1)$$

The electric field is a vector and its direction is the direction of force **F**. Its S.I. unit is N/C. Here **E** is the external field and not the field provided by the test charge. Further, it is assumed that the test charge q_0 is infinitesimally small, so that it does not disturb the field in which it is placed.

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The force **F** on the test charge q_0 indicates the existence of the electric field. From the value of q_0 and the measurement of force, the direction and magnitude of the electric field can be determined. The alternate unit of electric field is V/m.

The electric field **E** can be expressed as $\mathbf{E} = V/d$ volt/meter, where d is the distance in which potential difference V exists.

1.3 Magnetic Field

The magnetic field at a point is the vector field **B**, which exerts a force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ on the particle of charge q, moving with the velocity **v** through that point. The force **F** is perpendicular to the plane containing **v** and **B** and it is given by cross product of **v** and **B** as shown in Fig. 1.1.

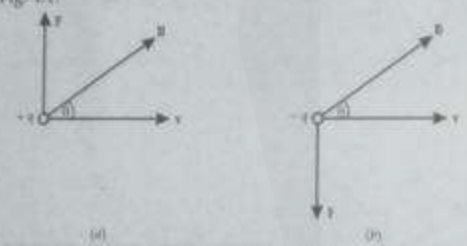

Fig. 1.1 Force on charged particles : (*a*) positive ; (*b*) negative.

The force on a negative charged particle is in a direction to that on a positive charged particle.

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) = qvB\sin\theta \quad ...(2)$$

where θ is the angle between **v** and **B**.

Eq. (2) states that

1. The magnetic force **F** is zero when the charge is stationary *i.e.* $\mathbf{v} = 0$.
2. The magnetic force **F** is zero when **v** and **B** are **either** parallel ($\theta = 0$) **or** antiparallel ($\theta = 180°$).
3. The magnetic force **F** is maximum, when **v** and **B** are perpendicular to each other ($\theta = 90°$).

The unit of magnetic field is

$$\mathbf{B} = \frac{\mathbf{F}}{q\mathbf{v}} = \frac{\text{Force}}{\text{Charge} \times \text{Velocity}}$$

$$= \frac{N}{C\,(m/s)}$$

$$= \frac{N}{A\,m} = \text{Tesla}(T)$$

Another unit of magnetic field is Gauss (G).

$$1T = 10^4 G$$

1.4 Lorentz Force

If we consider a point charge (q) is moving in an electromagnetic field, then the force **F** acting on the charge not only depends upon the position of this charge but also on the velocity of its charge. The force **F** is made up of two components :

(*a*) the electrostatic force $\mathbf{F}_e$ which does not depend upon velocity of charge.

(*b*) the magnetic force $\mathbf{F}_m$ which depends upon the velocity of charge.

If an electric charge at rest experiences force in a region of space, the region is said to be characterised by electric field. Let **E** be the intensity of the electric field at any point in the field, the electrostatic force which a test charge q experiences at that point in the field is given by :

$$\mathbf{F}_e = q\mathbf{E} \quad ...(3)$$

If a test charge q is moving with velocity **v** through a magnetic field, it experiences a deflecting force $\mathbf{F}_m$ and this force is given by

$$\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B}) \quad ...(4)$$

where **B** = Magnetic field induction or Magnetic field intensity

B is characteristic of given magnetic field and is defined as the force acting on a unit charge moving with unit velocity in direction at right angles to the magnetic field.

From Eq. (4), it is obvious that $\mathbf{F}_m$ is directed in a direction perpendicular to the plane formed by **v** and **B**. Thus total force acting on unit charge q is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

$$= q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) \quad ...(5)$$

This relation is known as *Lorentz force or Lorentz force equation.*

Eq. (5) is universal and holds good for constant as well as varying electric and magnetic fields.

Example 1.1. *A electron is moving with a velocity $(2i+3j)$ m/s in an electric field of intensity $(3\hat{i}+6\hat{j}+2\hat{k})$ and a magnetic field of $(2j+3k)$ T. Find the magnitude and direction of the Lorentz force acting on the electron.*

Solution. The Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$$\mathbf{F} = 1.6\times10^{-19}[(3\hat{i}+6\hat{j}+2\hat{k}) + [(2i+3j)\times(2j+3k)]]$$

$$= 1.6\times10^{-19}[3i+6j+2k+9i-6j+4k]$$

$$= 1.6\times10^{-19}[12i+6k] = 9.6\times10^{-19}(2i+k)$$

or $$|\mathbf{F}| = 9.6\times10^{-19}\sqrt{4+1} = 2.15\times10^{-18}\,\text{N}$$

It is in x-y plane an angle ϕ with x-axis

$$\phi = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

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1.5 Motion of Charged Particle in a Parallel Electric Field

Consider an electron is situated between two parallel plates of large surface area contained in an evacuated envelope. A difference of potential V is applied between two plates, the direction of the electric field between the plates is as shown in Fig. 1.2. The distance between two plates is d. If this distance (d) is small compared to the dimensions of the plates then the uniform electric field between the plates is

$$E = V/d \qquad ...(6)$$

and the lines of force are pointing along negative x-axis.

Electrons starting from the negative plate are attracted by the positive plate and since there is no force along y and z-directions, so the electrons emitted from negative plate are accelerated along x-direction towards the positive plate. The electric force acting on the electron of charge (e) and mass (m)

$$eE = ma_x \quad \text{or} \quad a_x = \frac{eE}{m} \qquad ...(7)$$

Hence

$$a_x = \frac{e}{m}\frac{V}{d} \qquad ...(8)$$

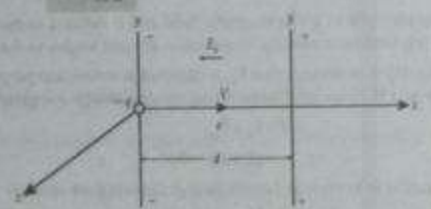

Fig. 1.2 Charged particle in parallel electric field.

Thus the acceleration of the electrons along x-axis is constant, for a particular value of V and d.

If the electron strikes the positive plate with a velocity v_x which is called as *terminal velocity*, then the kinetic energy (K.E.) of the electron will be

$$\text{K.E.} = \frac{1}{2}mv_x^2 \qquad ...(9)$$

which is equal to work done W, where

$$W = \text{Force} \times \text{displacement}$$
$$= eE \times d$$
$$= e\frac{V}{d} \times d = eV \qquad ...(10)$$

i.e., $\text{K.E.} = W$

or $\frac{1}{2}mv_x^2 = eV$

So the terminal velocity

$$v_x = \sqrt{\frac{2eV}{m}} \qquad ...(11)$$

Eq. (11) represents a new unit of energy or work called the electron volt (eV).

One electron volt is defined as the kinetic energy of a particle carrying a charge e Coulomb and accelerated through a potential difference of 1 volt.

$$1\text{eV} = 1.602 \times 10^{-19} \text{ Joule}$$

Example 1.2. *An electron starting from rest travels a distance of 10 mm, in a uniform electric field of 3×10^5 N/C. Calculate the speed of the electron.*

Solution. $d = 10 \text{ mm} = 10 \times 10^{-3}$ m, $E = 3 \times 10^5$ N/C, $e = 1.6 \times 10^{-19}$ C, $m = 9.1 \times 10^{-31}$ kg

The speed of electron $v = \sqrt{\frac{2eE.d}{m}}$; E.d = V = Potential difference

$$= \sqrt{\left[\frac{2 \times 1.6 \times 10^{-19} \times (3 \times 10^5) \times (10 \times 10^{-3})}{9.1 \times 10^{-31}}\right]} = 3.25 \times 10^8 \text{ m/s}.$$

1.6 Motion of Electron in Transverse Electric Field

Let us consider that an electron enters in the region between two parallel plates separated by a distance d (Fig. 1.3), with initial velocity (v_x) in positive x direction. An uniform electric field $\mathbf{E}_y$ is given between these two plates, the length of the field (E_y) is L. The direction of the electric field along negative y axis, no other fields existing in this region.

The motion of the charged particle under this transverse field E_y can be studied as follows :

Initial velocities along x, y and z directions are :

$$u_x = v_x, \quad u_y = 0, \quad u_z = 0 \quad \text{when } t = 0 \qquad ...(12)$$

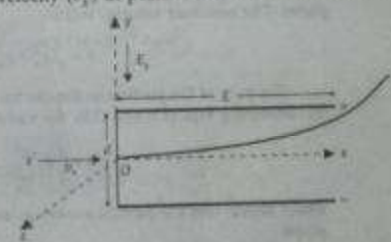

Fig. 1.3 Charged particle in transversed electric field.

Since there is no force along x and z directions, so velocity along z direction remains zero and the velocity along x-axis remains v_x and distance travelled along x-axis by any time t is

$$x = v_x t$$

So,

$$t = \frac{x}{v_x} \qquad ...(13)$$

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But there is constant electric field along y axis so there is a constant acceleration along y-axis and the velocity at any instant of time t is

$$v_y = u_y + a_y t$$

$$v_y = a_y t \qquad [\because u_y = 0]$$

$$v_y = \frac{eE_y}{m}\frac{x}{v_x} \qquad [\because ma_y = eE_y]$$

$$\Rightarrow \quad v_y = \frac{e}{m}\frac{V}{d}\frac{x}{v_x} \qquad ...(14)$$

The distance travelled along y-axis by the time t within the applied electric field (E_y) is

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\Rightarrow \quad y = \frac{1}{2}a_y t^2 \qquad [\because u_y = 0] \qquad ...(15)$$

When force acting on the electron of charge e along y-axis is

$$ma_y = eE_y$$

So

$$a_y = \frac{e}{m}E_y = \frac{e}{m}\frac{V}{d} \qquad ...(16)$$

These equations indicate that in the region between the plates the electron is accelerated upward with the velocity component v_y varying from point to point, whereas velocity component v_x remains unchanged in the passage of the electron between the plates. The resultant velocity will be

$$v = \sqrt{(v_x^2 + v_y^2)} \qquad ...(17)$$

The path of the particle in the electric field with respect to the point O is determined by combining Eqs. (13) and (15), the variable t being eliminated. This gives

$$y = \frac{1}{2}a_y\frac{x^2}{v_x^2} \qquad ...(18)$$

which shows that the particle moves in the *parabolic path* in the region between the charged plates.

The angular deflection of the electron from original path produced by the transversed electric field at any instant of time t given by

$$\tan\theta = \frac{v_y}{v_x} = \frac{eE_y}{m}\frac{x}{v_x^2}$$

$$\tan\theta = \frac{eV}{md}\frac{x}{v_x^2} \qquad ...(19)$$

when L is the length of the plates where the electric field is given, then the displacement of the electron due to transverse electric field, after travelling length $x = L$ *i.e.*, at the end point of the applied electric field area is obtained by Eqs. (16) and (18) as

$$y = \frac{1}{2}a_y t^2$$

$$\Rightarrow \quad y = \frac{1}{2}\left(\frac{eE_y}{m}\right)\left(\frac{L}{v_x}\right)^2 \qquad ...(20)$$

After the end of the applied electric field area, electron emerges out tangentially to the parabolic path at the point of emergence.

Example 1.3. *An electron is accelerated through a potential difference of 150 V. This electron is injected into a transverse electric field produced by the application of 20 V to a pair of parallel plates of length 10 cm and 1 cm apart. A screen is placed at 50 cm apart from the centre of the applied electric field. Find the deflection on screen.* **[GGSIPU, Feb 2008 14 marks]**

Solution. Given that $V_A = 150$ V, $V = 20$ V, $l = 10$ cm, $d = 1$ cm and $D = 50$ cm.

The velocity along x-axis be given by

$$v_x = \sqrt{\frac{2eV_A}{m}}$$

$$v_x = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 150}{9.1 \times 10^{-31}}} = 7.26 \times 10^6 \text{ m/s}$$

The acceleration along y-axis be a_y

$$a_y = \frac{e}{m}\frac{V}{d}$$

$$= \frac{1.6 \times 10^{-19}}{9.1 \times 10^{-31}} \times \frac{20}{1 \times 10^{-2}} = 3.516 \times 10^{14} \text{ m/s}^2$$

Now velocity along y-axis will be

$$v_y = a_y \cdot t = a_y \frac{l}{v_x}$$

$$= \frac{3.516 \times 10^{14} \times 10 \times 10^{-2}}{7.26 \times 10^6} = 4.8 \times 10^6 \text{ m/s}$$

then

$$\tan\theta = \frac{v_y}{v_x}$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{v_y}{v_x} = 33.47°$$

and deflection on screen

$$y = D\tan\theta = D\frac{v_y}{v_x}$$

$$= \frac{50 \times 10^{-2} \times 4.8 \times 10^6}{7.26 \times 10^6} = 0.33 \text{ m}$$

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1.7 Motion of Electron in Transversed Magnetic Field

Let us consider an electron with initial speed v entering a uniform transversed magnetic field acting downward along z axis, *i.e.*, perpendicular to the plane of the paper (Fig. 1.4). Since magnetic field is perpendicular to initial velocity v and so its motion at every instant same, no work is done on the electron. That means its kinetic energy will not increase, its speed remains unchanged within the magnetic field. The force acting on the electron due to transverse magnetic field is

$$F_m = Bev \qquad ...(21)$$

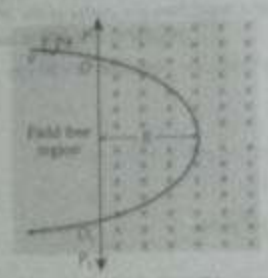

Fig. 1.4 Charged particle in transversed magnetic field.

Since B, e, v are constants in magnitude so F_m is constant in magnitude and perpendicular to the direction of motion in a circular path with constant speed within the transversed magnetic field. For the downward magnetic field, movement of the charged particle will be clockwise, whereas it will be anticlockwise for upward magnetic field. It is analogous to the problem of a mass tied to a rope and twirled around with constant speed. This force which is actually the tension of rope, remains constant in magnitude and is always directed towards the centre of circle, and so is normal to the motion, which is known as *centripetal force*.

So in the transversed magnetic field when the electron of mass m is moving in a circular path of radius R with constant speed v has a radial acceleration towards the centre, then the magnetic force acting on it is

$$F_m = \frac{mv^2}{R} \qquad ...(22)$$

i.e.

$$F_m = Bev = \frac{mv^2}{R}$$

or

$$R = \frac{mv}{eB} \qquad ...(23)$$

Eq. (23) shows that in transverse magnetic field, an electron will move in a circular path of radius R. After the magnetic field, the electron emerges out tangentially to the arc at the point of emergence.

From Eq. (23), we can get

$$\frac{e}{m} = \frac{v}{RB}$$

which can be used for determining the e/m ratio for electron.

If T = Time period of complete one revolution then

$$T = 2\pi R/v = \frac{2\pi Rm}{BeR}$$

$$T = \left(\frac{2\pi}{B}\right)\left(\frac{m}{e}\right)$$

So T is independent of v and R. The resonance frequency (ν) is given by,

$$\nu = \frac{1}{T} = \frac{Be}{2\pi m} \qquad ...(24)$$

Example 1.4. An electron accelerated by a potential difference of 1000 V enters at right angles into a uniform magnetic field of induction 1.19×10^{-3} Wb/m^2. Compute the radius of the electron trajectory in the magnetic field.

Solution. Given $V = 1000$ V = potential difference

B = magnetic field $= 1.19 \times 10^{-3}$ Wb/m^2

The K.E. of electron is given as

$$\frac{1}{2}mv^2 = eV \qquad ...(i)$$

The electron moves in a circular path of radius due to transverse magnetic field B

The F_m = magnetic force $= Bev = \frac{mv^2}{R}$ ...(ii)

From Eq. (i),

$$v = \sqrt{\frac{2eV}{m}} = \sqrt{\left[\frac{2 \times 1.6 \times 10^{-19} \times 1000}{9.1 \times 10^{-31}}\right]}$$

$$v = 18.752 \times 10^6 \text{ m/s}$$

From Eq. (ii),

$$R = \frac{mv}{eB}$$

$$= \frac{9.1 \times 10^{-31} \times 18.752 \times 10^6}{1.6 \times 10^{-19} \times 1.19 \times 10^{-3}} = 8.962 \times 10^{-3} \text{ m}$$

$$R = 8.962 \text{ cm}$$

1.8 Motion of Charged Particle in Combined Fields

When electric and magnetic fields are present simultaneously, the force experienced by a moving charged particle is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad ...(25)$$

When both the fields are present, there are many possibilities, *e.g.*, $\mathbf{E}$ and $\mathbf{B}$ are parallel, antiparallel, perpendicular or making arbitrary angle with each other.

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1.8.1 Electric and Magnetic Fields are Parallel and Antiparallel

Consider that **E** and **B** are parallel to each other ; along y-axis (Fig. 1.5). Let us assume that the velocity of the particle is making an angle θ with the magnetic and electric fields. It is convenient to consider motion in plane perpendicular to y-axis. We can resolve **v** into two $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ as shown in Fig. 1.5. In three dimensions velocity vector **v** may be in any plane ; $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ should be considered accordingly.

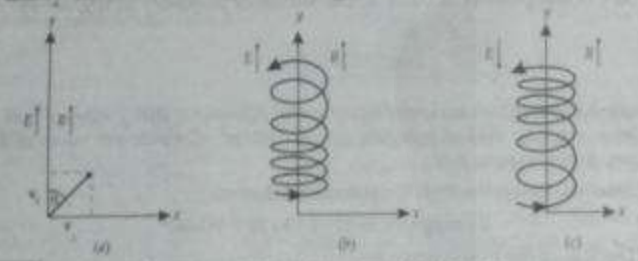

Fig. 1.5 Helical motion with varying pitch (*a*) directions of **E**, **B** and **v** (*b*) increasing pitch, (*c*) decreasing pitch.

In a plane perpendicular to y-axis, particle moves in a circular orbit of radius.

$$R = \frac{m\mathbf{v}_\perp}{q\mathbf{B}} \quad ...(26)$$

Along y-axis, the particle moves with uniform acceleration, or in case of antiparallel electric field, deceleration

$$\mathbf{a} = \frac{q\mathbf{E}}{m} \quad ...(27)$$

The resultant motion is superposition of circular motion and accelerating translational motion. It is a *helix* with varying pitch as shown in Fig. 1.5. This motion is similar to that of in uniform magnetic field. But due to presence of electric field, particle is accelerated (or deaccelerated) along the direction **E**, which results in varying the pitch of *helix*.

From the equation of kinematics, if $\mathbf{v}_\parallel$ is the original velocity then after time t it is

$$\mathbf{v}'_\parallel = \mathbf{v}_\parallel + \frac{q\mathbf{E}}{m}t \quad ...(28)$$

and the distance

$$y = \mathbf{a}t + \frac{1}{2}\mathbf{a}t^2$$

$$\Rightarrow \quad y = \mathbf{v}_\parallel t + \frac{1}{2}\frac{q\mathbf{E}}{m}t^2 \quad ...(29)$$

To remove t we can use $x = \mathbf{v}_\perp t$, therefore

$$y = \frac{\mathbf{v}_\parallel}{\mathbf{v}_\perp}x + \frac{1}{2}\frac{q\mathbf{E}}{m}\left(\frac{x}{v_\perp}\right)^2 \quad ...(30)$$

1.8.2 Motion of Charged Particle in Crossed Electric and Magnetic Field or Electric and Magnetic Fields are Perpendicular to each other

Consider an electron moving with an initial velocity **v** entering in a crossed electric and magnetic field. The initial direction of velocity of the electron is along x axis, electric field **E** applied along y axis. The direction of electric lines of force is along negative y axis. Magnetic field of flux density **B** is applied along z axis, the direction of magnetic lines of force along z axis, the circle represents the area of uniform magnetic field (Fig. 1.6). A screen is placed at a distance $D(=OS)$ from the centre of electric and magnetic field.

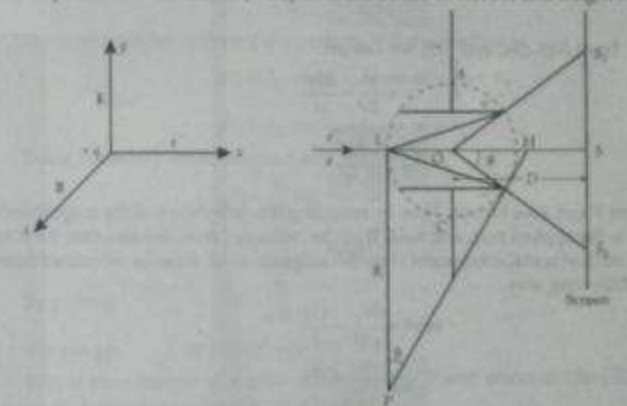

Fig. 1.6 Charged particle in crossed electric and magnetic field.

Within these crossed electric and magnetic fields if the electron is made to travel along x-direction then due to electric field alone, it will deflect upward along y-axis (Fig. 1.6). Whereas due to magnetic field alone it will deflect downward along y-axis. After crossing the magnetic field or electric field, electron will move tangentially to the arc at point of emergence. So if S is striking position of electron on the screen when there is no field, then due to electric field it will deflect by an amount SS_1 upward. Whereas due to magnetic field alone it will deflect by an amount SS_2 downward, just in the opposite direction. Now if force acting on the electron by electric and magnetic simultaneously are same, then there will not be any deflection of electron beam. Force acting on the electron due to electric field is eE. Whereas force acting due to magnetic field is Bev. If these two forces are equal then $SS_1 = SS_2$. So there will not be any deviation of the electron from its original path. In that condition

$$eE = Bev \quad ...(31)$$

i.e.,

$$v = E/B \quad ...(32)$$

But

$$E = \frac{V}{d} \quad ...(33)$$

where d is the distance between two plates and V the voltage applied between these two plates.

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So $$v = \frac{V}{Bd} \quad ...(34)$$

When the electron is moving in a transverse magnetic field then it will travel a circular path of radius R, then

$$\frac{mv^2}{R} = Bev$$

i.e., $$v = \frac{BRe}{m} \quad ...(35)$$

From Eqs. (34) and (35), we can get

$$v = \frac{V}{Bd} = \frac{BRe}{m}$$

$$\frac{e}{m} = \frac{V}{B^2 Rd} \quad ...(36)$$

where V and d are known. Now by recording the deflection θ of the magnetometer needle, due to the applied magnetic field, **B** can be calculated from the equation $B = H \tan\theta$, where H is the horizontal component of earth's magnetic field. R can be calculated from Fig. 1.6 in the following way

$$\tan\theta = \frac{SS_2}{OS} = \frac{LH}{R}$$

$$R = \frac{LH \times OS}{SS_2} \quad ...(37)$$

where OS is the distance between the centre of the deflecting plates and the screen. R is the radius of curvature of curved path due to magnetic field. LH is the distance between two magnets used to apply magnetic field. So, everything is known in Eq. (36), from which e/m can be determined. This principle is used in Thomson method for measuring e/m ratio for electron as discussed later in article 1.9.6.

Trajectory for Crossed Electric and Magnetic Fields

When electric and magnetic fields are perpendicular to each other, the motion is complex. Consider that **E** is along y-axis and **B** is along z-axis. The particle velocity is, say in xy plane (Fig. 1.7).

We can write equations of motion as

$$m\ddot{x} = q(\mathbf{v} \times \mathbf{B})_x = q\mathbf{B}\dot{y} \quad ...(38)$$

$$m\ddot{y} = q(\mathbf{v} \times \mathbf{B})_y + q\mathbf{E}_y = q\mathbf{E} - q\mathbf{B}\dot{x} \quad ...(39)$$

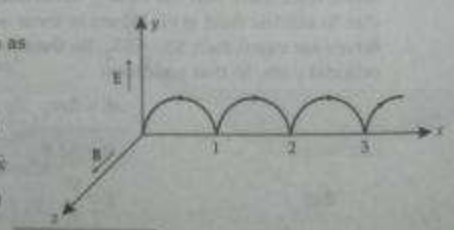

Fig. 1.7 Cycloid motion of charged particle.

where dot indicates time derivative and suffixes x, y indicate components along respecti directions. From $\left(\frac{q\mathbf{B}}{m}\right) = \omega$, we can write

$$\ddot{x} = \omega\dot{y} \quad ...(4$$

$$\ddot{y} = \frac{q\mathbf{E}}{m} - \omega\dot{x}$$

$$= \omega\left(\frac{\mathbf{E}}{\mathbf{B}} - \dot{x}\right) \quad ...(4$$

These are coupled differential equations and the solution is

$$x(t) = A_1 \cos\omega t + A_2 \sin\omega t + \frac{\mathbf{E}}{\mathbf{B}}t + A_3 \quad ...(4$$

$$y(t) = A_2 \cos\omega t - A_1 \sin\omega t + A_4 \quad ...(4$$

Using boundary conditions and solving, we get

$$x(t) = \frac{\mathbf{E}}{\omega\mathbf{B}}(\omega t - \sin\omega t) \quad ...(44$$

$$y(t) = \frac{\mathbf{E}}{\omega\mathbf{B}}(1 - \cos\omega t) \quad ...(45$$

By putting $$\mathbf{R} = \frac{\mathbf{E}}{\omega\mathbf{B}} \quad ...(46$$

We can get $$(x - R\omega t)^2 + (y - R)^2 = R^2 \quad ...(47$$

This is the equation of a circle having radius R and centre at $(R\omega t, R, 0)$, which is moving in x-direction with the constant speed

$$\mathbf{v} = \omega\mathbf{R} = \frac{\mathbf{E}}{\mathbf{B}} \quad ...(48)$$

Suppose the charged particle is at rest at the origin. The magnetic force is zero ; but the electric force accelerates the particle in y-direction. When it starts moving, a magnetic force is developed. This magnetic force pulls the particle in a circular path in the downward direction. The motion is now against the electric force, therefore, it slows down. The magnetic force also goes on decreasing. The particle comes at rest at point (Fig. 1.7). The entire process starts again which takes the particle to point 2 and it continues. This trajectory is called as **Cycloid**. The particle moves as if a point on the rim of wheel rolling down along x-axis.

If particle has initial velocity in xy-plane, then the resultant motion is superposition of the uniform motion along x-axis and the rotational motion in xy-plane. The trajectory is called as **trochoid**. As a special case, when initial velocity is zero the path is a **cycloid**.

Example 1.5. A proton is shot with a speed of 10×10^6 m/s along a line at an angle of 37° to the x-axis. If a uniform magnetic field of 0.2 Wb/m² exists parallel to the axis describe the motion of the proton.

Solution. Resolve the velocity into two components. In the x-direction the velocity is $v\cos 37° = v_x$ and along the y-direction $v\sin 37° = v_y$.

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Since the field is along x-direction parallel to v_x hence the particle will move with a constant speed

$$v_x = (10 \times 10^5)\cos 37° = 8 \times 10^5 \text{ m/s}$$

The y-component of velocity is perpendicular to the field, as a result the particle will describe a circle of radius given by the equation.

$$\frac{mv_y^2}{R} = qv_y B$$

or

$$R = \frac{mv_y}{qB}$$

$$= \frac{1.67399 \times 10^{-27} \times 10 \times 10^5 \sin 37°}{1.6 \times 10^{-19} \times 0.2} = 0.31 \text{ m}$$

it will take a time

$$t = \frac{2\pi R}{v_y}$$

$$= \frac{2 \times 3.14 \times 0.31}{10 \times 10^5 \times \sin 37°} = 32 \times 10^{-6} \text{ s}$$

for the proton to go around this circle. During this time it will have travelled with speed $v_x = 8 \times 10^5$ m/s parallel to x-axis, a distance

$$s = v_x t = 8 \times 10^5 \times 32 \times 10^{-6} = 2.6 \text{ m}$$

Hence the proton will follow a **helix** like path.

1.9 Applications of the Motion of Charged Particles in Electromagnetic Fields

1.9.1 Velocity Selector

In many experiments and instruments, a source of particles with a constant velocity is needed. This can be achieved by applying electric and magnetic fields, perpendicular to each other as shown in Fig. 1.8(a). When a positively charged particle enters the electric and magnetic fields, it experiences forces, due to the electric and magnetic fields in

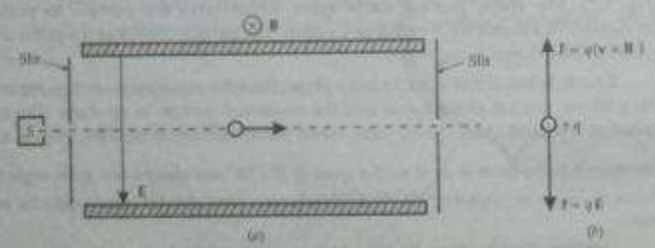

Fig. 1.8 (a) Velocity selector ; (b) Electric and magnetic force acting on a charged particle.

opposite directions, as shown in Fig. 1.8(b). When these two forces are balanced, the particle is unaffected and continues its motion in original direction

$$qvB = qE$$

$$\therefore \quad v = \frac{E}{B} \qquad \text{...(49)}$$

Thus particles having a velocity **v** (which is decided by **E** and **B**) will move in straight line and emerge from the slit. Particles with a velocity other than **v**, will move either up and down, depending upon the greater force. Thus, particles with a certain velocity can be selected, by changing **E** and/or **B**.

Example 1.6. *In a velocity selector, the electric field is 90 kV/m and the magnetic field is 0.4 T. Calculate the velocity of the undeflected particles.* **[GGSIPU Feb 2008 Reappear (2 Marks)]**

Solution. Given that, $E = 90$ kV/m $= 9.0 \times 10^4$ V/m, $B = 0.4$ T.

In a velocity selector, the electric and magnetic forces are balanced

i.e.,

$$qvB = qE$$

or

$$v = \frac{E}{B}$$

$$= \frac{90 \times 10^3}{0.4} = 2.25 \times 10^5 \text{ m/s}$$

$$v = 2.25 \times 10^5 \text{ m/s}$$

1.9.2 Cyclotron

The cyclotron is a heavy particle accelerator, particles such as protons, deutrons, or alpha particles are given multiple acceleration in resonance chamber. Referring to Fig. 1.9, two hollow D-shaped copper electrodes, or **dees**, are between the pole pieces of a large electromagnet. The region between the pole pieces is evacuated. The walls of the vacuum chamber are omitted. Two dees are connected to a high frequency source of alternating voltage. In the case of deutrons operation, ions in heavy hydrogen gas at low pressure are produced at the centre of the chamber by electrons from a filament.

Starting with the deutron at the point *P*, a negative potential on the right hand dee accelerates the deutron to the right. Entering the dee, the deutron is in a region free of electric field but still in the magnetic

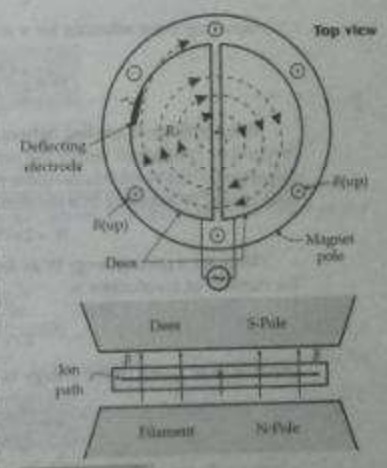

Fig. 1.9 Cyclotron.

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field between the pole pieces. Suppose that the lower pole is a north pole so that **B** is upward the duetron moves in a circle in a clockwise direction. If the timing is proper so that when the deutron again reaches the gap between the dees the electric field has reversed, it will be accelerated to the left. Having acquired additional energy, it moves in a circle of larger radius. By repetition of this process the energy, of the deutron is increased in steps until it reaches the periphery of the dees. Here a deflecting cathode at high negative potential pulls the deutron through an opening in the dees so that it can impinge on a specimen placed outside the dees.

The frequency in revolution per second of a particle moving normal to the magnetic field in the cyclotron is given by Eq. (24). That is

$$\nu = \frac{eB}{2\pi m} \qquad ...(50)$$

Provided that the particle velocity is small compared with light, m is substantially constant. Since e and **B** are also constant, ν is constant regardless of the velocity of the particles. However, from Eq. (23), the radius of the circular path is proportional to the particle velocity.

The final energy of a particle is determined by the radius R and the flux density **B**. From Eq. (23)

$$R = \frac{m\mathbf{v}}{e\mathbf{B}} \quad \text{or} \quad \mathbf{v} = \frac{Re\mathbf{B}}{m} \qquad ...(51)$$

The energy W of the particle is

$$W = \frac{1}{2}mv^2 \quad \text{or} \quad v = \sqrt{\frac{2W}{m}} \qquad ...(52)$$

Equating these relations for **v** and solving for the energy W of this particle

$$W = \frac{1}{2}\frac{(Re\mathbf{B})^2}{m} \qquad ...(53)$$

If the voltage V applied between the gap of the dees is small, a large number n of revolutions is required before the particle reaches the periphery. However, to reduce the dispersion of the beam it is desirable to make n small and, hence V as large as possible. The total energy W acquired in n revolutions is

$$W = 2nVe \qquad ...(54)$$

Thus for a given energy W as determined by Eq. (53) and for a particular voltage V, the number of revolutions is

$$n = \frac{W}{2eV} \qquad ...(55)$$

where W is in Joules. If the energy is expressed in electron volts W_{eV},

$$n = \frac{W_{eV}}{2eV} \times 1.6 \times 10^{-19} \qquad ...(56)$$

where e = charge of the particle (Coulomb) ; V = dees voltage (Volts)

Example 1.7. *A cyclotron accelerating proton has dees of 0.4 m radius and a radio-frequency supply of 12 MHz at 5000 V maximum value. Calculate (i) the strength of magnetic field necessary, (ii) the energy and velocity acquired by the protons ; (iii) the minimum of revolutions and time spent by a proton within the dees.*

[*Velocity of light* $(c) = 3 \times 10^8$ *m/s*, $m_p = 1.67 \times 10^{-27}$ kg, $q_p = 1.6 \times 10^{-19}$ C]

Solution. Given that, $R = 0.4$ m, $\nu = 12$ MHz $= 12 \times 10^7$ Hz, $V = 5000$ V, $c = 3 \times 10^8$ m/s, $m = 1.67 \times 10^{-27}$ kg and $q = 1.6 \times 10^{-19}$ C

(i) Let B be the magnetic field induction. The resonance condition for the cyclotron is

$$\nu = \frac{qB}{2\pi m}$$

Therefore $$B = \frac{2\pi m f}{q} = \frac{2 \times 3.14 \times 1.67 \times 10^{-27} \times 12 \times 10^8}{1.6 \times 10^{-19}}$$

$$B = 0.786 \text{ Wb/m}^2$$

(ii) The kinetic energy of a particle accelerated in a cyclotron is given by

$$W = \frac{q^2 B^2 R^2}{2m}, \text{ where } R \text{ is the dees radius}$$

Therefore, $$W = \frac{(1.6 \times 10^{-19})^2 \times (0.786)^2 \times (0.4)^2}{2 \times 1.67 \times 10^{-27}}$$

$$= 7.576 \times 10^{-13} \text{ Joules} = 4.73 \times 10^6 \text{ eV} = 4.73 \text{ MeV}.$$

If v is the velocity of the proton, then

$$K = W = \frac{1}{2}mv^2$$

or $$v = \sqrt{\frac{2W}{m}}$$

$$= \sqrt{\left[\frac{2 \times 7.576 \times 10^{-13}}{1.67 \times 10^{-27}}\right]} = \sqrt{(9 \times 10^{14})} = 3 \times 10^7 \text{ m/s}$$

(iii) Suppose the proton makes n revolutions to gain energy W. In each revolution it is pushed twice and each push gains energy qV, where V is the maximum value of the applied alternating voltage.

Thus $$n \times 2qV = W$$

Therefore $$n = \frac{W}{2qV} = \frac{7.576 \times 10^{-13}}{2 \times 1.6 \times 10^{-19} \times 5000} \approx 473$$

The time spent in one revolution is $\frac{2\pi m}{qB}$. At resonance it is $\frac{1}{\nu}$. Thus time required for 473 revolutions is

$$\frac{473}{\nu} = \frac{473}{12 \times 10^6} = 394 \times 10^{-6} \text{ s}$$

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1.9.3 Electrostatic Focusing

There is an analogy between the motion of charged particle in an electrostatic field and the propagation of light in a refractive medium. An electrostatic field can be represented by closely separated equipotential surfaces (Fig. 1.10). The potential difference between these equipotential surfaces is constant.

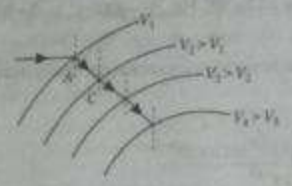

Fig. 1.10 Path of charged particle in an electric field.

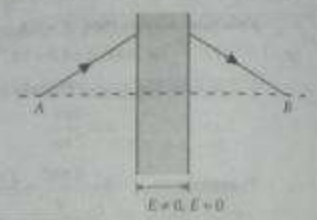

Fig. 1.11 A thin electron lens.

At the boundary at point A, the particle is accelerated and travels along the AC and this process continues. If the path is joined smoothly, it is curved. Thus, this technique of geometrical optics can be used to design electronic devices analogous to optical instruments. A thin electron lens can be formed as shown in Fig. 1.11. The dimensions of the lens *i.e.*, the region of electric field $E(E \neq 0)$, are small compared to the focal distances. The electron beam diverges from a point A and converges at a point C. The path inside the length is not shown in Fig. 1.11.

The geometrical optics of the electron beam is used in a cathode ray tube (CRT) for focusing the electron beam. Figs. 1.12 and 1.13 show the potential distributions for cylindrical electrodes of identical and different diameters respectively.

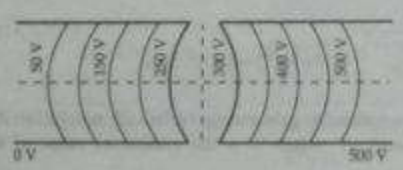

Fig. 1.12 Potential distribution for cylindrical electrodes with identical diameters.

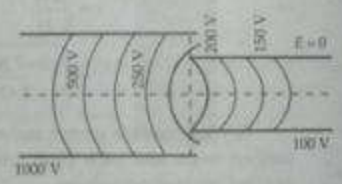

Fig. 1.13 Potential distribution for cylindrical electrodes with different diameters.

1.9.4 Magnetic Focusing

We have already seen that the motion of charged particles can be controlled by proper electric and magnetic fields. So it may also possible to focus the diverging beam of charged particles originating from the sources. These are essential features for some very important scientific equipments, such as cathode ray oscilloscope, mass spectroscope, Electron microscope etc. Now magnetic focusing is discussed below.

A uniform magnetic field has a focusing effect. Consider the charged particle in longitudinal magnetic field (Fig. 1.14).

Let us assume that a particle beam originates at O and a velocity v, makes a small angle θ at O. The particle performs helical motion with pitch

$$P = v_\parallel T$$

$$= v\cos\theta \frac{2\pi m}{eB} \quad ...(57)$$

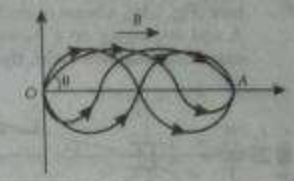

Fig. 1.14 Magnetic focusing longitudinal magnetic field.

For small angle θ

$$P = v\frac{2\pi m}{eB}\left(1-\frac{\theta^2}{2}\right) = \frac{2\pi m v}{eB} \quad ...(58)$$

When the angle of divergence is small, all particles are focused at point A. Now consider the motion of particles with the velocity perpendicular to the magnetic field originates at the point O (Fig. 1.15). If angular divergence is small, the beam is focused after a rotation of 180°. Suppose the apex angle of the beam is 2θ and the radius of the circular path of particle is R. We can write

$$AB = OB - OA = 2R - 2R\cos\theta$$

$$= 2R(1-\cos\theta)$$

$$= 4R\sin^2\theta/2 \quad ...(59)$$

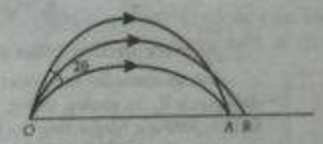

Fig. 1.15 Magnetic focusing transversed magnetic field

For small divergence angle of the beam, the focus width is

$$AB = R\theta^2 \quad ...(60)$$

The focusing effect of transverse uniform magnetic field is used in mass spectroscopy, electromagnetic separation of isotopes.

Example 1.8. A beam of charged particles is focused by 180° magnetic focusing method. The beam rotates in a circle of radius 30 cm. If the beam enters the slit with a divergence of angle 2° about the normal direction, find the spread of the focus point. **(GGSIPU Feb 2008 (2 Marks))**

Solution. Given R = radius of the circular path = 30 cm = 0.3 m

$$\theta = \text{divergence angle} = 2° = \frac{2\times\pi}{180}\text{ rad}$$

Then spread of the focus point

$$= R\theta^2$$

$$= 0.3\times\left(\frac{2\pi}{180}\right)^2 = \frac{0.3\times4\times3.14\times3.14}{180\times180}$$

$$= 0.000365\text{ m} = 3.65\times10^{-4}\text{ m}.$$

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1.9.5 The Cathode-Ray Tube

The cathode ray tube is a device for observing rapid voltage variations. In a cathode ray tube (Fig. 1.16), a beam of electrons is emitted from a cathode is accelerated by an electrode A and impinging on a fluorescent screen. By means of either a transverse electric or a transverse magnetic field, the beam may be deflected so that it strikes the screen at a distance y from the undeflected position. The spot on the screen is visible, and the particular usefulness of the cathode ray tube is that, because of the small inertia of electron beam, it can follow very rapid changes in the applied deflecting field. This is a somewhat over simplified discription of a cathode ray tube but will suffice for the brief analysis of some of its characteristics.

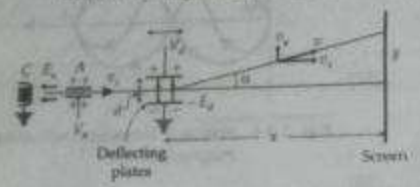

Fig. 1.16 Cathode ray tube with electrostatic deflection.

The positive accelerating voltage V_a is applied to the electrode A. This produces an accelerating field E_a that imparts a velocity v_x to the electrons. From Eq. (11),

$$v_x = \sqrt{\frac{2eV_a}{m}} \quad ...(62)$$

After the electron leaves the accelerating electrode, it maintains this velocity v_x.

Let us consider the effect of electrostatic deflection with two plates at a potential difference V_d as shown in Fig. 1.16. The path of an electron in the transverse deflecting field is *parabola*. Neglecting the fringing of the field at the edge of the plates, the electron is subjected to the deflecting field E_d for a distance l for a time $t = \frac{l}{v_x}$. The field E_d produces an acceleration a_y in the y-direction, which from Fig. 1.16 is

$$a_y = \frac{eV_d}{md} \quad ...(62)$$

Thus the electron acquires a velocity component v_y in the y-direction is given by

$$v_y = a_y \cdot t = \frac{eV_d l}{mv_x d} \quad ...(63)$$

The deflecting angle α [Fig. 1.16] is then

$$\alpha = \text{arc tan}\frac{v_y}{v_x} = \text{arc tan}\frac{eV_d l}{mv_x^2 d} \quad ...(64)$$

or

$$\alpha = \text{arc tan}\frac{V_d l}{2V_a d} \quad ...(65)$$

But from the tube geometry, assuming $x >> l$, the angle α is also given by

$$\alpha = \text{arc tan}\frac{y}{x} \quad ...(66)$$

and so, equating the argument in Eqs. (65) and (66)

$$y = \frac{V_d l x}{2V_a d} \text{ meters} \quad ...(67)$$

where y = deflection distance at the screen (meters)

V_d = deflecting potential (volts)

l = length of deflecting plates (meters)

x = distance from deflecting plate to screen (meters)

V_a = accelerating potential (volts)

d = spacing of deflecting plates (meters)

Solving for the volts per meter of deflection (ratio $\frac{V_d}{y}$), we have

$$\frac{V_d}{y} = \frac{2V_a d}{lx} \text{ volt/metre} \quad ...(68)$$

To increase the sensitivity, that is, to increase number of volts per meter of deflection, V_a and d should be decreased or an increase made in l or x.

Let us consider now the effect of a magnetic deflecting field. Suppose that the deflecting plates and electric field (Fig. 1.16) are replaced by a magnetic field of flux density B normal to the paper as in Fig. 1.17. The direction y is outward from this page. In this case the acceleration due to magnetic field is not in the y-direction but is normal to the circular path of the electron. Assume however, that l is so small that as an approximation the acceleration can be taken in the y-direction. Then the velocity component in the y-direction is

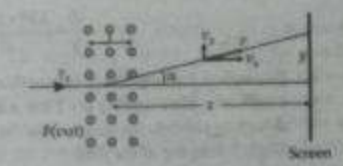

Fig. 1.17 Cathode ray tube with magnetic deflection.

$$v_y = a_y t = a_y \frac{l}{v_x} \quad ...(69)$$

Thus from equation $F = ma = ev_x B$

Then

$$v_y = \frac{ev_x B}{m}\frac{l}{v_x} = \frac{eBl}{m} \quad ...(70)$$

The deflecting angle α (Fig. 1.17) is

$$\alpha = \text{arc tan}\frac{v_y}{v_x} = \text{arc tan}\frac{eBl}{v_x m}$$

or

$$\alpha = \text{arc tan}\left(Bl\sqrt{\frac{e}{2mV_a}}\right) \quad ...(71)$$

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But we have also $\alpha = \text{arc}\tan\frac{y}{x}$, and so

$$y = x\,Bl\sqrt{\frac{e}{2mV_a}} \qquad ...(72)$$

where y = deflection distance at screen (metres)

x = distance from magnetic deflecting field to screen (meters)

B = flux density of deflecting field (weber/m^2)

e = charge on particle (Coulomb)

m = mass of particle (kg)

V_a = accelerating voltage (volts)

l = axial length of deflecting field (metres)

Solving for the flux density per metre deflection ratio (B/y), we have

$$\frac{B}{y} = \frac{1}{xl}\sqrt{\frac{2mV_a}{e}} \qquad ...(74)$$

where the ratio B/y is in weber per square metre per metre deflection. For an electron Eq. (74) becomes

$$\frac{B}{y} = \frac{3.38\times10^{-6}}{xl}\sqrt{V_a} \qquad ...(75)$$

Example 1.9 *A cathode ray tube with electrostatic deflection has an accelerating voltage V_a = 1500 V, a deflecting plate spacing d = 1 cm, a deflecting plate length l = 1 cm, and a distance x = 30 cm from deflecting plates to the screen. Find the voltage V_d required to deflect the spot by 1 cm on the screen. Neglect fringing of the field.*

Solution. From the relation, $\frac{V_d}{y} = \frac{2V_d d}{lx}$ V/m

$$= \frac{2\times1500\times1\times10^{-2}}{1\times10^{-2}\times30\times10^{-2}} = 10^4 \text{ V/m.}$$

Example 1.10 *A cathode ray tube with magnetic deflection has an accelerating voltage V_a = 1500 V, a magnetic deflecting field axial length l = 2 cm, and a distance x = 30 cm from the deflecting field to the screen. Find the magnetic field flux density B required to deflect the spot of an electron beam 1 cm on the screen.*

Solution. From the relation,

$$\frac{B}{y} = \frac{3.38\times10^{-6}}{xl}(V_a)^{1/2}$$

We have,

$$\frac{B}{y} = \frac{3.38\times10^{-6}\times(1500)^{1/2}}{30\times10^{-2}\times2\times10^{-2}} = 2.18\times10^{-2} \text{ Wb/m}^2\text{/m}$$

$$= 2.18\times10^{-4} \text{ Wb/m}^2\text{/cm.}$$

1.9.6 Thomson's Method for Measuring e/m for Electron

A discharge is struck in a rarefied gas in a glass cathode ray tube (CRT) G between the cathode C and anode A by means of source of a high voltage as shown in Fig. 1.18. C may be small aluminium disc, while A is brass cylinder penetrated with a fine bore along its axis. The cathode rays strike the anode A and some of them emerge through long, narrow hole in A as a fine, well collimated beam. The cathode ray beam, then continues its forward journey in the field free space with a uniform velocity v determined by the potential difference V between cathode C and anode A till it strikes normally the fluroscent screen S of zinc sulphide or barium platinum cyanide kept at the end of the tube.

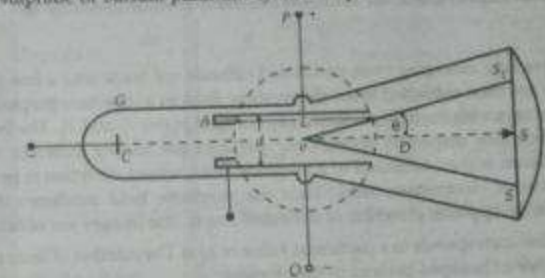

Fig. 1.18 Thomson's method for measuring e/m for electron.

P and Q are two parallel plates of length L between which electric field is applied. These plates are separated by a distance d. On applying electric field along y-axis perpendicular to the motion of the electron (along x-axis) the deflection of the spot occurs, which is marked by S_1. Now, by placing the CRT between two pole pieces of a magnet, a downward magnetic field can be applied simultaneously along z-axis, perpendicular to the motion of electron (along x-axis), as well as perpendicular to the direction of electric field (y-axis). The dashed circular area shown in Fig. 1.18 represents the region of uniform magnetic field. As stated earlier the direction of deflection will be downward. The crossed electric and magnetic fields are acting on the electron beam simultaneously. This will cancel each other's effect and the spot will come back to its original position S. Using results of article 1.7, one calculate e/m ratio of electron.

Example 1.11 *A particle moving with the speed 5×10^7 m/s in the magnetic field of 0.4 T. Describe a circle of radius 0.711 mm. Calculate the charge to mass ratio of the particle.*

Solution. Given that, $v = 5\times10^7$ m/s, $B = 0.4$ T and $R = 0.711$ mm $= 0.711\times10^{-3}$ m.

$$\frac{q}{m} = \frac{v}{BR} = \frac{5\times10^7}{0.4\times0.711\times10^{-3}}$$

$$= 17.58\times10^{10} \text{ C/kg.}$$

1.9.7 Mass Spectrograph

This is an instrument or device to measure the relative abundance and masses of isotopes present in the sample. Most of the measurements are taken with slightly charged positive ions. The spectrum is divided according to masses. When the ion beam is detected with the

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help of a photographic plate is called **mass spectrograph**. It is generally used for the precise measurement of the relative masses of the ions and for the determination of the masses of isotopes. When the beam is detected with the help of an electrometer, it is called a **mass spectrometer**. It is generally used for precise measurement of the relative abundance of ions.

1. Aston's Mass Spectrograph

This is an apparatus of high accuracy designed by F.W. Aston in 1919, which enables the measurement of mass of single atomic ions and is useful for investigation of isotopes. This method is an improvement on Thomson's method of mass spectrograph.

Principle

The positive rays emerging from perforated cathode are made into a fine pencil by using slits. They are then subjected to an electrostatic field in a direction perpendicular to the direction of rays with the help of electrically charged plates P_1 and P_2. The beam is not only deflected but also dispersed because the particles are having different velocities. The dispersed beam is then subjected to a magnetic field, whose direction is perpendicular to the direction of electrostatic field. Thus the magnetic field produces dispersion and deviation in an opposite direction of deflected beam, line images are obtained.

Each line corresponds to a particular value of q/m. The number of lines corresponding to the number of isotopes present in the element.

Theory

The different parts of the Aston's mass spectrograph are shown in Fig. 1.19. AO is the direction of positive rays before entering the electrostatic field is maintained by plates P_1 and P_2. The beam is deflected and dispersed downwards. Let θ and $d\theta$ be the angle of **deviation** and **dispersion**. Using a diaphragm D, some of the rays are selected and are allowed to pass between the poles of an electromagnet. The magnetic field being perpendicular to the plane of paper and inward. According to Fleming's left hand rule, the beam will be deflected upwards. This magnetic field annuls the dispersion produced by electric field and recombines the particles which are brought to focus in the form of sharp lines on a photographic plate CD. The lines are similar to those of spectral lines.

Let q' = charge on positive ray particle

m' = mass of each particle

E = electrostatic field

B = magnetic field strength

v = velocity of each particle

ϕ = angle of deviation produced by magnetic field

$d\phi$ = angle of dispersion produced by magnetic field.

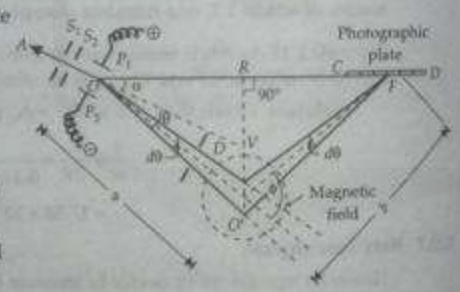

Fig. 1.19 Aston's mass spectrograph.

Considering that the deflection in electrostatic field is small, the curve near the vertex may be considered as circular of radius r, we have

$$Eq' = \frac{m'v^2}{r} \quad \text{or} \quad \frac{1}{r} = \frac{Eq'}{m'v^2} \qquad ...(76)$$

Hence the deflection θ, which is proportional to $\frac{1}{r}$ is given by

$$\theta = k\frac{Eq'}{m'v^2} = k_1\frac{q'}{m'v^2} \quad [\text{where } k_1 = kE, \text{ because } E \text{ and } k = \text{constant}]$$

$$\therefore \text{ Dispersion } \quad \frac{d\theta}{dv} = -2k_1\frac{q'}{m'v^3} = -2\frac{\theta}{v} \qquad ...(77)$$

If r' is the radius of curvature in magnetic field, then

$$Bq'v = \frac{m'v^2}{r'} \text{ or } \frac{1}{r'} = \frac{Bq'}{m'v}$$

$$\therefore \quad \phi = k'\frac{Bq'}{m'v} = k_2\frac{q'}{m'v} \quad [\text{where } k_2 = k'B, \text{ because } B \text{ and } k' = \text{constant}]$$

$$\text{Again dispersion} \quad \frac{d\phi}{dv} = -k_2\frac{q'}{m'v^2} = -\frac{\phi}{v} \qquad ...(78)$$

From Eqs. (77) and (78), one achieves

$$\frac{d\theta}{\theta} = 2\frac{d\phi}{\phi} \qquad ...(79)$$

Obviously, for a given deflection, the dispersion due to the electric field is twice that due to magnetic field. The small changes $d\theta$ and $d\phi$ refer to the particles with identical mass and charge but possessing velocities differing by dv.

In the absence of magnetic field, by dispersion produced in the beam for a distance $(a+b)$ is given by

$$= (a+b)d\theta \qquad ...(79)$$

where a = distance OO' and b = distance $O'F$

The magnetic field acts in a direction perpendicular to the electric field and produces the same dispersion in a distance b but in the opposite direction.

Dispersion produced by the magnetic field $= bd\phi$

As all the ions are focus to the same position

$$(a+b)d\theta = bd\phi$$

$$\text{and} \quad \frac{d\theta}{d\phi} = \frac{b}{(a+b)} \qquad ...(80)$$

$$\text{From Eq. (78),} \quad \frac{d\theta}{d\phi} = \frac{2\theta}{\phi}$$

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$$\frac{b}{(a+b)} = \frac{2\theta}{\phi}$$

or $$b\phi = (a+b)2\theta$$

or $$b(\phi - 2\theta) = 2a\theta \qquad ...(81)$$

This is condition of focusing.

Let $O'R$ be the perpendicular to the line CD produced and $\angle ROV = \alpha$. Then from $\Delta ROO'$, we have

$$RO' = OO' \sin(\alpha + \theta) = a \sin(\alpha + \theta)$$

In $\Delta RO'F$, $$RO' = O'F \sin(RFO')$$
$$= b \sin[180 - (\phi - \alpha - \theta)]$$
$$= b \sin(\phi - \alpha - \theta)$$

$$\therefore \quad a \sin(\alpha + \phi) = b \sin(\phi - \alpha - \theta) \qquad ...(82)$$

For small angles $a(\alpha + \theta) = b(\phi - \alpha - \theta)$

Comparing Eqs. (81) and (82), it is observed that two equations, are same when $\alpha = \theta$. Thus the focusing condition that the photographic plate must be placed at an angle θ with the direction of the incident positive ray beam.

Thus, we find that in Aston's apparatus,

(i) all particles of the same q/m are brought to the same focus irrespective of the velocities.

(ii) particles of different masses are brought from different foci.

Example 1.12 *A mass spectrometer is being used to monitor air pollutants. It is difficult to separate molecules with nearly equal mass such as CO (28.0106 u) and N_2 (28.0134 u). How large a radius of curvature must a spectrometer have to separate these molecules by 0.20 mm on the film ?*

Solution. Given that, $\Delta m = m_1 - m_2 = (28.0134 - 28.0106)\,u = 0.0028\,u$

$$\Delta d = 0.20 \text{ mm} = 0.20 \times 10^{-3} \text{ m}.$$

Two molecules of masses m_1 and m_2 follow the radii R_1 and R_2 respectively in the spectrometer. The two lines on the film are separated by a distance Δd given by

$$\Delta d = 2(R_1 - R_2) = 2\,\Delta R$$

In a spectrometer, the relation $\frac{mv^2}{R} = qvB$ and $\frac{\Delta R}{R} = \frac{\Delta m}{m}$

or $$R = \frac{\Delta R}{\Delta m} m = \frac{\Delta d \cdot m}{2\Delta m} = \frac{0.20 \times 10^{-3} \times 28.0120}{2 \times 0.0028}$$

Thus the spectrometer must have a radius of curvature of 1 m to separate the given molecules by 0.2 mm on the film.

2. Bainbridge's Mass Spectrograph

This is an elegant and advanced mass spectrograph for determining the atomic masses. A schematic diagram of Bainbridge's mass spectrograph is shown in Fig. 1.20. In it, a velocity selector is used to obtain ions of particular velocity v. The velocity selector allows a beam

of a positive ions having the same velocity v to pass undeviated through crossed electric and magnetic fields E and B. In this case forces due to these fields are equal and opposite.

i.e., Force due to electric field (qE) = Force due to magnetic field (Bqv)

or $$v = \frac{E}{B} \qquad ...(83)$$

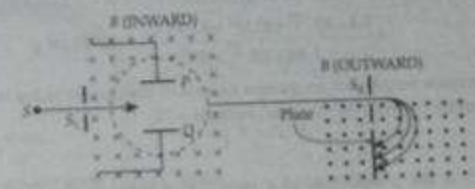

Fig. 1.20 Bainbridge's mass spectrograph.

All the ions having the same velocity $\left(v = \frac{E}{B}\right)$ enter the analysing chamber through the slit S_2. In this chamber another magnetic field B' is applied perpendicular to the plane of paper and in outward direction. Due to this, ions of different masses move in circles of different radii such that

$$\text{Magnetic force } qvB' = \text{centrifugal force } \frac{mv^2}{R}$$

or $$\frac{q}{m} = \frac{v}{B'R} = \frac{E}{BB'R} \qquad ...(84)$$

Thus we see the ions of different charges will follow different circular paths. Radius R of circular orbit is directly proportional to the mass m. The value of R is obtained from the position of the line on the photographic plate and hence $\frac{q}{m}$ is calculated using the above relation.

Example 1.13 *In a Bainbridge mass spectrograph the electric field between the plates of a velocity selector is 1000 V/cm and the magnetic induction in both magnetic fields is 0.5 Wb/m^2. A stream of singly charged neon ions moves in a circularly path of 8.74 cm radius in the magnetic field. Find the mass number of the neon isotope.*

Solution. Since the neon ions are singly charged, therefore

$$\text{charge } q = \text{proton charge} = 1.6 \times 10^{-19} \text{ C}.$$

Magnetic fields are equal, therefore,

$$B = B' = 0.5 \text{ Wb/m}^2$$

Electric field $E = 1000$ V/cm $= 100000$ V/m

Radius of the circular path

$$R = 8.74 \text{ cm} = 8.74 \times 10^{-2} \text{ m}.$$

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Substituting these values in the relation for q/m of an ion in Bainbridge mass spectrograph, we get

$$\frac{q}{m} = \frac{E}{BB'R} \quad \Rightarrow \quad m = \frac{qBB'R}{E}$$

$$m = \frac{1.6\times10^{-19}\times(0.5)^2\times8.74\times10^{-2}}{10^5}\ \text{kg}$$

$$= \frac{1.6\times10^{-19}\times(0.5)^2\times8.74\times10^{-2}}{1.66\times10^{-27}\times10^5}\ u = 21.06\ u$$

Since the mass number of an integer closest to the atomic mass in atomic mass scale, therefore the mass number of the neon isotope = 21.

1.9.8 The Hall Effect

The Hall effect is the production of a potential difference (the Hall voltage) across an electrical conductor in which an electric current flows in the presence of a magnetic field. Edwin Hall discovered this effect in 1879. The conductivity measurements are not sufficient for the determination of the number of conducting charges N and their mobility μ. Moreover, these measurements do not give any information about the sign of the prominant charge carrier. The Hall effect supplies the information of the sign of charge carrier.

DEFINITION

When a magnetic field is applied perpendicular to a conductor carrying current, a voltage is developed across the specimen in the direction perpendicular to both the current and magnetic field. This phenomenon is known as Hall effect.

Consider that an external electric field is applied along the axis of a specimen, then the electrons will drift in opposite direction. Again let a magnetic field be applied perpendicular to the one side. Of course, the electrons will not drift into space but a transverse electric field which causes a compensating drift such that the carriers remain in the specimen. The effect is known as **Hall effect**. *The Hall effect is thus observed when a magnetic field is applied at right angle to a conductor carrying a current.*

Consider a slab of material subjected to an external electric field E_x, along the x-direction and a magnetic field B_z along the z-direction as shown in Fig. 1.21. Due to the electric field a current density J_x will flow in the direction of E_x. Let us consider the case in which the current is carried by electrons of charge $-e$.

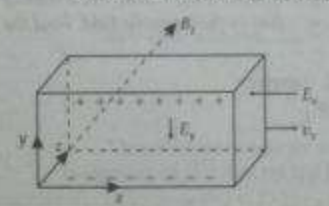

Fig. 1.21 The geometry of the electric and magnetic fields for a simple Hall effect calculation.

Under the influence of the magnetic field, the electron will be subjected to a **Lorentz force** such that the upper surface collects a positive charge, while the lower surface a negative charge. The accumulation of charge on the surface of the specimen continues until the force on moving charges due to the electric field associated with it is large enough to cancel the force exerted by the magnetic field. Ultimately a stationary state is reached when the current along y-axis vanishes, an electric field E_y is set up. If the charge carriers are holes then the case will be reversed *i.e.*, the upper surface would become

negative, while the lower surface as positive. Thus about the sign of charge carriers may be obtained. In this way, the measurement of Hall voltage gives the information about the charge carriers.

Hall Voltage and Hall Coefficient

The electric force on the electron having a charge $-e$ is $-eE$ and the force due to magnetic field B is $-\frac{e}{C}(\mathbf{v}\times\mathbf{B})$. Due to combined effect of electric and magnetic fields, the total force is given by

$$\mathbf{F} = -e\mathbf{E} - \left(-\frac{e}{C}(\mathbf{v}\times\mathbf{B})\right) \quad \text{...(84)}$$

In our case,

$$F_y = -eE_y + \frac{e}{C}v_x B_z \quad \text{...(85)}$$

In ths steady state $F_y = 0$, hence

$$0 = -eE_y + \frac{e}{C}v_x B_z$$

or

$$E_y = \frac{1}{C}v_x B_z \quad \text{...(86)}$$

This E_y is called as **Hall field**. It is also equal to V_y. The V_y is known as **Hall voltag** which is the potential difference across specimen along y direction (Fig. 1.21).

Now the current density is given by

$$J_x = N(-e)v_x$$

or

$$v_x = -\frac{J_x}{Ne} \quad \text{...(8}$$

From Eq. (87) substituting the value of v_x in Eq. (86), we have

$$E_y = -\frac{1}{C}\frac{J_x}{Ne}B_z$$

or

$$\frac{E_y}{J_x B_z} = \frac{-1}{CNe} = R_H \quad \text{...(}$$

$$\therefore \quad R_H = -\frac{1}{CNe} \quad \text{e.m.u.}$$

R_H is known as **Hall coefficient**. The Eq. (88) for Hall coefficient is in e.m.u. hence the value of the Hall coefficient in e.s.u. is given by

$$R_H = -\frac{1}{Ne} \quad \text{e.s.u.}$$

The Eq. (88) or (89) shows that the sign of Hall constant is the same as the sig carrier. Had our calculation been based upon holes, the sign of charge entering Lorentz force would have been positive and the corresponding Hall cons positive.

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Mobility and Hall Angle

The mobility (μ) is defined as the velocity acquired by the current carrying particles per unit electric field *i.e.*,

$$\mu = \frac{v_x}{E_x}$$

or

$$v_x = \mu E_x \quad \text{...(90)}$$

Substituting this value in Eq. (86)

$$E_y = \frac{1}{c}\mu E_x B_z \quad \text{...(91)}$$

Comparing Eqs. (91) and (88), we get

$$R_H \cdot I_x \cdot B_z = \frac{1}{c}\mu E_x B_z$$

or

$$\mu = \frac{R_H \cdot I_x}{E_x} \cdot C$$

$$\mu = R_H \cdot \sigma \cdot C \qquad \left[\because \frac{I_x}{E_x} = \sigma \Rightarrow I_x = \sigma E_x\right] \quad \text{...(92)}$$

In e.s.u.

$$\mu = R_H \cdot \sigma$$

$$= \frac{E_y \sigma}{I_x B_z}$$

$$= \frac{E_y \sigma}{\sigma E_x B_z} = \tan\phi \frac{1}{B_z}$$

where $\tan\phi = \frac{E_y}{E_x}$ is called Hall angle.

$$\therefore \quad \tan\phi = \mu B_z \Rightarrow \phi = \tan^{-1}(\mu B_z) \quad \text{...(93)}$$

Importance of Hall Effect

The measurement of the Hall effect gives the following important quantities :

(i) The sign of the current carrying charges is determined.

(ii) From the magnitude of Hall coefficient, the number of charge carriers per unit volume can be calculated.

(iii) The mobility is measured directly.

(iv) It can be used to decide whether a material is metal, semiconductor or insulator.

NOTE

Not all the metals have a negative Hall coefficient but some metals have a positive Hall coefficient (*i.e.*, charge carriers are holes) and if both holes and electrons contribute to conductivity, the R_H can be positive or negative depending upon the relative densities and mobilities of the carriers.

Example 1.14 *A semiconducting crystal 12 mm × 5 mm × 1 mm has a magnetic flux density of 0.5 T applied from front to back perpendicular to the largest faces. When 20 mA current flows lengthwise, the voltage measured across its width is 37 μV. Determine the Hall coefficient of this semiconductor.*

Solution. Given that $E_y = 37 \times 10^{-6}$ V

$b = 5$ mm $= 5 \times 10^{-3}$ m, $c = 1$ mm $= 10^{-3}$ m

$B_z = 0.5$ T

The Hall coefficient

$$R_H = \frac{1}{ne} = \frac{E_y}{B_z I_x} = \frac{E_y \cdot (b \times c)}{B_z}$$

$$R_H = \frac{37 \times 10^{-6} \times 5 \times 10^{-3} \times 10^{-3}}{0.5}$$

$$= 37 \times 10^{-6}\ \text{m}^3\text{C}^{-1}$$

Formulae at a Glance

1. Electric Field $\mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0}$

2. Magnetic Field

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

$$= qvB\sin\theta$$

or $\mathbf{B} = \frac{\mathbf{F}}{q\mathbf{v}}$

3. Lorentz Force

$$\mathbf{F}_e = q\mathbf{E}, \quad \mathbf{F}_m = q(\mathbf{v} \times \mathbf{B})$$

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

4. Motion of charged particle in a parallel electric field

$$a_x = \frac{eE}{m} = \frac{e}{m}\frac{V}{d}$$

Terminal velocity $v_x = \sqrt{\frac{2eV}{m}}$

5. Motion of charged particles in transverse electric field :

(i) $v_x = \frac{e}{m}\frac{V}{d}\frac{x}{v_x}$

(ii) Distance $y = \frac{1}{2}a_y t^2 = \frac{1}{2}a_y \frac{x^2}{v_x^2}$

(iii) Acceleration

$$a_y = \frac{e}{m}E_y = \frac{e}{m}\frac{V}{d}$$

(iv) Resultant velocity $v = \sqrt{v_x^2 + v_y^2}$

(v) $\tan\theta = \frac{v_y}{v_x} = \frac{eV}{mdv} = \frac{x}{v_x^2}$

(vi) $y = \frac{1}{2}\left(\frac{eE_y}{m}\right)\left(\frac{L}{v_x}\right)^2$

6. Motion of electron in transversed magnetic field :

(i) Magnetic force $F_m = Bev = \frac{mv^2}{R}$

(ii) Radius $R = \frac{mv}{eB}$

(iii) $\frac{e}{m} = \frac{v}{RB}$

(iv) Time period of complete one revolution

$$T = \frac{2\pi R}{v} = \frac{2\pi Rm}{BeR} = \frac{2\pi}{B}\left(\frac{m}{e}\right)$$

(v) Resonant frequency

$$f = \frac{1}{T} = \frac{Be}{2\pi m}$$

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7. Particle motion in combined fields

(a) Electric and magnetic fields are parallel and antiparallel

(i) Original velocity $\mathbf{v}_0$, then after time t it is

$$\mathbf{v}_t' = \mathbf{v}_0' + \frac{q\mathbf{E}}{m}t$$

(ii) Distance $y = \mathbf{v}_0 t + \frac{1}{2}\frac{q\mathbf{E}}{m}t^2$

$$= \frac{\mathbf{v}_y}{\mathbf{v}_x}x + \frac{1}{2}\frac{q\mathbf{E}}{\mathbf{m}}\left(\frac{x}{v_x}\right)^2$$

(b) Electric and magnetic fields are perpendicular to each other.

(i) $eE = Bev$ i.e., $v = \frac{E}{B}$

(ii) $E = \frac{V}{d}$

(iii) $v = \frac{V}{Bd}$

(iv) $v = \frac{BeR}{m}$

(v) $\frac{e}{m} = \frac{V}{B^2Rd}$

(vi) Trajectory

$$m\ddot{x} = q(\mathbf{v}\times\mathbf{B})_x = q\mathbf{B}\cdot\dot{y}$$

$$m\ddot{y} = q(\mathbf{v}\times\mathbf{B})_y + qE_y = -q\mathbf{E} - q\mathbf{B}\dot{x}$$

$$(x - R\omega t)^2 + (y - R)^2 = R^2 \quad \text{[cycloid]}$$

8. For velocity selector

(i) $q\mathbf{v}\mathbf{B} = q\mathbf{E}$

or $v = \frac{\mathbf{E}}{\mathbf{B}}$

9. For cyclotron

(i) Frequency of a particle moving normal to the magnetic field

$$f = \frac{e\mathbf{B}}{2\pi m}$$

(ii) The energy W of the particle is

$$W = \frac{1}{2}\frac{(Re\mathbf{B})^2}{m}$$

(iii) If n is number of revolutions

$$W = 2nVe$$

$$n = \frac{W_{eV}}{2eV} \times 1.6 \times 10^{-19}$$

10. Magnetic focusing

Pitch $P = v_\parallel T = v\cos\theta\frac{2\pi m}{eB}$

The focus width $= R\theta^2$

For small angle θ,

$$P = v\frac{2\pi m}{eB}\left(1 - \frac{\theta^2}{2}\right)$$

$$= \frac{2\pi m v}{eB}$$

11. Aston's mass spectrograph

(i) $\frac{1}{r} = \frac{Eq'}{m'v^2}$

(ii) $\theta = k_1\frac{q'}{m'v^2}$

[where $k_1 = kE$, because E = constant]

$$\frac{d\theta}{dv} = -2\frac{\theta}{v}$$

(iii) $\phi = k_2\frac{q'}{m'v}$

$$\frac{d\phi}{dv} = -\frac{\phi}{v}$$

(iv) $\frac{d\theta}{\theta} = 2\frac{d\phi}{\phi}$

(v) $b(\phi - 2\theta) = 2a\theta$

(vi) $a\sin(\alpha + \theta) = b\sin(\phi - \alpha - \theta)$

(vii) $a(\alpha + \theta) = b(\phi - \alpha - \theta)$

12. The Hall effect

(i) Hall coefficient

$$R_H = -\frac{1}{CNe} \text{ [in e.m.u.]}$$

$$R_H = -\frac{1}{Ne} \text{ [in e.s.u.]}$$

(ii) Mobility

$$\mu = R_H \cdot \sigma \cdot C \text{ (in e.m.u.)} \qquad \left[\because \frac{J_x}{E_x} = \sigma\right]$$

(iii) $\tan\phi = \mu B_z$

or $\phi = \tan^{-1}(\mu B_z)$

Miscellaneous Numerical Problems for Practice

Example 1. *An electron has a velocity of 10^4 m/s normal to a magnetic field of 0.1 Wb/m^2 flux density. Find the radius of the electron path and also its frequency.*

Solution. Given that $v = 10^4$ m/s, $B = 0.1$ Wb/m^2, $e = 1.6 \times 10^{-19}$ C and $m = 9.1 \times 10^{-31}$ kg.

From the relation for the radius

$$R = \frac{mv}{eB} = \frac{9.1 \times 10^{-31} \times 10^4}{1.6 \times 10^{-19} \times 0.1} = 5.7 \times 10^{-7} \text{ m}$$

This is very small circle. The frequency

$$\nu = \frac{v}{2\pi R} = \frac{eB}{2\pi m} = \frac{10^4}{2\pi \times 5.7 \times 10^{-7}} = 2.8 \times 10^9 \text{ Hz}.$$

Example 2. *An electron beam passes through a magnetic field of 2×10^{-3} Wb/m^2 and an electric field of 3.4×10^4 V/m, both acting simultaneously at the same point. If the path of electron remains undeviated, calculate the speed of electrons. If the electric field be removed, what will be the radius of electron path ? [Given, mass of electron $= 9.0 \times 10^{-31}$ kg and charge of electron $= 1.6 \times 10^{-19}$ C].*

Solution. Given that, $B = 2 \times 10^{-3}$ Wb/m^2, $E = 3.4 \times 10^4$ V/m,

$m = 9.0 \times 10^{-31}$ kg and $e = 1.6 \times 10^{-19}$ C.

At equilibrium, Force due to electric field (eE) = Force due to magnetic field (Bev)

or $$v = \frac{E}{B} = \frac{3.4 \times 10^4}{2 \times 10^{-3}} = 1.7 \times 10^7 \text{ m/s}$$

On removing the electric field,

Force due to magnetic field (Bev) = Centrifugal force ($\frac{mv^2}{R}$)

or $$R = \frac{9.0 \times 10^{-31} \times 1.7 \times 10^7}{2 \times 10^{-3} \times 1.6 \times 10^{-19}} = 4.78 \times 10^{-2} \text{ m}.$$

Example 3. *An electric field of 100 V/m is applied to a sample of n-type semiconductor, whose Hall coefficient is -0.0125 m^3/C. Determine the current density in the sample assuming $\mu = 0.36$ m^{2}V^{-1}s^{-1}.*

Solution. Given that, $E = 100$ V/m, $R_H = -0.0125$ m^3/C and mobility $\mu = 0.36$ m^{2}V^{-1}s^{-1}.

As Hall coefficient is given by

$$R_H = -\frac{1}{Ne}$$

$$N = -\frac{1}{R_H e} = -\frac{1}{(-0.0125)(1.6 \times 10^{-19})} = 5 \times 10^{20}/\text{m}^3$$

Again, $\sigma = Ne\mu$ or $\sigma = \frac{J_x}{E_y}$ or $J_x = \sigma E_y$

$$J_x = \sigma E_y = Ne\mu E_y$$

$$= 5 \times 10^{20} \times 1.6 \times 10^{-19} \times 0.36 \times 100 = 2880 \text{ A/m}^2$$

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Example 4. *The resistivity of a semiconductor was known to be 0.00893 Ωm at room temperature. The flux B in the Hall model was 0.5 Wb/m². Compute the Hall angle for a Hall coefficient of 3.66×10^{-4} m³/C.*

Solution. Given that $\rho = 0.00893\ \Omega m$, $B_z = 0.5\ Wb/m^2$, $R_H = 3.66 \times 10^{-4}\ m^3/C$, $\phi = ?$

As Hall angle $\tan\phi = \mu B_z$

$$\phi = \tan^{-1}(\mu B_z)$$

where $\mu = R_H \sigma$

$$\Rightarrow \quad \mu = \frac{R_H}{\rho} = \frac{3.66 \times 10^{-4}}{0.00893} = 0.041\ m^2 V^{-1} s^{-1}$$

$$\phi = \tan^{-1}(0.041 \times 0.5) = \tan^{-1}(0.0205)$$

$$\phi = 1.1744°$$

Example 5. *A stream of protons and deutrons in a vacuum chamber enters a uniform magnetic field. Both protons and deutrons have been subjected to the same accelerating potential, hence the kinetic energies of the particles are the same. If the magnetic field is perpendicular and the protons move in a circular path of radius 15 cm, find the radius of the path transversed by the deutrons.*

Solution. If m_1 and m_2 be the masses of proton and deutron respectively and v_1 and v_2 are their velocities, then

$$\text{Kinetic energy} = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2 \qquad ...(i)$$

If R_1 and R_2 be the radii of the paths of proton and deutron respectively in the kinetic field B then

$$\text{For proton} \quad \frac{m_1 v_1^2}{R_1} = Bqv_1 \quad \text{or} \quad v_1 = \frac{BqR_1}{m_1} \qquad ...(ii)$$

$$\text{and for deutron} \quad \frac{m_2 v_2^2}{R_2} = Bqv_2 \quad \text{or} \quad v_2 = \frac{BqR_2}{m_2} \qquad ...(iii)$$

Substituting these values in Eq. (i), we get

$$\frac{1}{2} m_1 (BqR_1/m_1)^2 = \frac{1}{2} m_2 (BqR_2/m_2)^2$$

or

$$\frac{R_1^2}{m_1} = \frac{R_2^2}{m_2}$$

As deutron is an isotope of hydrogen and consists of a proton and a neutron, hence $m_2 = 2m_1$. Therefore, we have

$$R_2 = R_1 \left(\frac{m_2}{m_1}\right)^{1/2} = 0.15(2)^{1/2}$$

$$R_2 = 0.212\ m$$

Example 6. *An electron is accelerated through a potential difference of 18 kV in a colour TV cathode ray tube. Calculate the kinetic energy and speed of the electron.*

Solution. Given that, $V = 18\ kV$, $e = 1.6 \times 10^{-19}$ C and $m = 9.1 \times 10^{-31}$ kg.

$$K.E. = eV = \frac{1}{2} mv^2$$

$$v = \sqrt{\left(\frac{2eV}{m}\right)} = \sqrt{\left(\frac{2 \times 1.6 \times 10^{-19} \times 18 \times 10^3}{9.1 \times 10^{-31}}\right)} = 7.95 \times 10^7\ m/s.$$

Example 7. *A proton with velocity 5×10^5 m/s enters an electric field of $-500j$ N/C at the origin making an angle of 42° with x axis. Determine the time required for the proton to reach the maximum height in electric field.*

Solution. Given that, $v = 5 \times 10^5$ m/s, $E = -500\ j$ N/C, $\theta = 42°$.

$$v = 0, \quad a = \frac{qE}{m} \text{ and } v = v_y = v \sin 42°$$

$$v = u + at, \quad \text{thus } t = -\frac{u}{a} = -\frac{um}{qE}$$

$$t = \frac{um}{eE} = \frac{5 \times 10^5 \times \sin 42° \times 1.67 \times 10^{-27}}{1.6 \times 10^{-19} \times 500}$$

$$= 6.98\ \mu s.$$

Example 8. *An α-particle with charge 3.2×10^{-19} C and mass 6.68×10^{-27} kg is injected into transversed magnetic field of 1.5 Wb/m² with velocity 7.263×10^6 m/s. Calculate*

(a) the force on the particle, and

(b) the periodic time and resonant frequency.

Solution. Given that, $q = 3.2 \times 10^{-19}$ C, $m = 6.68 \times 10^{-27}$ kg,

$B = 1.5\ Wb/m^2$ and $v = 7.263 \times 10^6$ m/s

Force acting on the particle is

$$F = Bqv = 1.5 \times 3.2 \times 10^{-19} \times 7.236 \times 10^6$$

$$= 34.86 \times 10^{-13}\ N$$

$$\text{Periodic Time } T = \frac{2\pi}{B} \cdot \frac{m}{q} = \frac{2 \times 3.14}{1.5} \times \frac{6.68 \times 10^{-27}}{3.2 \times 10^{-19}}$$

$$= 8.7396 \times 10^{-8}\ s$$

$$\text{Frequency } \nu = \frac{1}{T} = 1.44 \times 10^7\ Hz = 11.44\ MHz.$$

Example 9. *A proton is moving in a circular orbit of radius 20 cm in the magnetic field of 0.36 T. Determine the orbital speed of proton.*

Solution. Given that, $R = 20$ cm, $B = 0.36$ T, $q = 1.6 \times 10^{-19}$ C, $m = 1.67 \times 10^{-27}$ kg.

$$v = \frac{qBR}{m} = \frac{1.6 \times 10^{-19} \times 0.36 \times 0.2}{1.67 \times 10^{-27}} = 6.9 \times 10^6\ m/s.$$

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Example 10. In the velocity selector of Bainbridge mass spectrograph, the electric field is 250 V/cm and magnetic field is 0.18 T. A mixture of single charged ions of carbon isotopes C^{13} and C^{12} is analysed with this mass spectrograph. The magnetic field in the main chamber is 0.22 T. Calculate the separation on photographic plate.

Solution. Given that, $E = 250$ V/cm, $B = 0.18$ T, $B' = 0.22$ T

Then velocity $v = \frac{E}{B} = \frac{2.5 \times 10^4}{0.18} = 1.38 \times 10^5$ m/s

Separation $= 2(R_2 - R_1)$

$$= \frac{2v(m_2 - m_1)}{eB'} = \frac{2 \times 1.38 \times 10^5 \times (13-12) \times 1.67 \times 10^{-27}}{1.6 \times 10^{-19} \times 0.22} = 1.31 \text{ cm.}$$

Question Bank

Multiple Choice Questions

1. Electrons moving in a uniform magnetic field of 50 oersted have a velocity of 8.80×10^7 cm/s. What is the radius of circular path they follow :
   (a) 5 cm (b) 20 cm (c) 10 cm (d) 40 cm.
2. The path of a charged particle in crossed electric and magnetic fields inclined at an angle to each other is
   (a) circular (b) cycloidal (c) straight line (d) parabola.
3. An electron is projected in a magnetic field along the lines of force. How will its motion be affected ?
   (a) there will be no effect on its motion
   (b) the electron will travel along a circle and its speed will remain unchanged
   (c) the electron will follow the path of parabola and its speed will increase
   (d) the velocity will increase in magnitude but its direction will not change.
4. The path of a charged particle moving under magnetic field alone would be
   (a) helical (b) straight line (c) parabola (d) circular.
5. Two electron beams having velocities in the ratio of 1 : 2 are subjected separately identical magnetic fields. What is the ratio of the deflections produced ?
   (a) 4 : 1 (b) 1 : 4 (c) 2 : 1 (d) 1 : 2.
6. We know that cathode rays gain kinetic energy when they are accelerated by an electric field. If cathode rays are subjected to the action of magnetic field, then their
   (a) energy increases (b) momentum increases
   (c) potential energy increases (d) energy and momentum remain unaffected.
7. Aston's mass spectrograph is used
   (a) to produce cathode rays (b) to produce X-rays
   (c) for deflection of isotopes (d) to measure electronic charge.

8. The Hall effect is observed when
   (a) a magnetic field is applied at right angle to a conductor carrying a current
   (b) a electric field is applied at right angle to a conductor carrying a current
   (c) a magnetic field is applied parallel to a conductor carrying a current
   (d) a magnetic field is applied antiparallel to a conductor carrying a current.
9. In a cathode ray oscilloscope, an ac voltage is applied to the y plates. The pattern formed on the screen is
   (a) central spot (b) spot displaced in the vertical direction
   (c) sine curve (d) straight line.
10. In cyclotron very high velocity cannot be achieved because
    (a) magnetic field of required strength cannot be produced
    (b) electric field of required strength cannot be produced
    (c) electric field is always parallel to the direction of motion of accelerating particle
    (d) at high velocity resonant frequency does not remain constant.
11. For a cyclotron, the magnetic field $B = 1$ Weber and its radius is 0.5 m. The maximum energy which protons can acquire from this is nearly
    (a) 0.12 MeV (b) 1.2 MeV (c) 12 MeV (d) 120 MeV.
    [Take proton mass $= 1.6 \times 10^{-27}$ kg]
12. A cyclotron cannot be used to accelerate electrons to high energies because :
    (a) electrons carry negative charge (b) rest energy of electron is 0.511 eV
    (c) very high voltage will be required (d) radius of the dees will have to very large.
13. A cyclotron is operating at a frequency of 12×10^6 Hz. If mass of a deutron is 3.3×10^{-27} kg and the charge is 1.6×10^{-19} C, the necessarily magnetic field to accelerate it, is
    (a) 16 T (b) 1.6 T (c) 0.16 T (d) 0.016 T
14. There should be necessary conditions for a velocity selector
    (a) $v = \frac{E}{B}$ (b) $v = \frac{B}{E}$ (c) $v = \frac{F}{q}$ (d) $v = \frac{F}{qB}$.
15. If E_y, J_x and B are the Hall field, current density and magnetic field strength respectively, then the Hall constant is given by
    (a) $R_H = \frac{E_y / J_x}{B}$ (b) $R_H = \frac{J_x / B_y}{B}$ (c) $R_H = \frac{B}{E_y / J_x}$ (d) $R_H = \frac{B}{J_x / E_y}$.
16. The unit of Hall coefficient is
    (a) $Vm^3A^{-2}Wb$ (b) $Vm^3A^{-1}Wb^{-1}$ (c) $Vm^2A^{-1}Wb^{-1}$ (d) Vm^2AWb^{-1}.
17. The acceleration of an electron in an electric field of magnitude 50 V/cm, if e/m value of the electron is 176×10^{11} C / kg will be
    (a) 8.8×10^{14} m / s^2 (b) 6.2×10^{12} m / s^2 (c) 3.4×10^{12} m / s^2 (d) zero.

Answers

1. (c)	2. (b)	3. (a)	4. (d)	5. (d)	6. (d)
7. (c)	8. (a)	9. (a)	10. (d)	11. (c)	12. (c)
13. (b)	14. (b)	15. (a)	16. (c)	17. (b)	

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Theoretical Questions

1. The electric field is along y-axis and the magnetic field is along z-axis. If charge is released at the origin, describe its trajectory. [GGSIPU, Feb, 2006 (6 marks)]
2. Find the trajectory traced by the particle of mass 'm' and charge 'q' under the simultaneous application of crossed **E** and **B** fields. [GGSIPU, May, 2007 (5 marks)]
3. What is Lorentz force ? Show that the path followed by a charged particle in crossed electric and magnetic field is a cycloid. [GGSIPU, Feb, 2008(Reappear) (8 marks)]
4. Write short notes on
   (i) Magnetic focusing [GGSIPU, Feb, 2008 (5 marks)]
   (ii) The velocity selector technique and its applications. [GGSIPU, Feb, 2005 (2 marks)]
5. Show how the combination of an electric field and a magnetic field on a charged particle can be used as a velocity selector. How was this used in the historic e/m experiment of J.J. Thomson ? [GGSIPU, Feb, 2005 (12.5 marks)]
6. What type of motion do you expect, when the proton is placed in uniform electric field ?
7. An electron is moving with velocity **v** in uniform **E** and **B** field. If the velocity remains unchanged then (a) $\mathbf{v} \perp \mathbf{E}$ (b) $\mathbf{v} \perp \mathbf{B}$ (c) $\mathbf{v} \parallel \mathbf{B}$
8. Write short notes on : (i) cathode ray oscilloscope (ii) determination of e/m of electron (iii) Aston mass spectroscope.
9. The dees of a cyclotron are good conductors. Is there any effect of the alternating voltage applied on protons when they are inside the dee ?
10. What is the effect of magnetic field on a moving charge particle ? Describe any one application of this effect.
11. What is meant by Lorentz force ?
12. Describe the principle of electrostatic and magnetostatic focusing. Describe their applications in some instruments.
13. Discuss Aston and Bainbridge mass spectrograph. What is the use of velocity selector in mass spectrograph ?
14. Describe the effect of a perpendicular electric field motion of charged particles. Describe the appropriate formula for linear deflection.
15. When a particle enters a region perpendicular to the magnetic field, prove that, when it is in the region of the magnetic field, its kinetic energy is proportional to the square of the radius of its orbit.
16. Describe in short, the helical motion of a charged particle in a magnetic field.
17. What is Hall effect ? Find the expression for
    (a) Hall voltage (b) Hall coefficient
    (c) Mobility and (d) Hall angle.
18. Discuss the motion of charged particle in combined electric and magnetic fields.
19. Why the sun light is not deflected when passed through an electric and magnetic field ?
20. How can you prove that trajectory of an electron in a magnetic field acting at right angles to the direction of motion of electron is an arc of circle ?

21. What is the trajectory of an electron in an electric field acting at right angle to the direction of motion of an electron ?
22. Is it possible to measure the absolute value of charge on the electron by Thomson's method ? If not, why ?
23. What are crossed fields ? Why we use crossed fields in Thomson's method for determination of (e/m) of electron ?

Numerical Problems

1. Calculate the radius of the dees of a cyclotron capable of accelerating protons upto 3 MeV in presence of a magnetic field of 1.0 Tesla. [GGSIPU, April 2005(reappear) (3 marks)] [Ans. $R = 0.25$ m]
2. The radius of a cyclotron dee is 1 m and the applied magnetic field is 0.5 Wb/m^2. It is used to accelerate protons. Calculate the frequency of oscillating voltage needed to maintain resonance. What is the maximum energy acquired by the proton ? [GGSIPU, June 2007 (5 marks)] [Ans. $\nu = 7.62$ MHz, $W = 11.96$ MeV]
3. An electron has a velocity of 10^4 m/s normal to a magnetic field of 0.1 Wb/m^2 flux density. Compute the radius of the electron and also its frequencies. [$R = 5.7 \times 10^{-7}$ m, $\omega = 2.8 \times 10^{9}$ Hz]
4. Find the frequency for a deutron ($e = 1.6 \times 10^{-19}$ C, $m = 3.34 \times 10^{-27}$ kg) in a cyclotron with a flux density $B = 1.5$ Wb/m^2. [Ans. $f = 1.14 \times 10^{7}$ Hz]
5. A cyclotron has a maximum working radius $R = 50$ cm, a flux density $B = 1.5$ Wb/m^2. Find the energy which may be imported in deutron. [Ans. $W_0 = 1.33 \times 10^{7}$ eV]
6. An electron enters a region of magnetic field of flux density 5 Wb/m^2 with a velocity of 2×10^{8} m/s perpendicular to the field. Compute the magnetic force on the electron and compare it with weight of the electron. The charge on the electron is 1.6×10^{-19} C and its mass is 9.1×10^{-31} kg. [Ans. $F_m = 1.6 \times 10^{-12}$ N, $W = 8.9 \times 10^{-30}$ N, The magnetic force is 1.8×10^{17} times greater than the weight]
7. A 1.5 MeV proton is falling vertically downward. A field of magnetic induction of 2.5×10^{3} b/m^2 is acting horizontally from south to north. Find the magnitude and direction of the magnetic force exerted on the proton. The mass and charge of the proton are 1.7×10^{-27} kg and 1.6×10^{-19} C respectively. [Ans. $F = 6.72 \times 10^{-15}$ N, west to east]
8. An n-type germanium sample has a donor density of 10^{21} / m^3. It is arranged in a Hall experiment having magnetic field of 0.5 T and the current density is 500 A / m^2. Find the Hall voltage if the sample is 3 mm wide. [Ans. $V_H = 4.7$ mV]
9. The resistivity of a doped silicon sample is 8.9×10^{-3} Ωm. The Hall coefficient was measured to be 3.6×10^{-4} m^3 / C. Assuming single carrier conduction, find the mobility and density of charge carriers. [Ans. $N = 1.736 \times 10^{22}$ / m^3, $\mu = 0.0404$ m^{2}V^{-1}s^{-1}]
10. The Hall coefficient of a certain specimen of silicon was found to be -7.35×10^{-4} m^3/C from 100 to 400 K. Is this semiconductor instrinsic or extrinsic at room temperature and is it n-type or p-type ? The electrical conductivity at room temperature was found to be 200 Ωm^{-1}. Calculate the density and mobility of charge carriers at room temperature. [Ans. $N = 8.5 \times 10^{22}$ / m^3, $\mu = 1.47 \times 10^{-2}$ m^{2}V^{-1}s^{-1}]

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11. Find the acceleration of an electron in a 10 keV beam moving horizontally due to the vertical component of earth's magnetic field for which $B = 5 \times 10^{-5}$ Wb/m^2. ($m = 9.1 \times 10^{-31}$ kg and $e = 1.6 \times 10^{-19}$ C). [Ans. $a = 5.2 \times 10^{14}$ m/s^2]

12. A proton is to circulate the earth with a speed of 1.0×10^8 m/s. Find the minimum magnetic field, which should be created at the equator for the purpose. The mass and charge of the proton are 1.7×10^{-27} kg and 1.6×10^{-19} C respectively. The radius of the earth is 6.37×10^6 m. [Ans. $B = 1.668 \times 10^{-4}$ Wb / m^2]

13. The magnetic field in a certain region of space is given by $\mathbf{B} = 0.08\,\hat{i}$ T. a proton is shot into the field with velocity $(2 \times 10^5\,\hat{i} + 3 \times 10^5\,\hat{j})$ m/s. Calculate the radius and pitch for the helical path the proton follows. [Ans. $R = 0.039$ m, $P = 0.164$ m]

14. A proton is accelerated through a potential difference of 10^5 V, it then enters perpendicular to a magnetic field and follows a circle of 0.30 cm radius. Find the value of **B** in the field. [Ans. $B = 0.152$ T]

15. An α-particle of mass 6.65×10^{-27} kg and charge twice that of an electron travels at right angles to a magnetic field with speed 6×10^6 m/s. The flux density of the field is 0.2 Wb/m^2. Calculate :
    (i) the force acting on alpha particle
    (ii) its acceleration. [Ans. (i) $F = 3.84 \times 10^{-14}$ N, (ii) 5.77×10^{12} m/s^2]

16. Calculate the energy of an electron if it describes a circle of radius 0.303 m in magnetic field of 5×10^{-4} T. [Ans. 2.02×10^3 eV]

17. A cyclotron accelerates deutrons to 4 MeV. To what energy will the same cyclotron accelerate proton and α-particle ? [Ans. 8 MeV for both]

18. An electron travelling at 2×10^7 m/s enters a magnetic field of flux density 2×10^{-3} T. Show that the radius of the path described by it is 5.688×10^{-2} m.

19. For a cyclotron of 0.35 m radius and frequency of applied potential is 1.38×10^7 Hz, obtain the value of induction of magnetic field for synchronous acceleration of protons and hence calculate, the maximum energy of accelerated protons. Mass of proton = 1.67×10^{-27} kg. Charge of proton = 1.6×10^{-19} C. [Ans. $B = 0.9045$ Wb/m^2 ; $E = 0.048 \times 10^8$ eV]

20. An electron moving horizontally with a velocity of 1.7×10^7 m/s enters a vertical uniform electric field of 3.4×10^4 V/m acting in the electric field at a time when its horizontal displacement is 3 cm. Find the magnitude of B required to neutralise this vertical displacement. [Ans. $B = 2 \times 10^{-3}$ Wb/m^2]

21. A single ionsised Ge atom enters a Bainbridge mass spectrograph with a velocity 5×10^5 m/s. Calculate the radii of the paths followed by two most abundant isotopes of masses 72 and 79 when magnetic flux density 0.3 Wb/m^2. Calculate also the linear separation of the lines on photographic plate for these isotopes. [Ans. $2(R_{74} - R_{72}) = 0.033$ m]

22. In a cathode ray tube, a pair of vertically deflecting plates are 3 cm long and are separated by a distance of 0.75 cm. The screen is 20 cm away from the plates. The final anode voltage is 800 V. Determine the displacement produced by a deflecting voltage of 25 V, the angle made by the beam with the axis of the tube on emerging from the field and the velocity of the electrons (in the beam) on emerging from the electric field. [Ans. $y = 1.34$ cm, $\alpha = 3.58°$, $v = 1.68 \times 10^7$ m/s]

Chapter 2

Gauss's Law and Its Applications

2.1 Introduction

Karl Friedrich Gauss (1777-1855) gave a theorem, known after his name Gauss's theorem or Gauss's Law, that relates the integral of the normal component of electric field (*i.e.*, not outward electric) flux over any hypothetical closed surface (called a Gaussian surface) to the net charge enclosed by the surface.

2.2 Electric Flux : (ϕ_E)

The total number of electric lines of force emanating from a charge is called electric flux and denoted by Greek letter ϕ_E. A unit charge is supposed to emanate one flux. Thus in case of an isolated charge q coulomb the flux $\phi_E = q$ and is independent of the nature of medium. The unit of flux is that of charge i.e., Coulomb.

In order to explain electric flux, let us consider an electric field which may or may not be uniform as shown in Fig. 2.1. Assume in this field immersed an arbitrary closed surface. Let the surface divided into a number of elementary squares. These squares are assumed to be very small so that it may be considered as plane and the electric field over it may be taken as uniform. The each square on the surface may be represented by a vector ΔS or dS having magnitude equal to its area and its direction taken as outward normal drawn on this surface. Let the electric field vector acting on the surface be **E**. The scalar product *i.e.*, **E** dS is defined as the flux for the surface. The total electric flux ϕ_E through the total surface is given by

$$\phi_E = \oint \mathbf{E}\,d\mathbf{S} \quad \text{...(1)}$$

If the angle between **E** and d**S** is θ, then the scalar product will be given by

$$\mathbf{E}\,d\mathbf{S} = E\,dS\cos\theta \quad \text{...(2)}$$

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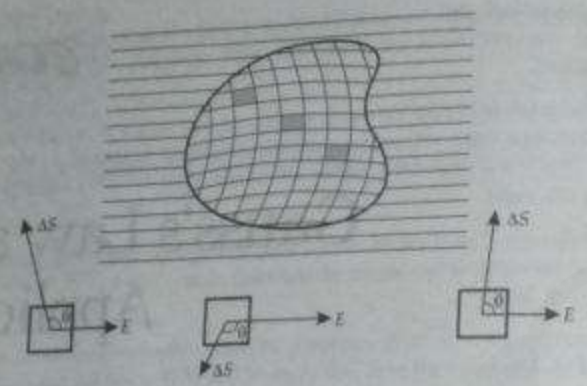

ectric Flux.

Hence the electric flux

$$\phi_E = \oint E\,dS\cos\theta$$
$$= E\cos\theta \oint dS \qquad ...(3)$$
$$\phi_E = E\cos\theta . A$$
$$= EA\cos\theta \qquad ...(4)$$

As $A = \oint dS$ = Area of surface]

or a closed flux (ϕ_E) is taken **positive** if the line of force point outward everywhere **gative** if the line of force are converging.

le 2.1. A point charge of 2×10^{-6} C is placed at the centre of a hypothetical cube with the side ach. Determine the electric flux through each face.

lution. A cube has six equal faces. The charges are situated at a centre of the cube. sly, equal amount of flux will be passing through each face.

ctric flux from charge q is equal to $\phi_E = \frac{q}{\varepsilon_0} = \frac{2\times10^{-6}}{\varepsilon_0}$

erefore electric flux from each face $= \frac{2\times10^{-6}}{6\times8.85\times10^{-12}} = 37.7\times10^{3}\ \text{Nm}^2/\text{C}$

2.2. If the electric field is given by $\mathbf{E} = 8\hat{i} + 4\hat{j} + 3\hat{k}$, calculate electric flux through the f area 100 units lying in x-y plane.

ution. In this problem $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vector along x, y and z axes respectively.

e electric flux is defined $\phi_E = \mathbf{E.S}$ The vector surface area in the x-y plane will be ection of the out drawn normal *i.e.*, in the z-direction.

efore $\quad \mathbf{S} = 100\hat{k}$

e $\quad \phi_E = \mathbf{E.S} = (8\hat{i} + 4\hat{j} + 3\hat{k}).(100\hat{k}) = 300$ units.

2.3 Electric Flux Density (D)

A point charge is surrounded by an electric field. Thus an isolated, positive point charge q has a radial field as indicated by the line radiating from q (Fig. 2.2). These lines indicated the direction of electric field *i.e.*, the direction of the force on a positive test charge.

The electric field intensity **E** at the radius r due to a charge q will be

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^3}\hat{r} \qquad ...(5)$$

where $\hat{r}$ is unit vector in radial direction. Multiplying Eq. (5) by ε we have

$$\varepsilon\mathbf{E} = \frac{q}{4\pi r^3}\hat{r} \qquad ...(6)$$

Fig. 2.2

From above it is clear that the quantity εE has the dimension of charge/area or surface charge density. The product $\varepsilon\mathbf{E}$ is designated by the symbol **D** called the electric flux density. Thus

$$\mathbf{D} = \varepsilon\mathbf{E} \qquad ...(7)$$

where ε = permittivity of the medium (F/m)

D = electric flux density or electric displacement vector (C/m^2)

E = electric field intensity (V/m)

The number of lines passing through a normal surface (Fig. 2.2) indicates the flux density **D** and the individual lines are flux lines.

According to Eq. (7) the flux density and field intensity are vectors having the same direction and flux density can be defined as the product of field intensity **E** and the permittivity of the medium ε.

As the integral of the normal component of a vector over that surface is defined as the flux of the vector over that surface. We can apply this definition to **D**, by integrating it over a surface S at a constant radius r from the point charge q (Fig. 2.3). As **D** is everywhere constant and is normal to S on the surface.

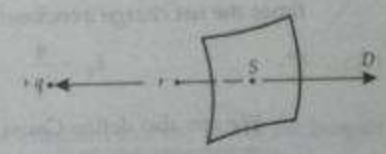

Fig. 2.3

So the flux of **D** over **S** is simply S times the magnitude of D, *i.e.*,

$$\text{Flux(of } D \text{ over } S) = SD$$

This flux (of **D** over a surface **S**) is called the **electric flux**, designated by ϕ_E.

Thus $\quad \phi_E$ = flux (of **D** over **S**) $= SD \qquad ...(8)$

and the flux of **D** has dimensions of charge. The electric flux density is designated by **D**. Thus from above relation

$$\mathbf{D} = \frac{\phi_E}{S} = \text{Flux density} \qquad ...(9)$$

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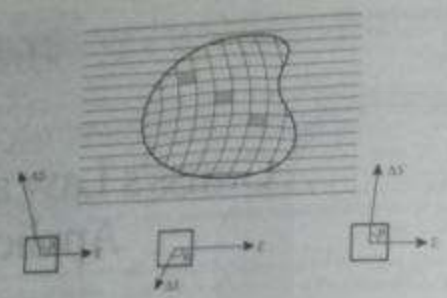

Fig. 2.1 Electric Flux.

Hence the electric flux

$$\phi_E = \oint E\,dS\cos\theta$$

$$= E\cos\theta \oint dS \qquad ...(3)$$

or

$$\phi_E = E\cos\theta\, A$$

$$= EA\cos\theta \qquad ...(4)$$

[As $A = \oint dS$ = Area of surface]

For a closed flux (ϕ_E) is taken **positive** if the line of force point outward everywhere and **negative** if the line of force are converging.

Example 2.1. *A point charge of 2×10^{-8} C is placed at the centre of a hypothetical cube with the side 0.1 m each. Determine the electric flux through each face.*

Solution. A cube has six equal faces. The charges are situated at a centre of the cube. Obviously, equal amount of flux will be passing through each face.

Electric flux from charge q is equal to $\phi_E = \frac{q}{\varepsilon_0} = \frac{2 \times 10^{-8}}{\varepsilon_0}$

Therefore electric flux from each face $= \frac{2 \times 10^{-8}}{6 \times 8.85 \times 10^{-12}} = 37.7 \times 10^{3}$ Nm^2/C

Example 2.2. *If the electric field is given by $\mathbf{E} = 8\hat{i} + 4\hat{j} + 3\hat{k}$, calculate electric flux through the surface of area 100 units lying in x-y plane.*

Solution. In this problem $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vector along x, y and z axes respectively.

Since electric flux is defined $\phi_E = \mathbf{E}.\mathbf{S}$. The vector surface area in the x-y plane will be in the direction of the out drawn normal *i.e.*, in the z-direction.

Therefore $\mathbf{S} = 100\hat{k}$

Hence $\phi_E = \mathbf{E}.\mathbf{S} = (8\hat{i} + 4\hat{j} + 3\hat{k})(100\hat{k}) = 300$ units.

2.3 Electric Flux Density (D)

A point charge is surrounded by an electric field. Thus an isolated, positive point charge q has a radial field as indicated by the line radiating from q (Fig. 2.2). These lines indicated the direction of electric field *i.e.*, the direction of the force on a positive test charge.

The electric field intensity $\mathbf{E}$ at the radius r due to a charge q will be

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\hat{r} \qquad ...(5)$$

where $\hat{r}$ is unit vector in radial direction. Multiplying Eq. (5) by ε we have

$$\varepsilon\mathbf{E} = \frac{q}{4\pi r^2}\hat{r} \qquad ...(6)$$

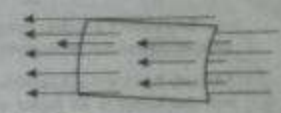

Fig. 2.2

From above it is clear that the quantity $\varepsilon\mathbf{E}$ has the dimension of charge/area or surface charge density. The product $\varepsilon\mathbf{E}$ is designated by the symbol $\mathbf{D}$ called the electric flux density. Thus

$$\mathbf{D} = \varepsilon\mathbf{E} \qquad ...(7)$$

where ε = permittivity of the medium (F/m)

$\mathbf{D}$ = electric flux density or electric displacement vector (C/m^2)

$\mathbf{E}$ = electric field intensity (V/m)

The number of lines passing through a normal surface (Fig. 2.2) indicates the flux density $\mathbf{D}$ and the individual lines are flux lines.

According to Eq. (7) the flux density and field intensity are vectors having the same direction and flux density can be defined as the product of field intensity $\mathbf{E}$ and the permittivity of the medium ε.

As the integral of the normal component of a vector over that surface is defined as the flux of the vector over that surface. We can apply this definition to $\mathbf{D}$, by integrating it over a surface S at a constant radius r from the point charge q (Fig. 2.3). As $\mathbf{D}$ is everywhere constant and is normal to S on the surface.

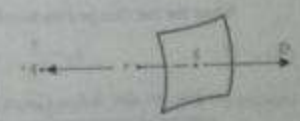

Fig. 2.3

So the flux of $\mathbf{D}$ over $\mathbf{S}$ is simply S times the magnitude of $\mathbf{D}$, *i.e.*,

$$\text{Flux (of } D \text{ over } \mathbf{S}) = SD$$

This flux (of $\mathbf{D}$ over a surface $\mathbf{S}$) is called the **electric flux** designated by ϕ_E.

Thus ϕ_E = flux (of $\mathbf{D}$ over $\mathbf{S}$) = SD ...(8)

and the flux of $\mathbf{D}$ has dimensions of charge. The electric flux density is designated by $\mathbf{D}$. Thus from above relation

$$\mathbf{D} = \frac{\phi_E}{S} = \text{Flux density} \qquad ...(9)$$

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Substituting **E** from Eq. (5) into Eq. (6), we obtain positive flux density **D** for a point charge q i.e.,

$$\mathbf{D} = \varepsilon \mathbf{E} = \frac{\varepsilon q}{4\pi\varepsilon r^3}\hat{r}$$

$$= \frac{q}{4\pi r^3}\hat{r} \qquad \text{...(10)}$$

From above it is clear that flux density depends upon the charge and the radius but is independent of the permittivity of the medium.

Example 2.3. *A rectangular surface 0.2m × 0.3m is placed in uniform electrostatic field 2000 N/C such that normal to the surface makes an angle of 30° with direction of electrostatic field. Find electric flux through this surface.*

Solution. The electric flux is given by

$$\phi_E = \mathbf{E} \cdot A\cos\theta$$

$$= 2000 \times 0.6 \times \cos 30^\circ$$

$$= 2000 \times 0.6 \times \frac{\sqrt{3}}{2} = 60\sqrt{3}$$

$$= 103.92 \ \frac{\text{Nm}^2}{\text{C}}$$

2.4 Gauss's Law or Gauss's Theorem

This theorem gives a relation between the total electric flux through any closed surface (known as Gaussian surface) and the net charge enclosed within it (surface).

According to this theorem, the electric flux ϕ_E through a closed surface is equal to $\frac{1}{\varepsilon_0}$ times the net charge q enclosed by the surface.

i.e., $\phi_E = \frac{q}{\varepsilon_0}$ where ε_0 = permittivity of free space.

We can also define Gauss's Theorem as :

> The surface integral of the normal component of the electric flux density **D** over closed surface equals the charge enclosed by closed surface.

Or

> The surface integral of the normal component of the electric field intensity **E** over a closed surface is equal to $\frac{1}{\varepsilon_0}$ times the total charge inside.

i.e., $\int \mathbf{E}.d\mathbf{S} = \frac{q}{\varepsilon_0}$

or $\int \mathbf{D}.d\mathbf{S} = q$

Proof of Gauss Theorem

Let us consider a closed surface surrounding a charge q. Let a very small area dS of this surface, at a distance r from the charge q subtended a solid angle $d\omega$ as shown in Fig. 2.4.

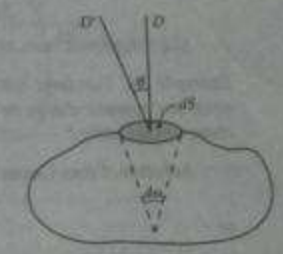

Fig. 2.4

Then the flux in the direction of radius vector through this surface is given by

$$d\phi'_E = \mathbf{E}.d\mathbf{S} \qquad \text{...(11)}$$

The component of flux in a direction normal to the surface will be

$$\phi_E = \phi'_E \cos\theta \qquad \text{...(12)}$$

where the θ = angle between E and dS, we know

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{...(13)}$$

$$\therefore \quad d\phi_E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dS\cos\theta \qquad \text{...(14)}$$

or

$$d\phi_E = \frac{q\cos\theta}{4\pi\varepsilon_0 r^2} \times dS \text{ Coulomb} \qquad \text{...(15)}$$

We know by definition solid angle subtended by a surface

$$d\omega = \frac{\text{Surface area}}{(\text{radius})^2}$$

$$\therefore \quad d\omega = \frac{dS\cos\theta}{r^2} \qquad \text{...(16)}$$

Substituting this value in Eq. (15), we have

$$d\phi = \frac{q\,d\omega}{4\pi\varepsilon_0} \qquad \text{...(17)}$$

The total flux ϕ, crossing the total surface of the imagined surface, will be the integral of $d\phi$ over the whole surface or the summation of $d\phi$ over the whole surface

$$\phi = \int d\phi = \frac{q}{4\pi\varepsilon_0}\int d\omega$$

$$= \frac{q}{4\pi\varepsilon_0} \times 4\pi \qquad \text{[As the angle subtended by a closed surface is } 4\pi\text{]}$$

$$= \frac{q}{\varepsilon_0} \qquad \text{...(18)}$$

In case there are a number of positive and negative charges such as $+q_1, -q_2, +q_3, +q_4, -q_5$ and $+q_6$ placed inside the surface, the total electric flux due to all charges will be

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$$\phi = \frac{1}{\varepsilon_0}[q_1 - q_2 + q_3 + q_4 - q_5 + q_6]$$

or
$$\phi = \frac{\Sigma q}{\varepsilon_0} \quad ...(19)$$

(Σq = algebraic sum of all the charges)

Example 2.4. *Two large metal plates of area 1 m² face each other. They are 5 cm apart and carry equal and opposite charge on their inner surfaces. If E between the plates is 55 N/C, what is the charge on the plate ?*

Solution. From Gauss law $E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0}$

or
$$q = EA\varepsilon_0$$
$$= 55 \times (1)^2 \times 8.85 \times 10^{-12} = 0.19 \times 10^{-9}\,\text{C}$$

2.5 Applications of Gauss's Law

To evaluate the electric field at any point using Gauss's law, an imaginary closed surface is chosen, such that the desired point lies on it. In fact, the ability to use the law to evaluate the electric field really depends on the proper choice of the closed surface so that the flux across it can be easily determined.

The electric field can then be obtained using Eqs. (11), (12) and (18) which can also be written as

$$\int_S E\,dS\cos\theta = \frac{q}{\varepsilon_0} \quad ...(20)$$

where θ is the angle between the directions of the vectors **E** and $d\mathbf{S}$. It is found that the integration can be carried out easily if the closed surface is chosen in such a way that every part of it satisfies one (or both) of the conditions given below :

1. The angle θ should have a value of either 90° or 270°.
2. Both θ and E should be constant.

Examples given below illustrate the use of Gauss's law in solving some important problems :

2.5.1 Electric Field Due to a Line Charge

Let us consider an infinitely long positive line charge of uniform linear charge density β. It is desired to find the electric field at a point P at a distance r from the line charge as shown in Fig. 2.5.

For the Gaussian surface, let us consider a closed cylinder of length l and the radius r so that the point P lies on its curved surface. The surface of the cylinder can be thought to consist of three parts-two flat surfaces S_1 and S_3 and one curved surface S_2, as shown in Fig. 2.5. For the surface S_1 and S_3, each element of area dS on them is at right angles to the

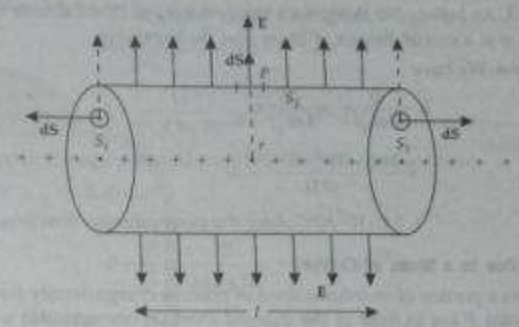

Fig. 2.5 Electric field due to a line charge.

electric field, which due to symmetry, is directed radially outward everywhere. Hence $\cos\theta$ being zero, the flux across the surfaces S_1 and S_3 is zero. On the surface of S_2, the electric field and area vector dS of each element are pointing radially outwards, hence, θ is 0° and $\cos\theta$ is unity. Besides, due to symmetry, the electric field at every point of the curved surface S_2 has the same magnitude.

We may thus write

$$\int_S \mathbf{E}\,d\mathbf{S} = \int_{S_1} \mathbf{E}\,d\mathbf{S} + \int_{S_2} \mathbf{E}\,d\mathbf{S} + \int_{S_3} \mathbf{E}\,d\mathbf{S}$$

$$= \int_{S_2} \mathbf{E}\,d\mathbf{S} = \int_{S_2} E\,dS\cos 0°$$

$$= E\int_{S_2} dS = E(2\pi rl) \quad ...(21)$$

because $2\pi rl$ is the area of the curved surface. The charge enclosed by the closed surface being βl, we obtain from Gauss's Law.

$$2\pi rlE = \frac{\beta l}{\varepsilon_0}$$

or
$$E = \frac{\beta}{2\pi\varepsilon_0 r} \quad ...(22)$$

However, for a finite length of the line charge, the electric field over the curved surface S_2 is neither constant nor does it act radially outwards everywhere. The evaluation of the integral, and hence the flux across the closed surface, cannot be easily carried out in this case. Though Gauss Law would still valid for the closed surface, it cannot be used to obtain the electric field at a point.

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Example 2.5. An infinite line charge has a charge density 3μ C/m. Calculate the electric field at a point which is at a normal distance of 10 cm from the line charge.

Solution. We have

$$\mathbf{E} = \frac{\beta}{2\pi\varepsilon_0 r}\hat{r} = \frac{2\beta}{4\pi\varepsilon_0 r}\hat{r}$$

$$= \frac{2\times 3\times 10^{-6}\times 9\times 10^{9}}{(0.1)}\hat{r}$$

$$= 5.4\times 10^{5}\ \text{N/C along the perpendicular direction of line charge.}$$

2.5.2 Electric Field Due to a Sheet of Charge

Fig. 2.6 shows a portion of an infinite sheet of positive charge density (*i.e.*, charge per unit area) σ. A point P lies in front of the charged sheet on one side. We want to obtain the electric field at the point P due to this charged sheet.

Let us consider a small cylinder of cross-sectional area S such that the point P lies on one face of the cylinder while the other face lies on the other side of the sheet over an area S, as shown in Fig. 2.6. Because of symmetry, the electric field E everywhere on either side of the sheet is constant and points outwards. Since the outward drawn normal to the curved area of the cylinder points at right angles to the direction of the electric field, the electric flux across the curved surface is zero. On the flat surfaces, both the electric field and the area vectors point outwards *i.e.*, along the same direction. The flux across each of the flat surfaces CD and FG is hence equal to ES. The charge enclosed by the closed cylinder being σS, we have, using Gauss's Law

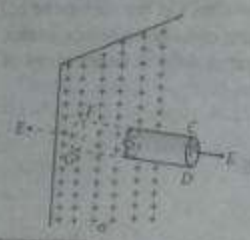

Fig. 2.6 Electric field due to a sheet of uniform charge distribution.

$$\int_{CD} \mathbf{E}\cdot d\mathbf{S} + \int_{FG} \mathbf{E}\cdot d\mathbf{S} = \frac{\sigma S}{\varepsilon_0} \qquad \text{...(23)}$$

or

$$ES + ES = \frac{\sigma S}{\varepsilon_0}$$

or

$$E = \frac{\sigma}{2\varepsilon_0} \qquad \text{...(24)}$$

If the charged sheet is of finite dimensions, the electric field around it is not constant and does not act outwards at every point. However, for points very close to the sheet of charge, it may still be regarded as infinite and the electric field can be calculated using Eq. (24).

Example 2.6. An infinite charge sheet has a surface charge density (σ) of 10^{-7} C/m². How far apart are the equipotential surfaces whose potential differs by 5 V ?

Solution. We know that for an infinite charge sheet

$$E = \frac{\sigma}{2\varepsilon_0}, \text{ Given } \sigma = 10^{-7}\ \text{C/m}^2$$

$$\therefore \quad E = \frac{10^{-7}}{2\times 8.9\times 10^{-12}} = 562\times 10^{3}\ \text{N/C}$$

Suppose, the equipotential surface is dl apart, then we have

$$E\,dl = dV$$

Given $\quad dV = 5\ V$

$$\therefore \quad dl = \frac{dV}{E} = \frac{5}{5.62\times 10^{3}} = 0.889\times 10^{-4}\ \text{m}.$$

2.5.3 Charge on a Conductor

It is well-known that charges reside on the outer surface of a conductor. This can be understood by using Gauss's law.

When a charge (say, positive) is given to a conductor C (Fig. 2.7) at a point, it establishes a field within it and the free electrons within the conductor tend to move towards the positive charge, thus leaving an excess of positive charge behind. The motion of electrons sets up electric currents within the conductor. These currents exist so long as an electric field exists within the conductor. It is found that the electric currents within the conductor exist for an extremely short interval of time ($\cong 10^{-8}$ s). One these cease *i.e.*, when the electrostatic conditions have been achieved, it can be concluded that there is no electric field within the conductor.

Fig. 2.7 Charge on a conductor.

Let us consider a closed surface S within the conductor lying very near the outer surface (*i.e.*, a few atomic distances inside) as shown in Fig. 2.7. Since the electric field within the conductor, under electrostatic conditions is zero, the electric flux across the surface S must be zero. Applying Gauss's law to the closed surface S, it follows that no charge exists within the conductor. Hence, the entire charge given to the conductor must lie on its surface.

2.5.4 Electric Field Due to a Charged Non-conducting Sphere

A non-conducting sphere of radius 'a' has a positive charge 'q' distributed uniformly within its volume. It is desired to determine the electric field due to the charged sphere at a point

(*a*) within the sphere ; (*b*) on the surface and (*c*) outside the sphere.

Since the charge is uniformly distributed throughout the volume of the sphere, the charge density (*i.e.*, charge per unit volume) ρ_v is given by

$$\rho_v = \frac{q}{\frac{4\pi}{3}a^3} \qquad \text{...(25)}$$

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(a) Within the sphere

Let us consider a point P lying inside the sphere at a distance r from the centre O, as shown in Fig. 2.8(a). To obtain the electric field E at the point P, we draw a sphere of radius r with O as its centre, so that the point P lies on the surface. The surface of this imaginary sphere is taken as the closed surface for applying Gauss's Law. Because of symmetry, the lines of electric field are radial and hence, the electric field $\mathbf{E}$ and the area vector $\mathbf{dS}$ both point radially outwards everywhere on the Gaussian surface, as shown in Fig 2.8(a). The magnitude of the electric field is a constant, say E_1 at every point on the surface. Hence the flux across the Gaussian surface of a sphere of radius r is

$$\phi_E = \int_S \mathbf{E}\cdot d\mathbf{S} = \int E_1 dS\cos 0^\circ = \int E_1 dS$$

$$= E_1 \int dS = E_1 4\pi r^2 \qquad \text{...(26)}$$

where $4\pi r^2$ denotes the surface area of the imaginary sphere. The charge enclosed by this Gaussian surface is equal to

$$\rho_v \frac{4\pi r^3}{3} = \frac{q(4\pi/3)r^3}{(4\pi/3)a^3} = q\frac{r^3}{a^3} \qquad \text{...(27)}$$

Applying Gauss Law to the surface of the sphere and using Eqs. (26) and (27), we have

$$E_1 4\pi r^2 = \frac{q}{\varepsilon_0}\frac{r^3}{a^3}$$

or

$$E_1 = \frac{qr}{4\pi\varepsilon_0 a^3} \qquad \text{...(28)}$$

where is the electric field at the point P, at a distance r from the centre of the sphere. The electric field at a point within the non-conducting sphere is, therefore, proportional to its distance from the centre of the sphere.

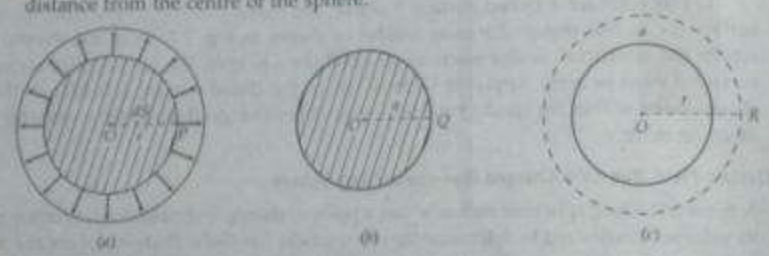

Fig. 2.8 Electric field due to a charged non-conducting sphere. Field (a) within the sphere, (b) on its surface, and (c) outside the sphere.

(b) On the Surface of the Sphere

Let us assume that a point Q lies on the surface of the sphere [Fig. 2.8(b)], which may also be considered as the Gaussian surface. If E_2 is the magnitude of the electric field on the surface of the sphere then, as has been pointed out above, the field vector $\mathbf{E}_2$ and the area

vector $\mathbf{dS}$ point along the same direction (radially outwards) at all points of the Gaussian surface. The electric flux ϕ_E across it can be written as

$$\phi_E = \int_S \mathbf{E}\cdot d\mathbf{S} = \int E_2 dS\cos 0^\circ$$

$$= E_2 \int dS = E_2(4\pi a^2) \qquad \text{...(29)}$$

Since the sphere encloses the entire charge q, we have by applying Gauss's Law to the surface of the sphere

$$E_2(4\pi a^2) = \frac{q}{\varepsilon_0}$$

or

$$E_2 = \frac{q}{4\pi\varepsilon_0 a^2} \qquad \text{...(30)}$$

(c) Outside the Sphere

A point R lies in space outside the sphere at a distance r from the centre O of the sphere [Fig. 2.8(c)]. To evaluate the electric field at R, we consider as a Gaussian surface, an imaginary sphere of radius r concentric with the given sphere. As mentioned earlier, the electric field, due to symmetry, is constant and point radially outwards and so does the area vector at every point of the imaginary sphere. Taking E_3 to be the electric field at any point on this sphere, the flux ϕ_E across the imaginary sphere is

$$\phi_E = \oint_S \mathbf{E}\cdot d\mathbf{S} = \int E_3 dS\cos 0^\circ$$

$$= E_3 \int dS = E_3 4\pi r^2 \qquad \text{...(31)}$$

$4\pi r^2$ being the area of sphere. The charge enclosed by the imaginary sphere in the present case is the total charge q on the given sphere. Using Gauss's Law, we obtain

$$E_3 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or

$$E_3 = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \text{...(32)}$$

as the electric field at point R outside the sphere at a distance r from its centre.

It is evident from Eqs. (30) and (32) that for points lying on the sphere and outside, the entire charge on the sphere may be regarded as a point charge (of the same magnitude) lying at the centre of sphere. The electric field at any point can then be evaluated by using Coulomb's Law.

Eqs. (28), (30) and (32) plotted together, give the variation of the electric field due to the charge on a non-conducting sphere (or a spherical distribution of charge) as one goes from the centre of the sphere outwards-this variation, on an arbitrary scale is represented in Fig. 2.9.

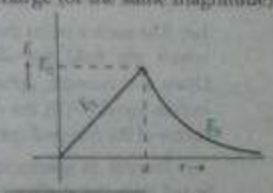

Fig. 2.9 Variation of the electric field due to a charged non-conducting sphere.

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2.5.5 Electric Field Around a Charged Conducting Sphere (Non-conducting Hollow Sphere)

Let us consider a uniformly charged sphere of radius a with a total charge Q

Case 1. *At a point outside the charged sphere.*

Let A be the point outside the charged sphere at which the field intensity is to be determined. Taking O as centre, draw a concentric sphere of radius equal to OA. The intensity E will be same at every point on this sphere and will be in the direction along the outward drawn radius OA as shown in Fig. 2.10. The total charge enclosed in this drawn sphere with a radius OA $(=r)$ will be equal to charge on the sphere of radius a *i.e.*, total charge on the sphere Q and the area of this drawn sphere will be $4\pi r^2$. According Gauss's theorem the total flux transversing an enclosed surface is equal to the total charge enclosed by the surface. Hence the total flux crossing this sphere of radius r will be

Fig. 2.10 Field due to uniformly charged sphere at a point outside the sphere.

$$\phi = q$$

$\therefore$ Flux density $\quad D = \text{Flux/area}$

$\therefore$ Flux density D at A $\quad = \dfrac{q}{4\pi r^2}$

Now we know intensity $\quad E_1 = \dfrac{D}{\varepsilon_0 \varepsilon_r}$

$\therefore$ Intensity at A $\quad = \dfrac{q}{4\pi r^2 \varepsilon_0 \varepsilon_r}$ ...(33)

Case 2. *At a point on the surface of the charged sphere*

As in this case the point is one the surface of the charged sphere the distance of the point is equal to radius of the sphere a. The intensity at a point on the surface of the charged sphere can be found by substituting $r = a$ in Eq. (5). Hence the electric intensity

$$E_2 = \frac{q}{4\pi\varepsilon_0 \varepsilon_r a^2} \quad ...(34)$$

Case 3. *At a point inside the surface of the charged sphere*

Let B be such a point inside the charged sphere at which the field intensity is to be determined. Draw a concentric sphere from centre O with a radius equal to OB, the distance of point from the centre of the charged sphere, as shown in Fig. 2.11.

In case of conductor as the charge is distributed over the surface of the conductor, the total charge Q given to the sphere will be distributed

Fig. 2.11 Electric field at a point inside the charged sphere.

over the sphere of radius a and no charge will be inside the drawn sphere with a radius equal to r. As a result the charge enclosed in the drawn sphere will be zero. Therefore the electric flux crossing the dotted sphere

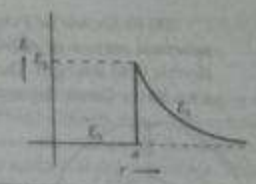

Fig. 2.12 Variation of E with r.

$$\phi = 0$$

$\therefore$ Flux density D at $B = 0$

$\therefore$ Intensity $\quad E_3 = D/\varepsilon_0 \varepsilon_r = 0$ ...(35)

$\therefore$ Intensity at $B = 0$

From above relation we find that the intensity at a point inside a charged sphere is zero.

Example 2.7. *A spherically symmetric charge distribution of radius a is characterised by the charge density function*

$$\rho(r) = \rho_0\left(1 - \frac{r^2}{a^2}\right) \quad \text{for } r \le a$$

$$= 0 \quad \text{for } r > a$$

Calculate :

(a) *the total amount of charge,*

(b) *the electric field strength at point of distance r from the centre if point is*

(i) *inside and* (ii) *outside the charge distribution*

(c) *the value of r for which the field is maximum.*

Solution. Consider a thin spherical shell of radius x and thickness dx concentric with spherical symmetric charge distribution.

The volume of the shell

$$dV = 4\pi x^2 dx$$

The charge on the shell

$$= 4\pi x^2 dx.\rho$$

$$= 4\pi x^2 \rho_0\left(1 - \frac{x^2}{a^2}\right)$$

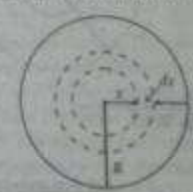

Fig. 2.13

(a) The amount of charge

$$= \int_0^R 4\pi x^2 . \rho_0\left(1 - \frac{x^2}{a^2}\right) dx = 4\pi\rho_0 \int_0^R x^2\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 4\pi\rho_0 \int_0^R \left(x^2 - \frac{x^4}{a^2}\right) dx = 4\pi\rho_0 \int_0^R \left[\frac{x^3}{3} - \frac{x^5}{5a^2}\right]_0^R$$

$$= 4\pi\rho_0\left(\frac{a^3}{3} - \frac{a^3}{5}\right) = 4\pi\rho_0 \frac{2a^3}{15}$$

$$= \frac{8}{15}\pi a^3 \rho_0 \text{ Coulomb.}$$

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(b) (i) *Electric field strength at a point inside the charge distribution.* Consider a Gaussian spherical surface of radius $r < a$, concentric with given distribution. In this case let E_1 be electric field strength. The charge enclosed by Gaussian surface will be from radius $x = 0$ to $x = r$. So by Gauss theorem

Fig. 2.14

$$4\pi r^2 E_1 = \frac{1}{\varepsilon_0}\int_0^r \rho 4\pi x^2 dx = \frac{1}{\varepsilon_0}\int_0^r 4\pi\rho_0\left(1-\frac{x^2}{a^2}\right)x^2 dx$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{x^3}{3}-\frac{x^5}{5a^2}\right]_0^r = \frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{r^3}{3}-\frac{r^5}{5a^2}\right]$$

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{4\pi\rho_0}{r^2}\left[r^3-\frac{r^5}{5a^2}\right] \text{ N/C}$$

(b) (ii) *Electric field strength at a point distance r from the centre outside the charge distribution (i.e., $r > a$).* Consider a spherical surface of radius $r > a$ concentric with spherical charge. The electric field strength E_0 at any point on this spherical surface may be calculated by Gauss theorem

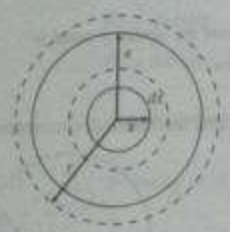

Fig. 2.15

i.e.,
$$\int_S \mathbf{E}\,d\mathbf{S} = \frac{1}{\varepsilon_0}\times \text{charge enclosed}$$

$$= \frac{1}{\varepsilon_0}\int_0^a \rho_0\left(1-\frac{x^2}{a^2}\right)4\pi x^2 dx$$

$$= \frac{1}{\varepsilon_0}\cdot\frac{8}{15}\pi a^3\rho_0$$

$$\Rightarrow \quad E_0 = \frac{8}{15}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{\pi a^3\rho_0}{r^2} = \frac{2}{15}\frac{a^3\rho_0}{\varepsilon_0 r^2} \text{ N/C}$$

(c) For maximum value of electric field $\frac{\partial E_1}{\partial r} = 0$

or
$$\frac{\partial}{\partial r}\left[\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)\right] = 0$$

or
$$\frac{\rho_0}{\varepsilon_0}\left(\frac{1}{3}-\frac{3r^2}{5a^2}\right) = 0$$

or
$$\frac{1}{3}-\frac{3r^2}{5a^2} = 0$$

or
$$r = \frac{\sqrt{5}a}{3}$$

2.6 Force on a Surface Charge and Energy Associated with an Electric Field

In a charged conductor, the total charge resides on its surface. Since similar charges repel each other, the charge on any part of the surface is repelled by the charge on the remaining surface of the conductor. This mutual repulsion between the charges of like sign on the surface of a charged conductor causes a tension on the surface of the conductor. Thus the charged conductor experiences a mechanical force whose magnitude per unit area, we have to determine.

Let XY be the surface of a charged conductor of charged density σ placed in free space (absolute permitivity ε_0). Let us consider a small element ab of area δA on the surface of the conductor (Fig. 2.16) and a point P just outside it. According to Coulomb's law, the electric field strength at P is normal to the surface and is given by

Fig. 2.16

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{...(36)}$$

It may be assumed that this field strength is partly due to charge on the element ab and partly due to the charge on the remaining surface. If E_1 and E_2 are these electric field strengths at P respectively, then

$$E = E_1 + E_2 = \frac{\sigma}{\varepsilon_0} \quad \text{...(37)}$$

Now consider a point P' just near the element ab inside the conductor. The electric field strength at P' may also be assumed to be partly due to the charge on element ab and partly due to remaining charge. The electric field strength at P' due to the element ab has the same magnitude (*i.e.*, E_1) as that at P; but opposite in direction; because P and P' are situated very close to each other; but opposite to the element ab. The electric field strength E_2 due to the charge on the remaining surface has the same magnitude and direction as that at P; but P and P' are very close to each other. Thus resultant intensity at P' is $E_2 - E_1$. As the resultant intensity at any point inside a charged conductor is zero, we must have

$$E_2 - E_1 = 0 \quad \text{or} \quad E_1 = E_2$$

Now from Eq. (37) gives

$$E_1 = E_2 = \frac{\sigma}{2\varepsilon_0} \quad \text{...(38)}$$

Thus the field strength on element ab due to the charge on the remaining surface is $E_2 = \frac{\sigma}{2\varepsilon_0}$. In other words this is the mechanical force per unit charge on element ab. As the charge on the small element ab is $\sigma\,\delta A$, the force on element ab of area δA

$$= \frac{\sigma}{2\varepsilon_0}\sigma\,\delta A = \frac{\sigma^2\,\delta A}{2\varepsilon_0}$$

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Thus the mechanical force per unit area, $\frac{F}{\delta A} = \frac{\sigma^2}{2\epsilon_0}$; this force acts normal to the surface.

As $E = \frac{\sigma}{\epsilon_0}$ *i.e.,* $\sigma = \epsilon_0 E$

$$\frac{F}{\delta A} = \frac{(\epsilon_0 E)^2}{2\epsilon_0} = \frac{1}{2}\epsilon_0 E^2$$

Therefore mechanical force per unit area on the surface of the charged conductor acting normal to the surface.

$$= \frac{1}{2}\epsilon_0 E^2 \quad ...(39)$$

Energy associated with an electric field

The outward mechanical force on a small elementary surface area δA is

$$= \frac{\epsilon_0 E}{2}.\delta A$$

If this small element is displaced normally through a distance δy in the opposite direction of the force, the work done during this operation is

$$\delta W = \left[\frac{\epsilon_0 E^2}{2}.\delta A\right].\delta y$$

The energy thus expended can be regarded as stored in the additional volume (increase in volume) of the electric field or medium which is

$$\delta V = \delta A \delta y$$

This work done increasing δV volume of the electric field,

$$\delta W = \frac{\epsilon_0 E^2}{2}(\delta A \delta y) = \frac{\epsilon_0 E^2}{2}.\delta V$$

Hence work done per unit volume of the field is

$$\frac{\delta W}{\delta V} = \frac{\epsilon_0 E^2}{2} \text{ Joules/m}^3$$

The work is stored as the energy of the strain in the field. This suggests that the energy per unit volume or energy density of an electric field is $\frac{\epsilon_0 E^2}{2}$ Joules / m^3.

Remark. If the space is a medium of absolute permittivity ϵ(or dielectric constant ϵ_r), then the energy density stored in the medium is

$$u = \frac{\epsilon E^2}{2} = \frac{\epsilon_r \epsilon_0 E^2}{2}$$

Example 2.8. *The surface density of charge on a conductor in dry air is 2.67×10^{-5} C/cm^2. What is the force per unit area due to this charge ?*

Solution. Given surface charge density

$$\sigma = 2.67 \times 10^{-5} \text{ Coul./cm}^2$$
$$= 2.67 \times 10^{-1} \text{ Coul./m}^2$$

Force per unit area

$$= \frac{\sigma^2}{2\epsilon_0}$$

$$= \frac{(2.67 \times 10^{-1})^2}{2 \times 8.86 \times 10^{-12}} = 4.02 \times 10^{9} \text{ Nm}^{-2}$$

2.7 Limitations of Gauss's Law

In the application of Gauss's law, the following points may be taken into account :

- Gauss's law depends on symmetry, if we can not show that symmetry exists then we cannot use Gauss's law.
- We have to look that in which coordinates of the field varies (**D** is a function of which variables)
- Which component of **D** is present.
- For closed surfaces **D** is either normal or tangential so that **D**.d**S** is not zero, then on the portion of closed surface D = constant.
- In a highly symmetrical distribution when Gauss's law is used, the integration most of time disappears.
- For even slightly complicated field it is not possible to find a suitable Gaussian surface, then Coulomb's law helps us to solve the problem and Gauss's law is not applicable.

Formulae at a Glance

1. Electric Flux : $\phi = \oint \mathbf{E}.d\mathbf{S}$

$$\mathbf{E}.d\mathbf{S} = EdS\cos\theta$$
$$\phi = EA\cos\theta$$

where $\oint dS = A$

2. Electric Flux Density

$$\mathbf{D} = \epsilon\mathbf{E} = \frac{q}{4\pi r^2}\hat{r}$$

3. Gauss's Law

$$\phi_E = \frac{q}{\epsilon_0}$$

$$\int \mathbf{E}.d\mathbf{S} = \frac{q}{\epsilon_0}$$

$$\int \mathbf{D}.d\mathbf{S} = q$$

4. Applications of Gauss's Law

(i) *Electric field due to a line charge*

$$E = \frac{\beta}{2\pi\epsilon_0 r}$$

β = charge per unit length

r = radius of the cylinder

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(ii) *Electric field due to sheet of charge*

$$E = \frac{\sigma}{2\varepsilon_0}$$

σ = charge density

(charge per unit area)

(iii) *Electric field due to charge on a conductor*

$$E = 0$$

(iv) *Electric field due to a charged non-conducting sphere.* Charge per unit volume (charge density)

$$\rho_v = \frac{q}{4\pi/3a^3}$$

(a) Within the sphere

$$E_1 = \frac{qr}{4\pi\varepsilon_0 a^3}$$

q = charge,

a = radius of sphere.

(b) On the surface of the sphere

$$E_2 = \frac{q}{4\pi\varepsilon_0 a^2}$$

(c) Outside the sphere

$$E_3 = \frac{q}{4\pi\varepsilon_0 r^2}$$

(v) Electric field around a charged conducting sphere (hollow sphere)

(a) Outside the sphere

$$E_1 = \frac{q}{4\pi r^2 \varepsilon_0 \varepsilon_r}$$

(b) On the surface

$$E_2 = \frac{q}{4\pi a^2 \varepsilon_0 \varepsilon_r}$$

(c) Within the sphere

$$E = \frac{D}{\varepsilon_0 \varepsilon_r} = 0$$

(vi) Mechanical force per unit area

$$F = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0 E^2.$$

(vii) Work done per unit volume

$$U = \frac{\delta W}{\delta V} = \frac{\varepsilon_0 E^2}{2} \text{ joules/m}^3$$

Miscellaneous Numerical Problems for Practice

Example 1. *5000 electric lines of force enter a given volume and 3000 leave it. Find the total charge contained in it.*

Solution. According to Gauss's theorem

Net electric flux $(\phi_E) = \frac{1}{\varepsilon_0} \times$ Charge enclosed (q)

Net flux ϕ_E = net no. of electric lines of force diverging from given surface

$$= -5000 + 3000 = -2000$$

Charge enclosed $q = \varepsilon_0 \phi_E = 8.86 \times 10^{-12} \times (-2000)$

$$= -17.72 \times 10^{-9}\text{ C} = -1.77 \times 10^{-8}\text{ C}$$

Example 2. *A uniformly charged sphere has a total charge of 300 μC and the radius of 8 cm. Find the electric intensity (a) at a point 16 m from the centre of sphere, (b) at a point on the surface of the sphere, (c) at a point 4 cm from the centre of the sphere i.e., at point inside the sphere.*

Solution. (a) We know that electric intensity at a point outside the charged sphere is given by

$$E = \frac{q}{4\pi\varepsilon_0 \varepsilon_r r^2}$$

In this problem

$$q = 300\,\mu\text{C},\ r = 16\text{ cm},\ \varepsilon_r = 1 \text{ (for air)},\ \varepsilon_0 = 8.854 \times 10^{-12}\text{ F/m}$$

$$\therefore \quad E = \frac{300 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times (0.16)^2} = 10.5 \times 10^7\text{ N/C}$$

(b) The intensity due to charged sphere at a point on the surface of the sphere of radius a is given by

$$E = \frac{q}{4\pi\varepsilon_0 \varepsilon_r a^2}$$

In this question $q = 300 \times 10^{-6}$ C, $R = 8 \times 10^{-2}$ m, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\varepsilon_r = 1$ (for air)

$$E = \frac{300 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times 1 \times (0.08)^2} = 42.13 \times 10^7\text{ N/C}$$

(c) The intensity at a point inside the charged sphere is zero. Hence the intensity at a point 4 cm from the centre of the sphere of radius 8 cm will be zero.

Example 3. *A charge 1μC is placed at the centre of a hollow cube. Calculate the electric flux diverging (i) through the centre (ii) through each face.*

Solution. Given $q = 1\mu\text{C} = 10^{-6}$ C

(i) Net electric flux diverging through the cube

$$\phi_E = \frac{1}{\varepsilon_0} q = \frac{1}{8.86 \times 10^{-12}} \times 10^{-6}\text{ V-m}$$

$$= 1.12 \times 10^5 \text{ volt-m}$$

(ii) As charge is placed symmetrically to all six faces of cube ; hence the electric flux through each of six faces is divided equally

$\therefore$ Electric flux through each face $\phi_1 = \frac{1}{6}\left(\frac{q}{\varepsilon_0}\right)$

$$= \frac{1}{6} \times 1.12 \times 10^5 = 5.6 \times 10^4 \text{ Volt-m}$$

Example 4. *A charge of 1500 μC is distributed over a very large sheet having surface area of 300 m². Calculate the electric field at a distance of 25 cm.*

Solution. Surface charge density $\sigma = \frac{\text{Total charge density}}{\text{Surface area}}$

$\therefore$ Surface charge density of the sheet

$$\sigma = \frac{1500 \times 10^{-6}}{300} = 5 \times 10^{-6}\text{ C/m}^2$$

Electric intensity due to charged sheet at a point is given by

$$E = \frac{\sigma}{\varepsilon_0 \varepsilon_r} = \frac{5 \times 10^{-6}}{8.854 \times 10^{-12} \times 1} = 56.472 \times 10^4\text{ N/C}$$

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Example 5. An electric field in a region is given by $\mathbf{E} = 3\hat{i} + 4\hat{j} - 5\hat{k}$. Calculate the electric flux through the surface $S = 2.0 \times 10^{-5}\,\hat{k}\,m^2$.

Solution. $\phi_E = \mathbf{E}.\mathbf{S} = (3\hat{i} + 4\hat{j} - 5\hat{k}).(2.0 \times 10^{-5}\,\hat{k})$

$$= 0 + 0 - 10 \times 10^{-5} = 1.0 \times 10^{-4}\ \text{V-m}$$

Example 6. Calculate the electric intensity required to just support an ion of mass 10^{-4} g and having a charge of 1.44 C in air.

Solution. The force required to just support the ion will be equal to the weight of the ion *i.e.*, mg and this much force must be experienced by the ion due to electrostatic field. If E is the intensity of the field, the force experienced will be the product of E and charge of ion. Therefore electrostatic force

$$F = E \times 1.44\ \text{N}$$

As stated above this must be equal to mg

$$\therefore \quad E \times 1.44 = 10^{-4} \times 10^{-3} \times 98$$

$$E = 681 \times 10^{-7}\ \text{N/C}$$

Example 7. Show that the potential difference between two points distant r_1 and r_2 from an infinitely long line charge of linear charge density (i.e., charge per unit length) is given by

$$\Delta\phi = \frac{1}{4\pi\varepsilon_0} 2\beta \log_e \frac{r_2}{r_1}$$

Solution. The field strength at a distance r from an infinite linear charge of density β is given by

$$E = \frac{1}{4\pi\varepsilon_0} \frac{2\beta}{r}$$

By definition electric potential at a distance r from the axis is

$$\phi_r = -\int_{r_{ref}}^{r} \mathbf{E}.d\mathbf{r}$$

Here r_{ref} denotes the reference distance for zero potential. Here reference distance can not be taken as infinity since the wire itself extends to infinity.

In this case we have to find the potential difference between two points distant r_1 and r_2 from the wire, which is given by

$$\therefore \quad \Delta\phi = -\int_{r_2}^{r_1} \frac{1}{4\pi\varepsilon_0} \frac{2\beta}{r} dr = \frac{1}{4\pi\varepsilon_0} 2\beta \log_e \frac{r_2}{r_1}$$

Example 8. 10 Coulomb of charge is uniformly distributed over the surface of a sphere of radius 10 cm. Find the electric intensity at : (a) a point distant 20 cm from the centre of the sphere. (b) a point on the surface of the sphere. (c) a point 5 cm from the centre of the sphere i.e., at a point inside the sphere is placed.

Solution. (*i*) We know that electric intensity at a point outside the charged sphere is given by

$$E = \frac{q}{4\pi\varepsilon_0\varepsilon_r r^2}$$

In this question

$q = 10$ Coulomb, $r = 20$ cm $= 0.2$ m, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\varepsilon_r = 1$ (for air)

$$E = \frac{10}{4\pi \times 8.854 \times 10^{-12} \times (0.2)^2} = 225 \times 10^{10}\ \text{N/C}$$

(*ii*) The intensity due to charged sphere at a point on the surface of sphere of radius a is given by

$$E = \frac{q}{4\pi\varepsilon_0\varepsilon_r a^2}; \qquad a = 10\ \text{cm} = 0.1\ \text{m}$$

$$E = \frac{10}{4\pi \times 8.854 \times 10^{-12} \times (0.1)^2} = 9 \times 10^{12}\ \text{N/C}$$

(*iii*) The intensity at a point inside the charged sphere in zero. Hence the intensity at a point 5 cm from the centre of sphere of radius 10 cm will be zero.

Example 9. Show that the potential at a point distance r_1 and r_2 respectively from centres of a long parallel pair of wires of negligible cross-section and having equal and opposite linear charge density β Coulomb per metre is given by

$$\phi = \frac{\beta}{4\pi\varepsilon_0} \log_e \frac{r_2}{r_1}$$

Solution. The field strength at distance r from the line charge is

$$E = \frac{1}{4\pi\varepsilon_0} \frac{2\beta}{r}$$

By definition, electric potential,

$$\phi = \int_{r_{ref}}^{r} \mathbf{E}.d\mathbf{r}$$

where r_{ref} denotes the reference point for zero potential.

Here reference point r_{ref} can not be taken to be infinity as charge itself extends to infinity. Therefore choosing reference point at a distance r_0 where potential is ϕ_0, we must have

Potential difference between r_0 and r

$$\phi - \phi_0 = -\int_{r_0}^{r} \mathbf{E}.d\mathbf{r} = -\int_{r_0}^{r} \frac{1}{4\pi\varepsilon_0} \frac{2\beta}{r} dr$$

$$= \frac{1}{4\pi\varepsilon_0} 2\beta \log_e \frac{r_0}{r} = \frac{\beta}{2\pi\varepsilon_0} \log_e \frac{r_0}{r}$$

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i.e., potential at distance r is

$$\phi = \frac{\beta}{2\pi\varepsilon_0}\log_e\frac{r_0}{r} + \phi_0 \qquad ...(i)$$

In case of two lines of charge of linear charge densities β and $(-\beta)$, if the point under consideration (say P) lies at distance r_1 and r_2 from wires, then potential (ϕ_1) at P due to one line charge of charge density $(+\beta)$ will be

$$\phi_1 = \frac{\beta}{2\pi\varepsilon_0}\log_e\frac{r_0}{r_1} + \phi_0 \qquad ...(ii)$$

While that due to other charge of density $(-\beta)$ will be

$$\phi_2 = \frac{-\beta}{2\pi\varepsilon_0}\log_e\frac{r_0}{r_2} + \phi_0 \qquad ...(iii)$$

Therefore the net potential due to both the wires will be

$$\phi = \phi_1 + \phi_2 = \frac{\beta}{2\pi\varepsilon_0}\log_e\frac{r_2}{r_1} + 2\phi_0 \qquad ...(iv)$$

If the potential at reference point r_0 is zero ; this is possible in this case if the reference point r_0 is at equal distance from both wires *i.e.*, choosing $\phi_0 = 0$, we have

$$\phi = \frac{\beta}{2\pi\varepsilon_0}\log_e\frac{r_2}{r_1} \qquad ...(v)$$

Hence proved.

Example 10. *A small sphere of mass 10^{-3} g and charge 4×10^{-8} C change from silk thread at 60° with a charged conducting sheet. Calculate the surface charge density for the sheet.*

Solution. Let us consider the situation as shown in Fig. 2.17. Now intensity near the charged conducting sheet

$$E = \frac{\sigma}{\varepsilon_0}$$

∴ Force exerted on the space

$$F = Eq$$

If the sphere is displaced through a distance r the work done in doing so will be

$$W = Eq\times r$$

Due to this work done the potential energy of the sphere increases by mgh

$$\therefore \quad mgh = Eqr = \frac{\sigma}{\varepsilon_0}qr$$

or

$$\sigma = \frac{mgh\,\varepsilon_0}{qr}$$

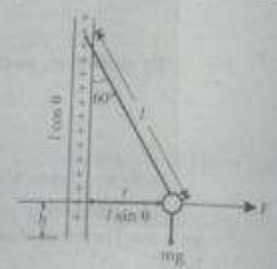

Fig. 2.17

If the length of the string is l then

$$h = l - l\cos\theta$$

$$r = l\sin\theta$$

Substituting these values we have

$$\sigma = \frac{mg(l - l\cos\theta)\varepsilon_0}{ql\sin\theta} = \frac{\varepsilon_0 mg(1-\cos 60°)}{q\sin 60°}$$

$$= \frac{8.854\times10^{-12}\times10^{-6}\times9.8(1-0.5)}{4\times10^{-8}\times0.866} = 1.252\times10^{-9}\ \text{C/m}^2$$

Question Bank

Multiple Choice Questions

1. Gauss's law states that the total electric flux through a closed surface is equal to
   (a) charge enclosed by the surface (b) $\frac{1}{\varepsilon_0}$ times the charge enclosed by the surface
   (c) charge that exists outside the surface (d) $\frac{1}{4\pi\varepsilon_0}$ times the charge enclosed by the surface.
2. The Gauss theorem in differential form is
   (a) div $\mathbf{E} = \frac{\rho}{\varepsilon_0}$ (b) div $\mathbf{E} = 0$
   (c) curl $\mathbf{E} = 0$ (d) none of the above.
3. A charge q is placed at the centre of the open end of a cylindrical vessel. The flux of electric field through the surface of vessel is
   (a) 0 (b) q/ε_0
   (c) $q/2\varepsilon_0$ (d) $2q/\varepsilon_0$
4. Inside a hollow charged spherical conductor, the potential
   (a) is constant
   (b) varies directly as the distance from the centre.
   (c) varies inversely as the distance from the centre
   (d) varies inversely as the square of the distance from the centre.
5. The electric field inside a spherical shell of uniform surface charge density is
   (a) zero (b) constant, less than zero
   (c) directly proportional to distance from centre (d) none of these.
6. A hollow sphere of charge q does not produce an electric field at any
   (a) external point (b) internal point
   (c) beyond 5 metres (d) beyond 10 metres.

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7. A hollow conducting sphere of radius R has a charge $(+q)$ on its surface. What is the electric potential within the sphere at a distance $r = \frac{R}{3}$ from its centre ?

(a) zero (b) $\frac{1}{4\pi\epsilon_0}\frac{q}{r}$

(c) $\frac{1}{4\pi\epsilon_0}\frac{q}{R}$ (d) $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$.

8. A spherical conductor of radius 2m is charged to a potential of 120V. It is now placed inside another hollow spherical conductor of radius 6m. Calculate the potential to which the bigger sphere would be raised.

(a) 20 V (b) 60 V (c) 80 V (d) 40 V.

9. The S.I. unit of electric flux is

(a) Weber (b) Newton per Coulomb

(c) Volt metre (d) Joule per Coulomb .

10. Two spheres A and B of radius a and b respectively are at same electric potential. The ratio of the surface charge densities of A and B is

(a) a/b (b) b/a (c) a^2/b^2 (d) b^2/a^2.

11. Electric field intensity at a point in between two parallel sheets due to like charges of same surface charge densities (σ) is

(a) $\frac{\sigma}{2\epsilon_0}$ (b) $\frac{\sigma}{\epsilon_0}$ (c) zero (d) $\frac{2\sigma}{\epsilon_0}$.

12. A hollow sphere of charge does not produce an electric field at any

(a) point beyond 5 metres (b) point beyond 10 metres

(c) interior point (d) outer point.

13. An infinite parallel plane sheet of a metal is charged to density σ C/m^2 in a medium of dielectric constant ϵ_r. Intensity of the electric field near the metallic surface will be

(a) $E = \frac{\sigma}{\epsilon_0\epsilon_r}$ (b) $E = \frac{\epsilon_r\sigma}{\epsilon_0}$

(c) $E = \frac{2\sigma}{\epsilon_0\epsilon_r}$ (d) $E = \frac{\sigma}{2\epsilon_0\epsilon_r}$.

14. The electric field inside a spherical shell of uniform surface charge density is

(a) directly proportional to distance from centre

(b) inversely proportional to distance from centre

(c) non-zero constant

(d) zero.

15. For a given surface, the Gauss's law is stated as $\oint \mathbf{E}.d\mathbf{S} = 0$. From this we conclude that :

(a) $\mathbf{E}$ is necessary zero as the surface

(b) $\mathbf{E}$ is perpendicular to surface at every point.

(c) the total flux, through the surface is zero

(d) the flux is only going out of the surface.

16. Two charges q_1 and q_2 are separated by a distance of 10 cm. The electric field is found to be zero at a point 4 cm from charge q_1, as shown in Fig. 2.14.

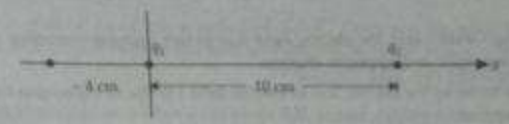

Fig. 2.18

(a) both charges are positive (b) q_1 is negative and q_2 is positive.

(c) q_2 is $\frac{49}{2}$ times greater than q_1

(d) the field cannot be zero except some point between the charges.

17. A hollow sphere of charge does not produce an electric field at

(a) interior point (b) external point

(c) surface point (d) at interior and external points.

18. Mathematical expression for Gauss theorem is

(a) $\oint \mathbf{E}.d\mathbf{S} = \epsilon\Sigma q_0$ (b) $\oint \mathbf{E}.d\mathbf{S} = \frac{\Sigma q}{\epsilon}$ (c) $\oint \mathbf{E}.d\mathbf{S} = \Sigma q$ (d) $\oint \mathbf{E}.d\mathbf{S} = \frac{\Sigma q}{4\pi\epsilon_0}$.

19. The mathematical expression for Gauss theorem in dielectrics is

(a) $\oint \mathbf{D}.d\mathbf{S} = \epsilon\Sigma q$ (b) $\oint \mathbf{D}.d\mathbf{S} = \frac{\Sigma q}{\epsilon}$ (c) $\oint_S \mathbf{D}.d\mathbf{S} = \Sigma q$ (d) $\oint_S \mathbf{D}.d\mathbf{S} = \frac{\Sigma q}{4\pi\epsilon}$.

20. A hollow metal sphere has radius r. If the potential difference between its surface and a point distance $3r$ from the centre is V, then the electric field intensity at a distance $3r$ from the centre

(a) $V/6r$ (b) $V/4r$ (c) $V/3r$ (d) $V/2r$.

Answers

1. (b)	2. (a)	3. (d)	4. (c)	5. (a)	6. (b
7. (c)	8. (d)	9. (c)	10. (c)	11. (a)	12. (c
13. (d)	14. (d)	15. (c)	16. (b) and (c)	17. (a)	18. (b
19. (c)	20. (a)				

Theoretical Questions

1. Use Gauss's law to find the electric field inside and outside of a uniformly charged sphere (σ density σ) of radius R. [GGSIPU, May 2008 (8.5 m

2. State Gauss's law and use it to show that any excess charge placed on an isolated conductor resides entirely on its surface. [GGSIPU, May 2005 (2.5 m

3. State Gauss's law in electrostatics. Apply it to find the electric field strength due to a sp charge distribution of radius 'a' at an external point P, which is at a distance r from the ce charge distribution. [GGSIPU, Feb. 2005 (4 m

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4. State and derive Gauss's law in electrostatics.
5. Calculate the intensity of electric field due to an infinite straight line charge having linear density of charge β.
6. Find by using Gauss's law, the electric field due to two uniform concentric shells of radii a and b ($b > a$) carrying equal and opposite charges.
7. What do you mean by the electric flux of electric field ? Write an expression for the electric flux over any closed surface.
8. State and prove Gauss's theorem. Apply it to find the electric field strength of a spherically symmetric charge distribution at (*i*) an external point, (*ii*) an internal point (*iii*) the surface of charge distribution.
9. Obtain an expression for the field intensity at any point due to an infinite cylindrical charge by Gauss theorem.
10. Obtain an expression for field intensity at any point due to (*i*) a conducting sphere (*ii*) a non-conducting sphere by Gauss law.
11. Explain the terms electric flux. Explain how the flux through a surface is related to the field intensity on the surface.
12. With the help of Gauss's law evaluate the electric field at a point.
    (*a*) due to an infinitely long line charge.
    (*b*) due to an uniformly charged infinite sheet.
13. A certain amount of charge is given to a good conductor. Explain, using Gauss's law, why the charges reside on the surface of the conductor.
14. Derive expressions for electric field due to a charged non-conducting sphere at a point.
    (*a*) within the sphere (*b*) on the surface of the sphere (*c*) outside the sphere
15. Derive expression for the electrostatic potential at a point :
    (*a*) due to a point charge (*b*) due to a line charge
    (*c*) due to a uniformly charged conducting sheet.
16. Using Gauss's law in integral from, obtain the electric field due to following charge distribution in spherical coordinates

    Charge density $\rho = \frac{k}{r^2}$ for $0 < r < a$

    $= 0$ for $a < r < \infty$

    where a is the radius of spherical volume, k is constant and r is radial distance.

    [GGSIPU, June 2006 (6 marks)]
17. A spherical rubber balloon carries a charge that is uniformly distributed over its surface. How does E vary for points (*a*) inside (*b*) on the surface and (*c*) outside the balloon, as the balloon is blown up ?
18. What is electric flux ? Compute the electric field intensity due to uniformly charged sphere, using Gauss's law, inside and outside the sphere.
19. Using Gauss's law to determine the electric intensity inside and outside a hollow spherical conductor, inside and outside radii being r_1 and r_2 carrying a charge q.
20. Find the electric field due to a non-conducting infinite plane sheet of charge with uniform charge density ρ C/m^2, using Gauss's law.
21. Give the limitations of Gauss's law.

Numerical Problems

1. A thin electric shell of metal has a radius of 0.25μC. Compute the electric intensity at a point (*i*) inside the shell (*ii*) just outside the shell and (*iii*) 3 m from the centre of the shell.

   [Ans. (*i*) zero, (*ii*) 2.88×10^4 N/C (*iii*) 200 N/C**]**
2. A thin spherical shell of metal has a radius of 0.25 m and carries a charge of 0.2μC. Evaluate the electric field at a point (*i*) 3 m from the centre of the shell (*ii*) just outside the shell and (*iii*) inside the shell. Given $\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$. **[Ans.** (*i*) 200 N/C (*ii*) 2.88×10^4 N/C (*iii*) zero**]**
3. The flux entering a closed surface is 2×10^3 Nm^2/C and flux leaving the closed surface is 8×10^3 Nm^2/C. Calculate the charge enclosed by the surface. **[Ans.** 5.31×10^{-8} C**]**
4. A point charge of 12 C is located at the centre of a cube of side 5 m. Calculate the electric flux through each of the surface. **[Ans.** 2 C**]**
5. A circular plane sheet of radius 1 m is placed in an electric field of 200 N/C making an angle 60° with the field. Compute the electric flux through the sheet. **[Ans.** $\phi_E = 2.78 \times 10^{15}$ C**]**
6. If a heavy atom can be pictured as a spherical nucleus with charge $+q$ and radius a embedded in a much larger sphere of negative charge, the electrons. This negative charge is distributed uniformly throughout this sphere of radius b. Find the electric field intensity outside the nucleus.

   [Ans. $E = \frac{q}{4\pi\epsilon_0 r^2}\left[1 - \left(\frac{r}{b}\right)^3\right]$**]**
7. A dielectric cylinder of radius a is infinitely long. It is non-uniformly charged such that volume charge density ρ varies directly as the distance from the axis of the cylinder. Calculate the electric intensity due to it, if ρ is zero at the axis and is ρ_s on the surface. **[Ans.** $E = \frac{\rho_s a}{3\epsilon_0}$**]**
8. A gold foil weighing 50 mg per sq cm placed on a horizontal charged plate. What would be the density of charge so that the foil may just rise ? **[Ans.** $\sigma = 9.313 \times 10^{-6}$ C/m^2**]**
9. A spherical shell of outer and inner radii a and b respectively is concentric to a metal sphere (of radius C) inside the shell. Find the net charge on the outer surface of the spherical shell, if the electric field at a point P outside the shell at a distance 40 cm from the centre is 200 N/C.

   [Ans. 3.55×10^{-9} C**]**
10. The inner surface of a non-conducting hemi-spherical bowl of radius a has uniformly spread charge of surface density σ over it. Find the electric field at the centre of the flat surface of the bowl.

    [Ans. $\frac{\sigma}{4\epsilon_0}$ N / C**]**
11. A thin circular ring of radius 20 cm is charged with a uniform charge density β C/m. A small section of 1 cm length is removed from the ring. Find the electric field intensity at the centre of the ring.

    [Ans. $2.25\beta \times 10^9$ N / C**]**
12. A circular ring of radius a carries a charge which varies as $\beta = \beta_0 \sin\theta$. Find the electric field at the centre of the ring.

    [Ans. $\frac{\beta_0}{4a\epsilon_0}$**]**
13. A uniform line charge with $\beta = 50$μC / m lies along the x-axis. Calculate the flux per unit length crosses the portion of the Z = -3m plane bounded by y = ±2m. **[Ans.** 6.36μC / m, 9.36 / ϵ_0 μWb / m**]**

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14. An electric charge q is uniformly distributed over the surface of a sphere of radius a. Show that the force on a small charge element dq is radial and outward and is given by $dF = (1/2)\,E\,dq$, where E is the electric field at the surface of the sphere.

15. A point charge q is at the origin of a spherical coordinate system. Find the which crosses the portion of a spherical shell described by $\alpha \le \theta \le \beta$. What will happen if $\alpha = 0$ and $\beta = \frac{\pi}{2}$.

[**Ans.** $\phi = \frac{q}{2\varepsilon_0}(\cos\alpha - \cos\beta)$, $\phi = \frac{q}{2\varepsilon_0}$]

16. Let $\rho_v = \begin{cases} \frac{10}{r^2}\ \text{mC/m}^3 & 1 < r < 4 \\ 0 & r > 4 \end{cases}$

(a) Find the net flux crossing surface $r = 2$ m and $r = 6$m.

(b) Determine D at $r = 1$m and $r = 5$m.

[**Ans.** (a) 251.3 mC, 502.6 mC ; (b) $10\hat{a}_r$ mC/m^2, $1.6\,\hat{a}_r$ mC/m^2]

17. In a spherical coordinate system, the volume charge density is $\rho_v = \rho_0\left(\frac{r}{a}\right)^{1/2}$ C/m^2

(i) How much charge lies in a sphere of radius 'a'?

(ii) Find the electric flux density at $r = a$. [**Ans.** $q = \frac{8}{9}\rho_0 \pi a^3$, $E = \frac{2a\rho_0}{9\varepsilon_0}\hat{a}_r$]

18. A positive charge q is uniformly distributed over an imaginary thin spherical shell of radius 'a' in air. Using Gauss's law, calculate the field intensity at any point inside and outside the shell. What shall be the field intensity at the surface of the shell. [**Ans.** $E = \frac{q}{4\pi\varepsilon_0 a^2}$]

19. A charged sphere of 7μC is placed in air. Find the electric intensity at a point 10 cm from the centre of the sphere. [**Ans.** 63×10^5 N/C]

20. 10C of charge is uniformly distributed over the surface of a sphere of radius 10cm. Find electric intensity at

(i) a point distant 20cm from the centre of the sphere

(ii) at a point on the surface of the sphere.

(iii) at a point 5cm from the centre of the sphere *i.e.*, at a point inside the sphere, the sphere is placed in air. [**Ans.** (i) 225×10^{10} N/C ; (ii) 9×10^{12} N/C ; (iii) zero]

23. Calculate the electric field intensity due to a spherical charge distribution given by

$$\rho = \rho_0\left(1 - \frac{r}{a}\right) \quad \text{where } r \le a$$
$$= 0 \quad \text{where } r > a$$

Find the values of r for which E is maximum. [**Ans.** $r = \frac{2a}{3}$]

Chapter 3

Scalar and Vector Fields

3.1 Introduction

Many physical quantities have different values at different points in space. For example, the temperature in a room is different at different points : high near a register, low near an open window and so on. The electric field around a point charge is large near the charge and decreases as we go away from the charge. Similarly, the gravitational force acting on satellite depends on its distance from the earth. The velocity of flow of water in a stream is large in rapids and in narrow channels and small over flat areas and where the stream is wide. In all these examples, there is a particular region of space which is of interest for the problem at hand ; at every point of the region some physical quantity has a value. The term **field** is used to mean both the region and the value of the physical quantity in the region (for example, electric field, gravitational field). If the physical quantity is a scalar (for example, temperature), we speak a **scalar field**. If the quantity is a vector (for example, electric field, force or velocity), we speak of a **vector field**.

A simple example of a scalar field is the gravitational potential energy near the earth ; its value is $V = mgz$ at every point of height z above some arbitrary reference level [which we take as the (x, y) plane]. Suppose that on a hill (Fig. 3.1) we mark a series of curves each corresponding to some value of z (curves of constant elevation, often called **contour lines** or **level lines**). Any curve or surface on which a potential is constant is called an **equipotential**. Thus these level lines are equipotentials of the gravitational field since along any one curve the value of the gravitational potential energy mgz is constant. The horizontal planes which intersect the hill in these curves are equipotential surfaces (or level surfaces) of the gravitational field.

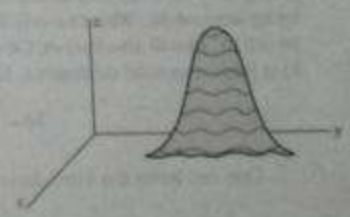

Fig. 3.1 Contour lines.

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As another example, let us ask for equipotential surfaces in the field of an electric point charge q. The potential is $V = 9 \times 10^9 \frac{q}{r}$ (in S.I. unit) at a point which is at a distance r from the charge. The potential V is constant if r is constant ; *i.e.*, the equipoteitials of the electric field are spheres with centres at the charge. Similarly we could imagine drawing a set of surfaces (probably very irregular) in a room so that at every point of a single surface the temperature would be constant. These surfaces would be like equipotentials, they are called isothermal when the constant quantity is the temperature.

3.2 Partial Derivative

Let a physical quantity ϕ is a position function of the coordinates (x, y, z) of the point of observation, where x, y and z independent variables, then

$$\phi = \phi(x, y, z) \qquad ...(1)$$

keeping y and z constant, let us change the value of x by an infinitesimal amount Δx and the value of ϕ changes by an amount of $\Delta\phi$ then for constant y and z the rate of change ϕ with respect to x, $\left(\frac{\Delta\phi}{\Delta x}\right)$ is known as partial derivative of ϕ with respect to x. Usually it is expressed as $\left(\frac{\partial\phi}{\partial x}\right)_{y,z}$ or simply $\frac{\partial\phi}{\partial x}$. Obviously, Δx is considered infinitely small *i.e.*, $\Delta x \to 0$, we have

$$\frac{\partial\phi}{\partial x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y, z) - \phi(x, y, z)}{\Delta x} \qquad ...(2)$$

Similarly, if one keeps x and z constant and the value of y is changed by an infinitesimal amount Δy, then

$$\frac{\partial\phi}{\partial y} = \lim_{\Delta y \to 0} \frac{\phi(x, y + \Delta y, z) - \phi(x, y, z)}{\Delta y} \qquad ...(3)$$

and if one keeps x and y constant, and the value of z is changed by an infinitesimal amount Δz, then

$$\frac{\partial\phi}{\partial z} = \lim_{\Delta z \to 0} \frac{\phi(x, y, z + \Delta z) - \phi(x, y, z)}{\Delta z} \qquad ...(4)$$

Now, assumed that all the three independent variables x, y and z are simultaneously changed by an infinitesimal amount δx, δy and δz respectively and the value of ϕ is changed by an amount $\delta\phi$. When $\delta x \to 0$, $\delta y \to 0$ and $\delta z \to 0$ *i.e.*, all the three vanish simultaneously $\delta\phi \to 0$ *i.e.*, $\delta\phi$ will also vanish. Obviously, in this state, the total change in the value of ϕ *i.e.*, $\delta\phi$ is known as total derivative. Thus

$$\delta\phi = \left(\frac{\partial\phi}{\partial x}\right)\delta x + \left(\frac{\partial\phi}{\partial y}\right)\delta y + \left(\frac{\partial\phi}{\partial z}\right)\delta z$$

One can write the total derivative for the limiting change as

$$d\phi = \left(\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial\phi}{\partial z}\right)dz \qquad ...(5)$$

The partial derivatives $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$, and $\frac{\partial\phi}{\partial z}$ are called **first partial derivatives** or partial derivative of first order. By differentiating these derivatives once more, one can obtain the **second partial derivatives** (or partial derivatives of the second order) *e.g.*, $\frac{\partial^2\phi}{\partial x^2}$, $\frac{\partial^2\phi}{\partial y\partial x}$, $\frac{\partial^2\phi}{\partial x\partial y}$, $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial z^2}$ etc. Obviously $\frac{\partial^2\phi}{\partial x^2}$ is the partial derivative of $\frac{\partial\phi}{\partial x}$ with respect to x, $\frac{\partial^2\phi}{\partial x\partial y}$ is the partial derivative of $\frac{\partial\phi}{\partial x}$ with respect to y etc. If all the derivatives concerned are continuous, then $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ *i.e.*, then the order of differentiation is immaterial. By differentiating the second partial derivatives again with respect to x, y and z respectively, one obtains the third partial derivatives and so on.

If $\mathbf{A}$ is vector depending on more than one scalar variables such as cartesian coordinates x, y, z of a point in space, then $\frac{\partial\mathbf{A}}{\partial x}$, $\frac{\partial\mathbf{A}}{\partial y}$ and $\frac{\partial\mathbf{A}}{\partial z}$ are partial derivatives of $\mathbf{A}$ with respect to x (when y and z remain constant), y (when x and z remain constant) and z (when x and y remain constant) respectively, if now x, y and z change simultaneously by small increments dx, dy and dz, then the total change or total derivative of $\mathbf{A}$ will be

$$d\mathbf{A} = \frac{\partial\mathbf{A}}{\partial x}dx + \frac{\partial\mathbf{A}}{\partial y}dy + \frac{\partial\mathbf{A}}{\partial z}dz$$

$$= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{A}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})\mathbf{A} \qquad ...(6)$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the direction of x-, y- and z-axes respectively. $\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ is defined as the vector differential operator, written as ∇ and read as 'del'.

Thus $$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \qquad ...(7)$$

Further if $\mathbf{r}$ is the position vector of the point (x, y, z) with respect to origin

$$\mathbf{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z \qquad ...(8)$$

So, that $$d\mathbf{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz \qquad ...(9)$$

The Eq. (6) can be expressed as

$$d\mathbf{A} = (\nabla \cdot d\mathbf{r})\mathbf{A} \qquad ...(10)$$

Here ∇ is a vector quantity possessing all the properties of an ordinary vector. Since it is an 'operator', its magnitude has no physical significance.

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The differential operator $\nabla \cdot \nabla = \nabla^2$ is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad ...(11)$$

∇^2 is a scalar and is known as 'Laplacian operator'.

3.3 Vector Integration

The integral, which we generally come across in vector algebra are the line integral, the surface integral and the volume integral.

3.3.1 Line Integral

Let PQ be any curve drawn in a vector field and $d\mathbf{l}$ an element of length along this curve at any point B. Let A represents the vector at B in the direction making an angle θ with $d\mathbf{l}$. In general if $\mathbf{A}$ varies in magnitude and direction from point to point along the curve PQ, the integral

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} = \int_P^Q A \cos\theta \, dl \quad ...(12)$$

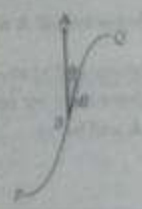

Fig. 3.2 Line integral.

is defined the line integral of $\mathbf{A}$ along PQ.

In terms of cartesian coordinate components

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} = \int_P^Q (A_x dx + A_y dy + A_z dz) \quad ...(13)$$

[where $\mathbf{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$; $d\mathbf{l} = \hat{i} dx + \hat{j} dy + \hat{k} dz$]

Therefore, the integration of a vector along a line or curve is known as its **line integral**.

Thus if a particle is moving under the influence of a force $\mathbf{F}$ along the curve PBQ then $\int_P^Q \mathbf{F} \cdot d\mathbf{l}$ will be equal to the total work done by the force on the particle for its entire path from P to Q.

If $\mathbf{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ and $d\mathbf{l} = \hat{i} dx + \hat{j} dy + \hat{k} dz$

then $$\int_P^Q \mathbf{F} \cdot d\mathbf{l} = \int_P^Q (F_x dx + F_y dy + F_z dz)$$

Example 3.1. If a force $\mathbf{F} = 2x^2 y \hat{i} + 3xy \hat{j}$ displaces a particle in the x-y plane from (0, 0) to (1, 4) along the curve $y = 4x^2$. Find the workdone.

Solution. Work, $W = \int_c \mathbf{F} \cdot d\mathbf{r}$

In x-y plane, $\mathbf{r} = x\hat{i} + y\hat{j} \Rightarrow d\mathbf{r} = dx\hat{i} + dy\hat{j}$

$$W = \int_c (2x^2 y \, dx + 3xy \, dy)$$

Given $y = 4x^2 \Rightarrow dy = 8x \, dx$

$$\therefore \quad W = \int_0^1 [2x^2 \cdot (4x^2) dx + (3x \cdot 4x^2)(8x \, dx)]$$

$$= 104 \int_0^1 x^4 \, dx = 104 \left[\frac{x^5}{5}\right]_0^1 = \frac{104}{5}$$

3.3.2 Surface Integral

Consider any element of infinitesimal area dS upon a surface S in vector field whose value is $\mathbf{A}$ at the middle point of the element, then

$$\iint_S \mathbf{A} \cdot d\mathbf{S} = \iint_S A \cos\theta \, dS \quad ...(14)$$

is defined as the surface integral or total flux of vector $\mathbf{A}$ through the whole surface.

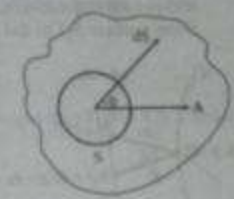

Fig. 3.3 Surface integral.

In cartesian coordinate components

$$\iint_S \mathbf{A} \cdot d\mathbf{S} = \iint_S (A_x dS_x + A_y dS_y + A_z dS_z) \quad ...(15)$$

[where $\mathbf{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$; $d\mathbf{S} = \hat{i} dS_x + \hat{j} dS_y + \hat{k} dS_z$]

The integration of a vector over the surface of the body is known as its surface integral.

3.3.3 Volume Integral

If we consider a closed surface in space enclosing a volume V.

Then,

$$\iiint_V \mathbf{A} \, dV = \iiint_{x\,y\,z} (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \, dx \, dy \, dz \quad ...(16)$$

[where $\mathbf{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $dV = dx \, dy \, dz$]

$$= \hat{i} \iiint_{x\,y\,z} A_x \, dx \, dy \, dz + \hat{j} \iiint_{x\,y\,z} A_y \, dx \, dy \, dz + \hat{k} \iiint_{x\,y\,z} A_z \, dx \, dy \, dz$$

3.4 Physical Significance of Gradient, Divergence and Curl

3.4.1 Gradient

If $\phi(x, y, z)$ be a scalar function of position in space (*i.e.*, of coordinates of x, y and z) then its partial derivatives along the three orthogonal axes are $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$.

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The gradient of the scalar function ϕ is defined by

$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \quad ...(16)$$

where $\nabla\,(del) = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ (∇ = Nabla or del)

> ∇ operator is a vector operator and when it operates with a scalar, it converts the scalar into vector.

Consider two level surfaces S_1 and S_2 very close together having constant value of ϕ and $(\phi + d\phi)$ respectively. If **dn** represents the displacement along the normal from the point P (on surface S_1) to the surface S_2, we may write

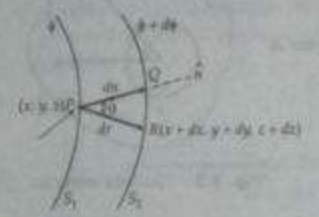

Fig. 3.4 Gradient of ϕ.

$$\mathbf{dn} = dr\cos\theta = \hat{n}\cdot\mathbf{dr}$$

where $\hat{n}$ is the normal unit vector to surface S_1 at P.

The rate of increase of ϕ at P in the direction of $\hat{n}$ is greatest and is equal to $\frac{\partial \phi}{\partial n}$.

Therefore,

$$d\phi = \frac{\partial \phi}{\partial n}dn = \frac{\partial \phi}{\partial n}\hat{n}\cdot d\mathbf{r} \quad ...(17)$$

But $\quad \nabla\phi\cdot d\mathbf{r} = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi$$

Hence, $\quad d\phi = \frac{\partial \phi}{\partial n}\cdot\hat{n}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r}$

Therefore, $\quad \nabla\phi = \left(\frac{\partial \phi}{\partial n}\right)\hat{n} \quad ...(18)$

It concludes that the gradient of a scalar function ϕ is a vector whose magnitude at any point is equal to maximum rate of change of scalar function with respect to distance along the direction of that change.

Example 3.2. *If $\phi(x, y, z) = 3(x^2y - y^2x)$; calculate gradient at the point (1, – 2, – 1).*

Solution. We know that,

$$\text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

Here $\quad \phi(x, y, z) = 3x^2y - 3y^2x$

$$\therefore \quad \text{grad } \phi(x, y, z) = \nabla\,(3x^2y - 3y^2x)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - 3y^2x)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - 3y^2x) + \hat{j}\frac{\partial}{\partial y}(3x^2y - 3y^2x) + \hat{k}\frac{\partial}{\partial z}(3x^2y - 3y^2x)$$

$$= (6xy - 3y^2)\hat{i} + (3x^2 - 6yx)\hat{j} + 0(k)$$

At points (1, –2, –1),

$$\text{grad } \phi = [6\times1\times(-2) - 3(-2)^2]\hat{i} + [3\times(1)^2 - 6(-2)(1)]\hat{j}$$

$$= (-12 - 12)i + (3 + 12)j = -24i + 15j.$$

3.4.2 Divergence

If $A(x, y, z)$ is a vector field, the scalar product of the vector operator ∇ and **A** is a scalar and is called the divergence of **A** *i.e.*, [Scalar Product]

In cartesian coordinates components,

$$\nabla\cdot\mathbf{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}\,A_x + \hat{j}\,A_y + \hat{k}\,A_z)$$

$$= \frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad ...(19)$$

Hence $\nabla\cdot\mathbf{A}$ is a scalar function. The vector field is called solenoidal if its divergence vanishes *i.e.*, $\nabla\cdot\mathbf{A} = 0$.

If the vector function **A** spread out (diverges) from a point, then it has a **positive** divergence at that point and acts as a **source** of the field **A**. On the other hand $\nabla\cdot\mathbf{A}$ is **negative** if the point acts as a **sink** of the field **A**.

Let **A** be the value of a vector field at the mid point $P(x, y, z)$ of a small rectangular parallelopiped with sides dx, dy and dz. The axial components of the vector **A** are A_x, A_y and A_z. We may think of **A** as the velocity of a fluid.

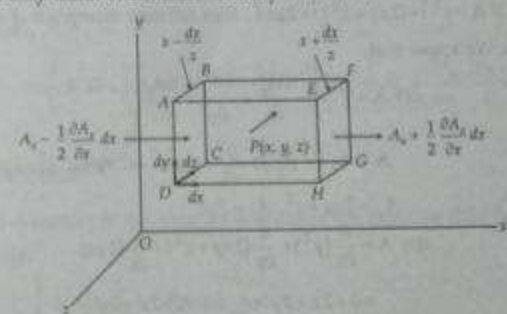

Fig. 3.5 Divergence of **A**.

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On the face $ABCD$ at the middle, the value of the x-component of the vector $\mathbf{A}$ is

$$A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx$$

As the face is infinitesimally small, this value can be taken as uniform over the face $ABCD$. Similarly for the face $EFGH$, the x-component is

$$A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx$$

But the **flux** through a face is defined as the *product of the area of the face and the normal component of the vector upon it.*

$\therefore$ Flux entering the face $ABCD = \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right)dy\,dz$

and flux leaving the face $EFGH = \left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right)dy\,dz$

Hence the excess of flux leaving the parallelopiped over that entering it in the x-direction is

$$\left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right)dy\,dz - \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right)dy\,dz = \frac{\partial A_x}{\partial x}dx\,dy\,dz$$

[In the Hydromechanical case, this represents the net volume of the fluid passing per second in the x-direction. Similarly, contribution parallel to y and z directions are $\frac{\partial A_y}{\partial y}dx\,dy\,dz$ and $\frac{\partial A_z}{\partial z}dx\,dy\,dz$.]

Hence the net flux diverging from the element,

$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)dx\,dy\,dz$$

$$= (\nabla \cdot \mathbf{A})dx\,dy\,dz = (\text{div } \mathbf{A})dx\,dy\,dz \qquad ...(20)$$

Thus $\nabla \cdot \mathbf{A}$ (or div $\mathbf{A}$) at a point gives the amount of flux diverging per unit volume from that point.

Example 3.3. *If* $\mathbf{A} = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$, *then calculate divergence of* $\mathbf{A}$ *at point (1, 2, 3).*

Solution. We know that,

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Here,

$$\mathbf{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$$

$\therefore A_x = y^2$, $A_y = 2xy + z^2$, $A_z = 2yz$

Hence,

$$\text{div } \mathbf{A} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= 0 + 2x + 2y \text{ or div } \mathbf{A} = 2x + 2y$$

At point (1, 2, 3); div $\mathbf{A} = 2(1) + 2(2) = 6$.

3.4.3 Curl

If $\mathbf{A}(x, y, z)$ is a vector field, the cross product of the operator ∇ and vector $\mathbf{A}$ is a **vector**. It is denoted by $\nabla \times \mathbf{A}$ and known as curl $\mathbf{A}$. We can obtain, [Vector Product]

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \text{[Rotation A]}$$

NOTE

If curl $\mathbf{A} = 0$ then $\mathbf{A}$ is known irrotational vector.

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \qquad ...(21)$$

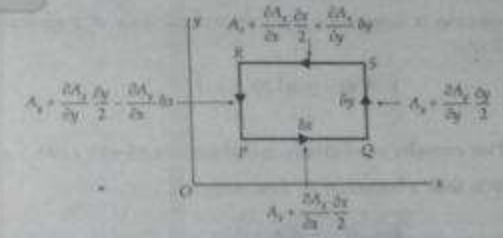

Fig. 3.6 Curl of **A**.

Line Integral of a Vector Field A Around Rectangle

Let the x and y components of $\mathbf{A}$ at the point P be A_x and A_y respectively.

The average value of the x-component of $\mathbf{A}$ over the path PQ can be taken as

$$A_x + \frac{\partial A_x}{\partial x}\frac{\delta x}{2}$$

and over the path RS as

$$A_x + \frac{\partial A_x}{\partial x}\frac{\delta x}{2} + \frac{\partial A_x}{\partial y}\delta y$$

Therefore,

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} + \int_R^S \mathbf{A} \cdot d\mathbf{l} = \left(A_x + \frac{\partial A_x}{\partial x}\frac{\delta x}{2}\right)\delta x - \left(A_x + \frac{\partial A_x}{\partial x}\frac{\delta x}{2} + \left(\frac{\partial A_x}{\partial y}\right)\delta y\right)\delta x$$

$$= -\frac{\partial A_x}{\partial y}\delta x\,\delta y \qquad ...(22)$$

because the direction of the path RS is towards the negative direction of x.

Similarly,

$$\int_Q^R \mathbf{A} \cdot d\mathbf{l} + \int_S^P \mathbf{A} \cdot d\mathbf{l} = +\frac{\partial A_y}{\partial x}\delta x\,\delta y \qquad ...(23)$$

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Adding Eqs. (22) and (23), we get the line integral

$$\oint_{PQRS} \mathbf{A}\cdot d\mathbf{l} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\delta x\,\delta y$$

But $\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$ is the z-component of curl$\mathbf{A}$ [$(\nabla \times \mathbf{A})$] and the infinitesimal area $\delta x\,\delta y$ may be regarded as the magnitude of the vector area $dS_z\,\hat{k}$. Thus,

$$\oint_{PQRS} \mathbf{A}\cdot d\mathbf{l} = (\text{curl } \mathbf{A})_z\,\delta S_z = (\text{curl } \mathbf{A})\cdot\delta S_z\,\hat{k}$$

This equation is correct for any infinitesimal area of magnitude δS_z at P, having boundary C, i.e.,

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = (\text{curl } \mathbf{A})\cdot\delta S_z\,\hat{k}$$

Now, if we consider any infinitesimal plane area $\delta\mathbf{S} = \delta S_x\,i + \delta S_y\,j + \delta S_z\,\hat{k}$ at P oriented in any direction with a boundary C, then evidently

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \text{curl } \mathbf{A}\cdot\delta S \qquad ...(24)$$

The maximum value of $\oint_C \mathbf{A}\cdot d\mathbf{l} = |\text{curl } \mathbf{A}|\,\delta S$

The magnitude of curl$\mathbf{A}$ at a point may be defined as the maximum value of the integral of $\mathbf{A}$ along a boundary C ($\oint \mathbf{A}\cdot d\mathbf{l}$) per unit area ; this boundary C is that of an infinitesimal plane area (δS) at the point under consideration.

Example 3.4. Given a vector $\mathbf{A} = \hat{i}(x^2-y^2) + \hat{j}(x+y) + \hat{k}(x^2-z^2)$, *calculate curl* $\mathbf{A}$ *at* $x = \frac{1}{2}$, $y = 3$ *and* $z = 1$.

Solution. We know that,

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or

$$\text{curl } \mathbf{A} = \hat{i}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{j}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) + \hat{k}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$$

Here

$$\mathbf{A} = \hat{i}(x^2-y^2) + \hat{j}(x+y) + \hat{k}(x^2-z^2)$$

$\therefore$

$$A_x = x^2 - y^2;\quad A_y = x + y \text{ and } A_z = x^2 - z^2$$

$\therefore$

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2) & (x+y) & (x^2-z^2) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x^2-z^2) - \frac{\partial}{\partial z}(x+y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x^2-y^2) - \frac{\partial}{\partial x}(x^2-z^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x^2-y^2)\right]$$

$$= \hat{i}[0-0] + \hat{j}[0-2x] + \hat{k}(1+2y)$$

$$\text{curl}\mathbf{A} = -2x\,\hat{j} + (1+2y)\hat{k}$$

At point $x = \frac{1}{2}$, $y = 3$ and $z = 1$,

$$\text{curl}\mathbf{A} = -2\times\frac{1}{2}\hat{j} + (1+2\times 3)\hat{k}$$

$$\Rightarrow \quad \text{curl}\mathbf{A} = -\hat{j} + 7\hat{k}$$

3.5 Gauss's Law, Gauss's Divergence Theorem or Gauss Transformation Formula :

The important Gauss-divergence theorem (Gauss's law) is oftenly used to transform a volume integral into surface integral. It states that, the *volume integral of the divergence of vector field* $\mathbf{A}$ *taken over any volume* V *bounded by a closed surface* S *is equal to the surface integral of* $\mathbf{A}$ *taken over the surface* S. Mathematically it can be expressed as,

$$\iiint_V \text{div } \mathbf{A}\,dV = \iint_S \mathbf{A}\cdot d\mathbf{S} \qquad ...(25)$$

Proof

To prove the theorem, consider a closed surface S of arbitrary shape in a vector field $\mathbf{A}$ (Fig. 3.7). Suppose the surface S encloses a volume V. Let $ABCDEFGH$ be a small cubical volume element of volume dV within the surface S. Since the amount of flux diverging per unit volume per second is represented by div$\mathbf{A}$, therefore the flux diverges from the volume element dV is div$\mathbf{A}\,dV$. Hence the total flux diverging through the entire volume V enclosed by the whose surface S is obtained by taking volume integral as

$$\iiint_V \text{div } \mathbf{A}\,dV \qquad ...(26)$$

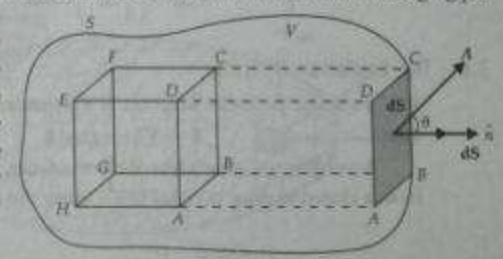

Fig. 3.7 Gauss's Theorem.

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Now, suppose dS be any surface element upon the surface S. If $\hat{n}$ be a unit vector along the outward drawn normal upon dS and θ be the angle between $\mathbf{A}$ at dS and $\hat{n}$, then the component of $\mathbf{A}$ along $\hat{n} = A\cos\theta = \mathbf{A}\cdot\hat{n}$

Therefore, the flux of $\mathbf{A}$ through the surface element dS.

$$= (\mathbf{A}\cdot\hat{n})dS = \mathbf{A}\cdot d\mathbf{S}$$

where $d\mathbf{S}$ is the area vector along $\hat{n}$

Hence, the total flux through the entire surface S is obtained by taking surface integral over the surface S as

$$\iint_S \mathbf{A}\cdot d\mathbf{S} \qquad ...(2$$

This total flux through the entire surface S must be equal to the total flux diverging from the volume V enclosed by the surface S. Therefore, Eqs. (26) and (27) are equal.

$$\iiint_V div\,\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot d\mathbf{S}$$

Hence, theorem is proved.

Example 3.5. Use Gauss divergence theorem to evaluate $\iint_S \mathbf{A}\cdot d\mathbf{S}$, *where* $\mathbf{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S *is the surface of sphere* $x^2+y^2+z^2=a^2$.

Solution. From Gauss divergence theorem,

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_V div\,\mathbf{A}\,dV$$

$$= \iiint_V \nabla\cdot\mathbf{A}\,dV = \iiint\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\,dV$$

$$= \iiint_N (3x^2 + 3y^2 + 3z^2)\,dV = 3\iiint (x^2+y^2+z^2)\,dV$$

$$= 3a^2\iiint dV = 3a^2 \times \text{Volume of sphere}$$

$$= 3a^2 \times \left(\frac{4}{3}\pi a^3\right) = 4\pi a^5$$

3.6 The Laplacian Operator

If ϕ be a scalar potential function. If ϕ represents an electric field $\mathbf{E}$, then

$$\mathbf{E} = -\nabla\phi = -\text{grad}\,\phi \qquad ...(28)$$

According to Gauss's law of electrostatics, the divergence of an electric field emerging from a closed surface is equal to $\frac{1}{\varepsilon_0}$ times the charge density enclosed in the surface *i.e.*,

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad ...(29)$$

Here ε is the permitivity of vacuum. Substituting Eq. (28) in Eq. (29), one obtains

$$-\nabla\cdot\nabla\phi = \frac{\rho}{\varepsilon_0} \qquad ...(30)$$

But $\nabla\cdot\nabla\phi = \text{div (grad }\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi$$

Thus $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is known as Laplacian operator or simply Laplacian $\nabla^2 = \nabla\cdot\nabla$ is usually pronounced as del squared.

3.7 Stoke's Theorem, Stoke's Curl Theorem or Stoke's Transformation Formula

Stoke's theorem is one of the important theorem which relates a line integral to a surface integral over any surface of which the line integral path is a boundary. Stoke's theorem may be stated as, *the surface integral of the curl of a vector field* $\mathbf{A}$ *taken over any surface* S *is equal to the line integral of* $\mathbf{A}$ *around a closed curve forming the periphery of the surface.*

Mathematically it can be expressed as

$$\iint_S (curl\,\mathbf{A})\,d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{l} \text{ or } \iint_S (\nabla\times\mathbf{A})\,d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{l} \qquad ...(31)$$

Proof

To prove the stoke's theorem, let us consider a closed curve in x-y plane in a vector field $\mathbf{A}$ as shown in Fig. 3.8. The line integral of $\mathbf{A}$ around the closed curve C forming the periphery of the surface traced counter clockwise is

$$\oint_C \mathbf{A}\cdot d\mathbf{l} \qquad ...(32)$$

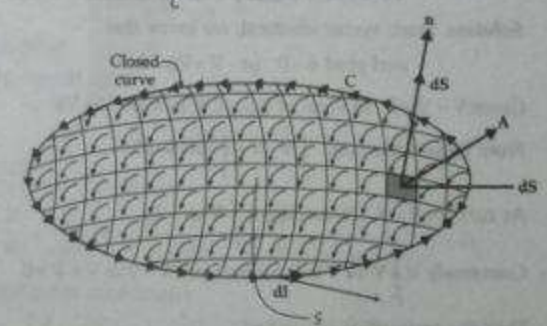

Fig. 3.8 Stoke's Theorem

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Let us divide the whole surface area S inside the curve C into large number of equal infinitesimal rectangular elements by a network of lines. Consider one such element of area $d\mathbf{S}$. If $\hat{n}$ is a unit vector along the outward drawn normal upon $d\mathbf{S}$, then the vector area of the element

$$\hat{n} \cdot d\mathbf{S} = d\mathbf{S} \qquad \text{...(33)}$$

According to the definition of curl of a (non-conservative field for which $\nabla \times \mathbf{A} \neq 0$) vector field, the curl of vector field $\mathbf{A}$ at any point in the field is the maximum line integral (it is in the direction normal to the surface) of the vector per unit area along the boundary of an infinitesimal area around that point. Thus, the line integral of vector $\mathbf{A}$ along the boundary of infinitesimal rectangular element of area $d\mathbf{S}$ is

$$(\text{curl } \mathbf{A}) \cdot d\mathbf{S} \qquad \text{...(34)}$$

Therefore, the total line integral of $\mathbf{A}$ around the closed curve C forming the periphery of the surface S is obtained by summing up the line integrals of $\mathbf{A}$ around the boundaries of all the elements, that is

$$\iint_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} \qquad \text{...(35)}$$

Since the side of every rectangular element inside the curve are common for two continuous elements and the line integrals along the common sides of the continuous elements mutually cancel each other because they transversed in opposite directions (as clear from Fig. 3.8), therefore, only the sides of elements that lie on the periphery of the surface contribute the line integral. Hence, the surface integral $\iint (\text{curl } \mathbf{A}) \cdot d\mathbf{S}$ represents the line integral of the vector $\mathbf{A}$ along the boundary C of the surface S and is equal to $\oint \mathbf{A} \cdot d\mathbf{l}$, that is

$$\iint_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Hence, theorem is proved.

Example 3.6 *Show that each of the following three equations*

$$\mathbf{V} = \nabla\phi,\ \nabla \times \mathbf{V} \text{ and } \oint \mathbf{V} \cdot d\mathbf{r} = 0 \text{ implies the other.}$$

Solution. From vector identical, we know that

$$\text{curl grad } \phi = 0 \quad \text{or} \quad \nabla \times \nabla\phi = 0 \qquad \text{...(i)}$$

Given $\nabla \times \mathbf{V} = 0$; in view of Eq. (i), it follows $\mathbf{V} = \nabla\phi$

From Stoke's theorem $\oint_C \mathbf{V} \cdot d\mathbf{r} = \oint_S \text{curl } \mathbf{V} \cdot d\mathbf{S}$

As curl $\mathbf{V} = \nabla \times \mathbf{V} = 0$, it follows $\oint_C \mathbf{V} \cdot d\mathbf{r} = 0$.

Conversely if $\oint_C \mathbf{V} \cdot d\mathbf{r} = 0$, it follows curl $\mathbf{V} = 0 \Rightarrow \nabla \times \mathbf{V} = 0$

Thus the preposition is verified.

3.8 Poisson's Equation

We know that the intensity of electric field at any point is the negative gradient of electric potential V at that point

$$\mathbf{E} = -\text{grad}V = -\nabla V \qquad \text{...(36)}$$

Also from the Gauss's law,

$$\text{div}\mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{...(37)}$$

On substituting the value of $\mathbf{E}$ from Eq. (36) in Eq. (37), we obtain,

$$\text{div}(-\text{grad}V) = \frac{\rho}{\varepsilon_0} \quad \text{or} \quad \text{div}(\text{grad}V) = -\frac{\rho}{\varepsilon_0}$$

or $$\nabla \cdot (\nabla V) = -\frac{\rho}{\varepsilon_0} \quad \text{or} \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad \text{...(38)}$$

This is Poisson's equation. In cartesian coordinates, we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial^2 y} + \frac{\partial^2 V}{dz^2} = -\frac{\rho}{\varepsilon_0} \qquad \text{...(39)}$$

In spherical polar coordinates it takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -\frac{\rho}{\varepsilon_0}$$

and in cylindrical polar coordinates, it may be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{...(41)}$$

3.9 Laplace's Equation

In a charge free region Poisson's equation changes to Laplace's equation. Since in a charge free region volume charge density is zero, the Laplace's equation is given by

$$\nabla^2 V = 0 \qquad \text{...(42)}$$

Eq. (42) represents as

(i) In cartesian coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{...(43)}$$

(ii) In spherical polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad \text{...(44)}$$

(iii) In cylindrical polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{...(45)}$$

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Formulae at a Glance

1. Partial Derivatives

(a) First order partial derivatives

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$$

(b) Second order partial derivatives

$$\frac{\partial^2\phi}{\partial x^2}, \frac{\partial^2\phi}{\partial y\partial x}, \frac{\partial^2\phi}{\partial x\partial y}, \frac{\partial^2\phi}{\partial y^2}, \frac{\partial^2\phi}{\partial z^2} \text{ etc.}$$

2. 'del' (differential) operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

3. Line integral

$$\int_P^Q \mathbf{A}\cdot d\mathbf{l} = \int_P^Q A\cos\theta\, dl$$

If $\mathbf{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$

$d\mathbf{l} = i\,dx + j\,dy + k\,dz$

Then

$$\int_P^Q \mathbf{A}\cdot d\mathbf{l} = \int_P^Q (A_x\,dx + A_y\,dy + A_z\,dz)$$

4. Surface integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S A\cos\theta\, dS$$

If $d\mathbf{S} = i\,dS_x + j\,dS_y + k\,dS_z$

Then

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S (A_x dS_x + A_y dS_y + A_z dS_z)$$

5. Volume Integral

$$\iiint_V \mathbf{A}\,dV = \iiint_{x\,y\,z} (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\,dV$$

$$= \iiint_{x\,y\,z} (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\,dx\,dy\,dz$$

6. Gradient of Scalar Function

(a) $\nabla\phi = \text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

(b) $\nabla\phi = \left(\frac{\partial\phi}{\partial n}\right)\hat{n}$

7. Divergence of a Vector Function

$\nabla\cdot\mathbf{A} = \text{div}\,\mathbf{A}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Net flux diverging from the element

$= (\text{div}\,\mathbf{A})\,dx\,dy\,dz$

8. Curl of a vector field

(a) $$\nabla\times\mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_z}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

(b) $\oint \mathbf{A}\cdot d\mathbf{l} = \text{curl}\,\mathbf{A}\cdot d\mathbf{S}$

9. Gauss's theorem

$$\iiint_V \text{div}\,\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot d\mathbf{S}$$

10. Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

11. Stoke's theorem

$$\iint (\text{curl}\,\mathbf{A})\,d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{S}$$

or $$\iint_S (\nabla\times\mathbf{A})\,d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{l}$$

12. Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}$$

13. Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Miscellaneous Numerical Problems for Practice

Example 1. A vector field **A** *is represented by the function* $\mathbf{A} = i(2x^2y - x^4) + j(yz^2) - k(xy^2)$, *then obtains the values of* $\frac{\partial\mathbf{A}}{\partial y}$, $\frac{\partial^2\mathbf{A}}{\partial x^2}$ *and* $\frac{\partial^2\mathbf{A}}{\partial y\partial z}$.

Solution. We have, $\mathbf{A} = \hat{i}(2x^2y - x^4) + \hat{j}yz^2 - \hat{k}xy^2$

$$\therefore \quad \frac{\partial\mathbf{A}}{\partial x} = \frac{\partial}{\partial x}[\hat{i}(2x^2y - x^4) + \hat{j}yz^2 - \hat{k}xy^2]$$

$$= \hat{i}(4xy - 4x^3) + \hat{j}0 - \hat{k}y^2$$

Now, $$\frac{\partial^2\mathbf{A}}{\partial x^2} = \frac{\partial\mathbf{A}}{\partial x}[\hat{i}(4xy - 4x^3) - \hat{k}y^2] = \hat{i}(4y - 12x^2) - \hat{k}$$

and $$\frac{\partial^2\mathbf{A}}{\partial y\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial^2\mathbf{A}}{\partial y}\right)$$

$$= \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\{\hat{i}(2x^2y - x^4) - \hat{j}yz^2 - \hat{k}xy^2\}\right]$$

$$= \frac{\partial}{\partial z}[\hat{i}(2x^2) + \hat{j}z^2 - \hat{k}(2xy)] = 2\hat{j}z$$

Example 2. Using Gauss-divergence theorem, evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, *where* $\mathbf{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ *and S is the surface of the cube bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution. According to the Gauss divergence theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \text{div}\mathbf{F}\,dV = \iiint\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dx\,dy\,dz$$

Here $\mathbf{F} = 4xz\,\hat{i} - y^2\,\hat{j} + 4z\,\hat{k}$

$$\therefore \quad \iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V\left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right]dx\,dy\,dz$$

$$= \iiint_V (4z - 2y + y)\,dx\,dy\,dz$$

The surface *S* of the cube is bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

$$\therefore \quad \iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\left(\frac{4z^2}{2} - yz\right)dx\,dy = \int_0^1\int_0^1 (2 - y)\,dx\,dy$$

$$= \int_0^1\left(2y - \frac{y^2}{2}\right)dx = \int_0^1 \frac{3}{2}\,dx = \frac{3}{2}[x]_0^1 = \frac{3}{2} = 1.5.$$

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Example 3. Verify Stoke's theorem for $\mathbf{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$, *where S is the upper half surface of the sphere* $x^2+y^2+z^2=1$ *and C is its boundary.*

Solution. Boundary of the sphere is a circle of radius one and centre at (0,0,0). This circle lies in the x-y plane. Suppose $x=\cos t$, $y=\sin t$, $z=0$ and $0\le t<2\pi$ be the parametric equations of the boundary.

Therefore, $\nabla\times\mathbf{F}=\hat{i}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)+\hat{j}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)+\hat{k}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$

Here $\mathbf{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$

$\therefore\ F_x=(2x-y),\ F_y=-yz^2$ and $F_z=-y^2z$

$$\therefore \nabla\times\mathbf{F}=\hat{i}\left[\frac{\partial}{\partial y}(-y^2z)-\frac{\partial}{\partial z}(-yz^2)\right]+\hat{j}\left[\frac{\partial}{\partial z}(2x-y)-\frac{\partial}{\partial x}(-y^2z)\right]+\hat{k}\left[\frac{\partial}{\partial x}(-yz^2)-\frac{\partial}{\partial y}(2x-y)\right]$$

$$=\hat{i}[-2yz+2yz]+\hat{j}(0-0)+\hat{k}(0+1)$$

$$=\hat{k}$$

and $\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C[(2x-y)dx-yz^2dy-y^2zdz]$ ...(i) $[\because \mathbf{F}\cdot d\mathbf{r}=F_x dx+F_y dy+F_z dz]$

Substituting $x=\cos t$, $y=\sin t$ and $z=0$, $dx=-\sin t\,dt$, $dy=\cos t\,dt$.

$$\therefore \oint \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(2\cos t-\sin t)(-\sin t\,dt)-0-0$$

$$=\int_0^{2\pi}(2\cos t-\sin t)(-\sin t\,dt)-0-0$$

$$=\int_0^{2\pi}\left[-\sin 2t+\frac{(1-\cos 2t)}{2}\right]dt$$

or $$\oint \mathbf{F}\cdot d\mathbf{r}=\left[\frac{\cos 2t}{2}+\frac{1}{2}t-\frac{\sin 2t}{4}\right]_0^{2\pi}=\pi$$ ...(ii)

$$\iint_S(\nabla\times\mathbf{F})\hat{n}\,dS=\iint_S \hat{k}\cdot\hat{n}\,dS=\iint_R dx\,dy$$ [From Eq. (i)]

where $R=dx\,dy$, it is the projection of surface S on x-y plane

$$\iint_R dx\,dy=\int_{x=-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}dy\,dx=4\int_0^1\int_0^{\sqrt{(1-x^2)}}dy\,dx$$

$$=4\int_0^1\int_0^{\sqrt{(1-x^2)}}y\,dx=4\int_0^1[y]_0^{\sqrt{1-x^2}}dx=4\int_0^1\sqrt{1-x^2}\,dx$$

Put $x=\sin\theta$, then $dx=\cos\theta\,d\theta$

$$\therefore \iint dx\,dy=4\int_0^{\pi/2}\cos\theta\cos\theta\,d\theta=4\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=4\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta=4\left[\frac{1}{2}\theta+\frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2}=\pi$$

Hence, the Stoke's theorem $\oint \mathbf{F}\cdot d\mathbf{r}=\iint_S(\nabla\times\mathbf{F})dS$ is verified.

Example 4. Find the gradients of the following functions at the point (2, –3, –4).

(a) $f(x,y,z)=x^2+y^3+z^4$ (b) $f(x,y,z)=x^2y^3z^4$

Solution. We know that $\text{grad}\,\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi$

(a) $\therefore$ grad $f(x,y,z)=$ grad $(x^2+y^3+z^4)$

$$=\hat{i}\frac{\partial}{\partial x}(x^2+y^3+z^4)+\hat{j}\frac{\partial}{\partial y}(x^2+y^3+z^4)+\hat{k}\frac{\partial}{\partial z}(x^2+y^3+z^4)$$

$$=2x\,\hat{i}+3y^2\,\hat{j}+4z^3\,\hat{k}$$

At the point, (2, –3, –4), we have

$$\text{grad}\,f=2(2)\hat{i}+3(-3)^2\hat{j}+4(-4)^3\hat{k}$$

$$=4\hat{i}+27\hat{j}+256\hat{k}$$

(b) Here $f(x,y,z)=x^2y^3z^4$

$$\therefore \text{grad}(x^2y^3z^4)=\hat{i}\frac{\partial}{\partial x}(x^2y^3z^4)+\hat{j}\frac{\partial}{\partial y}(x^2y^3z^4)+\hat{k}\frac{\partial}{\partial z}(x^2y^3z^4)$$

$$=2xy^3z^4\,\hat{i}+x^2(3y^2)z^4\,\hat{j}+x^2y^3(4z^3)\hat{k}$$

At the point (2, –3, –4), we have

$$\text{grad}(x^2y^3z^4)=2(2)(-3)^2(-4)^4\,\hat{i}+(2)^2\,3(-3)^2(-4)^4\,\hat{j}+(2)^2(-3)^3\,4(-4)^3\,\hat{k}$$

$$\text{grad}(x^2y^3z^4)=-27648\,\hat{i}+27648\,\hat{j}+27648\,\hat{k}$$

Example 5. Prove that

(i) grad $r=\hat{n}$, where $\hat{n}$ is a unit vector.

(ii) grad $\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}$, where $\mathbf{r}$ is a position vector.

Solution. According to the problem, $\mathbf{r}$ is a position vector, *i.e.*,

$\mathbf{r}=x\hat{i}+y\hat{j}+z\hat{k}$, therefore $|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$

[$\because$ If $\mathbf{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$, then $|A|=\sqrt{(A_x^2+A_y^2+A_z^2)}$]

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(i) To prove that **grad** $r = \hat{n}$

L.H.S. grad $r = \nabla (x^2+y^2+z^2)^{1/2}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}$$

or $$\text{grad}\, r = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$$

Considering first term, *i.e.*, $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}$

Differentiating it partially with respect to x, we get

$$\therefore \quad \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot(2x) = \frac{x}{\sqrt{(x^2+y^2+z^2)}}$$

Similarly, differentiating partially other terms with respect to y and z, we get

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2} = \frac{y}{\sqrt{(x^2+y^2+z^2)}}$$

and $$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2} = \frac{z}{\sqrt{(x^2+y^2+z^2)}}$$

Substituting these values in Eq. (i), we get

$$\text{grad}\, r = \frac{x\hat{i}}{\sqrt{(x^2+y^2+z^2)}} + \frac{y\hat{j}}{\sqrt{(x^2+y^2+z^2)}} + \frac{z\hat{k}}{\sqrt{(x^2+y^2+z^2)}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{(x^2+y^2+z^2)}} = \frac{\mathbf{r}}{r} = \hat{n} \quad \text{where } \hat{n} \text{ is a unit vector.}$$

= R.H.S.

(ii) To prove that **grad** $\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$

$$\therefore \quad \text{L.H.S.} = \text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\frac{1}{\sqrt{(x^2+y^2+z^2)}}\right]$$

or $$\text{grad}\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} \quad \text{...(i)}$$

Considering first term, *i.e.*, $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$

Differentiating partially with respect to x, we get

$$\therefore \quad \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-(1/2)-1}(2x) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

Similarly, differentiating partially other terms of Eq. (i) w.r. to y and z, we get

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} = -\frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Substituting these values in Eq. (i), we get

$$\text{grad}\left(\frac{1}{r}\right) = -\frac{x\hat{i}}{(x^2+y^2+z^2)^{3/2}} - \frac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} - \frac{z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\left[\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right] = -\frac{\mathbf{r}}{(r^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

$$\text{grad}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} = \text{R.H.S.}$$

Example 6. *If* $\mathbf{r}$ *is the position vector of a point then show that*

(i) *div* $\mathbf{r} = 3$ *and* (ii) *div* $\left(\frac{\mathbf{r}}{r^3}\right) = 0$.

Solution. According to the given problem, $\mathbf{r}$ is a position vector *i.e.*,

$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore,

$r = \sqrt{(x^2+y^2+z^2)}$

We know that, div $\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

(i) div $\mathbf{r} = 3$

Here $A_x = x$, $A_y = y$ and $A_z = z$

$$\therefore \quad \text{div}\, \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3.$$

(ii) div $\left(\frac{\mathbf{r}}{r^3}\right) = 0$

Here $\mathbf{A} = \frac{\mathbf{r}}{r^3} = \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{(x^2+y^2+z^2)^{3/2}}}$

$$\therefore \quad A_x = \frac{x}{(x^2+y^2+z^2)^{3/2}}, \quad A_y = \frac{y}{(x^2+y^2+z^2)^{3/2}} \text{ and } A_z = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

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$$\therefore \text{div } \mathbf{A} = \text{div}\left(\frac{\mathbf{r}}{r^3}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] \quad \text{...(i)}$$

Considering first term *i.e.*, $\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right]$

Differentiating it partially with respect to x, we get

$$\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{\partial}{\partial x}[x(x^2+y^2+z^2)^{-3/2}]$$

$$= x\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}(x)$$

$$= x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (x^2+y^2+z^2)^{-3/2}$$

$$= -3x^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2}$$

Similarly, differentiating partially other terms of Eq. (*i*) with respect to y and z, we get

$$\frac{\partial}{\partial y}[y(x^2+y^2+z^2)^{-3/2}] = -3y^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2}$$

and $$\frac{\partial}{\partial z}[z(x^2+y^2+z^2)^{-3/2}] = -3z^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2}$$

Substituting these value in Eq. (*i*), we get

$$\text{div}\left(\frac{\mathbf{r}}{r^3}\right) = -3x^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2} - 3y^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2} - 3z^2(x^2+y^2+z^2)^{-5/2}$$

or $$\text{div}\left(\frac{\mathbf{r}}{r^3}\right) = (-3x^2-3y^2-3z^2)(x^2+y^2+z^2)^{-5/2} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= -3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

This L.H.S. = R.H.S.

Example 7. *Find the workdone in moving a particle in a force field*

$\mathbf{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ *from point* $(1,-2,1)$ *to* $(3,1,4)$.

Solution. We know that, the workdone, $dW = \mathbf{F}.d\mathbf{r}$

Here $\mathbf{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$

and $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \quad d\mathbf{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Hence $dW = \mathbf{F}.d\mathbf{r} = [(2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}]\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

or $$dW = (2xy+z^3)dx + x^2dy + 3xz^2dz$$

Since, the particle moves from point $(1,-2,1)$ to $(3,1,4)$ the total workdone

$$W = \int dW = \int \mathbf{F}.d\mathbf{r}$$

or $$W = \int_{(1,-2,1)}^{(3,1,4)} [(2xy+z^3)dx + x^2dy + 3xz^3\,dz]$$

$$= \int_{(1,-2,1)}^{(3,1,4)} [(2xy\,dx + x^2\,dy)(z^3\,dx + 3xz^2\,dz)]$$

$$= \int_{(1,-2,1)}^{(3,1,4)} \{[d(x^2y)] + [d(xz^3)]\}$$

$$= [x^2y + xz^3]_{(1,-2,1)}^{(3,1,4)}$$

$$= [3^2(1) - 1^2(-2)] + [(3)(4)^3 - (1)(1)^3]$$

$$= [9+2] + [192-1] = 11 + 191 = 202$$

Example 8. *Show that force* $\mathbf{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$ *is conservative.*

Solution. We know that,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

or $$\text{curl } \mathbf{F} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Here $\mathbf{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

$\therefore$ $F_x = yz$, $F_y = xz$ and $F_z = xy$

$$\therefore \quad \nabla \times \mathbf{F} = \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right] + \hat{j}\left[\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right]$$

$$= (x-x)\hat{i} + (y-y)\hat{j} + (z-z)\hat{k} = 0.$$

Example 9. *The electric field due to a point charge is expressed as* $\mathbf{E} = \frac{q}{r^2}\hat{r}$. *Show that the divergence of electric field due to that point charge is zero.*

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Solution. $\nabla \cdot \mathbf{E} = \nabla \cdot \left(\frac{q}{r^3}\mathbf{r}\right) = \nabla \cdot \left(\frac{q}{r^3}\hat{r}\,r\right) = \nabla \cdot \left(\frac{q}{r^2}\hat{r}\right)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot q\left(\frac{\hat{i}x + \hat{j}y + \hat{k}z}{(x^2+y^2+z^2)^{3/2}}\right) \quad \left[\because \mathbf{r} = \hat{i}x + \hat{j}y + \hat{k}z \therefore |\mathbf{r}| = \sqrt{(x^2+y^2+z^2)}\right]$$

$$= q\left[\frac{\partial}{\partial x}\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\}\right] + q\left[\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\right] + q\left[\frac{\partial}{\partial z}\left\{\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\}\right]$$

$$= q\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{5x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{5y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{5z^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= 0.$$

Example 10. Show that curl grad $\phi = 0$, where ϕ is any scalar function.

Solution. $\nabla \times \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0 \qquad \left[\because \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y} \text{ and so on}\right]$$

Example 11. Find the constant 'a' for which the vector $\mathbf{A} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

Solution. Let $\mathbf{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$

$\Rightarrow \quad A_x = (x+3y),\ A_y = (y-2z)$ and $A_z = (x+az)$

For solenoidal $\nabla \cdot \mathbf{A} = 0$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) = 0$$

$$\Rightarrow \quad \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$\Rightarrow \quad 1 + 1 + a = 0 \quad \text{or} \quad a = -2$$

Question Bank

Multiple Choice Questions

1. The amount of flux diverging from a point per unit area per second is called
   (a) divergence of a vector field (b) divergence of a scalar field
   (c) gradient of a vector field (d) divergence of a scalar field
2. Choose the correct statement
   (a) gradient of scalar field is a scalar field (b) gradient of a scalar field is a scalar field
   (c) gradient of a vector field is a scalar field (d) gradient of a vector field is a vector field
3. $\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial x} + \frac{\partial A_z}{\partial z}$ represents
   (a) grad **A** (b) grad A (c) div **A** (d) div A
4. The maximum line integral of a vector per unit area along the boundary of an infinitesimal area is called
   (a) divergence of a vector field (b) divergence of a scalar field
   (c) curl of a vector field (d) curl of a scalar field.
5. The curl of a vector field can be represented as
   (a) curl A (b) $\nabla \times \mathbf{A}$ (c) $\nabla \cdot \mathbf{A}$ (d) ∇A
6. If $\mathbf{F} = x\hat{i} + y\hat{j} + z\hat{k}$, then its divergence is
   (a) $\hat{i} + \hat{j} + \hat{k}$ (b) 3 (c) $x + y + z$ (d) zero
7. If $\phi = yz$, then its gradient is
   (a) $z\hat{j} + y\hat{k}$ (b) 0 (c) $y\hat{j} + z\hat{k}$ (d) $\hat{i} + \hat{j} + \hat{k}$
8. The divergence of a position vector is
   (a) zero (b) 3 (c) 1 (d) 3^2
9. $\nabla^2\phi$ may be written as
   (a) $\nabla \times \nabla\phi$ (b) $\nabla \cdot \nabla\phi$ (c) $\nabla(\nabla \times \phi)$ (d) $\nabla \times (\nabla \times \phi)$
10. curl grad ϕ may also be written as
    (a) $\nabla \cdot \nabla\phi$ (b) $\nabla \times \nabla\phi$ (c) $\nabla^2\phi$ (d) $(\nabla\phi)\nabla$
11. div curl **A** may also be expressed as
    (a) $\nabla \times (\nabla \times \mathbf{A})$ (b) $\nabla \times (\mathbf{A} \times \nabla)$ (c) $\nabla(\nabla \times \mathbf{A})$ (d) $(\nabla \times \nabla)\mathbf{A}$
12. If $\mathbf{A} = \hat{i}x$ and $\mathbf{B} = \hat{j}y$ then $\nabla(\mathbf{A} \cdot \mathbf{B})$ is equal to
    (a) $\hat{i}y + \hat{j}x$ (b) zero (c) $\frac{1}{2}yx^2\hat{i} + \frac{1}{2}xy^2\hat{j}$ (d) 2
13. If the time integral of a vector field depends only upon the coordinates of two points in the field and independent of actual path taken between them, then the field is said to be
    (a) non-conservative (b) curled
    (c) non-lamellar (d) conservative.

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14. A theorem that relates surface integral with the volume integral is called
(a) Stoke's theorem (b) Gauss-divergence theorem
(c) Carnot's theorem (d) Green's theorem.

15. $\iint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_V \text{div}\,\mathbf{A}\,dV$ is the mathematical form of
(a) Gauss's divergence theorem (b) Stoke's theorem
(c) Green's theorem (d) none of the above.

16. A theorem which relates a line integral with surface integral is called
(a) Gauss-divergence theorem (b) Stoke's theorem
(c) Green's theorem (d) None of the above.

17. $\int_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{A})\,d\mathbf{S}$ is mathematical form of
(a) Gauss-divergence theorem (b) Green's theorem
(c) Stoke's theorem (d) None of these

18. The line integral of a vector taken around a closed curve, which bounds a surface, is equal to surface integral of the curl of that vector taken over that surface, is the statement of
(a) Gauss-divergence theorem (b) Stoke's theorem
(c) Green's theorem (d) None of the above.

19. The surface integral fo the normal component of a vector taken over a closed surface is equal to the volume integral of the divergence of that vector taken over the volume enclosed by the surface, is the statement of
(a) Gauss-divergence theorem (b) Stoke's theorem
(c) Green's theorem (d) Norton's theorem.

20. The Gauss-divergence theorem is
(a) $\iint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_V (\text{div}\,\mathbf{A})\,dV$ (b) $\iint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_V (\text{grad}\,\mathbf{A})\,dV$
(c) $\iint_S \mathbf{A}\cdot d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{r}$ (d) $\int \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{A})\cdot d\mathbf{S}$

21. The Stoke's theorem is
(a) $\iint_S \mathbf{A}\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{r}$ (b) $\oint \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{A})\,d\mathbf{S}$
(c) $\iint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_V (\text{div}\,\mathbf{A})\,dV$ (d) $\iint_S \mathbf{A}\cdot d\mathbf{S} = \iiint_V (\text{grad}\,\mathbf{A})\,dV$

Answers

1. (a)	2. (b)	3. (c)	4. (c)	5. (b)	6. (b)
7. (a)	8. (b)	9. (b)	10. (b)	11. (c)	12. (b)
13. (d)	14. (b)	15. (a)	16. (b)	17. (c)	18. (b)
19. (a)	20. (a)	21. (b)			

Theoretical Questions

1. What do you mean by scalar and vector fields ? Give at least one example of each.
2. Explain clearly what do you mean by divergence and curl of a vector field **F**.
3. What do you mean by gradient of scalar field ? Find an expression for the gradient of scalar field in terms of operator V.
4. Give physical significance of grad ϕ. Express grad ϕ in cartesian coordinates. Hence define V operator.
5. Give physical significance of divergence of a vector field. Express divergence of a vector in term of del operator.
6. What do you mean by divergence of a vector function ? Obtain an expression for divergence of a vector in cartesian coordinates.
7. Discuss in brief line integral, surface integral and volume integral.
8. Define curl of vector field. Give the physical significance of the curl of a vector field. Derive an expression for it. Prove that curl $\mathbf{A} = \nabla \times \mathbf{A}$.
9. What do you mean by curl of a vector function ? Obtain an expression for the curl of a vector function in cartesian coordinates.
10. State and prove Gauss law or Gauss divergence theorem. What is its importance ?
11. State and prove Stoke's curl theorem. What is its importance ?
12. What is main role of Laplacian in electrodynamics ?
13. Explain curl, divergence and gradient of a vector. Give the physical interpretation of divergence. State and prove Gauss's theorem.
14. If a vector field may be expressed as $\mathbf{A} = \theta\nabla\phi$ where θ and ϕ being the scalar function of position, show that with the help of Gauss's theorem.

$$\iiint_V (\theta\nabla^2\phi + \nabla\theta\cdot\nabla\phi)\,dV = \iint_S (\theta\nabla\phi)\,dS$$

Numerical Problems

1. A potential field is represented by the equation $\phi = 4yz^2 + 3xyz - z^2 + z$. Calculate the potential at the potential at the point (1, –1, –2). **[Ans. – 36 V]**
2. Show that function $\phi = x^2 - y^2$ satisfies the Laplace's equation.
3. The temperature of a body at any point changes according to the relation $T = 4x^2 + 3y^2 - 2z^2$. Show that the grad T at the point (1,2,3) is equal to $-8\hat{i} - 12\hat{j} - 12\hat{k}$.
4. If $\mathbf{a} = \alpha x\hat{i} + \beta y\hat{j} + \gamma z\hat{k}$, show that $\nabla(\mathbf{a}\cdot\mathbf{r}) = 2\mathbf{a}$
5. A vector function has the following components $E_x = 6xy$, $E_y = 3x^2 - 3y^2$ and $E_z = 0$, show that div $\mathbf{E} = 0$.
6. Calculate div **A**, when $A = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$ where $r^2 = x^2 + y^2 + z^2$. **[Ans. div $\mathbf{A} = 2r^{-1}$]**
7. Evaluate $\int \mathbf{F}\cdot d\mathbf{r}$, where $F = xy\hat{i} + (x^2 + y^2)\hat{j}$ and C is the arc of the curve, $y = x^2 - 4$ from (2,0) to (4,12). **[Ans. 732]**

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ove that curl $(\mathbf{A}\times\mathbf{r}) = 2\mathbf{A}$ where $\mathbf{A}$ is a constant vector.

rify divergence theorem for the vector, $\mathbf{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ taken over the cube $0 \le x, y, z \le 1$

rify Stoke's theorem for the vector $\mathbf{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$.

d the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$

culate the line integral of vector function $\mathbf{F} = \frac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ along the circle $x^2 + y^2 = 1$ transverse in

clockwise sense. [Ans. -2π]

$= y\hat{i} + (x^2 + y^2)\hat{j} + (yz + zx)\hat{k}$. Calculate : (a) div $\mathbf{F}$ and (b) curl $\mathbf{F}$

[Ans. (a) $x + 3y$ (b) $\hat{i}z - \hat{j}z + \hat{k}(2x - 1)$]

v that $\nabla\times(\mathbf{A} + \mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$

e that $\nabla\times(\phi\mathbf{A}) = \phi(\nabla\times\mathbf{A}) + (\nabla\phi)\times\mathbf{A}$

e that div(grad ϕ_1 × grad ϕ_2) = 0.

position vector of any point on the surface S, find $\iint_S \mathbf{r}\cdot d\mathbf{S}$.

[Ans. 3 V]

Chapter 4

Maxwell's Equations

4.1 Introduction

Electromagnetic theory of light was put founded by **James Clerk Maxwell** in 1864. According to him, light itself an electromagnetic waves transferring electric energy and magnetic energy in the given medium.

The variations of electric intensity and magnetic intensity are transverse in nature. The variation of **E** and **H** are perpendicular to each other and also the directions of wave propagation. The wave patterns of **E** and **H** for a traveling electromagnetic wave obey the fundamental equations, called **Maxwell's equations**. These equations are mathematical abstractions of experimental results.

4.2 Biot-Savart Law

Oersted's experimental investigation showed that a magnetic field is always established around a current carrying conductor just as it occurs around a magnet. Biot and Savart, in the same year 1820, performed a series of experiments to investigate the magnetic fields generated by various current carrying conductors and formulated a law to determine the magnitude and direction of the field so generated. This law is known as **Biot-Savart law**.

For the mathematical generalizations of experimental conclusions of Biot-Savart, consider a filamentary conductor of an arbitrary shape carrying a current i and P be a point at which the magnitude and direction of the magnetic field is to be determined (Fig. 4.1). Let us divide the conductor into infinitesimal current elements. According to Biot-Savart's observations the magnetic induction $d\mathbf{B}$ due to a current element $i\,d\mathbf{l}$ of a filamentary current carrying conductor at an external point P at a distant r from the element is given by

Fig. 4.1 Biot-Savart Law

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$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i d\mathbf{l} \times \mathbf{r}}{r^3} \quad ...(1)$$

where $\frac{\mu_0}{4\pi}$ is the permeability constant and μ_0 ($= 4\times10^{-7}$ Wb/Amp. or Newton/Ampere^2) the permeability of free space.

Eq. (1) is the vector form of Biot-Savart law. The magnitude of magnetic field $d\mathbf{B}$ at point P due to current element $i d\mathbf{l}$ is given by

$$dB = \frac{\mu_0}{4\pi}\frac{i dl \sin\theta}{r^2} \quad ...(2)$$

where θ is the angle between the current element $i d\mathbf{l}$ and radius $\mathbf{r}$.

As clear from Eq. (1) the magnetic induction $d\mathbf{B}$ is expressed in terms of vector product of $d\mathbf{l}$ and $\mathbf{r}$. Therefore the direction of $d\mathbf{B}$ is the direction of advancement of right hand screw whose axis is normal to the plane formed by $d\mathbf{l}$ and $\mathbf{r}$ and which is rotated from $d\mathbf{l}$ to $\mathbf{r}$. Thus $d\mathbf{B}$ at P is perpendicular to the plane of Fig. 4.1 and directed downward.

The resultant field $\mathbf{B}$ at P due to entire conductor is

$$\mathbf{B} = \int d\mathbf{B} \quad ...(3)$$

The magnetic induction $\mathbf{B}$ due to an entire current loop or a circuit configuration may be obtained by integrating Eq. (1) as

$$\mathbf{B} = \frac{\mu_0}{4\pi}\oint\frac{i d\mathbf{l} \times \mathbf{r}}{r^3} \quad ...(4)$$

Unit of $\mathbf{B}$ is **Tesla** or **Weber/m^2** (1 Tesla $= 10^4$ Gauss)

To express the law in more general form, consider volume distribution of current. If curve S encloses a region where currents are flowing (Fig. 4.2) and $\mathbf{J}$ is the current density at a point on the element of length $d\mathbf{l}$, area of cross-section dS and infinitesimal volume dV surrounding that point, then

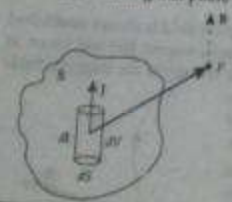

Fig. 4.2

$$i d\mathbf{l} = \mathbf{J}.d\mathbf{S}.d\mathbf{l} = \mathbf{J}dV \quad ...(5)$$

Hence, at a point P

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{(\mathbf{J}dV)\times \mathbf{r}}{r^3}\right)$$

or

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times \mathbf{r}}{r^3}dV \quad ...(6)$$

where $\mathbf{r}$ is directed from dV (the source point) to point P (The field point where $\mathbf{B}$ is evaluated).

Example 4.1. *A current element $i\Delta\mathbf{l} = 2\pi(0.6\hat{i} - 0.8\hat{j})\mu A$ is situated at a point (4, −2, 3). Find the incremental field $\Delta\mathbf{H}$ at a point (1,3,2).*

Solution. According to Biot-Savart's law,

$$\Delta\mathbf{H} = \frac{i\Delta\mathbf{l}\times\mathbf{r}}{4\pi r^3} = \frac{\Delta\mathbf{B}}{\mu} \quad ...(i)$$

Here $i\Delta\mathbf{l} = 2\pi(0.6\hat{i} - 0.8\hat{j})\mu A$

If the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is c then

$$\mathbf{r} = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$$

$$\therefore \quad \mathbf{r} = (1-4)\hat{i} + (3+2)\hat{j} + (2-3)\hat{k}$$

or

$$\mathbf{r} = -3\hat{i} + 5\hat{j} - \hat{k} \quad ...(ii)$$

$$r = \sqrt{[(-3)^2 + (5)^2 + (-1)^2]} = \sqrt{(9+25+1)}$$

$$= \sqrt{35} \quad ...(iii)$$

$$\therefore \quad i\Delta\mathbf{l}\times\mathbf{r} = 2\pi(0.6\hat{i} - 0.8\hat{j})\times(-3\hat{i} + 5\hat{j} - \hat{k})$$

We know that,

$$\mathbf{A}\times\mathbf{B} = \hat{i}(A_yB_z - B_yA_z) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

$$\therefore \quad i\Delta\mathbf{l}\times\mathbf{r} = 2\pi[\hat{i}(-0.8\times(-1) - 5\times 0) + \hat{j}(0\times(-3) - 0.6\times(-1)) + \hat{k}(0.6\times 5 - 0(-0.8))\hat{k}]$$

$$= 2\pi[0.8\hat{i} + 0.6\hat{j} + (3 - 2 - 4)\hat{k}]$$

or

$$i\Delta\mathbf{l}\times\mathbf{r} = 2\pi(0.8\hat{i} + 0.6\hat{j} + 0.6\hat{k}) \quad ...(iv)$$

Substituting these values of $i\Delta\mathbf{l}\times\mathbf{r}$ and r from relation (iii) and (iv) in relation (i), we get

$$\therefore \quad \Delta\mathbf{H} = \frac{2\pi(0.8\hat{i} + 0.6\hat{j} + 0.6\hat{k})}{4\pi(\sqrt{35})^3} = \frac{1}{70\sqrt{35}}(0.8\hat{i} + 0.6\hat{j} + 0.6\hat{k})\mu A/m$$

or

$$\Delta\mathbf{H} = \frac{(0.8\hat{i} + 0.6\hat{j} + 0.6\hat{k})}{414.124}$$

$$= 2.415\times10^{-3}(0.8\hat{i} + 0.6\hat{j} + 0.6\hat{k})$$

$$= 1.932\times10^{-3}\hat{i} + 1.449\times10^{-3}\hat{j} + 1.449\times10^{-3}\hat{k}$$

or

$$\Delta\mathbf{H} = (1.932\hat{i} + 1.449\hat{j} + 1.449\hat{k})\times10^{-3}\mu A/m$$

$$\Rightarrow \quad \Delta\mathbf{H} = (1.932\hat{i} + 1.449\hat{j} + 1.449\hat{k})\mu A/m$$

4.2.1 Magnetic Induction due to a Long Straight Conductor (Wire) Carrying Current

Let us consider an infinitely long straight conductor (or wire) placed in a free space and carrying a current of i Ampere (Fig. 4.3). Suppose P be a point distant d from the wire at which the magnetic induction $\mathbf{B}$ is to be determined and O be the foot of perpendicular from P to the wire ($OP = d$). Let us consider that the conductor is made up of large number of small current element and let $i d\mathbf{l}$ be one such current element at a point A on the wire. It

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current element $i d\mathbf{l}$ is at a distance l from O and at a distance r from P, then from Biot-Savart law, the magnetic induction $d\mathbf{B}$ at P due to current element $i d\mathbf{l}$ is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{(i d\mathbf{l} \times \mathbf{r})}{r^3} \quad ...(7)$$

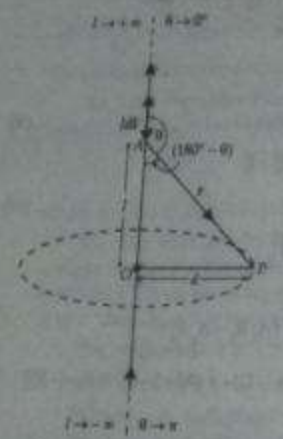

Fig. 4.3 B due to a long wire.

Its magnitude is

$$dB = \frac{\mu_0}{4\pi}\frac{i\,dl \sin\theta}{r^2} \quad ...(8)$$

From Fig. 4.3, we have

From ΔPAO

$$\tan(180° - \theta) = -\tan\theta = \frac{d}{l}$$

or $$l = -d\cot\theta$$

Therefore, $$dl = d\,\text{cosec}^2\,\theta\, d\theta \quad ...(9)$$

Again from ΔPAO

$$\sin(180° - \theta) = \frac{OP}{AP} = \frac{d}{r}$$

or $$\sin(180° - \theta) = \sin\theta = \frac{d}{r}$$

or $$r = d\,\text{cosec}\,\theta \quad ...(10)$$

Substituting the values of dl and r from Eq. (7) and (8) in Eq. (6)

$$dB = \frac{\mu_0}{4\pi}\frac{i(d\,\text{cosec}^2\theta\, d\theta)\sin\theta}{d^2\,\text{cosec}^2\theta}$$

or $$dB = \frac{\mu_0}{4\pi}\frac{i\sin\theta\, d\theta}{d} \quad ...(11)$$

The field at P due to entire wire is obtained by integrating Eq. (9) within the limit $\theta = 0$ (at the lower end) to $\theta = \pi$ (at the upper end). Therefore,

$$B = \frac{\mu_0 i}{4\pi d}\int_0^{\pi} \sin\theta\, d\theta$$

or $$B = \frac{\mu_0 i}{4\pi d}[-\cos\theta]_0^{\pi} = \frac{\mu_0 i}{4\pi d}(-\cos\pi + \cos 0°)$$

or $$B = \frac{\mu_0 i}{4\pi d}(1 + 1) \quad \text{or} \quad B = \frac{\mu_0 i}{2\pi d}\ \text{Weber/m}^2 \quad ...(12)$$

In terms of magnetic field intensity $\mathbf{H}$, Eq. (10) may be expressed as

$$H = \frac{B}{\mu_0} = \frac{i}{2\pi d}\ \text{Ampere/meter} \quad ...(13)$$

Example 4.2. A copper wire 0.254cm in diameter carries a current of 50Amp. Find the magnetic field induction B at the surface of the wire. The permeability constant $\mu_0 = 4\pi \times 10^{-7}$ Weber/Amp-m.

Solution. The magnitude of magnetic field induction **B** at a point distant d from a straight wire carrying a current of i Amp is given by

$$B = \frac{\mu_0 i}{2\pi d}$$

At the surface of the wire, $d = r$, the radius of wire

$$B = \frac{\mu_0 i}{2\pi r}$$

Here $r = \frac{0.254}{2} = 0.127$ m $= 0.127 \times 10^{-2}$ m, $i = 50$ Amp and $\mu_0 = 4\pi \times 10^{-7}$ Wb/A-m

$$\therefore \quad B = \frac{4\pi \times 10^{-7} \times 50}{2\pi \times 0.127 \times 10^{-2}} = 7.87 \times 10^{-3}\ \text{Wb/m}^2.$$

4.3 Ampere's Circuital Law

Ampere's law in magnetostatics is analogous to Gauss's law in electrostatics.

Statement : The line integral $\oint \mathbf{B}\cdot d\mathbf{l}$ of magnetic induction **B** for a closed path is numerically equal to μ_0 time the current through the area bounded by the path.

i.e., $$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i$$

where μ_0 = permeability constant.

Proof

Fig. 4.4. shows a long straight conductor carrying a current i in the direction shown.

Biot-Savart's law gives the magnitude of the magnetic field induction at a distance R from it

$$B = \frac{\mu_0 i}{2\pi R} \quad ...(14)$$

Its direction is given by tangent to a circle of radius R centered on the current carrying wire. The magnetic field **B** is constant in magnitude at all the points on the circle and its direction is parallel to the circle element $d\mathbf{l}$

Fig. 4.4

$$\therefore \quad \oint \mathbf{B}\cdot d\mathbf{l} = \oint \frac{\mu_0 i}{2\pi R} R\,d\theta = \frac{\mu_0 i}{2\pi}\oint d\theta$$

$$= \frac{\mu_0 i}{2\pi} \times 2\pi = \mu_0 i \quad ...(15)$$

Thus, the line integral $\oint \mathbf{B}\cdot d\mathbf{l}$ is equal to μ_0 times the current through the area bounded by the circles. It is Ampere's circuital law. It is equally applicable to any assembly of currents and for any closed curve.

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4.3.1 Applications

1. Magnetic Field Induction of a Solenoid

Fig. 4.5 shows a long solenoid carrying current i.

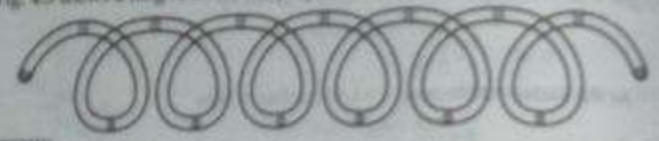

Fig. 4.5 Solenoid

The magnetic field outside the solenoid is very small in comparison with the field inside. Also the lines of induction inside the solenoid except near the edge, Fig. 4.6.

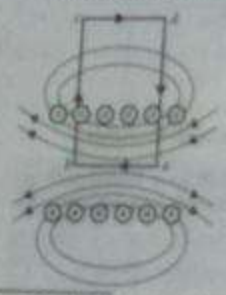

Fig. 4.6 M.F. of Solenoid

Fig. 4.6 shows a closed rectangular path abcd. The side ab is parallel to the axis of the solenoid ; and sides bc and da are very long so that the side cd is very far from the solenoid where the field is negligibly small. As the solenoid is very long and the rectangle abcd is not too near to either end of the solenoid it is right angles to the sides bc and da.

Ampere's law states that the integral for the rectangular path abcd must be equal to μ_0 times the current i enclosed within the path i.e.,

$$\oint_{abcd} \mathbf{B}.d\mathbf{l} = \mu_0 i \qquad ...(16)$$

We may break the integral

$$\oint_{abcd} \mathbf{B}.d\mathbf{l} = \int_a^b \mathbf{B}.d\mathbf{l} + \int_b^c \mathbf{B}.d\mathbf{l} + \int_c^d \mathbf{B}.d\mathbf{l} + \int_d^a \mathbf{B}.d\mathbf{l} \qquad ...(17)$$

But the field $\mathbf{B}$ is at right angles to $d\mathbf{l}$ along bc and da. Therefore, along the path bc and da, we get

$$\mathbf{B}.d\mathbf{l} = Bdl\cos 90° = 0 \quad \therefore \quad \int_b^c \mathbf{B}.d\mathbf{l} = \int_d^a \mathbf{B}.d\mathbf{l} = 0$$

Also the side cd lies outside the solenoid, where the field $\mathbf{B} = 0$, therefore,

$$\int_c^d \mathbf{B}.d\mathbf{l} = 0$$

$\therefore$ Eq. (17) may be written as $\oint_{abcd} \mathbf{B}.d\mathbf{l} = \int_a^b \mathbf{B}.d\mathbf{l}$

As $\mathbf{B}$ is parallel to $d\mathbf{l}$ along ab, therefore,

$$\int_a^b \mathbf{B}.d\mathbf{l} = B\int_a^b dl\cos 0° \qquad [\because B \text{ is constant inside the solenoid}]$$

$$= Bx \qquad [dl = x, \text{ length of } ab] \qquad ...(18)$$

If n is the number of turns per unit length of the solenoid, then the number of turns in a length x (or the number of turns passing through the rectangle abcd) is nx. The current in each turn is i_0.

Therefore, net current enclosed within the rectangle abcd is

$$i = nxi_0$$

Putting the value of $\oint_{abcd} \mathbf{B}.d\mathbf{l}$ from Eq. (18) and $i = nxi_0$ in Eq. (16)

$$Bx = \mu_0 nxi_0$$

$$B = \mu_0 ni_0 \qquad ...(19)$$

It is clear the magnetic field B is independent of the length and diameter of the solenoid. It is uniform over the cross-section of the solenoid.

2. Magnetic Field of an Endless Solenoid (or Toroid)

Consider a long solenoid bent round in the form of a closed ring, known as **toroid**. It is carrying a current i_0. The magnetic field $\mathbf{B}$ at any point within the toroid is along the tangent to a circle passing through that point and concentric with toroid. The magnitude of the field $\mathbf{B}$ is the same at all points on any such circle.

Fig. 4.7 shows a circle with centre O, of radius r through a point P within the toroid. Ampere's law states that the line integral for the circle must be equal to μ_0 times the current i enclosed within the circle.

Fig. 4.7 Toroid

i.e., $$\oint_{circle} \mathbf{B}.d\mathbf{l} = \mu_0 i \qquad ...(20)$$

But $$\oint_{circle} \mathbf{B}.d\mathbf{l} = \int Bdl = B\int dl \qquad [B \text{ and } dl \text{ are parallel}]$$

If N is the number of turns in the toroid, then the total current enclosed by the circle is $i = Ni_0$.

From Eq. (20), we get

$$B\int dl = B(2\pi r) = \mu_0 Ni_0$$

or $$B = \frac{\mu_0 Ni_0}{2\pi r}$$

i.e., the field varies with r. If the mean circumference of the toroid is l, then $l = 2\pi r$ and

$$B = \frac{\mu_0 Ni_0}{l}$$

The magnetic field due to toroid is zero at all points except within the core.

The magnetic field $\mathbf{B}$ at all points like Q is zero, because no current is enclosed by the circle through Q. The field at points like R is also zero because each turn of the toroid winding passes twice through the space enclosed by the circle through R, and equal currents are carried in opposite directions. Therefore, the net current enclosed by the circle through R is zero.

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Example 4.3. Derive an expression for **B** *at a distance x from the axis of a long cylindrical wire of radius a and carrying a current i_0.*

Solution. The cross-section of the wire, perpendicular to its length, is shown in Fig. 4.8. Assuming that the current i_0 is distributed uniformly over the cross-section of the wire. Now consider a circular path of radius x round the axis of wire. By symmetry, we see that **B** at any point over this path is tangential to it every where.

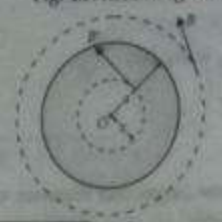

Fig. 4.8 Cross-section of wire ⊥ to its length

Thus, by Ampere's law

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 i$$

or

$$\oint \mathbf{B}.d\mathbf{l} = B(2\pi x) = \mu_0 \quad \text{(current enclosed by the path)}$$

There are three possible cases :

(i) $x > a$, current i = total current i_0 passing through the whole wire

$$B \cdot 2\pi x = \mu_0 i_0 \quad \text{or} \quad B = \frac{\mu_0 i_0}{2\pi x}$$

(ii) $x < a$, the fraction of the current that passes through the enclosed path contributes to **B**.

$$B \cdot 2\pi x = \mu_0 i = \mu_0 (i_0/\pi a^2)\pi x^2 \quad \text{or} \quad B = \frac{\mu_0 i_0 x}{2\pi a^2}$$

(iii) $x = a$, $i = i_0$ therefore, $B = \frac{\mu_0 i_0 a}{2\pi a^2} = \frac{\mu_0 i_0}{2\pi a}$

4.4 Equation of Continuity (Conservation of Charge)

We know that the net amount of charge in a isolated system remains constant. The movement of charge constitute the current *i.e.*,

$$i = \frac{dq}{dt} = \frac{d}{dt}\int \rho \, dV \qquad ...(22)$$

where ρ is charge density. The net amount of charge which crosses a unit area perpendicularly is defined as current density **J**. Now let a volume V is enclosed by a surface S then **J**.d**S** will represent the charge leaving volume V through an surface element d**S** in unit time (Fig. 4.9). Then for whose surface

Fig. 4.9 Illustration for Equation of continuity.

$$i = \oint \mathbf{J}.d\mathbf{S} \qquad ...(23)$$

Thus,

$$\oint \mathbf{J}.d\mathbf{S} = -\frac{dq}{dt} = -\frac{d}{dt}\int_V \rho \, dV$$

$$= -\int \frac{\partial \rho}{\partial t} dV \qquad ...(24)$$

[∵ Gauss divergence theorem $\oint \mathbf{J}.d\mathbf{S} = \int_V \text{div}\,\mathbf{J}\,dV$]

Then,

$$\int_V (\text{div}\,\mathbf{J})dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

$$\Rightarrow \quad \int_V \left(\text{div}\,\mathbf{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

or

$$\text{div}\,\mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad ...(25)$$

which is called **equation of continuity**. It states that *current flowing out of a given volume must be equal to the rate of decreases of charge within the volume.*

For stationary current ρ is constant.

So,

$$\frac{\partial \rho}{\partial t} = 0$$

$$\therefore \quad \text{div}\,\mathbf{J} = 0 \qquad ...(26)$$

4.5 Displacement Current Density

We know Ampere's circuital law that

$$\text{curl}\,\mathbf{B} = \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

or

$$\text{curl}\,\mathbf{H} = \mathbf{J} \qquad [\mathbf{B} = \mu_0 \mathbf{H}] \qquad ...(27)$$

Now, using continuity equation,

$$\text{div}(\text{curl}\,\mathbf{H}) = \text{div}\,\mathbf{J} = -\frac{\partial \rho}{\partial t}$$

or

$$0 = \frac{\partial \rho}{\partial t} \qquad ...(28)$$

which shows that Eq. (27) is valid for steady currents only and may not be valid for time varying fields. Thus for such cases the Ampere's law need modification. Maxwell suggested that the total current density will not be equal to **J** but something else. Now if we take it as $(\mathbf{J} + \mathbf{J}')$, then

$$\text{curl}\,\mathbf{H} = (\mathbf{J} + \mathbf{J}')$$

or

$$\text{div}\,(\text{curl}\,\mathbf{H}) = \text{div}\,(\mathbf{J} + \mathbf{J}')$$

$$0 = \text{div}\,\mathbf{J} + \text{div}\,\mathbf{J}' \qquad ...(29)$$

or

$$\text{div}\,\mathbf{J}' = -\text{div}\,\mathbf{J}$$

$$= \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D})$$

$$= \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \qquad ...(30)$$

$$= \text{div}\,\frac{\partial \mathbf{D}}{\partial t} \qquad ...(31)$$

or

$$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t}$$

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Then modified Ampere's law is

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(32)}$$

Eq. (32) is known as Maxwell's fourth equation.

Here $J = \frac{\partial \mathbf{D}}{\partial t}$ is called displacement current density which is equally effective in producing magnetic field as conduction current density. This displacement current is responsible for the production of magnetic field in empty space where the conduction current is zero.

Example 4.4 *Determine the conduction current and displacement current densities in a material having conductivity of 10^{-4} Siemen/m and relative permitivity $\epsilon_r = 2.25$. The electric field in the material is $E = 5\times10^{-6}\sin(9\times10^{9}t)$ Volt/m*

Solution. We know that, the conduction current density J is

$$\mathbf{J} = \sigma\mathbf{E}$$

or

$$J = \sigma E$$

where $\sigma = 10^{-4}$ Siemen/m and $E = 5\times10^{-6}\sin(9\times10^{9}t)$ Volt/m

$$\therefore \quad J = 10^{-4}\times5\times10^{-6}\sin(9\times10^{9}t)\ \text{Amp/m}^2$$

Electric displacement $\mathbf{D} = \varepsilon\mathbf{E}$ or $D = \varepsilon E$

We know that, permittivity $\varepsilon = \varepsilon_0\epsilon_r$

$$\therefore \quad D = \varepsilon_0\varepsilon_r E$$

The displacement current density $J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0\varepsilon_r E) = \varepsilon_0\varepsilon_r\frac{\partial E}{\partial t}$

Here, $\epsilon_r = 2.25$ and $\varepsilon_0 = \frac{1}{(4\pi\times9\times10^{9})}\ \text{C}^2/\text{Nm}^2$

$$\therefore \quad J_d = \frac{1}{4\pi\times9\times10^{9}}\times2.25\frac{\partial}{\partial t}[5\times10^{-6}\sin(9\times10^{9}t)]$$

$$= \frac{2.25\times5\times10^{-6}}{4\times3.14\times9\times10^{9}}\times9\times10^{9}\cos(9\times10^{9}t)$$

$$J_d = 8.95\times10^{-7}\cos(9\times10^{9}t)\ \text{A/m}^2$$

4.6 Basic Laws of Electricity and Magnetism

The basic laws of electricity and magnetism which form the basis of Maxwell's equations

1. The Gauss's law in Electrostatics is

The flux of electric displacement vector $\mathbf{D}$ is given by

$$\int_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dV$$

where S is surface integral over the surface enclosing volume V.

Applying Gauss divergence theorem to change surface integral into volume integral on L.H.S.

$$\int_V \text{div}\,\mathbf{D}\,dV = \int_V \rho\, dV$$

or

$$\int_V (\text{div}\,\mathbf{D} - \rho)\,dV = 0$$

As volume is arbitrary

$$\text{div}\,\mathbf{D} - \rho = 0$$

or

$$\text{div}\,\mathbf{D} = \rho \quad \text{...(33)}$$

2. The Gauss's law in magnetism is

The flux of magnetic flux density $\mathbf{B}$ is given by

$$\int \mathbf{B}\cdot d\mathbf{S} = 0$$

using Gauss divergence theorem to change surface integral into volume integral.

$$\int_V \text{div}\,\mathbf{B}\,dV = 0$$

As volume is arbitrary

$$\text{div}\,\mathbf{B} = 0 \quad \text{...(34)}$$

3. The universal form of Faraday's laws of electromagnetic induction is

$$\text{curl}\,\mathbf{B} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{...(35)}$$

4. The Ampere's law in magnetostatics is

$$\int_C \mathbf{H}\cdot d\mathbf{l} = \text{current enclosed by path}$$

$$= \int_V \mathbf{J}\cdot d\mathbf{S}$$

where $\mathbf{H}$ is magnetic field intensity and $\mathbf{J}$ is current density.

Using Stoke's theorem to change the line integral into surface on L.H.S., we get

$$\int_S \text{curl}\,\mathbf{H}\cdot d\mathbf{S} = \int_S \mathbf{J}\cdot d\mathbf{S}$$

or

$$\int (\text{curl}\,\mathbf{H} - \mathbf{J})\cdot d\mathbf{S} = 0$$

As surface is arbitrary, we get

$$\text{curl}\,\mathbf{H} - \mathbf{J} = 0 \quad \text{...(36)}$$

or

$$\text{curl}\,\mathbf{H} = \mathbf{J}$$

In addition these laws, we have

Ohm's Law. $\mathbf{J} = \sigma\mathbf{E}$ ...(37)

where σ is electrical conductivity and $\mathbf{E}$ is electric field strength.

$$\mathbf{D} = \varepsilon\mathbf{E} \quad \text{...(3}$$

and

$$\mathbf{B} = \mu\mathbf{H} \quad \text{...(3}$$

where ε = permitivity of medium

μ = permeability of medium

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4.7 Maxwell's Equations and Their Empirical Basis

There are four fundamental equations of electromagnetism known as Maxwell's equations, which may be written in differential form as :

(i) $\nabla \cdot \mathbf{D} = \rho$ (Differential form of Gauss's law in electrostatics)

(ii) $\nabla \cdot \mathbf{B} = 0$ (Differential form of Gauss's law in magnetostatics)

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Differential form of Faraday's law in electromagnetic induction)

(iv) $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ (Maxwell's modification in Ampere's law)

In above equations the notations have the following meanings :

$\mathbf{D}$ = electric displacement vector in C/m^2

ρ = charge density in C/m^3

$\mathbf{B}$ = magnetic induction in Wb/m^2 or Tesla

$\mathbf{E}$ = electric field intensity in V/m or N/C

$\mathbf{H}$ = magnetic field intensity or strength in A/turn-m

Each of the Maxwell's equations represent a generalisation of certain experimental observations :

- Eq.(i) represents the differential form of Gauss's law in electrostatics which in turn derives from Coulomb's law.
- Eq.(ii) represents Gauss's law in magnetostatics which is usually said to represent the fact that isolated magnetic poles do not exist in our physical world.
- Eq.(iii) represents differential form of Faraday's law of electromagnetic induction and
- Eq.(iv) represents Maxwell's modification of Ampere's law to include time varying field.

Maxwell's Equation in Integral form :

(i) $\int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV$ (ii) $\int_S \mathbf{B} \cdot d\mathbf{S} = 0$

(iii) $\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}$ (iv) $\int_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) d\mathbf{S}$

4.8 Derivation of Maxwell's Equations

1. Derivation of First Equation : div $\mathbf{D} = \nabla \cdot \mathbf{D} = \rho$

Let us consider a surface S bounding total charge consisting of free charge plus polarisation. If ρ and ρ_p are the charge densities of free charge and polarisation charge at a point in a small volume element dV, then Gauss's law can be expressed as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V (\rho + \rho_p) dV \quad ...(40)$$

But polarisation charge density $\rho_p = -\text{div}\,\mathbf{P}$, therefore above Eq. (40) takes the form

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V (\rho - \text{div}\,\mathbf{P}) dV$$

i.e.,

$$\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_V \rho \, dV - \int_V \text{div}\,\mathbf{P} \, dV$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \text{div}(\varepsilon_0 \mathbf{E}) dV = \int_V \rho \, dV - \int_V \text{div}\,\mathbf{P} \, dV$$

i.e.,

$$\int_V \text{div}(\varepsilon_0 \mathbf{E} + \mathbf{P}) dV = \int_V \rho \, dV$$

or

$$\int_V (\text{div}\mathbf{D} - \rho) dV = 0 \qquad [\because \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}]$$

Since this equation is true for all values, therefore, the integrand in this equation must vanish

i.e.,

$$(\text{div}\mathbf{D} - \rho) = 0$$

or

$$\text{div}\mathbf{D} = \rho$$

or

$$\nabla \cdot \mathbf{D} = \rho \quad ...(41)$$

2. Derivation of Second Equation div $\mathbf{B} = \nabla \cdot \mathbf{B} = 0$

Since isolated magnetic poles and magnetic currents due to them have no physical significance : therefore magnetic lines of force in general are either closed curves or go off to infinity. Consequently the number of magnetic lines of forces entering any arbitrary closed surface is exactly the same as leaving it. It means the flux of magnetic induction $\mathbf{B}$ across the closed surface is always zero.

i.e.,

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \text{div}\,\mathbf{B} \, dV = 0$$

As the volume bounding by surface is arbitrary therefore, this equation holds if the integrand vanishes

$$\text{div}\,\mathbf{B} = 0$$

or

$$\nabla \cdot \mathbf{B} = 0 \quad ...(43)$$

3. Derivation of Third Equation : $\left(\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\right)$

According to Faraday's law of electromagnetic induction it is known that e.m.f. induced in a closed loop is defined as negative rate of change of magnetic flux

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$$e = -\frac{d\phi}{dt} \quad \text{...(44)}$$

But magnetic flux $\phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ where S is any surface having loop as boundary.

$$e = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= -\int_S \frac{\partial \mathbf{B}}{\partial t} dS \quad \text{...(45)}$$

($\because$ Surface is fixed in space, hence only B changes with time)

But e.m.f. 'e' can also be computed by calculating the work done in carrying a unit charge round the closed loop C. Thus $\mathbf{E}$ is the electric field intensity at a small element $d\mathbf{l}$ of loop, we have

$$e = \int_C \mathbf{E} \cdot d\mathbf{l} \quad \text{...(46)}$$

Comparing Eqs. (45) and (46), we get

$$\int \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad \text{...(47)}$$

Using Stoke's theorem to change line integral into surface integral, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

or

$$\int_S \left[\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right] \cdot d\mathbf{S} = 0 \quad \text{...(48)}$$

Since surface is arbitrary, therefore, Eq. (48) holds only if the integrand vanishes.

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

or

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{or} \quad \text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

4. Derivation of fourth equation : $\left(\text{curl}\,H = J + \frac{\partial D}{\partial t}\right)$

Please see section 4.5 : Maxwell's displacement current density, Eq. (32) is known as Maxwell's 4th equation.

4.9 Maxwell's Equations in Integral Form

Physical Significance of Maxwell's Equations. By means of Gauss's theorem and Stoke's theorem, we can put the field equation in integral form and can obtain their physical significance.

1. Maxwell's First Equation : $(\nabla \cdot \mathbf{D} = \rho)$

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla \cdot \mathbf{D}\, dV = \int \rho\, dV$$

Changing volume integral in L.H.S. of above equation into surface integral by Gauss divergence theorem, we get

$$\int_S \mathbf{D}\, d\mathbf{S} = \int_V \rho\, dV \quad \text{...(49)}$$

where S is the surface, which bounds volume V. Eq. (49) represents Maxwell's first equation ($\nabla \cdot \mathbf{D} = \rho$) in integral form. Since $\int \rho\, dV = q$, the net charge contained in volume V, therefore, Maxwell's first equation signifies that :

The net outward flux of electric displacement vector through the surface enclosing a volume is equal to the net charge contained within that volume.

2. Maxwell's Second Equation is : $(\nabla \cdot \mathbf{B} = 0)$

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla \cdot \mathbf{B}\, dV = 0$$

Using Gauss divergence theorem to change volume integral into surface integral, we get

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{...(50)}$$

where S is the surface which bounds volume V. Eq. (50) represents Maxwell's second equation in integral form and signifies that the *net outward flux of magnetic induction* $\mathbf{B}$ *through any closed surface is equal to zero.*

3. Maxwell's Third Equation is : $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Integrating above equation over a surface S bounded by a curve C, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral along the boundary C, we get

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S} \quad \text{...(51)}$$

Eq. (51) represents Maxwell's third equation in integral form and signifies that

The electromotive force (emf $e = \int_C \mathbf{E} \cdot d\mathbf{l}$*) around a closed path is equal to negative rate of change of magnetic flux linked with the path* ($\because$ *magnetic flux* $\phi = \int_S \mathbf{B}\, d\mathbf{S}$).

4. Maxwell's Fourth Equation is : $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Taking surface integral over surface S bounded by curve C, we obtain

$$\int (\nabla \times \mathbf{H})\, d\mathbf{S} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) d\mathbf{S}$$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral, we get

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$$\oint \mathbf{H}.d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) d\mathbf{S} \qquad ...(52)$$

Eq. (52) represents Maxwell's 4th equation in integral form and signifies that, the *magnetomotive force (m.m.f. $= \oint \mathbf{H}.d\mathbf{l}$) around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.*

4.10 Maxwell's Equations in Some Particular Cases

Maxwell's equation in differential form are :

$$\left.\begin{aligned} &\nabla \cdot \mathbf{D} = \rho && ...(a)\\ &\nabla \cdot \mathbf{B} = 0 && ...(b)\\ &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && ...(c)\\ \text{and}\quad &\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && ...(d)\end{aligned}\right\} \qquad ...(53)$$

Case (i) Maxwell's Equations in Free Space. In free space the volume charge density ρ and current density $\mathbf{J}$ are zero. Hence Maxwell's Eqs. (53) take the form.

$$\left.\begin{aligned} &\nabla \cdot \mathbf{D} = 0 && ...(a)\\ &\nabla \cdot \mathbf{B} = 0 && ...(b)\\ &\nabla \cdot \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && ...(c)\\ \text{and}\quad &\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} && ...(d)\end{aligned}\right\} \qquad ...(54)$$

With $\mathbf{D} = \varepsilon_0 \mathbf{E}$

and $\mathbf{B} = \mu_0 \mathbf{H}$ ...(54)

where ε_0 and μ_0 are absolute permitivity and permeability of free space respectively.

Case (ii) Maxwell's equations is linear isotropic medium. In a linear isotropic medium, we have

$$\mathbf{D} = \varepsilon \mathbf{E} \text{ and } \mathbf{B} = \mu \mathbf{H} \qquad ...(56)$$

where ε and μ are absolute permitivity and permeability of medium respectively.

Using Eq. (56), Maxwell's Eqs. (53) for linear isotropic medium take the form

$$\left.\begin{aligned} &\nabla \cdot \mathbf{E} = 0 && ...(a)\\ &\nabla \cdot \mathbf{B} = 0 && ...(b)\\ &\nabla \cdot \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} && ...(c)\\ \text{and}\quad &\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} && ...(d)\end{aligned}\right\} \qquad ...(57)$$

Case (iii) Maxwell's equation for harmonically varying fields. If electromagnetic field vary harmonically with time that we may write

$$\mathbf{D} = \mathbf{D}_0 e^{i\omega t} \text{ and } \mathbf{B} = \mathbf{B}_0 e^{i\omega t} \qquad ...(58)$$

where $\mathbf{D}_0$ and $\mathbf{B}_0$ are peak values of $\mathbf{D}$ and $\mathbf{B}$ respectively Eq. (58) yield

$$\left.\begin{aligned} \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{D}_0 (i\omega) e^{i\omega t} = i\omega \mathbf{D}\\ \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{B}_0 (i\omega) e^{i\omega t} = i\omega \mathbf{B}\end{aligned}\right\} \qquad ...(59)$$

Using Eq. (59) Maxwell's Eqs. (53) take the form

$$\left.\begin{aligned} &\nabla \cdot \mathbf{D} = \rho && ...(a)\\ &\nabla \cdot \mathbf{B} = 0 && ...(b)\\ &\nabla \times \mathbf{E} + i\omega \mathbf{B} = 0 && ...(c)\\ &\nabla \times \mathbf{H} - i\omega \mathbf{D} = \mathbf{J} && ...(d)\end{aligned}\right\} \qquad ...(60)$$

Example 4.4. *Starting from Maxwell's equations, establish Coulomb's law.*

Solution. From Maxwell's first equation,

$$\text{div}\, \mathbf{D} = \rho$$

Taking its volume integral over a sphere of radius r

We get, $\int_V \text{div}\, \mathbf{D}\, dV = \int_V \rho\, dV$

As $\int_V \rho\, dV = \text{Net charge enclosed by sphere}$

$= q$ (say)

$\therefore \int_V \text{div}\, \mathbf{D}\, dV = q$

Changing volume integral into surface integral, we get

$$\int_S \mathbf{D} \cdot d\mathbf{S} = q$$

But $\mathbf{D} = \varepsilon \mathbf{E}$

$\therefore \int_S \varepsilon \mathbf{E}.d\mathbf{S} = q \Rightarrow \varepsilon \int \mathbf{E}.d\mathbf{S} = q$

$\Rightarrow \varepsilon E 4\pi r^2 = q \Rightarrow E = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}$

In vector form, $\mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{q\hat{r}}{r^3}$

Force on charge q_0 will be

$$\mathbf{F} = q_0 \mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{q q_0}{r^3}\hat{r}$$

This is Coulomb's law.

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Example 4.5. *Show that equation of continuity* $\text{div}\,\mathbf{J}+\frac{\partial\rho}{\partial t}=0$ *is contained in Maxwell's equations.*

Or

Starting from Maxwell's equations, establish the equation of continuity.

Solution. From Maxwell's 4th equation,

$$\text{curl}\,\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} \quad ...(i)$$

Taking divergence of either sides of Eq. (*i*), we get

$$\text{div}(\text{curl}\,\mathbf{H})=\text{div}\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right) \quad ...(ii)$$

But $\text{div}(\text{curl}\,\mathbf{H})=0$, since divergence of curl of any vector always vanishes, therefore Eq. (*ii*) gives

$$\text{div}\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right)=0$$

or
$$\text{div}\,\mathbf{J}+\text{div}\frac{\partial\mathbf{D}}{\partial t}=0$$

or
$$\text{div}\,\mathbf{J}+\frac{\partial}{\partial t}(\text{div}\,\mathbf{D})=0 \quad ...(iii)$$

(Since space and time operations are interchangeable)

Also, from Maxwell's first equation $\text{div}\,\mathbf{D}=\rho$, therefore Eq. (*iii*) gives

$$\text{div}\,\mathbf{J}+\frac{\partial\rho}{\partial t}=0 \quad ...(iv)$$

This is requires result.

Example 4.6. *Starting from Maxwell's equations*

$$\text{curl}\,\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \text{ and } \text{curl}\,\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$$

respectively, show that $\text{div}\,\mathbf{B}=0$ *and* $\text{div}\,\mathbf{D}=\rho$.

Solution. Given $\text{curl}\,\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$

Taking divergence of both sides

$$\text{div}\,\text{curl}\,\mathbf{E}=-\text{div}\left(\frac{\partial\mathbf{B}}{\partial t}\right)$$

As divergence curl of any vector is zero and space and time operations are interchangeable

$$\text{div}\frac{\partial\mathbf{B}}{\partial t}=0 \Rightarrow \frac{\partial}{\partial t}(\text{div}\,\mathbf{B})=0 \Rightarrow \text{div}\,\mathbf{B}=\text{constant}$$

As isolated magnetic poles do not exist in nature, therefore for $\text{div}\,\mathbf{B}=0$.

Given
$$\text{curl}\,\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$$

Taking divergence of both sides,

$$\text{div}\,\text{curl}\,\mathbf{H}=\text{div}\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right)$$

As $\text{div}\,\text{curl}\,\mathbf{H}=0$

$$\Rightarrow \quad \text{div}\,\mathbf{J}+\text{div}\frac{\partial\mathbf{D}}{\partial t}=0$$

$$\text{div}\,\mathbf{J}=-\text{div}\frac{\partial\mathbf{D}}{\partial t}$$

But
$$\text{div}\,\mathbf{J}+\frac{\partial\rho}{\partial t}=0 \quad \text{[Equation of continuity]}$$

$$\therefore \quad -\text{div}\left(\frac{\partial\mathbf{D}}{\partial t}\right)+\frac{\partial\rho}{\partial t}=0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\text{div}\,\mathbf{D})=\frac{\partial\rho}{\partial t}$$

On integrating $\text{div}\,\mathbf{D}=\rho$.

4.11 Electromagnetic Energy : Poynting Theorem

Since we know that,

Electrostatic potential energy

$$U_e=\frac{1}{2}\int_V \mathbf{E}.\mathbf{D}\,dV \quad ...(61)$$

and energy stored in magnetic field

$$U_m=\frac{1}{2}\int_V \mathbf{H}.\mathbf{B}\,dV \quad ...(62)$$

Now let us see whether these expressions apply to non-static situations.

Maxwell's equations in differential form are :

$$\left.\begin{array}{lll} \nabla\cdot\mathbf{D}=\rho & \text{or} & \text{div}\,\mathbf{D}=\rho \quad ...(a) \\ \nabla\cdot\mathbf{B}=0 & \text{or} & \text{div}\,\mathbf{B}=0 \quad ...(b) \\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} & \text{or} & \text{curl}\,\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad ...(c) \\ \text{and} \quad \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} & \text{or} & \text{curl}\,\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} \quad ...(d) \end{array}\right\} \quad ...(63)$$

Taking scalar product of Eq. 63(*c*) with **H** and Eq. 63 (*d*) with **E**, we get

$$\mathbf{H}\cdot\text{curl}\,\mathbf{E}=-\mathbf{H}.\frac{\partial\mathbf{B}}{\partial t} \quad ...(64)$$

and
$$\mathbf{E}.\text{curl}\,\mathbf{H}=\mathbf{E}.\mathbf{J}+\mathbf{E}.\frac{\partial\mathbf{D}}{\partial t} \quad ...(65)$$

Using vector identity,

$$\text{div}(\mathbf{E}\times\mathbf{H})=\mathbf{H}.\text{curl}\,\mathbf{E}-\mathbf{E}.\text{curl}\,\mathbf{H} \quad ...(66)$$

Now, subtracting Eq. (65) from Eq. (64), we get

$$\mathbf{H}.\text{curl}\,\mathbf{E}-\mathbf{E}.\text{curl}\,\mathbf{H}=-\mathbf{H}.\frac{\partial\mathbf{B}}{\partial t}-\mathbf{E}.\frac{\partial\mathbf{D}}{\partial t}-\mathbf{E}.\mathbf{J} \quad ...(67)$$

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From Eqs. (66) and (67), we get

$$\text{div}(\mathbf{E}\times\mathbf{H}) = -\left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right) - \mathbf{E}\cdot\mathbf{J} \qquad ...(68)$$

Now if medium is linear so that the relation,

$$\mathbf{B} = \mu\mathbf{H} \text{ and } \mathbf{D} = \varepsilon\mathbf{E} \qquad ...(69)$$

Apply, then we may write,

$$\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} = \mathbf{E}\cdot\frac{\partial}{\partial t}(\varepsilon\mathbf{E}) = \frac{1}{2}\varepsilon\frac{\partial}{\partial t}(\mathbf{E})^2 = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D}\right)$$

and

$$\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} = \mathbf{H}\cdot\frac{\partial}{\partial t}(\mu\mathbf{H}) = \frac{1}{2}\mu\frac{\partial}{\partial t}(\mathbf{H})^2 = \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{H}\cdot\mathbf{B}\right)$$

Using these relationship, Eq. (68) takes the form

$$\text{div}(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\right] - \mathbf{J}\cdot\mathbf{E} \qquad ...(70)$$

Each term in Eq. (70) has certain physical significance which may be seen by integrating Eq. (70) over a volume V bounded by surface S.

Thus,

$$\int_V \text{div}(\mathbf{E}\times\mathbf{H})\,dV = -\int_V\left\{\frac{\partial}{\partial t}\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\right\}dV - \int_V \mathbf{J}\cdot\mathbf{E}\,dV$$

Using Gauss div theorem to change volume integral into surface integral, we get

$$\oint(\mathbf{E}\times\mathbf{H})\,dS = -\frac{d}{dt}\int_V\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dV - \int_V \mathbf{J}\cdot\mathbf{E}\,dV$$

Rearranging this equation, we get

$$-\int \mathbf{J}\cdot\mathbf{E}\,dV = \frac{d}{dt}\int_V\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dV + \oint_S(\mathbf{E}\times\mathbf{H})\,d\mathbf{S} \qquad ...(71)$$

To understand the physical significance of above equation let us now interpret each term in it.

Interpretation of $\int_V \mathbf{J}\cdot\mathbf{E}\,dV$

To understand the meaning of this term let us consider a charge particle q moving with velocity $\mathbf{v}$ under the combined effect of mechanical, electric and magnetic forces.

The electromagnetic force due to field vector $\mathbf{E}$ and $\mathbf{B}$ acting on the charge particle is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

As the magnetic force $q(\mathbf{v}\times\mathbf{B})$ is always perpendicular to velocity, hence the magnetic field does not work. Therefore for a single charge q the rate of doing work by electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ is

$$\frac{\partial W}{\partial t} = \mathbf{F}\cdot\mathbf{v} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v} = q\mathbf{E}\cdot\mathbf{v}$$

If $\mathbf{F}_m$ is mechanical force, then workdone by *mechanical force against electromagnetic field vector* per unit time *i.e.*, the rate at which mechanical work is done on the particle as

$$\frac{\partial W_m}{\partial t} = \mathbf{F}_m\cdot\mathbf{v} = -\mathbf{F}\cdot\mathbf{v} = -q\mathbf{E}\cdot\mathbf{v} \qquad ...(72)$$

If electromagnetic field consists of a group of charges moving with different velocities *e.g.*, n_i charge carriers each of charge q_i moving with velocity v_i $(i = 1, 2, 3, 4, \ldots)$ the Eq. (72) may be written as

$$\frac{\partial W_m}{\partial t} = -\sum_i n_i q_i \mathbf{v}_i\cdot\mathbf{E}_i \qquad ...(73)$$

In this case current $\quad \mathbf{J} = \sum_i n_i q_i \mathbf{v}_i$

Using this substitution, Eq. (73) becomes,

$$\frac{\partial W_m}{\partial t} = -\sum_i \mathbf{J}_i\cdot\mathbf{E}_i = -\mathbf{J}\cdot\mathbf{E} \qquad ...(74)$$

This equation represents the *power density that is transferred into electromagnetic field.*

Therefore, the expression $-\int \mathbf{E}\cdot\mathbf{J}\,dV$ *represents rate of energy transferred into electromagnetic field through the motion of free charge in volume V.*

If there are no source of e.m.f. in volume V, then the term

$$-\int \mathbf{E}\cdot\mathbf{J}\,dV = -\int\frac{J^3}{\sigma}\,dV \qquad [\because \mathbf{J} = \sigma\mathbf{E}] \qquad ...(75)$$

is negative and represents negative of the rate of heat energy produced.

Interpretation of $\frac{d}{dt}\int\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dV$

We know $\quad \int_V \frac{1}{2}\mathbf{E}\cdot\mathbf{D}\,dV = U_e$ = Electrostatic potential energy in volume V

$\int_V \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\,dV = U_m$ = Magnetic energy in volume V

$$\therefore \text{ Obviously } U = U_e + U_m = \int_V \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dV \qquad ...(76)$$

represents some sort of potential energy of electromagnetic field. One need not ascribe this potential energy to the charged particles by consider this term as a **field energy**. This is known as **electromagnetic field energy** in volume V. *A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of electromagnetic theory.* Obviously $\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})$ represents energy density of the electromagnetic field *i.e.*,

$$U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}) \qquad ...(77)$$

Consequently the term $\frac{d}{dt}\int\frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B})\,dV$ represents the *rate of electromagnetic energy stored in volume V.*

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Interpretation of $\oint_S (\mathbf{E}\times\mathbf{H})\,d\mathbf{S}$

Since surface integral in this term involves only electric and magnetic fields, it is feasible to interpret this term as the rate of energy flow across the surface. It means that $(\mathbf{E}\times\mathbf{H})$ should represents the energy flow per unit time per unit area. The latter interpretation, however leads to certain difficulties ; the only interpretation which survives is that the surface **integral of $(\mathbf{E}\times\mathbf{H})$ over a closed surface represents the amount of electromagnetic energy crossing the closed surface per second**. The vector $(\mathbf{E}\times\mathbf{H})$ is known as the **Poynting vector** and usually represented by the symbol $\mathbf{S}$ *i.e.*,

$$\text{Poynting vector } \mathbf{S} = \mathbf{E}\times\mathbf{H} \qquad ...(78)$$

Interpretation Energy Eq. (70) or Eq. (71)

In view of above interpretation Eq. (70) may be expressed as

$$\boxed{-\mathbf{J}.\mathbf{E} = \frac{\partial U}{\partial t} + \nabla.\mathbf{S}} \qquad ...(79)$$

The physical meaning of this equation is that *the time rate of change of electromagnetic energy with certain volume plus time rate of energy flowing out through the boundary surface is equal to the power transferred into the electromagnetic field.*

This is the statement of conservation of energy in electromagnetism and known as **Poynting Theorem**.

4.12 Poynting Vector

In preceding section we have seen that

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \qquad ...(80)$$

is known as *Poynting vector* and is interpreted as the power flux *i.e.*, amount of energy crossing unit area placed perpendicular to the vector, per unit time. The conception of energy of the electromagnetic field as residing in the medium is very fundamental one and has great advantage in the development of the theory. Maxwell thought of the medium as resembling as elastic solid, the electrical energy representing the P.E. of strain of the medium, the magnetic energy the K.E. of motion. Though such a mechanical view no longer exists, still the energy is regarded as being localised in space and as travelling in the manner indicating by **Poynting Vector**. In a light wave there is certain energy per unit volume, proportional to the square of the amplitude ($\mathbf{E}$ or $\mathbf{H}$). This energy travels along it and Poynting vector is the vector that measures the rate of flow or the intensity of the wave. In a plane electromagnetic wave $\mathbf{E}$ and $\mathbf{H}$ are at right angles to each other and at right angles to the direction of flow ; thus $\mathbf{E}\times\mathbf{H}$ must be along the direction of flow. In more complicated waves as well, Poynting vector points along the direction of flow of radiation. For example, if we have a source of light and we wish to find at what rate it is emitting energy, we surround it by a closed surface and integrate the normal component of Poynting vector over the surface. The whole conception of energy being transported in the medium is fundamental to the electromagnetic theory of light.

In case of time varying field $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ gives the instantaneous value of the Poynting vector. Let us find the form of Poynting vector for such cases. Let the field $\mathbf{E}$ and $\mathbf{H}$ be given by real parts of complex exponentials of the form

$$\mathbf{E} = \mathbf{E}_0(r)e^{i\omega t} \qquad ...(81)$$

At a given point of space let us assume that $\mathbf{E}$ is given by the real part of $\mathbf{E}_0 e^{i\omega t}$ and $\mathbf{H}$ by real part $\mathbf{H}_0 e^{i\omega t}$ where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex vector functions of position. Let the real and imaginary parts of $\mathbf{E}_0$ be denoted by $\mathbf{E}_r$ and $\mathbf{E}_{im}$ respectively. Similarly real and imaginary parts of $\mathbf{H}_0$ are $\mathbf{H}_r$ and $\mathbf{H}_{im}$. Then

$$\mathbf{E} = \text{Real part of } (\mathbf{E}_0 e^{i\omega t}) = \text{Re}(\mathbf{E}_0 e^{i\omega t})$$

where Re denotes 'real part of'

$$\therefore \quad \mathbf{E} = \text{Re}(\mathbf{E}_r + i\mathbf{E}_{im})(\cos\omega t + i\sin\omega t)$$
$$= \mathbf{E}_r\cos\omega t - \mathbf{E}_{im}\sin\omega t$$

Similarly, $\quad \mathbf{H} = \mathbf{H}_r\cos\omega t - \mathbf{H}_{im}\sin\omega t$

Then Poynting vector is

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$
$$= (\mathbf{E}_r\times\mathbf{H}_r)\cos^2\omega t + (\mathbf{E}_{im}+\mathbf{H}_{im})\sin^2\omega t$$
$$-\{(\mathbf{E}_r\times\mathbf{H}_{im}) + (\mathbf{E}_{im}\times\mathbf{H}_r)\}\sin\omega t\cos\omega t$$

We notice that there are two types of terms in the above expressions ; the first two whose time average is different from zero, since $\sin^2\omega t$ and $\cos^2\omega t$ average to $\frac{1}{2}$, and the last term whose time average is zero. Thus the time average of **Poynting vector** (average being) taken over a complete cycle is

$$\langle\mathbf{S}\rangle = \langle\mathbf{E}\times\mathbf{H}\rangle = \frac{1}{2}\{(\mathbf{E}_r\times\mathbf{H}_r) + (\mathbf{E}_{im}\times\mathbf{H}_{im})\} \qquad ...(83)$$

This equation can be written in a convenient way by using the notation of complex conjugates, where the complex conjugate of a complex number is the number obtained from original one by changing the sign of i, wherever it appears is indicated by $*(\bullet)$ over the number. In terms of this notation

$$(\mathbf{E}\times\mathbf{H}^*) = (\mathbf{E}_0 e^{i\omega t})\times(\mathbf{H}^* e^{-i\omega t})$$
$$= (\mathbf{E}_0\times\mathbf{H}) = (\mathbf{E}_r + i\mathbf{E}_{im})\times(\mathbf{H}_r - i\mathbf{H}_{im})$$
$$= [(\mathbf{E}_r\times\mathbf{H}_r) + (\mathbf{E}_{im}\times\mathbf{H}_{im})] + i[\mathbf{E}_{im}\times\mathbf{H}_r - \mathbf{E}_r\times\mathbf{H}_{im}] \qquad ...(84)$$

Comparing Eq. (83) and (84), we note that, except for the factor $\frac{1}{2}$, the real part of Eq. (84) is just the same as the quantity appearing in Eq. (83), *i.e.*, we have

$$\langle\mathbf{S}\rangle = \langle\mathbf{E}\times\mathbf{H}\rangle = \frac{1}{2}\text{Re}(\mathbf{E}\times\mathbf{H}^*) \qquad ...(85)$$

where $\mathbf{E}$ and $\mathbf{H}$ appearing on the R.H.S. of above equation are the complex quantities whose real parts give the real $\mathbf{E}$ and $\mathbf{H}$ appearing on the L.H.S. of the equation.

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Similarly we may show that

Average electrostatic energy density $\langle U_e \rangle$

$$= \langle \frac{1}{2} \varepsilon E^2 \rangle = \frac{1}{4} \varepsilon \mathbf{E}.\mathbf{E}^* \quad ...(86)$$

and average magnetic energy density

$$\langle U_m \rangle = \langle \frac{1}{2} \mu H^2 \rangle = \frac{1}{4} \mu \mathbf{H}.\mathbf{H}^* \quad ...(87)$$

4.13 Plane Electromagnetic Waves in Free Space

Maxwell's equations are :

$$\left.\begin{aligned} &\text{div } \mathbf{D} = \nabla . \mathbf{D} = \rho && ...(a) \\ &\text{div } \mathbf{B} = \nabla . \mathbf{B} = 0 && ...(b) \\ &\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && ...(c) \\ \text{and} \quad &\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && ...(d) \end{aligned}\right\} \quad ...(88)$$

Free space is characterised by,

$$\rho = 0, \ \sigma = 0, \ \mu = \mu_0 \text{ and } \varepsilon = \varepsilon_0 \quad ...(89)$$

Therefore, Maxwell's Eqs. (88) reduce to

$$\left.\begin{aligned} &\text{div } \mathbf{E} = 0 && ...(a) \\ &\text{div } \mathbf{H} = 0 && ...(b) \\ &\text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} && ...(c) \\ \text{and} \quad &\text{curl } \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && ...(d) \end{aligned}\right\} \quad ...(90)$$

Taking curl of Eq. 90(*c*), we get

$$\text{curl curl } \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} (\text{curl } \mathbf{H})$$

Substituting curl H from Eq. 90(*d*), we get

$$\text{curl curl } \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

i.e.,

$$\text{curl curl } \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad ...(91)$$

Now, $\text{curl curl } \mathbf{E} = \text{grad div } \mathbf{E} - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$ [$\because$ div E = 0 from Eq. 90(*a*)]

Making substituting Eq. (91), becomes

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad ...(92)$$

Now taking curl of Eq. 90(*d*), we get

$$\text{curl curl } \mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t} \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t} \right)$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad ...(93)$$

Again using identity curl curl H = grad div H – ∇^2H and noting that div H = 0 from Eq. 90(*b*), we obtain

$$\text{curl curl } \mathbf{H} = -\nabla^2 \mathbf{H}$$

Making this substitution in Eq. (93), we get

$$\nabla^2 \mathbf{H} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad ...(94)$$

Eqs. (92) and (94) represent wave equation governing electric and magnetic fields (**E** and **H**) in free space. It may be noted that these equations may be obtained by using Eq. (89) in Eqs. (92) and (94) are vector equations of identical form, which means that each of six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 \phi - \mu_0 \varepsilon_0 \frac{\partial^2 \phi}{\partial t^2} = 0 \quad ...(95)$$

where ϕ is a scalar and can not stand for one of the components of **E** and **H**. it is obvious that Eq. (95) resembles with general wave equation

$$\nabla^2 \phi = \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} \quad ...(96)$$

where v is the velocity of wave.

Comparing Eqs. (95) and (96), we see that the field vector **E** and **H** are propagated in free space as waves at a speed equal to

$$v = \frac{1}{\sqrt{(\mu_0 \varepsilon_0)}} \quad ...(97)$$

$\because$ $\mu_0 = 4\pi \times 10^{-7}$ Wb/A-m

$\varepsilon_0 = 8.542 \times 10^{-12}$ F/m

So, that $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ m/F} = \sqrt{\left(\frac{4\pi}{\mu_0 4\pi\varepsilon_0} \right)}$

$$= \sqrt{\left(\frac{4\pi}{4\pi \times 10^{-7}} \times 9 \times 10^9 \right)}$$

$= 3 \times 10^8$ m/s = c = speed of light in vacuum.

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Therefore, it is reasonable to write c the speed of light in place of $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$, so Eqs. (92) and (94) take the form

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad ...(98)$$

$$\nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad ...(99)$$

and

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \quad ...(100)$$

Now, let us find the solution of above equations for plane electromagnetic waves. "*A plane wave is defined as a wave whose amplitude is the same at any point in a plane perpendicular to a specified direction*".

The plane wave solution of above equations in well known form may be written as

$$\mathbf{E}(r,t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(101)$$

$$\mathbf{H}(r,t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(102)$$

$$\phi(r,t) = \phi_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \quad ...(103)$$

where $\mathbf{E}_0$, $\mathbf{H}_0$ and ϕ_0 are complex amplitude which are constant in space and time while $\mathbf{k}$ is a wave propagation vector denoted as

$$\mathbf{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n} \quad ...(104)$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation. Now in order to apply the condition $\nabla \times \mathbf{E} = 0$ and $\nabla \times \mathbf{H} = 0$. Let us first find $\nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{H}$.

$$\nabla \cdot \mathbf{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot E_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= \left(\hat{i}\frac{\partial}{\partial t} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot [(\hat{i}\, E_{0x} + \hat{j}\, E_{0y} + \hat{k}\, E_{0z}) e^{i(k_x x + k_y y + k_z z) - i\omega t}]$$

$$[\because \hat{k}\cdot\hat{r} = (\hat{i}\, k_x + \hat{j}\, k_y + \hat{k}\, k_z)\cdot(\hat{i}\, x + \hat{j}\, y + \hat{k}\, z) = k_x x + k_y y + k_z z]$$

$$\therefore \quad \nabla \cdot \mathbf{E} = (E_{0x}\,\hat{i}\, k_x + E_{0y}\,\hat{i}\, k_y + E_{0z}\,\hat{i}\, k_z) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= \hat{i}\,(k_x E_{0x} + k_y E_{0y} + k_z E_{0z}) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= \hat{i}\,(\hat{i}\, k_x + \hat{i}\, k_y + \hat{k}\, k_z)\cdot(\hat{i}\, E_{0x} + \hat{j}\, E_{0y} + \hat{k}\, E_{0z}) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i\mathbf{k}\cdot\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$$

$$= i\mathbf{k}\cdot\mathbf{E}$$

Similarly, $\nabla \cdot \mathbf{H} = i\mathbf{k}\cdot\mathbf{H}$

Thus the requirement $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$ demand that

$$\mathbf{k}\cdot\mathbf{E} = 0 \text{ and } \mathbf{k}\cdot\mathbf{H} = 0 \quad ...(105)$$

This means that electromagnetic field vectors **E** and **H** are both perpendicular the direction of propagation vector **k**. This implies that **electromagnetic waves are transverse in character**. Further restrictions are provided by curl. Eqs. 90(*c*) and 90(*d*)

$$\text{curl}\,\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad \text{and} \quad \text{curl}\,\mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Using Eqs. (101) and (102), above equation yield

$$i\mathbf{k}\times\mathbf{E} = -\mu_0\,(-i\omega\mathbf{H}) \quad \text{or} \quad \mathbf{k}\times\mathbf{E} = \mu_0\omega\mathbf{H} \quad ...(106)$$

and

$$i\mathbf{k}\times\mathbf{H} = \epsilon_0\,(-i\omega\mathbf{E}) \quad \text{or} \quad \mathbf{k}\times\mathbf{H} = -\epsilon_0\omega\mathbf{E} \quad ...(107)$$

From Eq. (106), it is obvious that field vector **H** is perpendicular to both **k** and **E** and according to Eq. (107) **E** perpendicular to both **k** and **H**. This simply means that *field vectors E and H are mutually perpendicular and they are also perpendicular to the direction of propagation of wave*. This all in turn implies that in a plane electromagnetic wave, vectors (**E**, **H**, **k**) form a set of orthogonal vectors which form a right handed coordinate system in that order. (Fig. 4.10).

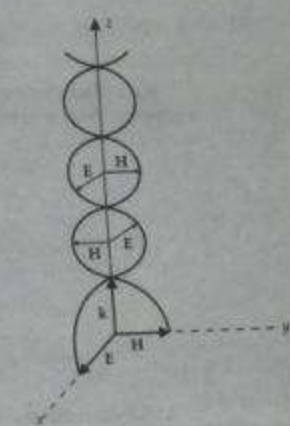

Fig. 4.10

Further from Eq. (106),

$$\mathbf{H} = \frac{1}{\mu_0\omega}(\mathbf{k}\times\mathbf{E}) = \frac{k}{\mu_0\omega}(\hat{n}\times\mathbf{E}) \quad [\because \mathbf{k} = k\hat{n}]$$

$$= \frac{1}{\mu_0 c}(\hat{n}\times\mathbf{E}) \quad ...(108)$$

This equation in terms of module

$$H = \frac{1}{\mu_0 c}E \quad \left[\because \frac{1}{\sqrt{\mu_0\epsilon_0}} = \frac{c}{1}\right]$$

Now the ratio of magnitude of **E** and magnitude of **H** symbolised as $\mathbf{Z}_0$ *i.e.*,

$$\mathbf{Z}_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \left|\frac{\mathbf{E}_0}{\mathbf{H}_0}\right| = \mu_0 c = \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)} \quad \left[\because c = \frac{1}{\sqrt{\mu_0\epsilon_0}}\right]$$

$$= \sqrt{\left(\frac{4\pi\times10^{-7}}{8.85\times10^{-12}}\right)} = 376.6 \text{ ohms} \quad ...(109)$$

where the units of Z_0 are most easily seen from the fact that it measures a ratio of E in V/m to H in Amp/m and there must equal V/A or Ohms. Because the units of E/H are the same as those of impedance, the value of $\mathbf{Z}_0$ is often referred to as the **wave impedance of free space**. Further since the ratio $\mathbf{Z}_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right|$ is real and positive ; this implies that field vectors **E** and **H** are in the same phase *i.e.*, they have the same relative magnitude at all points at all times. (Fig. 4.10).

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The Poynting vector (*i.e.*, energy flow per unit area per unit time) for a plane electromagnetic wave is given by

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \mathbf{E}\times\frac{\hat{n}\times\mathbf{E}}{\mu_0 c} \qquad \text{[using Eq. (108)]}$$

$$= \frac{1}{\mu_0 c}\mathbf{E}\times(\hat{n}\times\mathbf{E}) = \frac{1}{\mu_0 c} \qquad [(\mathbf{E}=E/\hat{n}\sim(\mathbf{E}\cdot\hat{n})\mathbf{E}]$$

$$= \frac{1}{\mu_0 c}E^2\hat{n} \qquad [\mathbf{E}\cdot\hat{n}=0;\ \mathbf{E}\text{ being }\perp\text{ to }\hat{n}]$$

$$= \frac{E^2}{Z_0}\hat{n} \qquad \text{[refer Eq. (109)]}$$

For a plane electromagnetic wave of angular frequency ω, the average value of $\mathbf{S}$ over a complete cycle is given by

$$\langle S\rangle = \frac{1}{Z_0}\langle E^2\rangle\hat{n} = \frac{1}{Z_0}\langle(E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-i\omega t)})^2\rangle_{real}\hat{n}$$

$$= \frac{1}{Z_0}E_0^2\langle\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\rangle\hat{n}$$

$$= \frac{1}{Z_0}\frac{E_0^2}{2}\hat{n} \qquad \left[\text{Average value of }\cos^2(k\cdot r-\omega t)=\frac{1}{2}\right]$$

$$= \frac{1}{Z_0}E_{rms}^2\hat{n} \qquad \text{...(110)} \qquad [\because E_{rms}=E_0/\sqrt{2}]$$

It is obvious that the direction of Poynting vector is along the direction of propagation of e.m.wave. This means that the **flow of energy in a plane electromagnetic wave in free space is along the direction of wave**.

Ratio of electrostatic and magnetic energy densities is given by

$$\frac{U_e}{U_m} = \frac{\frac{1}{2}\varepsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\varepsilon_0 E^2}{\mu_0 H^2} = \frac{\varepsilon_0}{\mu_0}\cdot\frac{\mu_0}{\varepsilon_0} = 1 \qquad \text{...(111)} \qquad \left[\because \frac{E}{H}=\sqrt{\frac{\mu_0}{\varepsilon_0}}\right]$$

i.e., the electromagnetic energy density is equal to magnetostatic energy density ($U_e = U_m$). Total electromagnetic energy density,

$$U = U_e + U_m = 2U_e = 2\times\frac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2$$

Time average of energy density,

$$\langle U\rangle = \langle\varepsilon_0 E^2\rangle = \varepsilon_0\langle(E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-i\omega t)})^2\rangle_{real}$$

$$= \varepsilon_0 E_0^2\langle\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\rangle = \frac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E_{rms}^2 \qquad \text{...(112)}$$

Dividing Eq. (110) by Eq. (112), we obtain

$$\frac{\langle\mathbf{S}\rangle}{\langle U\rangle} = \frac{1}{Z_0\varepsilon_0}\hat{\mathbf{n}} = \frac{1}{\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\varepsilon_0}\hat{\mathbf{n}} = \frac{\hat{\mathbf{n}}}{\sqrt{\mu_0\varepsilon_0}} = c\hat{\mathbf{n}} \qquad \text{...(113)}$$

Thus we obtain,

$$\left.\begin{aligned}\langle\mathbf{S}\rangle &= \langle U\rangle c\hat{\mathbf{n}} && \text{...(a)}\\ \text{i.e., Energy flux} &= \text{Energy density} \times \text{Velocity of light in vacuum} && \text{...(b)}\end{aligned}\right\} \qquad \text{...(114)}$$

The Eq. (114) implies that the *energy density associated with an e.m.wave in free space propagates with speed of light with which the field vectors do.*"

Summary

- In free space the e.m.waves travel with the speed of light.
- The e.m. field vectors **E** and **H** are mutually $\perp$ to the direction of propagation of e.m.waves. Thereby indicating the e.m.waves are transverse in nature.
- The field vectors **E** and **H** are in same phase.
- The direction of flow of e.m. energy is along the direction of wave propagation and the energy flow per unit area per second is represented by

$$\langle\mathbf{S}\rangle = \frac{E_{rms}^2}{Z_0}\hat{\mathbf{n}} = \langle U\rangle c\hat{\mathbf{n}}$$

- The electrostatic energy density is equal to the magnetic energy density and the energy density associated with the e.m. wave in free space propagates with the speed of light.

Example 4.7 *The electromagnetic wave intensity received on the surface of the earth from the sun is found to be 1.33kW/m². Find the amplitude of electric field vector associated with sunlight as received on earth surface. Assume sun's light to be mono-chromatic* ($\lambda = 6000$ Å).

[GGSIPU, Feb 2008 (3 marks)]

Solution. The energy transported by an electromagnetic wave per unit area per second during propagation is represented by Poynting vector **S** as

$$\mathbf{S} = (\mathbf{E}\times\mathbf{H})$$

The energy flux per unit area per second is

$$|\mathbf{S}| = |\mathbf{E}\times\mathbf{H}| = EH\sin 90^\circ = EH$$

The energy flux per unit area per second at the earth surface.

$$|\mathbf{S}| = 1.33\ \text{kW/m}^2 = 1.33\times10^3\ \text{Jm}^{-2}\text{s}^{-1} \qquad \text{...(i)}$$

$$\therefore \quad |\mathbf{S}| = 1330\ \text{Jm}^{-2}\text{s}^{-1}$$

We know that

$$Z_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}\ \text{Wb/A-m}}{8.854\times10^{-12}\ \text{C}^2/\text{Nm}^2}} = 376.72\ \text{Ohms} \qquad \text{...(ii)}$$

$$\frac{E}{H} = 376.72\ \text{Ohms}$$

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Multiplying Eqs. (*i*) and Eq. (*ii*), we get

$$EH \times \frac{E}{H} = 1330 \times 376.72$$

or $$\because E^2 = 501037.6, \quad E = 707.8 \text{ V/m}$$

Substituting this value in Eq. (*i*)

$$H = \frac{1330}{707.8} = 1.879 \text{ Amp/m.}$$

Therefore, the amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 707.8\sqrt{2} = 1000.8292 = 1000 \text{ V/m}$$

and $$H_0 = H\sqrt{2} = 1928\sqrt{2} = 2.73 \text{ Amp.turn/m.}$$

Example 4.8. *If the earth receives 2 Cal min^{-1} cm^{-2} solar energy, what are the amplitudes of electric and magnetic field of radiation ?*

Solution. As Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = EH \sin 90° = EH$$

Solar energy = 2 Cal min^{-1}cm^{-2}

$$= \frac{2 \times 4.18 \times 10^4}{60} \text{ Joule m}^{-2}\text{sec}^{-1}$$

Both are energy flux per unit area per second

Hence $$EH = \frac{2 \times 4.18 \times 10^4}{60} = 1400$$

But $$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi = 377$$

$$EH \times \frac{E}{H} = 1400 \times 377$$

$$E = \sqrt{1400 \times 377} = 726.5 \text{ V/m}$$

Now, $$H = \frac{E}{377} = 1.927 \text{ Amp/m}$$

Amplitudes of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 1024.3 \text{ V/m}$$

$$H_0 = H\sqrt{2} = 2.717 \text{ Amp/m}$$

4.14 Plane Electromagnetic Waves in an Isotropic (Simple) Dielectric (Plane e.m. waves in non-conducting Medium)

A non-conducting medium which has same properties in all directions is called an *isotropic dielectric*.

Maxwell's equations are :

$$\left.\begin{aligned} \text{div } \mathbf{D} &= \nabla \cdot \mathbf{D} = \rho && \text{...(a)} \\ \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 && \text{...(b)} \\ \text{curl } \mathbf{E} &= \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{...(c)} \\ \text{and} \quad \text{curl } \mathbf{H} &= \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && \text{...(d)} \end{aligned}\right\} \quad \text{...(115)}$$

In an isotropic dielectric (or non-conducting isotropic medium)

$$\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}, \ \mathbf{J} = \sigma \mathbf{E} = 0 \text{ and } \rho = 0 \quad \text{...(116)}$$

Therefore, Maxwell's equations in this case take the form

$$\left.\begin{aligned} \text{div } \mathbf{E} &= \nabla \cdot \mathbf{E} = 0 && \text{...(a)} \\ \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 && \text{...(b)} \\ \text{curl } \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} && \text{...(c)} \\ \text{and} \quad \text{curl } \mathbf{H} &= \varepsilon \frac{\partial \mathbf{E}}{\partial t} && \text{...(d)} \end{aligned}\right\} \quad \text{...(117)}$$

Taking curl of Eq. 117(*c*), we get

$$\text{curl curl } \mathbf{E} = -\mu \frac{\partial}{\partial t} (\text{curl } \mathbf{H})$$

Substituting curl **H** from Eq. 117(*d*) in above equation

$$\text{curl curl } \mathbf{E} = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e., $$\text{curl curl } \mathbf{E} = -\mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{...(118)}$$

Similarly if we take curl of Eq. 117(*d*) and substitute curl **E** from Eq. 117(*c*), we get

$$\text{curl curl } \mathbf{H} = -\mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{...(119)}$$

and keeping in mid Eqs. 117(*a*) and 117(*b*) *i.e.*, div **E** = 0 and div **H** = 0 Eqs. (118) and (119) give

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \text{...(120)}$$

and $$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad \text{...(121)}$$

These equations are vector equations of identical form which means that each of the six components of **E** and **H** separately satisfies the same scalar wave equation of the form

$$\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} = 0 \quad \text{...(122)}$$

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where ϕ is a scalar and a stand for any one of components of **E** and **H**. It is obvious that Eqs. (120), (121) resembles with general wave equation

$$\nabla^2\phi - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = 0 \qquad ...(123)$$

where v = speed of wave.

This means that the field vectors **E** and **H** are propagated in isotopic dielectric as waves with speed v given by

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0}} \qquad ...(124)$$

where K_m is relative permeability of medium and K_e is relative permittivity (or dielectric constant) of the medium.

As $\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$ speed of electromagnetic waves in free space.

$$\therefore \quad v = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \qquad ...(125)$$

Since $\mu_r > 1$ and $\mu_r > 1$, thereby indicating that *the speed of electromagnetic waves is in isotropic dielectric is less than the speed of electromagnetic in free space.*

We know the refractive index (n) is given as

$$n = \frac{c}{v} \quad i.e., \; v = \frac{c}{n} \qquad ...(126)$$

∴ Comparing Eqs. (125) and (126) we note that refractive index n in this particular case is

$$n = \sqrt{(\mu_r\varepsilon_r)} \qquad ...(127)$$

For a non-magnetic material $K_m = 1$, therefore,

$$n = \sqrt{\varepsilon_r} \quad i.e., \; n^2 = \varepsilon_r \qquad ...(128)$$

This is known as Maxwell's relation and has been verified by a number of experiments.

Replacing $\mu\varepsilon$ by $\frac{1}{v^2}$, wave Eqs. (120) and (121) may be expressed as

$$\nabla^2\mathbf{E} - \frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \qquad ...(129)$$

$$\nabla^2\mathbf{H} - \frac{1}{v^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0 \qquad ...(130)$$

The plane-wave solutions of Eqs. (129) and (130) is well known form may be written as

$$\mathbf{E}(r,t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(131)$$

$$\mathbf{H}(r,t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(132)$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex amplitudes which are constant in space and time, while **k** is wave propagation vector given by

$$\mathbf{k} = \hat{k}\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{\omega}{v}\hat{n} \qquad ...(133)$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation.

Relative Directions of E and H

The requirement $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{H} = 0$ demands that

$$\mathbf{k}\cdot\mathbf{E} = 0 \text{ and } \mathbf{k}\cdot\mathbf{H} = 0 \qquad ...(134)$$

Comparing Eqs. (122) and (123), we see

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad ...(135)$$

This means that the field vectors **E** and **H** are both perpendicular to the direction of propagation vector **k**. This implies that **electromagnetic waves in isotropic dielectric are transverse in nature**. Further restrictions are provided by curl Eqs. 117(c and 117(d)) viz.

$$\text{curl}\,\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \text{ and curl } \mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t} \qquad ...(136)$$

and

$$\mathbf{k}\times\mathbf{H} = -\varepsilon\omega\mathbf{E} \qquad ...(137)$$

From these equations it is obvious that field vectors **E** and **H** are mutually perpendicular and also they are perpendicular to the direction of propagation vector **k**. This in turn implies that a plane electromagnetic wave in isotropic dielectric, vector (**E**, **H**, **k**) form a set of orthogonal vectors which form a right handed coordinate system in a given order (Fig. 4.10).

Phase of E and H and Wave Impedance

From Eq. (136)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k}\times\mathbf{E}) = \frac{k}{\mu\omega}(\hat{n}\times\mathbf{E})$$

$$= \frac{1}{\mu v}(\hat{n}\times\mathbf{E}) = \sqrt{\frac{\varepsilon}{\mu}}(\hat{n}\times\mathbf{E}) \qquad \left[\because k = \frac{\omega}{v} \text{ and } v = \frac{1}{\sqrt{\mu\varepsilon}}\right] \qquad ...(138)$$

Now, the ratio of magnitude of **E** to the magnitude of **H** is symbolised by Z *i.e.*,

$$Z = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\left(\frac{\mu_r\mu_0}{\varepsilon_r\varepsilon_0}\right)} = \text{real quantity} \qquad ...(139)$$

This implies that the *field vectors E and H are in the same phase i.e., they have same relative magnitudes at all points at all times.*

The unit of Z come out to be Ohm, since

$$Z = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\text{Volt/m}}{\text{Amp/m}} = \frac{\text{Volt}}{\text{Amp}} = \text{Ohm};$$

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Hence the value Z is referred to as *wave impedance of isotropic dielectric medium. The wave of impedance of medium is related to that of free space by the relation*

$$Z = \sqrt{\left(\frac{\mu_r \mu_0}{\varepsilon_r \cdot \varepsilon_0}\right)} = \sqrt{\frac{\mu_r}{\varepsilon_r}} \cdot Z_0 \qquad ...(140)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the wave impedance of free space.

Poynting vector for a plane e.m.wave in an isotropic dielectric is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \left[\sqrt{\frac{\varepsilon}{\mu}}\, \hat{n} \times \mathbf{E}\right]$$

$$= \frac{\mathbf{E} \times (\hat{n} \times \mathbf{E})}{Z} \qquad \left[\because Z = \sqrt{\frac{\mu}{\varepsilon}}\right]$$

$$= \frac{(\mathbf{E} \cdot \mathbf{E})\hat{n} - (\mathbf{E} \cdot \hat{n})\mathbf{E}}{Z}$$

$$= \frac{E^2}{Z}\hat{n} \qquad [\because \mathbf{E} \cdot \hat{n} = 0 \text{ because } \mathbf{E} \perp \text{ to } \hat{n}]$$

The time average of Poynting vector is

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \left\langle \frac{E^2}{Z}\hat{n} \right\rangle = \frac{1}{Z}\langle E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - i\omega t)} \rangle_{real}\, \hat{n}$$

Since for finding actual physical fields we often take real parts of complex expotentials

$$\therefore \quad \langle \mathbf{S} \rangle = \frac{1}{Z} E_0^2 \langle \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t) \rangle\, \hat{n} = \frac{1}{Z} \cdot \frac{E_0^2}{2} \cdot \hat{n} = \frac{1}{2Z} E_0^2\, \hat{n}$$

$$= \frac{1}{Z} E_{rms}^2\, \hat{n} \qquad \left[\because E_{rms} = \frac{E_0}{\sqrt{2}}\right] \qquad ...(141)$$

$$= \sqrt{\left(\frac{\varepsilon_r}{\mu_r}\right)} \frac{1}{Z_0} E_{rms}^2\, \hat{n}$$

$$= \frac{\sqrt{\varepsilon_r \mu_r}}{\mu_r} \frac{1}{Z_0} E_{rms}^2\, \hat{n} = \frac{n}{\mu_r} \frac{1}{Z_0} E_{rms}^2\, \hat{n} \qquad ...(142)$$

[become refractive index $n = \sqrt{\mu_r \cdot \varepsilon_r}$]

$$= \frac{n}{\mu_r} \langle \mathbf{S} \rangle_{\text{free space}} \qquad \left[\text{become } \langle \mathbf{S} \rangle_{free} = \frac{1}{Z_0} E_{rms}^2\, \hat{n}\right]$$

Eqs. (141) and (142) show that the *flow of energy is along the direction of propagation of e.m. waves*. Eq. (141) shows the Poynting vector for electromagnetic waves in isotropic

dielectric is $\sqrt{\frac{\varepsilon_r}{\mu_r}}$ or $\frac{\hat{n}}{\mu_r}$ times of the Poynting vector if the same electromagnetic wave were propagated through free space. It may be noted that average of Poynting vector may also be obtained as

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2} \cdot \text{real part of } \langle \mathbf{E} \times \mathbf{H}^* \rangle$$

Power Flow and Energy Density

Let us find the ratio of electrostatic and magnetostatic energy densities in an electromagnetic wave field *i.e.*,

$$\frac{U_e}{U_m} = \frac{\frac{1}{2}\varepsilon E^2}{\frac{1}{2}\mu H^2} = \frac{\varepsilon}{\mu}\frac{E^2}{H^2} = \frac{\varepsilon}{\mu} Z^2 = \frac{\varepsilon}{\mu}\frac{\mu}{\varepsilon} = 1 \qquad \left[\because Z = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}\right] \qquad ...(144)$$

This implies that for the case of e.m.waves in an isotropic dielectric the electrostatic energy density (U_e) is equal to magnetostatic energy density (U_m).

Therefore, total electromagnetic energy density,

$$U = U_e + U_m = 2U_e \qquad [\because U_e = U_m]$$

$$= 2 \cdot \frac{1}{2}\varepsilon E^2 = \varepsilon E^2 \qquad ...(145)$$

Therefore time average of energy density

$$\langle U \rangle = \langle \varepsilon E^2 \rangle = \varepsilon \langle E^2 \rangle = \varepsilon \langle [E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - i\omega t)}]^2_{real} \rangle$$

$$= \varepsilon E^2 \langle \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t) \rangle = \frac{\varepsilon E_0^2}{2}$$

$$= \varepsilon E_{rms}^2 \qquad ...(146)$$

Dividing Eq. (139) with Eq. (146), we obtain

$$\frac{\langle \mathbf{S} \rangle}{\langle U \rangle} = \frac{(E_{rms}^2\, \hat{n}/Z)}{\varepsilon E_{rms}^2} = \frac{1}{Z\varepsilon}\hat{n} = \frac{1}{\sqrt{\left(\frac{\mu}{\varepsilon}\right)}\varepsilon}\hat{n} \qquad \left[\because Z = \sqrt{\frac{\mu}{\varepsilon}}\right]$$

$$= \frac{1}{\sqrt{\mu\varepsilon}}\hat{n} = v\hat{n} \qquad \left[\because v = \frac{1}{\sqrt{\mu\varepsilon}}\right]$$

Thus we obtain,

$$\langle \mathbf{S} \rangle = \langle U \rangle v\hat{n} \qquad ...(a)$$

or in words, Energy flux = $v \times$ Energy density ...(b) ...(147)

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This equation has a simple meaning. If the energy were flowing with velocity v (= phase velocity of e.m. wave with which e.m. field vectors propagated), in the direction of propagation of wave, all the energy contained in a cylinder of unit cross-section and height equal to v would cross unit cross-section per second, forming the flux. This in turn implies that the *energy density associated with an e.m.wave in a stationary homogeneous non-conducting medium propagates with the same speed with which the field vector do.*

Summary

- In isotropic dielectric the electromagnetic waves travels with a speed less than the speed of light.
- The e.m.field vectors **E** and **H** are mutually ⊥ and they are also perpendicular to the direction of propagation of electromagnetic wave. Thereby indicating the electromagnetic waves are transverse in nature.
- The field vectors **E** and **H** are in the same phase.
- The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is represented as

$$\langle \mathbf{S} \rangle = \frac{E_{rms}^2}{Z}\hat{n} = \langle U \rangle v \hat{n}$$

- The electrostatic energy density is equal to the magnetostatic energy density and the total energy density is given by

$$\langle U \rangle = \varepsilon E_{rms}^2$$

- The energy density associated with an e.m. wave propagates with the phase velocity of the wave.

Example 4.9. *The relative permittivity of distilled water is 81. Calculate refractive index and velocity of light in it.*

Solution. If μ and ε are permeability and permittivity of a medium then the velocity of light in that medium is given by

$$v = \frac{1}{\sqrt{\mu\varepsilon}}$$

where $\mu = \mu_r\mu_0$; μ_r = relative permeability and

μ_0 = permeability of free space

$\varepsilon = \varepsilon_0\varepsilon_r$;

ε_0 = permittivity of free space,

ε_r = relative permittivity

$$\therefore \quad v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \times \frac{1}{\sqrt{\mu_r \cdot \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \cdot \varepsilon_r}} \quad ...(i)$$

Here $\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$ the velocity of light in vacuum

$= 3 \times 10^8$ m/sec.

$\mu_r = 1$ [For non-magnetic material] and $\varepsilon_r = 81$ (given)

$$\therefore \quad v = \frac{3 \times 10^8}{\sqrt{81}} = 3.33 \times 10^7 \text{ m/s}$$

The refractive index of any medium is the ratio of the velocity of light in vacuum to the velocity of light in that medium.

$$\therefore \text{ Refractive index } (n) = \frac{c}{v} = \sqrt{\mu_r \cdot \varepsilon_r}$$

$$= \sqrt{81} = 9.0 \quad \text{[From Eq. (i)]}$$

4.15 Plane Electromagnetic Waves in Conducting Medium

Maxwell's equations are :

$$\left.\begin{aligned} \text{div } \mathbf{D} &= \nabla \cdot \mathbf{D} = \rho & ...(a) \\ \text{div } \mathbf{B} &= \nabla \cdot \mathbf{B} = 0 & ...(b) \\ \text{curl } \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & ...(c) \\ \text{curl } \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & ...(d) \end{aligned}\right\} \quad ...(148)$$

Let us assume that medium is linear and isotropic and is characterised by permittivity ε and permeability μ and conductivity σ, but not any charge or any current other from that determined by Ohm's law. Then

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{J} = \sigma\mathbf{E} \text{ and } \rho = 0 \quad ...(149)$$

So, that Maxwell's Eqs. (148) in this case take the form

$$\left.\begin{aligned} \text{div } \mathbf{E} &= 0 & ...(a) \\ \text{div } \mathbf{H} &= 0 & ...(b) \\ \text{curl } \mathbf{E} &= -\mu\frac{\partial \mathbf{H}}{\partial t} & ...(c) \\ \text{curl } \mathbf{H} &= \sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t} & ...(d) \end{aligned}\right\} \quad ...(150)$$

Taking curl of Eq. 150(*c*), we get

$$\text{curl curl } \mathbf{E} = -\mu\frac{\partial}{\partial t}(\text{curl } \mathbf{H})$$

Substituting curl **H** from Eq. 150(*d*), in above equation, we get

$$\text{curl curl } \mathbf{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.,

$$\text{curl curl } \mathbf{E} = -\sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad ...(151)$$

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Similarly, if we take the curl of Eq. 150(*d*) and substitute curl **E** from Eq. 150(*c*), we obtain

$$\text{curl curl}\,\mathbf{H} = -\sigma\mu\frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad ...(152)$$

Now, using vector identity

$$\text{curl curl}\,\mathbf{A} = \text{grad div}\,\mathbf{A} - \nabla^2\mathbf{A}$$

and keeping in view Eqs. 150(*a*) and 150(*b*) [*i.e.*, div **E** = 0 and div **H** = 0]

Eqs. (151) and (152) take the form

$$\nabla^2\mathbf{E} - \sigma\mu\frac{\partial \mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad ...(153)$$

$$\nabla^2\mathbf{H} - \sigma\mu\frac{\partial \mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \qquad ...(154)$$

These equations represent wave equations governing electromagnetic fields **E** and **H** is a homogenous conducting medium of conductivity σ. It is apparent that these equations are vector equations of **E** and **H** separately, satisfies the same scalar wave equation of the form.

$$\nabla^2\phi - \sigma\mu\frac{\partial \phi}{\partial t} - \varepsilon\mu\frac{\partial^2 \phi}{\partial t^2} = 0 \qquad ...(155)$$

where ϕ is a scalar and can stand for any one of components **E** and **H**.

In an isotropic dielectric we have seen that the time varying fields are transverse *i.e.*, the field vector, **E** and **H** are perpendicular to the direction in which the spatial variation occurs. In the limit of zero frequency we know from electrostatics and magnetostatics that static field in a dielectric are longitudinal in the sense that the fields are derivable from scalar potentials and so point in the direction of spatial variation.

If the conductivity is not zero, modifications are necessary. For simplicity, let us assume that field vary in only one spatial variable x_α. Therefore, decomposing the fields into longitudinal and transverse parts :

$$\left.\begin{aligned}\mathbf{E}(x_\alpha, t) &= \mathbf{E}_l(x_\alpha, t) + \mathbf{E}_t(x_\alpha, t)\\ \mathbf{H}(x_\alpha, t) &= \mathbf{H}_l(x_\alpha, t) + \mathbf{H}_t(x_\alpha, t)\end{aligned}\right\} \qquad ...(156)$$

where subscript *l* and *t* denoted longitudinal and transverse parts respectively. Then, because of properties of curl operation, we find that the transverse parts of **E** and **H** satisfy the two curl Eqs. 150(*c*) and 150(*d*) ; leading to transverse wave, while the longitudinal parts satisfy the equations :

$$\left.\begin{aligned}&\text{From Eq. 150(a)}, && \frac{\partial \mathbf{E}_l}{\partial x_\alpha} = 0 && ...(a)\\ &\text{From Eq. 150(b)}, && \frac{\partial \mathbf{H}_l}{\partial x_\alpha} = 0 && ...(b)\\ &\text{From Eq. 150(c)}, && \left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right)\mathbf{E}_l = 0 && ...(c)\ [\because \text{curl}\,\mathbf{E}_l = \text{curl grad}\,\phi_l = 0]\\ &\text{From Eq. 150(d)}, && \frac{\partial \mathbf{H}_l}{\partial t} = 0 && ...(d)\ [\because \text{curl}\,\mathbf{H}_l = \text{curl grad}\,\phi_l = 0]\end{aligned}\right\} ...(157)$$

Eqs. 150(*c*) and 150(*d*), show that the only longitudinal magnetic field is possible in a static uniform field. This is same situation as the case of a dielectric. But Eqs. 157(*a*) and 157(*d*) show that the longitudinal electric field is uniform in space, while possesses the time variation given by

$$\left(\frac{\partial}{\partial t} + \frac{\sigma}{\varepsilon}\right)E_l = 0 \qquad i.e., \qquad \frac{\partial E_l}{\partial t} = -\frac{\sigma}{\varepsilon}E_l$$

i.e.,

$$\frac{\partial E_l}{E_l} = -\frac{\sigma}{\varepsilon}dt \qquad ...(158)$$

Integrating, $\ln E_l = -\frac{\sigma}{\varepsilon}t + \ln E_0$ (where $\ln E_0$ is constant of integration)

i.e.,

$$E_l(x_\alpha, t) = E_0 e^{-(\sigma/\varepsilon)t} \qquad ...(159)$$

Consequently, **no static electric fields can exist in a conducting medium in the absence of an applied current density**. For good conductors like copper ; $\sigma \approx 10^7$ mho/m, so that disturbances are damped out in an extremely short time.

Therefore, we shall consider the transverse field in the conducting medium. Let us assume that the fields vary as $e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$, then solutions of Eqs. (153) to (155).

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(160)$$

$$\mathbf{H} = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(161)$$

$$\phi = \phi_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \qquad ...(162)$$

where **k** is a wave vector, may be complex, while $\mathbf{E}_0$, $\mathbf{H}_0$ and ϕ_0 are complex amplitudes which are constant in space and time.

Substituting value of ϕ from Eq. (162) in Eq. (155), we obtain

$$(-k^2 + i\sigma\omega + \mu\varepsilon\omega^2)\phi = 0$$

Since ϕ is arbitrary, therefore this equation holds only if

$$(-k^2 + i\sigma\mu\omega + \mu\varepsilon\omega^2) = 0$$

This means that the propagation wave vector **k** is complex given by

$$k^2 = \mu\varepsilon\omega^2\left(1 + \frac{i\sigma}{\omega\varepsilon}\right) \qquad ...(163)$$

In above equation the first term corresponds to the displacement current and the second to the conduction current contribution. As **k** is complex we may write assuming that σ is real

$$k = \alpha + i\beta \qquad ...(164)$$

So, that

$$k^2 = \alpha^2 - \beta^2 + 2i\alpha\beta \qquad ...(165)$$

Comparing Eqs. (163) and (165), we get

$$\left.\begin{aligned}\alpha^2 - \beta^2 &= \mu\varepsilon\omega^2\\ \text{and}\qquad 2\alpha\beta &= \mu\omega\sigma\end{aligned}\right\} \qquad ...(166)$$

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$$\left.\begin{aligned}\alpha &= \sqrt{\mu\varepsilon}\cdot\omega\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}+1}{2}\right]^{1/2}\\ \beta &= \sqrt{\mu\varepsilon}\cdot\omega\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}-1}{2}\right]^{1/2}\end{aligned}\right\} \quad ...(167)$$

Now, in terms of α and β, the field vector **E** and **H** take the form

$$\mathbf{E} = \mathbf{E}_0 e^{i(\alpha+i\beta)\hat{n}\cdot\mathbf{r}-i\omega t} = \mathbf{E}_0\, e^{-\beta\,\mathbf{n}\cdot\mathbf{r}} e^{i(\alpha\,\mathbf{n}\cdot\mathbf{r}-\omega t)} \quad ...(168)$$

and

$$\mathbf{H} = \mathbf{H}_0 e^{i(\alpha+i\beta)\hat{n}\cdot\mathbf{r}-i\omega t} = \mathbf{H}_0 e^{-\beta\,\mathbf{n}\cdot\mathbf{r}} e^{i(\alpha\,\hat{n}\cdot\mathbf{r}-\omega t)} \quad ...(169)$$

From Eqs. (168) and (169) it is obvious that *field amplitudes are spatially attenuated due to the presence of term* $e^{-\beta\,\mathbf{n}\cdot\mathbf{r}}$.

The quantity β is a measure of attenuation and is known as absorption coefficients. Also in last exponential terms in Eqs. (168) and (169) the usual notation **k** has been replaced by α, therefore, we conclude that the field vectors are propagated in the conducting medium with speed $\left(v = \frac{\omega}{k}\right)$ given by

$$v = \frac{\omega}{\alpha} = \frac{1}{\sqrt{\mu\varepsilon}}\left[\frac{\sqrt{\left\{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2\right\}}+1}{2}\right]^{-1/2} \quad \text{[by using Eq. (167)]} \quad ...(170)$$

Now let us consider the form of propagation vector $k = \alpha + i\beta$ in the two particular cases :

Case (i). For a poor conductor $\frac{\sigma}{\varepsilon} << 1$, then we get

$$\alpha = \sqrt{\mu\varepsilon}\,\omega \text{ and } \beta = 0$$

$$\therefore \quad k = \alpha + i\beta = \sqrt{\mu\varepsilon}\,\omega \quad ...(171)$$

This is correct to first order in $\sigma/\omega\varepsilon$. In this limit $\alpha >> \beta$ and the attenuation of wave determined by β is independent of frequency, aside from the possible variation of conductivity.

Case (ii). For a good conductor $\sigma/\omega\varepsilon >> 1$, so that α and β are approximately equal, *i.e.*,

$$\alpha = \beta = \sqrt{\mu\varepsilon}\cdot\omega\sqrt{\left(\frac{\sigma/\omega\varepsilon}{2}\right)} = \sqrt{\frac{\mu\sigma\omega}{2}} \quad ...(172)$$

$$\therefore \quad k = \alpha + i\beta = (1+i)\sqrt{\left(\frac{\mu\sigma\omega}{2}\right)} \quad ...(173)$$

where only lowest term in $\omega\varepsilon/\sigma$ have been kept.

4.16 Skin Depth or Penetration Depth

The wave given by Eqs. (168) and (169) show an exponential damping and attenuation with distance. Greater is the value of β, greater is the attenuation. The term $(1/\beta)$ measures, the depth at which electromagnetic wave entering a conductor is damped to $1/e = 0.369$ of its initial amplitude at the surface. The depth is known as the *skin depth* or the *penetration depth* and is usually represented by δ.

DEFINITION

The depth of penetration is defined as the depth in which the strength of electric field associated with the e.m.wave reduces to $1/e$ time of the initial values.

$$\left.\begin{aligned}\delta &= \frac{1}{\beta} = \frac{1}{\omega\sqrt{\mu\varepsilon}}\left[\frac{\sqrt{\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}}-1}{2}\right]^{-1/2} \quad ...(a)\\ &= \sqrt{\left(\frac{2}{\mu\sigma\omega}\right)} \quad \text{[For good conductor ...(b) characterised by } \frac{\sigma}{\omega\varepsilon} >> 1\text{]}\end{aligned}\right\} \quad ...(174)$$

NOTE

1. For silver $\sigma \approx 10^7$ mho/m at a typical microwave frequency $\approx 10^8$ c/sec, then skin depth $\approx 10^{-4}$ cm. Thus at microwave frequencies the skin depth in silver is very small and consequently performance of a pure a silver component and silver plated brass component would be expected to be indistinguishable.
2. For sea water $\sigma = 4.3$ mho/m at a frequency of 60kc/s ; so that $\delta = 1$ meter. That is why radio- communication with submerged submarine becomes increasing difficult at several skin depths.

It is obvious that skin depth decreases with increasing frequency. For copper at 60 cycles, δ is 0.86 cm, but at 1 megacycle, it has dropped to 0.0067. That is why in high frequency circuits current flows only on the surfaces of the conductors. The major importance of skin depth is that it measures the depth to which an e.m. wave can penetrate a conducting medium. Therefore, the conducting sheets which are used as electromagnetic shields must be thicker than the skin depth.

Relative Directions of E and H

Substituting **E** and **H** from Eqs. (160) and (161) Maxwell's divergence equation, we obtain

$$i\mathbf{k}\cdot\mathbf{E} = 0 \quad \text{or} \quad \mathbf{k}\cdot\mathbf{E} = 0 \quad ...(175)$$

$$i\mathbf{k}\cdot\mathbf{H} = 0 \quad \text{or} \quad \mathbf{k}\cdot\mathbf{H} = 0 \quad ...(176)$$

These equations imply that field vectors **E** and **H** are both perpendicular to the direction of propagation vector **k**. This implies that e.m.wave in a conducting medium is transverse. Further restrictions on **E** and **H** is imposed by curl equations. Using Eqs. (160) and (161) ; equations demand

$$i\mathbf{k}\times\mathbf{E} = i\mu\omega\mathbf{H} \quad \textit{i.e.,} \quad \mathbf{k}\times\mathbf{E} = \mu\omega\mathbf{H} \quad ...(177)$$

and

$$i\mathbf{k}\times\mathbf{H} = (\sigma - i\varepsilon\omega)\mathbf{E} \quad \textit{i.e.,} \quad \mathbf{k}\times\mathbf{H} = -(\varepsilon\omega + i\sigma)\mathbf{E} \quad ...(178)$$

These equations imply that e.m. field vectors **E** and **H** are mutually perpendicular and also they are perpendicular to the direction of propagation vector **k**.

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Phase of E and H

From Eq. (177)

$$\mathbf{H} = \frac{1}{\mu\omega}(\mathbf{k}\times\mathbf{E}) = \frac{1}{\mu\omega}\mathbf{k}(\hat{n}\times\mathbf{E})$$

$$= \frac{1}{\mu\omega}(\alpha + i\beta)(\hat{n}\times\mathbf{E}) \quad \text{...(a)}$$

This implies that $\left|\frac{\mathbf{H}}{\mathbf{E}}\right| = \frac{H_0}{E_0} = \frac{\alpha + i\beta}{\mu\omega} = \text{complex quantity}$ ...(b) ...(179)

i.e., field vectors **H** and **E** are out of phase in a conductor.

The magnitude and phase of complex **k** written as $\mathbf{k} = |\mathbf{k}|e^{i\phi}$, may be defined as

$$|\mathbf{k}| = |\alpha + i\beta| = \sqrt{(\alpha^2+\beta^2)} = \omega\sqrt{\mu\varepsilon}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} \quad \text{...(180)}$$

and $$\phi = \tan^{-1}\frac{\beta}{\alpha} = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right) \quad \text{...(181)}$$

So, Eq. (178) may be expressed as

$$\mathbf{H} = \frac{1}{\mu\omega}\omega\sqrt{\mu\varepsilon}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{i\phi}(\hat{n}\times\mathbf{E})$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{-i(-\phi)}(\hat{n}\times\mathbf{E}) \quad \text{...(182)}$$

This interpretation of this equation is that **H** lags behind **E** in time by phase angle ϕ gives by Eq. (181) and has a relative magnitude

$$\left|\frac{\mathbf{H}}{\mathbf{E}}\right| = \frac{H_0}{E_0} = \sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} \quad \text{...(183)}$$

Impedance of a conducting medium

The impedance of conducting medium is complex quantity and is given by

$$\mathbf{Z} = \frac{E}{H} = \frac{E}{\left|\frac{1}{\mu\omega}\mathbf{k}\times\mathbf{E}\right|} = \frac{\mu\omega}{k} = \frac{\mu\omega k}{k^2} = \frac{\mu\omega(\alpha+i\beta)}{\alpha^2+\beta^2} \quad \text{...(184)}$$

Poynting Vector

The Poynting vector is given by

$$\mathbf{S} = (\mathbf{E}\times\mathbf{H})$$

the time average of Poynting vector may expressed as

$$\mathbf{S}_{av} = \frac{1}{2}\text{Real part of}(\mathbf{E}\times\mathbf{H}^*) = \frac{1}{2}\text{Re}(\mathbf{E}\times\mathbf{H}^*)$$

where $\mathbf{H}^*$ (denotes complex conjugate of **H** and Re denotes real part)

$$\mathbf{S}_{av} = \frac{1}{2}\text{Re}\left[\mathbf{E}\times\left\{\sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} e^{-i\phi}\,\hat{n}\times\mathbf{E}^*\right\}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2}\text{Re}[\mathbf{E}\times(\hat{n}\times\mathbf{E}^*)e^{-i\phi}]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2}\text{Re}[\{(\mathbf{E}.\mathbf{E}^*)\hat{n}-(\mathbf{E}.\hat{n})\mathbf{E}^*\}e^{-i\phi}]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} E_0^2 e^{-2\alpha\hat{n}\cdot r}\,\hat{n}\cos\phi \quad \text{...(185)}$$

$[\because \mathbf{E}.\mathbf{E}^* = E_0^2 e^{-2\alpha\hat{n}\cdot r}$ and $\text{Re}(e^{-i\phi}) = \cos\phi]$

For good conductor $\frac{\sigma}{\omega\varepsilon} >> 1$ so that $\phi = \frac{\pi}{4}$ and also $E_{rms} = \frac{E_0}{\sqrt{2}}$ hence

$$\mathbf{S}_{av} = \sqrt{\left(\frac{\sigma}{2\mu\omega}\right)} E_{rms}^2 e^{-2\alpha\hat{n}\cdot r}\,\hat{n} \quad \text{...(186)}$$

Energy Density

The total energy density of e.m. field is given by ...(187)

$$U = U_e + U_m$$

where electrostatic energy,

$$U_e = \frac{1}{2}\text{Re}\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}^*) = \frac{1}{4}\varepsilon\,\text{Re}(\mathbf{E}\cdot\mathbf{E}^*)$$

$$= \frac{1}{4}\varepsilon E_0^2 e^{-2\alpha\hat{n}\cdot r} \quad \text{...(a)}$$

$$= \frac{1}{2}\varepsilon E_{rms}^2 e^{-2\alpha\hat{n}\cdot r} \quad \text{...(b)} \quad \text{...(188)}$$

and magnetic energy density

$$U_m = \frac{1}{2}\text{Re}\left(\frac{1}{2}\mathbf{H}\cdot\mathbf{B}^*\right) = \frac{1}{2}\mu\,\text{Re}(\mathbf{H}\cdot\mathbf{H}^*) = \frac{1}{4}\mu H_0^2 e^{-2\alpha\hat{n}\cdot r}$$

$$= \frac{1}{4}\mu\frac{\varepsilon}{\mu}\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}^{1/2} E_0^2 e^{-2\alpha\hat{n}\cdot r} \quad \text{[Using Eq. (183)]}$$

$$= \frac{1}{4}\varepsilon\left\{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right\}^{1/2} E_0^2 e^{-2\alpha\hat{n}\cdot r}$$

$$= \left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} U_e \quad \text{[Using 188(a)]} \quad \text{...(189)}$$

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∴ Total energy density,

$$U = U_e + U_m = U_e + \left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} U_e$$

$$= \left[1 + \left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2}\right] U_e$$

$$= \left[1 + \left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}\right]^{1/2} \times \frac{1}{2}\varepsilon E_{rms}^2 e^{-2\beta n \tau} \quad ...(190)$$

From Eqs. (186) and (190) it is obvious that *the energy flux and energy density are damped as the e.m.wave propagates in a conducting medium.* This energy lose is due to Joule heating of the medium. From Eqs. (188) and (189) it is also obvious that *in a conducting medium and electrostatic and magnetic energy densities are different ; the magnetic energy density being greater than electrostatic energy density.*

Summary

- The electromagnetic field vector **E** and **H** are mutually perpendicular and they are also perpendicular to the direction of propagation of electromagnetic wave. Thereby indicating the e.m.waves are **transverse** in nature.
- The field vectors **E** and **H** are damped exponentially as the wave penetrates the conducting medium.
- The field vectors **E** and **H** are not in phase, but **H** lags behind **E** by angle ϕ given by

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\varepsilon\omega}\right) \quad ...(190)$$

  and moreover the magnitude of **H** is much greater than **E**.
- The energy flow is along the direction of propagation of e.m.wave, but damped off exponentially.
- The magnetic energy density is much greater than electric energy density and both are damped off exponentially.

Example 4.10. *For silver $\sigma = 30$ MS/m. Calculate the frequency at which the depth of penetration is 1 mm. [Given $\mu_r = 1$].*

[GGSIPU, Feb. 2008, Reappear (2 marks)]

Solution. Since we know that $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{\mu\sigma(2\pi\nu)}}$

$$\delta^2 = \frac{1}{\pi\mu\sigma\nu} \quad \text{or} \quad \nu = \frac{1}{\pi\mu\sigma\delta^2}$$

$$= \frac{1}{3.14 \times 3.0 \times 10^8 \times 1 \times 1 \times 10^{-6}}$$

$$= 0.106 \text{ Hz} = 106 \text{ mHz}.$$

Example 4.11. *A plane e-m-wave travelling in the +ve z-direction in an unbounded lossless dielectric medium with relative permeability $\mu_r = 1$ and relative permitivity $\varepsilon_r = 3$ has an electric field intensity $E = 6$ Volts/metre. Find (i) speed of the em waves in the given medium (ii) impedance of the medium.*

[GGSIPU, Feb 2005 (2 marks)]

Solution. Given $\mu_r = 1$, $\varepsilon_r = 3$, $E_0 = 6$V/m

but $\varepsilon = \varepsilon_r \varepsilon_0$ and $\mu = \mu_r \mu_0$ where ε and μ are the permittivity and permeability of the medium and ε_0 and μ_0 corresponding constant for free space.

(*i*) The speed of the e.m. waves in given medium

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r \cdot \varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{1 \times 3}} = \sqrt{3} \times 10^8 = 1.732 \times 10^8 \text{ m/s}$$

(*ii*) The impedance Z of the medium is given by

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \cdot \mu_0}{\varepsilon_r \cdot \varepsilon_0}}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m and $\varepsilon_0 = 8.86 \times 10^{-12}$ F/m, we get

$$Z = \sqrt{\frac{4\pi \times 10^{-7} \times 1}{8.86 \times 10^{-12} \times 3}} = 2.17 \times 10^2 \text{ Ohms}.$$

Example 4.12. *A medium like copper conductor which is characterised by the parameter $\sigma = 5.8 \times 10^7$ mho/m, $\varepsilon_r = 1$, $\mu_r = 1$ supports a uniform plane wave of frequency 60Hz. Find the attenuation constant, propagation constant, intrinsic impedance, wavelength and phase velocity of the wave.*

Solution. Let us obtain the ratio

$$\frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 60 \times 8.854 \times 10^{-12}} = 173 \times 10^{14}$$

This is very-very greater than 1, therefore, it is a very good conductor.

Attenuation constant $\alpha = \left(\frac{\omega\mu\sigma}{2}\right)^{1/2} = 117.2$ per m

Phase constant $\beta = \left(\frac{\omega\mu\sigma}{2}\right)^{1/2} = 117.2$ per m

Propagation constant $k = \alpha + j\beta = 117.2 + j117.2$ per m

Wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{117.2} = 0.0536$ m.

Intrinsic impedance $Z = \sqrt{\frac{j\omega\mu}{\sigma}} = (2.022 + j2.002) \times 10^{-6}\ \Omega$

Phase velocity of wave

$$v = \lambda\nu = 0.0536 \times 60 = 3.216 \text{ m/s}$$

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Formulae at a Glance

1. Biot Savart Law

(i) $\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{i d\mathbf{l} \times \mathbf{r}}{r^3}$

$= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J} \times \mathbf{r}}{r^3} dV$

(ii) B due to long straight wire

$B = \frac{\mu_0 i}{2\pi d}$

$H = \frac{B}{\mu_0} = \frac{i}{2\pi d}$

2. Ampere's circuital law,

$\oint \mathbf{B}.d\mathbf{l} = \mu_0 i$

B = magnetic induction

μ_0 = permeability constant

i = current flowing

(i) Magnetic field induction of a solenoid

$B = \mu_0 n i$

n = No. of turns per unit length

(ii) Magnetic field of an endless solenoid (toroid)

$B = \frac{\mu_0 N i_0}{l}$

$l = 2\pi r$ = mean circumference of toroid

N = Number of toroid

3. Equation of continuity

$\text{div } \mathbf{J} + \frac{d\rho}{dt} = 0$

where J = current density

ρ = charge density

4. Displacement current density

$\text{curl } \mathbf{H} = \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$

$\mathbf{J}' = \frac{\partial \mathbf{D}}{\partial t}$ = displacement current density

5. Basic laws of Electromagnetics

(i) Gauss's law of electrostatics

$\text{div } \mathbf{D} = \rho$

or $\int \mathbf{D}.d\mathbf{S} = \int_V \rho dV$

(ii) The Gauss's law in magnetostatics

$\text{div } \mathbf{B} = 0$

$\int \mathbf{B}.d\mathbf{S} = 0$

(iii) Faraday's laws of e.m. induction

$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

(iv) Ampere's law in magnetostatics is

$\int \mathbf{H}.d\mathbf{l}$ = current enclosed

6. Maxwell's equations in differential form

(i) $\nabla.\mathbf{D} = \rho$

(ii) $\nabla.\mathbf{B} = 0$

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

(iv) $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

where D = electric displacement vector in C/m^2.

ρ = charge density in C/m^3

B = magnetic induction in Wb/m^2 or Tesla

E = electric field intensity in V/m or N/C

H = magnetic field intensity A/turn-m.

7. Maxwell's equation in integral form

(i) $\int_S \mathbf{D}.d\mathbf{S} = \int_V \rho dV$

(ii) $\int_S \mathbf{B}.d\mathbf{S} = 0$

(iii) $\oint \mathbf{E}.d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}.d\mathbf{S}$

(iv) $\oint \mathbf{H}.d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) d\mathbf{S}$

8. Maxwell's equations in free space

(i) $\nabla.\mathbf{D} = 0$

(ii) $\nabla.\mathbf{B} = 0$

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

(iv) $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$

with $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$

where ϵ_0 = absolute permittivity

and μ_0 = absolute permeability.

9. Maxwell's equation in linear isotropic medium

(i) $\nabla.\mathbf{E} = \rho/\epsilon$ (ii) $\nabla.\mathbf{B} = 0$

(iii) $\nabla \times \mathbf{E} = \mu \frac{\partial \mathbf{H}}{\partial t}$ (iv) $\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$

10. Maxwell's equation for harmonically varying field

(i) $\nabla.\mathbf{D} = \rho$ (ii) $\nabla.\mathbf{B} = 0$

(iii) $\nabla \times \mathbf{E} + i\omega\mathbf{B} = 0$ (iv) $\nabla \times \mathbf{H} - i\omega\mathbf{D} = \mathbf{J}$

11. (i) Electromagnetic energy

$U_e = \frac{1}{2}\int_V \mathbf{E}.\mathbf{B}$

(ii) Energy stored in magnetic field

$U_m = \frac{1}{2}\int_V \mathbf{H}.\mathbf{B}\, dV$

12. Electromagnetic Energy Theorem :

Poynting Theorem

$\oint (\mathbf{E} \times \mathbf{H}) dS = -\frac{d}{dt}\int_V \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B}) dV - \int_V \mathbf{J}.\mathbf{E} dV$

or $-\mathbf{J}.\mathbf{E} = \frac{\partial U}{\partial t} + \nabla.\mathbf{S}$

13. Poynting vector

$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{E^2}{Z_0}\hat{n}$

14. Velocity of plane electromagnet wave in free space

$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8$ m/s $= c$

= velocity of light in vacuum

$\mu_0 = 4\pi \times 10^{-7}$ Wb/A-m

$\epsilon_0 = 8.542 \times 10^{-12}$ F/m.

15. Wave impedance of free space

$Z_0 = \left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \left|\frac{E_0}{H_0}\right| = \mu_0 c$

$= \sqrt{\frac{\mu_0}{\epsilon_0}}$ $\left[c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}\right]$

$= 376.6$ ohm

16. $\langle \mathbf{S} \rangle = \frac{1}{Z_0} E_{rms}^2 \hat{n}$

17. Ratio of electrostatic and magnetic densities

$\frac{\langle \mathbf{S} \rangle}{\langle U \rangle} = c\hat{n}$

or $\langle \mathbf{S} \rangle = \langle U \rangle c\hat{n}$

18. Skin depth or penetration depth

$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$

Miscellaneous Numerical Problems For Practice

Example 1. Find the depth of penetration, δ of an EM wave in copper at ν = 60 Hz and ν = 100 Hz. For copper σ = 5.8 × 10⁷ mho/m, k_m = 1, k_e = 1.

Solution. For copper at ν = 60 Hz.

$$\frac{\sigma}{\omega\epsilon} = \frac{5.8 \times 10^7}{2\pi \times 60 \times 8.854 \times 10^{-12}}$$

$$= 174 \times 10^{14} >> 1$$

Therefore at ν = 60 Hz, copper is a very good conductor. The depth of penetration

$$\delta = \frac{1}{\beta} = \sqrt{\left(\frac{2}{\omega\mu\sigma}\right)}$$

$$= \sqrt{\frac{2}{2\pi \times 60 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}}$$

or $\delta = 8.53 \times 10^{-3}$ m.

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At $\nu = 100$ MHz

$$\frac{\sigma}{\omega\varepsilon} = \frac{9.8\times10^{7}}{2\pi\times8.854\times10^{-12}} = 0.10425\times10^{15}$$

$$= 10.425\times10^{8} >> 1$$

Copper is very good conductor at $\nu = 100$ MHz

The depth of penetration,

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi\times100\times4\pi\times10^{-7}\times58\times10^{7}}} = 6.6608\times10^{-8}\ \text{m}.$$

Example 2. *The light is generally characterised by electric vector, although it also possesses the magnetic vector, why ?*

Solution. Both electric and magnetic field vectors of the e.m. waves like light show their existence through the force exerted on the charged particles. The electric and magnetic forces respectively are given by

$$\mathbf{F}_e = q\mathbf{E},\ \mathbf{F}_m = q(\mathbf{v}\times\mathbf{B})$$

Their maximum magnitude are :

$$F_e = qE;\quad F_m = qvB = qv\mu_0 H$$

Here v is the velocity of the charged particles.

Now,
$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\therefore\quad F_m = \mu_0 qv\sqrt{\frac{\varepsilon}{\mu_0}}\,E = q\frac{v}{c}E$$

$$\frac{F_m}{F_e} = \frac{q(v/c)E}{qE} = \frac{v}{c}$$

Here c, the wave velocity is very-very large as compared to the particle velocity v. Hence, F_m is very-very small as compared to F_e (*i.e.* $v << c$, $\therefore$ $F_m << F_e$). Because of this reason We characterise light through the electric vector.

Example 3. *When the amplitude of magnetic field in a plane wave is 2 A/m (a) determine the magnitude of the electric field for the plane wave in free space. (b) determine the magnitude of electron field when the wave propagates in a medium which is characterised by $\sigma = 0$ and $\mu = \mu_0$ and $\varepsilon = 4\varepsilon_0$.*

Solution. (a) We have,

$$\frac{E}{H} = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= 120\ \pi\ \Omega \text{ for the free space}$$

But $H = 2$ A/m

$$E = Z_0 H$$

$$= 120\times2 = 240\ \pi\ \text{V/m}.$$

(b) $\sigma = 0$, $\varepsilon_r = 4$, $\mu_r = 1$

$$\eta_0 = \frac{\omega}{\beta} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{2}\times120\ \pi\ \Omega = 60\ \pi\ \Omega$$

$$E = Z_0 H = 60\pi\times2 = 120\ \pi\ \text{V/m}$$

$$E = 120\ \pi\ \text{V/m}.$$

Example 4. *Assuming that all the energy from a 1000W lamp is radiated uniformly, calculate the average values of intensities of electric and magnetic fields of radiations at a distance of 2m from the lamp.*

Solution. If the total power P_0 is radiated uniformly in all directions, then the power or energy flux per unit area per second at a distance r from the point source (*i.e.*, lamp) is

$$S_{av} = \frac{P_0}{4\pi r^2}$$

$$= \frac{1000}{4\pi(2)^2}\ \text{W/m}^2$$

From the definition of Poynting vector

$$|\mathbf{S}| = |(\mathbf{E}\times\mathbf{H})| = EH\sin 90^\circ \qquad (\because \mathbf{E} \text{ and } \mathbf{H} \text{ are } \perp \text{ to each other})$$

$$EH = \frac{1000}{16\pi}\ \text{W/m}^2 \qquad \text{...(i)}$$

But
$$Z = \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times10^{-7}}{8.854\times10^{-12}}} = 376.72\ \Omega \qquad \text{...(ii)}$$

Multiplying Eqs. (i) and (ii), we get

$$EH\frac{E}{H} = \frac{376.72\times1000}{16\pi}$$

or
$$E = \sqrt{\frac{376.72\times1000}{16\pi}} = 86.39\ \text{V/m}.$$

From Eq. (i),

$$H = \frac{1000}{16\pi E}$$

$$= \frac{1000}{16\times3.14\times86.39} = 0.23\ \text{A/m}.$$

Example 5. *Find out the current density and electric field intensity for aluminium conductor with conductivity $\sigma = 3.82\times10^{7}$ Siemens/m and the mobility of the electron $\mu_e = 0.0014\ \text{m}^2/(\text{Volt-sec})$ and drift velocity $= 9.2\times10^{-4}$ m/s.*

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Solution. The drift velocity of an electron is related with electric field intensity is

$$v_d = \mu_e E \quad \text{or} \quad E = \frac{v_d}{\mu_e}$$

where μ_e is the mobility of the electrons.

Here $v_d = 53\times10^{-4}$ m/s and $\mu_e = 0.0014\ m^2/(\text{Volt-sec})$

$$E = \frac{5.3\times10^{-4}}{0.0014} = 0.3786 \text{ V/m}.$$

Current density $J = \sigma E$

$$= 3.82\times10^{7}\times0.3786 = 1.446\times10^{7} \text{ Amp/m}^2$$

Example 6. *Calculate the skin depth for an e.m.wave of frequency 1MHz travelling through copper. Given $\sigma = 5.8\times10^{7}$ S/m and $\mu = 4\pi\times10^{-7}$.* **[GGSIPU, Feb 2005 (2 marks)]**

Solution. We know that,

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{\mu\sigma(2\pi\nu)}}$$

$$= \sqrt{\frac{2}{4\pi\times10^{-7}\times5.8\times10^{7}\times2\times\pi\times1\times10^{6}}} = 0.0660\times10^{-3}\text{m}$$

$$\delta = 66\mu\text{m}.$$

Example 7. *Calculate the penetration depth for 2 MHz e.m. wave through copper. Given : $\sigma = 5.8\times10^{7}$ S/m, $\mu = 4\pi\times10^{-7}$.* **[GGSIPU, May, June 2007 (1.5 marks)]**

Solution. We know that,

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{2}{\mu\sigma(2\pi\nu)}}$$

$$= \sqrt{\frac{2}{(4\pi\times10^{-7}\times5.8\times10^{7}\times2\times\pi\times2\times10^{6})}}$$

$$= 0.0467\times10^{-3} = 46.7\ \mu\text{m}.$$

Example 8. *Find the skin depth at a frequency 1.6 MHz in aluminium where $\sigma = 38.2$ MS/m and $\mu = 1$.* **[GGSIPU, Feb 2008 (2 marks)]**

Solution. We know that,

$$\delta = \sqrt{\left(\frac{2}{\mu\sigma\omega}\right)} = \sqrt{\frac{2}{\mu\sigma(2\pi\nu)}}$$

$$= \sqrt{\frac{2}{1\times38.2\times10^{6}\times2\times\pi\times1.6\times10^{6}}}$$

$$= 0.0721\times10^{-4}$$

$$= 7.21\times10^{-8}\text{m}.$$

Example 9. *The conduction current density in a lossy dielectric is given as $J_c = 0.02\sin(10^9 t)$ Amp/m^2, find the displacement current density if $\sigma = 10^3$ mho/m, $k_e = 6.5$ and $\varepsilon_0 = 8.854\times10^{-12}$.* **[GGSIPU, May 2006 (4.5 marks)]**

Solution. We have the conducting current density,

$$J_c = \sigma E$$

or

$$E = \frac{J_c}{\sigma} = \frac{0.02\sin(10^9 t)}{10^3}$$

$$E = 2.0\times10^{-5}\sin(10^9 t)\text{ V/m}$$

and Displacement current density

$$J_d = \varepsilon\frac{\partial E}{\partial t} = \varepsilon_0 k_e \frac{\partial}{\partial t}[2.0\times10^{-5}\sin(10^9 t)]$$

$$= 8.854\times10^{-12}\times6.5\times2.0\times10^{-5}\times10^{9}\cos(10^9 t)$$

$$= 1.151\times10^{-6}\cos(10^9 t)\text{ Amp/m}^2$$

$$J_d = 1.151\cos(10^9 t)\ \mu\text{A/m}^2$$

Example 10. *A parallel plate capacitor has circular plates, each of radius a. It is being charged. Show that the rate at which energy flow into the capacitor from the surrounding space is equal to the rate at which the stored electrical energy increases.*

Solution. Let E is the electric field at any instant within the capacitor during the charging process. Its direction is perpendicular to plates and directed from +ve plate and -ve plate.

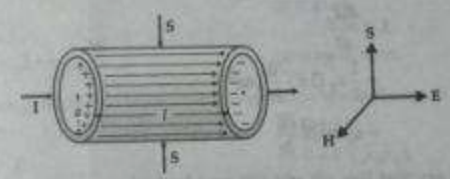

Fig. 4.11

As changing electric fields produce a magnetic field given by

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\varepsilon_0\frac{\partial\phi_E}{\partial t}$$

But $\quad \phi_E = \int_S \mathbf{E}\cdot d\mathbf{S}$ and $\mathbf{B} = \mu_0\mathbf{H}$ in free space

and $\quad \int \mu_0\mathbf{H}\cdot d\mathbf{l} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d\mathbf{S}$

$$\Rightarrow \quad \int \mathbf{H}\cdot d\mathbf{l} = \varepsilon_0\frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d\mathbf{S}$$

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As **H** is perpendicular to the length of plates, we have

$$H(2\pi a) = \varepsilon_0 \frac{\partial}{\partial t}(E\pi a^2)$$

$$\Rightarrow \quad H = \frac{1}{2}\varepsilon_0 a \frac{\partial E}{\partial t} \quad \text{...(i)}$$

The directions **S**, **E** and **H** are shown in Fig. 4.11

Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

In magnitude $S = EH \sin 90° = EH$

Using Eq. (i),

$$S = E\left(\frac{1}{2}\varepsilon_0 a \frac{\partial E}{\partial t}\right)$$

∴ The rate of flow of field energy into the capacitor through its cylindrical surface will be

$$P = -\int \mathbf{S}\cdot d\mathbf{A} = \int S\,dA \quad \text{(as } S \text{ and area element } dA \text{ are antiparallel)}$$

$$P = -\int_A \frac{1}{2}\varepsilon_0 a E \frac{\partial E}{\partial t} dA = \frac{1}{2}\varepsilon_0 a E \frac{\partial E}{\partial t}(2\pi a l)$$

$$= \varepsilon_0 \pi a^2 l E \frac{\partial E}{\partial t} \quad \text{...(ii)}$$

Energy stored in capacitor $U_e = \frac{1}{2}\varepsilon_0 E^2 \pi a^2 l$

Rate of increase of electrical energy,

$$R = \frac{dU_e}{dt}$$

$$= \frac{1}{2}\varepsilon_0 (2E)\frac{\partial E}{\partial t}(\pi a^2 l)$$

$$= \varepsilon_0 \pi a^2 l E \frac{\partial E}{\partial t} \quad \text{...(iii)}$$

Hence we say that Eqs. (ii) and (iii) are identical.

Question Bank

Multiple Choice Questions

1. The concept that a changing electric field in a conductor produces induced magnetic field was proposed by
   (a) Faraday (b) Biot-Savart (c) Maxwell (d) Oersted.
2. The concept of displacement current was proposed by
   (a) Maxwell (b) Faraday (c) Ampere (d) Gauss.

3. The displacement current can be represented as
   (a) $i_d = \mu_0 \frac{d\phi_E}{dt}$ (b) $i_d = \varepsilon_0 \frac{d\phi_E}{dt}$ (c) $i_d = \mu_0 \frac{d\phi_B}{dt}$ (d) $i_d = \varepsilon_0 \frac{d\phi_B}{dt}$
4. Maxwell's modified Ampere's law is valid
   (a) only when electric field does not change with time
   (b) only when electric field varies with time
   (c) in both of the above situations
   (d) none of the above.
5. Displacement current goes through the gap between the plates of a capacitor when the charge of a capacitor is
   (a) decreasing (b) increasing (c) remaining constant (d) zero.
6. The Maxwell's equation
   $\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$ is statement of
   (a) Gauss's law (b) Ampere's law
   (c) Faraday's law (d) Modified Ampere's law.
7. The Maxwell's equation which interprets that isolated magnetic poles do not exist is
   (a) $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$ (b) $\nabla\cdot\mathbf{B} = 0$
   (c) $\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (d) $\nabla\times\mathbf{B} = -\mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$
8. The Maxwell's equation which remains unchanged when a medium changes is
   (a) $\nabla\cdot\mathbf{B} = 0$ (b) $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$
   (c) $\nabla\times\mathbf{B} = -\mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$ (d) none of these.
9. In the following, which one is the consequence of the Maxwell's equations ?
   (a) $\mathbf{B} = \mu_0\mathbf{H}$ (b) $\mathbf{D} = \varepsilon_0\mathbf{E}$
   (c) $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ (d) $C = 1/(\sqrt{\varepsilon_0\mu_0})$
10. Which is the incorrect statement about the electromagnetic waves.
    (a) the electromagnetic field vectors **E** and **B** are mutually perpendicular and they are also perpendicular to the direction of propagation of the electromagnetic wave
    (b) the field vectors **E** and **H** are in same phase
    (c) the field vectors **E** and **H** are along the same direction
    (d) electromagnetic waves are transverse in nature.
11. Which one is the incorrect statement about the electromagnetic waves ?
    (a) In free space, the electromagnetic waves travel with the speed of light
    (b) The direction of flow of electromagnetic energy along the direction of propagation of wave
    (c) The electrostatic energy density is equal to magnetic energy density
    (d) The energy density associated with the electromagnetic wave in free space propagates with a speed less than the speed of light.

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12. A monochromatic electromagnetic wave means that
    (a) the field strength at a point varies with time according to sine and cosine function
    (b) the wave always travels in the same direction
    (c) electric field vector **E** lies in one direction only.
    (d) magnetic field vector **B** must be perpendicular to the direction of propagation.
13. A plane electromagnetic wave means that
    (a) the field strength at a point varies with time according to sine and cosine function
    (b) the wave always travels in the same direction
    (c) electric field vector **E** lies in one direction only
    (d) magnetic field vector **B** must be perpendicular to the direction of propagation.
14. "The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy which flowed out through the surface" is the statement of
    (a) Gauss's theorem (b) Stoke's theorem
    (c) Pythagoras theorem (d) Poynting theorem.
15. The energy per unit time, per unit area transported by the electromagnetic fields is expressed as
    (a) $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$ (b) $\mathbf{S} = \mathbf{E}\times\mathbf{B}$ (c) $\mathbf{S} = \mu_0(\mathbf{E}\times\mathbf{B})$ (d) $\mathbf{S} = \frac{1}{\epsilon_0}(\mathbf{E}\times\mathbf{H})$
16. The total energy stored in electromagnetic fields is
    (a) $\frac{\epsilon_0}{2}\int E^2 dV$ (b) $\frac{1}{2\mu_0}\int B^2 dV$
    (c) $\frac{1}{2}\int\left[\epsilon_0 E^2 + \left(\frac{1}{\mu_0}\right)B^2\right]dV$ (d) $\frac{\epsilon_0\mu_0}{2}\int E^2 B^2 dV$
17. A plane electromagnetic wave in a free space is represented by an equation
    (a) $\nabla^2\mathbf{E} - \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t} = 0$ (b) $\nabla^2\mathbf{B} - \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$
    (c) $\nabla^2\mathbf{E} - \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$ (d) $\nabla^2\mathbf{E} - \mu\sigma\frac{\partial^2\mathbf{E}}{\partial t} + \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$
18. For a static field in vacuum the correct form of the Maxwell's equation is
    (a) $\nabla\times\mathbf{E} = 0$ (b) $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
    (c) $\nabla\times\mathbf{B} = \mu_0\left[J + \epsilon_0\left(\frac{\partial\mathbf{E}}{\partial t}\right)\right]$ (d) $\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$
19. Choose the incorrect Maxwell's equation for harmonically time varying fields
    (a) $\nabla\times\mathbf{E} = -j\omega\mathbf{B}$ (b) $\nabla\times\mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E}$
    (c) $\oint \mathbf{H}.d\mathbf{l} = (\sigma + j\omega\epsilon)\int_S \mathbf{E}.d\mathbf{S}$ (d) $\int_C \mathbf{E}.d\mathbf{l} = -\int_S \frac{\partial\mathbf{B}}{\partial t}.d\mathbf{S}$
20. The depth of penetration is the depth in which the electromagnetic wave has been attenuated to
    (a) e of the original value (b) $\frac{1}{e}$ of the original value
    (c) 50% of the original value (d) 100% of the original value.

21. For a good conductor the skin depth is equal to
    (a) $\sqrt{\frac{\omega\epsilon\sigma}{2}}$ (b) $\sqrt{2\omega\mu\sigma}$
    (c) $\sqrt{\frac{2}{\omega\mu\sigma}}$ (d) $\frac{2}{\sqrt{\omega\mu\sigma}}$
22. The skin depth is
    (a) directly proportional to attenuation constant, (α)
    (b) inversely proportional to attenuation constant, (α)
    (c) directly proportional to phase constant, (β)
    (d) inversely proportional to phase constant (β).
23. A good dielectric medium is one for which
    (a) $\frac{\sigma}{\omega\epsilon} >> 1$ (b) $\frac{\sigma}{\omega\epsilon} << 1$ (c) $\frac{\sigma}{\omega\epsilon} = 1$ (d) $\frac{\sigma}{\omega\epsilon} = 0$
24. A good conducting medium is one for which
    (a) $\frac{\sigma}{\omega\epsilon} >> 1$ (b) $\frac{\sigma}{\omega\epsilon} << 1$ (c) $\frac{\sigma}{\omega\epsilon} = 1$ (d) $\frac{\sigma}{\omega\epsilon} = 0$
25. A quasi-conducting medium is one for which
    (a) $\frac{\sigma}{\omega\epsilon} >> 1$ (b) $\frac{\sigma}{\omega\epsilon} << 1$ (c) $\frac{\sigma}{\omega\epsilon} = 1$ (d) $\frac{\sigma}{\omega\epsilon} = 0$
26. When an electromagnetic wave is propagated in good conductors, the velocity of wave propagation is reduced to
    (a) $(2\omega/\mu\sigma)^{1/2}$ (b) $\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$ (c) $(\omega\mu\sigma/2)^{1/2}$ (d) $(\omega\mu/\sigma)^{1/2}$
27. The intrinsic impedance of good conductor is given by
    (a) $\sqrt{\frac{\mu}{\epsilon}}\left(1 + \frac{j\sigma}{2\omega\epsilon}\right)$ (b) $\sqrt{\frac{j\omega\mu}{\sigma}}$
    (c) $\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$ (d) $\sqrt{j\omega\mu\sigma\left(1 + j\frac{\omega\epsilon}{\sigma}\right)}$
28. The concept of displacement current was a major contribution attributed to
    (a) Faraday (b) Lenz
    (c) Maxwell (d) Lorentz.

Answers

1. (c)	2. (a)	3. (b)	4. (c)	5. (a) & (b)	6. (c)
7. (b)	8. (a)	9. (d)	10. (c)	11. (d)	12. (a)
13. (b)	14. (d)	15. (a)	16. (c)	17. (c)	18. (a)
19. (d)	20. (b)	21. (c)	22. (b)	23. (b)	24. (a)
25. (c)	26. (d)	27. (b)	28. (c)		

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eoretical Questions

What is the physical significance of equation of continuity ? [GGSIPU, May 2008 (2.5 marks)]

State Poynting theorem. [GGSIPU, May 2008 (4 marks)]

Write Maxwell's equation in integral form. Discuss in brief the physical meaning of each of these. [GGSIPU, May 2008 (6.25 marks)]

Discuss in brief the inconsistencies in the Ampere's law and describe how Maxwell fixed up this. Further discuss in brief the characteristics of displacement current. [GGSIPU, May 2008 (6.25 marks)]

State the Poynting theorem. Explain the term Poynting vector. [GGSIPU, May 2007 (2.5 marks)]

Give the equation of continuity of e.m. theory. Explain the inconsistency of Ampere's law for transient currents. How was the law modified in its generalized form to overcome the inconsistency. [GGSIPU, May 2007 (5 marks)]

Write the Maxwell's equations in differential form and give their physical significance. [GGSIPU, May 2007 (4 marks)]

Derive the wave equation for an isotropic dielectric medium. Prove the orthogonality of **E**, **H** and **k** vectors. Find the impedance of the medium. [GGSIPU, May 2007 (7 marks)]

What is Poynting vector ? How is the Poynting theorem derived from Maxwell's curl equations. Explain Poynting theorem. [GGSIPU, May 2006 (8 marks)]

Give the physical interpretation of differential form of Gauss law in magnetostatics. [GGSIPU, May 2006 (2 marks)]

State Poynting theorem. Give physical significance of Poynting vector. [GGSIPU, May 2006 (2 marks)]

Discuss the propagation of e.m. waves in dielectrics. Derive an expression for phase velocity and show that it is less than speed of light. [GGSIPU, Feb. 2008 (7 marks)]

State Ampere's law. Discuss why and how it was modified to include the displacement current. [GGSIPU, Feb. 2008 (7 marks)]

Discuss the propagation of e.m. waves through free space. Establish the transverse nature of the waves and find the wave impedance. [GGSIPU, Feb. 2008(Reappear) (8 marks)]

Write short note on Maxwell's equation and their physical significance. [GGSIPU, May 2008(Reappear) (5 marks)]

What do you understand by the term 'skin depth' ? [GGSIPU, Feb. 2005 (1 mark)]

Illustrate mathematically how and under what conditions does Ampere's circuital law fails. How did Maxwell modify Ampere's law to make it consistent under all conditions. Give the mathematical justification to prove its consistency. [GGSIPU, Feb. 2005 (4 marks)]

State the theorem showing conservation of energy in electromagnetism. Give the physical significance of the important vector quantity defined in the above mentioned theorem. [GGSIPU, Feb. 2005 (3 marks)]

Obtain the equation of propagation of plane e.m. wave in an isotropic dielectric (non-conducting) medium. Discuss the following for the above case.

i) relative direction of E and H.

ii) phase of E & H and impedance

iii) Poynting vector

[GGSIPU, Feb. 2005 (7 marks)]

20. Enumerate Maxwell's equations and show that they predict the existence of electromagnetic waves. [GGSIPU, Feb. 2005 (12.5 marks)]

21. Show that equation of continuity div $\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$ is contained in Maxwell's equations.

22. Derive the equation for e.m. wave propagation in terms of E by the use of Maxwell's equations. Show that E is perpendicular to *k*, the propagation vector.

23. Write Maxwell's equations (in differential form). Discuss integral form of Maxwell's equations and also their physical significance to electricity and magnetism.

24. What do you mean by equation of continuity and deduce an expression for this equation.

25. Derive electromagnetic wave equations for electric field (E) and magnetic field (H) in a dielectric medium.

26. State Faraday's laws of electromagnetic induction. Derive integral form of Faraday's law *i.e.*, $\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

27. State Biot-Savart law. Derive an expression of magnetic field (**B**) due to an infinite wire carrying a steady current.

28. Write Maxwell's equations in a free space. Derive wave equation for electric and magnetic fields in a source free region and prove that the speed of light is $1/\sqrt{\epsilon_0\mu_0}$.

29. (*a*) Write down the Maxwell's equations in free space. Explain the physical significance of each equations.

    (*b*) Show that, in absence of charge and current density, any solution of Maxwell's equations also satisfies the wave equation.

30. Prove that velocity of propagation of plane electromagnetic wave in vacuum space is $1/\sqrt{\epsilon_0\mu_0}$.

31. Find the relation between |E| and |H| for uniform plane polarised plane electromagnetic wave.

32. Using Maxwell's equations show that in free space the propagation of an electromagnetic wave is given by

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8 \text{ m/s}$$

33. State and interpret Maxwell's equations in free space. Show that in free space the speed of electromagnetic wave is constant.

34. What are the characteristics of the plane polarised plane wave solution of Maxwell's equation.

35. Obtain plane polarised plane wave solution of Maxwell's equations in simple dielectric, discuss briefly all simplifications used. Also discuss the characteristics of these waves.

36. Before the modification of the Ampere's circuital law, what was the anomaly and how it was removed by Maxwell ? Discuss the significance of displacement current and derive the wave equation for a plane monochromatic wave in a non-conducting homogeneous dielectric medium with no free charge.

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37. Write general Maxwell's equations. Obtain wave equations for electric and magnetic fields for the case of isotropic and homogeneous source free non-magnetic dielectric medium. How does the velocity of these waves depend on the dielectric constant.
38. Starting from the Biot-Savart law, prove that the magnetostatic field is solenoidal.
39. State Faraday's law for induced e.m.f. and obtain its differential form.

Numerical Problems

1. If the magnitude of **H** in a plane wave is 1.0Amp/m. Find the magnitude of **E** for a plane wave in free space. [Ans. |E|=377 Volts/m]
2. If the earth receives 2.0 Cal $min^{-1}cm^{-2}$ solar energy, what are the amplitudes of electric and magnetic fields of radiations. [Ans. $E_0 = 1024.3$ V/m, $H_0 = 2.717$Amp-turn/m]
3. Show that the energy stored in a magnetic field per unit volume in free space is $B^2/2\mu_0$.
4. Assuming that all the energy from a 1000 Watt lamp is radiated uniformly, calculate the average values of the intensities of electric and magnetic fields of radiation at a distance of 2m from the lamp. [Ans. E = 48.87 V/m, H = 0.4 Amp.-turn/m]
5. The ratio J/J_d (conduction current density to displacement current density) is very important at high frequencies. Calculate the ratio at 1 GHz for
   (a) distilled water ($\mu = \mu_0, \varepsilon = 81\varepsilon_0, \sigma = 2\times10^{-3}$ S/m)
   (b) sea water ($\mu = \mu_0, \varepsilon = 81\varepsilon_0, \sigma = 25$ S/m)
   (c) lime-stone ($\mu = \mu_0, \varepsilon = 5\varepsilon_0, \sigma = 2\times10^{-4}$ S/m)
6. Assume that dry soil has $\sigma = 10^{-4}$ S/m, $\varepsilon = 3\varepsilon_0$. Determine the frequency at which the ratio of the magnitudes of the conduction current density and the displacement current density is unity. [Ans. 600 kHz]
7. A conductor with cross-sectional area of 10 cm^2 carries a conduction current $0.2\sin 10^9 t$ mA and $K_e = 6$, calculate the magnitude of the displacement current density.
8. In a linear homogeneous and isotropic conductor, show that the charge density satisfies
$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$
9. In a certain material $\mu = \mu_0$, $\varepsilon = \varepsilon_0 K_e$ and $\sigma = 0$. If $\mathbf{H} = 10\sin(10^8 t - 2x)\mathbf{a}_z$ A/m, find J_d, E and ε_r.
10. In free space $E = \frac{80}{\rho}\cos(10^8 t - kz)\mathbf{a}_\rho$ V/m, find **k**, $\mathbf{J}_d$ and **H**.
$$\text{[Ans. } 0.333,\ \frac{-4.421\times10^{-2}}{\rho}\sin(10^8 t - kz)\mathbf{a}_\phi \text{ A/m and } \frac{2.5k}{2\pi\rho}\cos(10^8 t - kz)\mathbf{a}_\phi \text{ A/m]}$$
11. A certain material has $\sigma = 0$ and $K_e = 1$. If $\mathbf{H} = 4\sin(10^6 t - 0.01z)\mathbf{a}_y$ A/m, make use of Maxwell's equations to find (a) K_m (b) E.
12. Show that the skin depth in a poor conductor ($\sigma << \omega\varepsilon$) is $(2/\sigma)/\sqrt{\varepsilon/\mu}$ (independent frequency). Find the skin depth (in metres) for pure water.
13. Sea water at frequency $\nu = 4\times10^8$ Hz has permittivity $\varepsilon = 81\varepsilon_0$, permeability $\mu = \mu_0$ and resistivity $\rho = 0.23\Omega$m. What is the ratio of conduction current to displacement current ?

14. If the average distance between the sun and earth is 1.5×10^{11} m, show that the average solar energy incident on the earth is 2 cal/cm^2-min (called solar constant).
15. A plane electromagnetic wave travelling in positive z-direction in an unbounded lossless dielectric medium with relative permeability $K_m = 1$ and relative permittivity $K_e = 3$ has peak electric field intensity $E_0 = 6$ V/m.
   (i) the speed of the wave ; (ii) the intrinsic impedance of the medium ; (iii) the peak magnetic field intensity (H_0) and (iv) the peak Poynting vector S(z,t).
   [Ans. (i) 1.73×10^8 m/s ; (ii) 217.6Ω ; (iii) 2.76×10^{-2} A/m and (iv) s = 0.165 W/m
16. A 1000 W radio station sends power in all directions from its antenna. Find **E** and **B** in its wave at a distance of 10 km from the antenna (**B** and **E** transport same amount of energy).
   [Ans. E = 0.024 V/m, $B = 8.2\times10^{-11}$
17. At what frequency may earth be considered a perfect dielectric, if $\sigma = 5\times10^{-3}$ S/m, $K_m = 1$, $K_e =$ Calculate the value of attenuation constant at these frequencies.
   (take $\frac{\sigma}{\omega\varepsilon} \le 0.01$) [Ans. 1.13 GHz, 0.333 Np
18. A radio station P is 100 km from Q. It radiates an average power of 50kW. Assuming sphe radiation, find the value of S at Q. Calculate the values of E_0 and B_0 at an antenna located at
   [Ans. $S = 0.4\times10^{-6}$ Wb/m^2, $E_0 = 17\times10^{-2}$ V/m and $B_0 = 5.7\times10^{-}$
19. In the Bohr model of the hydrogen atom the electron circulates around the nucleus in a 5.1×10^{-11} m at a frequency of 6.8×10^{-15} Hz. What value of B is setup at the centre of the orb
   [Ans. B = 14 Wb
20. Show that in the electromagnetic wave, the electrostatic energy is equal to the magnetic e density. [Ans. U_e =
21. The electric field intensity of a uniform plane wave in air is 8000 V/m in the y-axis. The w propagating along x-axis at the frequency 10^{10} rad/s. Compute :
   (i) the wavelength λ, (ii) the frequency ν, (iii) the time period T and (iv) the amplitude of
   [Ans. (i) $\lambda = 18.8\times10^{-2}$ m, (ii) ν = 1591.3 MHz, (iii) T = 0.63 ns and (iv) 21.22 Am
22. Prove that (i) $\mathbf{S}_{av} = \mathbf{E}_{rms}\times\mathbf{H}_{rms}$ (ii) $I = S_{av} = \frac{1}{2}C\varepsilon_0 E_0^2 = \frac{1}{2}C\mu_0 H_0^2$
   where I is the intensity of the electromagnetic wave.
23. Calculate the Poynting vector at the surface of the sun. Given that the energy radiated per se 3.8×10^{26} J and radius of the sun is 0.7×10^9 m. Also calculate the amplitude of electric and m field vectors on the surface of earth. The distance of the earth from the sun is 0.15×10^{12} m
   [Ans. $S = 6.17\times10^7$ $Jm^{-1}s^{-1}$, $E_0 = 10^4$ V/m, $H_0 = 2.267$ A
24. Show that the energy flux in a plane polarised e.m. wave in free space is the energy densi the velocity of the wave. [Ans
25. Find the energy stored in one meter length of a laser beam operating 1mW. [Ans. u = 3.3×
26. Show that the energy density and Poynting vector of electromagnetic field are given by
$$U_{em} = \frac{1}{2}(\varepsilon_0 E^2 + \mu_0 H^2) \text{ and } \mathbf{S} = \mathbf{E}\times\mathbf{H}$$

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27. Find the conduction and displacement current densities in a material having conductivity of 10^{-3} S/m and $K_r = 2.5$ if the electric field in the material is

$$E = 5 \times 10^{-4} \sin 9.0 \times 10^{9} t \text{ V/m}$$

[**Ans.** $J_C = 5.0 \times 10^{-4} \sin 9.0 \times 10^{9} t$ Amp/m^2 and $J_D = 1.00 \times 10^{-6} \cos 9.0 \times 10^{9} t$ Amp/m^2]

28. A plane wave propagating in free space with a peak electric field of intensity 750 mV/m. Find average power through square area of 120cm on a side perpendicular to the direction of propagation. [**Ans.** $P_{av} = 10.7$ mW]

29. Show that in a conductor, the magnitude of the electric vector reduces to about 10% at a distance of $0.733\lambda_c$ where λ_c is the length of e.m. wave in the conductor.

30. The electric field in an electromagnetic wave is given by $E = E_0 \sin \omega(t - x/c)$, where $E_0 = 1000$ N/C. Find the energy contained in a cylinder of cross-section 10^{-3} m^2 and length 100 cm along the x-axis. [**Ans.** 4.425×10^{-11} J]

31. A plane monochromatic linearly polarised light wave is travelling eastward. The wave is polarised with E directed vertically up and down. Write expressions for E, H and B provided that $E_0 = 0.1$ V/m and frequency is 20 MHz. [**Ans.** $E_z = 0.1 \sin(4\pi \times 110^{-7} t - 0.419x)$, $B_y = \frac{-E_z}{C}$, $H_y = \frac{B_y}{\mu_0}$]

32. The electric field in an electromagnetic wave is given by $E = 50 \sin \omega(t - x/c)$ N/C. Find the energy contained in a cylinder of cross-section 10cm^2 and length 50cm along the x-axis. [**Ans.** 5.5×10^{-12} J]

33. Find the skin depth δ at a frequency of 3.0×10^{6} Hz in aluminium where $\sigma = 38 \times 10^{-6}$ S/m and $K_m = 1$. Also find out the propagation constant and wave velocity. [**Ans.** $\delta = 0.0416$ mm, $\gamma = 29986 \times 10^{3} \angle 45°$ m^{-1} and $v = 888.51$ m/s]

Chapter 5

Wave-Particle Dualism

5.1 Introduction

- Light is a wave : It spreads out over space.
- Light is also a particle (photon) that is at a point.

Phenomena such as the photoelectric effect and the compton effect offer iron-clad evidence that when light and matter interact, the light behaves as wave if it were composed of particles having energy $h\nu$ and momentum h/λ. But the phenomena of interference, diffraction and polarisation, radiant energy exhibit wave nature. Therefore electromagnetic radiation exhibits the dual property as **wave and particle**. In visible region, it is possible to observe both the particle characteristics and the wave characteristic of light. At higher frequencies, the momentum and energy of the photon increases. Consequently, the particle nature of light becomes more evident than its wave nature. For example, absorption of an X-ray photon is easily detected as a single event but wave effects are difficult to observe. X-rays are also diffracted by a crystal. Since diffraction is a wave phenomenon, X-rays also behave as waves.

In 1923, in his doctoral dissertation, Louis de Broglie postulated that *because photons have both wave and particle characteristics perhaps all forms of matter have both properties.* According to de Broglie, electrons just like light, have a dual particle-wave nature. Accopanying every electron in a wave (not an electromagnetic wave).

Louis de Broglie's hypothesis was based on the following facts :

(i) In universe, whole of the energy is in the form of radiation and matter. So both the forms of energy should possess similar characteristics.

(ii) Nature is symmetrical in many ways. As radiation has dual nature, matter should also possess dual nature.

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(iii) Ordinarily, periodicity is associated only with the wave phenomena like interference and diffraction, the patterns being governed by integral multiple of wavelength or *half* wavelength. But some intrinsic properties are associated with electrons in Bohr orbit.

(iv) Einstein in 1905 established a relation between mass and energy, $E = mc^2$. He said that both mass and energy were interconvertible. When energy E exhibits a dual aspect, mass namely matter also should exhibit a duality.

According to de Broglie, *a moving particle behaves sometimes as a wave and sometimes as a particle or a wave associated with moving particle.* The waves associated with material particles are called **matter waves** or **de Broglie waves**. They are seen with particles like electrons, protons, deutrons etc. The wavelength of matter waves is called **de Broglie wavelength** and is given by

$$\lambda = \frac{h}{p}$$

where h = Planck's constant

5.2 de Broglie's Hypothesis

A moving body behaves in certain ways as through it has a wave nature.

The dual nature of light possessing both wave and particle properties is clearly illustrated by combining Planck's relation for energy of a photon, $E = h\nu$ with Einstein mass energy relation $E = mc^2$.

Considering the photon to be a particle of mass m, moving with it is given by Einstein's mass-energy relation as

$$E = mc^2 \quad ...(1)$$

According to Planck's hypothesis, energy of photon of frequency ν is given by

$$E = h\nu \quad ...(2)$$

$$\therefore \quad h\nu = mc^2$$

or

$$mc = \frac{h\nu}{c}$$

Momentum of photon, $p = mc$

$$\therefore \quad p = \frac{h\nu}{c}$$

$$= \frac{h}{c/\nu} = \frac{h}{\lambda} \quad \left[\because \frac{c}{\nu} = \lambda\right] \quad ...(3)$$

As the matter also possesses dual nature, therefore, wavelength of the wave associated with a matter particle of mass m moving with velocity v is

$$\lambda = \frac{h}{p} = \frac{h}{mv} \quad ...(4)$$

This is called **de Broglie wave relation**. Hence, de-Broglie wavelength depends upon the mass of particle and its velocity.

5.3 de-Broglie Wavelength of an Electron

Consider an electron of mass m is under the potential difference of V volts. Its kinetic energy is $\frac{1}{2}mv^2$.

$$E = \frac{1}{2}mv^2 = eV$$

or

$$v = \sqrt{\left(\frac{2eV}{m}\right)} = \sqrt{\left(\frac{2E}{m}\right)} \quad ...(5)$$

NOTE

150 V is enough to produce electrons of wavelength 1 Å in contrast with X-rays which require about 12,000 V for 1 Å. Electrons generated by de Broglie waves are of short wavelengths and are quite useful for certain types of diffraction experiments. Because of the smallness of h, only for particles of atomic of nuclear size will the wave function be observable.

If λ is the de Broglie wavelength associated with an electron, then

$$\lambda = \frac{h}{mv} = \frac{h}{m\sqrt{\frac{2eV}{m}}} = \frac{h}{\sqrt{2meV}}$$

$$= \frac{h}{\sqrt{2mE}} \quad ...(6)$$

Substituting the value of h, m and e in Eq. (6), we get

$$\lambda = \frac{12.28}{\sqrt{V}} \text{ Å} \quad ...(7)$$

5.4 Relativistic Corrections in the Formula

In Newtonian Physics, mass is not a variable quantity while according to the theory of relativity, it is a variable quantity.

On account of the relativistic increase in the mass of the high velocity electrons, the necessary correction in Eq. (7) may be made. The relativistic mass is given by

$$m = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad ...(8)$$

where m_0 is the rest mass of the particle, v is its velocity and c the velocity of light.

Applying relativistic equation for kinetic energy K, viz

$$K = (m - m_0)c^2, \text{ we have}$$

$$(m - m_0)c^2 = Ve$$

or

$$(m - m_0) = \frac{Ve}{c^2}$$

or

$$m = m_0 + \frac{Ve}{c^2} = m_0\left(1 + \frac{Ve}{m_0c^2}\right)$$

or

$$m^{1/2} = m_0^{1/2}\left(1 + \frac{Ve}{m_0c^2}\right)^{1/2}$$

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Substituting the value of $m^{1/2}$ in Eq. (6), above, we get

$$\lambda = \frac{h}{\sqrt{2Vem_0}}\left(1+\frac{Ve}{m_0c^2}\right)^{-1/2} \quad \text{...(9)}$$

For the wavelength of electrons of known energy,

Now, Eq. (7)

$$\frac{h}{\sqrt{2Vem_0}} = \frac{12.28}{\sqrt{V}}\ \text{Å} \quad \text{...(10)}$$

Substitution in the above gives

$$\lambda = \frac{12.28}{\sqrt{V}}\left(1+\frac{Ve}{m_0c^2}\right)^{-1/2}\ \text{Å} \quad \text{...(11)}$$

Example 5.1. *Calculate the de Broglie wavelength of*

(i) a particle accelerated by a potential difference of 30,000 V, and

(ii) an electron moving with a velocity of 0.01 c, where c is the speed of light.

[GGSIPU, Feb. 2008, Reappear (6 marks)]

Solution. (i) The de Broglie wavelength is given by

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}} \qquad \left[\because E = \frac{1}{2}mv^2 \text{ so that } mv = \sqrt{2mE}\right]$$

Given : $E = Ve = 1.6\times10^{-19}$ V Joule

mass of electron $= 9.1\times10^{-31}$ kg

and Planck's constant $(h) = 6.63\times10^{-34}$ Js.

$$\therefore \quad \lambda = \frac{6.63\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times1.6\times10^{-19}\ V}} = \frac{12.28\times10^{-10}}{\sqrt{V}}$$

$$= \frac{12.28}{\sqrt{V}}\ \text{Å} = \frac{12.28}{173.2}\times10^{-10} = 7.09\times10^{-12}\ \text{m}.$$

(ii) The de Broglie wavelength is given by

$$\lambda = \frac{h}{mv} = \frac{h}{m\times(0.01c)}$$

$$= \frac{6.63\times10^{-34}}{9.1\times10^{-31}\times0.01\times3\times10^{8}}\ \text{m} = \frac{6.63\times10^{-11}}{9.1\times0.03}\ \text{m}$$

$$= 2.43\times10^{-10}\ \text{m} = 2.43\ \text{Å}$$

Example 5.2. *Calculate the de Broglie wavelength of virus particle of mass 1.0×10^{-15} kg moving at a speed of 2.0 mm/sec.*

[GGSIPU, May 2005 (2.5 marks)]

Solution. The de Broglie wavelength is given by

$$\lambda = \frac{h}{mv} = \frac{6.63\times10^{-34}}{1.0\times10^{-15}\times2.0\times10^{-3}} = 3.315\times10^{-16}\ \text{m}.$$

5.5 Characteristics of Matter Waves

1. The de Broglie wavelength of a particle of mass m moving with a velocity v is given by $\lambda = \frac{h}{\sqrt{2meV}}$. Larger the mass, shorter will be the de Broglie wavelength.
2. de Broglie waves are **pilot waves** *i.e.*, these waves guide the particle.
3. de Broglie waves are not electromagnetic waves.
4. Matter waves can not be observed. It is a wave model to describe and to study matter.
5. de Broglies waves are called **probability waves**. The amplitude of the wave reveals the probability of finding a particle in space at a particular instant. A large wave amplitude means a large probability to find the particle at that position as visualised in Fig. 5.1.

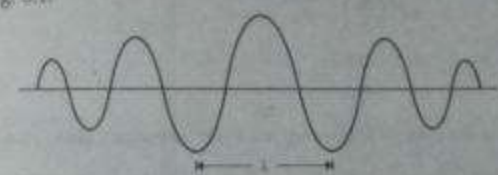

Fig. 5.1

6. The matter wave consists of a group of waves. The velocity of the group is different from that of the individual components of the wave. But it can be shown that the particle velocity v is same as that of the de Broglie groups associated with the particles. If the wave velocity and particle velocity is different, then the particle would soon be in the region where the amplitude of the wave is negligible and the wave would not have given the correct indication of the particle.
7. The phase velocity of the matter waves depends on the wavelength, even if the particle is moving in vacuum. But in vacuum all electromagnetic waves travels with the same velocity and is independent of the wavelength.

5.6 Phase Velocity

According to de Broglie, the matter wave consists of a *group of waves* or a *wavepacket*. The velocity with which each individual wave travels is called *phase velocity* or *wave velocity*. The de Broglie wave velocity or phase velocity is given by

$$v_p = v\lambda = 2\pi v\frac{\lambda}{2\pi} \qquad \left[\because k = \frac{2\pi}{\lambda}\right]$$

or

$$v_p = \frac{\omega}{k}$$

A plane simple harmonic wave travelling in the positive x-direction is given by

$$y = A\sin\omega\left(t-\frac{x}{v}\right) \quad \text{...(12)}$$

where A is amplitude, v the velocity of propagation of wave and ω the angular frequency $= 2\pi v$.

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The wave velocity is also phase velocity. The reason for this follows. In Eq. (12), the phase ϕ of the wave at position x and at time t is

$$\phi(x,t) = \omega\left(t - \frac{x}{v}\right) \quad ...(13)$$

Differentiating this equation with respect to t, we get

$$\frac{d\phi}{dt} = \omega\left(1 - \frac{1}{v}\frac{dx}{dt}\right)$$

For a point of constant phase,

$$\frac{d\phi}{dt} = 0$$

$$\therefore \quad 1 - \frac{1}{v}\left(\frac{dx}{dt}\right)_\phi = 0$$

or

$$\left(\frac{dx}{dt}\right)_\phi = v \quad ...(14)$$

$\left(\frac{dx}{dt}\right)_\phi$ is the velocity with which the displacement of a given phase moves forward.

Therefore, this quantity is called **phase velocity** and it is usually represented by v_p or v_ϕ. Thus the velocity of propagation or the velocity of a wave is the velocity with which a displacement of a given phase moves forward.

Replacing v by v_p in Eq. (12) and rewriting,

$$y = A\sin\omega\left(t - \frac{x}{v_p}\right) = A\sin\left(\omega t - \frac{\omega x}{v_p}\right)$$

$$= A\sin(\omega t - kx) \quad ...(15)$$

where $k = \frac{\omega}{v_p}$ or $v_p = \frac{\omega}{k}$

The quantity k is called the **propagation constant**, or the **phase constant** or the **propagation** or the **propagation number** of the wave.

5.7 Group Velocity and Wave Packets

When plane waves of slightly different wavelengths travels simultaneously in the same direction along a straight line, through a dispersive medium, (*i.e.*, a medium in which the phase velocity $v_p = \frac{\omega}{k}$ of a wave depends on its wave-length) successive groups of the waves are produced. (Fig. 5.2).

These wave groups are called **wave packets**. Each wave group travels with a velocity called **group velocity**. *"The velocity with which the resultant envelope of the group of waves travels is called group velocity"*, denoted by v_g. The velocity of the group is different from that of the individual components of the wave. The group velocity is different from the phase velocity of a wave.

The velocity with which resultant envelope moves is called group velocity and the velocity with which a point like P on the wave moves is called phase velocity.

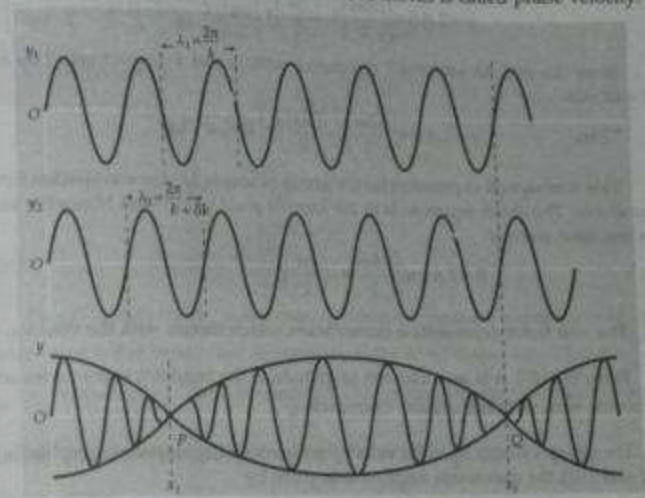

Fig. 5.2

Expression for Group Velocity

Let two plane simple harmonic waves of the same amplitude A but of slightly different wavelengths travelling simultaneously in the +ve x-direction in a dispersive medium be represented by

$$y_1 = A\sin(\omega t - kx) \quad ...(16)$$

$$y_2 = A\sin[(\omega + \Delta\omega)t - (k + \Delta k)x] \quad ...(17)$$

where y_1 and y_2 are instantaneous displacements, ω and $\omega + \Delta\omega$ are the angular velocities, k and $k + \Delta k$ are the wave propagation vectors or constants.

The resultant displacement y at time t and at position x is given by

$$y = y_1 + y_2$$

$$= A\sin(\omega t - kx) + A\sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

We know,

$$\sin A + \sin B = 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

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Making use of this equation, we get

$$y = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right]\sin\left[\left(\frac{2\omega+\Delta\omega}{2}\right)t - \left(\frac{2k+\Delta k}{2}\right)x\right] \quad ...(18)$$

Since $\Delta\omega$ and Δk are small compared with ω and k respectively $2\omega + \Delta\omega \approx 2\omega$ and $2k + \Delta k \approx 2k$.

Then,

$$y = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right]\sin(\omega t - kx) \quad ...(19)$$

This is analytical expression for the group of waves *i.e.*, the wavepacket formed by the two waves. The above equation is in the form of $y = A\sin(\omega t - kx)$. Hence the amplitude of the resultant wave is

$$R = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right] \quad ...(20)$$

The sine factor represents a carrier wave which travels with the velocity, $v_p = \frac{\omega}{k}$.

From Eq. (20), it is seen that the amplitude of the resultant wave varies according to the cosine term with the circular frequency $\frac{\Delta\omega}{2}$.

The group velocity v_g is the velocity with which the maximum amplitude moves. At $x = 0$ and $t = 0$, the maximum amplitude is given by

$$R_{max} = 2A \quad ...(21)$$

The group velocity is given by

$$v_g = \lim_{\Delta k \to 0}\frac{(\Delta\omega/2)}{(\Delta k/2)} = \lim_{\Delta k \to 0}\frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Hence

$$v_g = \frac{d\omega}{dk} \quad ...(22)$$

5.8 Relation Between Group Velocity and Phase Velocity

Phase velocity $\quad v_p = \frac{\omega}{k} \quad ...(23)$

Group velocity $\quad v_g = \frac{d\omega}{dk} \quad ...(24)$

where ω is the angular frequency of the wave and k is the wave vector.

From Eq. (23), we have

$$\omega = v_p k$$

$$\therefore \quad v_g = \frac{d}{dk}(v_p k) = v_p + k\frac{dv_p}{dk}$$

or

$$v_g = v_p + k\frac{dv_p}{d\lambda}\frac{d\lambda}{dk} \quad ...(25)$$

We know that propagation constant $k = \frac{2\pi}{\lambda}$ $\quad \left(\because \lambda = \frac{2\pi}{k}\right)$

But

$$\frac{d\lambda}{dk} = \frac{d}{dk}\left(\frac{2\pi}{k}\right) = -\frac{2\pi}{k^2} \quad ...(26)$$

Eq. (25) becomes,

$$v_g = v_p + k\frac{dv_p}{d\lambda}\left(\frac{-2\pi}{k^2}\right)$$

$$v_g = v_p - \frac{2\pi}{k}\frac{dv_p}{d\lambda}$$

or

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda} \quad \left[\because \lambda = \frac{2\pi}{k}\right] \quad ...(27)$$

Eq. (27) shows that v_g is less than v_p when the medium is dispersive *i.e.*, when v_p is a function of λ. For light wave in vacuum, there is no dispersion. Hence $\frac{dv_p}{d\lambda} = 0$, so that $v_g = v_p = c$ (velocity of light in vacuum). This is true for elastic waves in a homogeneous

5.9 Relation between Group Velocity and Particle Velocity

We know that

Group velocity, $\quad v_g = \frac{d\omega}{dk} \quad ...(28)$

Angular frequency $\quad \omega = 2\pi\nu = 2\pi\frac{E}{h} \quad ...(29)$

Differentiating, $\quad d\omega = \left(\frac{2\pi}{h}\right)dE \quad ...(30)$

Propagation constant $\quad k = \frac{2\pi}{\lambda} = 2\pi\frac{p}{h} \quad ...(31)$

For matter waves, $\quad \lambda = \frac{h}{p} \quad ...(32)$

where p = momentum of the particle.

Differentiating Eq. (32),

$$dk = \left(\frac{2\pi}{h}\right)dp \quad ...(33)$$

Dividing Eq. (30) by Eq. (33), we get

$$\frac{d\omega}{dk} = \frac{\left(\frac{2\pi}{h}\right)dE}{\left(\frac{2\pi}{h}\right)dp} = \frac{dE}{dp} \quad ...(34)$$

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We also know that

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{1}{2}\frac{p^2}{m}$$

or $$dE = \frac{1}{2}\frac{2p}{m}dp = \frac{p}{m}dp$$

or $$\frac{dE}{dp} = \frac{p}{m} \quad ...(38)$$

Since, $p = mv_{particle}$,

$$\frac{dE}{dp} = \frac{mv_{particle}}{m} = v_{particle} \quad ...(36)$$

From Eqs. (28), (34) and (36), we have

$$v_g = v_{particle} \quad ...(37)$$

Hence the group velocity of the wave packet is the same as the particle velocity.

5.10 Relation between Velocity of Light (c), Group Velocity (v_g) and Phase Velocity (v_p)

Phase velocity of the wave is given by,

$$v_p = \frac{\omega}{k} \quad ...(38)$$

Substituting the values of $\omega = 2\pi\nu = \frac{2\pi E}{h}$ and $k = \frac{2\pi}{\lambda} = 2\pi\frac{p}{h}$, we have

$$v_p = \frac{2\pi E/h}{2\pi p/h} = \frac{E}{p} = \frac{mc^2}{mv_{particle}} = \frac{c^2}{v_{particle}} \quad ...(39)$$

$\because v_g = v_{particle}$, we have

$$v_p = \frac{c^2}{v_g} \quad \text{or} \quad v_p v_g = c^2 \quad ...(40)$$

This is relation of velocity of light with phase velocity and group velocity.

5.11 Relation between v_g and v_p for a Non-relativistic Free particle

According to de Broglie hypothesis,

$$\lambda = \frac{h}{mv_g}$$

Total energy $$E = \frac{1}{2}mv_g^2$$

Also, $$E = h\nu$$

$$\nu = \frac{E}{h} = \frac{mv_g^2}{2h}$$

$\therefore$ Phase velocity,

$$v_p = \nu\lambda = \frac{mv_g^2}{2h} \times \frac{h}{mv_g} = \frac{v_g}{2}$$

$$v_p = \frac{v_g}{2} \quad ...(41)$$

Hence for a *non-relativistic free particle, the phase velocity is half of the group velocity.*

5.12 Expression for de Broglie Wavelength in terms of Group Velocity and Phase Velocity

According to de Broglie, a material particle in motion is associated with a system of plane waves, the superposition of a system of plane wave of slight different frequencies form a wave packet. The group velocity of wave packet is given by

$$v_g = \frac{d\omega}{dk} \quad ...(42)$$

We know that, $\omega = 2\pi\nu$ and $k = \frac{2\pi}{\lambda}$

Differentiating the above equations, we have

$$d\omega = 2\pi d\nu \text{ and } dk = 2\pi d\left(\frac{1}{\lambda}\right)$$

Substituting the value of $d\omega$ and dk in Eq. (42), we get

$$v_g = \frac{2\pi d\nu}{2\pi d\left(\frac{1}{\lambda}\right)} = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)} \quad ...(43)$$

Since the particle velocity is equal to group velocity, we can write

$$v_{particle} = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)} \quad \text{or} \quad d\left(\frac{1}{\lambda}\right) = \frac{d\nu}{v_{particle}} \quad ...(44)$$

Let E be the total energy and V the potential energy of the particle of mass m. If v is the velocity, then the total energy of the particle is given by

$$E = \frac{1}{2}mv_{particle}^2 + V \quad ...(45)$$

The relationship between the energy E of a photon and the frequency ν of the associated electromagnetic wave is given by the Planck Eisntein equation :

$$E = h\nu \quad ...(46)$$

Comparing Eqs. (45) and (46), we get

$$h\nu = \frac{1}{2}mv_{particle}^2 + V \quad ...(47)$$

Let the particle be moving under a constant electric potential *i.e.*, V is a constant quantity.

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Differentiating Eq. (47),

$$h\,dv = \frac{1}{2} m.2v_{particle}\,dv$$

$$= mv_{particle}\,dv \qquad ...(48)$$

Substituting the value of dv from Eq. (48) in Eq. (44), we get

$$d\left(\frac{1}{\lambda}\right) = \frac{m}{h}\frac{v_{particle}}{v_{particle}}\,dv = \left(\frac{m}{h}\right)dv$$

Integrating, $$\int d\left(\frac{1}{\lambda}\right) = \int\left(\frac{m}{h}\right)dv$$

$$\frac{1}{\lambda} = \left(\frac{m}{h}\right)v + C \qquad ...(49)$$

where C = constant of integration and is equal to zero.

$$\frac{1}{\lambda} = \frac{mv}{h} = \frac{p}{h} \qquad [\because\ p = \text{momentum} = mv]$$

or $$\lambda = \frac{h}{p} = \frac{h}{mv} \qquad ...(50)$$

Eq. (50) represents the wave-particle relation for photons and is called de-Broglie equation.

Example 5.3. *An electron has a de Broglie wavelength of 2 pm. Find its kinetic energy and phase & group velocities of its de Broglie waves. Rest mass energy of electron = 511 keV.*

Solution. Given $\lambda = 2$ pm $= 2\times10^{-12}$ m, $E_0 = 55$ keV.

According to Einstein's mass-energy relation

Rest mass energy $= m_0c^2$ or $511 \text{ keV} = m_0c^2$

$$\therefore \text{ Rest mass of electron } m_0 = \frac{511 \text{ keV}}{c^2}$$

$$= \frac{511\times10^3 \text{ eV}}{(3\times10^8)^2} = 9.1\times10^{-31} \text{ kg}$$

$$\text{Kinetic energy of the electron} = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

$$= \frac{(6.63\times10^{-34})^2}{2\times9.1\times10^{-31}\times(2\times10^{-12})^2} \text{ J}$$

$$\therefore \quad \frac{1}{2}mv^2 = 6.038\times10^{-14} \text{ J}$$

or $$v^2 = \frac{2\times6.038\times10^{-14}}{9.1\times10^{-31}} = 1.327\times10^{17} \frac{\text{m}^2}{\text{s}^2}$$

$$\therefore \text{ Particle velocity} = v_{particle} = \sqrt{1.327\times10^{17}} = 3.64\times10^8 \text{ m/s}.$$

$\because$ The particle velocity ($v_{particle}$) is equal to v_g (group velocity).

i.e., $$v_{particle} = v_g$$

$$v_g = 3.64\times10^8 \text{ m/s}$$

$$\text{Phase velocity } v_p = \frac{c^2}{v_g} = \frac{(3\times10^8)^2}{3.64\times10^8} = 2.47\times10^8 \text{ m/s}.$$

Example 5.4. *The phase velocity of ripples on a liquid surface is* $\sqrt{\left(\frac{2\pi S}{\lambda\rho}\right)}$, *where S is the surface tension and ρ the density of the liquid. Find the group velocity of the ripples.*

Solution. Since phase velocity $v_p = \sqrt{\frac{2\pi S}{\lambda\rho}}$

$$\therefore \quad v_p^2 = \frac{2\pi S}{\lambda\rho} \quad \text{or} \quad \lambda = \frac{2\pi S}{v_p^2\rho}$$

As wave number,

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{v_p} \qquad \text{(where } \omega \text{ is angular velocity)}$$

$$k = \frac{2\pi}{2\pi S}v_p^2\rho = \frac{v_p^2\rho}{S}$$

We know that group velocity is given by

$$v_g = \frac{d\omega}{dk}$$

As $$k = \frac{v_p^2\rho}{S} = \frac{\omega^2}{k^2}\frac{\rho}{S}$$

$$k^3 = \frac{\omega^2\rho}{S}$$

$$\omega^2 = \frac{k^3S}{\rho}, \quad \omega = \sqrt{\frac{S}{\rho}}k^{3/2}$$

$$v_g = \frac{d\omega}{dk} = \sqrt{\frac{S}{\rho}}\frac{3}{2}k^{1/2}$$

$$= \frac{3}{2}\sqrt{\left(\frac{Sk}{\rho}\right)} = \frac{3}{2}\sqrt{\frac{S}{\rho}\frac{v_p^2\rho}{S}}$$

$$v_g = \frac{3}{2}v_p$$

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5.13 Experimental Study of Matter Waves

An experiment that confirms the existence of de Broglie waves.

A wave effect with no analog in the behaviour of Newtonian particles is diffraction. In 1927, Clinton Davisson and Lester Germer in U.S.A. and G.P. Thomson in England independently confirmed de-Broglie's hypotheses by demonstrating that electron beams are diffracted when they are scattered by a regular arrays of crystals. It was found that there was considerable difference between X-rays and short waves produced by electrons in that the scattering coefficient of the electrons by atoms of the target substance is much higher than that of X-rays. In fact, atom scatter electrons much more efficiently for the energy involved and hence are preferred for structure investigations than X-rays. Electrons do not penetrate matter so well as X-rays, so that electron diffraction effect results from penetration only to small depths in the material. Evidently, surface would not be investigated by X-rays diffraction, because the patterns obtained are characteristic of the bulk material. Note that diffraction of electrons by gases required much shorter exposures than for X-rays diffraction. This is because of the relatively higher scattering coefficient of electrons by atoms.

5.13.1 Davisson and Germer's Experiment

Davisson and Germer in 1927 designed an apparatus to determine the wavelength associated with electrons.

The experimental arrangement used by Davisson and Germer is as visualised in Fig. 5.3. It consists of an electron gun which comprises of a tungsten filament F coated with zinc oxide heated by a low tension battery (L.T.B.) V_1 produces electrons. These electrons are accelerated by applying suitable potential from a high tension battery (H.T.B.) V_2. The accelerated electrons are collimated into a fine beam by allowing them to pass through pin holes H_1 and H_2 provided in the cylinder C.

The collimated beam of electrons falls on a large single crystal of nickel N. The crystal is capable of rotation about an axis parallel to the axis of the incident beam. The electrons are now scattered in all directions by the atoms of the crystal. The scattered electrons are collected by a Faraday chamber (D), called the collector. The collector also rotates about the sensitive galvanometer G whose deflection is proportional to the intensity of the electron beam entering the collector.

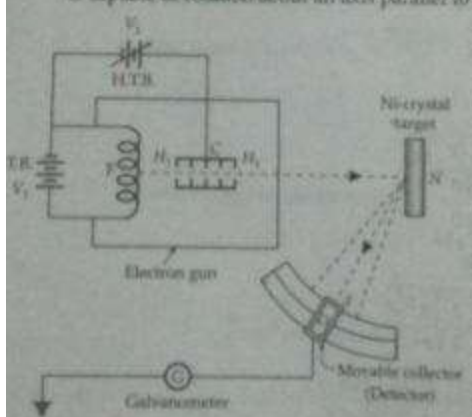

Fig. 5.3 Davisson and Germer Experiment.

It is possible that secondary electrons, produced by the impact of primary electrons on the nickel target may also enter the Faraday's chamber (D). This is prevented surrounding the collector by shielding chamber S, to which a retarding potential is applied. This potential is about nine-tenth of the accelerating voltage.

If a beam of electron incidents on the crystal, turned at any arbitrary azimuth and the distribution of the scattered beam is measured as a function of the colatitude – the angle between the incident beam and the beam entering the collector.

The whole instrument is kept in an evacuated chamber. The nickel crystal belongs to face-centered cubic (f.c.c.) type and it is so cut as to present a smooth reflection surface parallel to the lattice plane <111>.

This experiment can be performed by

- Normal incidence method and
- Oblique incident method.

(i) Normal Incidence Method

In this method voltage V_2 applied to the accelerating cylinder C is maintained at different constant value namely 36, 40, 44, 48, 54, 60, 64 and 68 V. If now the crystal is turned to the azimuth, the distribution curve for 44 V electrons [(Fig. 5.4c)] shows a slight hump at about colatitude 60°. With increasing voltage this hump moves upward and develops into a spur which becomes most prominent at 54 V [Fig. 5.4e] at which the colatitude of the spur is 50°. At higher voltages the spur gradually disappears.

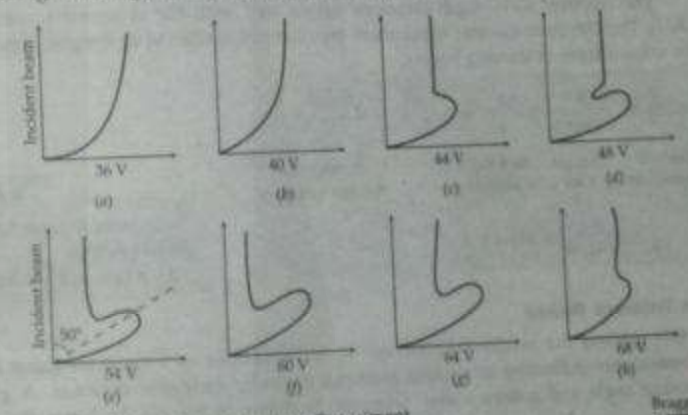

Fig. 5.4 Results of the Davisson Germer Experiment.

The intensity of the scattered beam is found to be maximum in a direction at 50° to the incident radiation. This is due to the constructive interference of electron waves scattered in this direction from the regularly spaced parallel planes in the crystal, which are rich in atoms.

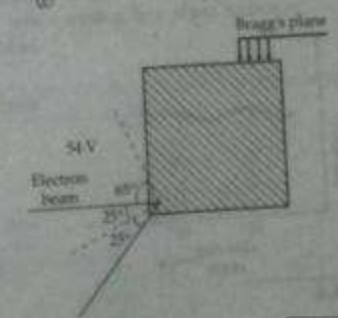

Fig. 5.5

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If the electron behaves as wave then according to Bragg's equation for maxima in the diffraction pattern

$$2d \sin\theta = n\lambda \quad ...(51)$$

where d is the interplanar distance for the (111) plane and n is the order of diffraction. For normal incident method, we have

$$2d \sin\theta = n\lambda \quad ...(52)$$

For nickel, $d = 0.91$ Å, distance between the Bragg planes, then

$$2 \times 0.91 \times 10^{-10} \sin 65° = 1 \times \lambda$$

or

$$\lambda = 1.65 \text{ Å} \quad ...(53)$$

Considering the electron to be a charged particle, the wavelength of the electron wave is given by

$$\lambda_e = \frac{12.28}{\sqrt{V}} \text{ Å} \quad ...(54)$$

Since, $V = 54$ V

$$\lambda_e = \frac{12.28}{\sqrt{54}} \text{ Å} = 1.66 \text{ Å} \quad ...(55)$$

The electron wavelength therefore agrees well with the observed wavelength of 1.66 Å. The Davisson-Germer experiment thus directly verifies to de Broglie hypothesis of the wave nature of moving bodies.

$$\frac{1}{\lambda} = \frac{n}{2d\sin\theta} = \frac{p}{h} = \frac{\sqrt{2mE}}{h} = \frac{\sqrt{2meV}}{h} \quad ...(56)$$

Electron wavelength | Bragg's Law | de Broglie relationship | Acceleration through voltage V

$$\frac{1}{\lambda} = \frac{n}{2d\sin\theta} = 0.815\sqrt{V}$$

NOTE

Interatomic distance for nickel is 2.15 Å. Thus

$d = 2.15 \sin 25° = 0.91$ Å

(ii) Oblique Incidence Method

In this method the accelerating voltage V_2 is maintained at a constant value and the galvanometer deflection at various glancing angles of incidence is noted. A graph of glancing angle and galvanometer deflection is shown in Fig. 5.6. It shows the various orders of maxima (humps).

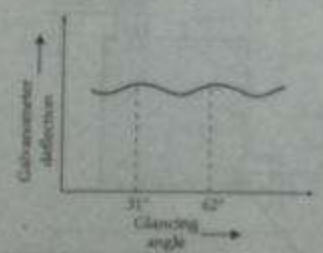

Fig. 5.6

Consider one such hump at 31° which occurs for 60 V accelerating potential. Another hump occurs at 62°.

According to Bragg's law, we have

$$2d \sin\theta_1 = n\lambda_e \quad ...(57)$$

and

$$2d \sin\theta_2 = (n+1)\lambda_e \quad ...(58)$$

Subtracting Eq. (57) from Eq. (58), we have

$$2d(\sin\theta_2 - \sin\theta_1) = \lambda_e \quad ...(59)$$

Substituting the value of d, θ_1 and θ_2 in Eq. (59), we get

$$\lambda_e = 2 \times 2.15(\sin 62° - \sin 31°) \text{ Å} = 4.30 \times 0.3679 \text{ Å}$$

$$= 2 \times 2.15(0.8829 - 0.5150) \text{ Å} = 4.30 \times 0.3679 \text{ Å}$$

$$\lambda_e = 1.582 \text{ Å} \quad ...(60)$$

We know,

$$\lambda = \frac{h}{\sqrt{2meV}} = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times 60}} = 1.55 \text{ Å}$$

This is in excellent agreement with observed value.

Example 5.5. An electron initially at rest is accelerated through a potential difference of 5000 V. Compute (i) momentum, (ii) the de Broglie wavelength, and (iii) the wave propagation vector of the electron. Also calculate the Bragg's angle for the first order reflection from the < 1 1 1 > planes of nickel which are 2.04 Å apart.

Solution. (i) Let v be the velocity of the electron then its K.E.

$$\frac{1}{2}mv^2 = Ve = 5000 \text{ eV} = 5000 \times 1.6 \times 10^{-19} \text{ J}$$

$$\therefore \quad (mv)^2 = 2m \times 5000 \times 1.6 \times 10^{-19} \text{ J}$$

and momentum of the electron

$$mv = \sqrt{(2m \times 5000 \times 1.6 \times 10^{-19} \text{ J})}$$

$$= \sqrt{2 \times 9.1 \times 10^{-31} \text{ kg} \times 5000 \times 1.6 \times 10^{-19} \text{ J}} \quad [\because m = 9.1 \times 10^{-31} \text{ kg}]$$

$$= \sqrt{9.1 \times 1.6 \times 10^{-46}} \text{ kg ms}^{-1}$$

$$= 10^{-23}\sqrt{9.1 \times 1.6} \text{ kg ms}^{-1} = 3.818 \times 10^{-23} \text{ kg ms}^{-1}$$

(ii) The de Broglie wavelength of the electron

$$\lambda = \frac{h}{mv} = \frac{6.625 \times 10^{-34} \text{ Js}}{3.818 \times 10^{-23} \text{ kg ms}^{-1}}$$

$$= 1.729 \times 10^{-11} \text{ m} = 0.1729 \text{ Å}$$

(iii) Here wave propagation vector is angular wave number is given by

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{1.729 \times 10^{-11} \text{ m}} = 36.34 \times 10^{10} \text{ m}^{-1}$$

(iv) Let θ be the Bragg's angle for the first order reflection from (1 1 1) plane of nickel (face centred cubic lattice). Here $d = 2.04$ Å ; $n = 1$ and $\lambda = 0.1729$ Å

Now, by Bragg's equation,

$$2d \sin\theta = n\lambda$$

$$2 \times 2.04 \text{ Å} \sin\theta = 1 \times 0.1729 \text{ Å}$$

or

$$\sin\theta = \frac{0.1729}{2 \times 2.04} = \frac{0.1729}{4.08} = 0.04237$$

$$\theta = 2°25'$$

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5.14 Heisenberg's Uncertainty Principle

We can not know the future because we can not know the present.

The new fundamental principle of indeterminacy also called uncertainty principle has been derived from the theoretical investigation of the wave probability associated with matter. We know that an electron in motion is associated with a wave whose wavelength is given by de Broglie relation. It can be considered as a wave packet formed by the superposition of a large number of waves of wavelengths slightly different from the wavelength of the associated wave. (The amplitude of the wave packet is zero everywhere in space except only over a region in which the particle (electron) is located). Since a wave packet is of finite width there will be uncertainty in specifying the position of electron. According to de Broglie's theory, the momentum of the electron will also be uncertain. Therefore, in general, it is not possible to determine position of the electron. If the momentum of electron is known, then by the de Broglie relation, the wavelength of the associated wave has a unique value i.e., the associated wave is monochromatic. In such a case the wavepacket has infinite length and hence the position of electron may be anywhere between minus infinity to plus infinity. On the basis of these considerations, Werner Heisenberg in 1927, enunciated the principle of Unbestimmtheit. This termed has been translated as uncertainty, indeterminacy or indefiniteness.

Proof of Heisenberg's Uncertainty Principle

Consider a particle moving along x-axis. It is considered as a wave packet. The envelope of the wave packet moves with a velocity equal to particle velocity. The wave packet moves with a velocity of equal to particle velocity. The wavelength is equal to the position of the particle. When the wave packet extends over a finite distance Δx along the axis of motion, then the two points at which the amplitude of the wave packet becomes zero successively will be separated by a distance Δx as shown in Fig. 5.7.

Fig. 5.7 Spatial extension of wave packet.

The points at which the amplitude of the wave packet becomes zero are called nodes. Due to the wave nature of matter, any measurement made to find the position of the particle will have a minimum error equal to the distance (Δx) between two successive nodes. Thus there is uncertainty in the position of the particle between two successive nodes.

The amplitude of the wave packet is given by

$$R = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right] \quad ...(61)$$

At nodes the,

amplitude of the wave packet is zero. Hence

$$0 = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right]$$

Since $2A \neq 0$.

Then

$$\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right] = 0$$

or

$$\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots (2n+1)\frac{\pi}{2}$$

If x_1 and x_2 are the two positions of two consecutive nodes, then

$$\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x_1 = (2n+1)\frac{\pi}{2} \quad ...(62)$$

and

$$\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x_2 = (2n+3)\frac{\pi}{2} = \pi + (2n+3)\frac{\pi}{2} \quad ...(63)$$

Subtracting Eq. (62) from Eq. (63), we get

$$\frac{\Delta k}{2}(x_2 - x_1) = \pi$$

or

$$(x_2 - x_1) = \frac{2\pi}{\Delta k} \quad \text{or} \quad \Delta x = \frac{2\pi}{\Delta k} \quad ...(64)$$

This is the fundamental error in the measurement of the position of the particle.

We know that propagation constant,

$$k = \frac{2\pi}{\lambda} = 2\pi\frac{p_x}{h} \quad \left[\because \lambda = \frac{h}{p_x}\right] \quad ...(65)$$

If Δp_x is the momentum of the particle along x-axis and h is Planck's constant, then

$$\Delta k = 2\pi\left(\frac{\Delta p_x}{h}\right) \quad ...(66)$$

where Δp_x is fundamental error in the measurement of the momentum of particle.

Substituting the value of Δk from Eq. (66) in the Eq. (64), we get

$$\Delta x = \frac{2\pi}{2\pi(\Delta p_x/h)} = \frac{h}{\Delta p_x} \quad ...(67)$$

According to Fourier analysis, the width Δx of a single wave represents the superposition of waves whose propagation constants vary in the range Δk.

It is given by

$$\Delta x = \frac{1}{\Delta k} \quad \text{or} \quad \Delta k = \frac{1}{\Delta x} \quad ...(68)$$

Eq. (68), is valid for superposition of only two wave trains. Hence Eq. (67) is also valid for a superposition of wave trains. Comparing Eqs. (66) and (68), we get

$$\frac{1}{\Delta x} = 2\pi\left(\frac{\Delta p_x}{h}\right)$$

or

$$\Delta x\,\Delta p_x = \frac{h}{2\pi} = \hbar \approx h \quad ...(69)$$

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If wave packets have shapes different from that visualised in Fig. 5.7, then the sign of equality is replaced by the sign ≥

$$\Delta x \, \Delta p_x \geq h$$

Hence it proves the uncertainty principle.

5.15 Statement of Heisenberg's Uncertainty Principle

Heisenberg was first, who pointed out the inherent relation in specifying the position and momentum of a particle regarded as a wave which is known as the **uncertainty principle**.

Heisenberg uncertainty principle states that *"In any simultaneous determination of the position and momentum of the particle, the product of the uncertainty Δx (or possible error) in the x-coordinate of a particle, in motion and uncertainty Δp_x in the x-component of momentum is of the order of or greater than $h(=1.054\times10^{-34}$ Js)"*

$$\Delta x . \Delta p_x \geq h$$

In other words, it is physically impossible to measure simultaneously the exact position and exact linear momentum of the particle.

5.16 Other forms of Heisenberg's Principle

The uncertainty relation can also be expressed in other forms.

If E is the energy of a quantum system at time t, then the uncertaintities ΔE and Δt are related by relation

$$\Delta E \, \Delta t \geq \frac{h}{2\pi} \qquad ...(70)$$

This is the energy-time uncertainty.

In terms of angular momentum L and angle θ, we can write

$$\Delta L \Delta\theta \geq \frac{h}{2\pi} \qquad ...(71)$$

This is the angular momentum angle uncertainty.

5.17 Experimental Proof of the Uncertainty Principle

The validity of Heisenberg's uncertainty principle may be illustrated by following hypothesis experiment.

(I) Determination of Position of a Particle by a Microscope :

Let us consider a process, where a particle, say an electron, is being observed by a microscope (Fig. 5.8). The minimum distance between two points which can be distinguished as separate by the microscope is given by

$$\Delta x = \frac{\lambda}{2\sin\theta} \qquad ...(72)$$

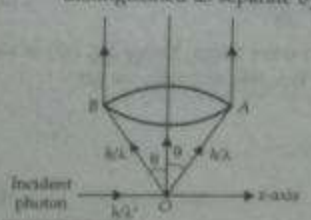

Fig. 5.8

Hence it represents the error or uncertainty in the measurement of position of the particle.

Now in order that the electron may be seen through the microscope, the incident photon must be scattered by the electron into the microscope objective. During scattering the electron recoils and suffers a change in

momentum. Let the incident photon has a wavelength λ' and hence a momentum h/λ'. It will scatter into the microscope with increased wavelength λ any where between OA and OB. In case it enters along OA, it's momentum along x-axis is given by $\frac{h}{\lambda}\sin\theta$. Hence the loss of momentum of the photon or the momentum imparted to the electron in the x-direction,

$$= \frac{h}{\lambda'} - \frac{h}{\lambda}\sin\theta \qquad ...(73)$$

If the scattered photon enters the microscope along OB, then the momentum imparted to the electron along x-direction

$$= \frac{h}{\lambda'} - \left(-\frac{h}{\lambda}\sin\theta\right)$$

$$= \frac{h}{\lambda'} + \frac{h}{\lambda}\sin\theta \qquad ...(74)$$

The momentum imparted to the electron can thus have any value between those given by Eqs. (73) and (74). Thus the error or the uncertainty in the measurement of momentum of the electron.

$$\Delta p = \left(\frac{h}{\lambda'} + \frac{h}{\lambda}\sin\theta\right) - \left(\frac{h}{\lambda'} - \frac{h}{\lambda}\sin\theta\right)$$

$$= \frac{2h}{\lambda}\sin\theta \qquad ...(75)$$

Multiplying Eqs. (72) and (75),

$$\Delta p \cdot \Delta x = \frac{\lambda}{2\sin\theta} \cdot \frac{2h}{\lambda}\sin\theta = h \qquad ...(76)$$

This is in accordance with Heisenberg's uncertainty principle.

(II) Diffraction of an Electron Beam by a Single Slit

Consider a narrow beam of electrons of momentum p passing through a narrow slit of width Δy (Fig. 5.9). Since the electron must pass through the slit, the width of the slit, Δy is a measure of uncertainty in the position of electron. Diffraction will occur and the diffraction pattern will be produced on screen as shown in Fig. 5.9. If we assume that the screen is enough for away relative to the width of the slit, the first minimum of the Fraunhofer diffraction pattern is obtained by putting $n=1$ in the equation $(d\sin\theta = n\lambda)$ describing the diffraction pattern due to single slit.

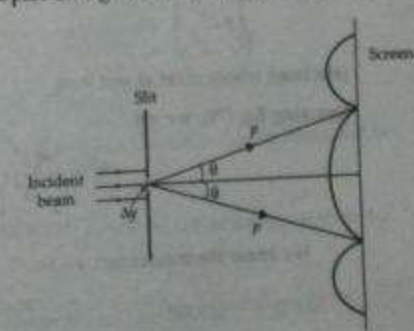

Fig. 5.9 Diffraction by a single slit.

Thus,

$$\Delta y \sin\theta = \lambda$$

or

$$\Delta y = \frac{\lambda}{\sin\theta} \qquad ...(77)$$

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The moving electrons in the beginning have no component of momentum along y-axis, since they are moving along x-axis. But at the slit, they deviate from their original path to form a pattern on the screen and hence have a component of momentum $p\sin\theta$ in the y-direction. Now as the electron may be anywhere within the pattern from angle $-\theta$ to $+\theta$, the y-component of momentum of electron may lie anywhere between $p\sin\theta$ and $-p\sin\theta$. Obviously, the uncertainty in the y-component of the momentum of electron is

$$\Delta p_y = p\sin\theta - (-p\sin\theta) = 2p\sin\theta$$

$$= 2\frac{h}{\lambda}\sin\theta \qquad ...(78)$$

Multiplying Eqs. (77) and (78), we have

$$\Delta y\,\Delta p_y = \frac{\lambda}{\sin\theta}\cdot\frac{2h}{\lambda}\sin\theta$$

$$= 2h$$

which is good agreement with uncertainity principle.

5.18 Applications of Uncertainty Principle

Many phenomena can be understood in terms of the uncertainty principle. A few of them are discussed here.

(i) Non-existing of Free Electrons in the Nucleus

According to theory of relativity, energy of a particle is given by the relation

$$E = mc^2 = \frac{m_0c^2}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} \qquad ...(79)$$

where $m = \dfrac{m_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$ is called relativistic mass and m_0c^2 is called rest energy of particle (electron) whose mass at rest is m_0.

Squaring Eq. (79), we get

$$E^2 = \frac{m_0^2c^4}{\left(1-\frac{v^2}{c^2}\right)} = \frac{m_0^2c^4}{c^2-v^2} \qquad ...(80)$$

We know the momentum $p = mv$

$$= \frac{m_0v}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} \qquad ...(81)$$

Squaring, we get

$$p^2 = \frac{m_0^2v^2}{\left(1-\frac{v^2}{c^2}\right)} = \frac{m_0^2v^2c^2}{(c^2-v^2)} \qquad ...(82)$$

Multiplying Eq. (82) by c^2, we get

$$p^2c^2 = \frac{m_0^2v^2c^4}{(c^2-v^2)} \qquad ...(83)$$

Substracting Eq. (83) from Eq. (80), we get

$$E^2 - p^2c^2 = \frac{m_0^2c^4}{c^2-v^2} - \frac{m_0^2v^2c^4}{c^2-v^2}$$

$$= \frac{m_0^2c^4(c^2-v^2)}{c^2-v^2} = m_0^2c^4$$

or

$$E^2 = p^2c^2 + m_0^2c^4 \qquad ...(84)$$

According to Heisenberg's uncertainty principle,

$$\Delta x\,\Delta p_x = \frac{h}{2\pi} \qquad ...(85)$$

The diameter of the nucleus is of the order of 10^{-14} m. If the electron exists in the nucleus, it can be anywhere within the diameter of the nucleus. Therefore, the maximum uncertainity Δx in the position of electron is the same as the diameter of the nucleus.

i.e.,

$$\Delta x = 10^{-14}\text{ m} \qquad ...(86)$$

∴ The minimum uncertainty in the momentum is given by

$$\Delta p_x = \frac{h}{2\pi\Delta x}$$

$$= \frac{6.63\times10^{-34}}{2\pi\times10^{-14}} = \frac{6.63\times10^{-20}}{2\pi}$$

$$= 1.055\times10^{-20}\text{ kg m/sec}$$

It means that if electron exists in the nucleus, its minimum momentum must be

$$p_{min} = 1.055\times10^{-20}\text{ kg m/s}$$

For the electron of the minimum momentum, the minimum energy is given by

$$E_{min}^2 = p_{min}^2c^2 + m_0^2c^4$$

$$= (1.055\times10^{-20}\times3\times10^8)^2 + (9.1\times10^{-31})^2(3\times10^8)^4$$

$$= (3\times10^8)^2[1.113\times10^{-40} + 7.4692\times10^{-44}]$$

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Since the second term in the bracket is much smaller than the first, it can be neglected, then,

$$E_{min} = 3\times10^8\sqrt{1.113\times10^{-40}}\ \text{J}$$

$$= 3\times10^8\times1.055\times10^{-20}\ \text{J} = 3.1649\times10^{-12}\ \text{J}$$

or

$$E_{min} = \frac{3.1649\times10^{-12}}{1.6\times10^{-19}}\ \text{eV} \simeq 20\ \text{MeV}.$$

Thus, if a free electron exists in the nucleus it must have a minimum energy of about 20 MeV.

The maximum K.E. which a β-particle, emitted from redioactive nuclei is of the order of 4 MeV. Therefore, this clearly shows that **electrons can not be present within the nucleus**.

ii) Spectral lines have a finite width

The average period that takes between the excitation of an atom and the time it radiates is 10^{-8} s. Thus the photon energy is uncertain by the amount

$$\Delta E = \frac{h}{\Delta t} = \frac{6.6\times10^{-34}\ \text{Js}}{10^{-8}\ \text{s}} = 6.6\times10^{-26}\ \text{J}$$

It means that excited energy levels have a finite energy spread. Thus when atoms fall back an excited level to ground level, the radiations emitted are not truely monochromatic.

Thus the energy levels of the atom must have a finite width or in other words the spectral lines can never be sharp but have a natural finite width.

iii) Radius of Bohr's First Orbit

If Δx and Δp are the uncertainty in determining the position and momentum of electron in Bohr's first orbit.

Then, $\Delta x\,\Delta p \approx \hbar$ or $\Delta p \approx \dfrac{\hbar}{\Delta x}$

Now kinetic energy,

$$K = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

Therefore, uncertainty in K.E.

$$\Delta K = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{2m(\Delta x)^2} \quad \text{where } \hbar = \frac{h}{2\pi}$$

Then uncertainty in P.E. of same electron,

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(-e)}{\Delta x} = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{\Delta x}$$

Therefore, uncertainty in total energy,

$$\Delta E = \Delta K + \Delta V = \frac{\hbar^2}{2m(\Delta x)^2} - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{\Delta x}$$

The value of energy E in fundamental state is minimum and the uncertainty in energy will be minimum when

$$\frac{d(\Delta E)}{d(\Delta x)} = 0$$

i.e.,

$$-\frac{\hbar^2}{m(\Delta x)^3} + \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{(\Delta x)^2} = 0 \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{(\Delta x)^2} = \frac{\hbar^2}{m(\Delta x)^3}$$

$$\therefore \quad \Delta x = 4\pi\varepsilon_0\frac{\hbar^2}{mZe^2}$$

Therefore, radius of first Bohr orbit,

$$r = \Delta x = 4\pi\varepsilon_0\frac{\hbar^2}{mZe^2} = 4\pi\varepsilon_0\frac{\left(\frac{h}{2\pi}\right)^2}{mZe^2}$$

$$r = \frac{\varepsilon_0 h^2}{\pi mZe^2} \qquad \text{...(87)}$$

Example 5.6. *Find the smallest possible uncertainty in position of the electron moving with velocity 3×10^7 m/s (Given $h = 6.63\times10^{-34}$ Js, $m_0 = 9.1\times10^{-31}$ kg).* **[GGSIPU, May 2007 (2.5 marks)]**

Solution. Given $v = 3\times10^7$ m/s

Let Δx_{min} be the minimum uncertainty in position of the electron and Δp the maximum uncertainty in the momentum of the electron.

Thus we have,

$$\Delta x_{min}\cdot\Delta p_{max} = \frac{h}{2\pi} \qquad \text{...(i)}$$

or

$$\Delta p_{max} = p = mv$$

$$\Delta p_{max} = \frac{m_0 v}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} \qquad \text{...(ii)}$$

$$\Delta x_{min} = \frac{h\sqrt{\left(1-\frac{v^2}{c^2}\right)}}{2\pi m_0 v}$$

$$= \frac{6.63\times10^{-34}\sqrt{1-\left(\frac{3\times10^7}{3\times10^8}\right)^2}}{2\times3.14\times9.1\times10^{-31}\times3\times10^7}\ \text{m}$$

$$= 0.03867\times0.9949\times10^{-10}\ \text{m}$$

$$= 3.8\times10^{-12}\ \text{m}.$$

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Example 5.7. A microscope, using photons, is employed to locate an electron in an atom within a distance of 0.2 Å. What is the uncertainty in the momentum of the electron located in this way ?

Solution. Given $\Delta x = 0.2$ Å $= 2 \times 10^{-11}$ m ; $\Delta p = ?$

Since we know that the uncertainty principle

$$\Delta x \Delta p \approx \frac{h}{2\pi} \quad \text{or} \quad \Delta p = \frac{h}{2\pi \Delta x}$$

$$\therefore \quad \Delta p = \frac{6.626 \times 10^{-34}}{2 \times 3.14 \times (0.2 \times 10^{-10})}$$

$$= 5.27 \times 10^{-24} \text{ kg ms}^{-1}.$$

Hence uncertainty in momentum $(\Delta p) = 5.27 \times 10^{-24}$ kg ms^{-1}.

Formulae at a Glance

1. Dual Nature of Matter.

The wavelength $(\lambda) = \frac{h}{p} = \frac{\text{Planck's constant}}{\text{momentum}}$

2. de Broglie wavelength of an electron

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{12.28}{\sqrt{V}} \text{ Å}$$

3. Phase Velocity,

$$v_p = \frac{\omega}{k}$$

$$= \frac{\text{Angular frequency } (2\pi\nu)}{\text{Propagation constant}}$$

4. Group Velocity,

$$v_g = \frac{d\omega}{dk}$$

5. Relation between v_p and v_g

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda}$$

6. Relation between v_g and particle velocity $v_{particle}$

$$v_g = v_{particle} = \frac{dE}{dp}$$

7. Relation between velocity of light (c), v_p and v_g

$$v_p v_g = c^2$$

8. In Davisson and Germer experiment,

(i) $2d \sin\theta = n\lambda$

$\Rightarrow \quad \lambda = 1.65$ Å ; d = interplanar distance

(ii) $\lambda_e = \frac{12.28}{\sqrt{V}}$ Å

$= 1.66$ Å.

9. Heisenberg's uncertainty principle,

(i) $\Delta x \Delta p_x \approx \frac{h}{2\pi} = \hbar$

$\Delta y \Delta p_y \approx \frac{h}{2\pi} = \hbar \qquad \Delta x \Delta p_x \geq \hbar$

$\Delta z \Delta p_z \approx \frac{h}{2\pi} = \hbar$

(ii) $\Delta E \cdot \Delta t \geq \hbar \qquad E$ = energy

(iii) $\Delta L \Delta\theta \geq \frac{h}{2\pi} \qquad L$ = angular mometum

$\Delta\theta$ = angular change

10. Experimental Proof of Uncertainty principle :

(i) Det. of position of particle by a microscope

$$\Delta x = \frac{\lambda}{2\sin\theta}; \quad \Delta p_x = \frac{2h}{\lambda}\sin\theta$$

$$\Delta p_x \cdot \Delta x = \frac{2h}{\lambda}\sin\theta \frac{\lambda}{2\sin\theta} = h$$

(ii) Diffraction of an electron beam by a single slit

$$\Delta y \sin\theta = \lambda$$

$$\Delta y = \frac{\lambda}{\sin\theta}$$

$$\Delta p_y = p\sin\theta - (-p\sin\theta)$$

$$= 2p\sin\theta$$

$$= 2\frac{h}{\lambda}\sin\theta$$

$$\Delta y \Delta p_y = \frac{\lambda}{\sin\theta} \cdot \frac{2h}{\lambda}\sin\theta$$

$$= 2h$$

11. Applications of Uncertainty Principle

(i) Non-existence of free electrons in the nucleus

$$E^2 = p^2c^2 + m_0^2c^4, \quad \Delta x \Delta p_x = \hbar$$

(ii) Spectral lines have a finite width

$$\Delta E = \frac{h}{\Delta t}$$

$$= 6.6 \times 10^{-28} \text{ J}$$

(iii) Radius of Bohr's first orbit

$$\Delta x \Delta p = \hbar$$

$$r = \Delta x = \frac{\varepsilon_0 h^2}{\pi m Z e^2}$$

Miscellaneous Numerical Problems for Practice

Example 1. The de Broglie wavelength of an electron is 73 Å. What is its velocity ? $m_e = 9.1 \times 10^{-31}$ kg and $h = 6.6 \times 10^{-34}$ Js.

Solution. $\lambda = \frac{h}{mv}$ or $v = \frac{\lambda}{mh}$

$$v = \frac{6.6 \times 10^{-34}}{9.1 \times 10^{-31} \times 73 \times 10^{-10}} \text{ m/s}$$

$$= 9.93 \times 10^4 \text{ m/s}$$

Example 2. Calculate the de Broglie wavelength in Å associated with proton of kinetic energy 100 eV. (Mass of proton $= 16725 \times 10^{-27}$ kg and $h = 6.625 \times 10^{-34}$ Js).

Solution. The kinetic energy is given by

$$\frac{1}{2}mv^2 = 100 \text{ eV}.$$

$$= 100 \times 1.6 \times 10^{-19} \text{ J}$$

$$v = \sqrt{\frac{2 \times 100 \times 1.6 \times 10^{-19} \text{ J}}{m}}$$

and momentum $\quad mv = \sqrt{200 \times 1.6 \times 10^{-19} \text{ Jm}}$

$$= \sqrt{200 \times 1.6 \times 10^{-19} \text{ J} \times 16725 \times 10^{-27} \text{ kg}}$$

$$= 10^{-23}\sqrt{320 \times 16725} \text{ kg ms}^{-1}.$$

Now, we know that

$$\lambda = \frac{h}{mv}$$

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Substituting the values of h and mv, we have

$$\lambda = \frac{6.625\times10^{-34}\ \text{J s}}{10^{-25}\sqrt{(320\times16725)}\ \text{kg ms}^{-1}}$$

$$= \frac{6.625\times10^{-10}}{\sqrt{320\times16725}} = \frac{6625}{2313}\times10^{-12}$$

$$= 2.864\times10^{-12}\ \text{m} = 0.02864\times10^{-10}\ \text{m}$$

$$\lambda = 0.0286\ \text{Å}$$

Example 3. *Calculate on the basis of wave mechanic theory the wavelength associated with (a) an electron moving with speed of the electron in the K-ring of the hydrogen atom, (b) a molecule of the hydrogen at 0° C (its velocity = 184×10^3 m/s).*

Solution. (*a*) The speed of the electron in the *K*-ring of hydrogen atom

$$v = 2.18\times10^6\ \text{m/s}$$

$$m = 9.1\times10^{-31}\ \text{kg}$$

$$\therefore \quad \lambda = \frac{h}{mv}$$

$$= \frac{6.625\times10^{-34}}{9.1\times10^{-31}\ \text{kg}\times2.18\times10^6\ \text{ms}^{-1}}$$

$$= 3.33\times10^{-10}\ \text{m} = 3.33\ \text{Å}$$

(*b*) $\lambda = \dfrac{h}{Mv}$

The mass of a hydrogen molecule

$$M = 2\times1.67\times10^{-27}\ \text{kg}$$

and $$v = 184\times10^3\ \text{m/s}$$

$$\therefore \quad \lambda = \frac{6.625\times10^{-34}\ \text{Js}}{2\times1.67\times10^{-27}\ \text{kg}\times1.84\times10^3\ \text{ms}^{-1}}$$

$$= \frac{6625}{334\times1.84}\times10^{-10}\ \text{m}$$

$$= 1.07\times10^{-10}\ \text{m} = 1.07\ \text{Å}$$

Example 4. *Find the de Broglie wavelength of a 10 keV electron, using SI units.*

Solution. Given $V = 10\ \text{keV} = 10^4\ \text{V}$

The kinetic energy of the electron,

$$K = \frac{1}{2}mv^2 = Ve$$

$$\therefore \quad m^2v^2 = 2mVe$$

or momentum $$mv = \sqrt{2mVe}$$

$$= \sqrt{2m\times10^4 e}$$

Substituting the values of m and eV, we get

$$mv = \sqrt{2\times9.1\times10^{-31}\ \text{kg}\times10^4\times1.6\times10^{-19}\ \text{J}}$$

$$= 10^{-23}\sqrt{2\times9.11\times16}\ \text{kg ms}^{-1}$$

∴ The de Broglie wavelength,

$$\lambda = \frac{h}{mv}$$

$$= \frac{6.625\times10^{-34}\ \text{Js}}{10^{-23}\sqrt{2\times9.1\times1.6}\ \text{kg ms}^{-1}}$$

$$= 1.228\times10^{-11}\ \text{m} = 0.1228\ \text{Å}$$

Example 5. *Show that the de Broglie wave velocity is a function of wavelength even in free space.*

Solution. If a particle of mass m is moving with a velocity v then its momentum $p = mv$ and the total energy $E = mc^2$, using relativistic relationship of energy and momentum

$$E^2 = p^2c^2 + m_0^2c^4$$

$$\Rightarrow \quad m_0^2 = \frac{E^2}{c^4} - \frac{p^2}{c^2}$$

From de Broglie wave equation $p = \dfrac{h}{\lambda}$ and $E = h\nu$

$$\therefore \quad m_0 = \frac{h}{c}\sqrt{\left(\frac{\nu^2}{c^2} - \frac{1}{\lambda^2}\right)} = \frac{h}{c}\sqrt{\frac{v_p^2}{c^2\lambda^2} - \frac{1}{\lambda^2}} \quad ; \quad v_p = \nu\cdot\lambda$$

or $$v_p = \sqrt{\left(1 + \frac{m_0^2c^2}{h^2}\lambda^2\right)}$$

This expression shows that for a particle of non-zero rest mass the wave velocity always greater than c and is a function of λ even in free space.

Example 6. *Can a photon and an electron of the same momentum have the same wavelength ? Compare their wavelengths of the two having the same energy.*

Solution. According to de Broglie wave equation,

$$\lambda = \frac{h}{p}$$

Therefore, for two particles electron and photon of same momentum, wavelength will be same.

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Now,

For photon $\lambda_p = \frac{h}{p} = \frac{h}{mc} = \frac{hc}{mc^2} = \frac{hc}{E_p}$ ...(i) $\left[\because E_p = h\nu = \frac{hc}{\lambda} = \frac{hc}{h/p} = \frac{hc}{h/mc} = mc^2\right]$

For electron, $\lambda_e = \frac{h}{mv} = \frac{h}{\sqrt{2mE_e}}$ ...(ii)

If the electron and photon have the same energy

i.e., $E_p = E_e = E$ (say)

Then Eqs. (i) and (ii),

$$\frac{\lambda_p}{\lambda_e} = \frac{hc}{E} \times \frac{\sqrt{2mE}}{h} = \sqrt{\frac{2mc^2}{E}}$$

Example 7. *A particle of rest mass m_0 has a kinetic energy K, what will be the value of λ if $K << m_0c^2$?*

Solution. The relativistic kinetic energy of a particle K is given by $K = E - E_0$, where E is the total energy given by $E = \sqrt{(p^2c^2 + m_0^2c^4)}$ and E_0 is the rest mass energy given by,

$$E_0 = m_0c^2.$$

$$\therefore \quad K = E - E_0 = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2$$

or $$p^2c^2 + m_0^2c^4 = (K + m_0c^2)^2 = K^2 + m_0^2c^4 + 2m_0c^2 \cdot K$$

or $$p = \frac{1}{c}\sqrt{K(K + 2m_0c^2)}$$

From de Broglie wave equation, we have

$$\lambda = \frac{h}{p}$$

or $$\lambda = \frac{hc}{\sqrt{K(K + 2m_0c^2)}}$$

Example 8. *A certain excited state of hydrogen atom is known to have a life of 2.5×10^{-4} s. What is the minimum error, with which the energy of the excited state can be measured.*

Solution. The energy time uncertainty relation is

$$\Delta E \cdot \Delta t \geq \hbar$$

$$\therefore \quad \Delta E = \frac{h}{2\pi\Delta t}$$

$$= \frac{6.62 \times 10^{-34}}{2 \times 3.14 \times 2.5 \times 10^{-14}} \text{ J} = 4.2 \times 10^{-21} \text{ J}$$

or $$\Delta E = \frac{4.2 \times 10^{-21}}{1.6 \times 10^{-19}}$$

$$\Delta E = 0.0262 \text{ eV}.$$

Example 9. *Compute the typical de Broglie wavelength of an electron in a metal at 27° C and compare it with the mean separation between two electrons in a metal which is about 2 Å.*

Solution. Given $T = 27°\text{C} = 300$ K

Energy of the electron at room temperatuie, $E = \frac{3}{2}k_BT$, where k_B is Boltzmann constant and $k_B = 1.38 \times 10^{-23}$ J mol^{-1} k^{-1}.

$$E = \frac{1}{2}mv^2 = \frac{3}{2}k_BT$$

or $$v = \sqrt{\frac{3k_BT}{m}}$$

de Broglie wavelength $\lambda = \frac{h}{mv} = \frac{h}{m\sqrt{(3k_BT)/m}} = \frac{h}{\sqrt{(3k_BTm)}}$

$$\therefore \quad \lambda = \frac{6.63 \times 10^{-34}}{\sqrt{3 \times 1.38 \times 10^{-23} \times 300 \times 9.1 \times 10^{-31}}}$$

$$= 6.26 \times 10^{-9} \text{ m} = 62.6 \text{ Å}$$

The separation between two electrons in metal $d = 2$ Å.

So, $\lambda >> d$.

Example 10. *Give an account of experimental evidence which demonstrates the wave like properties of moving electrons. Calculate the glancing angle at which electrons of energy 100 eV must be incident on the lattice planes of a metal crystal in order to give a strong Bragg reflection in the first order. Use the following data : lattice spacing = 2.15 Å, $m_e = 9.1 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C and $h = 6.63 \times 10^{-34}$ Js.*

Solution. Given $E = 100$ eV

Bragg's equation is $n\lambda = 2d \sin\theta$ ...(i)

de Broglie wavelength $\lambda = \frac{h}{\sqrt{2emV}}$ ...(ii)

$$\sin\theta = \frac{n\lambda}{2d} = \frac{n}{2d}\frac{h}{\sqrt{(2emV)}}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times 2.15 \times 10^{-10}\sqrt{2 \times 1.6 \times 10^{-19} \times 9.1 \times 10^{-31} \times 100}} = 0.2847$$

or $$\theta = 16°32'$$

Example 11. *An electron has a speed 500 m/s accurate to 0.01% with what fundamental accuracy can we locate the position of the electron ?*

Solution. Given : $v = 500$ m/s, $\Delta v = 0.01\%$ of v

$$\therefore \quad \Delta v = \frac{0.01}{100} \times 500 = 0.05 \text{ m/s}$$

$$= 5.0 \times 10^{-2} \text{ m/s}$$

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We know that the uncertainty relation is

$$\Delta x \cdot \Delta p_x \geq \frac{h}{2\pi}$$

∴ For the given uncertainty in speed, the minimum uncertainty in the position is given by

$$\Delta x = \frac{h}{2\pi p_x} = \frac{h}{2\pi m \Delta v}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 9.1 \times 10^{-31} \times 5 \times 10^{-2}} = 2.316 \times 10^{-3} \text{ m}.$$

Example 12. *Electrons of 400 eV are diffracted through a crystal and a second order maximum is observed where the angle between the diffracted beam and incident beam is 30°. Calculate :*

(i) the wavelength of the electron matter wave.

(ii) the interplanar distance of those lattice planes which are responsible for this maximum.

Solution. (i) $\lambda = \frac{h}{(2m_0 E)^{1/2}}$

$$= \frac{6.626 \times 10^{-34} \text{ J s}}{(2 \times 9.1 \times 10^{-31} \text{ kg} \times 400 \times 1.6 \times 10^{-19} \text{ J})^{1/2}}$$

$$= 0.61 \times 10^{-10} \text{ m} = 0.61 \text{ Å}.$$

(ii) $n\lambda = 2d \sin\theta$

$$n = 2, \ \theta = 30°, \ \lambda = 0.61 \text{ Å} = 0.61 \times 10^{-10} \text{ m}$$

$$\therefore \quad d = \frac{\lambda}{\sin\theta} = \frac{0.61 \times 10^{-10}}{\sin 30°} = 2 \times 0.61 \times 10^{-10} = 1.22 \times 10^{-10} \text{ m} = 1.22 \text{ Å}.$$

Example 13. *A ball of mass 10^{-3} kg moves with a velocity of 10^{-2} ms^{-1}. What is the de Broglie wavelength of the ball ?*

Solution. The de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.626 \times 10^{-34} \text{ Js}}{10^{-3} \text{kg} \times 10^{-2} \text{ ms}^{-1}} = 6.626 \times 10^{-29} \text{ m}.$$

Example 14. *Calculate the de Broglie wavelength of a 100 eV electron and a 1 MeV neutron, using the formula* $\lambda = \frac{h}{p} = \frac{h}{(2m_0 E)^{1/2}}$.

Solution. (i) The de Broglie wavelength of electron

$$\lambda_e = \frac{h}{(2m_0 E)^{1/2}} = \frac{6.626 \times 10^{-34} \text{ J s}}{\sqrt{(2 \times 9.1 \times 10^{-31} \text{ kg} \times 100 \times 1.6 \times 10^{-19} \text{ J})}}$$

$$= 1.228 \times 10^{-10} \text{ m} = 1.228 \text{ Å}.$$

(ii) The de Broglie wavelength of neutron,

$$\lambda_n = \frac{h}{(2m_0 E)^{1/2}}$$

$$= \frac{6.626 \times 10^{-34} \text{ J s}}{(2 \times 1.675 \times 10^{-27} \text{ kg} \times 10^6 \times 1.6 \times 10^{-19})^{1/2}}$$

$$= 2.862 \times 10^{-14} \text{ m}.$$

Example 15. *The energy of a free electron including its rest mass energy is 1 MeV. Calculate the group velocity and the phase velocity of the wave packet associated with the motion of the electron.*

Solution. Phase velocity $v_p = \frac{c^2}{v_g}$ and $v_g = \frac{p}{m}$

where $\quad p = (m^2c^2 - m_0^2c^2)^{1/2} = (m^2 - m_0^2)^{1/2} c$

and $\quad m = \frac{E}{c^2} = \frac{10^6 \times 1.6 \times 10^{-19} \text{ J}}{9 \times 10^{16} \text{ m}^2\text{s}^{-2}} = 1.778 \times 10^{-30} \text{ kg}.$

Therefore,

$$p = [(1.78)^2 - (9.1)^2]^{1/2} \times 10^{-31} \times 3 \times 10^8 \text{ kg ms}^{-1}$$

$$= 4.58 \times 10^{-22} \text{ kg ms}^{-1}.$$

$$\therefore \quad v_g = \frac{4.58 \times 10^{-22} \text{ kg ms}^{-1}}{1.778 \times 10^{-30} \text{ kg}} = 3.5 \times 10^8 \text{ ms}^{-1}$$

and $\quad v_p = \frac{9}{2.576} \times 10^8 \text{ ms}^{-1} = 3.5 \times 10^8 \text{ ms}^{-1}.$

Example 16. *(a) The average life time of an excited atom is about 10^{-8} s. What is the order of the natural width ($\Delta\nu$) of the line emitted by the atoms ?*

(b) The radius of an atomic nucleus is typically 5×10^{-15} m. What is the lower limit of the energy that an electron must have to be in the atomic nucleus ?

Solution. (a) The order of the natural line width is

$$\Delta\nu = \frac{1}{2\pi\Delta t} = \frac{10^8}{2\pi} \text{ Hz} = 16 \times 10^7 \text{ Hz}.$$

(b) The uncertainty in the electron's position is

$$\Delta x = 5 \times 10^{-15} \text{ m}.$$

Therefore, $\quad \Delta p \geq \frac{h}{\Delta x} \geq \frac{6.626 \times 10^{-34} \text{ Js}}{2\pi \times 5 \times 10^{-15} \text{ m}}$

$$\geq 2.11 \times 10^{-20} \text{ kg ms}^{-1}.$$

The momentum would also be of the same order if this is uncertainty in it. This suggests that the kinetic energy of the electron is far greater than its rest energy and we can write,

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K.E. = pc so that

$$K.E. = pc \geq (2.11 \times 10^{-20}\ \text{kg ms}^{-1})$$
$$\geq 633 \times 10^{-12}\ \text{J} \geq 39\ \text{MeV}.$$

Thus the K.E. of an electron must exceed 39 MeV for it to be a nuclear constituent.

Experiment indicate that electrons in an atom have only a fraction of this energy. Thus we can conclude that electrons are not present in atomic nuclei.

Example 17. *Show that the uncertainty principle can be expressed in the form $\Delta L \Delta\theta \geq h$, where ΔL is the uncertainty in angular momentum of the particle and $\Delta\theta$ is the uncertainty in its angular position.*

Solution. Consider a particle moving in a circle of radius r. If Δr is the arc length corresponding to angular position $\Delta\theta$, then we have

$$\Delta r \Delta p_r \geq h$$
$$\Rightarrow \quad r\Delta\theta m\Delta v \geq h$$
or
$$\Delta\theta m r \Delta v \geq h$$

But $L = mvr$ for the particle and $\Delta L = m\Delta vr$, since m and r are constant. Hence we obtain,

$$\Delta L \Delta\theta \geq h$$

Example 18. *Consider that a microscope object is moving along the x-axis and the uncertainties in its position are Δx_0 and Δx respectively, at $t = 0$ and $t = t$. Show that Δx is directly proportional to t and inversely proportional to Δx_0. From this problem what do you learn about the spreading of the waves associated with the motion of an object ?*

Solution. If v_g is the group velocity of the wave packet associated with the microscope particle then at time t

$$\Delta x = v_g t = \frac{p_0}{m} t = \frac{h}{\lambda_0 m} t$$

where λ_0 is the initial wavelength of the wavepacket at time $t = 0$. This is equal to Δx_0, the uncertainty in the particle's position at time $t = 0$. Thus we have,

$$\Delta x = \frac{h}{m} \frac{t}{\Delta x_0}$$

This result tells us that Δx *i.e.*, the spread of the wavepacket increases with time. The narrower the packet is initially, the quicker it spreads. This is hidden influence of the uncertainty principle. If the confinement length Δx is small, the uncertainty in its momentum and hence, its velocity is large $\left(\Delta v = \frac{h}{m\Delta x_0}\right)$. This means that the wavepacket will contain many waves of high velocity much greater than the average group velocity p_0/m. Due to the fluctuation in velocity, the distance covered by the particle will also be uncertain by an amount $\Delta x(t)$ *i.e.*, its spread will be large.

Example 19. *Derive Bohr's angular momentum quantization condition for the Bohr atom from de Broglie relation.*

Solution. de Broglie visualised that atomic electrons were confined waves and, therefore, produced a discrete stationary wave pattern. Then only those orbits would be

allowed for which an integral number of electron wavelengths could fit the circumference (Fig. 5.10). For example, one wavelength would fit the circumference of the lowest atomic orbit and two or more electron wavelengths would fit into higher orbits. Thus if de Broglie waves of wavelength λ fit a Bohr orbit or radius r to satisfy the stationary condition, we must have

$$2\pi r = n\lambda \qquad n = 1, 2, 3, \ldots$$

Since, $\lambda = \frac{h}{p} = \frac{h}{mv}$, we get

$$\frac{2\pi r m v}{h} = n$$

or
$$mvr = \frac{nh}{2\pi}$$

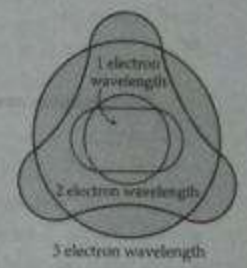

Fig. 5.10

which is the Bohr angular momentum quantisation.

Example 20. *A linear harmonic oscillator of mass m oscillates with a frequency $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is its force constant. Use the uncertainty principle to show that the minimum energy of the oscillator is $\frac{h\nu}{2}$.*

Solution. The energy of the linear harmonic oscillator is

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

This is a constant of motion. We can represent the constant value of E by means of averages of the kinetic and potential over a cycle of motion by writing.

$$E = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}k\langle x^2 \rangle$$

The average values of x and p should vanish for an oscillating particle. So, we can identify $\langle p^2 \rangle$ and $\langle x^2 \rangle$ with the squares of the corresponding uncertainties.

$$\langle x^2 \rangle = \langle x \rangle^2 + \langle \Delta x \rangle^2 = (\Delta x)^2$$

and
$$\langle p^2 \rangle = \langle p \rangle^2 + \langle \Delta p \rangle^2 = \left(\frac{h}{2\Delta x}\right)^2$$

Thus,
$$E = \frac{(\Delta p)^2}{2m} + \frac{1}{2}k(\Delta x)^2 = \frac{h^2}{8m(\Delta x)^2} + \frac{k}{2}(\Delta x)^2$$

Since from the uncertainty principle $\Delta x \Delta p \geq \frac{h}{2}$. To determine the minimum energy of the oscillator, we put

$$\frac{dE}{d(\Delta x)} = 0$$

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or $$-\frac{\hbar^2}{4m(\Delta x)^3}+k(\Delta x)=0$$

or $$(\Delta x)^2=\left(\frac{\hbar^2}{4mk}\right)^{1/2}$$

The minimum energy is

$$E_{min}=\frac{\hbar^2}{8m}\left(\frac{4mk}{\hbar^2}\right)^{1/2}+\frac{1}{2}k\left(\frac{\hbar^2}{4mk}\right)^{1/2}$$

$$=\frac{\hbar}{2}\left(\frac{k}{m}\right)^{1/2}+\frac{\hbar}{4}\left(\frac{k}{m}\right)^{1/2}$$

$$=\frac{\hbar}{2}\left(\frac{k}{m}\right)^{1/2}=\frac{h}{2(2\pi)}\left(\frac{k}{m}\right)^{1/2}$$

or $$E_{min}=\frac{h\nu}{2} \qquad \left[\because \nu=\frac{1}{2\pi}\left(\frac{k}{m}\right)^{1/2}\right]$$

Question Bank

Multiple Choice Questions

1. The de Broglie wavelength associated with an electron of mass m and accelerated by a potential V is
   (a) $\frac{h}{\sqrt{2mVe}}$ (b) $\frac{\sqrt{2mVe}}{h}$ (c) $\frac{h}{Vem}$ (d) $\frac{h}{2Vem}$
2. The wave property of large, massive objects is not observed because,
   (a) their acceleration of too small (b) their moments are too large
   (c) their speeds are too small (d) none of these.
3. Davisson-Germer experiment is
   (a) to show that the electron is negatively charged particle
   (b) to show that the cathode rays contain electrons.
   (c) to show that the wave nature of matter.
   (d) none of these.
4. In the Davisson-Germer experiment the hump was most prominent when the electron was accelerated by
   (a) 34 volts (b) 68 volts (c) 54 volts (d) none of these.
5. The de Broglie wavelength of the thermal neutron at T degree kelvin is
   (a) $\frac{2mk_BT}{h}$ (b) $\frac{h}{3mk_BT}$ (c) $\frac{\sqrt{(6mk_BT)}}{h}$ (d) $\frac{h}{\sqrt{(3mk_BT)}}$

6. An electron, a neutron and a proton have the same de Broglie wavelength ; which particle has greater velocity ?
   (a) electron (b) proton (c) neutron (d) none of these.
7. The wavelength associated with a moving particle
   (a) depends upon the charge associated with it
   (b) does not depend upon the charge associated with it.
   (c) depends upon the medium in which particle travels
   (d) none of these.
8. If momentum of an electron is doubled, the de Broglie wavelength is
   (a) halved (d) doubled (c) remain constant (d) none of these.
9. If an electron is accelerated by a potential of 100 V, then the wavelength associated with electron is
   (a) 1.1225 nm (b) 0.1225 Å (c) 12.25 nm (d) 0.01225 nm.
10. Davisson and Germer were the first to demonstrate
    (a) the straight line propagation of light. (b) the diffraction of photons
    (c) the effective mass of electrons (d) none of these.
11. The de Broglie wavelength λ associated with an electron taking into account the relativistic correction is
    (a) $\frac{h}{2m_0Ve}\left[1-\frac{Ve}{m_0c^2}\right]$ (b) $\frac{h}{\sqrt{2m_0Ve}}\left[1-\frac{Ve}{m_0c^2}\right]$
    (c) $\frac{2m_0Ve}{h}\left[1-\frac{Ve}{m_0c^2}\right]$ (d) $\frac{2m_0Ve}{\sqrt{h}}\left[1-\frac{Ve}{m_0c^2}\right]$
12. A proton and an α-particle have the same kinetic energy. If the mass of the α-particle is four times that of a proton, how do their de Broglie wavelengths compare ?
    (a) $\lambda_p=\frac{\lambda_\alpha}{2}$ (b) $\lambda_p=\frac{\lambda_\alpha}{2}$ (c) $\lambda_p=\lambda_\alpha$ (d) $\lambda_p=4\lambda_\alpha$
13. The accelerating potential V used to get an electron of wavelength λ is
    (a) $\frac{1}{2me}\left(\frac{h}{\lambda}\right)^2$ (b) $\frac{1}{2me}\left(\frac{h}{\lambda}\right)$ (c) $\frac{2me}{h}$ (d) $\frac{h}{2me}$
14. Simultaneous determination of exact position and momentum of a moving particle is impossible, it is
    (a) the uncertainty principle (b) the laser principle
    (c) the Einstein principle (d) none of these.
15. If Δx is the error in determining the position of an electron and Δp is the error in determining its momentum, then $(\Delta x)(\Delta p)$ is equal to
    (a) $\frac{2\pi}{h}$ (b) $(2\pi h)$
    (c) $\frac{h}{2\pi}$ (d) none of these.
16. Electron behaves as wave because they can be
    (a) deflected by an electric field. (b) deflected by a magnetic field.
    (c) diffracted by a crystal. (d) they ionise a gas.

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17. An electron has a speed of 100 m/s with an accuracy of 0.005%. The uncertainty in position will be
(a) 0.14 m (b) 1.4 m (c) 0.014 m (d) 14 m.

18. The duration of a laser pulse is 10^{-8} sec. The uncertainty in its energy will be
(a) 6.6×10^{-26} J (b) 6.6×10^{-34} J
(c) 6.6×10^{-42} J (d) $\frac{1}{6.6} \times 10^{28}$ J.

19. An electron of mass m, when accelerated through a potential difference V has de Broglie wavelength λ. The de Broglie wavelength associated with a proton of mass M accelerated through the same potential difference will be
(a) $\lambda(M/\mu)$ (b) $\lambda\sqrt{(m/M)}$
(c) $\lambda(M/m)$ (d) $\lambda\sqrt{(M/m)}$

20. A particle of mass m at rest decays into two particles of masses m_1 and m_2 having non-zero velocities. The ratio of de Broglie wavelength of the particles $\frac{\lambda_1}{\lambda_2}$ is
(a) $\frac{m_1}{m_2}$ (b) $\frac{m_2}{m_1}$
(c) $\sqrt{\frac{m_2}{m_1}}$ (d) 1.0

21. A moving particle is associated with a packet or group of waves. The group velocity is equal to
(a) velocity of particle (b) velocity of sound
(c) velocity of light (d) $\frac{1}{\text{particle velocity}}$

Answers

1. (a)	2. (b)	3. (c)	4. (c)	5. (d)	6. (d)
7. (b)	8. (c)	9. (a)	10. (d)	11. (b)	12. (d)
13. (a)	14. (a)	15. (c)	16. (c)	17. (d)	18. (a)
19. (b)	20. (d)	23. (a)			

Theoretical Questions

1. The uncertainty relation $\Delta x \Delta p_x \approx \hbar$ is correct or incorrect. Justify. [GGSIPU, May 2008 (2.5 marks)]
2. Discuss the main findings of Davisson-Germer experiment.
3. What is de Broglie hypothesis ? Show that the wavelength of an electron beam accelerated by a potential V volts is
$$\frac{h}{\sqrt{2Vem_0}}\left(1+\frac{Ve}{m_0c^2}\right)^{-1/2}$$ applying relativistic correction.
[GGSIPU, May 2006 (7 marks)]

4. How does Heisenberg's uncertainty principle help us to decide whether electron exist in the nucleus of not ? [GGSIPU, May 2006 (5.5 marks)]
5. State de Broglie theory of matter waves. Give the experimental verification for this hypothesis. [GGSIPU, May 2007 (4 marks)]
6. What do you understand by wave packet ? Explain the difference between phase velocity and group velocity. Show that the de Broglie group velocity associated with the wave packet is equal to the velocity of the particle. [GGSIPU, May 2007 (5 marks)]
7. State Heisenberg's uncertainty principle and use it to prove the non-existence of electron inside the nucleus. [GGSIPU, May 2007 (3.5 marks)]
8. Write a short note on 'experimental verification of uncertainty principle'. [GGSIPU, May 2007 (2.5 marks)]
9. Explain how matter waves differ from light and sound waves ? [GGSIPU, May 2007 (2 marks)]
10. State Heisenberg's unceratainty principle. Give its experimental verification and use it to prove the non-existence of electrons in the nucleus. [GGSIPU, Feb. 2008 (7 marks)]
11. Explain de Broglie hypothesis. Describe an experiment for the confirmation of wave nature of electrons. [GGSIPU, Feb., 2008 (Reappear) 6 marks]
12. Write short note on uncertainty principle. [GGSIPU, Feb. 2008 (Reappear (4 marks))]
13. The de Broglie matter waves travel with a wave velocity greater than the speed of light. Does this violate Einstein's special theory of relativity. Justify your answer. [GGSIPU, Feb. 2005 (2 marks)]
14. Describe the de Broglie hypothesis of matter waves. Who gave an experimental verification for this hypothesis ? [GGSIPU, Feb. 2005 (2 marks)]
15. What is a wavepacket ? Give the physical significance of Max Born approximation. [GGSIPU, Feb. 2005 (2 marks)]
16. Write down the expression for the uncertainty principle. [GGSIPU, May 2005 (1.25 Marks)]
17. Explain de Broglie's concept of matter waves. How was it experimentally verified by Davisson and Germer ? Why is the wave nature of matter not easily recognisable in our daily life ?
18. Derive the relation among phase velocity v_p ; group velocity v_g and velocity of light 'c' for a de Broglie wave in free space.
19. Explain Heisenberg's uncertainity principle.
20. Describe the Davisson-Germer experiment to test the validity of the de Broglie hypothesis for matter wave.
21. Starting from the concept of a wavegroup, show that the uncertainty principle can be derived as a naturaly consequence of wave-particle duality.
22. Describe Davisson-Germer experiment for experimental confirmation of the existence of matter wave.
23. Name the experiments and give physical interpretation to their results which support the following hypothesis.
(a) Electromagnetic radiations have particle nature.
(b) Electrons have an associated wave nature.
(Discuss qualitatively only one experiment for each hypothesis without deriving any formula)

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24. State the uncertainty principle and using it derive expressions of the radius and energy of the electron in ground state of H-atom.
25. What do you understand by a wave group ? Show that the concept of a wave group leads to Heisenberg uncertainty principle.
26. Write a short note on Davisson-Germer experiment.
27. Using wave equation and relativity, show that the velocity of de Broglie or matter wave is greater than the velocity of light waves.
28. The phase velocity of ocean waves is $\frac{\sqrt{gh}}{2\pi}$, where g is accleration due to gravity. Find the group velocity of ocean waves.
29. Define phase velocity and group velocity. Under what conditions will they be identical ?
30. Describe the Davisson-Germer experiment in support of existence of matter waves.
31. Give de Broglie theory of matter waves. Derive an expression for de Broglie wavelength of matter wave. Show that the electrons accelerated through a potential diffience of V volt have $\frac{12.28}{\sqrt{V}}$ Å associated with them.
32. Define phase velocity and group velocity of waves and obtain a relationship between them.
33. Distinguish between phase velocity and group velocity. Show that group velocity of the de Broglie wave is the same as the velocity of the particle. What is wave packet ?

Numerical Problems

1. A 150 eV increases in an electron's energy changes its de Broglie wavelength by a factor of two. Calculate the initial de Broglie wavelength by a factor of two. Calculate the initial de Broglie wavelength of the electron. [**Ans.** $\lambda = 1.73$ Å]
2. Calculate the de Broglie wavelength and the kinetic energy of electrons which undergo first order Bragg diffraction by a nickel crystal at an angle of 30°. For nickel $d = 2.15$ Å.
[**Ans.** $\lambda = 2.15$ Å, $K = 32.5$ eV]
3. High energy protons of 200 GeV (1 GeV = 10^9 eV) are diffracted by a hydrogen target at an angle θ is given by
$$p\sin\theta = \frac{1.2}{c}\text{GeV}$$
Note that the proton are moving at relativistic energies. Estimate the radius of the proton.
[**Ans.** $R = 2.5 \times 10^{-16}$ m]
4. The radius of a hydrogen atom is 5.3×10^{-11} m. Estimate the minimum kinetic energy of the electron in this atom using the uncertainty principle.
[**Ans.** K.E. = 13.7 eV]
5. An atom remain in an excited state for 10^{-9} s. Calculate the uncertainty in its energy.
[**Ans.** $\Delta E = 1.054 \times 10^{-25}$ J]
6. The most rapidly moving electron in metallic sodium, at the absolute zero of temperature, has a kinetic energy 3 eV. Show that its de Broglie wavelength of 7Å.

7. Compute the de Broglie wavelength of the following
   (a) A 1000 kg auto mobile travelling at 100 m/s.
   (b) A 10 kg bullet travelling at 500 m/s.
   (c) A smoke particle of mass 10^{-8} g moving at 1 cm/sec.
   (d) An electron with a kinetic energy at 1 eV.
[**Ans.** (a) 6.6×10^{-39} m (b) 1.3×10^{-34} m (c) 6.63×10^{-23} m (d) 1.2×10^{-9} m]
8. Compare the momentum, the total energy and the kinetic energy of an electron with a de Broglie wavelength of 2Å with that of a photon of same wavelength.
$$\left[\textbf{Ans.}\ (i)\ \frac{p_E}{p_p} = 1\ (ii)\ \frac{E_E}{E_p} = 82.35\ (iii)\ \frac{K_E}{K_p} = 0.006\right]$$
9. An electron of energy 250 eV is passed through a circular hole of radius 10^{-5} m. What is the uncertainty introduced in the angle of emergence ? [**Ans.** $\theta = 0.062 \times 10^{-4}$ rad $\cong 1.42^{11}$ s]
10. An electron has a speed of 800 ms^{-1} with an accuracy of 0.004%. Calculate the certainty with which we can locate the position of the electron. [**Ans.** $\Delta x = 3.6 \times 10^{-3}$ m]
11. Using the Heisenberg uncertainty relation calculate the kinetic energy of an electron in a hydrogen atom. [**Ans.** $K = 0.78$ eV]
12. The speed of electron is measured to within an uncertainty of 2.0×10^4 ms^{-1}. What is the minimum space required by the electron to be confined in an atom. Interpret the results.
[**Ans.** $\Delta x = 0.563$ Å]
13. A bullet of mass 25 g is moving with a speed of 400 ms^{-1}. The speed is measured accurate upto 0.02%. Calculate the uncertainty with which the position of the bullet can be located.
[**Ans.** 3.31×10^{-31} m]
14. An electron has a wavelength 166×10^{-10} m. Find the kinetic energy, phase velocity, and group velocity of the de Broglie wave. [**Ans.** 54.63 eV, 2.05×10^{10} ms^{-1}, 4.39×10^6 ms^{-1}]
15. An electron of rest energy 511 keV has de Broglie wavelength of 2 pm. Find its kinetic energy and group velocity. [**Ans.** 6.031×10^{-14} J, 3.639×10^8 m/s]
16. (a) The energy state of nucleus is uncertain by an amount 1 eV. What is the minimum limit on its life time ?
    (b) What is the de Broglie wavelength of 1 MeV electron ?
17. Find the de Broglie wavelength of a deutron accelerated through a potential of 4000 V. Assume $m_p = m_n$.
18. Estimate the energy of a nucleon confined to a region of size 10^{-14} m. Take nucleon mass to be 1840 times the electron mass.
19. Calculate the lowest energy of an electron confined inside an atom of dimension 1Å on the basis of uncertainty principle.
20. A proton is confined to a nucleus of radius 5×10^{-5} m. Calculate the minimum uncertainty in its momentum and the minimum kinetic energy, the proton should have.
21. Calculate the uncertainty in the energy of a nucleus whose life time is 10^{-9} sec.
22. The position and momentum of 1.5 keV electron are simultaneously estimate. If the position is located to within 1.5 nm. Obtain the percentage of uncertainty in its momentum. [**Ans.** 2.1%]

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h what fundamental accuracy one can locate the position of an electron that has a spread of 10^6 accurate to 0.01%. [Ans. $\Delta x = 0.72 \times 10^{-5}$ m]

mate the wavelength associated with an electron having kinetic energy equal to 0.8 MeV. [Ans. $\lambda = 0.0011$ nm]

ulate the time required for the a line of wavelength 500 nm and width 10^{-10} cm. [Ans. $\Delta t = 8.2 \times 10^{-8}$ s]

ulate the smallest possible uncertainty in the position of an electron moving with velocity 10^7 m/s. [Ans. $(\Delta x)_{min} = 0$]

energy associated with an atomic particle of the order of $k_B T$ where k_B Boltzmann's constant, ulate the wavelength of thermal neutrons. [Ans. $\lambda = 0.177$ nm]

that the de Broglie wavelength for a material particle of rest mass m_0 and charge e accelerate rest through a potential difference of V volt relativistically is given by

$$\lambda = \frac{h}{2m_0 eV\left(1 + \frac{eV}{2m_0 c^2}\right)}$$

cited atom has an average life time of 10^{-8} s. That is, during this time it emits a photon and s to the ground state. What is minimum uncertainty in the frequency of this photon ? [Ans. $\Delta\nu \geq 796 \times 10^6\ s^{-1}$]

tain excited state of hydrogen atom is known to have a life time of 2.5×10^{-14} sec. What is the mum uncertainty in the measurement of energy of this excited state ? If the centre of hydrogen be located with a precision of 0.01 Å, what is the corresponding uncertainty in its velocity ? mass of hydrogen atom = 1672×10^{-27} kg. [Ans. 2.1×10^{-21} J, 3.2×10^4 m/s]

is the minimum kinetic energy of an electron confined in a region of width $a \approx 0.1$ nm, the size mall atom ? [Ans. $\langle k \rangle = 4\,eV$]

osition x of a 0.01 g pellet has been carefully measured and is known within $\pm 0.5\,\mu$m. ding to the uncertainty principle, what are the minimum uncertainties in its momentum and ty, consistent with our knowledge of x ? [Ans. $\Delta p \geq 10^{-28}$ kg–ms ; $\Delta v \geq 10^{-23}$ m/s]

Chapter 6

The Equation of Motion of Matter Waves : Schrödinger's Equation

6.1 Introduction

In chapter 5, we have seen that a microscopic particle is essentially represented by a matter wave with its wavelength given by de Broglie relation. How do we describe the motion of such a particle or a system of such particles ? Clearly, we can not make use of Newton's laws of motion for this purpose. So a new theoretical description is needed for the motion of quantum mechanical particles. The new theory should be consistent with wave nature of particles. It should also reduce to Newtonian mechanics for macroscopic particles. Recall this condition is similar to the special theory of relativity which reduces to Newtonian mechanics at velocities much smaller than the velocity of light.

In this connection, I would like to recount a story. At the end of a seminar, in 1926, on de Broglie waves, the physicist Peter Debye said to another physicist that matter is a wave, there should be a wave equation to describe a matter wave. Debye promptly forgot about it but other physicist, Erwin Schrödinger proceeded to discover the wave equation for matter waves. This equation is named after him as the **Schrödinger equation**.

6.2 Wave Function

As we know, the height of water surface varies periodically in water waves, the pressure of gas varies periodically in sound waves and the electric and magnetic fields vary periodically in light waves but what is the quantity which varies periodically in case of matter waves ?

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Answer is **wave function**, the quantity whose variations make up the matter waves. So the amplitude of matter wave is described by wave function, represented by a Greek letter ψ consists of real and imaginary parts :

$$\varphi = A + iB$$

Conjugate of φ is $\quad \varphi^* = A - iB$

and $\quad \varphi\varphi^* = |\varphi|^2 = A^2 + B^2 \qquad [\because i^2 = -1]$

$|\varphi|^2$ at a particular place at a particular time is proportional to the probability of finding the particle there at that time.

$$\therefore \quad \text{Probability density} = |\varphi|^2 = \varphi^*\varphi$$

Let us assume that wavefunction φ is specified in x-direction by a wave equation

$$\varphi = Ae^{-i\omega\left(t-\frac{x}{v}\right)}$$

where $\omega = 2\pi\nu$ and $v = \nu\lambda$

$$\therefore \quad \varphi = Ae^{-2\pi\left(\nu t-\frac{x}{\lambda}\right)} \qquad ...(1)$$

As $\quad E = h\nu = 2\pi\hbar\nu$ and $\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p} \qquad ...(2)$

Putting the values of E and λ in Eq. (1), we get

$$\varphi = Ae^{-\frac{i}{\hbar}(Et-px)} \qquad ...(3)$$

Eq. (3) is the wave equation for a free particle.

Partially differentiate Eq. (3) with respect to t, we get

$$\frac{\partial\varphi}{\partial t} = -\frac{iE}{\hbar}Ae^{-\frac{i}{\hbar}(Et-px)}$$

$$= -\frac{iE}{\hbar}\varphi$$

or $\quad E\varphi = i\hbar\frac{\partial\varphi}{\partial t} \qquad [i^2 = -1] \qquad ...(4)$

Here Energy operator $\quad E = i\hbar\frac{\partial}{\partial t} \qquad ...(5)$

Now, partially differentiate Eq. (3), with respect to x, we get

$$\frac{\partial\varphi}{\partial x} = \frac{ip}{\hbar}Ae^{-\frac{i}{\hbar}(Et-px)}$$

or $\quad \frac{\partial\varphi}{\partial x} = \frac{ip}{\hbar}\varphi$

or $\quad p\varphi = \frac{\hbar}{i}\frac{\partial\varphi}{\partial x} \qquad ...(6)$

Hence Momentum operator $\quad p = \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x} \qquad ...(7)$

An operator tells us what operation to carry out on the quantity that follows it.

The operator $i\hbar\frac{\partial}{\partial t}$ (as Eq. (5)) instructs us to take partial derivative of what comes after it with respect to t and multiply the result by $i\hbar$.

From Eq. (5), consider the eigen-value equation.

$$i\hbar\frac{\partial\varphi}{\partial t} = E\varphi$$

Hence φ is said to be *eigen function* of the operator $i\hbar\frac{\partial}{\partial t}$ and E is called the corresponding *energy eigen value.*

6.3 Requirement that are Imposed on a Physically Acceptable Wave Function or Properties of the Wave Function

1. φ must be single valued, finite and continuous for all values of x.
2. $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ and $\frac{\partial\varphi}{\partial z}$ must be finite and continuous for all values of x, y and z respectively, except at those points where $V \to \infty$. At these points $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ and $\frac{\partial\varphi}{\partial z}$ have a finite discontinuity but φ remains continuous.
3. φ must be normalised, which means that φ must go to zero as $x \to \pm\infty$, $y \to \pm\infty$, $z \to \pm\infty$ in order that $\int|\varphi|^2 dV$ overall space be a finite constant.

   If $\int_{-\infty}^{+\infty}|\varphi|^2 dV = 0$ *i.e.*, the particle does not exist but $|\varphi|^2$ overall space must be finite *i.e.*, the body is to be somewhere.

   $$\therefore \quad \int_{-\infty}^{+\infty}|\varphi|^2 dV = 0,\ \infty,\ -\text{ve or complex} \quad [\text{which is not possible}]$$

   Therefore only possibility left is that integral be a finite quantity.

Normalisation

Normalisation condition is

$$\therefore \quad \int_{-\infty}^{+\infty}|\varphi|^2 dV = 1 \qquad ...(8)$$

[Wave functions which satisfy this condition [Eq. (8)] are said to be normalised]

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As $\quad |\psi|^2 = \psi^*\psi =$ probability density $= P$

Therefore, probability between the limits x_1 and x_2 is given as in one-dimensional case.

Probability

$$P_{x_1x_2} = \int_{x_1}^{x_2} |\psi|^2 \, dx \qquad \text{...(9)}$$

Expectation Value

To correlate experiment and theory we define the expectation value of any parameter :

$$\langle x \rangle = \frac{\int_{-\infty}^{+\infty} x|\psi|^2 \, dx}{\int_{-\infty}^{+\infty} |\psi|^2 \, dx} = \frac{\int_{-\infty}^{+\infty} \psi^* x \psi \, dx}{\int_{-\infty}^{+\infty} \psi^* \psi \, dx}$$

If ψ is a normalised wave function, then $\int_{-\infty}^{+\infty} \psi^* \psi \, dx = 1$.

$$\therefore \quad \langle x \rangle = \int_{-\infty}^{+\infty} x|\psi|^2 \, dx \qquad \text{...(10)}$$

Orthogonal Wave Functions

If ψ_a and ψ_b are the wave functions such that

$$\int \psi_a^* \psi_b \, dV = 0 \quad \text{or} \quad \int \psi_b^* \psi_a \, dV = 0 \text{ for } a \neq b \qquad \text{...(11)}$$

then the wave functions ψ_a and ψ_b are known as **orthogonal** to each other. The wave functions which satisfy normalisation condition *i.e.*,

$$\int \psi_a^* \psi_a \, dV = 1 \qquad \text{...(12)}$$

and orthogonality condition Eq. (11) are called orthonormal functions and condition for orthonormality is

$$\int \psi_a^* \psi_b \, dV = \delta_{ab} \qquad \text{...(13)}$$

where δ_{ab} = kronecker delta function, it is defined as

$$\delta_{ab} = 0 \quad \text{if } a \neq b$$
$$= 1 \quad \text{if } a = b$$

Example 6.1. A particle limited to the x-axis has the wave function $\psi = ax$ between $x = 0$ and $x = L$, $\psi = 0$ elsewhere. Find (a) the probability that particle can found between $x = 0.45$ and $\lambda = 0.55$. (b) The expectation value $\langle x \rangle$ of the particle's position. **[GGSIPU, Feb. 2008, April 2007(2 marks)]**

Solution. (*a*) The probability is

$$\int_{x_1}^{x_2} |\psi|^2 \, dx = a^2 \int_{0.45}^{0.55} x^2 \, dx = a^2 \left[\frac{x^3}{3}\right]_{0.45}^{0.55} = 0.0251 \, a^2$$

(*b*) The expectation value is

$$\langle x \rangle = \int_0^1 x|\psi|^2 \, dx = a^2 \int_0^1 x^3 \, dx = a^2 \left[\frac{x^4}{4}\right]_0^1 = \frac{a^2}{4}$$

Example 6.2. The wave function of a particle is given by

$$\psi(x) = Ce^{-a^2x^2}, \quad -\infty < x < \infty$$

where C and a are the some constants. Calculate the probability of finding the particle in the region $0 < x < \infty$.

Solution. The probability of finding the particle in the region $0 < x < \infty$ is

$$P = \int_0^\infty |\psi(x)|^2 \, dx = \int_0^\infty C^2 e^{-2a^2x^2} \, dx$$

[Also from standard integral $\int_0^\infty e^{-ax^2} \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$]

We get, $\quad P = C^2 \int_0^\infty e^{-2a^2x^2} \, dx = \frac{1}{2} C^2 \sqrt{\left(\frac{\pi}{2a^2}\right)}$

Applying normalising condition *i.e.*,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} C^2 e^{-2a^2x^2} \, dx = 1$$

$$\Rightarrow \quad C^2 \sqrt{\frac{\pi}{2a^2}} = 1$$

Which gives $\quad C^2 = \sqrt{\frac{2a^2}{\pi}}$

$$\therefore \quad P = \sqrt{\frac{2a^2}{\pi}} \cdot \frac{1}{2} \sqrt{\frac{\pi}{2a^2}} = \frac{1}{2}$$

Example 6.3. Show that function $\psi_0 = \left(\frac{b}{4}\right)^{1/8} \exp\left(-\frac{bx^2}{2}\right)$ and $\psi_1 = \left(\frac{4b^3}{4}\right)^{1/8} \exp\left(-\frac{bx^2}{2}\right) x$ are normalised over the interval $-\infty < x < \infty$.

Solution. (*a*) $\int_{-\infty}^{\infty} \psi_0 \psi_0^* \, dx = \int_{-\infty}^{\infty} \left(\frac{b}{4}\right)^{1/2} \exp(-bx^2) dx$

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Let $\quad xb^{1/2} = y;\ dx = \frac{dy}{\sqrt{(b)}}$

Now, $$\int_{-\infty}^{+\infty} \varphi_0 \varphi_0^* dx = \int_{-\infty}^{\infty} |\varphi_0|^2 dx = 2\left(\frac{b}{\pi}\right)^{1/2} \int_0^{\infty} \exp\cdot(-bx^2) dx$$

$$= 2\left(\frac{b}{\pi}\right)^{1/2} \int_0^{\infty} \frac{\exp\cdot(-y^2) dy}{\sqrt{b}}$$

$$= 2\left(\frac{b}{\pi}\right)^{1/2} \cdot \frac{1}{2}\left(\frac{\pi}{4b}\right)^{1/2} = 1$$

(b) $$\int_{-\infty}^{\infty} \varphi_1 \varphi_1^* dx = 2\left(\frac{4b^3}{\pi}\right)^{1/2} \int_0^{\infty} x^2 \exp\cdot(-bx^2) dx \qquad \left[\because \int_0^{\infty} x^2 \exp\cdot(-\alpha x^2) dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}\right]$$

We get,

$$\int_0^{\infty} |\varphi| dx = 2\left(\frac{4b^3}{\pi}\right)^{1/2} \left[\frac{\pi}{16b^3}\right]^{1/2} = 1.$$

6.4 Postulates of Quantum Mechanics

(i) Description of the System

Every particle (or a system of particles) is represented by a "**wave function**", which is a function of space coordinates and time. The wave function determines all that can be known about the system it represents.

(ii) Time Evolution of a System

The time dependent schrödinger equation for one-dimensional motion of a particle of mass m moving in a potential $V(x,t)$ is given by

$$i\hbar \frac{\partial \varphi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi(x,t)}{\partial x^2} + V(x,t)\varphi(x,t) \qquad ...(14)$$

(iii) Probabilistic Interpretation of the Wave function

If at any instant t, a measurement is made to locate the object represented by the wave function $\varphi(x,t)$, then the probability $P(x,t)dx$ that the object will be found between x and $(x+dx)$,

$$P(x,t)dx = \varphi^*(x,t)\varphi(x,t)dx = |\varphi(x,t)|^2 dx \qquad ...(15)$$

(iv) Description of Physical Quantities

Every physical observable is associated with an operator which acts on the wave function.

(v) The Measurement Postulate

The average of the measured value of x is given by

$$\langle x \rangle = \frac{\int \varphi^* x \varphi dV}{\int \varphi^* \varphi dV} = \frac{(\varphi, x\varphi)}{(\varphi, \varphi)} \qquad ...(16)$$

$\langle x \rangle$ is known as the expectation value of x.

6.5 Schrödinger Wave Equation

In 1926, Erwin Schrödinger presented a famous wave equation as a development of de Broglie ideas of the wave properties of matter. The Schrödinger wave equation is the fundamental equation of wave mechanics in the same as the Newton's second law of motion of classical mechanics. It is differential equation of the de Broglie waves associated with particles and describes the motion of particles.

6.5.1 Schrödinger's Time Dependent Wave Equation

Let us assume that φ for a particle moving freely in positive x-direction is

$$\varphi = Ae^{-i\omega\left(t - \frac{x}{v}\right)} \qquad ...(17)$$

as $\quad \omega = 2\pi\nu \qquad v = \nu\lambda$

$$\varphi = Ae^{-2\pi i\left(\nu t - \frac{x}{\lambda}\right)} \qquad ...(18)$$

As $\quad E = h\nu = 2\pi\hbar\nu \quad$ and $\quad \lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}$

$\therefore$ For a free particle wave equation becomes

$$\varphi = Ae^{-\frac{i}{\hbar}(Et - px)} \qquad ...(19)$$

As it is calculated in article 6.2

$$E\varphi = i\hbar \frac{\partial \varphi}{\partial t} \quad \text{and} \quad p\varphi = \frac{\hbar}{i}\frac{\partial \varphi}{\partial t} \qquad ...(a)$$

As total energy, $E = \text{K.E.} + V$ (Potential energy) $\qquad ...(20)$

Now, $\quad \text{K.E.} = \frac{p^2}{2m}$

$\therefore$ Eq. (20) in terms of wave function φ can be written as

$$E\varphi = \left(\frac{p^2}{2m}\right)\varphi + V\varphi \qquad ...(21)$$

Putting the values of $E\varphi$ and $P\varphi$ from Eq. (a) in Eq. (21), we have

$$i\hbar \frac{\partial \varphi}{\partial t} = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \frac{1}{2m}\varphi + V\varphi$$

or $$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \varphi}{\partial x^2} + V\varphi \qquad ...(22)$$

Eq. (22) is *Schrödinger's time dependent wave equation in one-dimension.*

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In three-dimensional form,

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\right]+V\varphi \quad ...(23)$$

or

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\varphi+V\varphi \quad \left[\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right] \quad ...(24)$$

or

$$-\frac{\hbar^2}{2m}\nabla^2\varphi+V\varphi = i\frac{\partial\varphi}{\partial t} \quad ...(25)$$

or

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\varphi = i\hbar\frac{\partial\varphi}{\partial t} \quad ...(26)$$

The Eq. (26), contains time and hence is called *time dependent Schrödinger equation.*

The operator $\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)$ is called **Hamiltonian** and is represented by H ; while operator $i\hbar\frac{\partial}{\partial t}$, operated on φ, gives E which may be seen by Eq. (5). Eq. (26) may be written as

$$H\varphi = E\varphi \quad ...(27)$$

The above forms of the Schrödinger's equation describe the motion of a *non-relativistic material particle.*

6.5.2 Schrödinger's Time independent wave equation

Again consider the Eq. (17),

$$\varphi = Ae^{-\frac{i}{\hbar}(Et-px)} = Ae^{-\frac{i}{\hbar}Et}.e^{\frac{i}{\hbar}px}$$

or

$$\varphi = \varphi_0 e^{-\frac{i}{\hbar}Et} \quad ...(28)$$

Differentiate partially with respect to t, we get

$$\frac{\partial\varphi}{\partial t} = -\frac{iE}{\hbar}\varphi_0 e^{-\frac{i}{\hbar}Et} \quad ...(29)$$

and

$$\frac{\partial^2\varphi}{\partial x^2} = +\frac{\partial^2\varphi_0}{\partial x^2}e^{-\frac{i}{\hbar}Et} \quad ...(30)$$

Putting Eqs. (28), (29) and (30), we get

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}+V\varphi$$

$$i\hbar\left(-\frac{iE}{\hbar}\right)\varphi_0 e^{-\frac{i}{\hbar}Et} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi_0}{\partial x^2}e^{-\frac{i}{\hbar}Et}+V\varphi_0 e^{-\frac{i}{\hbar}Et}$$

$$\Rightarrow \quad E\varphi_0 = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi_0}{\partial x^2}+V\varphi_0 \quad ...(31)$$

or

$$\frac{\partial^2\varphi_0}{\partial x^2}+\frac{2m}{\hbar^2}(E-V)\varphi_0 = 0 \quad ...(32)$$

This Schrodinger's time independent wave equation in one dimension.

In three dimensional case,

$$\nabla^2\varphi_0+\frac{2m}{\hbar^2}(E-V)\varphi_0 = 0 \quad ...(33)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

6.6 Physical significance of wave function (φ)

φ can not be interpreted in terms of an experiment. The probability that something be in certain place at a given time must lie between 0 (the object is not definitely there) and 1 (the object is definitely there). An intermediate probability , say 0.2 means, there is a 20% chance of finding the object. But the amplitude of a wave can be negative as well as positive and negative probability is meaningless. Hence φ by itself cannot be an observed quantity.

The probability of experimentally finding the body described by the wave function φ at the point (x, y, z) at the time t is proportional to the value of $|\varphi|^2$ there at t. A large value of $|\varphi|^2$ means the strong possibility of the body's presence, while a small value of $|\varphi|^2$ means the slight possibility of its presence. As long as $|\varphi|^2$ is not actually zero somewhere, there is a definite change however small, of detecting it there.

6.7 Application of Schrödinger's Wave Equation

6.7.1 The Free Particle

When a particle is not subjected to any external force, so that it moves in a region in which its potential energy is constant, it is said to be a free particle. Such a particle has a definite value of total energy and definite value of momentum. But the position of the particle is completely unknown.

Suppose a particle of mass m is in motion along the x-axis. Suppose no force is acting on the particles so that the potential energy of the particle is constant. For convenience, the constant potential energy is taken to be zero *i.e.*, $V = 0$.

Hence Schrödinger equation becomes,

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{2m}{\hbar^2}E\varphi = 0 \quad ...(34)$$

or

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{8\pi^2 m}{h^2}E\varphi = 0 \quad ...(35)$$

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Multiplying by $\frac{h^2}{8\pi^2 m}$ throughout, we get

$$\frac{h^2}{8\pi^2 m}\frac{\partial^2\varphi}{\partial x^2} + E\varphi = 0 \quad \text{or} \quad \frac{-h^2}{8\pi^2 m}\frac{\partial^2\varphi}{\partial x^2} = E\varphi \qquad ...(36)$$

Since the particle is moving freely with zero potential energy its total energy E is the kinetic energy, given by

$$E = \frac{p_x^2}{2m} \qquad ...(37)$$

where p_x is the momentum of the particle.

Rewriting Eq. (34),

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{8\pi^2 m}{h^2}E\varphi = 0 \quad \text{or} \quad \frac{\partial^2\varphi}{\partial x^2} + K^2\varphi = 0 \qquad ...(38)$$

where $K^2 = \frac{8\pi^2 mE}{h^2}$

So that,

$$E = \frac{K^2h^2}{8\pi^2 m} \qquad ...(39)$$

In this case energy E is not quantised. Hence when a particle is not bound in a system it does not have quantised energy state.

6.7.2 Particle in box (Infinite Square Well Potential)

Consider a particle moving inside a box along the x-direction. The particle is bouncing back and forth between the walls of the box a is the width of the box.

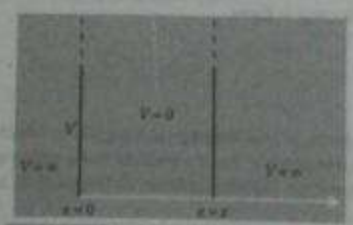

Fig. 6.1 Particle in box.

The potential energy V of the particle is infinite on both sides of the box.

$$V = 0 \quad \text{for } 0 < x < a$$
$$V = \infty \quad \text{for } x \le 0 \text{ and } x \ge a$$

The particle can not exist outside the box, so its wave function φ is 0 for $x \le 0$ and $x \ge a$.

Within the box, the Schrödinger's equation becomes

$$\frac{d^2\varphi}{dx^2} + \frac{2m}{h^2}E\varphi = 0 \qquad [\because V \text{ is 0 for the free particle}] \qquad ...(40)$$

$$\Rightarrow \quad \frac{d^2\varphi}{dx^2} + K^2\varphi = 0 \qquad \text{where } K = \sqrt{\frac{2mE}{h}} \qquad ...(41)$$

The general solution of this equation is

$$\varphi = A\sin Kx + B\cos Kx$$

Using boundary conditions

$$\varphi = 0 \text{ at } x = 0$$
$$0 = A\sin 0 + B$$
$$B = 0$$

and

$$\varphi = 0 \text{ at } x = a$$
$$0 = A\sin Ka$$
$$\sin n\pi = \sin Ka$$
$$K = \frac{n\pi}{a}$$

Wave function,

$$\varphi_n(x) = A\sin\frac{n\pi x}{a} \qquad \text{where } n = 1, 2, 3... \qquad ...(42)$$

Energy level,

$$E_n = \frac{K^2h^2}{2m} \qquad \left[\hbar = \frac{h}{2\pi}\right]$$

$\therefore$ Eigen value

$$E_n = \left(\frac{n\pi}{a}\right)^2\left(\frac{h}{2\pi}\right)^2\frac{1}{2m}$$

$$E_n = \frac{n^2h^2}{8ma^2} \qquad \text{where } n = 1, 2, 3, 4, ... \qquad ...(43)$$

Therefore, it is clear from Eq. (43) that inside an infinitely deep potential well (or in an infinite square well), the particle cannot have an arbitrary energy, but can have only certain discrete energy corresponding to $n = 1, 2, 3, ...$ Each permitted energy level energy is called *eigen-value* of the particle and constitutes the energy level of the system. The wave function φ corresponding to each eigen value are called *eigen functions*.

To find eigen-functions of the particle using Eq. (42) and applying normalisation condition,

$$\int_{-\infty}^{+\infty} |\varphi_n(x)|^2 dx = 1$$

$$\Rightarrow \quad \int_0^a |\varphi_n(x)|^2 dx = 1$$

$$\Rightarrow \quad A^2\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = 1$$

$$\Rightarrow \quad \frac{A^2}{2}\int_0^a\left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right]dx = 1$$

$$\Rightarrow \quad \frac{A^2}{2}\left[\int_0^a dx - \int_0^a \cos\left(\frac{2n\pi x}{a}\right)dx\right] = 1$$

$$\Rightarrow \quad \frac{A^2}{2}[a] = 1 \quad \Rightarrow \quad A = \sqrt{\left(\frac{2}{a}\right)}$$

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∴ Eigen function

$$\varphi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \qquad \text{where } n = 1, 2, 3, \ldots \qquad \ldots(44)$$

Although φ_n may be negative as well as positive, $|\varphi_n|^2$ is always positive and since ψ_n is normalised, its square value at a given x is equal to the probability of finding the particle. The first three eigen functions $\varphi_1, \varphi_2, \varphi_3$ together with probability densities $|\varphi_1|^2$, $|\varphi_2|^2$, $|\varphi_3|^2$ are shown in Figs. 6.2(*a*) and (*b*) respectively.

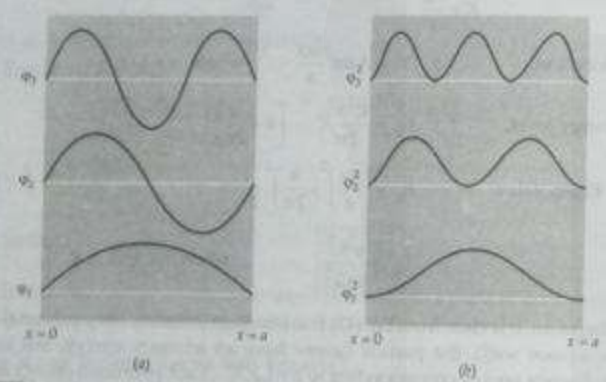

Fig. 6.2 Wave functions and probability densities of a particle confined to a box with rigid walls.

Example 6.4. *A particle is moving in a one-dimensional box (of infinite height) of width 2.5 nm at the centre of the box when it is in its state of least energy.*

Solution. Since we know that,

$$\varphi_n(x) = \sqrt{\left(\frac{2}{a}\right)}\left[\sin \frac{n\pi x}{a}\right]$$

When the particle is in least energy state ($n = 1$), the wave function becomes

$$\varphi_1(x) = \sqrt{\left(\frac{2}{a}\right)}\left[\sin \frac{\pi x}{a}\right]$$

At the centre of the box, $x = \frac{a}{2}$, the probability of finding the particle at midpoint of the box is

$$|\varphi_1(x)|^2 = \left(\frac{2}{a}\right)\left[\sin^2 \frac{\pi x}{a}\right]$$

The probability 'P' in the interval Δx is given by

$$P = |\varphi_1(x)|^2 (\Delta x) = \left(\frac{2}{a}\right)\Delta x \text{ at } x = \frac{a}{2}.$$

With

(*i*) $a = 2.5$ nm (*ii*) $\Delta x = 0.5$ nm

Thus, $$P = \left(\frac{2}{a}\right)\Delta x = \frac{2}{2.5} \times (0.5) = 0.4$$

$$P = 0.4$$

Example 6.5. *Find the energy of an electron moving in one dimension in an infinitely high potential box of width 1 Å.*

Solution. The eigen value of energy

$$E_n = \frac{n^2 h^2}{8ma^2}$$

when the particle is in the least energy state ($n = 1$), the energy

$$E_1 = \frac{h^2}{8ma^2}$$

with (*i*) $h = 6.62 \times 10^{-34}$ Js (*ii*) $m = 9.1 \times 10^{-31}$ kg (*iii*) $a = 1 \times 10^{-10}$ m

i.e., $$E_1 = \frac{(6.62 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (1 \times 10^{-10})^2}$$

$$= \frac{6.62}{8 \times 9.1} \times 10^{-68} \times 10^{+51} = \frac{6.62}{72.8} \times 10^{-17}$$

$$= \frac{6620}{728} \times 10^{-19} \text{ J} = 9.093 \times 10^{-19} \text{ J} = 5.68 \text{ eV}.$$

6.7.3 Potential Step

The potential function of a potential step is defined by

$$V(x) = 0 \qquad x < 0 \qquad \ldots(45)$$
$$= V_0 \qquad x > 0$$

Let the electrons of energy E move from left to right. Schrödinger's wave equations for the regions I and II are :

Region I. $$\frac{d^2\varphi_1}{dx^2} + \frac{2m}{\hbar^2} E\varphi_1 = 0 \qquad \ldots(46)$$

$[\because V = 0]$

Region II. $$\frac{d^2\varphi_2}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\varphi_2 = 0 \qquad \ldots(47)$$

where φ_1 and φ_2 are the wave functions associated with particle (electron) in region I and region II respectively.

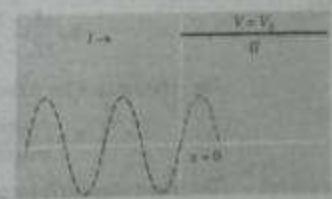

Fig. 6.3 Potential step.

Let $K_1 = \frac{\sqrt{2mE}}{\hbar}$ and $K_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$

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Eqs. (46) and (47), may be written as,

$$\frac{d^2\varphi_1}{dx^2} + K_1^2\varphi_1 = 0 \qquad ...(48)$$

$$\frac{d^2\varphi_2}{dx^2} + K_2^2\varphi_2 = 0 \qquad ...(49)$$

The solutions of these equations are :

$$\varphi_1 = Ae^{iK_1x} + Be^{-iK_1x} \qquad ...(50)$$

$$\varphi_2 = Ce^{iK_2x} + De^{-iK_2x} \qquad ...(51)$$

For $x > 0$, no particle can flow to the left, hence $D = 0$.

$$\varphi_2 = Ce^{iK_2x} \qquad ...(52)$$

As the boundary at $x = 0$,

We have, $\varphi_1 = \varphi_2 \Rightarrow [Ae^{iK_1x} + Be^{-iK_1x} = Ce^{iK_2x}]_{x=0}$

$\Rightarrow$ $A + B = C$ ...(i)

We also have,

$$\frac{d\varphi_1}{dx} = \frac{d\varphi_2}{dx} \Rightarrow [AiK_1e^{iK_1x} - BiK_1e^{-iK_1x} = CiK_2e^{iK_2x}]_{x=0}$$

$\Rightarrow$ $A - B = \frac{K_2}{K_1}C$ ...(ii)

From Eqs. (i) and (ii), we obtain

This gives, $2A = \left(1 + \frac{K_2}{K_1}\right)C$ or $A = \frac{K_1 + K_2}{2K_1}C$

or $$C = \frac{2K_1}{K_1 + K_2}A \qquad ...(53)$$

Also, $$2B = \left(1 - \frac{K_2}{K_1}\right)C = \frac{(K_1 - K_2)}{K_1}\frac{2K_1}{(K_1 + K_2)}A$$

$$B = \frac{(K_1 - K_2)}{(K_1 + K_2)}A \qquad ...(54)$$

The reflectance R (or reflectivity or reflection coefficient)

$$R = \frac{\text{magnitude of reflected current}}{\text{magnitude of incident current}}$$

The transmittance (T) (or transmission or transmittivity coefficient)

$$T = \frac{\text{magnitude of transmitted current}}{\text{magnitude of incident current}}$$

Now two cases arise : $E < V_0$ and $E > V_0$

Case 1. $E > V_0$

In this case, $K_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$ is real.

Probability density, $[J_x]_I = \frac{\hbar}{2im}\left[\varphi_1^*\frac{d\varphi_1}{dx} - \varphi_1\frac{d\varphi_1^*}{dx}\right]$

$$= \frac{\hbar}{2im}[(A^*e^{-iK_1x} + B^*e^{+iK_1x})\times(iK_1)(Ae^{iK_1x} - Be^{-iK_1x})$$

$$-(Ae^{iK_1x} + Be^{-iK_1x})\times(-iK_1)(A^*e^{-iK_1x} - B^*e^{iK_1x})]$$

$$= \frac{K_1\hbar(AA^* - B^*B)}{m} = \frac{K_1\hbar}{m}[|A|^2 - |B|^2] \qquad ...(a)$$

From Eq. (a), it is evident that the current in the region I is equal to the difference between two terms, of which the first one is proportional to $K_1|A|^2$ represents the incident beam travelling from left to right, whereas the second one which is proportional to $K_1|B|^2$ represents the reflected beam travelling from right to left.

$\therefore$ Probability of current of the incident beam $= |A|^2\frac{K_1\hbar}{m}$ ...(b)

and the probability of current of the reflected beam $= |B|^2\frac{K_1}{m}\hbar$ ...(c)

Now, $[J_x]_{II} = \frac{\hbar}{2im}\left[\varphi_2^*\frac{d\varphi_2}{dx} - \varphi_2\frac{d\varphi_2^*}{dx}\right]$

$$\Rightarrow \quad [J_x]_{II} = \frac{\hbar}{2im}[(Ce^{-iK_2x})(iK_2)Ce^{iK_2x} - (Ce^{iK_2x})(-iK_2)Ce^{-iK_2x}]$$

$$= \frac{K_2}{2m}[CC^* + CC^*] = \frac{K_2}{m}(CC^*) = \frac{|C|^2K_2\hbar}{m}$$

This gives, $[J_x]_{II} = \frac{|C|^2K_2\hbar}{m}$

Now, we find the expressions for reflectance and transmittance for the case $E > V_0$.

$$R = \frac{|B|^2\frac{K_1\hbar}{m}}{|A|^2\frac{K_1\hbar}{m}} = \left|\frac{B}{A}\right|^2 = \left(\frac{K_1 - K_2}{K_1 + K_2}\right)^2 \qquad ...(55)$$

$$T = \frac{|C|^2\frac{K_2\hbar}{m}}{|A|^2\frac{K_1\hbar}{m}} = \left|\frac{C}{A}\right|^2\frac{K_2}{K_1} = \left(\frac{2K_1}{K_1 + K_2}\right)^2\frac{K_2}{K_1} = \frac{4K_1K_2}{(K_1 + K_2)^2} \qquad ...(56)$$

Eq. (55) shows that R is a positive quantity. This implies that there is a chance of reflection at the boundary yet the particle has large energy than barrier. The incident wave also advances in region II due to positive T value.

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Case II. $E < V_0$

In this case K_2 is imaginary *i.e.*, $K_2 = i\gamma$, hence $\psi_2 = Ce^{i\gamma x} = Ce^{-\gamma x}$

In this case the probability current is given by

$$J_x = \frac{\hbar}{2im}\left[\psi_2^* \frac{d\psi_2}{dx} - \psi_2 \frac{\partial \psi_2^*}{\partial x}\right]$$

$$= \frac{\hbar}{2im}\left[C^* e^{-\gamma x} \frac{d}{dx}(Ce^{-\gamma x}) - Ce^{-\gamma x} \frac{d}{dx}(C^* e^{-\gamma x})\right]$$

$$= \frac{\hbar}{2im}\{-C^* C \cdot \gamma e^{-2\gamma x} + C^* C \cdot \gamma e^{-2\gamma x}\}$$

$$J_x = 0$$

The transmittance $(T) = \dfrac{\text{Magnitude of transmittance current}}{\text{Magnitude of incident current}} = 0$

Now reflectance $(R) = \dfrac{\text{Magnitude of reflected current}}{\text{Magnitude of incident current}}$

$$\Rightarrow \quad R = \frac{|B|^2 \frac{K_1}{m}\hbar}{|A|^2 \frac{K_1}{m}\hbar} = \left|\frac{B}{A}\right|^2 = \frac{BB^*}{AA^*}$$

$$R = \frac{\left(\frac{K_1 - K_2}{K_1 + K_2}\right)A\left(\frac{K_1 - K_2}{K_1 + K_2}\right)^* A^*}{AA^*} = \frac{(K_1 - K_2)(K_1 - K_2^*)}{(K_1 + K_2)(K_1 + K_2^*)}$$

[By using complex conjugate of Eq. (54)]

$$R = \left(\frac{K_1 - K_2}{K_1 + K_2}\right)\left(\frac{K_1 - K_2^*}{K_1 + K_2^*}\right)$$

$$= \frac{(K_1 - K_2)(K_1 + K_2)}{(K_1 + K_2)(K_1 - K_2)} = 1 \qquad [\because K_2^* = -K_2]$$

This gives $T = 0$ and $R = 1$

Thus in this case, no wave is transmitted in the region II, that is, probability of finding particle in region II is zero and all the wave is reflected in region I.

The sum of reflectivity and transmittivity gives

Case I. When $E > V_0$ $\quad R + T = \dfrac{(K_1 - K_2)^2 + 4K_1K_2}{(K_1 + K_2)^2} = \dfrac{(K_1 + K_2)^2}{(K_1 + K_2)^2} = 1$

$$R + T = 1$$

Case II. When $E < V_0$ $\quad R + T = 1 + 0$

$$R + T = 1$$

Hence, the sum of reflection and transmission coefficients is unity.

Example 6.6. If a beam of electrons impinges on an energy barrier of height 0.030 eV and of infinite width, find the fraction of electrons reflected and transmitted at the barrier if the energy of the impinging electron is : (a) 0.040 eV (b) 0.025 eV (c) 0.030 eV.

Solution. (*a*) Given $E = 0.040\,\text{eV} = 0.040 \times 1.6 \times 10^{-19}\,\text{J}$

$$= 0.064 \times 10^{-19}\,\text{J}$$

and $V_0 = 0.030\,\text{eV} = 0.030 \times 1.6 \times 10^{-19}\,\text{J}$

$$= 0.048 \times 10^{-19}\,\text{J}$$

i.e., $E > V_0$

Then the fraction of electrons reflected, *i.e.*, the reflectance is given by $R = \dfrac{(K_1 - K_2)^2}{(K_1 + K_2)^2}$

and the fraction of electrons transmitted $T = \dfrac{4K_1K_2}{(K_1 + K_2)^2}$

We know that,

$$K_1 = \sqrt{2mE} = \sqrt{(2 \times 9.1 \times 10^{-31} \times 0.064 \times 10^{-19})}$$

$$= 1.07 \times 10^{-25}\,\text{kg m/sec.}$$

and $K_2 = \sqrt{[2m(E - V_0)]} = \sqrt{2 \times 9.0 \times 10^{-31} \times (0.40 - 0.30) \times 1.6 \times 10^{-19}}$

$$= \sqrt{(2 \times 9 \times 10^{-31} \times 0.10 \times 16 \times 10^{-19})} = 5.37 \times 10^{-26}\,\text{kg m/sec}$$

Then reflectance,

$$R = \frac{(K_1 - K_2)^2}{(K_1 + K_2)^2} = \left[\frac{1.07 \times 10^{-25} - 5.37 \times 10^{-26}}{1.07 \times 10^{-25} + 5.37 \times 10^{-26}}\right]^2 = 0.1$$

i.e., fraction of electron reflected = 0.1.

The fraction of electron transmitted

$$T = \frac{4K_1K_2}{(K_1 + K_2)^2} = \frac{4 \times 1.07 \times 10^{-25} \times 5.37 \times 10^{-26}}{(1.07 \times 10^{-25} + 5.37 \times 10^{-26})^2} = 0.9$$

(*b*) In this case $E < V_0$

$\therefore$ The transmission coefficient = 0 *i.e.*, the fraction of electrons transmitted = 0.

and the reflection coefficient = 1 *i.e.*, the fraction of electron reflected = 100%.

(*c*) In this case $E = V_0$

$$K_1 = \sqrt{2mE} = \sqrt{(2 \times 9.1 \times 10^{-31} \times 0.030 \times 1.6 \times 10^{-19})}\,\text{kg m/sec.}$$

$\therefore$ The fraction of electron reflected $= \dfrac{(K_1 - K_2)^2}{(K_1 + K_2)^2} = 1$

The fraction of electrons transmitted $= \dfrac{4K_1K_2}{(K_1 + K_2)^2} = 0$

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6.7.4 Barrier Potential (Tunneling Effect)

Let us consider the one-dimensional problem, where the potential function is defined as

$$\left.\begin{aligned} V(x) &= 0 \quad \text{for} \quad x < 0 \\ &= V_0 \quad \text{for} \quad 0 < x < a \\ &= 0 \quad \text{for} \quad x > a \end{aligned}\right\} \qquad ...(57)$$

Here, we have a potential between $x = 0$ and $x = a$.

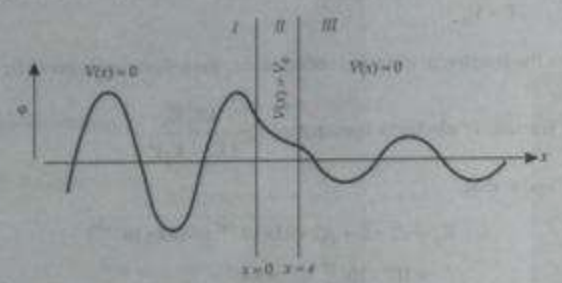

Fig. 6.4 Barrier Potential.

If a particle having energy less than V_0 *i.e.*, $E < V_0$ approaches this barrier from the left *i.e.*, from first region, classically the particle will always be reflected and hence will not penetrate the barrier. However the quantum mechanics predicts that the particle has some probability of penetrating to region III, the probability of penetration being greater if $(V_0 - E)$ and a are smaller. Moreover, if $E > V_0$, classical mechanics predicts that the particle will always be transmitted ; while according to quantum-mechanics, the particle has a finite probability of transmission as well as reflection.

To solve the problem, let us write three Schrödinger equation in three regions.

Region I. $$\frac{d^2\varphi_1}{dx^2} + \frac{2mE\varphi_1}{\hbar^2} = 0 \quad \text{or} \quad \frac{d^2\varphi_1}{dx^2} + K^2\varphi_1 = 0 \qquad ...(58)$$

Region II. $$\frac{d^2\varphi_2}{dx^2} + \frac{2m(E - V_0)\varphi_2}{\hbar^2} = 0 \quad \text{or} \quad \frac{d^2\varphi_2}{dx^2} - \beta^2\varphi_2 = 0 \qquad ...(59)$$

Region III. $$\frac{d^2\varphi_3}{dX^2} + K^2\varphi_3 = 0 \qquad ...(60)$$

where $K = \dfrac{\sqrt{2mE}}{\hbar}$ and $\beta = \dfrac{\sqrt{2m(V_0 - E)}}{\hbar}$

Solutions for Eqs. (58), (59) and (60), will be

$$\varphi_1 = Ae^{iKx} + Be^{-iKx} \qquad ...(61)$$

$$\varphi_2 = Ce^{\beta x} + De^{-\beta x} \qquad ...(62)$$

$$\varphi_3 = Ge^{iKx} + He^{-iKx} \qquad ...(63)$$

$$= Ge^{iKx} \qquad ...(64)$$

$$H = 0 \qquad \text{[Because no wave travels back from infinity in III region]}$$

Applying boundary condition to particle,

(*i*) At $x = 0$, we have

$$\varphi_1 = \varphi_2$$

$$\Rightarrow \quad [Ae^{iKx} + Be^{-iKx} = Ce^{\beta x} + De^{-\beta x}]_{x=0}$$

$$A + B = C + D \qquad ...(65)$$

We also have,

$$\frac{d\varphi_1}{dx} = \frac{d\varphi_2}{dx}$$

$$\Rightarrow \quad [iKAe^{iKx} - iKBe^{-iKx} = \beta Ce^{\beta x} - \beta De^{-\beta x}]_{x=0}$$

$$\Rightarrow \quad A - B = \frac{\beta}{iK}(C - D) \qquad ...(66)$$

From Eqs. (65) and (66), one obtains

$$A = \frac{1}{2}\left[\left(1 + \frac{\beta}{iK}\right)C + \left(1 - \frac{\beta}{iK}\right)D\right] \qquad ...(67)$$

and $$B = \frac{1}{2}\left[\left(1 - \frac{\beta}{iK}\right)C + \left(1 + \frac{\beta}{iK}\right)D\right] \qquad ...(68)$$

(*ii*) At $x = a$, we have

$$\varphi_2 = \varphi_3$$

$$\Rightarrow \quad [Ce^{\beta x} + De^{-\beta x} = Ge^{iKx}]_{x=a} \qquad ...(69)$$

$$\Rightarrow \quad Ce^{\beta a} + De^{-\beta a} = Ge^{iKa}$$

We also have,

$$\frac{d\varphi_2}{dx} = \frac{d\varphi_3}{dx}$$

$$\Rightarrow \quad [C\beta e^{\beta x} - D\beta e^{-\beta x} = GiKe^{iKx}]_{x=a}$$

$$\Rightarrow \quad C\beta e^{\beta a} - D\beta e^{-\beta a} = GiKe^{iKa} \qquad ...(70)$$

$$\Rightarrow \quad Ce^{\beta a} - De^{-\beta a} = \frac{iK}{\beta}Ge^{iKa}$$

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From Eqs. (71) and (72), one gets

$$C = \frac{1}{2}\left(1+\frac{iK}{\beta}\right)e^{-\beta a}\, Ge^{iKa} \qquad ...(71)$$

and

$$D = \frac{1}{2}\left(1+\frac{iK}{2}\right)e^{\beta a}\, Ge^{iKa} \qquad ...(72)$$

Putting value of C and D from Eqs. (71) and (72) in Eq. (67), we get

$$A = \frac{1}{4}\left(1+\frac{iK}{\beta}\right)Ge^{(iK-\beta)a}\left(1+\frac{\beta}{iK}\right)+\frac{1}{4}\left(1-\frac{iK}{\beta}\right)\left(1-\frac{\beta}{ik}\right)Ge^{(iK+\beta)a}$$

$$= \left[\left(\frac{e^{\beta a}+e^{-\beta a}}{2}\right)-\frac{1}{2}\left(\frac{\beta}{iK}+\frac{iK}{\beta}\right)\left(\frac{e^{\beta a}-e^{-\beta a}}{2}\right)\right]Ge^{iKa}$$

$$= \left[\cosh\beta a+\frac{i}{2}\left(\frac{\beta}{K}-\frac{K}{\beta}\right)\sinh\beta a\right]Ge^{iKa}$$

Now,

$$T = \frac{GG^*}{AA^*}\frac{v_1}{v_1} = \left|\frac{G}{A}\right|^2$$

Here,

$$\frac{A}{G} = \cosh\beta a+\frac{i}{2}\left(\frac{\beta}{K}-\frac{K}{\beta}\right)\sin\beta a$$

$$\left|\frac{A}{G}\right|^2 = \left(\frac{A}{G}\right)\left(\frac{A}{G}\right)^*$$

$$\left|\frac{A}{G}\right|^2 = \cosh^2\beta a+\frac{i}{4}\left(\frac{\beta}{K}-\frac{K}{\beta}\right)\sinh^2\beta a$$

$$= 1+\left[1+\frac{1}{4}\left(\frac{\beta}{K}-\frac{K}{\beta}\right)^2\right]\sinh^2\beta a = 1+\frac{1}{4}\left[\left(\frac{\beta}{K}-\frac{K}{\beta}\right)^2+4\right]\sinh^2\beta a$$

$$K = \frac{\sqrt{2mE}}{\hbar} \text{ and } \beta = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\frac{\beta}{K}-\frac{K}{\beta} = \sqrt{\frac{V_0-E}{E}}-\sqrt{\frac{E}{V_0-E}} = \frac{V_0-E-E}{\sqrt{E(V_0-E)}} = \frac{V_0-2E}{\sqrt{E(V_0-E)}}$$

$$\left(\frac{\beta}{K}-\frac{K}{\beta}\right)^2+4 = \frac{(V_0-2E)^2}{E(V_0-E)}+4 = \frac{V_0^2+4E^2-4EV_0+4EV_0-4E^2}{E(V_0-E)} = \frac{V_0^2}{E(V_0-E)}$$

$$\frac{1}{T} = 1+\frac{1}{4}\left[\frac{V_0^2}{E(V_0-E)}\right]$$

$$T = \frac{1}{1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\beta a} \qquad ...(73)$$

If $e^{\beta a} >> 1$

$$\sinh^2\beta a = \left(\frac{e^{\beta a}-e^{-\beta a}}{2}\right)^2 = \frac{e^{2\beta a}}{4} \qquad [e^{\beta a} \text{ is very large ; } e^{-\beta a} \text{ is very small}]$$

$$\frac{1}{T} = 1+\frac{V_0^2}{4E(V_0-E)}\frac{e^{2\beta a}}{4} = 1+\frac{V_0^2}{16E(V_0-E)}e^{2\beta a}$$

$$\frac{1}{T} = \frac{V_0^2}{16E(V_0-E)}e^{2\beta a}$$

$$T = \left[\frac{16E(V_0-E)}{V_0^2}\right]e^{-2\beta a}$$

$$T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a}$$

$$\beta = \frac{\sqrt{2m(V_0-E)}}{\hbar} \; ; \; T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-\frac{2\sqrt{2m(V_0-E)}a}{\hbar}}$$

$$T \approx e^{-2\beta a} \qquad ...(74)$$

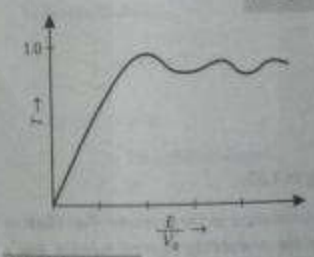

Fig. 6.5

If a particle with energy E incidents on a thin energy barrier of height greater than E, there is a finite probability of the particle penetrating the barrier. This phenomenon is called the **tunnel effect**.

This effect was used by George Gamow in 1928 to explain the phenomenon of α-decay exhibited by radioactive nuclei.

Applications of Tunnel Effect

1. Emission of α-particles from radioactive nuclei or α-decay. The average energy of an α-particle formed within the nucleus is less than the height of the potential barrier around the nucleus which is formed by the nuclear binding forces. Classically, the α-particle can not escape from the nucleus, but quantum mechanically it tunnels through the barrier. This tunneling constitutes radioactive α-decay.
2. Cold emission of electron : emission of electron by application of external electric field, not by heating material.
3. Transfer of electron from metal or metal to semiconductor.

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Example 6.8. *An electron in a metal encounters a barrier layer of height 6 eV and thickness 0.5 nm. If the electron energy of 5 eV, what is the probability of tunneling through the barrier ?*

Solution. We know,

$$T = \frac{16E}{V_0}\left[1 - \frac{E}{V}\right]\exp(-2\beta a)$$

With,

(i) $E = 5\,\text{eV}$ (ii) $V_0 = 16\,\text{eV}$

Thus, $2\beta a = \dfrac{2a[2m(V_0 - E)]^{1/2}}{h/2\pi}$

with

(i) $a = 0.5 \times 10^{-9}$ m. (ii) $m = 9.1 \times 10^{-31}$ kg

(iii) $V_0 = 6\,eV$ (iv) $E = 5V$

(v) $\dfrac{h}{2\pi} = \hbar = 1.054 \times 10^{-34}$ J.s

$$2\beta a = \frac{2 \times 0.5 \times 10^{-9}[2 \times 9.1 \times 10^{-31} \times (6-5) \times 16 \times 10^{-19}]^{1/2}}{1.054 \times 10^{-34}}$$

i.e., $2\beta a = 5.12$

Now, $T = \dfrac{16E}{V_0}\left[1 - \dfrac{E}{V_0}\right]\exp(-2\beta a)$

$$= \frac{16 \times 5}{6}\left[1 - \frac{5}{6}\right]\exp(-5.12) = 0.0133$$

Hence the percentage of probability of tunneling is 1.33.

Example 6.9. *The potential barrier problem is a good approximation to the problem of an electron trapped inside but near the surface of the metal. Calculate the probability of transmission that a 1.0 eV electron will penetrate a potential barrier of 4.0 eV when the barrier width is 2 Å.*

Solution. Since, we know that

$$T = \frac{16E}{V_0}\left(1 - \frac{E}{V_0}\right)\exp\left[-\frac{2a}{\hbar}(2m(V_0 - E))\right]$$

$$= 16\left(\frac{1.0\,\text{eV}}{4.0\,\text{eV}}\right)\left(1 - \frac{1.0\,\text{eV}}{4.0\,\text{eV}}\right)$$

$$\exp\left[-\frac{2 \times 2 \times 10^{-10}\,\text{m}}{1.05 \times 10^{-34}\,\text{Js}}\sqrt{2 \times (9.1 \times 10^{-31}\,\text{kg})(4-1) \times (1.6 \times 10^{-19})\,\text{J}}\right]$$

$$\approx 0.084.$$

Thus only about eight, 1 eV electrons, out of every hundred, penetrate the barrier.

Formulae at a Glance

1. Wave equation for a free particle

$$\varphi = Ae^{\frac{-1}{\hbar}(Et - px)}$$

where φ = wave function

E = Energy,

p = momentum

2. Energy operator

$$E = i\hbar\frac{\partial}{\partial t}$$

3. Momentum operator

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

4. $i\hbar\dfrac{\partial\varphi}{\partial t} = E\varphi$

φ = Eigen function of the operator $i\hbar\dfrac{\partial}{\partial t}$

E = energy eigen value

5. Normalisation

$$\int_{-\infty}^{\infty}|\varphi|^2 dV = 1$$

As $|\varphi|^2 = \varphi^*\varphi$

= Probability density

$= \rho$

6. Probability

$$P_{x_1x_2} = \int_{x_1}^{x_2}|\varphi|^2 dx$$

7. Expectation value

$$\langle x\rangle = \frac{\int_{-\infty}^{\infty}\varphi^*\varphi\, dx}{\int_{-\infty}^{\infty}\varphi^*\varphi\, dx}$$

For normalised function the

$$\int_{-\infty}^{\infty}\varphi^*\varphi\, dx = 1,$$

$$\langle x\rangle = \int_{-\infty}^{\infty}x|\varphi|^2 dx$$

8. Orthogonal wave function

$$\int\varphi_a^*\varphi_b\, dV = 0$$

or $\int\varphi_b^*\varphi_a\, dV = 0$ for $a \neq b$

9. Orthonormality

$$\int\varphi_a^*\varphi_b\, dV = \delta_{ab}$$

$\delta_{ab} = 0$ if $a \neq b$

$= 1$ if $a = b$

10. Schrodinger's time dependent wave Equation

For 1 – D,

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2} + V\varphi$$

For 3 – D,

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\varphi + V\varphi$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$H\varphi = E\varphi.$$

11. Schrodinger's time independent wave equation

For (1 – D), $\dfrac{\partial^2\varphi_0}{\partial x^2} + \dfrac{2m}{\hbar^2}(E - V)\varphi_0 = 0$

For (3 – D), $\nabla^2\varphi_0 + \dfrac{2m}{\hbar^2}(E - V)\varphi_0 = 0$

12. Schrodinger's wave equation for free particle ($V = 0$)

$$\nabla^2\varphi + \frac{2m}{\hbar^2}E\varphi = 0$$

or $E = \dfrac{\hbar^2k^2}{8\pi^2m}$

13. Particle in box

(i) Eigen value

$$E_n = \frac{n^2h^2}{8ma^2}$$

(ii) Eigen Function

$$\varphi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

where a is the width of the box.

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14. Potential step

$$K_1 = \frac{\sqrt{2mE}}{\hbar} \text{ and } K_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Reflectance (R)

$$= \frac{\text{magnitude of reflected current}}{\text{magnitude of incident current}}$$

Transmittance (T)

$$= \frac{\text{magnitude of transmitted current}}{\text{magnitude of incident current}}$$

Case (i) **When $E > V_0$**

$$R = \frac{(K_1 - K_2)^2}{(K_1 + K_2)^2}, \quad T = \frac{4K_1K_2}{(K_1 + K_2)^2} \qquad R + T = 1$$

Case (ii) **When $E < V_0$**

$$T = 0, \quad R = 1, \quad R + T = 1$$

15. Barrier Potential

$$K = \frac{\sqrt{2mE}}{\hbar} \text{ and } \beta = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \beta a}$$

Case (i) $e^{\beta a} >> a$

$$T = \frac{16E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-\frac{2\sqrt{2m(V_0 - E)}\,a}{\hbar}}$$

Miscellaneous Numerical Problems for Practice

Example 1. *Calculate the ground state energy of an electron confined to a box 1 Å wide.*

[GGSIPU, April 2008 (2 marks)]

Solution. We have, $m = 9.1 \times 10^{-31}$ kg, $a = 1 \times 10^{-10}$ m

Since energy of electron in nth state is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$= \frac{n^2h^2}{8ma^2}$$

For ground state,

$$E_1 = \frac{h^2}{8ma^2}$$

$$= \frac{(6.625 \times 10^{-34})^2}{8}$$

$$E_1 = 6.016 \times 10^{-18} \text{ J}$$

Example 2. *An electron is confined to move between two rigid walls separated by 10^{-9} m. Find the de Broglie wavelengths representing the first three allowed energy states of the electron and the corresponding energies (electron mass is 9.1×10^{-31} kg and $h = 6.63 \times 10^{-34}$ Js).*

Solution. An electron moving back and forth between rigid walls will form a stationary wave pattern with nodes at the walls. For this, the distance between the walls must be an integral multiple of the de-Broglie half wavelengths.

Thus, $$a = n\frac{\lambda}{2}$$

where $n = 1, 2, 3, \ldots$

or $$\lambda = \frac{2a}{n}$$

Here $a = 10^{-9}$ m $= 10$ Å

$$\lambda = \frac{2 \times 10}{n} \text{ Å}$$

$$\lambda = \frac{2 \times 10 \text{ Å}}{n} \quad (n = 1, 2, 3, \ldots)$$

$$= 20 \text{ Å}, 10 \text{ Å}, 6.7 \text{ Å}, \ldots$$

The corresponding energies are :

$$E_n = \frac{n^2h^2}{8ma^2}$$

Here $m = 9.1 \times 10^{-31}$ kg and $a = 10^{-9}$ m

$$E_n = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-9})^2} \text{ J}$$

$$= 6.04 \times 10^{-20}\, n^2 \text{ J}$$

$$= \frac{6.04 \times 10^{-20}\, n^2}{1.6 \times 10^{-19}} \text{ eV}$$

$$= 0.38\, n^2 \text{ eV}$$

For $n = 1, 2, 3, \ldots$ we have

$$E_1 = 0.38 \text{ eV}, \; E_2 = 1.52 \text{ eV}, \; E_3 = 3.42 \text{ eV}, \ldots$$

Example 3. *An electron is confined to a one-dimensional of side 1 Å. Obtain the first four eigen value of energy in eV of the electron.*

Solution. $E_n = \dfrac{n^2h^2}{8ma^2}$

$\Rightarrow$ $E_n = 38\, n^2 \text{eV}, \; a = 1 \text{ Å}$

$E_1 = 38$ eV, $E_2 = 152$ eV, $E_3 = 342$ eV, $E_4 = 608$ eV.

Example 4. *A particle of mass m is moving in an infinitesimal deep square well potential extending from $x = 0$ to $x = a$. Show that $\langle E \rangle = \dfrac{n^2h^2}{8ma^2}$.*

Or

Prove that the value of energy obtained for a particle of mass m moving in a one dimensional box can also be obtained with the help of relation used to obtain expectation value.

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Solution. The normalised wave function for a particle of mass m in one-dimensional box is given by

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

Now the energy operator $E = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

So, that $$\langle E \rangle = \int_0^a \varphi_n^*(x)\, \hat{E}\, \varphi_n(x)\, dx$$

$$= -\frac{\hbar^2}{2m}\int_0^a \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} \times \frac{d^2}{dx^2}\left(\sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}\right) dx$$

$$= -\frac{\hbar^2}{2m} \times \frac{2}{a} \int_0^a \sin\frac{n\pi x}{a}\left(-\frac{n^2\pi^2}{a^2}\right)\sin\frac{n\pi x}{a}\, dx$$

$$= \frac{n^2\pi^2\hbar^2}{ma^3}\int_0^a \sin^2\frac{n\pi x}{a}\, dx$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^3}\int_0^a \left[1 - \cos\frac{2n\pi x}{a}\right] dx$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^3} \times a = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{n^2h^2}{8ma^2}$$

Example 5. *A proton or a neutron in a nucleus can be roughly regarded as a particle in a box ; the surface of the nucleus plays the role of the wall of the box, and the proton moves more or less freely between these walls. What is the energy released when a proton makes a transition from first excited state to the ground state of a box of nuclear size, say 10×10^{-14} m ?*

Solution. With $a = 1 \times 10^{-14}$ m and $m = m_p = 167 \times 10^{-27}$ kg.

The energy of the ground state (for $n = 1$) is

$$E_1 = \frac{\pi^2\hbar^2}{2m_p a^2} = 33 \times 10^{-13}\ \text{J}.$$

The energy of the first excited state is four times as large

$$E_2 = 4E_1 = 132 \times 10^{-13}\ \text{J}$$

Hence the energy released in the difference

$$(E_2 - E_1) = 132 \times 10^{-13}\ \text{J} - 33 \times 10^{-13}\ \text{J} = 99 \times 10^{-13}\ \text{J} = 62\ \text{MeV}.$$

Example 6. *At a certain time, the normalised wave function of a particle moving along the x-axis has the form*

$$\psi(x) = x + \beta \quad \text{for} \quad -\beta < x < 0$$
$$= -x + \beta \quad \text{for} \quad 0 < x < \beta$$

and zero elsewhere. Find the value of β and the probability that the particles position is between $x = \beta/2$ and $x = \beta$.

Solution. The probability of finding the particle is given by

$$\int |\psi|^2\, dx = 1$$

For the present case, above equation can be modified as

$$-\int_\beta^0 (x+\beta)^2\, dx + \int_0^\beta (-x+\beta)^2\, dx = 1$$

or $$-\int_\beta^0 (x^2+\beta^2+2\beta x)\, dx + \int_0^\beta (x^2+\beta^2-2\beta x)\, dx = 1$$

or $$\left[\frac{x^3}{3} + \beta^2 x + 2\beta\frac{x^2}{2}\right]_{-\beta}^{0} + \left[\frac{x^3}{3} + \beta^2 x - 2\beta\frac{x^2}{2}\right]_0^\beta = 1$$

or $$\frac{\beta^3}{3} + \frac{\beta^3}{3} = 1 \quad \Rightarrow \quad \beta = \left(\frac{3}{2}\right)^{1/3}$$

Now, the probability of finding the particle between $\beta/2$ and β is

$$P = \int_{\beta/2}^{\beta} |\psi|^2\, dx = \int_{\beta/2}^{\beta} (-x+\beta)^2\, dx$$

or $$P = \int_{\beta/2}^{\beta} (x^2+\beta^2-2\beta x)\, dx = \left[\frac{x^3}{3} + \beta^2 x - 2\beta\frac{x^2}{2}\right]_{\beta/2}^{\beta} = \frac{7\beta^3}{24}$$

$$\because \quad \beta = \left(\frac{3}{2}\right)^{1/3} \quad \therefore \quad P = \frac{7 \times 3}{24 \times 2} = 0.43$$

or $$P = 43\%.$$

Example 7. *A particle is moving in a one dimensional box (of infinite height) of width 10 Å. Calculate the probability of finding the particle within an interval of 1 Å at the centre the box, when it is in its state of least energy.*

Solution. The wave function of the particle in the ground state ($n = 1$) is

$$\varphi_1 = \sqrt{\left(\frac{2}{a}\right)} \sin\frac{\pi x}{a}$$

The probability of finding the particle in unit interval at the centre of the box $\left(x = \frac{a}{2}\right)$ is

$$P = |\varphi_1(x)|^2\, \Delta x = \left(\frac{2}{a}\right)\Delta x \text{ at } x = \frac{a}{2}.$$

With

(i) $a = 10$ Å (ii) $\Delta x = 1$ Å

Thus, $P = \frac{2}{10} \times 1 = 0.2$

$$P = 0.2$$

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Example 8. *A particle is moving in a one-dimensional potential box of infinite height. What is the probability of finding the particle in a small interval Δx at the centre of the box when it is in the energy state, next to the least energy state.*

Solution. The wavefunction of the particle in the first excited state ($n = 2$) is

$$\varphi_2 = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}$$

At the centre of the box $\left(x = \frac{a}{2}\right)$, then $\varphi_2 = 0$

and probability $\quad P = |\varphi_2(x)|^2 \Delta x = 0$

$$P = 0.$$

Example 9. *The normalised state of a free particle is represented by a wave function*

$$\varphi(x) = Ne^{-(x^2/2a^2)} e^{ikx}$$

(a) *Find :*

(i) *the factor N* (ii) *the expectation value of position and momentum.*

(b) *What region of space, the particle is most likely to be found ?*

Solution. (a) (i) This factor N, being the normalisation constant can be found from the normalisation condition as follows :

$$\int_{-\infty}^{\infty} \varphi^* \varphi \, dx = 1$$

or $\quad |N|^2 \int_{-\infty}^{\infty} (e^{-x^2/2a^2} e^{-ikx})(e^{-x^2/2a^2} e^{-ikx}) dx = 1 \qquad$ (where $|N|^2 = NN^*$)

or $\quad |N|^2 \int_{-\infty}^{\infty} e^{-x^2/a^2} dx = 1$

or $\quad |N|^2 . a\sqrt{\pi} = 1$

or $\quad |N| = \frac{1}{a^{1/2} \pi^{1/4}}$

or $\quad N = \frac{1}{a^{1/2} \pi^{1/4}} e^{i\phi}$

where ϕ is a factor, which is arbitrary.

(ii) The expectation value position coordinate x is given by

$$\langle x \rangle = \int_{-\infty}^{\infty} \varphi^* x \varphi dx = \int_{-\infty}^{\infty} N^* e^{-(x^2/2a^2)-ikx} . x . Ne^{-(x^2/2a^2)+ikx} dx$$

$$= NN^* \int_{-\infty}^{+\infty} e^{-(x^2/a^2)} x \, dx = 0$$

The expectation value of momentum coordinate p_x is given by

$$\langle p_x \rangle = \int_{-\infty}^{\infty} \varphi^* p_x \varphi dx$$

$$= \int_{-\infty}^{\infty} N^* e^{-(x^2/2a^2)-ikx} \left(\frac{\hbar}{i} \frac{\partial}{\partial x}\right) \{Ne^{-(x^2/2a^2)+ikx}\} dx$$

$$= \int_{-\infty}^{\infty} N^* e^{-(x^2/2a^2)-ikx} \left(\frac{\hbar}{i}\right) Ne^{-(x^2/2a^2)+ikx} \left(-\frac{2x}{2a^2} + ik\right) dx$$

$$= NN^* \frac{i\hbar}{a^2} \int_{-\infty}^{\infty} e^{-x^2/a^2} x \, dx + NN^* \int_{-\infty}^{\infty} \hbar k e^{-x^2/a^2} dx$$

$$= 0 + \hbar k = \hbar k$$

So, that $\quad \langle p_x \rangle = \hbar k$ and $\langle x \rangle = 0$.

(b) The probability density is given by

$$P(x) = |\varphi(x)|^2 = \varphi^* x \varphi(x) = |N|^2 e^{-x^2/a^2} = \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$

According to this equation, the probability is maximum for $x = 0$ and decreases exponentially for $|x| > a$. So, the particle is most likely to be found in a region with width 'a' on either side of the origin $x = 0$.

Example 10. *Normalise the wave function*

$$\phi(x) = e^{-|x|} \sin \alpha x$$

Solution. Let the normalised wave function be

$$\varphi(x) = A\phi(x)$$

where A is the normalisation constant.

From the normalisation property, we have

$$\int_{-\infty}^{\infty} \varphi \varphi^* dx = 1$$

i.e., $\quad \int_{-\infty}^{\infty} A^* \phi^*(x) A\phi(x) dx = 1$

or $\quad AA^* \int_{-\infty}^{\infty} \phi^*(x) \phi(x) dx = 1$

or $\quad |A|^2 \int_{-\infty}^{\infty} e^{-|x|} \sin \alpha x \, e^{-|x|} \sin \alpha x \, dx = 1$

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or $$\frac{|A|^2}{2}\int_{-\infty}^{\infty} e^{-2|x|}\, 2\sin^2 \alpha x\, dx = 1$$

or $$\frac{|A|^2}{2}\int_{-\infty}^{\infty} e^{-2|x|}(1-\cos 2\alpha x)\, dx = 1$$

or $$\frac{|A|^2}{2}\left[\int_{-\infty}^{\infty} e^{-2|x|} dx - \int_{-\infty}^{\infty} e^{-2|x|}\cos 2\alpha x\, dx\right] = 1$$

or $$\frac{|A|^2}{2}\left[\int_{-\infty}^{\infty} e^{-2|x|} dx - \int_{-\infty}^{\infty} e^{-2|x|}\cos 2\alpha x\, dx\right] = 1$$

or $$\frac{|A|^2}{2}\left[0-\left(-\frac{\alpha^2}{1+\alpha^2}\right)\right] = 1$$

or $$\frac{|A|^2}{2}\,\frac{\alpha^2}{1+\alpha^2} = 1$$

So, that $$|A|^2 = \sqrt{\left[\frac{2(1+\alpha)^2}{\alpha^2}\right]}$$

So, the normalised wave function is given by

$$\varphi(x) = A\phi(x) = \sqrt{\left[\frac{2(1+\alpha^2)}{\alpha^2}\right]}\, e^{-|x|}\sin \alpha x$$

Example 11. *An eigen-function of an operator $\frac{d^2}{dx^2}$ is $\varphi = e^{\alpha x}$. Find the corresponding eigen value.*

Solution. Let $G = \frac{d^2}{dx^2}$

$$G\varphi = \frac{d^2}{dx^2}(e^{\alpha x}) = \frac{d}{dx}\left[\frac{d}{dx}(e^{\alpha x})\right]$$

$$= \frac{d}{dx}(\alpha e^{\alpha x}) = \alpha^2 e^{\alpha x}$$

But $e^{\alpha x} = \varphi$

$\therefore$ $G\varphi = \alpha^2\varphi$

Hence eigen value $G = \alpha^2$.

Example 12. *Find the value of normalisation constant A for the wave function $\varphi = Axe^{-x^2/2}$.*

[Given that $\int_0^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{4}\sqrt{\frac{4}{\alpha^3}}$]

Solution. Normalisation : $\int_{-\infty}^{\infty} |\varphi|^2 dx = 1$

Here $\varphi = Axe^{-x^2/2}$

$$A^2\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = 1$$

$\Rightarrow$ $$2A^2\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = 1 \qquad \text{[due to even function]}$$

Given that $\int_0^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$

Here $\alpha = 1$ $$2A^2\,\frac{1}{4}\,\frac{\sqrt{\pi}}{1} = 1$$

$$A^2 = \frac{2}{\sqrt{\pi}} \quad \text{or} \quad A = \frac{\sqrt{2}}{(\pi)^{1/4}}$$

Example 13. *The wave function of a certain particle is $\varphi = A\cos^2 x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$*

(a) Find the value of A (b) Find the probability that the particle be found between $x = 0$ and $x = \frac{\pi}{4}$.

Solution. As $\varphi = A\cos^2 x$ $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

(a) $$\int_{-\pi/2}^{\pi/2} |\varphi|^2 dx = 1$$

$\Rightarrow$ $$2A^2\int_0^{\pi/2} \cos^4 x\, dx = 1$$

or $$2A^2\,\frac{3\pi}{16} = 1$$

$$\frac{3\pi}{8}A^2 = 1 \quad \therefore \quad A = \sqrt{\frac{8}{3\pi}}$$

(b) $$P = \int_0^{\pi/4} |\varphi|^2 dx = A^2\int_0^{\pi/4} \cos^4 x\, dx \quad \text{as} \quad A = \sqrt{\frac{8}{3\pi}}$$

$$P = \frac{8}{3\pi}\int_0^{\pi/4} \cos^4 x\, dx$$

$$P = 0.462$$

Example 14. *An electron and a proton with the same energy E approach a barrier potential whose height V is greater than E, which have larger probability to be the another side of barrier ?*

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Solution. The transmittance in case of tunneling effect is given by following equation

$$T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)\exp\left(-\frac{2a\sqrt{[2m(V_0-E)]}}{\hbar}\right)$$

Here
$$T_e = 16\left(\frac{E}{V}\right)\left(1-\frac{E}{V}\right)\exp\left(\frac{-2a\sqrt{[2m_e(V-E)]}}{\hbar}\right)$$

and
$$T_p = 16\left(\frac{E}{V}\right)\left(1-\frac{E}{V}\right)\exp\left(\frac{-2a\sqrt{[2m_p(V-E)]}}{\hbar}\right)$$

$$\Rightarrow \quad \frac{T_e}{T_p} = \frac{\exp\left(-\frac{2a}{\hbar}\sqrt{2m_e(V-E)}\right)}{\exp\left(-\frac{2a}{\hbar}\sqrt{2m_p(V-E)}\right)}$$

$\because \quad m_e < m_p$

$\Rightarrow \quad T_e > T_p$

Hence electron possesses greater probability.

Example 15. *An electron is in a box of 0.01 nm. Find its permitted energy.*

[GGSIPU, May 2008 (2.5 marks)]

Solution. $m = 9.1\times10^{-31}$ kg, $a = 0.01$ nm $= 10\times10^{-11}$ m

$E_n = ?$

$$E_n = \frac{n^2h^2}{8ma^2} = \frac{n^2\times(6.626\times10^{-34})^2}{8\times9.1\times10^{-22}}$$

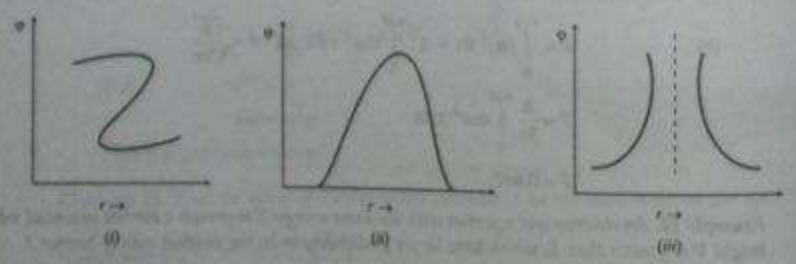
$$= 6\times10^{-16}\,n^2 \text{ joule} = \frac{6\times10^{-16}}{16\times10^{-19}}\text{ eV} = \frac{60000}{16}n^2 \text{ eV}.$$

For $n = 1, 2, 3, \ldots$ the energy eigen value $E_1, E_2, E_3, \ldots$

$E_1 = 3750$ eV, $E_2 = 15000$ eV, $E_3 = 33750$ eV, ...

Example 16. *Which of the following wave functions may represent a physical system ?*

[GGSIPU, May 2008 (2 marks)]



Solution. A function is called the correct physical function, if it is finite, single valued and continuous.

The function (*i*) has multivalues at single position and (*ii*) goes to infinity and also discontinuous, hence these are not correct physical functions.

Only (*iii*) function represents the physical system.

Example 17. *If $\varphi_1(x,t)$ and $\varphi_2(x,t)$ are two solutions of the Schrödinger's equation*

$$i\hbar\frac{\partial\varphi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi(x,t)}{\partial x^2} + V(x,t)\varphi(x,t)$$

Show that (a) $a\varphi_1$ and (b) $a\varphi_1 + b\varphi_2$ are also solutions of this equation, where a and b are arbitrary constant.

Solution. (*a*) Substituting $\varphi = a\varphi_1$ in given equation, we obtain

$$i\hbar a\frac{\partial\varphi_1}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi_1}{\partial x^2}\right) + V(x,t)a\varphi_1(x,t)$$

or
$$i\hbar\frac{\partial\varphi_1}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi_1}{\partial x^2} + V(x,t)\varphi_1(x,t) \quad \text{...(i)}$$

since φ_1 satisfies given equation.

(*b*) Now substituting $\varphi = a\varphi_1 + b\varphi_2$ in the given equation

$$i\hbar\left(a\frac{\partial\varphi_1}{\partial t} + b\frac{\partial\varphi_2}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(a\frac{\partial^2\varphi_1}{\partial x^2} + b\frac{\partial^2\varphi_2}{\partial x^2}\right) + Va\varphi_1 + Vb\varphi_2$$

or
$$a\left(i\hbar\frac{\partial\varphi_1}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\varphi_1}{\partial x} - V\varphi_1\right) + b\left(i\hbar\frac{\partial\varphi_2}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\varphi_2}{\partial x^2} - V\varphi_2\right) = 0 \quad \text{...(ii)}$$

Since φ_1 and φ_2 satisfy the given equation, the Eq. (*ii*) is identically zero. Hence φ satisfies the given equation.

Question Bank

Multiple Choice Questions

1. The wave function associated with a free electron is

   (*a*) $\varphi = A\exp(-iEt/\hbar)$ (*b*) $\varphi = A\exp(-\varphi_x/\hbar)$

   (*c*) $\varphi = A\exp\left[\frac{-i}{\hbar}(Et - px)\right]$ (*d*) none of these

2. If E_1 is the energy of the electron in the ground state of a one-dimensional potential box of length *a*, E_2 is the energy of the electron in the second energy state of a one dimentional potential box of length $a/3$, then $E_1 - E_2$ is

   (*a*) $-\frac{h^2}{ma^2}$ (*b*) $-\frac{10h^2}{ma^2}$ (*c*) $\frac{h^2}{ma^2}$ (*d*) none of these.

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4. The wave function for the motion of a particle in a one dimensional potential box of length a is given by $\varphi_n = A\sin\left(\frac{n\pi x}{a}\right)$, where A is the normalization constant. Its value is

(a) $\left(\frac{2}{a}\right)^{1/2}$ (b) $\frac{2}{a}$ (c) a (d) $\frac{1}{a}$

5. If several combinations of the same set of quantum numbers gives different energy states, but with the same energy value, then the state is said to be

(a) degenerate energy state (b) non-degenerate energy state
(c) excited energy state (d) none of these.

6. The expression for the discrete energy values of the electrons in a cubical box of lattice parameter a is $\frac{n^2h^2}{8ma^2}$, where n is

(a) sum of the square of three integers. (b) an integer
(c) square root of an integer (d) none of these.

7. The energy of the lowest state in one dimensional potential box of length a is

(a) zero (b) $\frac{2h^2}{8ma^2}$ (c) $\frac{h^2}{8ma^2}$ (d) $\frac{h}{8ma^2}$

8. The normalised wave function for a particle in a rectangular box of dimension a, b and c is

(a) $\varphi_n = \left(\sqrt{\frac{4}{abc}}\right)\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{b}\right)\sin\left(\frac{n_z\pi z}{c}\right)$ (b) $\varphi_n = \left(\sqrt{\left(\frac{8}{a}\right)}\right)\sin\left(\frac{n_x\pi x}{a}\right)$

(c) $\varphi_n = \left[\sqrt{\left(\frac{2}{abc}\right)}\right]\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{b}\right)\sin\left(\frac{n_z\pi z}{c}\right)$ (d) none of these.

9. The spacing between nth energy level and the next higher level in a one-dimensional potential box increase by

(a) $(2n-1)$ (b) $(2n+1)$ (c) $(n-1)$ (d) $(n+1)$

10. If E_1 is the energy of the electron in the lowest state of a one-dimensional potential box of length a and E_2 is the energy of the electron in the first and second energy state respectively, then $\frac{\varphi_1}{\varphi_2}$ is

(a) 0 (b) ∞ (c) 1 (d) 2

11. If φ_n is the wave function associated with an electron in a one dimensional potential box of length a, then $|\varphi_n|$ can not be used to represent the probability of finding the electron at a point x, because

(a) φ_n has only positive values. (b) φ_n has only negative values.
(c) φ_n has both negative and positive values. (d) none of these.

12. The wave function φ of a particle is given by

$$\varphi = Ae^{-Kx} \quad \text{for} \quad 0 < x < \infty$$
$$= 0 \quad \text{for} \quad -\infty < x < 0$$

[GGSIPU, May 2006 (2.5 marks)]

The value of A

(a) $\sqrt{\frac{2}{K}}$ (b) $\sqrt{\frac{K}{2}}$ (c) $\sqrt{K}$ (d) $\sqrt{2K}$

13. If a one dimensional box of length a, the probability of finding a particle will be maximum at

(a) $\frac{a}{2}$ (b) a (c) zero (d) $\frac{a}{4}$

14. In case of potential step of height V_0. If a particle of energy of energy $E < V_0$ the transmittance is :

(a) zero (b) finite non-zero (c) infinite (d) 1.

15. The energies of a particle in a box are given by

(a) continuous energy spectrum (b) $\frac{n^2\pi^2\hbar^2}{2ma^2}$

(c) $\frac{\pi^2\hbar^2}{2ma^2n^2}$ (d) $n\frac{h}{2\pi}$

16. According to quantum mechanics, a free particle can possess

(a) discrete energies (b) continuous energies
(c) only one single value of energy (d) none of these.

17. The expectation value of momentum p is

(a) $\int\varphi^*\varphi\left(\frac{\hbar}{i}\nabla\right)d\tau$ (b) $\int\frac{\hbar}{i}\nabla(\varphi^*\varphi)d\tau$ (c) $\int\varphi^*\left(\frac{\hbar}{i}\nabla\right)\varphi d\tau$ (d) $\int\varphi\left(\frac{\hbar}{i}\nabla\right)\varphi^* d\tau$

18. The expression $|\varphi(r,t)|^2$ stand for

(a) position (b) position probability density
(c) normalisation (d) time probability density.

19. According to Sohrödinger a particle is equivalent to a :

(a) a single wave (b) a wave packet
(c) a light wave (d) can not behave as wave.

20. Electrons with energy E are incident on a potential step of height V such that $E > V$. Which of the following is wave mechanically true ?

(a) Electrons are transmitted
(b) Electrons will be reflected
(c) Partial reflection and partial transmission will take place.
(d) none of these.

21. The allowed energy values of a particle in a box of length a are

(a) $\frac{nh}{2\pi}$ (b) $\frac{n^2\pi^2\hbar^2}{2ma^2}$ (c) $\frac{\pi^2\hbar^2}{2ma^2n^2}$ (d) $\frac{n\pi\hbar}{2ma}$

22. Each expectation value is

(a) an estimated value (b) a real value
(c) an observable quantity (d) none of above is fully correct.

23. Schrödinger equation can be written as

(a) $H\varphi = 0$ (b) $E\varphi = 0$ (c) $H\varphi - E\varphi = 0$ (d) $H(\varphi) + E(\varphi) = 0$

24. An acceptable wave function φ associated with a moving material particle must be

(a) Infinite, single valued and continuous. (b) Finite, single valued and continuous.
(c) Finite, multiple value and continuous. (d) Finite, multiple value and discontinuous.

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Answers

1. (d)	2. (a)	3. (d)	4. (a)	5. (a)	6. (a)
7. (d)	8. (b)	9. (b)	10. (b)	11. (c)	12. (a)
13. (a)	14. (a)	15. (b)	16. (a)	17. (c)	18. (b)
19. (b)	20. (c)	21. (b)	22. (a)	23. (a)	24. (b)

Theoretical Questions

1. Give the properties of an accepted wave function. Derive the steady state of Schrödinger's wave equation for a non-relativistic particles. **[GGSIPU, May 2007 (4.5 marks)]**
2. A particle moves in a one-dimensional potential whose function is given by

$$V(x) = 0 \quad \text{for} \quad x < 0$$
$$= V_0 \quad \text{for} \quad x \geq 0$$

Write down the Schrödinger wave equation for the particle and find its solution. Find the reflection and transmission coefficient for $E > V_0$ and $E < V_0$, where E is the total energy of the particle. **[GGSIPU, May 2007 (5 marks)]**

3. Explain the requirement that are imposed on a physical acceptable wave function. Why is a wave function said to be normalised ? What do you understand by "Eigen value" and "Eigen function" ? **[GGSIPU, May 2006 (5 marks)]**
4. Write a short note on "Energy quantization for particle in a one-dimensional potential box" **[GGSIPU, May 2007 (2.5 marks)]**
5. Discuss the properties of wave functions in a quantum mechanical system. **[GGSIPU, April 2007 (2 marks)]**
6. Obtain time dependent Schrödinger equation and hence derive steady state form. Give significance of the equation. **[GGSIPU, April 2007, (5 marks)]**
7. What is potential barrier and tunnel effect ? Calculate the transmission probability for rectangular barrier for the condition of $E < V_0$, where E is energy of particle and V_0 is barrier height. **[GGSIPU, April 2007 (5 marks)]**
8. The time dependent Schrödinger's equation is written as $H\psi = E\psi$. Explain the physical significance of H. Find the solution of H. **[GGSIPU, Feb 2008 (5 marks)]**
9. A particle of mass 'm' and energy 'E' is approaching a single step barrier given by $V = 0$ for $x < 0$ and $V = V_0$ for $x \geq 0$. Calculate the transmission and reflection coefficients for the case when $E < V_0$. How do the quantum mechanical results vary from the classical predictions in this case ? Explain quantum mechanical tunneling. **[GGSIPU, May 2005 (2.5 marks)]**
10. Write down the expression for the time dependent Schrödinger's equation. **[GGSIPU, May 2005 (12.5 marks)]**
11. Set up the Schrödinger equation for a particle in an infinite well (one-dimensional). Solve it for eigen values and eigen functions and plot the first three eigen functions ψ_1, ψ_2 and ψ_3. **[GGSIPU, May 2005 (12.5 marks)]**

12. Why wave functions are to be normalised ? **[GGSIPU, May 2008 (2.5 marks)]**
13. Derive one-dimensional time dependent Schrödinger wave equation for non-relativistic particle. Give the physical interpretation of the Hamiltonian operator H. **[GGSIPU, April 2008 (6 marks)]**
14. Write down the Schrödinger's equation for a free particle confined to move in a plane and find out the allowed energy values.
15. Which of the following forms of the wave function are acceptable in quantum mechanics.

    (i) $\sin x$ (ii) $\tan x$ (iii) e^{-x} (iv) e^{-x^2}
16. A beam of particles of mass m and energy E is incident on a step potential of height V_0 from the left. Discuss the solution for $E < V_0$ from the left and explain how tunneling can be understood with violation of energy conservation principle. Give two practical examples of quantum tunnel effect.
17. Find the eigen values and eigen functions of a particle moving in a one dimensional square well potential with infinitely high walls.
18. What is expectation value ?
19. Explain what is meant by quantum tunneling. How can it be understood on the basis of uncertainty principle.
20. Find the eigen values and functions of a particle moving in one-dimensional square well potential with infinitely high walls. Can the particle in a such a potential over have zero total energy ? not, then explain clearly, why not ?
21. Write down the postulates of quantum mechanics.
22. Find one-dimension motion of a particle of mass m is described by the following equation

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \varphi = E\psi$$

where all symbols have their usual meaning. Find the energy eigen values of the particle.

23. Establish Schrödinger time independent wave equation.
24. What are the orthogonal wave functions ? The wave function of a particle is given by $\varphi = C\exp(-x^2\alpha^2)$ in the interval $-\infty < x < +\infty$, where C and α are constant. Calculate the probability of finding the particle in the region $0 < x < \infty$.
25. What is the meaning of normalisation of wave function ? How would you normalise a given wave function $\varphi(x) = x^{-1/4}\sin\alpha x$ in the interval $-\infty \leq x \leq \infty$.
26. Derive Schrödinger's time dependent and time independent wave equations.
27. Derive Schrödinger's time independent wave equation. A particle is in motion along a line between $x = 0$ and $x = a$ with zero potential energy. At points for which $x < 0$ and $x > a$, the potential energy is infinite. Solving Schrödinger's equation for the particle.
28. Derive Schrödinger's time-independent and time dependent equations for matter waves. What is physical significance of the wave function ?
29. Using Schrödinger wave equation, calculate the energies and the wave functions of the particle in a box. (Treat one-dimensional case)
30. Set up the Schrödinger wave function for particle in a box. Solve this equation to find out its wave function and energy. Plot the wave functions for the ground state, first and second excited states.

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31. A particle is in motion along a line between $x = 0$ and $x = a$ with zero potential energy. At points for which $x < 0$ and $x > a$, the potential energy is infinite. The wave function for the particle in the nth state is given by $\varphi_n = A\sin\frac{n\pi x}{a}$. Find the expression for the normalised wave function.

32. Write the Schrödinger wave equation for the particle in a box and solve it to obtain the eigen values and eigen functions.

33. (a) What is significance of wave function ?

(b) Define an operator, eigen function and eigen values ?

34. The potential function for a particle moving along the positive x-direction is given by $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$. Calculate transmittance and reflectance at the potential discontinuity and show that $R + T = 1$.

Numerical Problems

1. A grain of sand of mass 1×10^{-7} kg is confined in a potential well with a dimension $a = 1.0$mm. Calculate the smallest energy which is the grain would possess in electron volt.

[Ans. 3.1×10^{-42} eV]

2. A series of measurements on a one-dimensional system leaves the system with the wave function $\varphi = A\sin\frac{\pi x}{a}$ for $0 \le x \le L$ and zero else where.

(i) Find the normalisation constant A. (ii) Plot the wave function.

(iii) Find the probability that the particle trapped in the system can be found between 0.45 a and 0.55 a.

[GGSIPU, April 2008 (4 marks)]

3. Write an expression for the eigen values of a particle of mass 100 gms confined in a 1 - D box of side 10 cm with perfectly rigid walls ($h = 6.63 \times 10^{-34}$ Js). State whether the permitted energy levels are discrete or continuous.

[GGSIPU, May 2007 (2.5 marks)]

4. Electrons with energies of 1.0 eV and 2.0 eV are incident on a barrier 10.0 eV high and 0.50 nm wide. (a) Find the respective transmission probabilities (b) How are these affected if the barrier is double in width ? [Ans. (a) $T_1 = 1.1 \times 10^{-7}$, $T_2 = 2.4 \times 10^{-7}$ (b) $T_1' = 1.3 \times 10^{-14}$, $T_2' = 5.1 \times 10^{-4}$]

5. An α-particle is trapped in a nucleus whose radius is $r_0 = 14 \times 10^{-15}$ m. What is the probability that an α-particle will escape from the nucleus of its energy is 2 MeV ? The potential barrier at the surface of the nucleus is 4 MeV.

[Ans. 0.124]

6. Calculate the lowest energy that a neutron can have in a box of rigid ball of length 10^{-14} m.

[Ans. 3.28×10^{-13} J]

7. The wave function for a particle in one-dimensional box is given by $\varphi(x) = \sqrt{\left(\frac{2}{a}\right)}\sin\frac{n\pi x}{a}$. Show that the wave function for two different states are orthogonal for such a particle.

[GGSIPU, May 2007 (2.5 marks)]

8. A stream of electrons, each of energy $E = 3$eV is incident on a potential barrier of height $V_0 = 4$eV. The width of barrier is 20 Å. Calculate the percentage transmission of the beam through this barrier.

[Ans. $T = 3.797 \times 10^{-9}$, %transmission $= 3.797 \times 10^{-7}$%]

9. A beam of monoenergetic electrons strikes the surface of a metal at normal incidence. Calculate the reflection probability of these electrons if $E = 0.5$eV and $V_0 = 8$eV.

[Ans. 2 eV]

10. An α-particle having energy 10 MeV approaches a potential barrier of height equal to 30 MeV. Determine the width of the potential barrier if the transmission coefficient is 2×10^{-3}. Given mass of α-particle $= 4 \times 1.67 \times 10^{-27}$ kg and $\hbar = 1.054 \times 10^{-34}$ Js

[Ans. $a = 1.91 \times 10^{-15}$ m]

11. The wave function φ of a particle is given by

$$\varphi = N\exp\left(-\frac{x^2}{2\beta}\right) \text{ for } -\infty < x < \infty. \text{ Find } N.$$

[Ans. $N = \left(\frac{1}{\pi\beta}\right)^{1/4}$]

12. Find the width of the potential barrier for an α-particle emitted with a kinetic energy 5.5 MeV from ${}_{86}Rn^{222}$. $r_0 = 1.5 \times 10^{-15}(M_A)^{1/3}$ m.

[Ans. $a = 34.88 \times 10^{-15}$ m]

13. An electron is trapped in an one-dimensional box of length 0.1 nm. Calculate the energy required to excite the electron from its ground state to the fifth excited state.

[Ans. $E_0 = 6.022 \times 10^{-18}$ J, $E_1, E_2, E_3 \ldots E_5 = 2.168 \times 10^{-16}$ J]

14. Show taht $\phi(x) = e^{iKx}$ is acceptable eigen function where K is some finite constant. For a region $-a \le x \le a$ normalise the given eigen function.

[Ans. $\varphi(x) = \frac{1}{\sqrt{2a}}e^{iKx}$]

15. If a dust particle of 1 μg requires 100 s to cross a distance of 1 mm which is the separation between two rigid walls of the potential, determine the quantum numbers described by it. [Ans. $n = 3 \times 10^{16}$]

16. An electron is bound by a potential box of infinite height having width of 2.5 Å. Calculate the lowest three permissible energies that the electron can have.

[Ans. 6 eV, 24 eV, 54 eV]

17. The wavelength of laser is 632.8 nm. Assuming this light is due to transition from $n = 4$ state to $n = 3$ state of an electron in a box, determine the width of the box.

[Ans. $a = 1.16$ nm]

18. Consider an α-particle in a nucleus as a particle moving in a box of width 10^{-15} m. Estimate the energy of the α-particle in its lowest energy state. The mass of the α-particle is $4 \times 1.66 \times 10^{-27}$ kg.

[Ans. 51. MeV]

19. A particle constrained to move along the x-axis is described by the wave function

$$\varphi(x) = 2x \quad 0 < x < 1$$
$$= 0 \quad \text{elsewhere}$$

Calculate the probability of finding the particle within the interval (0, 0. 4). [Ans. 0.085]

20. Find the lowest three energies in eV, for an electron in a one-dimensional box of length $a = 0.2$nm (about the size of an atom). [Ans. 9.4, 37.5, 84.3 eV]

21. What is spacing, in eV, between the lowest two levels of an electron confined in a one-dimensional wire of length 1 cm ? [Ans. 1.1×10^{-14} eV]

22. Consider an electron, mass $m = 0.9 \times 10^{-30}$ kg in an infinite wall that is 2 cm wide. For what value of n will the electron have an energy of 1 eV ? [Ans. 3×10^7]

23. A beam of electrons is incident on a barrier 6.00 eV high and 0.200 nm wide. Find the energy they should have if 1.00 percent of them are to get through the barrier. [Ans. 5.95 eV]

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24. The wave function for a steady state is given by

$$\varphi(x) = N\exp\left(-\frac{x^2}{2}\right)$$

Calculate the value of the normalisation constant N. [**Ans.** $N = \left(\frac{1}{\pi}\right)^{1/4}$]

25. A certain function is given by

$$\varphi(x) = N(1+ix)\exp(-x) \quad \text{for } x > 1$$
$$= 0 \quad \text{for } x < 1$$

Obtain the value of the normalisation constant N. Why can it not represent a physical system ?

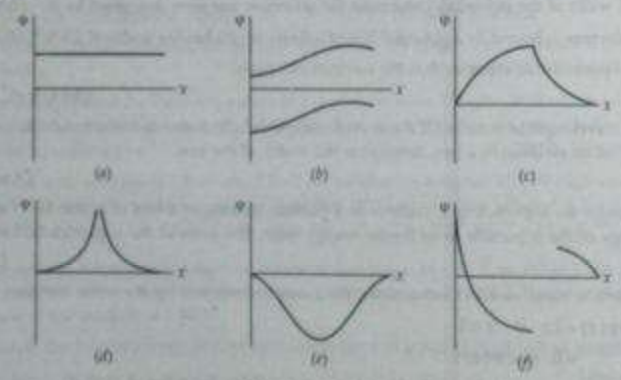
[**Ans.** $|N| = \frac{2e}{\sqrt{7}}$]

[The wave function can not represent a physical system because it is discontinuous at $x = 1$.]

26. A state of a particle of mass m is given by $e^{-\alpha x^2}$. Normalise the wave function and calculate the expectation value of kinetic energy of the particle.

[**Ans.** $N = \left(\frac{2\alpha}{\pi}\right)^{1/4}$, $\langle K.E.\rangle = \frac{1}{2}\frac{\alpha\hbar^2}{m}$]

27. Which of the wave functions in Fig. cannot have physical significance in the interval shown ? Why not ?



[**Ans.** b is double valued ; c has a discontinuous derivative ; d goes to infinity ; f is discontinuous.]

28. Which of the following wave function can not be solutions of Schrodinger's equation for all values of x ? Why not ?

(a) $\varphi = A\sec x$ (b) $\varphi = A\tan x$ (c) $\varphi = Ae^{x^2}$ (d) $\varphi = Ae^{-x^2}$

[**Ans.** a and b are discontinuous and become infinite at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$; c becomes as infinite as x goes to $\pm\infty$.]

Chapter 7

Statistical Mechanics

7.1 Introduction

- Statistical mechanics is a formalism, which aims at explaining the physical properties of matter in bulk on the basis of the dynamical behaviour of its microscopic constituents. The scope of formalism is almost unlimited as the very large range of natural phenomena for in principle it is applicable to matter in any state whatsoever.
- The branch of physics in which statistical methods are applied to the microscopic constituents of a system in order to predict its microscopic properties. The earliest application of this method was Boltzmann's attempt to explain the thermodynamic properties of gases on the basis of the statistical properties of large assemblies of molecules.

In classical statistical mechanics, each particle is regarded as occupying a point in **phase space** *i.e.*, to have an exact position and momentum at any particular instant. The probability that this point will occupy any small volume of the phase space is taken to be proportional to the volume. The Maxwell-Boltzmann law gives the most probable distribution of the particles in phase space.

With the advent of quantum theory, the exactness of the these premises was disturbed (by Heisenberg Uncertainty Principle). In the quantum statistics that evolved as a result, the phase space is divided into cells, each having volume h^f, where h is the Planck's constant and f is the number of degrees of freedom of the particles. This new concept led to Bose-Einstein statistics, and for particles obeying the Pauli exclusion principle, to Fermi-Dirac statistics.

7.2 Some Important Definitions

7.2.1 System

- **Static System.** The system, in which its composing particles are at rest.
- **Dynamic system.** The system, in which its composing particles are in constant state of motion.

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7.2.2 Phase Space

In classical mechanics, the instantaneous dynamical state of a particle is completely specified by its three *position co-ordinates* (x, y, z) ; and the corresponding *momentum components* (p_x, p_y, p_z). Thus six coordinates are needed to specify a one particle system completely. *Gibbs* suggested that any instantaneous state (position and momentum) of the particle may be conveniently represented by some points in an imaginary six-dimensional space in which the six coordinates (x, y, z, p_x, p_y, p_z) are marked along six mutually perpendicular axes in space. The six dimension space is known as '*phase space*' or '*μ-space*'. The point in the phase space representing instantaneous state of the particle is called the '*phase point*'. As the time progresses, the phase point moves in the phase space. The path of the phase point represents a 'trajectory' of the particle.

If the system contains a large number of particles, then every particle is completely specified by a point in the phase space. Thus, the instantaneous state of a system of particles is represented by the corresponding distribution of phase points in the phase space.

Division of Phase Space into Cells

The conception of a "point" in the phase space (position-momentum space) is to be considered in the light of uncertainty principle. The phase space is divided into tiny six-dimensional cells, whose sides are $\delta_x, \delta_y, \delta_z, \delta_{px}, \delta_{py}, \delta_{pz}$. such cells are called 'phase cells'. The volume of each of these cells is

$$\delta V = \delta_x \delta_y \delta_z \delta_{px} \delta_{py} \delta_{pz}$$

Classically there is no restriction on the volume of the phase cell : it may be reduced to any extent, tending to zero also, without affecting the classical results. However according to uncertainty principle

$$\delta_z \delta_{pz} \geq \hbar \qquad \left[\because \hbar = \frac{h}{2\pi}\right]$$

$$\delta_y \delta_{py} \geq \hbar$$

$$\delta_z \delta_{pz} \geq \hbar$$

Hence we see that $$\delta V \geq (\hbar)^3 \qquad ...(1)$$

Thus a "point" in the phase space is actually a cell, whose minimum volume is of the order of $\hbar^3$. It means that a particle in phase space cannot be considered exactly located at the point (x, y, z, p_x, p_y, p_z); but can only be found some where within a phase cell centred at that point.

Now, the state of system can be described by specifying the distribution of the particles of the system among the phase cells. We can determine the probabilities of occurrence of all possible distributions that are permitted by the nature of the system. Out of these we can select the most probable one. *The state of the system when it is in thermal equilibrium corresponds to the most probable distribution of particles in the phase space.*

7.2.3 Microstates and Macrostates of a System

Let us consider a system consisting of a large number of gas molecules in the phase space which has been divided into tiny cells. Each cell represents a small region of position and

momentum. Each molecule may be specified by a point (phase point) lying somewhere inside one of these cells. *The microstate of the system at a particular instant be defined when we specify its to which particular cell each molecule of the system belongs at that instant.* The deep information is, however, unnecessary to determine the observable properties of the system (gas). For example, the density is uniform if the number of molecules in each cell is same, regardless of which particular cell.

A macrostate of the system, on the other hand, can be defined by just giving the number of molecules in each cell ; such as n_1 molecules are in cell 1, n_2 are in cell 2, and so on.

There may be a large number of microstate corresponding to the same macrostate. As an example, let us consider a system of three molecules only (for simplicity) named α, β, γ, which are to be distributed in two halves of a box, the left half L and the right half R.

There are *four* possible distributions

(*i*) 3 molecules in L and zero in R

(*ii*) 2 molecules in L and 1 in R

(*iii*) 1 molecules in L and 2 in R

(*iv*) Zero molecules in L and 3 in R.

Let us called these distribution as (3, 0) ; (2, 1) ; (1, 2) ; (1, 3) respectively.

Now, the distributions (3, 0) and (0, 3) can occur in one way only ; while the distributions (2, 1) and (1, 2) can each occur in three ways as illustrated in the following table.

Table 7.1

Distribution	Left	Right
(3, 0)	$\alpha\beta\gamma$	–
(2, 1)	$\alpha\beta$ $\beta\gamma$	γ α
(1, 2)	α β γ	$\beta\gamma$ $\alpha\gamma$ $\alpha\beta$
(0, 3)	0	$\alpha\beta\gamma$

Thus, the total number of ways in which three molecules can occupy two halves of the box are $1+3+3+1=8(=2^3)$ corresponding to four different distribution. Each way of arrangement of molecules in a microstate of the system, while each different distribution of molecules is *macrostate*. Thus, there are eight microstate and four macrostates of the system. There is only one microstate $(\alpha\beta\gamma, -)$ corresponding to the macrostate (3, 0) ; three microstates $(\alpha\beta, \gamma)$; $(\alpha\gamma, \beta)$; $(\beta\gamma, \alpha)$ corresponding to the macrostate (2, 1) ; and so on.

In a gaseous system there is a very large number of molecules which are in random motion. Hence there exists a very large number of microstate corresponding to a given macrostate. If at two instant t and t' (say) the experimentally measurable quantities pressure, volume and temperature are same, then the system is in the same macroscopic state at these instants. However, the microscopic states may be (and are) different because the positions and velocities (momenta) of the individual molecules are constantly changing.

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7.2.4 Constraints and Accessible States

The restrictions imposed by physical laws on the distribution of molecules among the cells in the phase space are called the 'constraints' of the system. For example, the total number of molecules N (say) in a system remains unchanged. If there are n_1 molecules in cell 1, n_2 in cell 2, ... and n_r in cell r, then the constraint requires

$$\sum_{i=1}^{r} n_i = N \qquad ...(2)$$

Similarly, if the total energy E of the system is constant and if n_1 molecules have energy E_1 each, n_2 molecules have E_2 each, ... and so on, then we have the constant :

$$\sum_{i=1}^{r} n_i E_i = E \qquad ...(3)$$

The microstates which are permitted under the constrained imposed upon the system are called **'accessible microstates'**.

A fundamental postulate of statistical mechanics is that *for a system in equilibrium, all accessible microstates corresponding to a given macrostate are equally probable.* This is known as the postulate of **'equal a priori probability'**.

7.3 Classification of Statistical Mechanics

Every solid, liquid or gas in an assembly of an enormous number of microscopic particles. Likewise, radiation is an assembly of photons. Obviously, the actual motions or interactions of individual particles cannot be investigated. However, the macroscopic properties of such assemblies can be explained in terms of the statistical distribution of the individuals among different possible states and their most probable behaviour.

For example, from Maxwell distribution of speeds among the molecules of a gas we can calculate mean speed (which is related to the momentum carried by the molecules), mean square speed (which is related to the energy of the molecules) and so on. From the *average* quantities we calculate observable properties like pressure and temperature of the gas.

Usually we consider how a fixed amount of energy is distributed among the various identical particles of an assembly. Now there are three kinds of identical particles

(*i*) Identical particles of any spin which are so much separated in the assembly that they can be distinguished from one another. The molecules of a gas are particles of this kind.

(*ii*) Identical particles of zero or integral spin which cannot be distinguished from one another. These are called **Bose particles** (or bosons) and do not obey Pauli's exclusion principle. Photon, phonons and α-particle are of this kind.

(*iii*) Identical particles of half integral spin which cannot be distinguished from one another. These are called **Fermi particles** (or fermions) and do obey Pauli's exclusion principle. Electrons, protons, neutrons are particles of this kind.

The first kind of the particles are the classical particles and obey the Maxwell-Boltzmann energy distribution law. The second and third kinds of particles are quantum particles and energy distribution laws for them can be derived by methods of quantum statistics (Bose-Einstein and Fermi-Dirac statistics) only.

7.4 Maxwell-Boltzmann Statistics

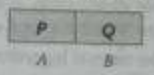
(*i*) This statistics is applicable to the identical, distinguishable particles of any spin. The molecules of a gas are the particles of this kind.

(*ii*) The particles are distinguishable, because they have specified position and momentum. Wave function associated with particle do not overlap to each other or overlapping is upto negligible extent.

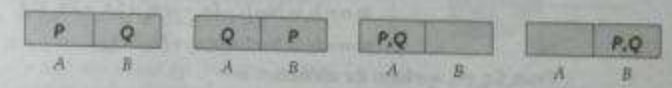
(*iii*) The particle do not follow any symmetric condition or restriction. They have any spin.

If P and Q are occupied in states A and B ; the wave function φ for system can be written as

$$\varphi = \varphi_A(P)\varphi_B(Q)$$



(*iv*) The occupancy of cell or states are 0, 1, 2, 3, ... , n_i, *i.e.*, if P and Q are two particles which have filled into two quantum states or cells, then they are filled by four ways :



Let us consider a system of N distinguishable molecules of a gas. Suppose n_1 of them have energy E_1, n_2 of them have energy E_2.....,n_i have E_i and so on. Thus entire assembly of molecules can be divided into different energy states with energies $E_1, E_2, E_3, ..., E_i, ...$ and having $n_1, n_2, n_3, ..., n_i, ...$ molecules.

(*a*) The total number of molecules N is constant. Hence,

$$N = n_1 + n_2 + n_3 + ... + n_i + ... = \text{constant}$$

or $$\delta N = \delta n_1 + \delta n_2 + \delta n_3 + ... + \delta n_i + ... = ...0$$

i.e., $$\sum_i \delta n_i = 0 \qquad ...(4)$$

(*b*) The total energy E of the gas molecules is constant. Hence

$$E = n_1 E_1 + n_2 E_2 + n_3 E_3 + ... + n_i E_i + ... = \text{constant}$$

or $$\delta E = E_1 \delta n_1 + E_2 \delta n_2 + E_3 \delta n_3 + ... + E_n \delta n_i ... + = ...= 0$$

i.e., $$\sum_i E_i \delta n_i = 0 \qquad ...(5)$$

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(c) Suppose there are g_i cells with energy E_i. The total number of ways in which n_i molecules can have the energy E_i is $(g_i)^{n_i}$. Hence the total number of ways in which N molecules can be distributed among the various energies is

$$\omega_1 = (g_1)^{n_1}(g_2)^{n_2}(g_3)^{n_3}\ldots\ldots(g_i)^{n_i}\ldots \quad \ldots(6)$$

The number of ways in which the groups of $n_1, n_2, n_3, \ldots\ldots n_i \ldots$ particles can be chosen from N particles is given by

$$\omega_2 = \frac{N!}{n_1!\,n_2!\,n_3!\ldots\ldots} \quad \ldots(7)$$

The number of distinguishable ways in which N molecules can be distributed among the possible energy level in

$$\omega = \omega_1\omega_2 = \frac{N!}{n_1!\,n_2!\,n_3!\ldots\ldots}(g_1)^{n_1}(g_2)^{n_2}(g_3)^{n_3} \quad \ldots(8)$$

The quantity ω is called the **thermodynamic probability** for the system. For the most probable distribution, ω is a maximum subject to the restriction that the total number of particles N and the total energy E are constant.

The natural logarithms of Eq. (8) is

$$\ln\omega = \ln N! - \Sigma\ln n_i! + \Sigma n_i \ln g_i$$

Using Stirling's theorem,

$$\ln n! = n\ln n - n$$

$$\ln\omega = N\ln N - N - \Sigma n_i - \ln n_i + \Sigma n_i + \Sigma n_i \ln g_i$$

$$\Rightarrow \quad \ln\omega = N\ln N - \Sigma n_i \ln n_i + \Sigma n_i \ln g_i \quad \ldots(9)$$

From Eq. (9), we have for maximum ω

$$\delta\ln\omega_{max} = -\Sigma n_i \frac{1}{n_i}\delta n_i - \Sigma(\ln n_i)\delta n_i + \Sigma(\ln g_i)\delta n_i = 0$$

or $$-\Sigma(\ln n_i)\delta n_i + \Sigma(\ln g_i)\delta n_i = 0 \qquad [\because \Sigma\delta n_i = 0] \quad \ldots(10)$$

Eqs. (4) and (5) can be incorporated into Eq. (10) by making use of Lagrange's method of undetermined multipliers. Multiplying Eq. (4) by $(-\alpha)$ and Eq. (5) by $(-\beta)$ and adding to Eq. (10), we get

$$\Sigma(-\ln n_i + \ln g_i - \alpha - \beta E_i)\delta n_i = 0 \quad \ldots(11)$$

or $$-\ln n_i + \ln g_i - \alpha - \beta E_i = 0$$

or $$n_i = g_i e^{-\alpha} e^{-\beta E_i} \quad \ldots(12)$$

Eq. (12) is called Maxwell-Boltzmann distribution law.

M – B distribution in terms of temperature

It can be shown that $\beta = \frac{1}{k_B T}$ and $\alpha = -\frac{\mu}{k_B T}$ here μ = chemical potential,

where k_B = Boltzmann's constant and T = absolute temperature of the gas

$$n_i = \frac{g_i}{e^{(\alpha+E_i/k_BT)}} = \frac{g_i}{e^{(E_i-\mu)/k_BT)}} \quad \ldots(13)$$

7.5 Applications of Maxwell-Boltzmann Distribution Law

7.5.1 Molecular Energies in an Ideal Gas

The Maxwell-Boltzmann distribution law is

$$n_i = \frac{g_i}{e^{(\alpha+E_i/k_BT)}} \quad \ldots(14)$$

Consider an ideal gas that contains N molecules. Consider a continuous distribution of molecular energies instead of the discrete set $E_1, E_2, E_3, \ldots\ldots$ then Eq. (14) becomes,

$$n(E)dE = g(E)e^{-\alpha}e^{-(E/k_BT)}dE \quad \ldots(15)$$

Here $n(E)dE$ represents the number of molecules having energies between E and $(E+dE)$. $g(E)dE$ represents the number of states that have energies between E and $E+dE$.

Let us first find $g(E)dE$. A molecule of energy E has a momentum p whose magnitude is

$$p = \sqrt{2mE} = \sqrt{p_x^2 + p_y^2 + p_z^2} \quad \ldots(16)$$

Each set of momentum components p_x, p_y, p_z specifies a different state of motion. Let us imagine a *momentum space* whose coordinate axes are p_x, p_y and p_z (Fig. 7.1). The number of momentum states available to a particle with a momentum whose magnitude is between p and $(p+dp)$ is proportional to the volume of a spherical shell in momentum space of radius p and thickness dp. Volume of this spherical shell is $4\pi p^2 dp$.

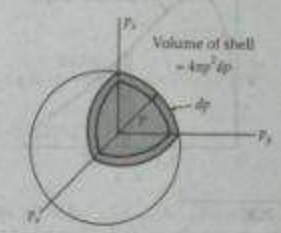

Fig. 7.1 Momentum Space.

Hence

$$g(p)dp = Bp^2 dp \quad \ldots(17)$$

where B is some constant. Since each momentum magnitude p corresponds to a single energy E, the number of energy states $g(E)dE$ between E and $(E+dE)$ is the same as the number of momentum states $g(p)dp$ between p and $(p+dp)$.

$$\therefore \quad g(E)dE = Bp^2 dp \quad \ldots(18)$$

But $p^2 = 2mE$ and $dp = \frac{2mdE}{\sqrt{2mE}}$

$$\therefore \quad g(E)dE = \sqrt{2}m^{3/2}B\sqrt{E}dE \quad \ldots(19)$$

Substituting for $g(E)dE$ in Eq. (15)

$$n(E)dE = C\sqrt{E}e^{-(E/k_BT)}dE \quad \ldots(20)$$

where $C = \sqrt{2}m^{3/2}Be^{-\alpha}$ = constant.

To find C we make use of the normalisation condition that the total number of molecules is N.

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Therefore $$N = \int_0^\infty n(E)dE = C\int_0^\infty \sqrt{E}e^{-(E/k_BT)}dE \quad ...(21)$$

From definite integrals as given below

$$\int_0^\infty \sqrt{x}e^{-ax}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

We find that $$N = C\frac{1}{2(1/k_BT)}\sqrt{\frac{\pi}{(1/k_BT)}} = C\times\frac{\sqrt{\pi}}{2}\times(k_BT)^{3/2}$$

or $$C = \frac{2\pi N}{(\pi k_BT)^{3/2}} \quad ...(22)$$

$$\therefore \quad n(E)dE = \frac{2\pi N}{(\pi k_BT)^{3/2}}\sqrt{E}e^{-E/k_BT}dE \quad ...(23)$$

Eq. (23) is plotted in Fig. 7.2 in terms of k_BT

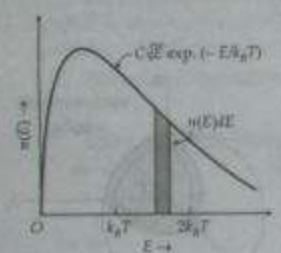

Fig. 7.2

The total internal energy of the system is

$$E = \int_0^\infty E\,n(E)dE = \frac{2\pi N}{(\pi k_BT)^{3/2}}\int_0^\infty E^{3/2}e^{-E/k_BT}dE$$

The value of the definite integral is $\frac{3}{4}(k_BT)^2\sqrt{\pi k_BT}$

$$\therefore \quad E = \frac{2\pi N}{(\pi k_BT)^{3/2}}\times\frac{3}{4}(k_BT)^2\sqrt{(\pi k_BT)} = \frac{3}{2}Nk_BT \quad ...(24)$$

The average energy of an ideal gas molecule is (E/N)

$$\therefore \quad E = \frac{3}{2}k_BT \quad ...(25)$$

7.5.2 Maxwell-Boltzmann Velocity Distribution Law

Substitute $E = \frac{1}{2}mv^2$ and $dE = mvdv$ in Eq. (23). Then we get

$$\therefore \quad n(v)dv = \frac{\sqrt{2}\pi Nm^{3/2}}{(\pi k_BT)^{3/2}}v^2e^{-mv^2/2k_BT}dv \quad ...(26)$$

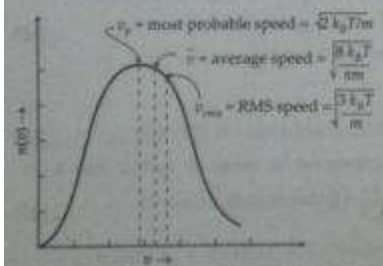

Fig. 7.3

Eq. (26) represents the number of molecules with speed between v and $(v + dv)$ in an assembly of ideal gas containing N molecules at absolute temperature T. This formula is plotted in Fig. 7.3.

Example 7.1. At what temperature would one in a thousand of atom in a gas of atom hydrogen be in $n = 2$ energy level ?

Solution. For hydrogen $n(E) = -\frac{13.6}{n^2}$ eV

$$n(E_1) = n_1 = -13.6\text{ eV}$$
$$n(E_2) = n_2 = -3.5\text{ eV}$$

$g(E)$ = no. of state form $= 2n^2$

$$g(E_1) = g_1 = 2 \text{ and } g(E_2) = g_2 = 8$$

For Maxwell-Boltzmann distribution is

$$\frac{n_2}{n_1} = \frac{g_2\,e^{-E_2/k_BT}}{g_2\,e^{-E_1/k_BT}} = \frac{8}{2}e^{-(E_2-E_1)/k_BT}$$

or $$\frac{1}{10^3} = 4e^{-(E_2-E_1)/k_BT} \quad \text{or} \quad e^{(E_2-E_1)/k_BT} = \frac{10^4}{2.5} = 4000$$

Taking logarithms both the sides

$$\frac{E_2-E_1}{k_BT} = \ln 4000$$

$$k_BT = \frac{(E_2-E_1)}{\ln 4000} = \frac{10.2\text{ eV}}{8.29}$$

$$T = \frac{10.2\times16\times10^{-19}}{8.29\times1.38\times10^{-23}}\text{ K} \approx 1.43\times10^4\text{ K}$$

$$T = 14300\text{ K} = 14300 - 273 = 14027^\circ\text{C}$$

7.6 Exchange Symmetry of Wave Functions

Let there be N particles described by the wave function $\varphi(1,2,3,\ldots,r,\ldots,s,\ldots,N)$ where 1 stands for all coordinates (position and spin) of particle 1 and so on. If the wave function is operated by an exchange operator $\hat{P}$ then

$$\hat{P}_{r,s}\,\varphi(1,2,3,\ldots,r,\ldots,s,\ldots,N) = P\varphi(1,2,3,\ldots,s,\ldots,r,\ldots,N) \quad ...(27)$$

The effect of the operator $\hat{P}_{r,s}$ is to interchange the particles r and s i.e., formerly called particle 'r' is now called 's' and vice versa. For two particles system, we write

$$\hat{P}_{12}\,\varphi(1,2) = P\varphi(2,1) \quad ...(28)$$

Eq. (27) is the eigen value equation of the operator $\hat{P}_{r,s}$ and P stands for the eigen value of the operator $\hat{P}_{r,s}$.

There are two kinds of wave function φ depending on value of P symmetric wave function and antisymmetric wave functions.

(i) Symmetric wave function (φ_S)

A wave function is said to be symmetric if the interchange of any pair of particles among its arguments, leave the wave function unchanged.

i.e., $$P_{1,2}\varphi_S(1,2) = +\varphi_S(2,1) \quad \text{(here } P = 1) \quad ...(29)$$

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(ii) Antisymmetric wave function (φ_A)

The wave function is antisymmetric if the interchange of any pair of particles among its arguments, changes the sign of the wave function.

$$P_{1,2}\varphi_A(1,2) = -\varphi_A(2,1); \text{ here } P = -1 \quad ...(30)$$

The exchange symmetry of the wave functions has close relation with the intrinsic angular momentum of the particle. The relationship is listed as follows :

(a) The identical particles having an integral quantum number of their intrinsic spins are described by symmetric wave functions, such as

$$\varphi_S(1,2) = +\varphi(2,1) \quad ...(31)$$

The type of particles described by symmetric wave functions are known as BOSONS as they obey Bose-Einstein statistics.

(b) The particles which have half integral quantum number for their intrinsic spins are described by the antisymmetric wave functions, such as

$$\varphi_A(1,2) = -\varphi(2,1) \quad ...(32)$$

The type of particles described by antisymmetric wave functions are called FERMIONS as they obey Fermi-Dirac statistics.

7.7 Bose-Einstein Statistics

Bose-Einstein statistics is obeyed by indistinguishable particle of integral spin that have symmetric wave functions and does not obey Pauli exclusion principle.

Consider an assembly of N Bosons. They are identical and indistinguishable. No restriction is imposed as to the number of particles that may occupy a given cell. Let us now consider a box divided into n_i sections by $(g_i - 1)$ partitions and n_i indistinguishable particles to be distributed among these sections. The permutations of n_i particles and $(g_i - 1)$ partitions simultaneously is given by $(n_i + g_i - 1)!$. But this includes also the permutations of n_i particles among themselves and $(g_i - 1)$ partitions among themselves, as both these groups are internally indistinguishable. Hence the actual number of ways in which n_i particles are to be distributed in g_i sublevels is

$$\frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$$

Therefore, the total number of distinguishable and distinct ways of arranging N particles in all the variable energy states is given by

$$\omega = \prod \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \quad ...(33)$$

n_i and g_i are large numbers. Hence we may neglect 1 from above expression

$$\therefore \quad \omega = \prod \frac{(n_i + g_i)!}{n_i!(g_i)!} \quad ...(34)$$

Taking natural logarithms of above equation

$$\ln \omega = \Sigma[\ln(n_i + g_i)! - \ln n_i! - \ln(g_i)!] \quad ...(35)$$

As n_i and g_i are large numbers, we can use Stirling's approximation i.e.,

$$\ln n! = n \ln - n$$

$$\ln \omega = \Sigma[(n_i + g_i)\ln(n_i + g_i) - (n_i + g_i) - n_i \ln n_i + n_i - g_i \ln g_i + g_i] \quad ...(36)$$

Here g_i is not subjected to variation and n_i varies continuously. For most probable distribution

$$\delta(\ln \omega_{max}) = 0 \quad ...(37)$$

Hence if the ω of Eq. (37) represents a maximum

$$\delta(\ln \omega_{max}) = \Sigma[\ln(n_i + g_i) - \ln n_i]\delta n_i = 0 \quad ...(38)$$

The total number of particle and total energy are constant

$$\Sigma \delta n_i = 0 \quad ...(39)$$

$$\Sigma E \delta n_i = 0 \quad ...(40)$$

Multiplying Eq. (39) by $(-\alpha)$ and Eq. (40) by $(-\beta)$ and adding to Eq. (38)

$$\Sigma[\ln(n_i + g_i) - \ln n_i - \alpha - \beta E_i]\delta n_i = 0$$

The variation of δn_i are independent of each other. Hence we get

$$\ln\left(\frac{n_i + g_i}{n_i}\right) - \alpha - \beta E_i = 0 \quad ...(41)$$

or $$n_i = \frac{g_i}{(e^{\alpha + \beta E_i} - 1)} \quad \text{or} \quad n_i = \frac{g_i}{(e^{\alpha} e^{\beta E_i} - 1)}$$

$$\Rightarrow \quad n_i = \frac{g_i}{(e^{\alpha} e^{E_i/k_B T} - 1)} \quad ...(42)$$

This is known as expression for Bose-Einstein statistics.

7.8 Applications of Bose-Einstein Statistics

7.8.1 Planck's Radiation Law and Rayleigh Jeans Formula

Radiator or photon gas considered as Bose-Einstein gas. Each photon have energy $h\nu$.

Let dn be the number of photon distributed in frequency internal ν to $(\nu + d\nu)$

$$dn = n(\nu)d\nu = g(\nu) f(\nu) d\nu \quad ...(43)$$

$$\because \quad g(\nu)d\nu = \frac{8\pi h \nu^2}{c^3} d\nu \quad ...(44)$$

and $$f(\nu) = \frac{1}{(e^{h\nu/k_B T} - 1)} \quad [\text{as } \alpha = 0 \text{ and } E = h\nu] \quad ...(45)$$

putting these values in Eq. (43), we get

$$dn = \frac{8\pi h \nu^2}{c^3} \cdot \frac{1}{(e^{h\nu/k_B T} - 1)} d\nu \quad ...(46)$$

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Let dE be the energy distributed in frequency interval ν and $(\nu + d\nu)$

$$dE = E\,dn = h\nu\,dn \qquad ...(47)$$

From Eqs. (46) and (47), we obtain

$$dE = E(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\cdot\frac{1}{(e^{h\nu/k_BT}-1)}d\nu \qquad ...(48)$$

Eq. (48) is known as **Planck's radiation formula** for the spectral energy density of black body radiation, which agrees with experimental curve (Fig. 7.4.)

For low frequency or high wave length

$$(e^{h\nu/k_BT}-1) = 1 + \frac{h\nu}{k_BT} - 1 = \frac{h\nu}{k_BT}$$

The Eq. (48) becomes as

$$dE = E(\nu)d\nu = \frac{8\pi\nu^2 kT}{c^3}d\nu \qquad ...(49)$$

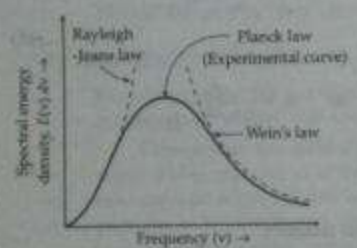

Fig. 7.4 ν vs. $E(\nu)d\nu$ curve.

Eq. (49) is termed as *Rayleigh-Jeans law*. This laws is hold good for low frequencies and high wave-lengths. Experimental agreement is shown in Fig. 7.4. *For high frequency low frequency :*

Eq. (48) can be expressed in terms of wavelength of radiation as follows :

$$E_\lambda\,d\lambda = \frac{8\pi ch}{\lambda^5}\left(\frac{1}{e^{[(hc/\lambda k_BT)-1]}}\right)d\lambda$$

or

$$E_\lambda = 8\pi hc(\lambda^{-5})\left[\exp\left(\frac{hc}{\lambda k_BT}-1\right)\right]^{-1}$$

Let λ_{max} be the wavelength whose energy density is the greatest.

Then, $\dfrac{dE_\lambda}{d\lambda} = 0$

Hence $\lambda_{max}T = \dfrac{hc}{4.965\,K} = 2.898\times10^{-3}$ mK

This is *Wien's displacement law.*

The peak in the black body spectrum shifts to progressively shorter wavelength (higher frequencies) as temperature is increased.

Spectral energy density decreases with frequency at high frequency. This is called as *Wein's law of energy distribution for black body radiation.*

Hence, Rayleigh-Jeans and Wein's Laws are unable to explain the complete energy distribution for black body radiation, while Planck's law explains it completely, which is based on Bose-Einstein statistics.

7.8.2 Bose-Einstein Condensation

According to Bose-Einstein distribution function, we know that

$$n_i = \frac{g_i}{(e^{\alpha+E_i/k_BT}-1)} \qquad ...(50)$$

The number of gaseous molecules distributed in energy interval E to $(E + dE)$ is written as

$$dn(E) = \frac{V}{h^3}(2\pi mk_BT)^{3/2}\frac{2}{\sqrt{\pi}}\times\frac{x^{1/2}dx}{(e^{\varphi+x}-1)} \qquad ...(51)$$

Then total number of particle

$$n = \int dn(E) = \frac{V}{\sqrt{\pi}}(2\pi mk_BT)f(\varphi) \qquad ...(52)$$

$$\left[\text{where } f_1(\varphi) = \frac{2}{\sqrt{\pi}}\int_0^\infty\frac{x^{1/2}dx}{(e^{\varphi+x}-1)}\right]$$

and total energy $E = \int E\,dn(E) = \dfrac{3}{2}\dfrac{V}{h^3}(2\pi mk_BT)^{3/2}k_BTf_2(\varphi)$ ...(53)

$$\left[\text{where } f_2(\varphi) = \frac{4}{3\sqrt{\pi}}\int_0^\infty\frac{x^{3/2}}{e^{\varphi+x}-1}\right]$$

The integral functions can be expanded and they both come out equal to

$$f_1(\varphi) = f_2(\varphi) = A = \frac{n}{V}\frac{h^3}{(2\pi k_BT)^{3/2}} \qquad ...(54)$$

If the temperature is decreased or the density of the particles (n/V) increased, the value of A increases, the gas under there condition are known as *'degenerate'* and *'A'* the *degeneracy parameter.*

If temperature is low, then the degree of degeneracy will be large ; particle density also be large and mass of each boson is small.

The maximum value of $f_1(\varphi)$ is 2.612

$$\therefore \left(\frac{n}{V}\right)_{max} = \frac{(2\pi mk_BT)^{3/2}}{h^3} \qquad ...(55)$$

In terms of critical temperature $T = T_C$ the Eq. (55) becomes

$$\left(\frac{n}{V}\right) = \frac{(2\pi mk_BT_C)^{3/2}}{h^3} \qquad ...(56)$$

or

$$T_C = \frac{h^3}{2\pi mk_B}\left(\frac{1}{2.612}\frac{n}{V}\right)^{2/3} \qquad ...(57)$$

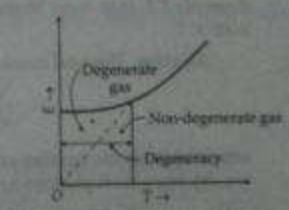

Fig. 7.5 B-E condensation.

So T_C *is that particular temperature at which degeneracy level starts.* E_0 be the zero point energy (Fig. 7.5)

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Again according to Bose-Einstein statistics

$$n_i = \frac{g_i}{(e^{\alpha+E_i/k_BT}-1)} \quad ...(58)$$

For ground state $E_i = E_0 = 0$ and $g_i = 1$

Then number of particles in the ground state $n \to n_0$

$$n_0 = \frac{1}{e^{\alpha}-1} \quad ...(59)$$

So total number of particle in the degenerate case is given as

$$n = n_0 + n' \quad ...(60)$$

where

$$n' = \frac{4\pi mV}{h^3}\sqrt{2m}\int_0^\infty \frac{E^{1/2}dE}{(e^{\alpha+E/k_BT}-1)} \quad ...(61)$$

$$n' = n\left(\frac{T}{T_C}\right)\frac{f_1(\varphi)}{f_1(0)} \quad [\text{where } f_1(0)=2.612] \quad ...(62)$$

$$\Rightarrow \quad n' = n\left(\frac{T}{T_C}\right)^{3/2} \quad [\text{As } f_1(\varphi) < f_1(0) \text{ for } T < T_C] \quad ...(63)$$

and the rest of particles

$$n_0 = n - n' = n\left[1 - \Sigma\left(\frac{T}{T_C}\right)^{3/2}\right] \quad (\text{for } T < T_C) \quad ...(64)$$

Condense into ground state.

So this swift increases the particles in the ground state below T_C to form a **Bose-Einstein (B – E) gas**, is known as **Bose-Einstein Condensation**. At the ground state $E = 0$, the particle of Bose-Einstein degenerate gas condense into ground state and do not contribute to energy. *Example* : Superfluidity of liquid ^{4}He.

Example 7.2. *How many photons are present in 100 cm^3 of radiation in thermal equilibrium at 1000 K ?*

Solution. The total number of photons per unit volume is given by

$$\frac{N}{V} = \int_0^\infty n(\nu)d\nu$$

where $n(\nu)d\nu$ is the number of photons per unit volume with frequencies between ν and $(\nu + d\nu)$. Since such photons have energies of $h\nu$.

$$n(\nu)d\nu = \frac{E(\nu)d\nu}{h\nu}$$

with $E(\nu)d\nu$ being the energy density given by Planck's formula. Hence the total number of photons in the volume V is

$$N = \int_0^\infty \frac{E(\nu)d\nu}{h\nu} = \frac{8\pi V}{c^3}\int_0^\infty \frac{\nu^2 d\nu}{(e^{h\nu/k_BT}-1)}$$

If we let $\frac{h\nu}{k_BT} = x$, then $\nu = \frac{k_BTx}{h}$ and $d\nu = \left(\frac{k_BT}{h}\right)dx$

So that, $$N = 8\pi V\left(\frac{k_BT}{hc}\right)^3\int_0^\infty \frac{x^2\,dx}{(e^x-1)}$$

The definite integral is a standard one equal to 2.404. Inserting numerical values of the other quantities with $V = 100\text{ cc} = 100\times10^{-6}\text{ m}^3$, we get

$$N = 2.03\times10^{10} \text{ photons}$$

7.9 Fermi-Dirac Statistics

Fermi Dirac statistics is obeyed by indistinguishable particles of half integral spin that have antisymmetric wave functions and obey Pauli exclusion principle. Consider N fermions with the total energy E. Suppose that n_1 particles occupy the first energy level with energy E_1, n_2 particles occupy the second energy level with energy E_2 and so on. Let us now first out the total number of ways in which n_i particles can be distributed in g_i cells having the same energy E_i.

The number of distinguishable arrangements of n_i particles in g_i cells is

$$\frac{g_i!}{n_i!(g_i-n_i)!}$$

The total number of eigen states for the whole system is given by

$$\omega = \Pi\frac{g_i!}{n_i!(g_i-n_i)!} \quad ...(65)$$

Taking the natural logarithms of both sides

$$\ln\omega = \Sigma[\ln g_i! - \ln n_i! - \ln(g_i-n_i)!]$$

Applying Stirlling formula [*i.e.*, $\ln n! = n\ln n - n$]

$$\ln\omega = \Sigma[g_i\ln g_i - g_i - n_i\ln n_i + n_i - (g_i-n_i)\ln(g_i-n_i) + (g_i-n_i)]$$

or

$$\ln\omega = \Sigma[g_i\ln g_i - n_i\ln n_i - (g_i-n_i)\ln(g_i-n_i)] \quad ...(66)$$

Here g_i is not subject to variation and n_i varies continuously. For most probably distribution $\delta(\ln\omega_{max}) = 0$

$$\delta(\ln\omega_{max}) = \Sigma[-\ln n_i + \ln(g_i-n_i)]\delta n_i = 0 \quad ...(67)$$

But

$$\Sigma\delta n_i = 0 \quad ...(68)$$

$$\Sigma E\delta n_i = 0 \quad ...(69)$$

Multiplying Eq. (68) by $(-\alpha)$ and Eq. (69) by $(-\beta)$ and adding in Eq. (67)

$$\Sigma[-\ln n_i + \ln(g_i-n_i) - \alpha - \beta E_i]\delta n_i = 0$$

As the variation δn_i are independent of each other, we get

$$\ln\left(\frac{g_i-n_i}{n_i}\right) - \alpha - \beta E_i = 0$$

or

$$n_i = \frac{g_i}{(e^{\alpha}e^{\beta E_i}+1)} \quad ...(70)$$

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This is known as Fermi-Dirac distribution law

$$\beta = \frac{1}{k_B T} \quad \text{and} \quad \alpha = -E_F / k_B T$$

where E_F is called Fermi energy $E_F = -\alpha k_B T$

$$n_i = \frac{g_i}{(e^{\alpha} e^{E_i/k_B T}) + 1} \qquad ...(71)$$

Definition of Fermi energy. At the absolute zero of temperature the maximum kinetic energy that a free electron can have is called the *fermi energy* E_F.

$$n_i = \frac{g_i}{[e^{(E_i - E_F)/k_B T}] + 1} \qquad ...(72)$$

7.10 Applications of Fermi-Dirac Statistics

7.10.1 Fermi Energy

Consider a system of N fermions enclosed in a volume V. We know that because of Pauli's principle, only one fermion can be accommodated in a given state. It is also known to us that the highest energy possessed by a fermion at $T = 0$ is called ***Fermi Energy***, E_F. Let us now derive an expression for E_F.

We know that the number of quantum states of a particle with momentum in the interval p and $(p + dp)$ is $\frac{4\pi V}{h^3} p^2 dp$. We multiply this number by $(2s+1)$. For electron, $s = \frac{1}{2}$ so that the required number of state is $\frac{8\pi V}{h^3} p^2 dp$. Denoting the highest momentum p_F, we have

$$N = \frac{8\pi V}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi V}{h^3} \frac{p_F^3}{3} \qquad ...(73)$$

This yield an expression for *Fermi momentum*, p_F

$$p_F = \left(\frac{3N}{8\pi V}\right)^{1/3} h \qquad ...(74)$$

and the Fermi energy

$$E_F = \frac{p_F^2}{2m} = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} \qquad ...(75)$$

If we draw a sphere with radius p_F; all the particles will be found inside the sphere. This is called **Fermi surface**. In case of the alkali and the noble atoms; the surface is sphere. In other cases, the shape can be quite complicated.

We define what is known as Fermi temperature, T_F, through the relation

$$T_F = \frac{E_F}{k_B} \qquad ...(76)$$

The values of E_F range from about 2 eV to 15 eV. It is the lowest for Cs(1.58 eV) and highest for *Be*(14.14 eV). The corresponding Fermi temperatures are of order of 10^4 K – 10^5 K.

The ground state energy is given by

$$E_0 = \frac{8\pi V}{h^3} \int_0^{p_F} \frac{p^2}{2m} p^2 dp = \frac{8\pi V}{h^3} \frac{p_F^5}{10\,m} \qquad ...(77)$$

Using Eq. (74), we obtain

$$E_0 = \frac{3}{5} N E_F \qquad ...(78)$$

The mean energy per fermion for a completely degenerate electron gas is given by

$$\bar{E} = \frac{E_0}{N} = \frac{3}{5} E_F \qquad ...(79)$$

7.10.2 Electronic Heat capacity

The heat capacity of metals (solids) is given as

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3R = 24.9 \text{ J mol}^{-1}\text{K}^{-1}. \qquad ...(80)$$

The correct explanation of heat capacity of metals remained a puzzle for a long time. Of course, it should be no surprise to us that classical statistics fails to give the right answer because an assembly of electron (electron gas) obey Fermi-Dirac statistics. We can easily show, using Fermi-Dirac statistics that electronic heat capacity varies with temperature. Moreover, heat capacity of a metal at low temperatures is the sum of an electronic contribution which is proportional to T, and the lattice contribution which is proportional to T^3.

Experiments reveal that the contribution of electronic heat capacity is about 1% of the total. To show this we assume that only those electrons which occupy energy states upto $k_B T$ of the Fermi level participate in thermal processes. Hence the fraction of particles thermally excited is proportional to $(k_B T / E_F)$. Since the thermal energy per excited particle is $k_B T$

$$E \sim (k_B T)\frac{k_B T}{E_F} N = \frac{N k_B^2 T^2}{E_F}$$

Hence

$$(C_V)_{El} = \left(\frac{\partial E}{\partial T}\right)_V \frac{N k_B^2 T^2}{E_F} = Nk\left(\frac{T}{T_F}\right) \qquad ...(81)$$

That is, for $T << T_F$, the electronic heat capacity of fermions varies linearly with temperature. At room tempetature

$$\frac{T}{T_F} = \frac{300}{10^4} \sim O(10^{-2}) \qquad ...(82)$$

A more exact, but some what difficult, calculation gives the following result :

$$(C_V)_{El} = \frac{N k_B \pi^2}{2T_F} T = aT \qquad ...(83)$$

where

$$a = \frac{N k_B \pi^2}{2T_F} = \frac{N k_B^2 \pi^2}{2E_F} \qquad ...(84)$$

is known as **Sommerfeld constant**.

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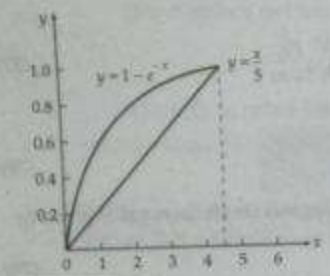

Fig. 7.6 Plot of Eq. (86) as a function of T^2.

The total heat capacity of a metal is made up of two parts. The electronic contribution dominates at low temperatures. But around room temperature, the electronic contribution is a small fraction of total

$$(C_V)_{Total} = aT + bT^3 \quad ...(85)$$

or

$$\frac{(C_V)_{Total}}{T} = a + bT^2 \quad ...(86)$$

A plot of Eq. (86) is visualised is Fig. 7.6 as a function for potassium. The agreement is seen to be excellent. The intercept gives the value of a. For potassium, sodium and copper the typical values are 2.08, 1.38, 0.695 respectively.

Example 7.3. *Find the Fermi Energy in copper on the assumption that each copper atom contributes one free electrons to the electron gas. The density of copper is $8.94 \times 10^3\ kg/m^3$ and its atomic mass is 63.5 u.*

Solution. The energy density $\frac{N}{V}$ in copper is equal to the number of copper atoms per unit volume.

Since $1u = 1.66 \times 10^{-27}$ kg

$$\frac{N}{V} = \frac{\text{atoms}}{m^3} = \frac{\text{mass}/m^3}{\text{mass/atom}} = \frac{8.94 \times 10^3\ kg/m^3}{(63.54) \times (1.66 \times 10^{-27}\ kg/u}$$

$$= 8.48 \times 10^{28}\ \text{atoms}/m^3 = 8.43 \times 10^{28}\ \text{electrons}/m^3$$

The corresponding Fermi energy is given by

$$E_F = \frac{h^2}{2m}\left(\frac{2N}{8\pi V}\right)^{2/3}$$

$$E_F = \frac{(6.63 \times 10^{-34}\ J \cdot s)^2}{(2 \times 9.11 \times 10^{-31}\ kg/\text{electron})}\left[\frac{3 \times (8.48 \times 10^{28}\ \text{electrons}/m^3)}{8\pi}\right]^{2/3}$$

$$= 1.13 \times 10^{-18}\ J = 7.04\ eV.$$

At absolute zero, T = 0 K, there should be electrons with energy upto 7.04 eV in copper, by contrast, all the molecules in an ideal gas at 0 K would have zero energy. The electron gas in a metal is said to be 'degenerate'.

7.11 Comparison of the Three Distribution Laws

Let us define

$$f(E_i) = \frac{n_i}{g_i} \quad ...(87)$$

$f(E_i)$ is know as "occupation index" of a state of energy E_i. It represents the average number of particles in each of the states of that energy. Thus

$$f_{MB} = \frac{1}{e^{\alpha} e^{E_i/k_B T}} \quad ...(88)$$

$$f_{BE} = \frac{1}{(e^{\alpha} e^{E_i/k_B T} - 1)} \quad ...(89)$$

and

$$f_{FD} = \frac{1}{(e^{\alpha} e^{E_i/k_B T} + 1)} \quad ...(90)$$

In the following diagrams we plot each occupation index $f(E_i)$ against energy E_i for two different values of T and α.

In Maxwell-Boltzmann distribution (Fig. 7.7) the occupation index $f(E_i)$ falls purely exponentially with increase in energy E_i. In fact, it falls by a factor of $1/e$ for each increase of $k_B T$ in the energy.

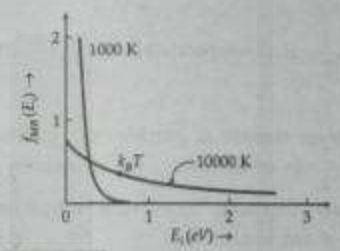

Fig. 7.7 M.B. distribution curve.

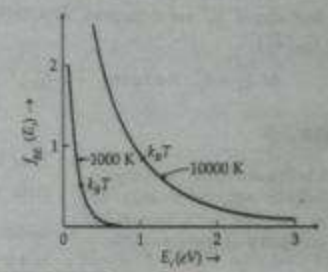

Fig. 7.8 B.E. distribution curve.

The Bose-Einstein index against energy is plotted in Fig. 7.8 for temperature of 1000 K and 10000 K in each case for $\alpha = 0$ (corresponding to a "gas" of photons). For $E_i >> k_B T$, the Bose-Einstein distribution approaches the exponential form is characteristics of the M.B. distribution. In this region the average number of particles per quantum state is much less than 1. However, for $E_i << k_B T$, the −1 term in the denominator causes the occupation index much greater compared to that of the M.B. distribution. This means that for energies small compared to $k_B T$, the number of particles per quantum state is greater for the B.E. distribution than the M.B. distribution.

The F.D. distribution is plotted in Fig.7.9 for three different values of T and α. In this distribution the occupation index never goes above 1. This signifies that we cannot have more than 1 particle per quantum state as required by Pauli's exclusive principle which applies in this case. Further, in this distribution the parameter α is strongly dependent on temperature T, and we write

$$\alpha = \frac{-E_F}{k_B T} \quad ...(91)$$

so that the Fermi-Dirac occupation index becomes

$$f_{FD}(E_i) = \frac{1}{(e^{(E_i - E_F)/k_B T} + 1)} \quad ...(92)$$

where E_F is called Fermi energy.

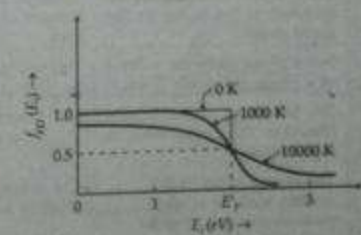

Fig. 7.9 F.D. distribution curve.

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Let us consider the situation at the absolute zero of temperature. At $T=0$, $(E_i - E_F)/k_BT = -\infty$ (for $E_i < E_F$) and $(E_i - E_F)/k_BT = +\infty$ (for $E > E_F$). Therefore

for $E_i < E_F$ $\quad f_{FD}(E_i) = \frac{1}{e^{-\infty}+1} = 1 \quad (\because e^{-\infty} = 0)$ ...(93)

and for $E_i > E_F$ $\quad f_{FD}(E_i) = \frac{1}{e^{\infty}+1} = 0 \quad (\because e^{\infty} = \infty)$ ...(94)

Thus at $T=0$, all energy states from $E_i = 0$ to $E_i = E_F$ are occupied because $f_{FD}(E_i) = 1$, while all states above E_F are vacant.

As the temperature rises, some of the states just below E_F become vacant, while some just above E_F are occupied. The higher the temperature, the more in the spreading in $f_{FD}(E_i)$.

At $E_i = E_F$, we have $\quad f_{FD}(E_i) = \frac{1}{e^0+1} = \frac{1}{2}$, at all temperatures. ...(95)

NOTE

At high temperature, the Fermi energy varies slightly with temperature. In the Fig. 7.9, the Fermi energy for curve at 10000 K is denoted by E_F'.

That is, the average number of particles per quantum state is exactly $\frac{1}{2}$. In other words, the probability of finding an electron with energy equal to the Fermi energy in a a metal is $\frac{1}{2}$ at any temperature.

MB · BE · FD

7.12 Comparison of M.B., B.E. and F.D. Statistics

	Maxwell-Boltzmann statistics (classical statistics)	Bose-Einstein statistics (Quantum statistics)	Fermi-Dirac statistics (Quantum statistics)
1.	Applies to systems of distinguishable particles, such as gases.	Applies to systems of indistinguishable particles not obeying Pauli's exclusion principle, such as photons, phonons and liquid helium at low temperature	Applies to indistinguishable particle obeying Pauli exclusion principle, such as free electrons in metal electrons in a star (white dwarf star)
2.	The distribution law is $n_i = \frac{g_i}{e^{\alpha} e^{E_i/k_BT}}$	The distribution law is $n_i = \frac{g_i}{(e^{\alpha} e^{E_i/k_BT} - 1)}$	The distribution law is $n_i = \frac{g_i}{(e^{\alpha} e^{E_i/k_BT} + 1)}$
3.	There is no restriction on the number of particles in the given state.	There is no restriction on the number of particles in a given state.	Only one particle in a given quantum state is allowed.
4.	In the phase space the volume of a phase cell is not fixed.	The volume of phase cell is of the order of h^3.	The volume of phase cell is of order of h^3.
5.	The behaviour of distribution function $f(E_i)$ against E_i is exponential.	For $E_i >> k_BT$, exponential for $E_i << k_BT$, lies above M.B.	For $E_i >> k_BT$, exponential where $E_i >> E_F$. If $E_F >> k_BT$ decreases abruptly near E_F.
6.	The energy of an M.B. system, such as in ideal monoatomic gas, may be zero at absolute zero.	The energy may be zero at absolute zero.	The energy at absolute zero, cannot be zero because all particles cannot come down to ground state due to Pauli's exclusive principle.

7.13 Limits When both B-E and F-D Approach M-B Statistics

The distribution law of three statistics are under

$$\frac{g_i}{n_i} = e^{\alpha} e^{E_i/k_BT} \quad ...(96) \quad \text{[For } M-B\text{]}$$

$$\frac{g_i}{n_i} = e^{\alpha} e^{E_i/k_BT} - 1 \quad ...(97) \quad \text{[For } B-E\text{]}$$

and

$$\frac{g_i}{n_i} = e^{\alpha} e^{E_i/k_BT} + 1 \quad ...(98) \quad \text{[For } F-D\text{]}$$

If $\frac{g_i}{n_i} >> 1$ then $\frac{g_i}{n_i} \approx \left(\frac{g_i}{n_i}+1\right) \approx \left(\frac{g_i}{n_i}-1\right)$ In this limit both B.E. and F.D. distributions are identical with M.B. distribution. This limit $\left(\frac{g_i}{n_i} >> 1\right)$ occurs when the temperature is not too low and pressure (or density) is not too high.

Formulae at a Glance

1. Volume of each phase cell

$$\delta V = \delta_x \delta_y \delta_z \delta p_x \delta p_y \delta p_z$$

$\delta_x \delta p_x \geq \hbar$, $\delta_y \delta p_y \geq \hbar$ and $\delta_z \delta p_z \geq \hbar$

$$\delta V = h^3 \qquad \left[\because \hbar = \frac{h}{2\pi}\right]$$

2. Maxwell-Boltzmann Statistics

(i) The distribution law $n_i = \frac{g_i}{e^{\alpha + E_i/k_BT}}$

(ii) Molecular energy in an ideal gas

$$E = \frac{2\pi N}{(\pi k_BT)^{3/2}} \frac{3}{4}(k_BT)^2 \sqrt{\pi k_BT} = \frac{3}{2} N k_BT$$

and $\bar{E} = \frac{3}{2} k_BT$

(iii) Maxwell-Boltzmann velocity distribution law

$$n(v)dv = \frac{\sqrt{2} N m^{3/2}}{(\pi k_BT)^{3/2}} v^2 e^{-\frac{mv^2}{2k_BT}} dv$$

(a) RMS velocity $v_{rms} = \sqrt{\frac{3k_BT}{m}}$

(b) Average velocity $= \bar{v} = \sqrt{\frac{8k_BT}{\pi m}}$

(c) Most probable velocity $(v_p) = \sqrt{\frac{2k_BT}{m}}$

3. (i) Symmetric wave function

$$\varphi_s(1,2) = +\varphi(2,1)$$

(ii) Antisymmetric wave function

$$\varphi_A(1,2) = -\varphi(2,1)$$

4. Bose-Einstein statistics

(i) distribution law $n_i = \frac{g_i}{(e^{\alpha + E_i/k_BT} - 1)}$

(ii) Planck's radiation law

$$dE = E(v)dv = \frac{8\pi h v^3}{c^3} \frac{1}{(e^{hv/k_BT} - 1)} dv$$

(iii) Rayleigh-Jeans Law $E(v)dv = \frac{8\pi v^2 k_BT}{c^3} dv$

(iv) Wein's displacement law

$$\lambda_{max} T = \text{Constant}$$

(v) **Bose-Einstein Condensation**

(a) Critical temperature

$$T_C = \frac{h^2}{2\pi m k_B T} \left(\frac{1}{2.612} \frac{n}{V}\right)^{2/3}$$

(b) Rest of particles

$$n_0 = n\left[1 - \left(\frac{T}{T_C}\right)^{3/2}\right] \text{ for } T < T_C$$

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5. Fermi Dirac Statistics

(i) Distribution law $n_i = \dfrac{g_i}{(e^{\alpha+E_i/k_BT}+1)}$

(ii) Fermi momentum $p_F = \left(\dfrac{3N}{8\pi V}\right)^{1/3} h$

(iii) Fermi Energy $E_F = \dfrac{p_F^2}{2m} = \dfrac{h^2}{2m}\left(\dfrac{3N}{8\pi V}\right)^{2/3}$

(iv) Ground state energy

$$E_0 = \frac{8\pi V}{h^3}\frac{p_F^5}{10m} = \frac{3}{5}NE_F$$

(v) Mean energy of Fermion

$$\bar{E} = \frac{E_0}{N} = \frac{3}{5}E_F$$

(vi) Electronic heat capacity

$$(C_v)_{El} = \frac{\pi^2}{2}Nk_B\left(\frac{T}{T_F}\right) = aT$$

(vii) Sommerfeld constant

$$a = \frac{Nk_B^2\pi^2}{2E_F}$$

Miscellaneous Numerical Problems for Practice

Example 1. *Show that on increasing temperature, the number of atoms in excited state increases.*

Solution. $n(E) = g(E).\ f(E)$

For Maxwell-Boltzmann distribution

$$n(E_1) = \frac{g(E_1)}{e^{(\alpha+E_1/k_BT)}} = g(E_1)Ae^{-E_1/k_BT}$$

Then $\quad n(E_1) = g(E_1)\,Ae^{-E_1/k_BT}$

and $\quad n(E_2) = g(E_2)\,Ae^{-E_2/k_BT}$

$$\Rightarrow \quad \frac{n(E_2)}{n(E_1)} = \frac{g(E_2)}{g(E_1)}e^{-(E_2-E_1)/k_BT}$$

On increasing temperature $\Rightarrow$ $n(E_2) > n(E_1)$.

Example 2. *(a) Show that at $E = E_F$ and $T \neq 0K$ The occupancy is equal to 0.5. (b) Show that at $E < E_F$ and $T = 0K$. The occupancy of state is perpendicular and at $E > E_F$ and $T = 0K$, the occupancy of state is zero.*

Solution. From Fermi-Dirac statistics

$$n_i = \frac{g_i}{e^{\alpha+E_i/k_BT}+1}$$

This is written as

$$n(E) = \frac{g(E)}{e^{(E-E_F)/k_BT}+1} = g(E)\,f(E)$$

Then occupancy $\quad f(E) = \dfrac{1}{(e^{(E-E_F)/k_BT}+1)}$

(a) When $T \neq 0$ and $E = E_F$ then $f(E) = \dfrac{1}{e^0+1} = \dfrac{1}{1+1} = \dfrac{1}{2} = 0.5$

(b) When $T = 0$ and $E < E_F$ then $f(E) = \dfrac{1}{e^{-\infty}+1} = \dfrac{1}{0+1} = 1$

Example 3. *Aluminium is a fcc crystal with lattice constant 4.05 Å and the metal has 3 free electrons per atom. Calculate the Fermi enegy in eV for the metal. Also evaluate its Fermi factor at an ambient temperature 300 K for an energy value 0.1 eV higher than E_F.*

Solution. The crystal structure of *fcc*

$\therefore$ Number of atoms/unit cell in Al is 4

Lattice constant $(a) = 4.05 \times 10^{-10}$ m

Number of free electron per atom = 3

$T = 300\ K,\ \ E - E_F = 0.1\,eV;\ E_F = ?$ and $f(E) = ?$

We know that Fermi energy (E_F) is given by $E_F = \dfrac{h^2}{2m}\left(\dfrac{3n_c}{8\pi}\right)^{2/3}$

where n_c is the number of electrons per unit volume

$$n_c = \frac{N}{V} = \frac{\text{Number of free electrons / atom} \times \text{Number of atoms / unit}}{a^3}$$

$$= \frac{3\times 4}{(4.05\times 10^{-10})^3} = 1.806\times 10^{28}\ \text{electrons / m}^3$$

Hence $\quad E_F = \dfrac{(6.625\times 10^{-34}\ \text{J.s})^2}{(2\times 9.1\times 10^{-31}\ \text{kg / electron})}\times\left(\dfrac{3}{8\times 3.14}\times 1.806\times 10^{28}\ \text{electrons / m}^3\right)^{2/3}$

$$= 1.867\times 10^{-18}\ \text{J} = 11.66\ \text{eV}$$

(ii) Fermi function $f(E) = \dfrac{1}{[e^{(E-E_F/k_BT)}+1]} = \dfrac{1}{e^{\frac{0.1\times 1.6\times 10^{-19}}{300\times 1.38\times 10^{-23}}}+1}$

$$= \frac{1}{(e^{3.665}+1)} = \frac{1}{47.691+1} = \frac{1}{48.691}$$

$$E_F = 0.021\ \text{eV}$$

Example 4. *The Fermi level in potassium is 2.1 eV at a particular temperature. Calculate the number of free electrons per unit volume in potassium at the same temperature.*

Solution. Given $E_F = 2.1\ \text{eV} = 2.1\times 1.6\times 10^{-19}$ J. $\quad n_c = ?$

The Fermi energy is given $E_F = \dfrac{h^2}{2m}\left(\dfrac{3n_c}{8\pi}\right)^{2/3}$

$$\therefore \quad n_c = \left(\frac{2m}{h^2}E_F\right)^{3/2}\times\frac{8\pi}{3} = \left[\frac{2\times 9.1\times 10^{-31}\times 2.1\times 16\times 10^{-19}}{(6.625\times 10^{-34})^2}\right]^{3/2}\times\frac{3.14\times 8}{3}$$

$$= (5.379\times 10^{18})\times 1.047 = 1.379\times 10^{28}\ \text{electrons / m}^3.$$

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Example 5. The density of zinc is 7.13×10^3 kg/m^3 and its atomic weight is 65.4. Calculate the Fermi energy and the mean energy at $T=0$ K.

Solution. Given $\rho=7.13\times10^3$ kg m^{-3}, $M=65.4$

Since we know that

$$E_F=\frac{h^2}{2m}\left(\frac{3n_c}{8\pi}\right)^{2/3}$$

or $$E_F=3.65\times10^{-19}\,n_c^{2/3}\,\text{eV} \qquad \text{(on putting the value } h, m, \pi)$$

and $$n_c=\frac{2pN}{M}=\frac{2\times\text{density}\times\text{Avogadro's No.}}{\text{Molecular weight}}$$

$$n_c=\frac{2\times7.13\times6.023\times10^{26}}{65.4}=1313\times10^{26}$$

$\therefore$ $$E_F=3.65\times10^{-19}\times(1313\times10^{26})^{2/3}=11.1\,\text{eV}$$

and Mean energy $(\bar{E})=\frac{3}{5}E_F=\frac{3}{5}\times11.1\,\text{eV}=6.66\,\text{eV}.$

Example 6. At what temperature can we expect a 10% probability that electrons in a metal will have an energy which is 1% above E_F ? The Fermi energy of the metal is 5.5 eV.

Solution. Given $f(E)=10\%$, $E=E_F+1\%$ of E_F, $E_F=5.5$ eV, $T=?$

$$f(E)=\frac{1}{(e^{(E-E_F)/k_BT}+1)}$$

$$E=5.5+\frac{5.5}{100}=5.5+0.555;\quad E-E_F=0.555.$$

$$0.1=\frac{1}{\left(\exp.\frac{0.555\times1.6\times10^{-19}}{1.38\times10^{-23}T}\right)+1}=\frac{1}{\exp.\left(\frac{637.7}{T}\right)+1}$$

or $$T=290.2\,\text{K}$$

Example 7. Assume that a large number of particles each of mass m say are lying in a box at an equilibrium temperature T. Is it possible for any one of them to spontaneously fly to a height h ? Assume that we can apply the Maxwell-Boltzmann law to the particles.

Solution. The Maxwell-Boltzmann law says that the probability of a particle having an energy E is proportional to $\exp.\left(-\frac{E}{k_BT}\right)$.

When the particle flies through a height h, it acquires an additional P.E. = mgh. The probability of the particle acquiring this additional energy is proportional to

$$\exp.\left(-\frac{E}{k_BT}\right)=\exp\left(-\frac{mgh}{k_BT}\right)$$

Usually this factor will be so small that we can assume it to be practically zero. Thus, though the possibility mentioned is not strictly impossibility, it is an extremely improbable occurrence.

Example 8. Suppose that the maximum temperature in an atomic bomb explosion is 10^7 K. What is the corresponding wave length of maximum energy ?

Solution. $\lambda_{max}T=0.289$ cm K

$$\lambda_{max}=\frac{0.2892}{T}=\frac{0.2892}{10^7}=2.9\times10^{-8}\,\text{cm}\approx2.9\,\text{Å}$$

Example 9. Calculate the surface temperature of the sun and moon given that $\lambda_m=4573$ Å and 14 μ respectively, λ_m being the wavelength of the maximum intensity of emission.

Solution. We know that $\lambda_{max}T=0.2892$

For Sun $\lambda_{max}=4573\times10^{-10}$ m or 4573×10^{-8} cm

$$4573\times10^{-8}\times T=0.2892$$

or $$T=6324\,\text{K} \quad\text{or}\quad 6051°\text{C}$$

For moon $\lambda_{max}=14\times10^{-4}$ m or 14×10^{-4} cm

$\therefore$ $$1400\times10^{-4}\times T=0.2892$$

$$T=\frac{0.2892\times10^4}{1400}=206.57\,\text{K}$$

Example 10. Verify that rms speed of an ideal gas molecular is about 9% greater than its average speed.

Solution. The equation $n(v)dv=4\pi N\left(\frac{m}{2\pi k_BT}\right)^{3/2}v^2e^{-\frac{mv^2}{2k_BT}}dv$

gives the number of molecules with speeds between v and $(v+dv)$ in a sample of N molecules. To find their average speed $\bar{v}$, we multiply $n(v)dv$ by v, integrate over all values of v from 0 to ∞, and then divide by N, we get

$$\bar{v}=\frac{1}{N}\int_0^\infty v\,n(v)dv=4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_0^\infty v^3e^{-\frac{mv^2}{2k_BT}}dv \qquad ...(i)$$

If we let $a=\frac{m}{2k_BT}$ we see that the integral is the standard one

$$\int_0^\infty x^3e^{-ax^2}dx=\frac{1}{2a^2}$$

ans so $$\bar{v}=\left[4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}\right]\left[\frac{1}{2}\left(\frac{2k_BT}{m}\right)^2\right]=\sqrt{\frac{8k_BT}{\pi m}} \qquad ...(ii)$$

and we know that $$v_{rms}=\sqrt{\frac{3k_BT}{m}} \qquad ...(iii)$$

Comparing Eqs. (ii) and (iii), we get

$$v_{rms}=\sqrt{\frac{3k_BT}{m}}=\sqrt{\frac{3\pi}{8}}\,\bar{v}=1.09\,\bar{v}$$

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Example 11. *Radiation from the Big Bang has been Doppler-shifted to longer wavelengths by the expansion of the universe and today has a spectrum corresponding to that of a black body at 2.7 K. Find the wavelength at which the energy density of this radiation is a maximum. In what region of the spectrum is this radiation.*

Solution. We know that

$$\lambda_{max} T = 2.892 \times 10^{-3}\,\text{mK}$$

$$\lambda_{max} = \frac{2.892 \times 10^{-3}\,\text{mK}}{T}$$

$$= \frac{2.892 \times 10^{-3}\,\text{mK}}{2.7\,\text{K}} = 1.1 \times 10^{-3}\,\text{m} = 1.1\,\text{mm}$$

The wavelength is in the microwave region.

Note. The radiation was first detected in a microwave survey of the sky in 1964

Example 12. *As the temperature rises the Fermi-Dirac distribution spreads away from the steplike form it takes at $T = 0$. More particles in a gas of fermions take on energies greater than E_F, and fewer take on energies less than E_F. Consider a system with $E_F = 5.0\,eV$. For what temperature is the number of particles at $E_F/2$ equal to 90% of the zero temperature number at $E_F/2$?*

Solution. According to the Fermi-Dirac distribution, the number of particles at $E < E_F$ for $T = 0$ is 2. (The normalisation does not concern us here since we are giving to compare the values of the distribution at two different values of T.) Thus we want the value of T for which

$$\frac{2}{\exp\left[\frac{(E_F/2 - E_F)}{k_B T}\right]+1} = \frac{2}{\exp\left[-\frac{E_F/2}{k_B T}\right]+1} = (0.9) \times 2 = 1.8$$

This means that $\exp\left(\frac{-E_F}{2k_B T}\right) = \frac{2-1.8}{1.8} = 0.11$

we can now take the logarithms of this equation to find T.

We obtain

$$\ln\left[\exp\left(-\frac{E_F}{2k_B T}\right)\right] = -\frac{E_F}{2k_B T} = \ln(0.11)$$

or $$T = \frac{-E_F}{2k_B \ln(0.11)} = -E_F/[2k(-2.2)] = 0.23\,E_F/k$$

Using $k_B = [1.38 \times 10^{-23}\,\text{J/K}]/(1.6 \times 10^{-19}\,\text{J/eV})$

$= 8.63 \times 10^{-5}\,\text{eV/K}$, we have

$$T = \frac{(0.23)(5.0\,\text{eV})}{(8.63 \times 10^{-5}\,\text{eV/K})} = 1.33 \times 10^{4}\,\text{K}.$$

Example 13. *Calculate the relative number of atoms of hydrogen gas present in ground state and first excited state at room temperature and at very high temperature of 5000 K. Assume M.B. statistics to hold good for hydrogen gas.*

Solution. Assuming Maxwell-Boltzmann statistics for hydrogen gas the required ratio can be calculated using the expression

$$\frac{n(E_2)}{n(E_1)} = \frac{g(E_2)}{g(E_1)} e^{(E_1 - E_2)/k_B T}$$

For hydrogen atom in ground state ($n = 1$), there are two possible electron configuration in this state *i.e.*, $g(E_1) = 2$; whereas in the first excited state ($n = 2$). These are eight possible configurations (*i.e.*, $2n^2$) or $g(E_2) = 8$.

Further $(E_1 - E_2) = -10.2$ (from Bohr model)

The the ratio $n(E_2)/n(E_1)$ becomes

$$\frac{n(E_2)}{n(E_1)} = \frac{8}{2} e^{(-10.2)/k_B T}$$

(*i*) At room temperature $T = 300\,\text{K}$, therefore

$$\frac{n(E_2)}{n(E_1)} = 4e^{(-404)} \approx 10^{-175}$$

(*ii*) At $T = 5000\,\text{K}$

$$\frac{n(E_2)}{n(E_1)} = 4e^{(-23.7)} \approx 2 \times 10^{-10}.$$

Example 14. *Calculate the Fermi temperature for (i) liquid ^{3}He and (ii) electrons in a white a dwarf star using the known experimental data on the two system.*

Solution. We have

$$T_F = \frac{E_F}{k_B} = \frac{h^2}{2mk}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

(*i*) For liquid ^{3}He

$$\frac{V}{N} = 63\,\text{Å atom}^{-1} = 6.3 \times 10^{-24}\,\text{cm}^{-3}\,\text{atom}^{-1}$$

$$\frac{N}{V} = \frac{10^{24}}{63}\,\text{atom cm}^{-3}$$

$$m = 5.01 \times 10^{-24}\,\text{g}$$

This yields, $T_F = 4.9\,\text{K}$

(*ii*) For white drafts $\rho = 10^7\,\text{g cm}^{-3}$

$$n = \frac{N}{V} = 10^{30}\,\text{electrons cm}^{-3}$$

This yields $T_F = 10^{10}\,\text{K}$

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Question Bank

Multiple Choice Questions

1. A body absorbs all the radiations incident over it is called :
   (a) good absorber (b) black body
   (c) good emitter (d) perfectly black body.
2. In case of a black body for distribution of energy radiated by a black body, Planck's formula reduces to Rayleigh Jeans formula for
   (a) long wavelength region (b) short wavelength region
   (c) equal wavelength of radiation (d) none of the above.
3. The wavelength corresponding to the maximum as per Wein's law is
   (a) directly proportional to the fourth power of temperature
   (b) directly proportional to the 1/4 power of temperature
   (c) inversely proportional to the temperature
   (d) inversely proportional to 1/4 power of temperature.
4. The phenomenon known as Bose-Einstein condensation, was invoked by F. London to explain
   (a) the free electron theory (b) to super-fluidity exhibited by liquid ^{4}He
   (c) the Planck's radiation law (d) none of these.
5. The highest energy possessed by a fermion at $T = 0$ is called
   (a) zero-point energy (b) radiation energy
   (c) Fermi energy (d) Bose energy.
6. Which is of the following statistics under classical statistics ?
   (a) Maxwell-Boltzmann statistics (b) Bose-Einstein statistics
   (c) Fermi-Dirac statistics (d) none of these.
7. The correct expression for Fermi energy (E_F) is
   (a) $E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$ (b) $E_F = \frac{2m}{h^2}\left(\frac{8\pi V}{3N}\right)^{2/3}$
   (c) $E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$ (d) none of these.
8. The electronic contribution to the heat capacity of a metal is given by
   $(C_V)_{el} = aT$, where a is known as
   (a) Planck's constant (b) Wein's constant
   (c) Einstein's constant (d) Sommerfield's constant.
9. The point in the phase space representing instantaneous state of particle is called
   (a) boiling point (b) melting point
   (c) phase point (d)none of these.

10. Volume of each phase cell is equal and greater tan to
    (a) $(h)^3$ (b) h^3
    (c) $\frac{h^2}{c}$ (d) $\frac{c^2}{h^2}$
    where symbol are having their usual meanings
11. Suppose (A) is applicable to the identical, distinguishable particles of any spin. Then (A) is
    (a) Maxwell-Boltzman statistics
    (b) Bose-Einstein statistics
    (c) Fermi-Dirac statistics
    (d) none of these.
12. Suppose (B) is applicable to the identical, indistinguishable particles of zero or integral spin. Then (B) is
    (a) Maxwell-Boltzmann statistics (b) Bose-Einstein statistics
    (c) Fermi-Dirac statistics (d) none of these.
13. Suppose (C) is applicable to the indistinguishable particles of half integral spin. Then (C) is
    (a) Maxwell-Boltzmann statistics (b) Bose-Einstein statistics
    (c) Fermi-Dirac statistics (d) none of these.
14. The type of particles described by symmetric wave functions are known as
    (a) Magnon (b) Phonon
    (c) Boson (d) Fermions.
15. The type of particles described by anti-symmetric wave functions are called as
    (a) Fermions (b) Bosons
    (c) Magnons (d) Phonons.
16. The mean energy per fermion for a completely degenerate electron gas is given by
    (a) $\bar{E} = \frac{3}{5}(E - E_F)$ (b) $\bar{E} = \frac{3}{5}E_F$
    (c) $\bar{E} = \frac{1}{2}mv^2$ (d) $\bar{E} = mgh$
17. In case of F.D. statistics. The energy at absolute zero, cannot be zero because all particle cannot come down to ground state due to
    (a) Pauli's exclusive principle (b) Heisenberg's principle
    (c) Gibb's principle (d) none of these.
18. Molecular energy in an ideal gas is given by
    (a) $\bar{E} = \frac{3}{2}k_BT$ (b) $\bar{E} = \frac{5}{2}k_BT$
    (c) $\bar{E} = \frac{k_BT}{2}$ (d) $\bar{E} = \frac{\sqrt{3}}{\sqrt{2}}k_BT$.
19. Wein's law of radiation is
    (a) $\lambda_m T^{-1}$ = constant (b) $\lambda_m T$ = constant
    (c) $\lambda_m^{-1} T^3$ = constant (d) $\lambda_m^{-1} T^2$ = constant.

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20. Relation between the colour and the temperature of a star is given by
(a) Wein's displacement law (b) Planck's law
(c) Hubble's law (d) Fraunhofer's diffraction law.

21. If the temperature of a black body is doubled, the wavelength at which the spectral radiancy has its maximum is
(a) Quadrupled (b) Eight times
(c) Doubled (d) Halved.

22. A black body has maximum wavelength λ_m at 2000 K. Its corresponding wavelength at 3000 K will be
(a) $\frac{16}{81}\lambda_m$ (b) $\frac{81}{16}\lambda_m$
(c) $\frac{3}{2}\lambda_m$ (d) $\frac{2}{3}\lambda_m$.

23. Wein's displacement law fails at
(a) low temperature (b) high temperature
(c) long wavelengths (d) short wavelengths.

24. Two stars X and Y emit lights of yellow and blue colours respectively. Of the two, the temperature will be greater for
(a) X (b) Y
(c) neither of them (d) uncertain result.

25. The ratio of temperatures of a planet and moon, if the wavelengths of their maximum emission radiations are 140 Å and 4200 Å respectively is
(a) 3 : 1 (b) 1 : 3
(c) 30 : 1 (d) 1 : 30.

26. Planck's radiation law verifies the following nature of radiation
(a) electromagnetic nature (b) wave mechanical nature
(c) thermodynamic nature (d) quantum nature.

27. Four slabs of iron heated to different temperatures show different colours listed below. The temperature is highest for :
(a) yellow slab (b) red slab
(c) white slab (d) green slab.

28. The wavelength of radiation emitted by a body depends upon
(a) nature of its surface (b) the area of its surface
(c) the temperature of the surface (d) all of above.

29. The general expression for the Fermi energy of a metal at 0 K is
(a) $3.65 \times 10^{19} \times n^{2/3}$ eV (b) $3.65 \times 10^{19} \times n^{2}$ eV
(c) $3.65 \times 10^{19} \times n^{1/3}$ eV (d) $3.65 \times 10^{19} \times n^{3/2}$ eV.
where n is the number of free electrons/m^3.

30. If the Fermi energy of silver at 0 K is 5 eV, the mean energy of electron in silver at 0 K is
(a) 6 eV (b) 12 eV
(c) 1.5 eV (d) 3 eV.

31. If the Fermi energy of a metal is 1.4 eV, the Fermi temperature of the metal is approximately.
(a) 1.6×10^3 K (b) 1.6×10^4 K
(c) 1.6×10^5 K (d) 16×10^6 K

Answers

1. (b)	2. (b)	3. (c)	4. (b)	5. (c)	6. (a)
7. (a)	8. (d)	9. (c)	10. (a)	11. (a)	12. (b)
13. (c)	14. (c)	15. (a)	16. (b)	17. (a)	18. (a)
19. (b)	20. (a)	21. (d)	22. (d)	23. (c)	24. (b)
25. (c)	26. (d)	27. (c)	28. (c)	29. (a)	30. (d)
31. (b)					

Theoretical Questions

1. Which type of the statistics shall be applicable for a gas of photons ? Justify your answer. **[GGSIPU, May 2008 (2.5 marks)]**

2. Compare the salient features of Maxwell-Boltzmann (MB), Bose-Einstein (BE) and Fermi-Dirac (FD) statistics. **[GGSIPU, May 2008 (6.5 marks)]**

3. Use the distribution function to show the Bose-Einstein (BE) and Fermi-Dirac (FD) distributions at very high temperature goes to Maxwell-Boltzmann (MB) distribution. **[GGSIPU, May 2008 (4 marks)]**

4. Distinguish between Maxwell-Bolzmann, Bose-Einstein and Fermi-Dirac statistics on the basis of their functions. **[GGSIPU, April 2006 ; May 2007 (3 marks)]**

5. Compare the qualitative features of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac statistics. **[GGSIPU, April 2007 (5 marks)]**

6. Compare the salient features of the three statistical distribution functions : Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein. Given one example of particles in each of these categories. **[GGSIPU, May 2005 (12.5 marks)]**

7. Distinguish between a Boson and Fermion ? **[GGSIPU, April 2008 (2 marks)]**

8. Explain the terms : position space, momentum space and phase space. **[GGSIPU, April 2008 (2.5 marks)]**

9. Explain Fermi-Dirac distribution function. Plot this function (with energy) for various temperatures including O K. **[GGSIPU, May 2006 (3.5 marks)]**

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10. Obtain an expression for the Bose-Einstein distribution law. Deduce the expression for distribution of energy among particles according to Bose-Einstein statistics. [GGSIPU, April 2007 (5 marks)]

11. The Fermi-Dirac distribution function is given by $f(E) = \frac{1}{[1 + e^{(E - E_F)/k_B T}]}$. Find the occupation probability of a state having energy $E = E_F$. [GGSIPU, April 2005 (5 marks)]

12. Deduce expressions for distribution functions in the Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein statistics and mention the cases where respective distribution functions are applicable.

13. Show that the energy density per unit volume of a photon gas is given by

$$E(\nu)d\nu = \left(\frac{8\pi h}{c^3}\right)\frac{\nu^3 d\nu}{(e^{h\nu/k_B T} - 1)}$$

where k_B is Boltzmann constant.

14. What do you understand by phase space ? What is its significance ?

15. Write short notes on

(i) Exchange symmetry of wave functions

(ii) Quantum statistics of identical particles.

16. Give a comparison of M.B., B.E. and F.D. statistics, under what limits B.E. and F.D. statistics approach M.B. statistics ?

17. In classical statistics, the number of ways in which n_i particles can be distributed among g_i states in $g_i^{n_i}$. Divide this by $n_i!$ and obtain the Maxwell Boltzmann distribution.

18. Discuss the distribution of two particles in two cells in terms of (i) M.B. (ii) B.E. and (iii) F.D statistics

19. What is the significance of the Fermi energy in a fermion system at 0 K ? At $T > 0$K ?

20. At the same temperature, will a glass of classical molecules, a gas of bosons, or a gas of fermions exert the greatest pressure ? The least pressure ? Why ?

21. Give the main points of difference between classical and quantum statistics. What are the two types of quantum statistics and how do they differ from one each other ?

22. Where do we use the Bose-Einstein statistics ? Derive an expression for the probability of occurrence, of a state of a given energy, in this statistics.

23. Show that quantum statistical distribution functions gives same results as classical statistical distribution function if the inter particle distance is much larger than the de Broglie wavelength of the wave associated with the particle under study.

24. Deduce Rayleigh-Jeans Law and Wein's displacement law from Planck's law.

25. In case of Maxwell-Boltzmann statistics, one considers the identical gas molecules as distinguishable. Comment on this statement.

26. What was the need to introduce quantum statistics ?

27. Show that the Rayleigh-Jean's formula and the Wein's displacement formula are special cases of the Planck's law of black body radiation.

28. Explain the difference between macrostate or microstate with the help of an example.

29. Obtain an expression for E_F, the fermi energy at $T = 0$K, for an electron gas in a metal.

Numerical Problems

1. Prove that the average energy of electron at $T = 0$K is equal to $\frac{3}{5}E_F$, where E_F is the Fermi energy at 0 K.

2. Show that Bose-Einstein (BE) and Fermi-Dirac (FD) statistics reduces to Maxwell-Boltzmann (MB) statistics at high temperature.

3. The atomic density at 0°C and at atmospheric pressure is 2.7×10^{25} per m^3 for hydrogen. Prove that in first excited state of hydrogen at 10000°C, the number of atom will be 10^{22}.

4. The number of conduction electrons per cm^3 is 2.42×10^{22} in beryllium and 0.91×10^{22} in cesium. If Fermi energy of conduction electrons in Be is 14.14 eV, calculate its value in the case of Cs.
[Ans. 1.6 eV]

5. Cesium metal has one conduction electron per atom. Calculate the Fermi energy in eV for cesium, given that, it is a bcc crystal with lattice constant of 6.14 Å. [Ans. $E_F = 1.537$ eV]

6. Use the Fermi function to obtain the value of f(E) for $E - E_F = 0.1$eV. [Ans. 0.0205]

7. The fermi level in silver is 5.5 eV, what are the energies for which the probabilities of occupancy at 300 K are 0.99, 0.01 and 0.5. (Given $1J = 6.24 \times 10^{18}$ eV).
[Ans. $E_1 = 5.38$ eV, $E_2 = 5.50$ eV, $E_3 = 5.62$ eV]

8. Fermi energy for gold is 5.54 eV. Calculate the Fermi temperature, given $k_B = 1.38 \times 10^{-23}$ JK^{-1}.
[Ans. $T_F = 6.42 \times 10^4$]

9. The Fermi energy of silver is 5.51 eV. What is the average energy of a free electron at 0 K ?
[Ans. $\bar{E} = 3.306$ eV]

10. Evaluate the temperature at which there is 1% probability that a state with energy 0.5 eV above the Fermi energy will be occupied by an electron. (Given $E_F = 5$ eV) [Ans. 1264 K]

11. Find the rms speed of oxygen molecules at 0° C. [Ans. 461 m/s]

12. Show that the de Broglie wavelength of an oxygen molecule in thermal equilibrium in the atmosphere at 20°C is smaller than its diameter of about 4×10^{-10} m.

13. How many independent standing waves with wavelength between 9.5 and 10.5 mm can occur in a cubical cavity 1 m on a side ? How many with wavelengths between 99.5 and 100.5 mm ?
[Ans. 2.5×10^6, 2.5×10^2]

14. An object is at a temperature of 400°C. At what temperature would it radiate energy twice as fast ?
[Ans. 527°C]

15. The brightest part of the spectrum of the star Sirius is located at a wavelength of about 290 mm. What is the surface temperature of Sirius. [Ans. 1×10^4 K]

16. Find the specific heat at constant volume of 100cm^3 of radiation in thermal equilibrium at 1000 K.
[Ans. 3.03×10^{12} J/K]

17. Show that in a gas of non-interacting fermions the average energy per particle in the high temperature limit is the classical result $\langle E \rangle = \frac{3k_B T}{2}$.

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18. A water droplet 1 micron (10^{-6}m) in diameter is in equilibrium with air molecules at room temperature (295 K). What is the root-mean-square (rms) speed of the droplet. [**Ans.** 4.8 mm/s]

19. What is the wavelength at which the human body radiates maximum energy ? [**Ans.** $\lambda_m = 9322$ nm]

20. What is the wavelength of solar radiation to which human eye is most sensitive. Assume the temperature of sun to be 5700 K. [**Ans.** $\lambda_m = 5070$ Å]

21. For O_2 gas at room temperature, calculate the average, most probable and rms velocities. Given $k_B = 1.38 \times 10^{-16}$ ergs/K and $N = 6.023 \times 10^{23}$ mol^{-1}. [**Ans.** $v_{mp} = 3.09 \times 10^4$ cm / s, $v_{rms} = 4.8 \times 10^4$ cm / s and $\bar{v} = 4.4 \times 10^4$ cm / s]

22. Metallic potassium has a density of 0.86×10^3 kg/m^3 and an atomic weight of 39. Find the Fermi energy for the electrons in the metal if each potassium atom denotes one electron to the electron gas. [**Ans.** $E_F = 2.05$ eV]

23. At $T = 300$ K it is found that the Fermi energy of a certain material is reduced from its value at 0 K by 12×10^{-4} eV. What is the Fermi energy of this material at 0 K ? [**Ans.** 4.58 eV]

Chapter 8

Semiconductor Physics

8.1 Introduction

Solids can be classified on several bases. One basis is its electrical resistivity or conductivity. They are classified as conductors, insulators or semiconductors if their resistivities are very low, very high or of intermediate value respectively. The typical values of the resistivities for three classes are of the order of $10^{-8}\Omega$m, $10^{10}\Omega$m and 1Ωm respectively. Another essential distinguishing feature between a conductor and a semi- conductor is the sign of their temperature coefficient of resistance. **Conductors** have a positive temperature coefficient of resistance at all temperatures and their resistance increases approximately linear with temperature.

An exceptional behaviour is shown by a **superconductor** for which the resistance sharply drops to very low values as the temperature is decreased below a critical temperature.

Semiconductors, on the other hand, have a negative temperature coefficient of resistance over an appreciable range of temperature the resistance decreasing with temperature exponentially. This is shown in Fig. 8.1.

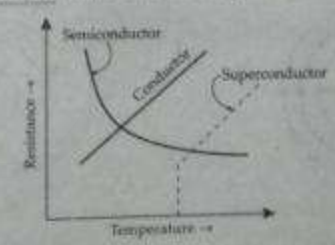

Fig. 8.1 Temperature dependence of resistance for a typical conductor, a semiconductor and a superconductor. The three curves have different scale of resistance.

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The conductivities of solids can be explained as follows : A solid is made up of a large number of atoms arranged in a regular manner. The atoms are composed of a positively charged nucleus around which electrons revolve in certain well defined orbits. An electron in a orbit represents a bound electrons ; it is constrained to move in that particular orbit under the electrostatic attraction of the nucleus. Such an electron cannot carry current through the solid because it is not free to move along the direction of an external electric field. Therefore, a solid containing only the electrons that are bound in atoms will be an insulator. In order that a current might flow through the solid, it must contain some electrons which are free to move under the action of an external electric field without much hinderance. As we know that, a metal has a large number of such free electrons. Their movement in the absence of any electric field is random ; as such it does not result in a non-zero average current in any direction. However, when electric field is applied, they acquire an non-zero average velocity in the direction of field and result in non-zero current through the solid. Thus a perfect insulator does not have any free electron, good conductor has a very large number of them and we expect that, semiconductor will have some of them.

8.2 Bohr's Atomic Model

In 1913, Neil Bohr, Danish Physicist gave clear explanation of atomic structure. According to Bohr :

(i) An atom consists of a positively charged nucleus around which negatively charged electrons revolve in different circular orbits.

(ii) The electrons can revolve round the nucleus only in certain permitted orbits *i.e.*, orbits of certain radii are allowed.

(iii) The electron in each permitted orbit have a certain fixed amount of energy. The larger the orbit (*i.e.*, larger radius) the greater is the energy of electrons.

(iv) If electron is given addition energy (*e.g.*, heat, light etc.) it is lifted to higher orbit. The atom is said to be in state of excitation. This state does not last long, because electron soon falls back to the original lower orbit. As it falls, it gives back the acquired energy in the form of heat, light or other radiation.

Fig. 8.2 shows the structure of silicon atom. It has 14 electrons.

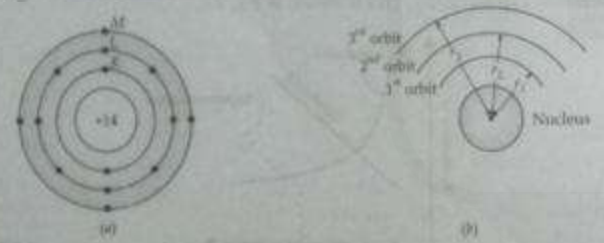

Fig. 8.2

Two electrons revolve in the first orbit, 8 in the second orbit and 4 in the third orbit. The first, second, third orbits etc. are also known as *K*, *L*, *M* orbits respectively.

These electrons can revolve only in permitted orbits (*i.e.*, orbits or radii[1] r_1, r_2 and r_3) and not in any arbitrary orbit. Thus, all radii between r_1 and r_2 or between r_2 and r_3 are forbidden. Each orbit has fixed amount of energy associated with it. If an electron in the first orbit is to be lifted to the second orbit, just the[2] right amount of energy should be supplied to it. When this electron jumps from the second orbit to first, it will give back the acquired energy in the form of electromagnetic radiations.

8.3 Energy Levels

It has already been discussed that each orbit has fixed amount of energy associated with it. The electrons moving in a particular orbit possess the energy of that orbit. The larger the orbit, the greater is its energy. It becomes clear that outer orbit electrons possess more energy than the inner orbit electrons.

A convenient way of representing the energy of different orbits is shown in Fig. 8.3(*ii*). This is known as *energy level diagram*, the first orbit represents *first energy level*, the second orbit indicates the *second energy level* and so on. The larger the orbit of an electron, the greater is the energy and higher is the energy level.

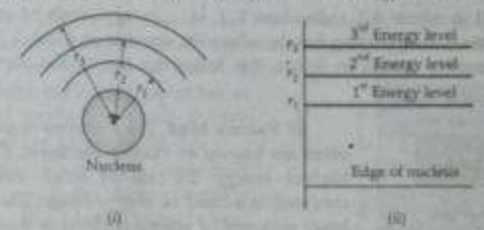

Fig. 8.3

8.4 Energy Band

In case of a single isolated atom, the electrons in any orbit possess definite energy. However, an atom in a solid is greatly influenced by the closely-packed neighbouring atoms. The result is that the electron in any orbit of such an atom can have a range of energies rather than a single energy. This is known as *energy band*.

"The range of energies possessed by an electron in a solid is known as energy band."

The concept of energy band can be easily understood by referring to Fig. 8.4. Fig. 8.4(*iii*) shows the energy levels of a single isolated atom of silicon. Each orbit of an atom has a single energy. Therefore, an electron can have only single energy corresponding to the orbit in which it exists. However, when the atom is in a solid, the electron in any orbit can have a range of energies. For instance, electrons in the first orbit have slightly different energies because no two electrons in this orbit see exactly the same charge environment. Since there are millions of first orbit electrons, the slightly different energy levels form a band, called 1st energy band [Fig. 8.4(*iii*)]. The electrons in the first orbit can

1. The value of radii are determined from quantum consideration.
2. So that its total energy is equal to that of second orbit.

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have any energy range in this band. Similarly, second orbit electrons form second energy band and so on.

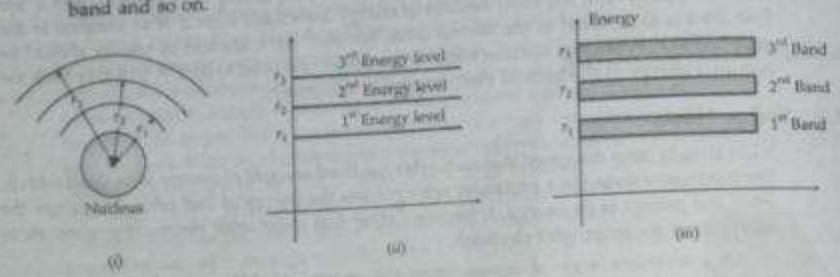

Fig. 8.4

8.5 Important Energy Band in Solids

As discussed in article 8.2, individual *K, L, M,...* energy levels of an isolated atom are converted into corresponding bands, when the atom is in a solid. Though there are a number of energy bands in solids, the following are of the particular importance. [Fig. 8.5].

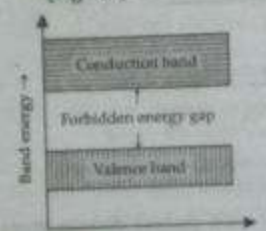

Fig. 8.5

(i) ***Valence band.*** The electrons in outermost orbit of an atom are known as valence electrons. These electrons have highest energy. In case of solid, valence electrons are confined in a band or energy range. *The range of energy (i.e., band) possessed by valence electrons is known as* ***valence band.*** This band may be completely or partially filled. In short, we can say that *the uppermost energy band of solid which is partially or completely filled by the electrons, is called as* ***valence band.*** For instance, in case of inert gases, the valence band is full whereas for other materials, it is only partially filled. The partially filled band can accommodate more electrons.

(ii) ***Conduction band.*** In certain materials (*e.g.*, metals), the valence electrons are loosely attached to the nucleus. Even at ordinary temperature, some of the valence electrons may get detached to become free electrons. In fact, it is these free electrons which are responsible for the conduction of current in a conductor. For this reason, they are called conduction electrons.

> The range of energy (*i.e.*, band) possessed by conduction electrons is known as **conduction band.**

All electrons in the conduction band are free electrons. If a substance has empty conduction band, it means current conduction is not possible in that substance. Generally, insulators have empty conduction band. On the other hand, it is partially filled for conductors.

(iii) ***Forbidden energy gap.*** *The separation between conduction band and valence band on the energy level diagram is known as forbidden energy gap.* No electron of a solid can stay in a forbidden energy gap as there is no allowed energy state in this region. The width of the forbidden energy gap is measure of the boundage of valence electrons to the atom. The greater the energy gap, more tightly the valence electrons are bound to the nucleus. In order to push an electron from valence band to conduction band (*i.e.*, to make the valence electron free), external energy equal to forbidden energy gap must be supplied.

8.6 Classification of Solids on the Basis of Energy Bands

We know that some solids are good conductors of electricity while other are insulators. There is also an intermediate class of semiconductors. The difference in the behaviour of solids as regards their electrical conductivity can be beautifully explained in terms of energy bands. The electrons in the lower energy band are tightly bound to the nucleus and play no part in the conduction process. However, the valence and conduction bands are of particular importance in ascertaining the electrical behaviour of various solids.

(a) Insulators or Non-Conductors or Dielectrics

Insulators (*e.g.*, wood, glass etc.) are those substances which do not allow the passage of electric current through them. In terms of energy band, the valence band is completely filled while the conduction band is empty. Further, the energy gap between valence and conduction bands is very large (≥ 15 eV) as shown in Fig. 8.6.

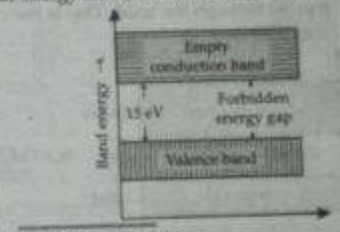

Fig. 8.6 Insulator

Therefore, a very high electric field is required to push the valence electrons to the conduction band. For this reason, the electrical conductivity of such materials is extremely small and may be regarded as nil under ordinary conditions.

At room temperature, the valence electrons of the insulators do not have enough energy to cross over the conduction band. However when the temperature is raised, some of the valence electrons may acquire enough energy to cross over to the conduction band. Hence the resistance of an insulator decreases with increase in temperature *i.e.*, an insulator has negative temperature coefficient of resistance.

(b) Conductors

A second possibility is that of a solid in which the valence band is completely filled with electrons but there is no forbidden energy gap *i.e.*, the valence and conduction bands overlap (Fig. 8.7). In such a solid, vacant energy levels are available adjacent to the filled levels. Therefore, it is possible to change the energy of the valence electrons by accelerating them by external electric

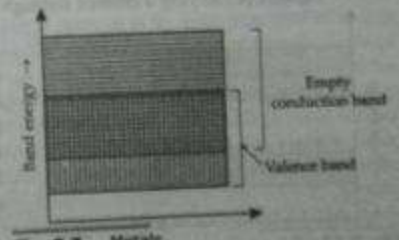

Fig. 8.7 Metals

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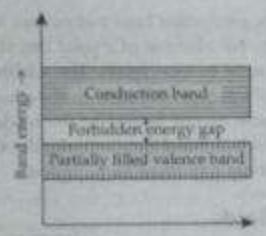

Fig. 8.8

field ; the solid can carry a current. It will a good conductor, for example : metals (copper, aluminium etc.)

The third possibility is that of a solid in which the valence band is only partially filled (Fig. 8.8), so that unfilled levels are available adjacent to the filled levels and external electronic field is able to accelerate the valence electrons and make them carry a current. The solid is, thus a good conductor. This is a situation of alkali metals.

(c) Semiconductors

(*i*) ***Intrinsic Semiconductors.*** If the valence band is completely filled, conduction band empty but the forbidden energy is small, of the order of about 1 eV [Fig. 8.9], the solid will be insulator at 0 K. At higher temperatures such that the thermal energy $k_B T$ is more than the gap energy ΔE_g, some of the valence electrons are able to jump over the gap into the conduction band. This creates free electrons in the conduction band and free holes in valence band. Thus the electron-hole pairs are thermally generated and can carry a small current through the solid. This is the case of an *intrinsic semiconductor*.

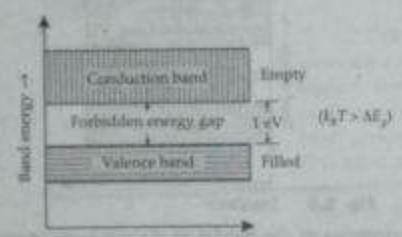

Fig. 8.9

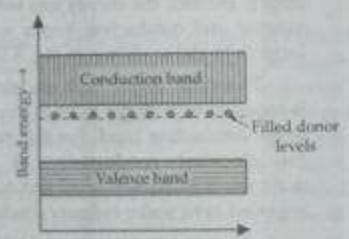

Fig. 8.10 *N*-type semiconductor.

(*ii*) ***Extrinsic Semiconductors.*** In an *N-type extrinsic semiconductors*, the energy levels of the valence electrons of the impurity atoms lie just below the bottom of the conduction band (Fig. 8.10). These electrons are very easily excited into the conduction band and are then capable of carrying a current through the solid.

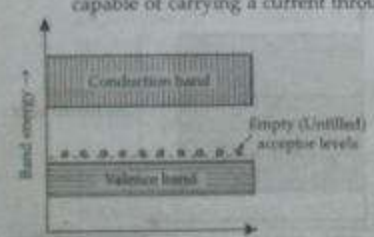

Fig. 8.11 *P*-type semiconductor.

Likewise, in a *P-type semiconductors*, the impurity atoms have unfilled energy levels, which lie just above the top of the valence band (Fig. 8.11). Electrons from the valence band are easily excited into these vacant levels leaving behind vacancies or free holes in the valence band which can carry a current through the solid.

8.7 Intrinsic Semiconductor

When the conductivity in crystalline semiconductor is only due to breaking of covalent bonds of the electrons in the valence band, then the substance is said to be an intrinsic semiconductor.

8.7.1 Electrons and Positive holes in a Semiconductor

When a covalent bond of Si or Ge crystal breaks, then a free electron (n) moves from valence band to conduction band of a semiconductor and it leaves behind an unfilled electronic state in the valence band (Fig. 8.12). The absence of an electron in valence band is called positive hole (p). The word "hole" in a semiconductor therefore refers to the empty energy levels in valence band. So whenever there is an extra electron in the conduction band there is a simultaneous production of a hole in the valence band [Fig. 8.12(*a*)]. The number of electrons in C.B. (conduction band) and simultaneously, number of holes in

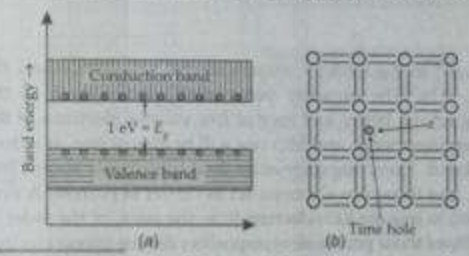

Fig. 8.12 Formation of holes in V.B. and movement of holes in V.B. which give rise to hole-current.

Fig. 8.13

valence band will increase with increase of temperature. But for intrinsic semiconductor number of electrons in conduction band (n) and number of holes in valence band (p) are always same. So mathematically, we can say

$$n = p \quad \text{...(1)}$$

$$\Rightarrow \quad n = p = n_i \quad \text{...(2)}$$

where n_i is the number of charge carriers in intrinsic semiconductor. The positive hole, regarded as the active particle in the valence band, in the same way that the electron is considered to be active particle in conduction band. This leads to the *concept of conductivity in semiconductors is caused not only by the motion of electrons in the conduction band which give rise to conduction current but also by the motion of the positive hole which gives rise to hole current in valence band*. The movement of hole in the valence band is understood from Figs. 8.12, 8.13. Hole positions can be changed from one place to other by interchanging their positions with electrons (Figs. 8.12, 8.13). This positive holes are also physical entities whose movement constitute a flow of current, which is called "hole current". Since movement of hole is opposite to that of electron, so the direction of hole current is same as direction of current flow. So in *semiconductors, there are two charge carriers, electron and hole. But in metals electrons are the only charge carriers, because there is no question of breaking bonds and thereby forming the holes.*

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8.8 Extrinsic Semiconductors

The electrical conductivity of intrinsic semiconductor is very small. To increases the conductivity of intrinsic semiconductor a small percentage of trivalent or pentavalent atom (impurities) are added to the pure semiconductor in the process of crystallification, which is called **doping** and results the impure semiconductor being called *extrinsic semiconductor*. The conductivity of extrinsic semiconductor is much higher, say for example 12 times than intrinsic semiconductor when an impurity is added 1 part in 10^8. The impurity atom has a size which is almost of the same order of the host lattice. Since percentage of impurity atoms is very small so every impurity atom is surrounded by a normal lattice site. So basic structure of crystal will not get altered after doping. There are two types of extrinsic semiconductors :

- *N*-type semiconductor
- *P*-type semiconductor.

8.8.1 *N*-Type Semiconductor

If a pentavalent atom is introduced in the intrinsic semiconductor say germanium the situation will as shown in Fig. 8.14. The impurity pentavalent atoms, some of the germanium atoms in the original crystal lattice and four of five valence electrons of the impurity atom will occupy four covalent bonds and fifth one will be almost free, as it does not find a place in the covalent bond. Every impurity atom thus contributes almost free electrons without creating a hole, and those free electrons act as carrier of current. A very small amount of energy is required to free the fifth electron from the atom, of the order of 0.01 eV for Ge and 0.05 eV for Si. Since those pentavalent impurities donate excess electron carrier, so they are called *N*-type impurity, and the crystal doped with donor impurity is called *N*-type semiconductor. Each ionized donor atom has a net charge of $(+e)$ and is bound or immobile (Fig. 8.15).

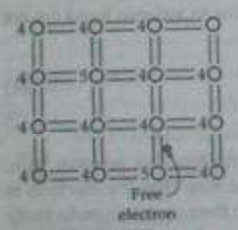

Fig. 8.14

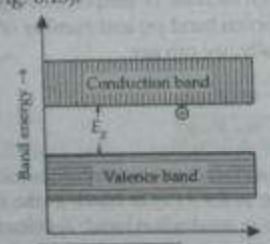

Fig. 8.15

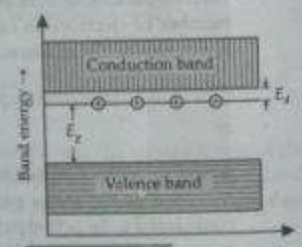

Fig. 8.16

When a donor impurities are doped to an intrinsic semiconductor Ge or Si, then additional discrete levels are introduced just below the conduction band in the forbidden gap [Fig.8.16]. These new additional level will be discrete and known as donor level E_d (Fig. 8.16) because of additional impurity atoms are situated far apart in the crystal structure and hence their interaction is small. Since very small amount of energy (0.01 eV for Ge and 0.05 eV for Si) is required to free the electron from donor level to conduction band, therefore almost all the fifth electron of donor materials are raised to the conduction band at room temperature.

8.8.2 *P*-Type Semiconductor

When a trivalent impurity (boron Z = 65, gallium Z = 31 or indium Z = 49) is added to the intrinsic semiconductor, only three of the covalent bond can be filled up and the vacancy that exists in the fourth bond constitute a hole (Fig. 8.20) without making an electron free. These holes can accept electrons and thus make available positive carriers in the valence band. So these trivalent impurities are called *acceptor* because every trivalent atom creates a *hole*, which accept electron and the intrinsic semiconductor with acceptor impurity is called *P*-type semiconductor. Each acceptor atom has a net charge $(-e)$ and is bound or immobile (Fig. 8.18).

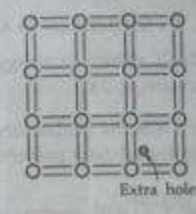

Fig. 8.17

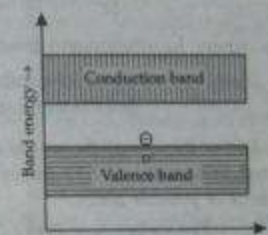

Fig. 8.18

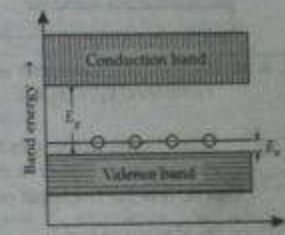

Fig. 8.19

When acceptor impurity is doped to an intrinsic semiconductor, Ge or Si, then the additional discrete energy levels are introduced. Just above the valence band (Fig.8.19) in the forbidden gap, which is called acceptor level (E_a). Since a very small amount of energy is necessary for an electron to leave the valence band and occupy the acceptor level, it thus creates a large number of holes in the valence band in the extrinsic semiconductor.

Thus by doping the impurity atoms in the intrinsic semiconductor not only conductivity is increased but it also serves to produce a conductor in which the electric carriers are mainly electrons or mainly holes.

Electrons are majority carriers and holes are minority carriers in N-type semiconductor whereas in P-type semiconductor the majority carriers are holes and minority carriers are electrons.

8.9 Fermi Level : Effect of Temperature on Intrinsic Semiconductor

As we know that at absolute zero all the electronic states of the valence band are full and those of the conduction band are empty, so a semiconductor becomes an insulator at 0 K. But as the temperature increases some electrons from the valence band get sufficient energy and become free. They move to the conduction band and take part in conduction and give rise to conductivity in the semiconductor (Fig. 8.12).

Now we want to discuss the phenomenon with quantum mechanical point of view. Classically all electrons have zero energy at 0 K, but quantum mechanically electrons can not have zero energy at 0 K. *The maximum energy that electrons may possess at 0 K is the Fermi energy* (E_F). So quantum mechanically the electrons actually have energies extending from 0 to E_F at absolute zero temperature (Fig. 8.20).

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Now in order to know how many of the electronic energy states in the valence band and conduction band will be occupied at different temperatures, we introduce a Fermi-factor [$f(E)$], which is the number that expresses probability that a state of a given energy (E) is occupied by an electron under condition of thermal equilibrium. This number has a value between zero and unity and is a function of energy and temperature (or distribution function for Fermi-Dirac statistics) as

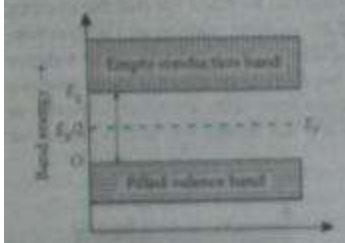

Fig. 8.20 Intrinsic semiconductor with Fermi level $E_F = (E_g/2)$.

$$f(E) = \frac{1}{[e^{(E-E_F)/(k_B T)} + 1]} \quad ...(3)$$

where k_B = Boltzmann constant, T = temperature in K

and E_F = Fermi level of energy in eV, maximum energy that electron may possess at 0 K.

The Fermi factor is independent of the energy density of states; it is the probability that the states occupied at that level irrespective of the number of states actually present. That is, it is the fractional occupancy of possible states.

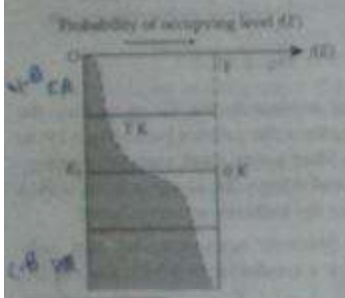

Fig. 8.21 Variation of Fermi factor $f(E)$ with temperature.

The variation of Fermi factor $f(E)$ with energy at different temperature for intrinsic semiconductor is as shown in Fig. 8.21, where full line curve shows $f(E)$ at 0 K and dotted line curve at T K.

When $T = 0$ K, $f(E)$ can take two values

(i) $E > E_F$

$$f(E) = \frac{1}{1 + \exp(\infty)} = 0 \quad \text{(because } e^{\infty} = \infty\text{)}$$

(ii) $E < E_F$

$$f(E) = \frac{1}{1 + \exp(-\infty)} = 1 \quad \text{(because } e^{-\infty} = 0\text{)}$$

Thus at $T = 0$ K, $f(E) = 1$, when $E < E_F$, which means all the level below E_F *i.e.*, valence band are filled up by electron and $f(E) = 0$, when $E > E_F$ *i.e.*, all level above E_F *i.e.*, conduction bands are empty. Because at 0 K, no heat energy is present, so no covalent bonds are being broken and the semiconductor behaves as an insulator.

But when $t = T$ K, then at $E = E_F$, $f(E)$ becomes

$$f(E) = \frac{1}{e^0 + 1} = \frac{1}{2} \quad [\because e^0 = 1]$$

which means that when the temperature is not 0 K but some higher value say $T = 1000$ K, then some covalent bonds will break and some electrons will be available in conduction band whereas some electron vacancy *i.e.*, hole will be available at valence band. This is shown by the dotted line in Fig. 8.21.

8.10 Fermi Level in an Intrinsic Semiconductor

The position of Fermi level in an intrinsic semiconductor can be calculated in the following ways ; it should be noted the Fermi level will be somewhere in the forbidden gap, since nearly all the valence band states are occupied and very few electrons will be found in the conduction band, at room temperature.

Now the total number of occupied states [$n(E)$] by electrons, with energy between E and $(E + dE)$ is equal to the product of the available number of states [$g(E)$] and the probability of their occupancy [$f(E)$], that is,

$$n(E)dE = g(E)\, f(E)\, dE \quad ...(4)$$

where $n(E)$ = total number of occupied states in conduction band by electrons *i.e.*, the density of electrons in conduction band.

$g(E)$ = Number of available states and

$f(E)$ = Probability of occupancy of those states

Similarly the number of holes in the valence band within the energy E and $(E + dE)$ is

$$p(E)dE = g(E)[1 - f(E)]dE \quad ...(5)$$

Consider energy level E_C in conduction band and E_V in valence band which are symmetrically placed about the centre of the energy gap (E_g) [Fig. 8.22]. For an energy range dE, the number of electrons in the conduction band is

$$n(E_C)dE = g(E_C)\, f(E_C)\, dE \quad ...(6)$$

Similarly, the number of holes in valence band is

$$p(E_V)dE = g(E_V)[1 - f(E_V)]dE \quad ...(7)$$

Assuming for simplicity $g(E_C) = g(E_V)$, the number of available states in conduction bands and valence bands are almost same. Hence

$$\frac{n(E_C)}{p(E_V)} = \frac{f(E_C)}{[1 - f(E_V)]} \quad ...(8)$$

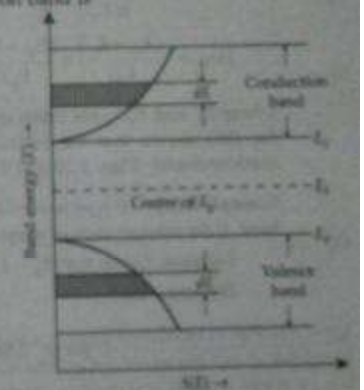

Fig. 8.22 Variation of available number of states [$g(E)$] with energy(E) in valence and conduction band

Under certain condition, the following approximation to the Fermi factor can be made. Assuming that at room temperature energy level E_C in conduction band is far above the Fermi-level (E_C), say at 300 K, the factor

$$\exp\left(\frac{E_C - E_F}{k_B T}\right) >> 1$$

so

$$f(E_C) = \frac{1}{1 + e^{(E_C - E_F)/(k_B T)}} = e^{-(E_C - E_F)/(k_B T)} \quad ...(9)$$

or

$$f(E_C) = e^{(E_F - E_C)/(k_B T)} \quad ...(10)$$

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Similarly assuming E_V level in valence band, which is far below from Fermi level, then

$$1-f(E_V)=1-\frac{1}{1+e^{(E_V-E_F)/k_BT}} \quad ...(11)$$

where $E_F-E_V >> k_BT$ at room temperature, so the above expression becomes

$$[1-f(E_V)]=1-\left[1-\exp.\left(\frac{E_V-E_F}{k_BT}\right)\right]$$

$$=\exp.\left(\frac{E_V-E_F}{k_BT}\right) \quad ...(12)$$

Thus above expression becomes

$$\frac{n(E_C)}{p(E_V)}=\frac{\exp.[(E_F-E_C)/k_BT]}{\exp.[(E_V-E_F)/k_BT]} \quad ...(13)$$

by assuming $(E_C-E_F)>>k_BT$ and $(E_F-E_V)>>k_BT$ at room temperature.

For intrinsic semiconductor the number of electrons is equal to the number of holes.

i.e., $$n_i=p_i \quad ...(14)$$

So $$\frac{n(E_C)}{p(E_V)}=\frac{n_i}{p_i}=1$$

Hence $$\frac{(E_F-E_C)}{k_BT}=\frac{(E_V-E_F)}{k_BT} \quad \text{or} \quad E_F=\frac{E_C+E_V}{2} \quad ...(15)$$

where E_C and E_V were taken symmetrical about the centre of forbidden gap (E_g). Above Eq. (15) shows that the Fermi level lies at the centre of forbidden gap for intrinsic semiconductor (Figs. 8.20 and 8.22) and it is independent of the temperature.

Example 8.1. *In a P-type semiconductor at T = 300 K, the Fermi level lies 0.4 eV above the valence band. If the concentration of acceptor atoms is doubled, find the new position of the Fermi level.*

Solution. Given that $E_F-E_V=0.4$ eV, $N'_a=2N_a$

We know that

$$p=N_a=N_V e^{-(E_F-E_V)/k_BT}.$$

Then $$p'=N'a=2N_a=2N_V\exp.\left(\frac{E'_F-E_V}{k_BT}\right) \quad ...(i)$$

$$p=N_a=N_V\exp.\left(-\frac{E_F-E_V}{k_BT}\right) \quad ...(ii)$$

Divide Eq. (i) by Eq. (ii), we get

$$\frac{p'}{p}=2=\exp.\left[\frac{(E_F-E_V)-(E'_F-E_V)}{k_BT}\right]$$

Taking logarithms on both sides

$$\ln 2=\frac{(E_F-E_V)-(E'_F-E_V)}{k_BT}$$

$$(E'_F-E_V)=(E_F-E_V)-k_BT\ln 2$$

$$=0.4-\frac{1.38\times10^{-23}\times300}{1.6\times10^{-19}}\ln 2=0.4-0.02584\ln 2$$

$$=0.4-0.0077=0.3923\text{ eV}.$$

The Fermi level will be 0.3923 eV from the valence band.

Example 8.2. *In an N-type semiconductor the Fermi level lies 0.3 eV below the conduction band at 300 K. If the temperature is increased to 330 K. Find the new position of the Fermi level.*

Solution. At 300 K E_F lies below 0.3 eV below E_F

We know that

$$E_C-E_F=k_BT\ln\frac{N_C}{N_d}$$

$$\Rightarrow \quad 0.3\text{ eV}=300k_B\ln\frac{N_C}{N_d} \quad ...(i)$$

At 330 K, $$E_C-E_F=330k_B\ln\frac{N_C}{N_d} \quad ...(ii)$$

Dividing Eq. (ii) by Eq. (i)

$$\frac{E_C-E_F}{0.3}=\frac{330}{300} \quad \text{or} \quad E_C-E_F=\frac{330\times3}{300}=0.33\text{ eV}.$$

Thus the Fermi level lies 0.33 eV below the conduction band.

8.11 Fermi Level in an Extrinsic Semiconductor

In an intrinsic semiconductor the number of electrons is equal to number of holes ($n_i=p_i$). But in N-type extrinsic semiconductor number of electrons are increased due to doping of pentavalent atom ($n_e>n_i$) and number of holes are decreased ($p_e<p_i$) than which would be available for intrinsic semiconductor. The number of holes are decreased because of the large number of electrons present which increase the rate of recombination of electrons and holes. With similar argument it can be shown that in P-type extrinsic semiconductor the number of holes are increased due to doping of trivalent impurity atom ($p_e>p_i$) and number of electrons decreased ($n_e<n_i$) than which would be available for intrinsic semiconductor. *Now since the Fermi level (E_F) is a measure of the probability of occupancy of the allowed energy states by the electron,* so for an intrinsic semiconductor when $n_i=p_i$: Fermi level is at the centre of the forbidden gap (Fig. 8.22). But for N-type semiconductor, since

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$n_e > p_e$, it is clear that E_F must move closer to conduction band [Fig. 8.23(b)], to indicate that many of the energy states in the band are filled by the donor electrons, and of few holes exist in the valence band. Similarly, for *P*-type semiconductor, since $p_e > n_e$ so E_F must move from the centre of the forbidden gap closer to the valence band [Fig. 8.23(c)].

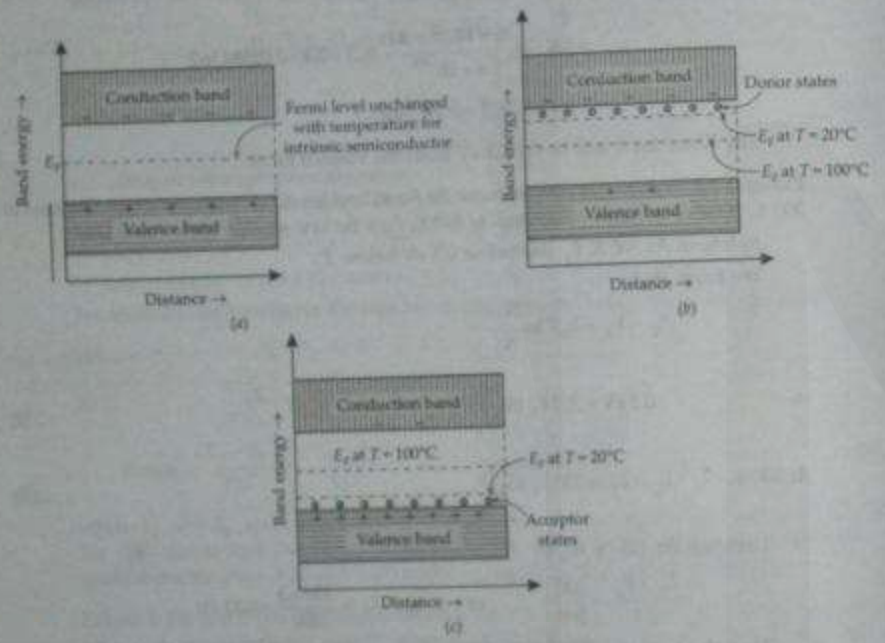

Fig. 8.23 Position of Fermi level (E_F) for (a) intrinsic semiconductor ; (b) *N*-type semiconductor ; (c) *P*-type semiconductor.

8.12 Variation of Fermi Level with Temperature in Extrinsic Semiconductors

For an intrinsic semiconductor $n_i = p_i$ and as temperature increases both n_i and p_i will increase. Thus the Fermi level (E_F) will remain approximately at the centre of the forbidden gap. Thus for intrinsic semiconductor Fermi level is independent of temperature [Fig. 8.23(a)].

But in an extrinsic semiconductor it is different. Say for *N*-type material the electrons are coming from two different sources. Some electrons come from donor state, which are easily separated from their parent atom and they do not vary much as the temperature is increased. The other electrons in the conduction band are present because of the breaking

of a covalent band. Such intrinsic effect will increase in number as temperature raised. So proportion of the total number of conduction electrons will increase from valence band. Thus *as the temperature rises the material becomes more and more intrinsic and Fermi level moves closer to the intrinsic position, i.e., at the centre of the forbidden gap* [Fig. 8.23(b)].

Similarly, for *P*-type semiconductor as temperature increases the material also becomes increasingly intrinsic and the Fermi-level rises until it approaches the centre of the gap as in the intrinsic semiconductor [Fig. 8.23(c)]. Thus both *N*-type and *P*-type materials become more and more like intrinsic material at high temperature. This places a limit on the operating temperature of an extrinsic semiconductor device.

8.13 Densities of Charge Carriers in a Semiconductor

In any semiconductor, the processes of thermal generation of new electron-hole pairs and recombination of the existing electrons and holes take place continuously. The number of electron-hole pairs created per unit volume per second is known as the *rate of generation* (g). It is a function of the energy gap (ΔE_g) and the temperature T, varying directly as T and inversely as ΔE_g.

$$g = f(\Delta E_g, T) \quad \text{...(16)}$$

The recombination rate r depends on the rate at which electrons and holes collide with each other. This, in turn, is directly proportional to the number densities n and p of the electrons and holes respectively in the semiconductor, thus

$$r \propto np \quad \text{...(17)}$$

Equilibrium is reached when the rate of generation just equal the rate of recombination resulting in constant carrier densities.

In an intrinsic semiconductor, the electrons and holes are present in equal numbers so that $n_i = p_i$, the subscript i indicating the intrinsic material. The product $n_i p_i = n_i^2$ depends on the energy gap ΔE_g, hence it is a characteristic of the material and increases exponentially with temperature for a given material. Typical values of n_i at 300 K are 2.5×10^{19} per m^3 for germanium and 1.5×10^{16} per m^3 for silicon.

In an *N*-type semiconductor, in addition to the thermally generated electron-hole pairs, extra electrons are injected into the material from the impurities, the increasing the electron density. This enhances the probability of hole colliding with an electron and recombining with it. Thus recombination rate increases so that the hole density goes below the intrinsic levels. Thus the electron density goes up while the hole density goes down such that their product remains unchanged at the intrinsic level :

$$n_n p_n = n_i^2 \quad \text{...(18)}$$

where the subscript n indicates *N*-type material. Likewise in a *P*-type semiconductor because of the extra holes injected from the acceptor impurities, the hole density goes up while the electron density goes down in such a way that

$$n_p p_p = n_i^2 \quad \text{...(19)}$$

In order to calculate the carrier densities, we need to consider the balance of positive and negative charges in the semiconductor material which, as a whole, is electrically neutral. The positive charge is provided by the free holes and the donor atoms that have

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donated one electron and have become ionized. The negative charge is provided by the free electrons and the acceptor atoms that have already taken up an electron from the host crystal. For electrical neutrality, we must have

$$p + N_D = n + N_A \quad ...(20)$$

where N_D and N_A are the number of donor and the acceptor atoms per unit volume respectively.

For an *N*-type semiconductor, $N_A = 0$ and the hole density is small as compared to the donor density, hence the electron density is approximately equal to the donor density :

$$n_n = N_D \quad ...(21)$$

The hole density can then be obtained from Eq. (18) as

$$p_n = \frac{n_i^2}{N_D} \quad ...(22)$$

The hole and electron densities for a *P*-type semiconductor can be obtained in a similar manner. These are

$$p_p = N_A, \quad n_p = n_i^2 / N_A \quad ...(23)$$

14 Drift of Charge Carriers in External Fields

If a semiconductor is placed in an external field, its charge carriers experiences a force and move in the direction of the field. This is known as the **drift** and it results in a drift current through the semiconductor.

It is often more convenient to talk in terms of the current density **J** rather then the total current *I* through a conductor. It is defined as the *current through a unit cross-sectional area normal to the direction of flow of charges*. Let ρ be the density of free charges in a conductor, consider a cylindrical region with its axis in the direction of *v*, its length equal to *v* and its cross-section unity as shown in Fig. 8.24. The charge contained in this cylinder is ρv. All this charge but no more will pass the cross-section in one second. Hence, the current through the unit cross-section *i.e.*, the current density **J** will be :

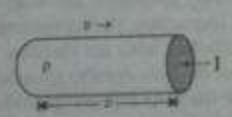

Fig. 8.24

$$\mathbf{J} = \rho \mathbf{v} \quad ...(24)$$

Now, let us consider a semiconductor placed in an external electric field **E**, so that the electrons and holes acquire average drift velocities equal to $\mathbf{v}_e$ and $\mathbf{v}_h$ respectively. Then from Eq. (24), the current density $\mathbf{J}_e$ due to the drift of electron is

$$\mathbf{J}_e = ne\mathbf{v}_e \quad ...(25)$$

and that due to the hole drift is

$$\mathbf{J}_h = pe\mathbf{v}_h \quad ...(26)$$

where *n* and *p* are respectively, the electron and hole densities in the semiconductor. In Eq. (25), the negative sign of the electronic charge has been left out because the electron velocity $\mathbf{v}_e$ will also be in opposite direction. Thus the net current density $\mathbf{J}_{drift}$ through the semiconductor becomes

$$\mathbf{J}_{drift} = e(n\mathbf{v}_e + p\mathbf{v}_h) \quad ...(27)$$

The drift velocities are directly proportional to the field **E**, hence it is convenient to introduce a new parameter, the mobility (μ), defined as *the drift velocity per unit electric field*(**E**). Thus

$$\mu_e = \frac{\mathbf{v}_e}{\mathbf{E}} \quad \text{and} \quad \mu_h = \frac{\mathbf{v}_h}{\mathbf{E}} \quad ...(28)$$

Combining Eqs. (27) and(28), the current density $\mathbf{J}_{drift}$ can be written in terms of the field **E** and the electron and hole mobilities as

$$\mathbf{J}_{drift} = e\mathbf{E}(n\mu_e + p\mu_h) \quad ...(29)$$

Therefore, the conductivity (σ) of the semiconductor, which is equal to the current density per unit electric field, is given by

$$\sigma = e(n\mu_e + p\mu_h) \quad ...(30)$$

The resistivity ρ in the inverse of the conductivity and is given by

$$\rho = \frac{1}{e(n\mu_e + p\mu_h)} \quad ...(31)$$

Example 8.3. *Show that the ratio of maximum resistivity to intrinsic resistivity is expressed as*

$$\frac{\rho_{max}}{\rho_i} = \frac{\mu_e + \mu_h}{2\sqrt{\mu_e \mu_h}}.$$

Solution. Intrinsic conductivity $\sigma_i = n_i e(\mu_e + \mu_h)$

The conductivity at any other doping level is expressed as

$$\sigma = e(n_e\mu_e + n_h\mu_h) \quad [\because n_e n_h = n_i^2]$$

Hence

$$\sigma = e\left[n_e\mu_e + \frac{n_i^2\mu_h}{n_e}\right]$$

For resistivity to be maximum, conductivity has to be minimum, for that

$$\frac{d\sigma}{dn_e} = 0 = e\left(\mu_e - \frac{n_i^2\mu_h}{n_e^2}\right)$$

or

$$n_e^2 = \frac{n_i\mu_h}{\mu_e} \quad \text{or} \quad n_e = n_i\sqrt{\frac{\mu_h}{\mu_e}}$$

and

$$n_h = n_i\sqrt{\frac{\mu_e}{\mu_h}}$$

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Hence $$\sigma_{max} = e(\sqrt{\mu_e} - \mu_h + \sqrt{\mu_e} + \mu_h)\, n_i = 2en_i\sqrt{\mu_e\mu_h}$$

Therefore $$\frac{\sigma_i}{\sigma_{max}} = \frac{\rho_{max}}{\rho_i} = \frac{en_i(\mu_e + \mu_h)}{2en_i\sqrt{\mu_e\mu_h}} = \frac{(\mu_e + \mu_h)}{2\sqrt{\mu_e\mu_h}}$$

Example 8.4. *An intrinsic silicon specimen at approximately 300 K has a conductivity of $4.3 \times 10^{-4}\,\Omega^{-1}m^{-1}$. What is the intrinsic carrier concentration ? If a current is passed through a specimen what proportion of it is carried by the electrons ? The same specimen is now doped to make N-type. The donor concentration is $10^{21}/m^3$. Find the hole density of the doped specimen and also the proportion of current that would now be carried by electrons. Assume that the mobilities are substantially unchanged by the doping process.*

Given μ_e at 300 K $= 0.135\, m^2/V\text{-}s$; μ_h at 300 K $= 0.048\, m^2/V\text{-}s$.

Solution. For intrinsic semiconductor

$$\sigma = e(n\mu_e + p\mu_h) \text{ with } n = p = n_i$$

Hence $$n_i = \frac{\sigma}{e(\mu_e + \mu_h)}$$

Now $$n_i = \frac{4.3 \times 10^{-4}}{1.6 \times 10^{-19} \times (0.135 + 0.048)} = 1.47 \times 10^{16}/m^3$$

The proportion of current carried by electrons is given by

$$\frac{n\mu_e}{n(\mu_e + \mu_h)} = \frac{\mu_e}{(\mu_e + \mu_h)} = \frac{0.135}{0.183} = 0.738$$

For extrinsic N-type material, we have

$$p = \frac{n_i^2}{N_d} \text{ with } n_i = 1.47 \times 10^{16}/m^3 \text{ and } N_d = 10^{21}/m^3.$$

Hence the concentration of holes in the N-type material is

$$p = \frac{n_i^2}{N_d} = \frac{(1.47 \times 10^{16})}{10^{21}} = 2.16 \times 10^{11}/m^3$$

i.e., $$p = 2.16 \times 10^{11}/m^3.$$

Hence the proportion of current in this case is

$$\frac{n\mu_e}{n\mu_e + p\mu_h} = \frac{1}{\left(1 + \frac{p\mu_h}{n\mu_e}\right)} = \left(1 + \frac{p\mu_h}{n\mu_e}\right)^{-1} = \left(1 - \frac{p\mu_h}{n\mu_e}\right)$$

$$= \left[1 - \frac{2.16 \times 10^{11} \times 0.048}{10^{21} \times 0.135}\right]$$

i.e., $$= [1 - 10^{-10}] \simeq 1$$

and so the current is mainly due to electrons.

8.15 P-N Junction : Diffusion of Charge Carriers

If a junction is formed between a P-type and an N-type semiconductor, the electron density charges by several orders of magnitude as we go from N-side having electrons as majority charge carriers to the P-side, where the electrons are in majority. The statement is true for hole density also as we go from the P-side to the N-side. Thus there exist steep concentration gradients $\frac{dp}{dx}$ and $\frac{dn}{dx}$ in the junction region as shown in Fig. 8.25.

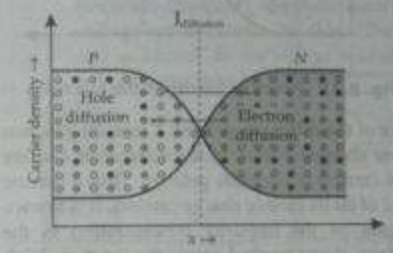

Fig. 8.25 Concentration gradients in a P-N junction.

This sets in a diffusion process from the regions of higher concentration to the region of lower concentration. Thus electrons diffuse from N-side to P-side and holes diffuse from P-side to N-side resulting in diffusion currents through the junction. The rate of diffusion is proportional to the magnitude of the concentration gradient. This rate multiplied by the charge of carrier gives the diffusion current density. Thus

$$J_h = -eD_h \frac{dp}{dx}$$

and $$J_e = eD_e \frac{dn}{dx} \quad \text{...(32)}$$

where D_h and D_e are known as the diffusion constants for the hole and electron, respectively. Since the electron and hole movements are in opposite directions, their currents add up to give the net diffusion current through the junction as

$$I_{diff} = e\left(D_e \frac{dn}{dx} + D_h \frac{dp}{dx}\right) \quad \text{...(33)}$$

8.16 Potential Barrier at a P-N Junction

The diffusion process discussed in the previous section is a self-limiting process due to the fact that it disturbs the electrical neutrality of both sides of the junction. When the electrons cross-over from the N-side to P-side, they recombine with the excess hole present there and are immobilized. This leaves N-side deficient in negative charge and P-side having an excess of negative charge. Similarly, the diffusion of holes from P side to N side leave P side deficient in positive charge and N-side having an excess of positive charge. As a result, two sides of the junction are no longer electrically neutral ; P-side acquires an excess of negative charge, while N-side acquires an excess of positive charge as shown in Fig. 8.26. This creates a potential difference at the

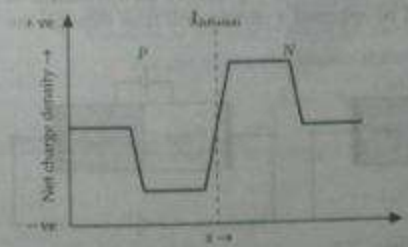

Fig. 8.26 Effect of diffusion on the net charge density.

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junction as shown in Fig. 8.27. The potential difference has the right polarity to oppose further diffusion of charge carriers across the junction. Equilibrium is reached when enough number of carriers have diffused so as to create a potential difference that is just sufficient to stop further diffusion. this potential difference is known as *junction potential barrier* (V_B) or the contact potential. It is of the order of a few tenths of a volt for most *P-N* junctions.

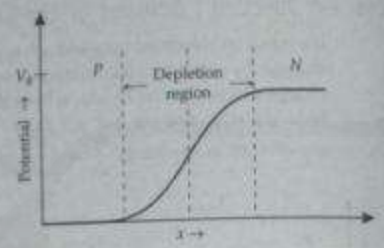

Fig. 8.27 Potential barrier at the junction.

There is another interesting consequence of the diffusion process. Close the junction on either side of it, nearly all the mobile free charges diffuse to the other side and are immobilized there by recombination with the carriers of opposite polarity. Thus a narrow region exists at the junction which is depleted of all its mobile charge carriers. It is known as the *depletion region*. Its thickness depends on the impurity concentration in the semiconductors and also on the external voltage, if any applied across the junction. With no external voltage, the thickness is typically of the order of *one micron*. Since the depletion region does not have any free charges, it behaves like a dielectric medium between two conducting regions.

Thus it forms a capacitor, whose capacitance is known as the *junction capacitance*. Its typical values are of the order of 10 pf. The junction capacitance plays an important role in circuits using high frequency alternating currents.

8.17 *P-N* Junction Diode

Most semiconductor diodes consists of a *P-N* junction. The circuit symbol for a normal diode is shown in Fig. 8.28 the arrow indicates the forward direction of the diode as explained below. As discussed in the previous articles 8.15 and 8.16, the diffusion of the majority charge carriers across the *P-N* junction sets up a potential barrier at the junction with *P* side as negative and *N*-side as positive. This barrier is just sufficient to stop the movement of carriers across the junction. If now an external potential difference with a polarity so as to oppose the barrier is applied as shown in Fig. 8.29(*a*), the movement of majority carriers across the junction would be resumed a current will flow through the

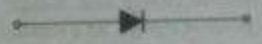

Fig. 8.28 Circuit symbol of a normal semiconductor diode.

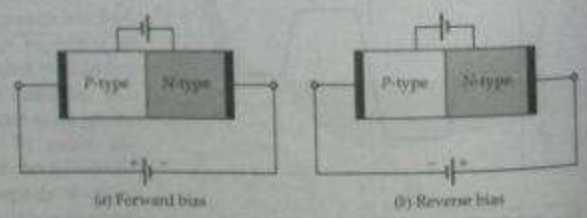

Fig. 8.29 Biasing of a *P-N* junction.

junction in the *P* to *N* direction. The external voltage is then known as the *forward bias voltage*, the resulting current as the *forward current* and the direction *P* to *N* as the *forward direction* of the diode. As the forward bias voltage is increased, the current *I* through the junction increases as shown in Fig. 8.30. The increase in current is nearly exponential for the most part and it follows an equation known as the **diode equation**.

$$I = I_S[e^{eV/k_BT} - 1] \quad ...(34)$$

where I_S is equal to the saturation current under reverse bias as discussed below, *e* is the electronic charge, k_B the Boltzmann's constant, *T* the temperature in *K* and *V* is the bias voltage applied across the diode. The term k_BT/e in Eq. (34) has the dimensions of voltage and is known as the *voltage equivalent of temperature*, denoted by V_T. Now the diode equation can than be written as

$$I = I_S[e^{V/V_T} - 1] \quad ...(35)$$

in forward bias, $V > V_T$, thus $e^{V/VT} >> 1$ and $I = I_S e^{V/V_T}$

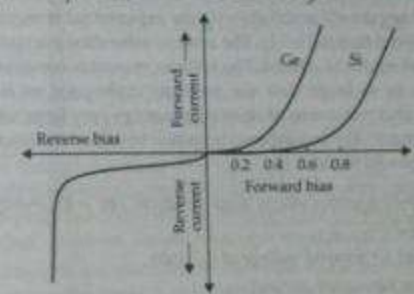

Fig. 8.30 Characteristic curve of a *P-N* junction diode.

At room temperature, V_T becomes about 26 mV. As shown in characteristic curve, the diode current is negligible until the voltage exceeds a critical value known as **cut-in-voltage** or the **threshold voltage**. The cut in voltage is about 0.2 volts for Ge diodes and about 0.6 volts for Si diodes.

Another important phenomenon occurs in the forward bias condition. As the movement of charge carriers is resumed, the depletion layer narrows down and finally vanishes as the bias exceeds the cut-in-voltage.

We can define the *forward dynamic resistance* (r_f) of the diode as follows :

$$r_f = \left(\frac{\partial V}{\partial I}\right) \quad ...(36)$$

Its value can be obtained from Eq. (34) or (35). Differentiating with respect to *I*, we get

$$I = I_S e^{V/V_T} \quad \frac{1}{V_T}\frac{\partial V}{\partial I}$$

or

$$\frac{\partial V}{\partial I} = \frac{V_T}{(I + I_S)} \approx \frac{V_T}{I} \quad ...(37)$$

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provided that the bias voltage $V >> V_T$ so that $I >> I_S$, substituting for V_T at the room temperature, the dynamic resistance is given by

$$r_f = \frac{0.026}{I} \text{ volts} \quad ...(38)$$

and is only 1 ohm at a current of 26 mA.

If the polarity of the external bias is reverse as shown in Fig. 8.29(*b*), it has two effects. One that the reverse bias pulls the majority carriers further away from the junction so that the depletion layer widens. This may be important in some later discussion. The other effect of the reverse bias is that it effectively raises the potential barrier at the junction. Therefore the current through the diode is expected to be zero for all reverse bias voltages. However, the reverse bias has the right polarity for the movement of minority charge carriers which will carry a small but non-zero reverse current through the diode. As the reverse bias is increased, the reverse current increases and quickly saturates because the minority carriers are in short supply. This is shown in Fig. 830. It can be verified from Eq. (35) also. For negative V with $|V| >> V_T$, the exponential term will be small compared to 1 and can be ignored that is $I = -I_S$. The reverse saturation current I_S is of the order of μA for Ge diodes and nA for Si diodes. The reverse current is constant at the value of I_S until the reverse bias is so large that the junction undergoes an *electrical breakdown*. It is characterised by sharp increase of reverse current to very large values unless it is limited by the external circuit. The breakdown is caused by either *Avalanche effect* or *Zener effect* as discussed in *articles* 8.18 and 8.19 :

Example 8.5. *A P-N junction silicon diode conducts 240 mA current when a forward voltage 0.8 volts is applied. Find :*

(*i*) *Current for forward voltage of 0.7 volts.*

(*ii*) *Reverse saturation current.*

Solution. (*i*) From junction diode equation

$$I = I_S[e^{V/\eta k_B T} - 1] = I_S[e^{V/\eta V_T} - 1]$$

where V_T (volt equivalent of temperature) is given by

$$V_T = \frac{k_B T}{e} = \frac{1.38 \times 10^{-23} \times 300}{1.6 \times 10^{-19}} \text{ Volt}$$

at room temperature (300 K)

$$V_T = 0.026 \text{ volt} \qquad \because \qquad \eta = 2 \text{ for silicon}$$

$$I = I_S e^{[0.8/(2 \times 0.026)]} \qquad \text{and} \qquad I' = I_S[e^{0.7/(2 \times 0.026)}]$$

$$\therefore \qquad \frac{I'}{I} = \exp\left[\frac{0.7 - 0.8}{2 \times 0.026}\right] = 35 \text{ mA}$$

(*ii*) From *diode equation*

$$240 \times 10^{-3} = I_S[e^{0.8/0.052} - 1]$$

or

$$I_S = \frac{240 \times 10^{-3}}{4.8 \times 10^6} = 50 \times 10^{-9} \text{ A} = 50 \text{ nA}.$$

8.18 Avalanche Effect

In the reverse-biased condition, the velocity of minority carriers increases as the bias voltage is increased. If this is sufficiently large (of the order of 100 volts for a typical *PN* junction), the minority carriers acquire high velocities and are able to further ionize the fixed ions of the crystal with which they collide. This generates new electron-hole pairs. They also move in opposite directions and acquire sufficient velocities to ionize other ions fixed in the crystal. The process is thus regenerative and very soon, a very large number of new carriers are generated, which carry a large current through the junction. This is known as *Avalanche breakdown* of the junction. This current will increase excessively and might damage the diode by the excessive heat produced unless it is limited by external circuit to safe value.

8.19 Zener Effect

In some junction diodes, the breakdown is obtained at very low reverse bias voltages (of the order of about 5 V). This happens in diodes with very thin depletion layers about 100 – 200 Å in thickness. The electric field across the depletion layer is approximately equal to the bias voltage divided by the thickness of the layer. Thus a thin layer results in strong electric fields of the order of 10^6 V/cm even for low bias voltages. This field is so strong that it is able to pull out more electrons from the fixed ions in the depletion region by the process of field emission. A very large number of new carriers are thus created which carry a large reverse current through the diode. This is known as *Zener effect*. As pointed out previously, the thickness of the depletion layer depends on the impurity concentration, the thinner is the layer for high doping. Thus Zener effect is observed only in the diodes having high concentration of impurity atoms ; in others, the mechanism of breakdown is the Avalanche effect.

Both mechanisms are reversible in the sense that if the voltage is reduced, the excess carriers recombine, the breakdown vanishes and the diode comes back in original condition.

8.20 Zener Diode

A specially designed silicon diode, which optimises in the breakdown region is known Zener Diode[3].

The ordinary rectifier and small signal diodes are never intentionally operated in the breakdown region because this may damage them. On the other hand, Zener diodes are operated in the breakdown region. Therefore, Zener diodes are cryptically design to have a sharp breakdown voltage. By varying the doping level of silicon diode, a manufacturer can produce Zener diodes with breakdown voltages from 2 to 200 V.

Fig. 8.31 shows the schematic symbol of Zener diode. It is similar to the symbol of an ordinary crystal diode except that its bar is just turned into Z-shape.

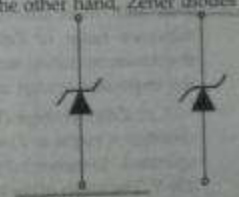

Fig. 8.31

3. The satisfactory of breakdown region (or breakdown voltage of junction) was first given by the American physicist C. Zener hence, the name Zener diode.

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8.20.1 CHARACTERISTICS OF A ZENER DIODE

The V-I characteristics of a Zener diode are shown in Fig. 8.32. the following points are worth noting :

(i) Its characteristics are similar to an ordinary rectifier diode with the exception that it has a sharp (or distinct) breakdown voltage called Zener voltage (V_Z).

(ii) It can be operated in any of the three regions *i.e.*, forward[4], leakage or breakdown. But usually it is operated in the breakdown region as shown in Fig. 8.35.

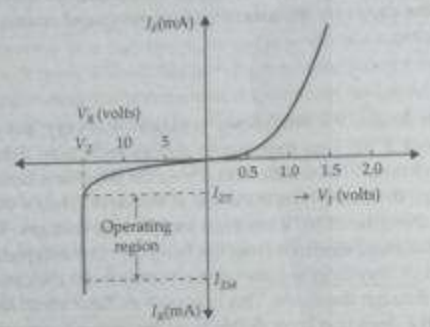

Fig. 8.32

(iii) The voltage is almost constant (V_Z) over the operating region.

(iv) Usually, the value of V_Z at a particular test current I_{ZT} is specified in the data sheet.

(v) During operation it will not burn as long as the external circuit limits the current flowing through it below the *burn out value i.e.*, I_{ZM} (the maximum rated Zener current).

8.20.2 Rating of Zener diode

Manufacturers provide a data sheet for various Zener diodes bearing different members. This data sheet shows the ratings of different diodes which include Zener voltage, tolerance range of Zener voltage, Zener current limits, maximum power dissipation, maximum operating temperature, maximum Zener resistance (or impedance) etc. Some of the important ratings are discussed below :

(i) ***Zener voltage (V_Z).*** *The voltage at which a Zener breakdown occurs in the reverse bias condition is known as Zener voltage.* In fact, it is the voltage at which the Zener diode is to be operated. Commercially available Zener diodes are having Zener voltage from 3 V to 200 V. The value of breakdown (or Zener) voltage depends upon doping-more the doping, lesser the breakdown voltage.

4. Although, a Zener diode can be operated in forward region like an ordinary diode and can be applied for rectification. But it is not used for rectification because to its heavy cost as compared to an ordinary diode.

(ii) ***Tolerance.*** *The range of voltages about the breakdown voltage in which a Zener diode conducts in reverse direction is called tolerance.*

In fact, during manufacturing, it is very difficult to have exact doping for every Zener of the same number (*i.e.*, type). Therefore, breakdown voltages of Zeners of the same number also differ slightly, the range of breakdown voltages for the same type of Zener diode is described as tolerance. For example, consider a particular type of Zeners marked 9 V, 10% tolerance. The Zener diodes may have break down voltage from 8.1 V(9 – 0.09) to 9.9 V (9 + 09) instead of sharp 9 V for all.

Effect of temperature. It may be noted that the breakdown voltage of Zener diode depends upon operating temperature. It decreases with increase in junction temperature. This is because of the increased reverse current (*i.e.*, increase in minority carriers) that flows with increasing temperature. The decrease is about 2 mV/°C rise in temperature.

(iii) ***Power rating (P_{ZM}).*** *The maximum power which Zener diode can dissipate (or handle) without damage is known as its power rating (P_{ZM}).* Commercially available zener diodes have power ratings from $\frac{1}{4}$ W to more than 50 W.

Power rating is the product of maximum current (I_{ZM}) which a Zener diode can handle and the rated or operated voltage of Zener diode (V_Z)

$$\therefore \quad P_{ZM} = I_{ZM} V_Z \qquad ...(39)$$

Sometimes, a datasheet includes the maximum current rating of a Zener diode.

(iv) ***Maximum current rating (I_{ZM}).*** *The maximum value of current which a Zener diode can handle at its rated voltage without damage is known as its maximum current rating (I_{ZM}).*

(v) ***Zener resistance (R_{ZT}).*** When a Zener diode is operated in the breakdown region, as increase in current produces a slight increase in voltage. This shows that a Zener diode has a small a.c. resistance called Zener resistance (often called Zener impedance Z_{ZT}). Sometimes, manufacturers specify this resistance in their data sheets.

The opposition offered to the current flowing through the Zener diode in the operating region is known as ***Zener resistance (R_Z)*** or ***Zener impedance (Z_Z).***

The Zener resistance at test current I_{ZT} is represented as R_{ZT}.

8.20.3 Equivalent Circuit of an Ideal Zener Diode and Actual Zener Diode

(i) ***Ideal Zener diode.*** For ideal Zener diode, in the V-I graph, the breakdown region is considered to be vertical as shown in Fig. 8.33. This shows that voltage is constant even if the current changes. Hence the Zener resistance is neglected. This means that a Zener diode operating in the breakdown region ideally acts like a battery and its equivalent circuit is shown in Fig. 8.33. It means, in a circuit, mentally an ideal Zener diode can be replaced by a voltage source V_Z, provided the Zener diode is operating in the breakdown region.

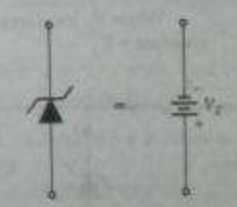

Fig. 8.33

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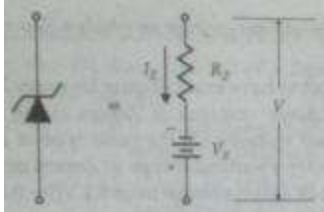


8.34

(ii) Actual Zener diode. For an actual Zener diode, in the *V-I* graph, the breakdown region is not quite vertical. It shows the Zener diode has a Zener resistance. Hence, an actual Zener diode is visualised as if a resistance R_Z is connected in series with a battery of voltage V_Z as shown in Fig. 8.34. The voltage across the Zener will be

$$V = V_Z + I_Z R_Z \quad ...(40)$$

.4 Applications of Zener Diode

Zener diodes find wide commercial and industrial applications. Some important applicants are discussed below :

ner Diode as Voltage Stabiliser

A Zener diode can be used as a voltage regulator to provide a constant voltage may vary over a sufficient range. For this purpose, a Zener diode of Zener voltage V_Z is reverse connected across the load R_L across which constant output is desired. This combination is connected to source with a series resistance R. The series resistance absorbs the output voltage fluctuations so as to maintain voltage across load. The source is such that $V \geq V_Z$.

8.35

When $V < V_Z$, then Zener is in OFF state. This output voltage have any value *i.e.*, $V_0 = IR_L$

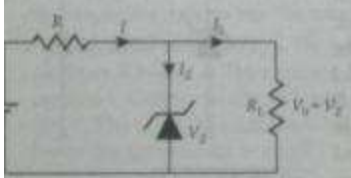
But when $V \geq V_Z$, then Zener goes in ON-state, since it is connected parallel to R_L, thus output voltage becomes equal to V_Z

$$E_0 = V_Z = V_0 \quad ...(41)$$

The working of regulation can be understood in following two points :

(*i*) ***When Input is variable* (If V_i varies and R_L is fixed).** When input voltage (V_i) increases, the potential drop across R increases, due to which total current I increases. At this the Zener current increases to maintain the excess current in R and hence load current remains constant. Thus the output voltage remains constant, equal to Zener voltage.

When V_i increases, then I and I_Z increase but I_L remains fixed. Then $E_0 = V_0 =$ constant $= V_Z$



8.36

Let $(V_i)_{min}$ and $(V_i)_{max}$ are the minimum and maximum input voltages for which output voltage is equal to V_Z.

(*a*) *Calculation of* $(V_i)_{min}$. At this voltage (input voltage) the Zener turns in ON condition only and voltage across load becomes equal to V_Z.

i.e., $V_Z = E_0 = V_0$

[just ON state of Zener $I_Z = 0 \neq 0$]

in OFF state of Zener $I = I_L$, $I_Z = 0$

$$E_0 = V_0 = IR_L = \frac{V_i}{R + R_L} \cdot R_L = \frac{R_L}{R + R_L} V_i$$

Thus $$V_Z = \frac{R_L}{R + R_L} \cdot (V_i)_{min}$$

or $$(V_i)_{min} = \frac{R + R_L}{R_L} \cdot V_Z \quad ...(42)$$

(*b*) *Calculation of* $(V_i)_{max}$. At maximum input voltage, the Zener goes in deep breakdown and high current through Zener flows.

$$I_Z = \frac{E_0}{R_L} = \frac{V_Z}{R_L}$$

$$I_{max} = I_Z + I_L$$

$$\therefore \quad V_i = IR + V_0 = IR + E_0$$

or $$(V_i)_{max} = I_{max} R + V_Z \quad ...(43)$$

(*ii*) ***When load is variable* (V_i = *constant and* R_L = *variable*).** When load resistance decreases, the load current increases. The extra current can not come from source. The additional load current is compensated by decrease in Zener current at Zener voltage *i.e.*, thus output voltage stays at constant value V_Z. When R decreases, then I_L increases.

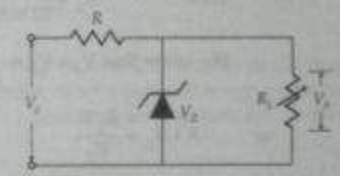


Fig. 8.37

$\because \quad I = I_L + I_Z$

and I is constant, then I_Z decreases so that $V_0 = V_Z$.

For a definite range of R_L, the output voltage remains constant. Let $(R_L)_{min}$ and $(R_L)_{max}$ are the minimum and maximum value of load resistance for which output voltage is equal to Zener voltage.

(*a*) *Calculation of* $(R_L)_{min}$ *and* $(I_L)_{max}$: Once the Zener is in 'ON' state, load voltage $V_0 (= V_Z)$ is constant. As a result, when load resistance is minimum (*i.e.*, $(R_L)_{min}$, load current will be maximum $\left((I_L)_{max} = \frac{V_0}{R_L}\right)$, the Zener current have minimum value $I_Z = I - I_L$. Thus for constant output voltage, the Zener should be in 'ON' state

i.e., $$V_0 = V_Z = \frac{R_L}{R + R_L} V_i \quad \Rightarrow \quad \frac{R + R_L}{R_L} = \frac{V_i}{V_Z}$$

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or $$\frac{R}{R_L}+1=\frac{V_i}{V_Z} \Rightarrow \frac{R}{R_L}=\frac{V_i-V_Z}{V_Z}$$

$$R_L=(R_L)_{min}=\frac{RV_Z}{(V_i-V_Z)} \quad ...(44)$$

(b) *Calculation of* $(R_L)_{max}$ *or* $(I_L)_{min}$: $(R_L)_{max}$ or $(I_L)_{min}$, the Zener is in deep breakdown region and Zener current is large, less than burn out current $(I_Z)_{max}=I-I_L$.

At this, the Zener is in ON state and output voltage is equal to Zener voltage.

$$(I_L)_{min}=I-(I_Z)_{max}$$

and $$(R_L)_{max}=\frac{V_o}{(I_L)_{min}}=\frac{V_Z}{(I_L)_{min}}$$

$$(R_L)_{max}=\frac{V_Z}{(I_L)_{min}}. \quad ...(45)$$

Example 8.6. *A Zener diode shunt regulator has to supply a load current that change from 0 – 200 mA at 10 V. The input voltage to the regulator circuit varies 15 V – 20 V. Zener diode stabilizes at a minimum current of 10 mA. Find out the series resistance.* **[GGSIPU April 2007 (5 Marks)]**

Solution. Given that $V_Z=10\text{ V}$, $I_L=0.200\text{ mA}$, $V_i=15\text{ V}-20\text{ V}$, $(I_L)_{min}=10\text{ mA}$

Then $$R_L=\frac{1.0\text{ V}}{10\text{ mA}}=1\text{ k}\Omega$$

We know that $$V_o=V_Z=\frac{R_L V_i}{R+R_L}$$

or $$R+R_L=\frac{R_L V_i}{V_Z}$$

$$R=\frac{R_L\times V_i}{V_Z}-R_L=\frac{1\times10^3\times15}{10}-10^3=0.5\times10^3\,\Omega=500\,\Omega.$$

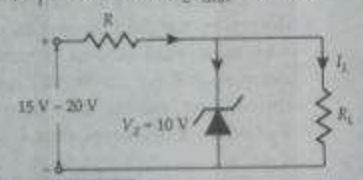

Fig. 8.38

Example 8.7. *A 9 V stabilized voltage supply is required to run a car stereo system from the car's 12 V battery. A Zener diode with* $V_Z=9\text{ V}$ *and* $P_{max}=0.27\text{ W}$ *is used as voltage regulator. Find the value of series resistor. If the load resistance is 450Ω, find the diode current.*

Solution. Let I = current drawn from the battery

I_Z = current through Zener diode

I_L = current through load resistance (Fig. 8.39).

Voltage drop across

$$R=V_i-V_L=V_i-V_Z=12-9=3\text{ V}$$

Potential difference across

$$R_L=V_Z=V_o=9\text{ V}$$

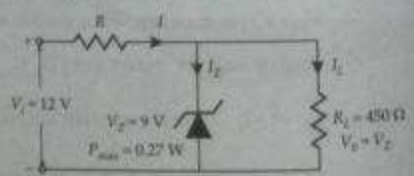

Fig. 8.39

∴ Load current $$I_L=\frac{\text{p.d. across } R_L}{\text{load reistance}}=\frac{9}{450}=0.02\text{ A}$$

Zener current $$I_Z=\frac{P_{max}}{V_{max}}=\frac{0.27}{9}=0.03\text{ A}$$

∴ The current drawn from the battery

$$I=I_Z+I_L=0.03\text{ A}+0.02\text{ A}=0.05\text{ A}$$

∴ The value of series resistor

$$R=\frac{\text{Voltage drop across } R}{\text{Current through } R}=\frac{3}{0.05}=60\,\Omega$$

2. Zener Diode for meter protection

Zener diodes are generally employed in multimeters to protect the meter movement against damage from accidental overloads. For the protection of meter, the Zener diode is connected in parallel with it as shown in Fig. 8.40.

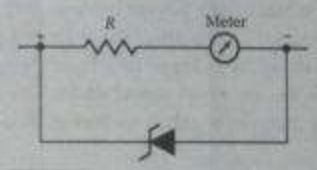

Fig. 8.40

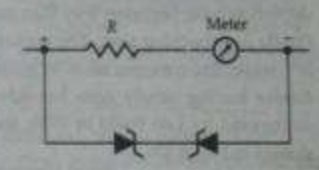

Fig. 8.41

In case of accidental overload, most of current will pass through the zener diode, thus protecting the meter movement from damage. When the meter movement is required to be protected, regardless of the applied polarity (*i.e.*, a.c.), the circuit arrangement is modified as shown in Fig. 8.41.

3. Zener diode for wave shaping

Zener diodes are also used to convert sine wave into almost square wave. For this, the circuit is arranged as shown in Fig. 8.42. During positive as well as negative half cycle, when the voltage across the diodes is below Zener value they offer a high resistance path and input voltage appears across the output terminals. However as the input voltage increases beyond Zener value, the Zener diodes offers a low resistance path and conduct

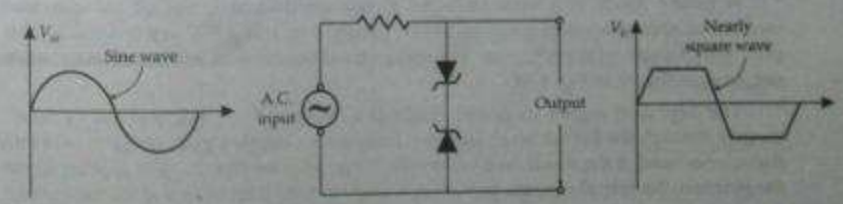

Fig. 8.42

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large current. This results in the heavy voltage drop across series resistor R and hence peaks of the input wave are clipped-off when appears at the output as shown in Fig. 8.42. Thus an input sine wave is clipped off at the peaks and almost a square wave appears at the output.

8.20.5 Distinguish a Zener Diode from an Ordinary Diode

Ordinary junction diodes are operated within the breakdown voltage in reverse biased condition. Operation beyond breakdown voltage may damage them. But a Zener diode is specially designed for operation beyond breakdown voltage. This breakdown voltage is called *Zener Breakdown Potential*.

Ordinary diode is usually used for rectification while Zener diode is used for voltage regulation.

8.21 Tunnel Diode

If the impurity concentration is increased well beyond that in the Zener diode, the depletion layer becomes very thin and the reverse breakdown voltage becomes very small, finally approaching zero. The concentrations required for this are of the order of 1 part in 10^3 while the concentration in normal diodes are of the order of 10^8. The *P-N* junction diodes having nearly zero breakdown voltage are known as *tunnel diodes*. They were discovered by Leo Easki in 1958. the schematic diagram (symbol) for tunnel diode is as shown in Fig. 8.43.

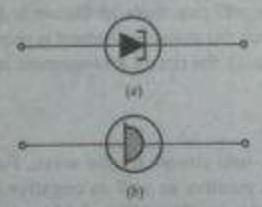

Fig. 8.43

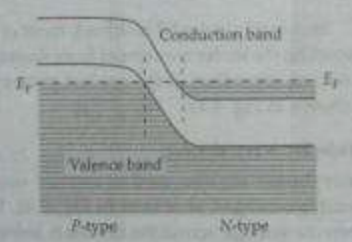

Fig. 8.44

A tunnel diode is a semiconductor device which makes use of the quantum mechanical phenomenon of potential barrier penetration. It is a *P-N* junction made from *quite heavily doped* semiconductors. The energy band structure of such a junction, when unbiased, is shown in Fig. 8.44.

The depletion region in tunnel diode is narrow 10^{-6} m, electron can "tunnel" directly through the forbidden band there, from the conduction band of the *N*-side into the valence band of the *P*-side, and vice versa. When no external voltage is applied across the junction, the rate of electron tunneling is same in both directions and the net current is zero.

(*i*) When a small forward bias voltage is applied to the diode, the energy level in the *N*-type semiconductor move up relative to those in the *P*-type semiconductor (Fig. 8.45). Now the number of electrons tunneling from the *N*-side to the *P*-side is increased because these electrons find directly before them empty energy states in the *P*-side valence band ; whereas the number of electrons tunneling in the opposite direction is decreased. Hence there is a net electron flow to the left, which corresponds to a conventional current to the right.

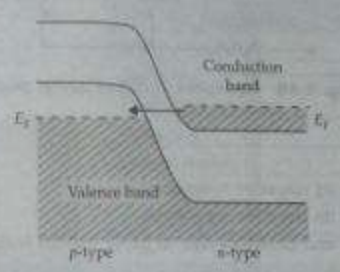

Fig. 8.45

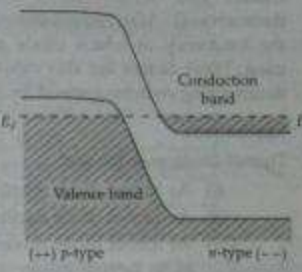

Fig. 8.46

(*ii*) As the applied voltage continues to be increased, the net current at certain voltage begins to decrease because the number of empty states in the *P*-side valence band available for the electrons in the *N*-side conduction band decreases. In Fig. 8.46, the net current is reduced almost to zero because electrons in the *N*-side conduction band find no empty states into which they could tunnel directly.

(*iii*) With still higher applied voltage, the current follows the characteristic of an ordinary *P-N* junction diode. The net current is now due to the 'diffusion' of majority charge carriers (electrons in the conduction band on the *N*-side and holes in the valence band on the *P*-side) across the junctions.

Fig. 8.47 shows the voltage-current characteristic curve of a tunnel diode. The points *a*, *b* and *c* corresponds to the parts of previous figures 8.44, 8.45 and 8.46 respectively. (The dashed curve indicate the behaviour of ordinary junction diode).

In the region between points *b* and *c* the tunnel diode has a *negative differential resistance*, *i.e.*, the current decreases with increasing voltage. The feature makes it specially useful in the switching circuits of computers.

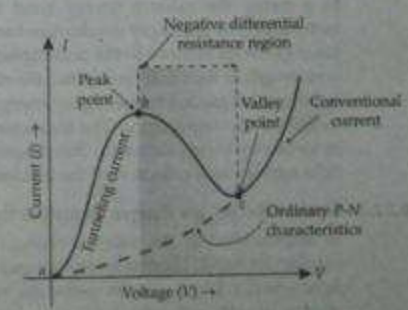

Fig. 8.47 *V-I* characteristic curve of tunnel diode.

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1 Equivalent Circuit

The equivalent circuit of a tunnel diode is visualised in Fig. 8.48. The capacitance C and resistance ($-R_N$) are the junction diffusion capacitance (1 to 10 pF) and the negative resistance respectively. The induction L_s is due to terminal leads (0.1 to 4 nH). The resistance R_s is due to the leads, ohmic contact and semiconductor materials themselves (1 – 5Ω). These factors decides the frequency at which diode may be used. These factors are also valuable for determining switching speed limits.

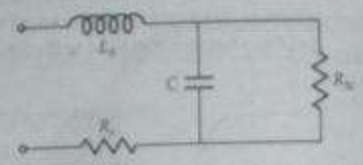

Fig. 8.48 Equivalent circuit of tunnel diode.

2 Applications

Tunnel diode may be used :

(i) As tunnel diode amplifiers due to its negative resistance.

(ii) As tunnel diode oscillators due to its negative resistance.

(iii) As tunnel diode mixers, converter and detectors due to triple valued feature of its current curve.

(iv) Tunneling is much faster than normal crossing which enables a tunnel diode to switch ON and OFF state much faster than other diode. Thus it is used as ultrahigh speed switches.

3 Advantages and Disadvantages of Tunnel Diodes

The advantages of tunnel diodes are low cost, low noise, simplicity, high speed of operation, ease of operation, low power and environmental immunity.

The disadvantage of the tunnel diodes are its low output voltage swing and being a two terminal device, it cannot provide isolation between the input and output circuits.

2 Some Important Explanations on the Basis of band Theory

2.1 Metal are Opaque to Light

In a metal, the valence energy band is either partially filled or overlapped (if filled completely) by an empty conduction band. It can result in acquiring additional energy by some electrons to move to the unoccupied energy levels of the same band. Hence, when any wavelength of light falls on a metal, the electron absorb the photon (energy packet) of visible light and are excited to the higher energy (unoccupied) levels. That is the metal is opaque to light of all wavelengths. The characteristic lustre of metals is due to re-radiation of light absorbed by their valence (or free) electrons. If the metal surface is smooth, the re-radiated light appears as a reflection of the incident light.

2.2 Semiconductors are Opaque to visible light but transparent to Infra-red

In semiconductor, the valence energy band is filled above which there is an empty conduction band. There is an energy gap of about 1 eV in between them. Photons of visible light (energy 1 to 3 eV) can excite the valence electrons to the conduction band, while a photon of infra-red light can not do so because of its in sufficient energy. Thus infrared light is passed *i.e.*, the semiconductor is transparent to infrared light.

8.22.3 Most of the insulator are transparent to visible light

In insualtors, there is a large energy gap so that photons of visible light can not excite the electrons in the valence electrons to reach the conduction band. Thus insulators are transparent to visible light. Many insulators are however, opaque to ultra violet light whose 'high energy' photons are absorbed by valence electrons to reach the conduction band.

Table 8.1 *Electric current conduction mechanisms in semiconductor devices.*

Mechanism	Examples
Drift	Resistor
Diffusion	P-N Junctions
Thermionic emission	Schottky barrier diodes
Tunneling	Tunnel diodes
Recombination	LEDs and semiconductor lasers
Generation	Solar cells
Avalanche	IMPATT diodes

Formulae at a Glance

1. Band theory of solids

$$E = \frac{p^2}{2m} = \frac{\hbar^2}{2m}k_B^2 \quad \text{or} \quad k_B^2 = \frac{2m}{\hbar^2}E$$

k = wave propagation constant.

2. Distribution Function for Fermi Dirac statistics

$$F(E) = \frac{1}{[e^{(E-E_F)/k_BT} + 1]}$$

3. Fermi Energy (Intrinsic Semiconductor)

$$E_F = \frac{E_C + E_V}{2}$$

E_C = energy level in conduction band

E_V = energy level in valence band.

4. Recombination rate $r = np$

n = electrons density

p = holes density

5. (i) $n_n p_n = n_i^2$ [N-type semiconductor]

(ii) $n_p p_p = n_i^2$ [P-type semiconductor]

6. Current density $J = \rho v$

ρ = charge density

7. $J_{drift} = e(nv_n + pv_p) = eE(n\mu_n + p\mu_p)$

μ_n = mobility of electrons

μ_p = mobility of holes

8. Conductivity $\sigma = e(n\mu_n + p\mu_p)$

9. Resistivity $\rho = \dfrac{1}{e(n\mu_n + p\mu_p)}$

10. $J_{diffusion} = e\left(D_n\dfrac{dn}{dx} + D_p\dfrac{dp}{dx}\right)$

11. Diode equation

$$I = I_0[e^{eV/k_BT} - 1]$$

$$r_f = \left(\frac{\partial V}{\partial I}\right) = \frac{0.026}{I}\text{ Volts}$$

12. Power rating

$$P_{ZM} = I_{ZM}V_Z$$

13. Zener Diode as voltage stabilizer

(i) When input is variable

(a) $(V_i)_{min} = \dfrac{R + R_L}{R_L}V_Z$

(b) $(V_i)_{max} = I_{max}R + V_Z$

(ii) When load is variable

(a) $(R_L)_{min} = \dfrac{RV_Z}{(V_i - V_Z)}$

(b) $(R_L)_{max} = \dfrac{V_Z}{(I_L)_{min}}$

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Miscellaneous Solved Examples for Practice

Example 1. *The saturation current density of a P-N junction germanium diode is 200 mA/m² at 27°C. Find the voltage to be applied across the junction to have forward current density of 10^4 A/m² to flow.*

Solution. The diode current for germanium at a temperature T is given by

$$I = I_S\left[\exp\left(\frac{eV}{k_BT}\right)-1\right]$$

$$J = J_S\left[\exp\left(\frac{eV}{k_BT}\right)-1\right]$$

or $$\frac{J}{J_S} = \exp\left(\frac{eV}{k_BT}\right)-1 \quad \text{or} \quad \frac{eV}{k_BT} = \ln\left(\frac{J}{J_S}\right)$$

With $J = 10^4\ \text{A/m}^2$, $J_S = 200\times10^{-3}\ \text{A/m}^2$, $e = 1.6\times10^{-19}$ C, $k_B = 1.38\times10^{-23}$ J/K, $T = 27°\text{C} = 300$ K

i.e., $$V = \frac{k_BT}{e}\ln\left(\frac{J}{J_S}\right)$$

$$= \frac{1.38\times10^{-23}\times300}{1.6\times10^{-19}}\ln\frac{10^4}{200\times10^{-3}} = 0.28\ \text{V}.$$

Example 2. *Find the mobility of electrons in copper assuming that each atom contributes one free electron for conduction. [Resistivity of copper is $1.7\times10^{-6}\,\Omega$ cm, atomic weight 63.54, density 8.96 g/cc, avogadro's number 6.025×10^{23} and electronic charge is 1.6×10^{-19} C.]*

Solution. We shall first find the number of free electrons per unit volume and is given by

$$\frac{\text{Avogadro's number} \times \text{density}}{\text{Atomic weight}} = \frac{6.025\times10^{23}\times8.96}{63.54}$$

$$= 8.50\times10^{22}$$

Therefore mobility of electron is given by

$$\mu = \frac{\sigma_n}{n_e}$$

where σ_n is the conductivity. If ρ_n be the resistivity then $\rho_n = \frac{1}{\sigma_n}$, so that $\mu = \frac{1}{\rho_n n_e}$

$$\mu = \frac{1}{(1.7\times10^{-6}\times8.49\times10^{22}\times1.6\times10^{-19})}$$

$$= 43.28\ \text{cm}^2/\text{volt sec}.$$

Example 3. *For copper at 1000 K, find the energy at which the probability $f(E)$ that a conduction electron state will be occupied is 0.90.*

Solution. The probability $f(E)$ of a state corresponding to energy E being occupied by an electron at temperature T is given by

$$f(E) = \frac{1}{1+\exp\left(\frac{E-E_F}{k_BT}\right)} = \frac{1}{(1+e^{\Delta E/k_BT})}$$

Therefore $$0.90 = \frac{1}{1+e^{\Delta E/k_BT}}$$

or $$\frac{\Delta E}{k_BT} = -2.20$$

Thus $$\Delta E = -2.20\,k_BT$$

$$= -2.20\times(8.63\times10^{-5}\ \text{eV/K})\times(1000\ \text{K})$$

$$= -0.19\ \text{eV}.$$

For copper, $E_F = 7.00$ eV so that

$$E = E_F + \Delta E$$

$$= 7.00\ \text{eV} - 0.19\ \text{eV} = 6.81\ \text{eV}$$

Example 4. *A bar of N-type Si with a cross-sectional area of 1 mm×1 mm and a length of 1 cm is connected to a 2 V battery supply and is carrying a current of 2 mA at 300 K.*

(a) *Calculate the thermal equilibrium electron-hole densities in the bar.*

(b) *Calculate dopant concentration presuming that the only donors with an energy level of 0.148 eV below the conduction band.*

(c) *Determine the temperature at which the bar becomes intrinsic.*

Solution. (a) The resistance R of the bar is given by

$$R = \rho\left(\frac{l}{A}\right) = \rho\left[\frac{10^{-2}}{10^{-3}\times10^{-3}}\right] = 10^4\rho \qquad \text{...(i)}$$

Also $$R = \left(\frac{V}{I}\right) = \frac{2}{(2\times10^{-3})} = 10^3\,\Omega \qquad \text{...(ii)}$$

Comparing (i) and (ii), we find

$$\frac{1}{\rho} = e(\mu_e n + \mu_h p) = 10 \qquad \text{...(iii)}$$

$\because$ The sample is N-type, $n >> p$.

Therefore $$e\mu_e n = 10 \qquad \text{...(iv)}$$

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Taking $\mu_e = 1350\ cm^2/Vs = 0.135\ m^2/Vs$, we get from (iv)

$$n = \frac{10}{(1.5\times10^{-19}\times0.135)} = 4.63\times10^{20}/m^3$$

but $$p = \left(\frac{n_i^2}{n}\right) = \frac{(1.5\times10^{16})^2}{4.63\times10^{20}} = 4.86\times10^{11}/m^3$$

Hence $n = 4.63\times10^{14}/cm^3$ and $p = 4.86\times10^5/cm^3$.

(b) The electron density is given by

$$n = N_C \exp\left[\frac{-(E_C - E_F)}{k_B T}\right] \quad ...(v)$$

where for Si,

$$N_c = 2[2\pi m_n^* k_B T/h^2]^{3/2} = 2.8\times10^{19}/cm^3.$$

Therefore, from Eq. (v)

$$(E_C - E_F) = k_B T \ln\left(\frac{N_C}{n}\right)$$

$$= 0.026 \ln[2.8\times10^{19}/4.63\times10^{14}] = 0.286\ eV$$

But it is given that

$$(E_C - E_d) = 0.148\ eV$$

We may now write

$$(E_d - E_F) = (E_C - E_F) - (E_C - E_d)$$

$$= 0.286 - 0.148 = 0.138\ eV.$$

Therefore, the number of electrons in the donor state is

$$n_d = \frac{N_d}{\left[1+\exp.\left\{\frac{E_d - E_F}{k_B T}\right\}\right]} = \frac{N_d}{\left\{1+\exp.\left(\frac{0.138}{0.026}\right)\right\}} = \frac{N_d}{1+\exp.(5.3)}$$

But $N_d = n + n_d$ or $n = n_d/\{1+\exp.(-5.318)\} = 4.63\times10^{20}$

Solving for N_d, one finds

$$N_d = 4.65\times10^{20}/m^3.$$

(c) The semiconductor specimen becomes intrinsic at a temperature T, when $n_i(T) = N_d$, that is

$$(N_C N_V)^{1/2} \exp\left[-\left(\frac{E_g}{2k_B T}\right)\right] = N_d$$

or $$N_c \exp[-(E_g/2k_B T)] = N_d$$

Taking $E_g = 1.12$ eV and $k_B = 1.38\times10^{-23}$ J/K and solving for T, one obtains, $T = 606.8$ K or $t = 333.8°C$. Therefore the bar becomes intrinsic at a temperature 333.8° C.

Example 5. In a voltage regulator a 12 V Zener diode is connected in series with a resistance of 150Ω. A load resistance of 1 kΩ is connected in parallel with diode. If minimum Zener current is zero and maximum Zener current is 20 mA. Calculate the operating range of input voltage.

[GGSIPU April 2008 (4 marks)]

Solution. $V_Z = 12$ V, $R = 150\Omega$, $R_L = 1k\Omega$, $(I_Z)_{min} = 0$, $(I_Z)_{max} = 20$ mA, V_i = Range = ?

This is case of fixed R_L and variable V_i

(i) For $(V_i)_{min}$: $V_o = V_Z = \frac{R_L V_i}{R + R_L}$

$$(V_i)_{min} = \frac{(R+R_L)V_Z}{R_L}$$

$$= \frac{(150+1000)\times12}{1000}$$

$$= 1.150\times12 = 13.800 = 13.8\ V$$

(ii) For $(V_i)_{max}$:

$$I_L = \frac{V_o}{R_L} = \frac{V_Z}{R_L} = \frac{12}{1000} = 12\ mV$$

is fixed, the value of I will maximum when Zener current is maximum

$$I_{max} = I_{ZM} + I_L$$

Now $$V_i = IR + V_o \quad \because \quad V_o = V_Z$$

$$(V_i)_{max} = I_{max} R + V_Z$$

$$= (12+20)\times10^{-3}\times150 + 12$$

$$= 32\times50\times10^{-3} + 12$$

$$= 4800 + 12 = 16.8\ V$$

$\therefore$ Operating range of input voltage = 13.8 V to 16.8 V.

Example 6. For what voltage will the reverse current in a P-N junction germanium diode attain a value of 90% of its saturation value at room temperature ?

Solution. Give $I = 90\%$ of I_S, $T = 300$ K, V = ?

The general form of the rectifier equation is

$$I = I_S\left[\exp.\left(\frac{eV}{k_B T}\right) - 1\right]$$

or $$\frac{I}{I_S} = \exp.\left(\frac{eV}{k_B T}\right) - 1$$

or $$0.9 = \exp.\left(\frac{eV}{k_B T}\right)$$

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or $$\frac{eV}{k_B T} = \ln 0.9$$

$$\frac{eV}{k_B T} = \ln 0.9$$

or $$V = \frac{\ln 0.9 \times k_B T}{e} = \frac{\ln 0.9 \times 1.38 \times 10^{-23} \times 300}{1.6 \times 10^{-19}} = 0.017 \text{ volt.}$$

Example 7. *In Fig. 8.49, assume that the supply voltage V_i is 9 V and that the Zener voltage V_0 is 6 V. If the maximum Zener current that can safely flow be 20 mA, determine the value of the Zener resistance R. If a load resistance R_L of 1 kΩ be connected across the Zener diode, calculate the load current and the Zener current. Also, calculate the maximum value of R_L, that can be employed.*

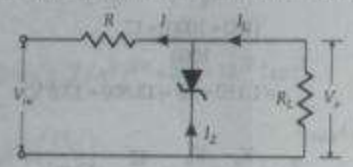

Fig. 8.49

Solution. I_Z is a maximum when $R_L = \infty$

$$R = \frac{V_i - V_o}{I} = \frac{9-6}{20 \times 10^{-3}} = 150\,\Omega$$

The current through the load resistance $R_L = 1\,\text{k}\Omega$ is

$$I_L = \frac{V_o}{R_L} = \frac{6}{10^3} = 6\text{ mA}$$

The Zener current is $I_Z = I - I_L = 20 - 6 = 14$ mA. As R_L is decreased, the current through it increases, and the limiting case, I_L becomes equal to I *i.e.*, 20 mA. Hence minimum value of R_L is

$$R_L = \frac{6\text{ V}}{20\text{ mA}} = 300\,\Omega$$

Example 8. *Across a load of 1000 Ω, a voltage of 5 V is to be maintained. Using a 5 V Zener diode, design a regulated power supply. The minimum Zener current required is 5 mA. Draw the circuit diagram of the regulated power supply. The supply voltage variation is 7.5 V to 15 V.*

Solution. Fig. 8.50, shows the circuit diagram of the regulated power supply.

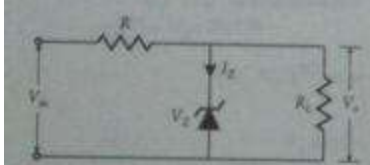

Fig. 8.50

Minimum Zener current is 5 mA

$$R = \frac{V_i - V_o}{I_Z}$$

$(V_i - V_o)$ is to vary from $(15-5) = 10$ to $(7.5-5) = 2.5$ volts

$$I_L = \frac{5}{1000} = 5\text{ mA}$$

The minimum current through load resistance is

$$= 5 + 5 = 10\text{ mA}$$

$$\therefore \text{ Load resistance} = \frac{2.5}{10 \times 10^{-3}} = 250\,\Omega$$

$\therefore$ Maximum Zener current $= (40 - 5) = 35$ mA.

Example 9. *Find the maximum and minimum value of Zener diode current in the Fig. 8.51 shown.*

Solution. Current through R

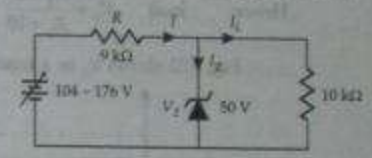

Fig. 8.51

Input current $$I = \frac{176 - 50}{9 \times 10^3}$$

$$= \frac{126}{9 \times 10^3} = 14\text{ mA}$$

Load Current $$I = \frac{50}{10 \times 10^3} = 5\text{ mA}$$

$\therefore$ Maximum Zener current

$$(I_Z)_{(max)} = 14 - 5 = 9\text{ mA}$$

For maximum input voltage, the Zener current will be minimum *i.e.*,

$$I = \frac{104 - 50}{9 \times 10^3} = \frac{54}{9 \times 10^3} = 6\text{ mA}$$

Maximum Zener current

$$(I_Z)_{min} = 6 - 5 = 1\text{ mA.}$$

Example 10. *In Fig. 8.52, R_L is varied from 1 kΩ to 100 kΩ. Derive and draw the output voltage V_0 as the function of the resistance value R_L.*

Solution. Input current $I = I_Z + I_L$

V_0 is fairly constant at 5 V (V_Z)

Now,

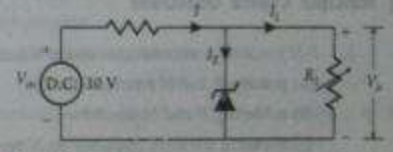

Fig. 8.52

for $R_L = 100\,\text{k}\Omega$,

$$I_L = \frac{5}{100 \times 10^3} = 50\,\mu\text{A}$$

for $R_L = 1\,\text{k}\Omega$,

$$I_L = \frac{5}{1 \times 10^3} = 5\text{ mA}$$

The input current,

$$I = \frac{V_i - V_o}{R} = \frac{10 - 5}{10 \times 10^3} = 0.5\text{ mA}$$

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The Zener diode will conduct only when V_Z is more than 5 V, which can be achieved for a load of 10 kΩ

i.e., $$\frac{10 \times 10}{10+10} = 5\text{ V}$$

Hence, upto 10 kΩ, the Zener diode current I_Z is negligible Beyond this value, say, 20 kΩ, it will carry the excess current supplied by D.C. source to maintain the load approximately constant upto 10 kΩ.

Hence load $$V_0 = \frac{R_L \times 10}{R_L + 10}$$

Fig. 8.53 shows V_0 as a function of R_L changes from 1 to 10 kΩ.

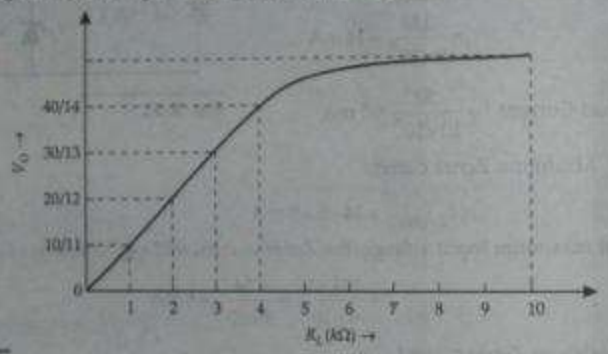

Fig. 8.53

Question Bank

Multiple Choice Questions

1. P-N junction semiconductor diode is formed by
   (a) putting P and N junction closely
   (b) soldering P and N junctions
   (c) allowing trivalent impurity on N-type silicon
   (d) connecting P and N materials.
2. With increase in temperature
   (a) resistance decreases
   (b) resistance increases
   (c) energy of the atom is increased
   (d) energy of the atoms is decreased.
3. Resistivity in a semiconductor depend on
   (a) the shape of the semiconductor
   (b) the size of the semiconductor
   (c) the weight of the semiconductor
   (d) the atomic nature of the semiconductor.

4. The barrier potential, in a semiconductor diode offers opposition to
   (a) minority carriers in both region
   (b) majority carriers in both region
   (c) electrons in N-region
   (d) holes in the P-region.
5. The forbidden energy gap in a semiconductor
   (a) lies just below the valence band
   (b) is the same as the valence band
   (c) lies between conduction band and valence band
   (d) lies above the conduction band
6. An electron in the conduction band
   (a) is bound to its parent atom
   (b) has no charge
   (c) is located near the top of the crystal
   (d) has a higher energy than an electron in the valance band
7. In a semiconductor, the movements of holes is due to
   (a) movement of holes in conduction band
   (b) movement of electrons in conduction band
   (c) movement of electrons in valence band
   (d) movement of holes in valence band.
8. Silicon devices are preferred at high temperature operations as compared to germanium because
   (a) silicon is more thermally stable
   (b) silicon can dissipate more power
   (c) reverse saturation current is less in case of silicon
   (d) all of these.
9. The P-N junction behaves like a
   (a) linear
   (b) triode
   (c) tetrode
   (d) pentode.
10. An intrinsic semiconductor at absolute zero temperature
    (a) acts as an insulator
    (b) acts as a conductor
    (c) has large number of holes
    (d) has large number of electrons.
11. Consider a single crystal of an intrinsic semiconductor. The number of free carriers at the Fermi-level at room temperature is
    (a) half the total number of electrons in the crystal
    (b) half the number of free electrons in the crystal
    (c) half the number of atoms in the crystal
    (d) zero.
12. Forbidden energy gap at 0 K for Si and Ge respectively are
    (a) 1.21 eV and 0.786 eV
    (b) 1.1 eV
    (c) 0.785 eV and 1.21 eV
    (d) 0.72 eV and 1.21 eV.

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13. In a semiconductor diode avalanche breakdown occurs when
(a) reverse bias exceeds Zener voltage
(b) potential barrier is eliminated
(c) forward current generates sufficient heat
(d) reverse current generates sufficient heat

14. Breaking of a covalent bond produces a free electron, which moves about the lattice in a random manner
(a) true (b) false
(c) incorrect (d) none of these

15. The electronic configuration of silicon is
(a) $1s^2, 2s^2 2p^6, 3s^2 3p^4$ (b) $1s^2, 2s^2 2p^6, 3s^1 3p^6$
(c) $1s^2, 2s^2 2p^6, 3s^2 3p^6$ (d) $1s^2, 2s^2 2p^6, 3s^2 3p^2$

16. The density of carriers in a pure semiconductor is proportional to
(a) $\exp(-E_g / k_B T)$ (b) $\exp\left(-\frac{2E_g}{k_B T}\right)$
(c) $\exp\left(\frac{E_g}{k_B T^2}\right)$ (d) $\exp(-E_g / 2k_B T)$

17. When a free electron recombines with a hole, these results
(a) generation of energy (b) release of energy
(c) no change of energy (d) none of the above.

18. The diffusion current is proportional to
(a) square of the applied electric field (b) applied electric field
(c) concentration gradient of charge carriers (d) none of the above.

19. Room temperature resistivity of pure germanium in ohm meter is
(a) 47 (b) 4.7 (c) 0.47 (d) 0.047

20. The forbidden energy gap of carbon in diamond structure is
(a) 0.7 eV (b) 1 eV
(c) 0.01 eV (d) none of these

21. A semiconductor exhibits
(a) positive temperature coefficient of resistivity
(b) negative temperature coefficient of resistivity
(c) positive and negative temperature coefficient of resistivity
(d) none of these.

22. A life time of electrons in a P-type semiconductor is
(a) the average time it takes them to diffuse to the contact
(b) the average time before they recombine
(c) the average time they take to cross the depletion region
(d) none of these.

23. In a tunnel diode, electron can tunnel through the P-N junction mainly because
(a) impurity level is low (b) they have high energy
(c) barrier potential is very low (d) depletion layer is extremely thin.

24. Mark the wrong statement for a tunnel diode :
(a) V-I characteristics of tunnel diode is made up of hills and valleys
(b) a tunnel diode is very heavily doped junctions diode
(c) a tunnel diode is always biased in the reverse direction
(d) a tunnel diode can function as an oscillator

25. A Zener diode
(a) is always diode (b) is connected in series
(c) has a sharp breakdown at low reverse voltage
(d) has a negative resistance

26. The tunnel diode has thickness of depletion layer approximately meters.
(a) 10×10^{8} (b) 10×10^{-9} (c) 0.1×10^{-5} (d) 8.1×10^{-4}

27. In a P-N junction diode, holes diffuse from the P-region to the N-region because
(a) the free electrons in the N-region attract them
(b) they are swept across the junction by the potential difference
(c) there is greater concentration of holes in the P-region as compared to N-region
(d) none of the above.

28. In P-N junction diode, if the junction current is zero, this means that
(a) the potential barrier has disappeared
(b) there are no carriers crossing the junction
(c) the number of majority carriers crossing the junction equals the number of minority carriers crossing the junction
(d) the number of holes diffusing from the P-region equals the number of electrons diffusing from the N-region.

29. In a semiconductor diode, the barrier potential offers opposition to only
(a) majority carriers in both regions (b) minority carriers in both region
(c) free electrons in the N-region (d) hole in the P-region

30. When we apply reverse bias to a junction diode, it
(a) lowers the potential barrier
(b) raises the potential barrier
(c) greatly increases the minority carrier current
(d) greatly increases the majority carrier current.

31. The number of minority carriers crossing the junction of a diode depends primary on the
(a) concentration of doping impurities
(b) magnitude of the potential barrier
(c) magnitude of the forward bias voltage
(d) rate of thermal generation of electron-hole pairs

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32. When forward bias is applied to a junction, it
    (a) increases the potential barrier
    (b) decreases the potential barrier current
    (c) reduces the majority-carrier to zero
    (d) reduces the majority-carrier current to zero.
33. The depletion or space-charge region in a junction diode contains charges that are
    (a) mostly majority carriers
    (b) mostly minority carriers
    (c) mobile donor and acceptor ions
    (d) fixed donor and acceptor ions.

Answers

1. (c)	2. (c)	3. (d)	4. (b)	5. (b)	6. (d)
7. (c)	8. (d)	9. (a)	10. (a)	11. (b)	12. (b)
13. (a)	14. (a)	15. (c)	16. (d)	17. (d)	18. (c)
19. (c)	20. (d)	21. (b)	22. (c)	23. (a)	24. (c)
25. (c)	26. (c)	27. (c)	28. (c)	29. (a)	30. (b)
31. (d)	32. (b)	33. (d)			

Theoretical Questions

1. Draw the labeled energy band diagram including the Fermi energy for a tunnel diode. [GGSIPU, May 2008 (2.5 marks)]
2. What is Fermi energy ? What is its physical significance ? [GGSIPU, May 2008 (6.25 marks)]
3. Distinguish between avalanche multiplication and Zener breakdown. [GGSIPU, May 2008 (6.25 marks)]
4. Draw the energy band diagram of an unbiased *P-N* junction. [GGSIPU, May 2007 (5 marks)]
5. Explain the terms : (*i*) barrier energy (*ii*) barrier potential and (*iii*) depletion layer across a *P-N* junction.
6. Explain qualitatively how a semiconductor diode can be used as rectifier.
7. What is a Zener diode ? Draw an equivalent circuit of an ideal Zener in breakdown region. What are its ON and OFF states ? Explain the formation of potential barrier in a *P-N* junction. [GGSIPU, May 2006 (8 marks)]
8. In a *P*-type semiconductor, what are the majority charge carriers ? Is the semiconductor positively charged, negatively charged or neutral ? Explain. [GGSIPU, May 2005 (2.5 marks)]
9. Explain the construction and working of a tunnel diode. Illustrate your answer with appropriate energy level diagrams and a voltage versus current plot. Name one application of a tunnel diode. [GGSIPU, May 2005 (6 marks)]
10. On the basis of band theory explain why diamond is transparent to visible light. Given forbidden gap in diamond is 6 eV. [GGSIPU, April 2005 (2 marks)]

11. Enunciate the difference between semiconductors and normal conductors in terms of their energy gap. [GGSIPU, April 2005 (2 marks)]
12. Discuss the effect of temperature and doping concentration on the position of Fermi level in a *N*-type semiconductor. [GGSIPU, April 2008 (2 marks)]
13. What is called barrier tunneling ? Explain tunnel diode operation with a neat diagram. [GGSIPU, April 2008 (2 marks)]
14. Write short notes on
    (*i*) Voltage regulation using Zener diode
    (*ii*) Fermi level and Fermi energy.
    (*iii*) P-N junction diode
    (*iv*) Zener diode
    (*v*) Tunneling effect [GGSIPU, May 2007 (2.5 marks)]
    (*vi*) Energy level diagram for intrinsic and doped semiconductors [GGSIPU, April 2007 (2.5 marks)]
15. Why do energy bands arise in solids. Distinguish between conductors, semiconductors and insulators on the basis of band theory. [GGSIPU, May 2007 (3 marks)]
16. Describe the doping features of a tunnel diode. Plot its *I-V* characteristics and explain the nature of curve. [GGSIPU, May 2007 (3 marks)]
17. Explain the working of a Zener diode as a voltage stabiliser. [GGSIPU, May 2007 (3 marks)]
18. Explain the origin of formation of energy bands. How does the band theory of solids lead to the classification of solid into conductors, semiconductors and insulators. [GGSIPU, May 2006 (3 marks)]
19. What is the significance of Fermi energy in a fermion system at 0 K ; At $T > 0$K ? [GGSIPU, April 2008 (3 marks)]
20. Explain why a semiconductor acts as an insulator at 0 K ? [GGSIPU, April 2008 (3 marks)]
21. What are *P*-type and *N*-type semiconductors ?
22. Explain the effect of biasing on Fermi level in *P-N* junction.
23. How you will classify the materials as conductor, semiconductor and insulator on the basis of band theory of solid ? Explain each state in detail with the help of suitable diagrams.
24. What are intrinsic and extrinsic semiconductors ? Discuss the location of Fermi levels under suitable limiting conditions and give the necessary theory.
25. Explain the terms knee voltage, forward resistance and reverse resistance as applied to a semiconductor diode.
26. State the cause of Zener breakdown in a *P-N* junction diode.
27. Explain the difference between Zener breakdown and avalanche breakdown.
28. Explain the formation of depletion region at the P-N junction. Discuss its role in the electrical breakdown of the junction.
29. Give the circuit symbols for the following
    (*i*) P-N junction diode
    (*ii*) Zener diode
    (*iii*) Tunnel diode

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30. Give the energy band diagram of :
    (i) an intrinsic semiconductor (ii) a P-type semiconductor
    (iii) a N-type semiconductor (iv) Zener diode
    (v) Tunnel diode.
31. Explain following terms for a P-N junction
    (i) Potential barrier
    (ii) depletion region
    (iii) Avalanche and Zener breakdown.
32. Explain current flow mechanism across a P-N junction diode and draw its I-V curve.
33. Explain using a suitable circuit the working of a Zener diode as a voltage regulator.
34. What are N-type semiconductor ? What is the effect of temperature on extrinsic semiconductor ? How does a P-N junction work with forward bias and reverse bias.
35. Explain the classifications of solids. What are metals, insulator and semiconductors ?
36. What do you mean by semiconductors ? Explain the term intrinsic and extrinsic semiconductors.
37. What are P-and N -type semiconductors ? P-and N-type semiconductors have holes and electrons respectively. Explain why they are electrically neutral ?
38. Draw typical band model diagram for a semiconductor showing Fermi level, what do you mean by intrinsic and extrinsic semiconductors ?
39. How are the band structures of insulators and semiconductors similar ? How are they different ?
40. (a) When germanium is doped with aluminium, is the result an N-type or a P-type semiconductors ?
    (b) Why ?
41. Show that the probability of occupancy of the state of energy E above E_F by an electron in a semiconductor is the same as the probability of occupancy of the state of energy E below E_F by a hole.
42. Classify each of the following as N-type or P-type semiconductors
    (a) Doped by acceptor atoms
    (b) Crystal with pentavalent impurities
    (c) Majority carriers are holes
    (d) Donor atoms were added to crystal
    (e) Minority carriers are free electrons
43. Describe with a neat circuit diagram the working of a Zener diode based voltage regulator.
44. Draw the equivalent circuit of an ideal Zener diode in the breakdown region.
45. What is Zener effect ? Explain the function of a Zener diode and draw its characteristics.
46. Why is Zener diode used as a voltage regulator.
47. Explain how Zener diode maintains constant voltage across the load.
48. State and explain the characteristic of a Zener diode. How it can be used as voltage regulator ?
49. How does a tunnel diode differ from a conventional P-N junction diode ? With the help of its energy band diagram explain its principle of operation in forward and reverse bias conditions. Sketch its V-I characteristics and mark negative resistance on it.

Numerical Problems

1. Two Ge diodes, one with $I_{S1} = 10^{-12}$ A and another with $I_{S2} = 10^{-10}$ A, are connected in parallel. If the total current I_T in the diodes is 2 mA, find the voltage across the diodes. [**Ans.** $V = 0.437$ volt]
2. The saturation current density of P-N junction Ge diode is 200 Amp / m^2 at 27°C. Find the voltage to be applied across the junction to have a forward current density of 10^4 Amp / m^2 to flow. [**Ans.** 0.28 volt]
3. Assuming the carrier mobilities to be independent of doping concentrations, calculate the maximum possible resistivities of Si and GaAs crystals at 300 K. Compare these values with corresponding intrinsic resistivities. [**Ans.** $3.6 \times 10^5\ \Omega$ cm, 1.58, $9.47 \times 10^8\ \Omega$ cm, 2.41]
4. Calculate the Fermi level position in Si doped with 10^{15} phosphorus atoms/cm^3 at room temperature assuming complete ionisation. From the calculated Fermi level position, check the assumption of complete ionization of carriers. Given that $N_C = 2.8 \times 109$ / cm^3, and $(E_C - E_d) = 0.148$ eV. [**Ans.** 99%]
5. A silicon sample, having 5×10^{28} atoms/m^3, has antimony doped to a concentration of one atom of Sb for every 2×10^9 atoms of Si. Assuming acceptor density to be zero and $n_i = 1.5 \times 10^{16}$ per cubic meter at 300 K, calculate the carrier densities at this temperature. [**Ans.** 2.5×10^{19} m^3, 9×10^{12} / m^3]
6. The current in a Ge P-N junction diode is 25 mA for a bias voltage of 0.25 volts at an operating temperature of 300 K. Calculate the reverse saturation current for the diode at this temperature. [**Ans.** 1.67 μA]
7. A Zener diode is specified as having a breakdown voltage of 9.1 V, with a maximum power dissipation of 364 mW. What is the maximum current the diode can handle ? [**Ans.** 40 mA]
8. Determine the electron (minority carrier) and hole (majority carrier) concentration in P-type Si with an acceptor ion concentration of 1×10^{16} / cm^3, when operating at 270 K. [**Ans.** $p = 10^6$/cm^3, $n = 89.2$ / cm^3]
9. Find the electron (majority carrier) and hole (minority carrier) concentration in N-type Si with a donor ion concentration of 5×10^9 / m^3 when operating at 270 K. [**Ans.** $p = 7.27 \times 10^8$ / cm^3, $n = 12.27 \times 10^9$ / cm^3]
10. A Ge diode has a saturation current $I_S = 10$ nA at $T = 300$ K. Find the forward current i_D if forward voltage drop V_D is 0.5 V. [**Ans.** $i_D = 2.47$ A]
11. Determine the maximum and minimum value of Zener current if in the circuit shown in Fig. 8.54, the load resistance $R_L = 5$kΩ, series resistance is 2kΩ, input voltage varies between 80 V and 100 V and the output voltage is 50 V. [**Ans.** 15 mA, 5 mA]

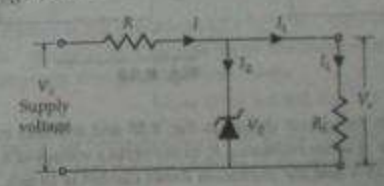

Fig. 8.54

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12. In the circuit shown in Fig. 8.54, the series resistance is 2 kΩ, load resistance is 5 kΩ. Input voltage is 60 V and output voltage is 35 V. Determine (*i*) voltage drop across series resistance (*ii*) Load current (*iii*) current through series resistance (*iv*) Zener diode current.

[**Ans.** (*i*) 25 V, (*ii*) 7 mA, (*iii*) 12.5 mA, (*iv*) 5.5 mA]

13. Determine (*a*) output voltage, (*b*) voltage across R_L and (*c*) the current through Zener diode shown in Fig. 8.54. The load resistance $R_L = 10\text{k}\Omega$, series resistance is 5kΩ, supply voltage $V_s = 12\text{V}$ and Zener voltage V_Z is 8 V.

[**Ans.** (*a*) 8 V, (*b*) 4 V, (*c*) 0]

14. For the circuit shown in Fig. 8.55, find the maximum and minimum values of Zener diode current.

[**Ans.** $(I_Z)_{max} = 9\text{mA}$, $(I_Z)_{min} = 1\text{mA}$]

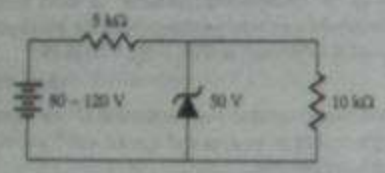

Fig. 8.55

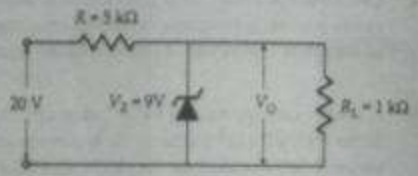

Fig. 8.56

15. For the circuit shown in Fig. 8.56, determine (*a*) the out voltage, (*b*) voltage drop across the series resistance (*iii*) current through the Zener diode. [**Ans.** (*i*) 9 V ; (*ii*) 11 V (*iii*) $I_L = 13\text{mA}$]

16. A Zener diode of Fig. 8.57 has $V_Z = 12\text{V}$, determine the minimum and maximum Zener current as well as the output voltage (*i*) when the Zener diode is considered to be ideal one, (*ii*) when the Zener resistance of Zener diode is 7 Ω. Comment on the result.

[**Ans.** (*i*) $(I_Z)_{min} = 9.57$ mA, $(I_Z)_{max} = 46.34$ mA, $V_0 = 12$ V ;
(*ii*) $(I_Z)_{min} = 9.67$ mA, $V_o = 12.068$ V, $(I_Z)_{max} = 45.95$ mA, $V_o = 12.32$ V.
Comment : The output voltage is almost constant, although input voltage change nearly 2.5 times.]

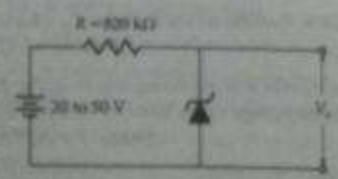

Fig. 8.57

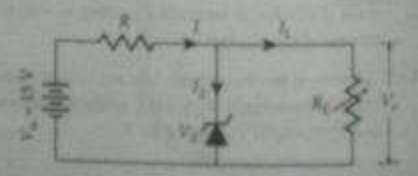

Fig. 8.58

17. A 6 V zener is used in the circuit shown in Fig. 8.58 and the load current is to vary from 20 to 100 mA. Find the value of series resistance R_s to maintain a voltage of 6 V across the load. The input voltage is constant at 15 V and the minimum Zener current is 15 mA. [**Ans.** $R_s = 78.26\,\Omega$]

18. Calculate the current through load resistor and zener diodes shown in Fig. 8.59. Assume the Zener diodes to ideal.

[**Ans.** 1 mA, 199 mA]

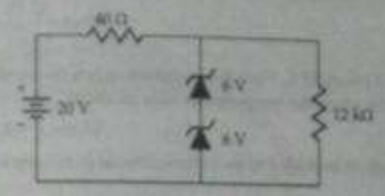

Fig. 8.59

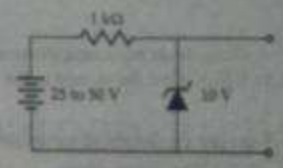

Fig. 8.60

19. Determine the minimum and maximum value of current flowing through Zener diode shown in Fig. 8.60 ; (*i*) when the diode is ideal, (*ii*) when the diode has a Zener resistance of 5Ω.

[**Ans.** 15 mA ; 40 mA ; 14.89 mA, 39.72 mA]

20. A 10 V Zener diode is used to regulate the voltage across a variable load resistor taking current 10 to 85 mA. The input voltage varies between 13 V to 16 V. Calculate the value of series resistor if minimum Zener current is 15 mA. [**Ans.** 30 Ω]

21. Determine the range of input voltage applied to a Zener circuit to maintain 30 V across a load resistor of 2kΩ. Assume that the value of series resistance $R_s = 200\Omega$ and Zener current rating 25 mA. [**Ans.** 33 V to 38 V]

22. The forbidden gap in pure silicon is 1.1 eV. Compare the number of conduction electrons at temperature 37°C and 27°C. (Given $k_B = 8.6 \times 10^{-5}$ eV / K) [**Ans.** $\frac{n_2}{n_1} = 2.96$]

23. Find the resistivity of intrinsic germanium at 300 K. Given that the intrinsic density of carriers is $2.5 \times 10^{19} / \text{m}^3$. [**Ans.** $\rho = 0.43\ \Omega\text{m}$]

24. The following data are given for an intrinsic semiconductor at 27°C : $n_i = 2.4 \times 10^{19} / \text{m}^3$, $\mu_e = 0.39\ \text{m}^2 V^{-1} s^{-1}$ and $\mu_h = 0.19\ \text{m}^2 V^{-1} s^{-1}$. Calculate the conductivity of the intrinsic semiconductor. [**Ans.** $2.22\ \Omega^{-1}\text{m}^{-1}$]

25. In an *N*-type semiconductor the Fermi level lies 0.4 eV below the conduction band. If the concentration of donor atoms is double, find the new position of Fermi level. Assume $k_B T = 0.03$eV.

[**Ans.** 0.379 eV]

26. The potential difference of 0.35 V is applied across a germanium diode at 300 K. Calculate the forward current in terms of the reverse saturation current.

[**Ans.** $I_f = 7.52 \times 10^5\, I_s$]

27. The intrinsic carrier density at 300 K in silicon $1.5 \times 10^{16} / \text{m}^3$. If the electron and hole mobilities are 0.13 and 0.05 $\text{m}^2 V^{-1} s^{-1}$, respectively. Calculate the conductivity of (*a*) intrinsic silicon and (*b*) silicon containing 1 donor impurity atom per 10^8 silicon atoms.

[**Ans.** (*a*) $\sigma_i = 0.432 \times 10^{-3}\ \Omega^{-1}\text{m}^{-1}$ (*b*) $\sigma_{ext} = 10.4\ \Omega^{-1}\text{m}^{-1}$]

28. In a *P*-type semiconductor, the Fermi level lies 0.4 eV above the valence band. If the concentration of the acceptor atom is tripled, find the new position of the Fermi level.

[**Ans.** $E_v - E_F = 0.367$ eV]

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29. An intrinsic semiconductor material P has an energy gap 0.36 eV while material Q has energy gap 0.72 eV. Compare the intrinsic carrier densities in these two materials at 300 K. Assume that the effective masses of all the electrons and holes are equal to the free electron mass.

[**Ans.** $\frac{n_i(P)}{n_i(Q)} = 1015$]

30. A silicon diode has a saturation current of 0.1 pA at 20°C. Find its current when it is forward biased by 0.55 V. Find the current in the same diode when the temperature rises to 100°C.

[**Ans.** $I' = 0.51$ nA]

31. Calculate the density of impurity atoms that must be added to an intrinsic crystal to convert it to:

(i) $10^{-2}\Omega$m P-type silicon.

(ii) $10^{-2}\Omega$m N-type silicon. [**Ans.** $N_a = 4.63 \times 10^{21}\ m^{-3}$, (ii) $N_d = 1.3 \times 10^{22}/m^3$]

32. A 10 V Zener diode is used to regulate the voltage across a variable load resistor. The input voltage varies between 13 V and 16 V and the load current varies between 10 mA and 85 mA. The minimum Zener current is 15 mA. Calculate the value of resistance. [**GGSIPU May 2007 (3.5 marks)**]

33. In an N-type semiconductor, the Fermi level lies 0.4 eV below the conduction ba at 300 K. If the temperature is increased to 350 K, find the new position of the Fermi level. [**Ans.** 0.47 eV]

Chapter 9

Superconductivity

9.1 Introduction

A number of materials have been found to exhibit phenomenon of superconductivity. As the temperature is decreased, the resistivity at first decreases regularly, like that of any metal. At the so called **critical temperature**, a phase transition occurs and the resistivity suddenly drops to zero, as shown in Fig. 9.1.

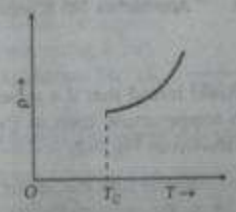

Fig. 9.1 Resistivity variation with temperature for a superconductor.

This property of superconduction was first discovered by Kammerlingh Onnes in 1911 while studying mercury at liquid helium temperature. He observed that at 4.2 K, the resistance of mercury suddenly vanishes. Later on this phenomenon was observed in many other metals and compounds. Till 1986, the maximum superconducting transition temperature was observed in Nb_3Ge at 23.2 K. In 1986, Bednorz and Müller discovered that the superconducting transition temperature in case of $La_{1.85}Sr_{0.15}CuO_4$ was 40 K and this landmark discovery resulted in increasing the superconducting transition temperature as of date upto 150 K. We will discuss Meissner effect, type I and type II superconductor, BCS theory, London's equations, properties of superconductors and applications of them in this chapter. A list of superconductors with transition temperature is given in Table 9.1.

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9.1 Superconducting transition temperatures of typical superconducting materials

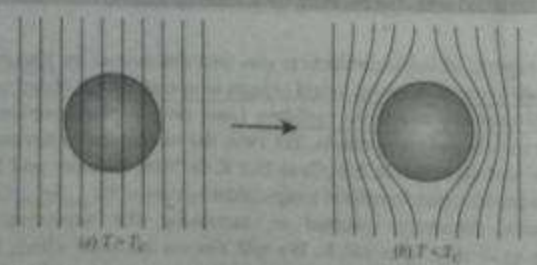
Compound	$T_C(K)$	Compound	$T_C(K)$
I_3[a]	1.4	K_3C_{60}	18.0
	4.2	Nb_3Sn	18.05
	6.0	Nb_3Ge	23.2
C	7.0	$Nd_{1.85}Ce_{0.15}CuO_4$	30/25
	9.5	$Ba_{0.7}K_{0.3}BiO_3$	35
	10.4	$La_{1.8}Sr_{0.2}CuO_4$	35
$_6B_4$	11.7	Rb_2CsC_6	35
(Hf)Cl	12 – 24	$TlSrLaCuO_5$	40
$_4C$	13.0	$YBa_2Cu_3O_7$	90
$_3Pb_{0.75}O_3$	13.0	$Bi_2(CaSr)_3Cu_2O_8$	90
$Mo_6S_{7.5}$	15.2	$Bi_2(CaSr)_4Cu_3O_{10}$	110
	16.0	$Tl_2Ca_2Ba_2Cu_3O_{10}$	125
	17.1	$HgCa_2Ba_2Cu_3O_8$	133[b]

[a] -ethylenedithiotetrathiafulvalene. [b] Approaches 165 K under pressure.

Meissner Effect

In 1933 Meissner and Ochensfeld found that if a superconductor is cooled in a magnetic field to below the transition temperature, then at the transition, the lines of magnetic induction are pushed out as shown in Fig. 9.2.

"The exclusion (expulsion) of magnetic flux from the interior of the superconductor as the material undergoes the transition to the superconducting phase is known as Meissner effect."



Meissner effect in a superconducting sphere cooled in a constant applied magnetic field ; on passing below the transition temperature the lines of induction **B** are ejected from the sphere.

Superconductors are perfectly diamagnetic with magnetic susceptibility

$$\chi = \frac{M}{H} = -1 \; [B = \mu_0(H + M) = 0 \quad \text{or} \quad H + M = 0 \quad \text{or} \quad M = -H \quad \text{or} \quad \frac{M}{H} = -1] \; ; \qquad ...(1)$$

no magnetic field can exist inside them under any circumstances.

The Meisser effect can be demonstrated as follows :

Consider two coils, a primary and a secondary, are wound on a superconducting material, which is kept above its critical temperature. The primary coil is connected to dc battery with plug key K. The secondary coil S in connected to a Ballistic Galvanometer (B.G.) (Fig. 9.3).

When the key K is closed, current flows in the primary coil which sets up a magnetic field in it. The magnetic flux links with the secondary coil. The change in magnetic flux of the secondary coil causes a momentary current flows through B.G. which shows a deflection. When the current in the primary coil reaches a steady value, the magnetic flux also becomes steady. Hence there will be change in the magnetic flux linked with the secondary coil at normal temperature. Now the temperature of the 'superconductor' is gradually decreased. As soon as the temperature reaches below the threshold temperature, the B.G. suddenly shows a momentary deflection, indicating a change in the magnetic flux linked with the secondary coil. This change in magnetic flux can be attributed to the expulsion of the magnetic flux from the body of the superconducting material as shown in Fig. 9.2(b). The magnetic expulsion continues for all values of $T < T_C$.

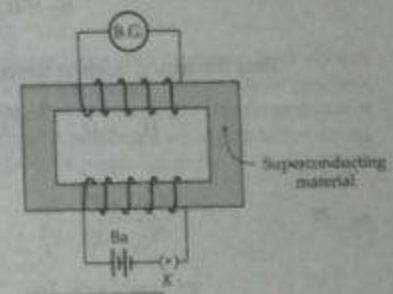


Fig. 9.3 Meissner effect exp.

Meissner effect shows that in the superconductor $B = 0$. The superconductor acts as an ideal diamagnet.

9.3 Magnetic Effects

In 1913, Kammerlingh Onnes observed that superconductivity vanishes if a sufficiently strong magnetic field is applied. The minimum magnetic field which is necessary to regain the normal resistivity is called **Critical Magnetic Field (H_c)**. When the applied magnetic field exceeds the critical value H_c, the superconducting state is destroyed and the material goes into normal state. Obviously the value of H_c varies with temperature. Fig. 9.4 shows the dependence of H_c on temperature ($T < T_c$), which indicates that the material remains superconducting until a corresponding critical magnetic field is applied.

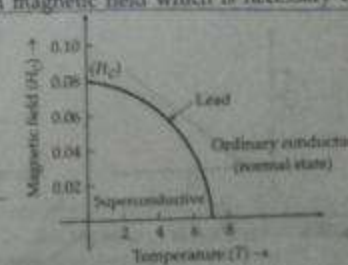


Fig. 9.4 Dependence of H_c on T.

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The magnetic field that causes a transition from the superconductive to the normal state at a given temperature is called the critical field H_C.

The critical field is a function of temperature and obeys the following parabolic law :

$$H_C = H_0\left(1-\frac{T^2}{T_C^2}\right) \quad ...(2)$$

where H_0 is the critical magnetic field at 0 K.

Example 9.1. The superconducting behaviour of a specimen is destroyed at field values of 1.0×10^5 and 2.0×10^5 Amp/m at 10 K and 12 K respectively. Calculate the transition temperature of the specimen. **[GGSIPU, May 2007 (2-5 marks)]**

Solution. The temperature dependence of H_C is given by

$$H_C = H_0\left[1-\left(\frac{T}{T_C}\right)^2\right]$$

Thus the critical fields at temperatures T_1 and T_2 are

$$H_{C_1} = H_0\left(\frac{T_C^2-T_1^2}{T_C^2}\right) \quad \text{and} \quad H_{C_2} = H_0\left(\frac{T_C^2-T_2^2}{T_C^2}\right)$$

or $$\frac{H_{C_1}}{H_{C_2}} = \frac{T_C^2-T_1^2}{T_C^2-T_2^2} \quad \text{or} \quad \frac{1.0\times10^5}{2.0\times10^5} = \frac{T_C^2-(10)^2}{T_C^2-(12)^2}$$

$$T_C^2-144 = 2T_C^2-200$$

$$T_C^2 = 56 \quad \Rightarrow \quad T_C = 7.5\,\text{K}$$

9.4 Isotope Effect

It has been observed that the critical temperature of superconductors decreases with increasing isotopic mass M. It is given by

$$T_C \propto \frac{1}{\sqrt{M}}$$

$$T_C\sqrt{M} = \text{a constant} \quad ...(3)$$

NOTE
Eq. (3) may also be written as $T_C M^{\beta}$ = constant and $0 < \beta < 0.5$]

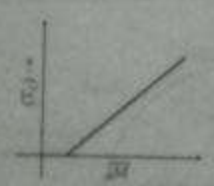

Fig. 9.5 Variation of T_C with $\sqrt{M}$.

The variation of critical temperature of an element with its isotopic mass is known as isotopic effect.

The variation of T_C with $\sqrt{M}$ is shown in Fig. 9.5.

Table 9.2 gives specific values of T_C for three different isotopes of Hg.

Table 9.2

Isotopes of Hg	Critical Temperature T_C (K)
199 Hg	4.164
200 Hg	4.153
204 Hg	4.126

For example, T_C for mercury varies from 4.185 K to 4.16 K when M changes from 199.5 a.m.u. to 203.4 a.m.u. The mean square of amplitude of atomic vibrations is proportional to $\sqrt{M}$ at low temperatures. The above equation suggests that the lattice vibrations are involved to cause superconductivity. The natural frequency of vibration of any body is given by the relation.

$$\omega = \sqrt{\left(\frac{\mu}{M}\right)} \quad ...(4)$$

where μ is the force constant. When a superconductor undergoes transition from normal state to the superconducting state, there will be a sudden change in the manner in which the current flows. This transition occurs at the critical temperature T_C. The dependence of T_C on the isotomic mass M has been experimentally confirmed which provides strong support for the electron-phonon interaction model of superconductivity.

The dependency of T_C on M is given by

$$T_C \propto M^{-\beta} \quad ...(5)$$

where β varies from 0 (Rubidium) to 0.5 (Cadmium).

Example 9.2. The critical temperature, T_C for Hg with isotopic mass 199.5 is 4.185 K. Calculate its critical temperature when its isotopic mass changes to 203.4.

Solution. The critical temperature varies with isotopic mass

$$T_C M^{1/2} = \text{constant}$$

where T_C is critical temperature and M is isotopic mass.

$$\therefore \quad T_{C_1} M_1^{1/2} = T_{C_2} M_2^{1/2}$$

We are given $T_{C_1} = 4.185$ K, $M_1 = 199.5$, $T_{C_2} = ?$

$M_2 = 203.4$

$T_{C_2} = 4.14$ K

9.5 Silsbee's Rule

When a current through the superconductor is increased beyond a critical value $I_C(T)$, the superconductor again becomes a normal conductor *i.e.*, the magnetic field which causes a superconductor to become normal from a superconducting state is not necessarily an external magnetic field, it may arise as a result of electric current flow in the conductor, the

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superconductivity may be destroyed, when the current exceeds the critical value I_C, which at the surface of the wire will produce a critical field H_C given

$$I_C = 2\pi r H_C \quad ...(6)$$

This is known as Silsbee's rule, r being radius of wire.

Example 9.3. *Calculate the critical current, which can flow through a long thin superconducting wire of aluminium of diameter 10^{-3} m. The critical magnetic field for Al is 7.9×10^5 Amp/m*

[GGSIPU April 2007 (2 Marks)]

Solution. We know from Silsbee's rule that

$$I_C = 2\pi r H_C$$

$$I_C = \frac{8\pi \times 10^{-3} \times 7.9 \times 10^5}{8} = 3.14 \times 7.9 \text{ Ampere} = 24.806 \text{ Ampere.}$$

9.6 Persistent Current

When a superconducting material is placed in a magnetic field at transition temperature, then an induced current flows in the material such that it opposes the applied field and ejects external magnetic lines of forces outside the specimen. (Fig. 9.6).

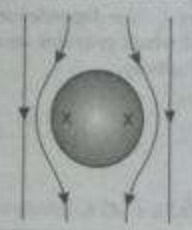

Fig. 9.6

Since resistivity of superconductor in superconducting phase is zero, thus the current due to induction in specimen persists as long as the field is on. Hence, due to induced current, specimen acquire a negative magnetic moment, which proves the diamagnetic property of a superconductor or Meissner effect.

The induced current (which is permanent and continuous due to zero resistance) produced in superconductor at transition temperature in presence of field, is called as **persistent current**.

9.7 Magnetic Levitation

According to Meissner's effect, a superconducting material in superconducting state shows a diamagnetic property *i.e.*, a negative magnetic moment develops in superconducting

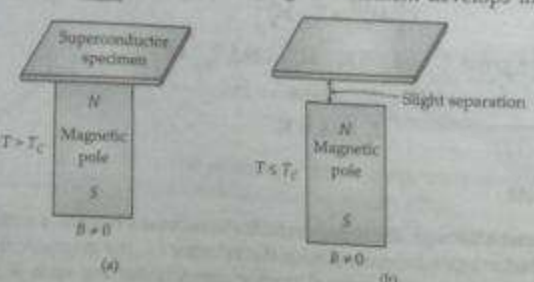

Fig. 9.7

specimen, when it is placed in magnetic field at transition temperature. Thus in superconducting state, the specimen moves towards lower field in non-uniform field.

Thus, when a superconducting specimen is placed on a magnetic pole and its temperature is cooled below critical temperature, then due to Meissner effect, negative magnetic moment is induced in specimen, which causes a slight separation between specimen and pole. This effect is called **magnetic levitation**. The maglev train is based on this effect.

9.8 Specific Heat of a Superconductor

The specific heat of normal metal varies with the absolute temperature

$$C_n(\gamma) = \gamma T + \beta T^3 \quad ...(7)$$

Thus the graph between C_n and T for normal metal is straight line. Below the critical temperature, in superconducting state, the electronic specific heat decreases with decreasing in temperature.

$$C_S = Ae^{-\Delta/k_B T}$$

$$\because \quad \Delta = bk_B T_C \; ; b = \text{constant}$$

$$C_S = Ae^{-(bT_C/T)} \quad ...(8)$$

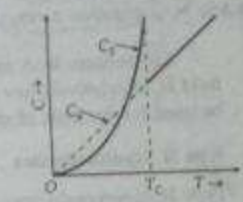

Fig. 9.8

9.9 Types of Superconductors

Superconductors are classified into two categories, namely type I superconductors and type II superconductors.

9.9.1 Type I Superconductors

Type I superconductors are also known as soft superconductors exhibit complete Meissner's effect. In type I superconductors, the transition from superconducting state to normal state in presence of a magnetic field occurs sharply at the critical value H_C, as illustrated in Fig. 9.9.

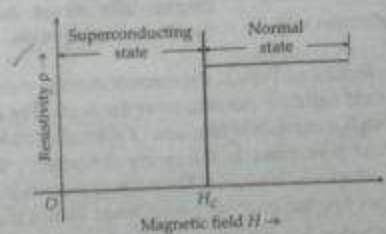

Fig. 9.9 Variation of ρ of a type I as a function of H.

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They are perfectly diamagnetic below H_C and completely expel the magnetic field from interior of the superconducting phase upto the critical field strength H_C. The magnetisation of material grows in proportion to the external magnetic field H and then abruptly drops to zero at the transition to the normally superconducting state as shown in Fig. 9.10.

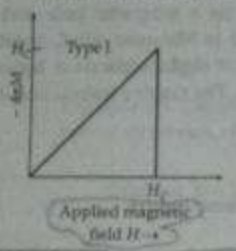

Fig. 9.10. Magnetization curve.

The magnetic field can penetrate only the surface layer and current can only flow in this layer. The resistance increases from zero to some value as in the case of a normal conductor. Consequently, type I superconductors are poor carriers of electrical current. Type I superconductors exist only in two states, normal and superconducting.

Aluminium, lead and indium are examples of type I superconductors. The critical field H_C is relatively low for type I superconductors. Hence type I superconductors cannot be used for the coils of strong electromagnets.

9.9.2 Type II Superconductors

Type II superconductors, also known as hard superconductors are quite different from type I superconductors. Type II superconductors are characterised by two critical fields H_{C_1} and H_{C_2} as shown in Fig. 9.11. For values of the field less than H_{C_1}, the material behave like a perfect diamagnet and the external magnetic flux is completely expelled from the interior of the material. At lower critical field H_{C_1}, the magnetic field lines begin penetrating the material.

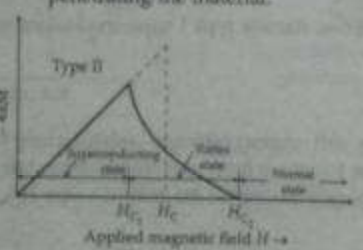

Fig. 9.11 Variation of magnetization with H.

As the magnetic field increases further, the penetration of external magnetic line of forces through the material increases. At the upper critical field H_{C_2} the induced magnetization vanishes completely and destroyed the super-conductivity. However, in the region between H_{C_1} and H_{C_2}, a type II superconductor is a mixed state known as **Vortex State** *i.e.*, the material is in a magnetically mixed state but electrically it is a superconducting state.

A distinguishing feature of Type II superconductors is that super currents arising in an external magnetic field can flow not only over the surface of a conductor but also in its bulk. It behaves as though it consists of filament of normal material serve as the path along which the magnetic field penetrates. In the centre of filaments there are no Cooper pairs and therefore superconductivity is absent. The normal regions are surrounded by vortices of super currents. A flux line together with its current vortex is called a **Fluxoid**. At H_{C_1}, fluxoids appear in the material and increase in number as the magnetic field is increased. At H_{C_2}, the fluxoids fill the entire specimen and superconductivity disappears.

Type II superconductors can carry larger currents when the magnetic field is between H_{C_1} and H_{C_2}.

Type II materials are very useful in application of creating a very high magnetic fields. Transition metals and alloys consisting of niobium, aluminium, silicon and vanadium exhibit type II superconductivity. Ceramic superconductors also belong to this category.

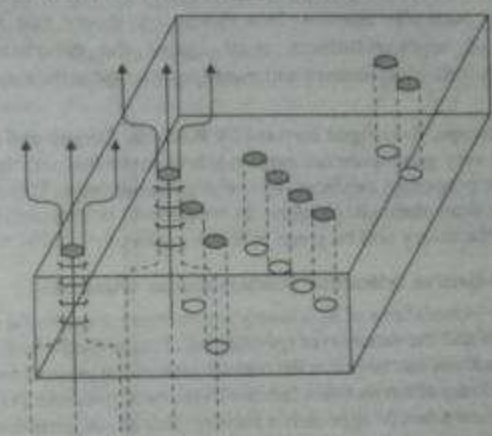

Fig. 9.12

A material can change from type I to type II on the substitution of some impurities. For example Pb is a type I superconductor with H_{C_1} = 600 G at 4 K. When 2 wt% indium is added to it, it becomes a type II superconductor with H_{C_1} = 400 G and H_{C_2} = 1000 G.

9.9.3 Comparison Between Type I and Type II Superconductors

S.No.	Type I Superconductor	Type II Superconductor
1.	These superconductors are called as soft superconductors	These superconductors are called hard superconductors
2.	The critical field value is very low.	The critical field value is very high.
3.	Only one critical field (H_C) exists for these superconductors.	Two critical fields H_{C_1} (lower critical field) and H_{C_2} (upper critical field) exist for these superconductors.
4.	These superconductors exhibit complete Meissner's effect.	These do not exhibit a perfect and complete Meissner's effect.
5.	These materials have limited technical applications because of very lower field strength.	These materials have wider technological applications because of very higher field strength value.
6.	Examples : Pb, Hg, Zn etc.	Examples : Nb_3Ga, Nb_3Si etc.

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9 BCS Theory of Superconductivity (Qualitative)

It is interesting to note that in the experimental survey of superconducting state, a sufficient qualitative information has been given without any reference in particular to any superconducting element *e.g.*, we have discussed magnetic properties (Meissner's effect), thermodynamic effect (isotopic effect), for all the superconducting elements without specifying any particular element. This means any theory that aims to explain the phenomenon of superconductivity must ignore the complicated feature which characterise any individual element and must be presented in the form of idealised model of metal.

The microscopic theory put forward by **Bardeen, Cooper** and **Schrieffer** (BCS) in 1957 provide a very good quantum explanation of superconductivity and accounts very well for all the properties exhibited by the superconductors. This theory involves the electron interactions through phonons as mediators. In this article only **qualitative description** of the theory will be given in following steps :

1 Electron-Lattice-Electron Interaction (Electron-phonon interaction)

The basis of the formulations of BCS theory are the two experimental conclusions namely the isotope effect and the variation of specific heat of superconductors. From isotope effect $T_C M^{1/2}$ = constant, one can infer that the transition resulting in superconducting state must involve the dynamics of ion motions, lattice vibrations or phonons. Further we note that T_c attains a value zero when M approaches infinity. This all suggests that *non-zero transition temperature is a consequence of the finite mass of the ions which can contribute phonons by their vibrations.* Frohlich and Bardeen in 1950, pointed out that an electron moving through a crystal lattice has a self energy accompanied by "virtual" phonons. This means that an electron moving through the lattice distorts the lattice and lattice, in turn, acts on the electron by virtue of electrostatic forces between them. The oscillatory distortion of the lattice is quantised in terms of phonons and so one can interpret the interaction between the lattice and electron as the constant emission and re-absorption (creation and annihilation) of phonons by the later.

> "BCS showed that the basic interaction responsible for superconductivity appears to be that of a pair of electrons by means of an interchange of virtual phonons".

This is explained below :

Suppose an electron approaches a positive ion core. It suffers attractive Coulomb interaction. Due to this attraction ion core is set in motion and consequently distorts the lattice. Smaller the mass of positive ion core, greater will be the distortion. Suppose towards that side another electron comes and sees this distorted lattice. Then the interaction between the two, the electron and the distorted lattice occurs, which in its effects lowers the energy of second electron. Thus we interpret the two electrons interact via the lattice distortion or the phonon field resulting in the lowering of energy for the electrons. The lowering of energy for the electron energy implies that the force between the two electrons is attractive. This type of interaction is called electron-lattice-electron interaction. This interaction is strongest when the two electrons have equal and opposite momenta and spins.

Since the oscillatory distortion of lattice, as pointed out earlier, is quantised in terms of phonons and therefore above interaction can also be interpreted as electron-electron interaction via phonons the mediator. Suppose, as shown in Fig. 9.13 an electron of wave vector **K** emits a virtual phonon **q** which is absorbed by an electron **K'**. **K** is thus altered as (**K**−**q**) and **K'** as (**K'** + **q**). The process being a virtual one energy need not conserved (the phonons because, as a consequence of uncertainty principle, their very short life time renders it unnecessary to conserve the energy in the process). In fact the nature of resulting electron-electron interaction depends on the relative magnitudes of the electronic energy ; the interaction is attractive.

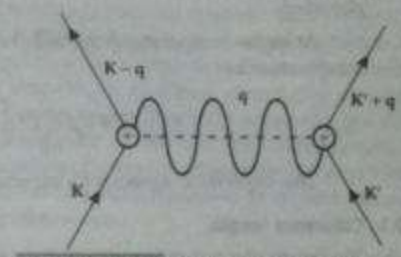

Fig. 9.13 Electron-phonon interaction.

9.10.2 Cooper Pair

The fundamental postulate of BCS theory is that the superconductivity occurs when an attractive interaction, mentioned above, between two electrons, by means of a phonon exchange, dominate the usual repulsive Coulomb interaction. ***Two such electrons which interact alternatively in the phonon field are called a Cooper Pair.*** These cooper pairs have certain aspects of single particles. The energy of the pair of electrons in the bound state is less than the energy of the pair in free state (electron separated). The difference of the energy of the two states is the binding energy of the Cooper pair and should, therefore, be supplied if the pair is broken. At temperature less than critical temperature, electron-lattice-electron interaction is stronger than electron-electron Coulomb interaction, and so the valence electrons tend to pair up. Pairing is complete at $T = 0$ K and is completely broken at critical temperature.

9.10.3 Existence of Energy Gap

The energy difference between the free state of the electron (*i.e.*, energy of individual electron-a case of normal state) and the paired state (the energy of paired electron-a case of superconducting state) appears as the **energy gap at the Fermi surface**. The normal electron states are above the energy gap and superconducting electron states are below the energy gap at the Fermi surface. Energy gap is a function of temperature unlike to case of constant energy gap in semiconductors and insulators. Since pairing is complete at 0 K, the difference in the energy of free and paired electron states (*i.e.*, normal and superconducting states) is maximum or in other words energy gap is maximum at absolute zero. At $T = T_C$, pairing is dissolved and energy gap reduces to zero.

Across the energy gap, there are many excited states for superconducting Cooper pairs. BCS theory thus predicts many electron ground states as well as excited states for the superconductor in the range 0 to T_c and in these states Cooper pairs are supposed to be in condensed state with a definite phase coherence. At critical temperature, this coherence disappears and the pairs are broken resulting in the transition of superconducting state to normal state.

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BCS theory predicted that the energy gap in superconductors at $T = 0$ K is $E_g(0) = 3.54\, k_B T_C$. This prediction can be verified by studying the absorption of e.m. radiation by superconductors. Only photons with energies more than or equal to E_g are absorbed.

At higher temperatures the BCS theory accounts for the variation of energy gap with temperature as

$$E_g \cong 1.74\, E_g(0)\left[1 - \frac{T}{T_c}\right]^{1/2} \quad \text{...(9)}$$

This expression agrees with experimental data on energy gap for various elements.

9.10.3 Coherence length

The paired electrons (Cooper pairs) are not scattered because of their peculiar property of smoothly riding over the lattice imperfections without over exchanging energy with them. Consequently, they can maintain their coupled motion upto a certain distance called Coherence length. The later is found to be of the order of 0.01 nm.

Definition

The maximum distance upto which the electron pairs are co-related to produced super-conductivity is called coherence length. → 0.01 nm

The property of a superconductor depends on the co-relation of electrons within a volume of ξ_0^3.

The ratio of London's penetration depth[1] to the coherence length is given by

$$R = \frac{\lambda}{\xi_0}$$

From BCS theory, a co-relation exists between energy gap and intrinsic coherence length as

$$\xi_0 = \frac{\hbar v}{2\Delta}$$

where 2Δ is the energy gap.

Example 9.4. *Calculate the energy gap (E_g) for Nb (T_C = 9.25 K) at $T = 0$ K and find the minimum photon wavelength needed to break the cooper pair.*

Solution. Since we know that at 0 K, energy gap

$$E_g = 3.54\, k_B T_c$$
$$= 3.54 \times 1.38 \times 10^{-23} \times 9.25 \text{ J}$$
$$= 4.25 \times 10^{-22} \text{ J}$$

and
$$\lambda_{min} = \frac{hc}{E_g} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{4.22 \times 10^{-22}} \text{ m}$$
$$= 4.29 \times 10^{-4} \text{ m}$$

1. London penetration depth will be discussed in article 9.11.3.

9.11 Electrodynamics of Superconductors : London Equations

As we know that the magnetic properties of a superconductor can be explained in ge terms by considering the superconductor to be perfect diamagnetic. Accordir Meissner's effect, a superconductor will completely eject out magnetic lines yet we sh account for the penetration of magnetic field into the surface of **superconductor**. I really been found that the penetration depths are much larger than the atomic distanc order to explain this penetration, it is desirable to modify a consecutive equati electrodynamics, say Ohm's law, rather than to modify Maxwell's equations. We alr know that Maxwell's equations are inadequate to explain the electrodynami superconductors *i.e.*, to account for the condition **E** = 0 and **B** = 0 together. London bro (F.London and H.London), in 1935 derived two field equations to explain the s conducting state of matter by modifying Ohm's Law.

The way in which the entropy varies with temperature suggests that the numb normal conduction electrons decreases below transition temperature, superconducting electrons increase. Therefore London brothers, basing on assum that there are two types of conduction electrons in a superconductor-namely *super-electrons* and *normal electrons*, put forward the idea that *at any temperature the s super electrons and the normal electrons is equal to the conduction electrons density in mate the normal state*. Further the super electrons are not subjected to any lattice scatterin are merely accelerated in an electric field.

As pointed out above, at 0 K superconductor consists only super electrons but temperature increases the proportion of the normal electrons compared to super ele increases until at transition temperature, all electrons are normal.

9.11.1 First London Equation

Let n_s be the super electron density at a temperature less than the transition temper Then the super current density will be

$$\mathbf{J}_S = -en_S\mathbf{v}_S$$

where $\mathbf{v}_S$ is the drift velocity of superelectrons. Since super electrons are not subjec any lattice scattering, they are continuously accelerated by the electric field. With a on super electron being $e\mathbf{E}$, we can write the equation of motion

$$m\frac{d\mathbf{v}_S}{dt} = -e\mathbf{E}$$

Differentiating Eq. (10), with respect to time

$$\frac{d\mathbf{J}_S}{dt} = -en_S\frac{d\mathbf{v}_S}{dt}$$

which gives with Eq. (11)

$$\frac{d\mathbf{J}_S}{dt} = \frac{n_S e^2}{m}\mathbf{E}$$

This is the **first London equation**, which describes the absence of resistanc equation shows that it is possible to have steady currents in the absence of electric fie

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$\mathbf{E} = 0$, $\mathbf{J}_s$ is finite and steady) which is the phenomenon of superconductivity. The corresponding expression for normal current density is

$$\mathbf{J}_N = \sigma \mathbf{E} \qquad ...(14)$$

which shows that no current is possible in the absence an electric field which is in line with the behaviour of the material in the normal state.

9.11.2 Second London's Equation

From Maxwell's third equation, we write

$$\text{Curl } \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

or

$$\nabla \times \mathbf{E} = -\mu_0 \frac{d\mathbf{H}}{dt} \qquad ...(15)$$

Taking curl of Eq. (13), we can write

$$\nabla \times \frac{d\mathbf{J}_S}{dt} = \frac{n_S e^2}{m} (\nabla \times \mathbf{E})$$

$$= -\frac{\mu_0 n_S e^2}{m} \frac{d\mathbf{H}}{dt} \qquad \text{[On applying Eq. (10)]} \qquad ...(16)$$

On integrating Eq. (15), we write as

$$\text{Curl } \mathbf{J}_s = -\frac{\mu_0 n_S e^2}{m} (\mathbf{H} - \mathbf{H}_0) \qquad ...(17)$$

where $\mathbf{H}_0$ is a constant of integration. Since Meissner effect prohibits magnetic fields inside the superconductor, $\mathbf{H}_0$ must be zero. Therefore

$$\text{Curl } \mathbf{J}_s = -\frac{\mu_0 n_s e^2}{m} \mathbf{H} \qquad ...(18)$$

which is **second London equation**.

9.11.3 Calculation of penetration depth

Eq. (18) explains Meissner effect. We shall now see whether equation predicts the penetration of super-current and magnetic flux in a super conductor or not.

Taking curl of Eq. (18)

$$\text{curl curl } \mathbf{J}_s = -\frac{\mu_0 n_S e^2}{m} \text{curl } \mathbf{H}$$

$$\text{grad div } \mathbf{J}_S - \nabla^2 \mathbf{J}_S = -\frac{\mu_0 n_S e^2}{m} \text{curl } \mathbf{H}$$

since div $\mathbf{J}_s$ is zero, we find that

$$\nabla^2 \mathbf{J}_S = \frac{\mu_0 n_S e^2}{m} \text{curl } \mathbf{H} \qquad ...(19)$$

From Maxwell's 4th equation

$$\text{curl } \mathbf{H} = \mathbf{J}_N + \mathbf{J}_S + \frac{d\mathbf{D}}{dt}$$

where $\mathbf{D}$ is electric displacement vector. For direct current case, under consideration, we can neglect $\mathbf{J}_N$ and $\frac{d\mathbf{D}}{dt}$ (displacement current) so that

$$\text{curl } \mathbf{H} = \mathbf{J}_S \qquad ...(20)$$

From Eqs. (19) and (20), we find

$$\nabla^2 \mathbf{J}_S = \frac{\mu_0 n_S e^2}{m} \mathbf{J}_S$$

or

$$\nabla^2 \mathbf{J}_S = \frac{\mathbf{J}_S}{\lambda^2} \qquad ...(21)$$

where

$$\lambda^2 = \frac{m}{\mu_0 n_S e^2} \qquad ...(22)$$

The parameter λ has the dimension of length and is called **penetration depth**. In one-dimension, Eq. (21) can be expressed as

$$\frac{d^2 \mathbf{J}_S}{dx^2} = \frac{\mathbf{J}_S}{\lambda^2} \qquad ...(23)$$

the solution of which can be written as

$$\mathbf{J}_S = A_1 e^{x/\lambda} + A_2 e^{-x/\lambda} \qquad ...(24)$$

where A and B are the two constants. x represents the distance into the metal from the surface. The first factor $A_1 e^{x/\lambda}$ of above equation predicts that $\mathbf{J}_s$ increases with x which is contrary to the fact. Therefore only second factor of Eq. (24) would lead appropriate solution. Thus

$$\mathbf{J}_s = A_2 e^{-x/\lambda}$$

or $x = 0$, let $\mathbf{J}_S = \mathbf{J}_0$

then

$$\mathbf{J}_S = \mathbf{J}_0 e^{-x/\lambda}$$

when

$$x = \lambda, \frac{\mathbf{J}_S}{\mathbf{J}_0} = \frac{1}{e} \qquad ...(25)$$

which defines the penetration depth as the distance into the superconductor at which the current value falls to $\left(\frac{1}{e}\right)$ of it value at the surface $\mathbf{J}_0$.

The order of penetration depth of magnetic field is about 10 – 100 nm.

The penetration depth also varies with temperature as visualised in Fig. 9.14.

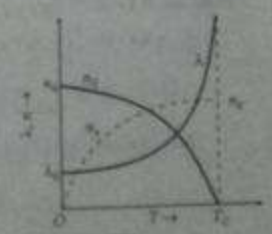

Fig. 9.14

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The variation can be expressed as

$$\lambda(T) = \frac{\lambda(0)}{\left(1-\frac{T^4}{T_c^4}\right)^{1/2}} \quad ...(26)$$

where $\lambda(T)$ is the value at temperature T and $\lambda(0)$ is the value at absolute zero.

Since the London penetration depth and number of superelectrons n_s is inversely related to each other and is also temperature dependent. Therefore an equation similar to Eq. (26) can be obtained for superelectrons *i.e.*,

$$n_s = n_0\left(1-\frac{T^4}{T_c^4}\right) \quad ...(27)$$

The variation of superelectron (n_s) and normal electrons with temperature is shown in Fig. 9.14.

Drawback of the Theory

Though the London's theory explains the two conditions $\mathbf{E}=0$ and $\mathbf{B}=0$, characterising the superconducting state but it does not gives any insight into the underlying electronic processes in superconductors. BCS theory gives better understanding of superconducting phenomenon.

Example 9.5. *The penetration depth of mercury at 3.5 K as 750 Å. Estimate the penetration depth at 0 K.* **[GGSIPU May 2007 (2.5 Marks)]**

Solution. Given $\lambda(3.5) = 750$ Å, $T = 3.5$ K, $\lambda(0) = ?$

We know that $T_c = 4.12$ K (for mercury)

Then $$\lambda(0) = \lambda(T)\left[1-\left(\frac{T}{T_c}\right)^4\right]^{1/2} = 750\left[1-\left(\frac{3.5}{4.12}\right)^4\right]^{1/2} \text{ Å} = 1033 \text{ Å}$$

Example 9.6. *Find London's penetration depth for lead having superconducting electron density of $3\times10^{28}\,m^{-3}$ at 5 K. The transition temperature for lead is 7.22 K.* **[GGSIPU April 2008 (2 Marks)]**

Solution. We know that $\lambda^2 = \frac{m}{\mu_0 n_s e^2}$

with $m = 9.1\times10^{-31}$ kg; $n_s = 3\times10^{28}\,m^{-3}$

$\mu_0 = 4\pi\times10^{-7}$ H/m; $e = 1.6\times10^{-19}$ C

$$\lambda^2 = \frac{9.1\times10^{-31}}{4\pi\times10^{-7}\times3\times10^{28}\times(1.6\times10^{-19})^2} \Rightarrow \lambda \simeq 30 \text{ nm}, \lambda(0) \simeq 30 \text{ nm}$$

For evaluating $\lambda(T)$ at any temperature, we will use

$$\lambda(T) = \lambda(0)\left[1-\left(\frac{T}{T_c}\right)^4\right]^{-1/2} = 30\text{ nm}\left[1-\left(\frac{5}{722}\right)^4\right]^{-1/2} = 34.18 \text{ nm}.$$

9.12 Properties of Superconductors

1. At room temperature, superconducting material have greater resistivity than other.
2. The presence of trace amount of pure magnetic substance in the superconductors severally lowers the transition temperature.
3. By increasing the pressure, the transition temperature can be increased for ultra low transition temperature specimen.
4. In superconducting state there is no change in the crystal structure, as found by X-rays diffraction method.
5. Entropy increases on going from superconducting state to normal state because entropy is a measure of the disorder and superconducting state is more ordered than the normal state.
6. For an ideal or type I superconductor, there is a marked drop in thermal conductivity when the superconductivity state sets in. For non-ideal or type II superconductors, in some cases there is an increase of thermal conductivity in superconductivity state.
7. It has been observed that critical temperatures of superconductor varies with isotopic mass. The relation is

$$T_C M^{1/2} = \text{constant} \qquad \text{or} \qquad T_C \propto \frac{1}{M^{1/2}}$$

where M is isotopic mass. The dependence of T_C on the isotopic mass verifies that lattice vibration and hence electron-lattice-electron are deeply involved in superconductivity.

9.13 Applications of Superconductors

Table 9.3 Main applications of superconductivity.

Magnets	❑ High-field magnet applications ❑ Nuclear magnetic resonance (NMR), medical diagnostics and spectroscopy ❑ Ore refining (magnetic separators) ❑ Magnetic levitation ❑ Magnetic shielding ❑ Large physics machines (collider, fusion confinement, r.f. cavities)
Energy-related	❑ Production by magnetic fusion and magneto-hydrodynamics ❑ Energy storage (magnetic) ❑ Electrical power transmission
Transportation	❑ High-speed trains ❑ Ship-drive systems

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Electronics and small devices	❑ SQUIDS (Superconducting quantum interference devices) ❑ Josephson devices (square-law detector, parametric amplifier, mixer) ❑ Bolometer ❑ Electromagnetic shielding
Computers and information processing	❑ Semiconductor-superconductor hybrids (A-D converters) ❑ Active superconducting elements (FETs) ❑ Voltage standard ❑ Optoelectronics ❑ Matched filters

9.13.1 Superconducting Quantum Interference Device (SQUID)

It is basically a superconducting loop with a "weak link" to measure magnetic flux changes within the loop. SQUID instruments used to study tiny magnetic signals in the brain and elsewhere would be useless without superconducting magnetic shields.

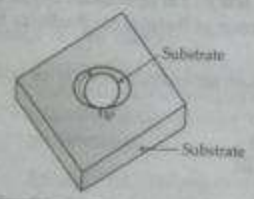

Fig. 9.15 Structure of typical SQUID.

SQUID consists of a thin film of superconducting material deposited on substrate. A weak link is formed by scraping away a portion of the thin film. It is region that has much lower critical current than the rest of the super-conducting ring. When the current in the link exceeds the critical currents, the link becomes normal. It allows fluxons penetrate the link, the current falls to critical value and the link reverts the superconducting state. Thus the weak link acts as a gate. It can be prepared such that it allows only single fluxon.

The critical current in the weak link changes periodically as the total flux through the area enclosed by superconducting loop changes. The current executes one cycle each time the flux changes by one fluxon. The periodic variations in the SQUID current can be sensed by a nearby coil. The changing magnetic field due to changes in SQUID current induced e.m.f. and current in the sensing coil. The e.m.f. induced in the coil can be used to drive an electronic counting circuit.

It is expected that SQUIDs made from oxide superconductors and operating at liquid nitrogen temperature (77 K) will be soon available commercially.

SQUIDs are the basis of most sensitive galvanometers. They are used in a variety of electrical and magnetic measurement applications. SQUIDS can detect small fractions of a flux quantum. SQUID magnetometers are used to detect the paramagnetic response in the liver, giving to doctors an accurate read out of iron in the body. They can measure the feeble magnetic fields of the order of 10^{-14} to 10^{-15} Tesla which are produced by electrical currents in heart and brain.

9.13.2 Maglev Vehicles

Vehicles which are magnetically levitated are called Maglev vehicles. Maglev vehicles (coaches) do not slide over steel rails but float on a four inch air cushion over a strongly magnetised track. Magnetic levitation is based on *Meissner Effect*.

Maglev coaches consist of superconducting magnets built into its base The vehicle floats on aluminium guideway by magnetic *levitation*. The magnetic levitation is produced by enormous magnetic repulsion between superconducting magnetic inside the vehicle and aluminium guideway. That is superconducting coils produce the magnetic repulsion in order to levitate the coaches. The currents in the aluminium guideway not only produce magnetic field to levitate the vehicle but also help in propelling the vehicle forward. Maglev coaches are provided with retractable wheels. The vehicle runs on the guideway with these wheels as aeroplane runs during its take off. Once the vehicle is levitated in air, the wheels are retracted into the body. As there does not exist mechanical friction, speeds upto 500 km/hr can be easily achieved.

9.13.3 Superconducting Magnets

Superconducting magnet is an electromagnet made by using coils of superconducting wires or cables. In the superconducting state the current flows without any resistive loss. The superconductors which are used for the application should have high critical fields. The most widely used materials are therefore type II superconductor like Nb-Ti and Nb_3Sn compounds. Other materials which may be useful in the future are $V_3(H_{c_2} = 350 \times 10^3$ Oersted) and $V_3Si(H_{c_2} = 350 \times 10^3$ Oersted). Only Nb-Zr and Nb-Ti alloys with critical fields of the order of 100 kilo Oersted are normally ductile and can be easily be wound into coils. The hard superconductors are used for all applications of superconducting coils and magnets.

Uses of Superconducting Magnets

1. Superconducting magnets are used in magnetic resonance imaging (MRI) which is technique employed to generate images of body cross-sections.
2. Superconducting magnets are used in NMR (Nuclear Magnetic Resonance) imaging in the medical field in the form of NMR tomography.
3. Superconducting magnets are used for containment of high temperature plasma in a closed region for controlled nuclear fusion.
4. Superconducting magnets are used in chemical industries for magnetic separation and refining.
5. They are used in large synchronous generators and motors.

9.13.4 Cryotrons

They are used in computers. A cryotron has a core '*A*' and a winding '*B*' made from different superconductors. For example, the core may be of Ta(tantalum) metal and winding may be either Nb(Niobium) or lead(Pb) as visualised in Fig. 9.16.

The temperature is initially kept lower than T_c for both materials. Now when a current of high value passes through coil *B* it produces a critical magnetic field for core *A*(tantalum). The core becomes a normal material with some resistance and hence the current in the core

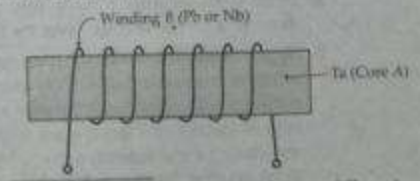

Fig. 9.16 A cryotron containing two different superconductors.

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decreases. Thus the current in core *A* may be controlled by the current in coil *B*. This combination of circumstances makes it possible to construct a flip-flop circuit which locks itself to one of two possible conduction modes. Such a device can be used in computers because it can triggered from one state to other by adding two more cryotrons. Circuit composed of networks of cryotron flip-flops have an extremely low impedance and are operated from a constant current source.

9.13.5 Microwave Filters for Cellular Telephone Networks

The approximate zero resistance of a superconductor is also playing important role in microwave application. High performance filters can be designed with lower loss and sharper "wings" than can be achieved using conventional metals like copper or gold even when they are too cooled to liquid nitrogen temperature. The characteristic of high temperature superconductor (HTS) microwave filters can also be rapidly switched by optical inputs, making them particularly useful for rapidly switched multiband operations. Switchable HTS microwave filters have already been tested in the USA to communicate on thirty two channels. The increased flexibility and security, that such filters can provide are likely to be just as important for commercial and civil information and communication technologies in general.

HTS microwave filters are already being used to increase the performance of cellular telephone networks in USA. It is demonstrated that the combination of a HTS thin film microwave resonator and a cryogenic low noise amplifier can improve the sensitivity of the network by a factor of two. The higher sensitivity achievable with superconducting HTS filters and cryogenic low-noise amplifiers in a city environment could reduce the microwave power output needed from hand held telephones which could give important health and environment advantages. Cellular telephone network systems are expected to be particularly important for developing countries since such system obviate the need for setting up an expensive infrastructure of individual telephone lines. If performance of microwave filter is better, then there will be more the telephone conversation, that can be handled simultaneously, and the fewer the transmitter stations that need to be built. All these advantages can easily into higher profit.

The use of superconductors to enhance systems performance is IT and communication technology is an obvious target area of HTS applications. Such applications will become increasingly important with the inexorable growth in future of IT requirement for broad bandwidth with speed in communication networks.

9.13.6 Other Applications

1. Since superconductor have zero resistance, electrical power transmission can be done at a lower voltage level by using wires made of superconducting materials.
2. Superconducting coils in transformers and electrical machines generates much stronger magnetic fields. Core made of ferri or ferromagnetic materials will not be required and eddy current losses and hysteresis losses will be eliminated. Hence the size of motors and generators will be drastically reduced. Hence, superconductors are likely to revolutionize the whole range of rotating electrical machines, making them smaller, lighter and highly efficient.

3. The Meissner effect can be embodies in bearings that would operate without frictional losses in all kinds of rotating machines.
4. Superconductors will radically changes the IC fabrication. Use of superconductors will make it possible to cram more circuits in a given area. As a result, electronic appliances and computers will be still smaller.
5. Several medical diagnostic equipments are now employing SQUIDS which detect very minute changes in the magnetic fields of a human brain or body.
6. The use of HTCs for generation, storage, transmission and transformation electrical power in electrical industry is being assessed by major industries all over the world. The three parameters, viz. critical temperature T_C, critical current I_C and critical field H_C are continually being improved to develop the power industry.
7. The semiconductor logic elements based on Josephson junction can operate at a speed of few micro seconds. Josephson junctions are therefore expected to increase the speed of superconductors.

Formulae at a Glance

1. Superconductor are perfectly diamagnetic with magnetic susceptibility

$$\chi = \frac{M}{H} = -1$$

$$[B = \mu_0(H + M) = 0 \text{ or } H + M = 0 \text{ or } M = -H]$$

2. Magnetic effects

$$H_C = H_0\left(1 - \frac{T^2}{T_C^2}\right)$$

H_C = critical field at any temperature

H_0 = critical field at zero K temperature

T_C = critical temperature

3. Isotope effect

$T_C M^{\beta}$ = constant and ; $0 < \beta < 0.5$

$\omega = \sqrt{\frac{\mu}{M}}$; μ = force constant

$T_C \propto M^{-\beta}$

$\beta = 0$ (Rubidium) and 0.5 (Cadmium)

4. Specific heat of superconductor $C_s = Ae^{-(\gamma T_C/T)}$

5. Silsbee's rule $I_C = 2\pi r H_C$

6. $E_g(o) = 3.54 k_B T_C$

7. $E_g(T) \approx 1.74 E_g(o)\left[1 - \frac{T}{T_C}\right]^{1/2}$

8. First London equation

$$\frac{dJ_s}{dt} = \frac{n_s e^2}{m} E$$

9. Second London equation

$$\text{Curl } J_s = -\frac{n_s e^2}{m} H$$

10. Penetration depth

(a) $\lambda = \left(\frac{m}{\mu_0 n_s e^2}\right)^{1/2}$

(b) $\lambda(T) = \lambda(0)\left[1 - \left(\frac{T}{T_C}\right)^4\right]^{-1/2}$

11. $n_s = n_0\left[1 - \left(\frac{T}{T_C}\right)^4\right]$

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Miscellaneous Numerical Problems for Practice

Example 1. The London's penetration depth for a superconductor at 3 K and 7.1 K are 39.6 nm and 173 nm respectively. Determine superconducting transition temperature.

Solution. Given $\lambda(T_1) = 39.6$ nm, $\lambda(T_2) = 173$ nm, $T_1 = 3$ K, $T_2 = 7.1$ K

We know that,
$$\frac{\lambda(T)}{\lambda(0)} = \left[1 - \left(\frac{T}{T_C}\right)^4\right]^{-1/2}$$

or
$$\left[\frac{\lambda(T_1)}{\lambda(T_2)}\right]^2 = \frac{T_C^4 - T_2^4}{T_C^4 - T_1^4}$$

or
$$\left(\frac{39.5}{173}\right)^2 = \frac{T_C^4 - (7.1)^4}{T_C^4 - (3)^4} \quad \text{or} \quad T_C = 7.194 \text{ K}$$

Example 2. Compute the superconducting electron density of mercury at 3.5 K. Given transition temperature of mercury is 4.22 K.

Solution. The normal current density in mercury can be found in terms of molecular weight and density, therefore

$$n_0 = \frac{N\rho}{M} = \frac{6.023 \times 10^{26} \times 13.55 \times 10^3}{2006} = 4.06 \times 10^{28} / \text{m}^3$$

$$n_s = n_0\left[1 - \left(\frac{T}{T_C}\right)^4\right] = 4.06 \times 10^{28}\left[1 - \frac{(3.5)^4}{(4.22)^4}\right] = 2.138 \times 10^{28} / \text{m}^3.$$

Example 3. Prove that Meissner effect contradicts the Maxwell's equation.

Solution. The resistivity of a superconductor is zero below T_C

i.e.,
$$\rho = 0 \qquad ...(i)$$

from Ohm's law, we have

$$\mathbf{J} = \sigma \mathbf{E} \Rightarrow \mathbf{E} = \frac{1}{\sigma}\mathbf{J} = \rho \mathbf{J} \qquad ...(ii)$$

From Eqs. (i) and (ii), we have

$$\mathbf{E} = 0 \quad \text{or} \quad |\mathbf{E}| = E = 0 \qquad ...(iii)$$

From Maxwell's equation, we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad ...(iv)$$

From Eqs. (iii) and (iv), we have

$$\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\mathbf{B} = \text{constant} \qquad ...(v)$$

But according to Meissner's effect for superconductor $\mathbf{B} = 0$. Hence, Meissner effect contradicts the Maxwell's equation.

Example 4. The lead material works as superconductor at a temperature of $T_C = 7.2$ K. If the constant characteristic of the lead material at 0 K is $H_0 = \frac{8 \times 10^5}{4\pi}$ A/m, what is the value of the magnetic field in the lead at 5 K?

Solution.
$$H_C = H_0\left[1 - \left(\frac{T}{T_C}\right)^2\right]$$

$$= \frac{8 \times 10^5}{4\pi}\left[1 - \left(\frac{5}{7.26}\right)^2\right] = \frac{4 \times 10^5}{4\pi} = 3.18 \times 10^4 \text{ A/m}$$

Example 5. Calculate the critical current for a superconducting wire of lead having a diameter of 1 mm at 4.2 K. Critical temperature for lead is 7.18 K and $H_C(0) = 6.5 \times 10^4$.

Solution. Given $H_0 = 6.5 \times 10^4$ A/m, $T_C = 7.18$ K, $r = 0.5 \times 10^{-3}$ m, $I_C = ?$, $H_C = ?$

Since we know that

$$H_C = H_0\left[1 - \frac{T^2}{T_C^2}\right] = 6.5 \times 10^4\left[1 - \frac{4.2}{7.18^2}\right]$$

$$= 6.5 \times 10^4 (1 - 0.34217) = 6.5 \times 10^4 \times 0.65783$$

$$H_C = 42.758 \text{ kA/m}.$$

$$I_C = 2\pi r H_c = 2 \times 3.14 \times 0.5 \times 10^{-3} \times 42.758 \times 10^3 \text{ A}$$

$$I_C = 134.26 \text{ A}.$$

Example 6. The critical temperature for a metal with isotopic mass 199.5 is 4.185 K. Calculate the isotopic mass if the critical temperature falls to 4.133 K.

Solution. The critical temperature varies with isotopic mass

$$T_C M^{1/2} = \text{constant}$$

Then
$$T_{C_1} M_1^{1/2} = T_{C_2} M_2^{1/2}$$

or
$$M_2^{1/2} = (199.5)^{1/2} \times 1.013$$

$$\Rightarrow \quad M_2 = 199.5 \times (1.013)^2$$

$$\Rightarrow \quad M_2 = 204.55.$$

Example 7. The critical temperature of Nb is 9.15 K. At zero kelvin the critical field is 0.196 T. Calculate the critical field at 6 K.

Solution. We know that

$$H_C = H_0\left(1 - \frac{T^2}{T_C^2}\right)$$

$$= 0.196\left[\left(1 - \frac{6}{9.15}\right)^2\right] = 0.196[1 - 0.43] = 0.196 \times 0.57$$

$$H_C = 0.112 \text{ T}.$$

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Example 8. For a super conducting specimen, the values of critical fields are 1.4×10^5 and 4.2×10^5 A/m for 14 K and 13 K. Calculate the transition temperature and critical fields at 0 K and 4.2 K.

Solution. Since, we know that

$$H_{C_1} = H_0\left(1-\frac{T_1^2}{T_C^2}\right) \quad \text{--(i)}$$

$$H_{C_2} = H_0\left(1-\frac{T_2^2}{T_C^2}\right) \quad \text{--(ii)}$$

$$\frac{H_{C_1}}{H_{C_2}} = \frac{T_C^2-T_1^2}{T_C^2-T_2^2}$$

$$\Rightarrow \quad \frac{1.4\times10^5}{4.2\times10^5} = \frac{T_C^2-14^2}{T_C^2-13^2}$$

$$\Rightarrow \quad \frac{1}{3} = \frac{T_C^2-196}{T_C^2-169}$$

$$\Rightarrow \quad T_C^2-169 = 3T_C^2-588$$

$$\Rightarrow \quad 2T_C^2 = 419$$

$$\Rightarrow \quad T_C^2 = \frac{419}{2} \quad \Rightarrow \quad T_C = 14.5\,\text{K}$$

From Eq. (i),

$$H_0 = \frac{H_{C_1}}{\left(1-\frac{T_1^2}{T_C^2}\right)} = \frac{1.4\times10^5}{\left[1-\left(\frac{14}{14.5}\right)^2\right]} = 20.7\times10^5\ \text{A/m}$$

Critical fields at 0 K and 4.2 K are calculated as

At 0 K $\quad H_C = H_0\left(1-\frac{T}{T_C}\right)^2 = H_0(1-0) = H_0 = 20.7\times10^5$ A/m

At 4.2 K $\quad H_C = 20.7\times10^5\left[1-\frac{(4.2)^2}{(4.5)^2}\right] = 18.9\times10^5$ A/m.

Example 9. Calculate the critical current density for 1 mm diameter wire of lead at 4.2 K. A parabolic dependence of H_C on T may be assumed. Give T_C for lead = 7.18 K and H_C for lead = 6.5×10^5 A/m.

Solution. Since we know that $H_C = H_0\left(1-\frac{T^2}{T_C^2}\right)$

$$H_C = 6.5\times10^5\left[1-\left(\frac{4.2}{7.18}\right)^2\right] = 4.28\times10^4\ \text{Amp/m}$$

Now $\quad I_C = 2\pi r H_C = \pi\times1\times10^{-3}\times4.28\times10^4$ Amp.

Critical current density

$$J_C = \frac{I_C}{A} = \frac{\pi\times1\times10^{-3}\times4.28\times10^4}{\pi\times\left(\frac{1}{2}\right)^2\times(10^{-3})^2}\ \text{Amp/m}^2$$

$$J_C = 1.71\times10^8\ \text{Amp/m}^2.$$

Example 10. Critical temperature of a superconductor when no magnetic field is present is T_C. Compute the temperature at which the critical field becomes half of its value at 0 K ?

Solution. Given that, $H_C = \frac{H_0}{2}$

We know that $H_C = H_0\left[1-\frac{T^2}{T_C^2}\right] \qquad H_C = \frac{H_0}{2}$

$$\frac{H_0}{2} = H_0\left[1-\frac{T^2}{T_C^2}\right]$$

$$\Rightarrow \quad \left(\frac{T}{T_C}\right)^2 = \frac{1}{2}$$

$$T = \frac{T_C}{\sqrt{2}} = 0.24\,T_C.$$

Example 11. The penetration depth of Sn at absolute zero is 3.4×10^{-6} cm. Find the value of penetration depth and superelectron density (n_S) at 3.5 K. Transition temperature of Sn is 3.72 K.

Solution. Given, $\mu_0 = 4\pi\times10^{-7}$ H/m, $m = 9.1\times10^{-31}$ kg, $e = 1.6\times10^{-19}$ C,

$\lambda(0) = 3.4^{-6}$ cm $= 3.4\times10^{-8}$ m, $T_C = 3.72$ K, $T = 3.5$ K

We have

$$\left[\frac{\lambda(T)}{\lambda(0)}\right]^2 = \left[1-\left(\frac{T}{T_C}\right)^4\right]^{-1}$$

$$\lambda(T) = \lambda(0)\left[1-\left(\frac{T}{T_C}\right)^4\right]^{-1/2} = 3.4\times10^{-8}\left[1-\left(\frac{3.5}{3.72}\right)^4\right]^{-1/2}$$

$$\Rightarrow \quad \lambda(T) = 7.30\times10^{-8}\,\text{m}.$$

Again, we have

$$\lambda^2 = \frac{m}{\mu_0 n_S e^2}$$

$$\Rightarrow \quad n_S = \frac{9.1\times10}{4\pi\times10^{-7}\times(7.30\times10^{-8}\times1.6\times10^{-19})^2} = 5.30\times10^{27}/\text{m}^3.$$

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Question Bank

Multiple Choice Questions

1. Two essential and independent properties of superconducting materials are
   (a) zero resistivity and diamagnetic (b) high resistivity and paramagnetic
   (c) zero resistivity and ferromagnetic (d) high resistivity and paramagnetic.
   [GGSIPU, May 2006 (2.5 marks)]
2. A superconductor has
   (a) a positive susceptibility
   (b) a negative susceptibility
   (c) becomes quite small as temperature approaches the transition temperature
   (d) become quite large as temperature approaches the transition temperature.
3. Critical magnetic field
   (a) does not depend on temperature
   (b) increases if temperature increases
   (c) increases if temperature decreases
   (d) does not depend on superconducting transition temperature.
4. Superconducting electron density
   (a) at absolute zero is finite
   (b) at absolute zero is zero.
   (c) at absolute zero is infinite.
   (d) increases as temperature increases from absolute zero to transition temperature.
5. Which of the following can be type I superconductor ?
   (a) Pb (b) Nb (c) Nb_3Sn (d) Nb_3Ge
6. Which of the following is a multiple connected superconductor ?
   (a) a solid sphere (b) a solid cylinder (c) a rod (d) a ring.
7. The isotope effect
   (a) is zero (b) is generally in the range 0.2-0.6
   (c) is generally in the range 0.5-1 (d) is generally greater than 1.
8. The temperature at which a conductor becomes a superconductor is called
   (a) superconducting temperature (b) Curie temperature
   (c) Onne's temperature (d) transition temperature.
9. Superconductors are generally
   (a) ferromagnetic and antiferromagnetic metals
   (b) monovalent metals
   (c) amorphous thin films of Be and Bi
   (d) thin film of barium titanate.

10. Hard superconductors observe
    (a) breakdown of Silsbee's rule
    (b) incomplete Meissner effect
    (c) high critical field and transition temperature
    (d) all the above.
11. Soft superconductors observe
    (a) Meissner effect (b) Silsbee's rule
    (c) both (a) and (b) (d) none of these.
12. In superconducting state
    (a) entropy increases and thermal conductivity decreases
    (b) entropy and thermal conductivity decreases
    (c) entropy and thermal conductivity increases
    (d) entropy decreases and thermal conductivity increases
13. Superconductivity results due to
    (a) crystal structure having infinite atomic vibrations at 0 K
    (b) crystal structure having no atomic vibrations at 0 K
    (c) all electrons having Fermi energy at 0 K
    (d) all electrons interacting in the superconducting state.
14. Transition temperature T_C and critical field H_C for a super conductor are related as
    (a) $H_C = H_0\left[1-\left(\frac{T}{T_C}\right)^2\right]$ (b) $T_C = T_0\left[\left(1-\frac{H_0}{H_C}\right)^2\right]$
    (c) $H_C = H_0(T_C - 1)$ (d) $H_C = H_0(T_C + 1)$
15. The favourable condition both for superconductivity and high resistance is
    (a) a weak electron-phonon interaction (b) a strong electron-phonon interaction
    (c) a weak phonon-phonon interaction (d) none of these.
16. The width of the energy gap of a superconductor is maximum at
    (a) 0 K (b) transition temperature
    (c) room temperature (d) none of these.
17. As isotopic mass of mercury decreases
    (a) the critical temperature decreases slightly (b) the critical temperature increases slightly
    (c) both (a) and (b) (d) remains constant.
18. Cooper pairs are formed
    (a) at very low temperature as the thermal energy is sufficient to form the Cooper pair
    (b) at very low temperature as the thermal energy is not sufficient to disrupt the binding
    (c) both (a) and (b)
    (d) none of these.

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19. The London's penetration depth is given by

(a) $\lambda = \left(\frac{m}{n_s \mu_0 e^2}\right)^{1/2}$ (b) $\lambda = \left(\frac{n_s}{m \mu_0 e^2}\right)^{1/2}$

(c) $\lambda = (n_s m \mu_0 e^2)^{1/2}$ (d) $\lambda = \left(\frac{n_s \mu_0 e^2}{m}\right)^{1/2}$.

20. Meissner effect is related to
(a) optical properties of superconductors
(b) magnetic properties of superconductors
(c) thermal properties of superconductors
(d) electrical properties of superconductors.

21. A superconducting material when placed in a magnetic field will
(a) attract the magnetic field toward its centre
(b) repel all the magnetic lines of forces passing through it
(c) attract the magnetic field but transfer it into a concentrated zone
(d) not influence the magnetic field.

22. If the external field is increased beyond a certain value, called critical field H_C a superconducting material changes to
(a) normal state (b) remains unaffected
(c) ferromagnetic state (d) antiferromagnetic state.

23. For a material to be considered as a superconductor, it has to exhibit
(a) only zero resistivity ($\rho = 0$)
(b) only Meissner effect ($\mathbf{B} = 0$ inside the superconductor)
(c) zero resistivity and Meissner effect both
(d) only Josephson effect.

24. The superconducting materials which exhibit two critical fields (lower H_{C_1} and upper H_{C_2}) and a mixed state are called :
(a) Type I or soft superconductors (b) Type II or hard superconductors
(c) Type I as well as Type II superconductors (d) None of the above.

25. The characteristic lengths of a superconductor are :
(a) only penetration depth (λ)
(b) only coherence length (ξ)
(c) penetration depth as well as coherence length
(d) None of above.

26. Between H_{C_1} and H_{C_2}, flux enters into type II superconductor and forms triangular lattice of flux lines, flux is called :
(a) Meissner's effect (b) Vortex lattice
(c) Josephson effect (d) Reentrant superconductivity.

27. T_C depends on the isotopic mass M, of the element for some of the superconductors, then relation between T_C and M is
(a) $T_C \propto \frac{1}{M\omega}$ (b) $T_C \propto M^{-\beta}$
(c) $T_C \propto \frac{M}{\omega}$ (d) $T_C \propto \frac{4\pi}{T} M^{\beta}$.

28. In a BCS theory of superconductivity, electron pair responsible for superconducting state is called :
(a) Boson (b) Fermion
(c) Cooper pair (d) None of the above.

Answers

1. (a)	2. (b)	3. (c)	4. (a)	5. (a)	6. (d)
7. (b)	8. (d)	9. (c)	10. (b)	11. (c)	12. (b)
13. (b)	14. (a)	15. (b)	16. (a)	17. (b)	18. (b)
19. (a)	20. (b)	21. (b)	22. (a)	23. (c)	24. (b)
25. (c)	26. (b)	27. (b)	28. (c)		

Theoretical Questions

1. Explain the phenomenon behind the process of Magnetic levitation. [GGSIPU, April 2005, 2007 (2.5 marks)]

2. Write short notes on :
(i) Applications of superconductors in medicine. [GGSIPU, April 2005, 2007 (2.5 marks)]
(ii) Type I and Type II superconductor. [GGSIPU, May 2007 (2.5 marks)]
(iii) Isotope effect
(iv) Coherence length
(v) Mixed state
(vi) Energy gap
(vii) Superconductivity
(viii) Superconducting magnets
(ix) Persistent current in superconductor.

3. Enunciate the difference between superconductors and normal conductors in terms of their energy gap. [GGSIPU, April 2005 (2 marks)]

4. Explain Meissner effect and flux penetration in superconductors. How were these phenomena explained on the basis of London's equation ? [GGSIPU, April 2005 Reappear (7 marks)]

5. Give a few applications of superconductors. [GGSIPU, April 2005 Reappear (3 marks)]

6. Why is superconductor called a "perfect diamagnet" ? [GGSIPU, May 2005 (2.5 marks)]

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7. In the context of superconductivity, briefly explain following :
   (i) Meissner effect
   (ii) Type I and Type II superconductors
   (iii) Josephson function [GGSIPU, May 2005 (2.5 marks)]
8. What is superconductivity ? What is the effect of magnetic field on a superconductor ? Explain the use of superconductor in a switching device. [GGSIPU, May 2006 (4.5 marks)]
9. Discuss in very brief two important applications of superconductors. [GGSIPU, May 2008 (2 marks)]
10. Distinguish between type I and type II superconductors. [GGSIPU, May 2008 (7.5 marks)]
11. What are Cooper pairs ? [GGSIPU, May 2008 (5 marks)]
12. Give the basic ideas of BCS theory. Explain how this theory provides adequate explanation of the phenomenon of superconductivity. [GGSIPU, May 2007 (6 marks)]
13. State and explain Meissner's effect. How do London's equations account for this effect ? [GGSIPU, May 2007 (7 marks)]
14. How cooper pairs formed ? Explain the BCS theory of superconductivity and discuss the existence of energy gap based on this theory. [GGSIPU, April 2008 (7 marks)]
15. Give justification. In a thin film Meissner effect is incomplete whereas a bulk superconductor exhibits complete Meissner effect. [GGSIPU, April 2008 (3 marks)]
16. Using Ohm's law show that zero resistivity does not lead to perfect diamagnetism in superconductivity. [GGSIPU, April 2007 (3 marks)]
17. Draw a well labeled magnetization curves for type I and type II superconductors. [GGSIPU, April 2007 (2 marks)]
18. Show that London's equation leads to Meissner effect. Hence explain that thin films do not exhibit Meissner effect. [GGSIPU, April 2007 (5 marks)]
19. On the basis of electron lattice interaction explain the phenomenon of Meissner effect and zero resistance in superconducting materials. How you will get perpetual current in a coil using the property of superconductivity
20. How the various observable phenomena of superconductors such as zero resistance and Meissner effect can be explained by BCS theory ? How will you use the property of superconductors in calibration of very small voltages ?
21. What are soft and hard superconductors ? How the energy gap concept is explained in superconductivity ?
22. Explain isotopic effect on superconductors.
23. Explain why good conductors can not be superconductors.
24. Give some important applications of superconductors.
25. What is SQUID ? Give its importance in superconductivity.
26. What are superconducting magnets ? Give their uses.
27. Describe how BCS theory explains super conductivity. Explain different types of superconductors. What are the applications of superconductivity ?
28. Discuss the highlights of BCS theory. How does a superconductor differ from a normal conductor ?
29. Compare the dependence of resistance on temperature of superconductor with that of a normal conductor. Describe briefly the formation of Cooper pairs.

30. Discuss in brief BCS theory of superconductivity. Explain the magnetic behaviour of Type I and Type II superconductors.
31. Define the critical magnetic field for a superconductor. How is it related to the transition temperature ?
32. What is Meissner effect ? Prove that Meissner effect and the disappearance of resistivity in a superconductor are mutually consistent.
33. Show that Meissner effect implies perfect conductivity, but vice versa is not true.
34. What are type I and type II superconducting materials ? Give three examples of each. Why type II materials are preferred for applications of superconductivity.
35. Discuss London's theory of superconductivity.

Numerical Problems

1. A superconducting lead has a critical temperature of 7.26 K at zero magnetic field and a critical field of 8×10^5 A / m at 0 K. Find critical field at 5 K. [Ans. 4.2×10^5 A / m]
2. Estimate the London penetration depth from the following data :

   Critical temperature = 3.7 K

   Super electrons density = 7.3 cm^{-3}

   Atomic weight = 118.7

   Effective mass $M^* = 1.9$ m

   [where m = mass of free electron] [Ans. $\lambda = 381$ Å]
3. The penetration depth λ of Hg at 3.5 K is about 750 Å. Estimate the value of (a) λ (b) n_s (superconducting electron density) as T leads to 0 K. [Ans. (a) 530 Å, (b) 10^{28} / m^3]
4. A superconducting tin has a critical temperature of 3.7 K is zero magnetic field and a critical field of 0.0306 T at 0 K. Find the critical field at 2 K. [Ans. 0.0217 T]
5. Using normal isotopic effect, calculate the transition temperature for samples of ^{201}Hg, ^{202}Hg and ^{204}Hg given that the transition temperature for ordinary mercury ; of an average atomic mass 200.59 is 4.153 K. [Ans. 4.149 K, 4.138 K, 4.118 K]
6. The London penetration depths for lead at 3 K and 7.1 K are respectively 396 Å and 1730 Å. Calculate the penetration depth at 0 K. [Ans. 7.193 K, 390 Å]
7. A specimen of V_3Ga has a transition temperature 14.5 K. At 14 K, the critical field is 14×10^5 Amp. turns/meter and at 13K, 4.2×10^5 Amp. turns/meter. Determine critical field at 4.2 K. [Ans. 14.5 K, 20×10^5 and 18.3 Amp turns/m]
8. The magnetic field intensity in the tin material is zero at 3.69 K and $\frac{3 \times 10^5}{4\pi}$ at 0 K. Compute the temperature of the superconductor if the field intensity is measured as $\frac{2 \times 10^5}{4\pi}$. [Ans. T = 2.13 K]
9. The magnetic field intensity in the mercury at T = 0 K is $\frac{4.1 \times 10^5}{4\pi}$ A / m and $\frac{3 \times 10^5}{4\pi}$ A / m at 2.3 K. Calculate critical temperature. [Ans. 4.15 K]

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10. The critical temperature for lead is 7.1 K. At 6 K the superconducting property disappear if it is subjected to a magnetic field of 4.67×10^4 A / m. Find H_0. [Ans. $H_o = 163.37 \times 10^3$ A / m]

11. The isotopes of lead of atomic mass 206 and 210 have T_C values 7.193 K and 7.125 K respectively. Calculate the value of β. [Ans. β = 0.4962]

12. Superconducting Nb has a critical temperature of 9.15 K at zero magnetic field and a critical field of 0.196 T at 0 K. Find critical field at 5 K. [Ans. H_C = 0.138 T]

13. The critical field for vanadium is 10^5 A / m at 8.58 K and 2×10^5 A / m at 0 K. Determine the T_C value. [Ans. T_C = 12.133 K]

14. Compute the critical current through a long thin superconducting wire of radius 0.5 mm. The critical magnetic field is 7.2kA / m. [Ans. I_C = 22.608 A]

15. Superconducting *Sn* has a critical temperature of 3.7 K at zero magnetic field of 0.0306 T at 0 K. Find the critical field at 2 K. [Ans. H_C = 0.22 T]

16. For mercury of mass number 202, the β value is 0.50, and T_C is 4.2 K. Find the transition temperature for the isotope of mercury of mass number 200. [Ans. T_{C_2} = 4.22 K]

17. Experiments on a sample show that at 12 K a critical field of 15 T exist and at 10 K the critical field is 18 T. Calculate the transition temperature and critical field at 0 K.

18. Critical temperature of sample with isotopic mass of 204.87. [Ans. T_C = 18.58 K]

19. London's penetration depth for a sample at 5 K and 7 K are 41.2 nm and 180.3 nm respectively. Find its transition temperature and penetration depth at 0 K. [Ans. λ(0) = 29.5 nm]

20. A lead wire has a critical magnetic field of 6.5×10^5 Amp / m at 0 K. The critical temperature is 7.18 K. At what temperature the critical field would be dropped to 4.5×10^5 Amp / m ? The diameter of the wire is 2 mm. What is the critical current density at that temperature ? [Ans. T = 5.19 K, $J_C = 9.0 \times 10^6$ Amp / m^2]

21. The critical temperature of mercury is 4.2 K.
    (a) Calculate the energy gap in electron volt at $T = 0$.
    (b) Calculate the wavelength of a photon whose energy is just sufficient to break up cooper pairs in mercury at $T = 0$. In what region of the electromagnetic spectrum are such photons found ? [Ans. (a) 1.8×10^{-22} J (b) $\lambda = 1.1 \times 10^{-3}$ m, microwave region]

22. How much current can a lead wire, 1 mm in diameter carry in its superconducting state at 4.2 K ? (Given T_C for Pb is 7.2 K and $H_C(o) = 0.08$(03 T). [Ans. I_C = 137 Ampere]

23. For a superconductor T_C = 3K and $n_s = 10^{28}$ / m^3. Find the penetration depth at 0 K and 1 K. [Ans. λ(0) = 530 Å, λ(1) = 531 Å]

Chapter 10

X-Rays

10.1 Introduction

The subject of X-rays is one of the best examples of the importance of research in pure science. In fact, with three months after Röntgen's discovery, X-rays were being put to practical use in hospitals in connection with surgical operations.

The original discovery of X-rays was made by German physicist, *Wilhelm Röntgen* on November 8, 1895. Röntgen found that a piece of barium platinocyanide placed at some distance from the discharge tube exhibits *fluorescence*. He was soon able to show that the observed fluorescence of barium platinocyanide is neither due to ultraviolet light nor due to cathode rays. It was due to a new kind of rays. These rays originated at the point where glass walls were struck by the cathode ray stream in tube. However, Röentgen did not succeed in ascertaining the nature of new-radiation and therefore, designed these rays as X-rays (X → unknown).

10.2 Production of X-Rays

Röntgen's early investigation showed that X-rays originate from that spot where the cathode rays strike the glass wall. This point was established when it was found that if the cathode rays are deflected by means of a magnet, the X-rays proceed from another spot, a new terminus of the cathode rays. Now classical electromagnetic theory tells us that a sudden deceleration of a beam of energetic electrons would result in the production of radiation of very short wavelength. We, therefore, realise that Röntgen's early observation can be generalised to the statement : *X-rays are produced whenever high speed electrons are stopped abruptly by allowing them to impinge upon a suitable target.* [*When a beam of fast moving electrons strikes upon a solid target, an invisible high penetrating radiation (electromagnetic radiation X-rays) is produced.*] X-rays tube therefore, consists essentially of a source of electrons, some means of accelerating these electrons and a target on which the electrons strike, X-rays are given out from the point where the electrons strike this target.

These are *two* general kinds of X-rays tubes :

- the gas tube, and
- the Coolidge tube.

We now briefly talk about these two.

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10.2.1 The Gas Tube

The gas tube is basically a low pressure discharge tube. It has a concave cathode C. The target T is placed at the centre of curvature. The gas in the tube contains a few electrons and positive ions strike the cathode and release electrons. The electrons coverage towards the target. On the striking the target, the electrons energy is converted into X-rays.

The gas tube has two important drawbacks. The intensity and frequency of X-rays obtained can not be easily and independently controlled. These depend on the gas pressure. This is so because the gas pressure determines the voltage needed to maintain discharge. This voltage in turn determines the energy of electrons striking the target. The electron energy determines the frequency of the emitted X-rays.

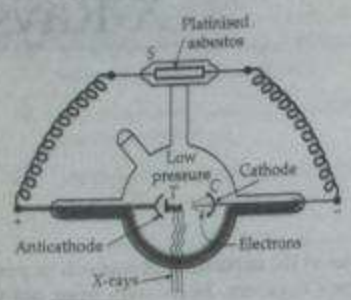

Fig. 10.1 Gas Tube

It is found that due to **absorption** of gases by the walls of the discharge tube, the tube becomes increasingly difficult to operate. A piece of platinised *asbestos* is made to release the absorbed gases. Because of the above mentioned drawbacks, this tube has become almost obsolete and is not in practice use any more.

10.2.2 The Coolidge Tube

The modern type of X-rays tube designed by Coolidge is shown in Fig. 10.2.

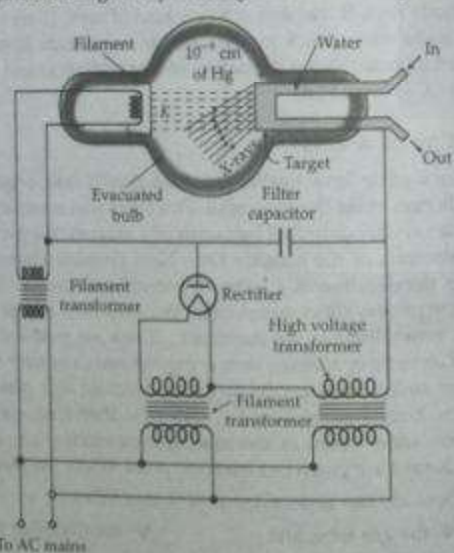

Fig. 10.2 Coolidge Tube

The source of electrons in this tube is a *heated filament*. The tube is evacuated to the possible (pressure $\approx 10^{-6}$ cm of Hg). The electrons emitted from the filament are accelerated towards the anode by a high potential difference. Within the anode block a target is placed. The electrons are focused on this target. Due to absorption of electrons the anode temperature rises. The anode is usually, therefore, cooled by circulating water around it.

The high voltage between the cathode and the anode is applied with the help of a step-up-transformer.

In the Coolidge tube, the intensity and the frequency of X-rays can be easily controlled. The intensity of X-rays depends on the number of striking the target per second. This number can be increased or decreased by increasing or decreasing the filament voltage. The frequency of X-rays emitted depends on the voltage between the anode and cathode. We can easily write down expression for the minimum wavelength of X-rays emitted.

Let V be the tube voltage. The maximum energy of electrons striking the target is eV. At best, the whole of this converted into a quanta of frequency ν_{max}. Therefore

$$E_{max} = eV = h\nu_{max} \quad \text{or} \quad eV = \frac{hc}{\lambda_{min}}$$

or

$$\lambda_{min} = \frac{hc}{eV} \qquad ...(1)$$

Thus the minimum wavelength of X-rays, provided by a Coolidge tube, is inversely proportional to the accelerating voltage applied between the cathode and the anode.

Example 10.1. *An X-rays tube operated at 40 kV emits continuous X-rays spectrum with a short wavelength limit $\lambda_{min} = 0.310$ Å. Calculate the value of Planck's constant.*

[GGSIPU, May 2007 (2.5 marks)]

Solution. We have $\lambda_{min} = \frac{hc}{eV} \Rightarrow h = \frac{eV\lambda_{min}}{c}$

given $e = 1.6 \times 10^{-19}$ C, $V = 40$ kV $= 4.0 \times 10^4$ V

$\lambda_{min} = 0.310$ Å $= 0.310 \times 10^{-10}$ m, $c = 3 \times 10^8$ m/s

$$h = \frac{(1.6 \times 10^{-19}\,\text{C}) \times (4.0 \times 10^4\,\text{V}) \times (0.31 \times 10^{-10}\,\text{m})}{(3 \times 10^8\,\text{m/s})} = 6.61 \times 10^{-34}\,\text{J s}.$$

Example 10.2. *A Coolidge tube operates at 50 kV. Find :*

(i) the maximum velocity of electrons striking the anticathode,

(ii) minimum wavelength of X-rays generated. Given $e = 1.6 \times 10^{-19}$ C and $m = 9.1 \times 10^{-31}$ kg.

Solution. (i) $\frac{1}{2}mv_{max}^2 = eV$ or $v_{max} = \sqrt{\frac{2eV}{m}}$

$$\therefore \quad v_{max} = \left[\frac{2 \times 1.6 \times 10^{-19} \times 50 \times 10^3}{9.1 \times 10^{-31}}\right]^{1/2} = 1.326 \times 10^8\,\text{m/s}$$

(ii) $\lambda_{min} = \frac{12400}{V} = \frac{12400}{50 \times 10^3} = 0.24$ Å.

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10.3 Continuous and Characteristic X-Rays

When radiations emitted from an X-rays tube are analysed in terms of wavelengths and curves are plotted between intensity of radiation and wavelength, the curves similar to as visualised in Fig. 10.3 are obtained.

It is evident from Fig. 10.3, that at lower values of V (potential difference between cathode and anode), smooth and continuous curves starting from a minimum wavelength (λ_{min}) and extending upto very high wavelength are obtained. As V is increased, λ_{min} shifts towards lower wavelength side. *This continuous spectrum of emitted radiation is termed as* ***continuous X-rays****. When V is further increased, bumps start appearing at fixed wavelengths in I versus λ curve. These bumps at fixed wavelength are called the* ***characteristic X-rays****. The study of the continuous X-rays spectrum is more important in the field of medicine, surgery and industry because the greater part of the energy is to be found in that part, while study of characteristic X-rays is important scientific and research point of view.*

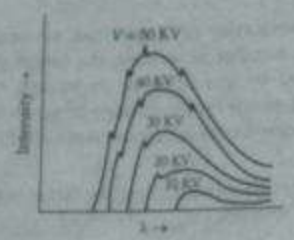

Fig. 10.3 Intensity variation with wavelength of emitted

10.3.1 Continuous X-Rays

The radiation emitted from X-rays tube which has smooth variation of intensity with wavelength is called continuous X-rays. The shape of continuous X-rays is independent of the material of the target. These curves have a limit at short wavelength (λ_{min}). The value of λ_{min} depends upon to potential difference (V) between cathode and target, but does not depend upon the material of the target. The value of λ_{min} varies with V, according to following relation

$$\lambda_{min} = \frac{K}{V} \quad ...(2)$$

Therefore a plot between λ_{min} and $\frac{1}{V}$ is a straight line as visualised in Fig. 10.4.

Eq. (2) was first obtained from experimental data by Duane and Hunt in 1915 and is referred to as **Duane-Hunt Law.**

The mechanism of continuous X-rays emission is as follows. It is well known that when electron or any charged particle is decelerated, they emit electromagnetic radiations. In X-rays tube, electrons are emitted from cathode and proceed toward target. While moving, they are continuously accelerated by potential given to target which is several kilo volts. By the time these electrons reach the target, they acquire an energy (E), which is given by the expression

$$E = eV \quad ...(3)$$

where V is the potential difference between the cathode and the target. Whole of this energy of

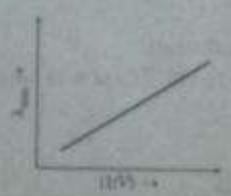

Fig. 10.4 Variation of minimum X-rays wavelength with target potential in the X-rays tube.

electrons is in the form of kinetic energy. The target is a metal of high atomic number. A metallic solid can be pictured as sea of conduction electrons in which ions are placed at specific points. The microscopic arrangement near the surface of the target is similar to one shown schematically in Fig. 10.5.

When these electrons strikes the target, they penetrate into it. While penetrating into the target, they have to pass through the cloud of conduction electrons. Further when an electron strikes on ion, it penetrates into electronic orbits surrounding the nucleus. In this process they encounter the repulsive force electrons. Thus they are decelerated and lose their energy. This energy is emitted in the form of X-rays. The frequency of emitted X-rays photon (ν) will be

$$\nu = \frac{E - E'}{h} \quad ...(4)$$

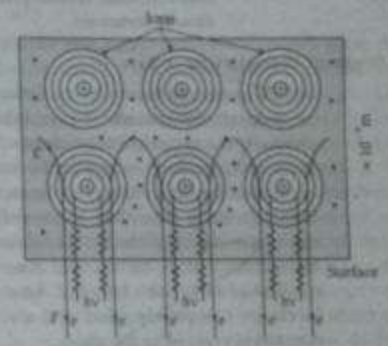

Fig. 10.5 Emission of continuous X-rays.

where E and E' are the energies of the electron when it enters the target and comes out of the charge cloud surrounding the nucleus as shown in Fig. 10.5. Depending upon the deflected path of the electron, its energy E' can vary from 0 to E in different processes. Thus frequency of emitted photon can vary from lowest frequency falling in the X-rays range. ν_{max} is obtained when E' becomes zero. Thus,

$$\nu_{max} = \frac{E}{h} = \frac{eV}{h} \quad ...(5)$$

and

$$\lambda_{min} = \frac{c}{\nu_{max}} = \frac{ch}{eV}$$

$$= \frac{K}{V} = \frac{12400}{V} \text{ Å} \quad ...(6)$$

where $K = \frac{ch}{e} = 12.400 \times 10^{-10}$ V-m, which obtained by putting values of c, h and e. For $V = 50$ kV, $\lambda_{min} = 0.248$ Å $= 2.48 \times 10^{-9}$ m.

Eqs. (5) and (6) are mathematical form of Duane-Hunt law.

10.3.2 Characteristic X-Rays

It has been pointed out earlier, in the radiation from X-rays tube, bumps appear at definite wavelengths in the background of smooth continuous X-rays spectra. These bumps are referred to as *characteristic X-rays*.

The two important points about these bumps are :

(i) They start appearing only when the value of the potential difference across the tube is higher than certain critical value (V_c). This critical potential is highly dependent on the material of anticathode.

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(ii) The definite wavelengths or frequencies of characteristic X-rays do not depend in any way on the applied potential difference but are intimately connected with the material of the anticathode. In fact, these line spectra are characteristic features of the anticathode material and are related with its atomic structure.

Origin of Characteristic X-Rays

Bohr's quantum theory of radiation gives a simple but adequate explanation for the emission of characteristic X-rays. When high velocity electrons from the cathode strike the atoms of the target element, most of the electrons are retarded by outer electrons of the atom and lose their energy in the form of continuous X-rays and are deflected back. But because of the high velocity of the incident electron (1/10th of the velocity of light), some of these electrons (less than a percent) enter deep into the atom and collide against an electron in the inner shell. This collision is inelastic in nature and as a result the inner orbit takes energy from the incident electron and is knocked out from the shell. The incident electron as well as this knocked out electron, do not find a place to stay in the atom as outer levels are completely filled and are thus deflected out of the atom. Thus a vacancy of electron is created in the atom.

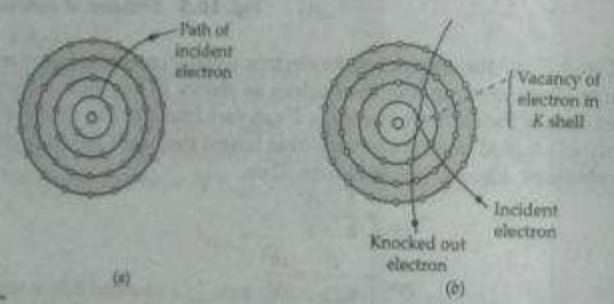

Fig. 10.6 Emission of characteristic X-Rays.

To create this vacancy, which is prerequisite for a characteristic X-rays emission, it is essential that the incident electron must have energy equal to or greater than the binding energy (W) of the electron in the shell. The critical limit of voltage (V_C) for characteristic X-rays emission, as observed experimentally, is set up when the energy of the incident electron (eV_C) is just equal to this binding energy. Thus

$$eV_C = W \quad \text{...(7)}$$

or

$$V_C = \frac{W}{e} \quad \text{...(8)}$$

As estimate of the critical potential can be obtained from the Bohr's theory of hydrogen like atom. According to this, binding energy of electron in hydrogen like atom is given by the expression,

$$W = \frac{13.6\, eZ^2}{n^2} \text{ (in eV)} \quad \text{...(9)}$$

where Z is the atomic number of element used as target and n is the principal quantum number of the shell in which the electron is

$$\text{The critical potential} \quad V_C = \frac{13.6\, Z^2}{n^2} \text{ volts} \quad \text{...(10)}$$

Thus critical exciting potential for characteristic X-rays emission depends upon the element used as target, being proportional to square of its atomic number. This equation, however, does not gives the correct value of V_C, as all electrons cannot be hydrogen like atom, but at least it gives correct order of magnitude. As has already been said incident electron creates a vacancy in the inner shell. An electron from the outer shells immediately fills this created vacancy. Due to transition of the electron, photon of energy equal to the energy difference between two electronic states is emitted in accordance with the Bohr's theory. As an electron from the outer orbit jumps to fill the place of the knocked out electron, a vacancy is created in its place, which in turn is filled by an electron from one of the outer orbits, with a simultaneous emission of another photon of lesser energy ; this state of affair continues till a vacancy is created in the outermost orbit and the atom comes to an ionized state. In this span of time, it (the atom) captures an electron and returns to normal atomic state. In this way, a single atom emits a series of photons of definite wavelengths. It should be pointed out at this stage, since there are hundreds and thousands of atoms in the target in which vacancies in the inner cell are created, there is a finite probability for these vacancies to be filled by electrons of any outer orbit and thus a series of lines characterising the shell in which vacancy is created are emitted. For example, if a vacancy is created in K shell, its place may be filled by electrons from L, M, N etc. shells and thus K_α, K_β, K_γ, etc. lines are obtained. Similar argument could be forwarded for L, M, P series and so on. The situation is best represented by Kossel's diagram shown in Fig. 10.7.

Fig. 10.7 Kossel's diagram.

The characteristic X-rays spectra can also be interpreted in terms of energy level diagram as shown in Fig. 10.8. Lines indicating transitions between X-rays levels on such diagrams are often drawn with an arrow head at each end, to indicate that as an electron moves upward, the atom moves downward. Thus in the emission of K lines, an electron goes from L to K shell, while the atom drops from the higher K level to the lower L level.

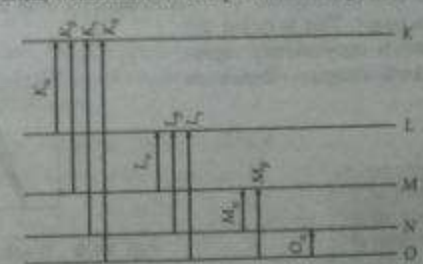

Fig. 10.8 X-rays level diagram.

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According to Bohr's theory of atomic structure, electron in a particular shell possesses definite energy whose magnitude depends upon the atomic number of the element. The electron in K, L etc. shells of atom possess energies denoted by $E_k(Z)$, $E_l(Z)$, $E_m(Z)$ and so on. Hence one can calculate the frequency of different lines. Thus frequency of K line ($\nu_{K\alpha}$) is

$$\nu_{K\alpha} = \frac{E_k(Z) - E_l(Z)}{h} \quad ...(11)$$

Similarly,

$$\nu_{K\beta} = \frac{E_k(Z) - E_m(Z)}{h} \quad ...(12)$$

and

$$\nu_{K\gamma} = \frac{E_l(Z) - E_n(Z)}{h} \quad ...(13)$$

In general

$$\nu = \frac{E_i(Z) - E_j(Z)}{h} \quad \text{for } i > j \quad ...(14)$$

It is evident from these, the difference of energy states for particular element is constant and hence frequency is fixed. It will vary with the element. Higher the atomic number of the emitting element, higher the frequency.

As a remark, it may be pointed here that K, L and to some extent M series from atoms of higher atomic number fall in the X-rays region (1 Å to 10^{-3} Å) others, *e.g.*, P and Q lie in the vacuum ultraviolet or ultraviolet region.

Characteristic X-rays are also emitted when X-rays from hard X-rays tube are made to fall on an element, provided the primary X-rays are harder than the characteristic X-rays. The reason for this can be easily understood *from the quantum theory of radiation*, according to which X-rays are nothing but a stream of energy packets or photon of high energy. When these photons fall on the element, they penetrate deep into the atoms and knock out electrons from inner shell by an inelastic collision and rest for characteristic X-rays emission follows as discussed earlier.

10.4 Moseley's Law

A systematic and comprehensive study of the characteristic X-rays spectra of large number of elements was done by Moseley during the years 1913-15. He found that for a particular line, the root of frequency ($\sqrt{\nu}$) varies linearly with the atomic number (Z) of element used as a target. This is called the **Moseley's law** and is conveniently represented by Moseley's diagram shows in Fig. 10.9.

STATEMENT

The square root of frequency of a spectral line in an X-ray spectrum is proportional to the atomic number of the element emitting it.

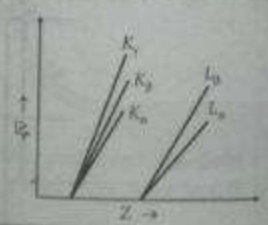

Fig. 10.9 Graph representing Mosley's Law.

The equation of a particular straight line can be represented is

$$\sqrt{\nu} = \sqrt{a}[(Z-b)] \quad ...(15)$$

where $\sqrt{a}$ is the slope of the line which can be as

$$\nu = a(Z-b)^2 \quad ...(16)$$

This is the mathematical representation of Moseley's law.

The constant a is different for different X-rays lines, but b is approximately same for a particular series. Theoretical quantitative explanation of Moseley's law is obtained from Bohr theory for the spectral line combined with a correction for nuclear charge for the so called *Screening effect*. Screening effect is a small correction for the effective charge of the nuclei due to presence of other electrons. For an electron in the higher orbit, the nucleus is seen as screened by inner orbit electrons *i.e.*, the electric force with which an electron of the outer shell is attracted is not $\frac{Ze^2}{r^2}$, the force due to actual charge of nucleus, but $(Z-b)e^2/r^2$, where b is constant, called **screening constant**. This is known as screening effect and could not be taken into account in the Bohr's theory, because it was primarily meant for hydrogen like atoms. Thus if a vacancy of electron is created in K shell, an electron from L shell sees the effective nuclear charge as equal to $(Z-1)e$ and not Ze; on account of the single electron still remaining in K shell. It is obvious that the screening constant will be approximately same for particular series *e.g.*, b is 1 for K series lines, 6.4 for L series and so on. Taking this into account [*i.e.*, replacing Z by $(Z-b)$], one can write general formula for the frequency (ν) of X-rays line for any element using Bohr's theory as follows :

$$\nu = R(Z-b)^2\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right] \quad ...(2)$$

where R is Rydberg constant ((109700 cm^{-1} or 3.291×10^{15} Hz), Z is the atomic number of element emitting the X-rays line, b is the screening constant and n_1 and n_2 are principal quantum numbers of electronic states involved in the transition. Since for a particular line, n_1 and n_2 is constant (*e.g.*, for K line $n_1 = 1$, $n_2 = 2$ and for L line $n_1 = 2$, $n_2 = 3$ and so on), one can write

$$\nu = a(Z-b)^2 \quad ...(18)$$

where

$$a = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \quad ...(19)$$

This Moseley Law, the quantitative agreement between the values a and b given by above theory is reasonably good as obtained by Moseley. Following are some examples which justify the explanation.

(i) For K line from Cu, Moseley found $\lambda = 1.54$ Å

Thus experimental value of frequency (ν) is

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8}{1.54 \times 10^{-10}} = 1.95 \times 10^8 \text{ Hz}$$

Now for the K_α line $n_1 = 1$, $n_2 = 2$ and $b = 1$

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$$\nu_{k\alpha} = R(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right)=\frac{3}{4}R(Z-1)^2$$

But for Cu, $Z=29$. Therefore

$$(\nu_{k\alpha})_{Cu}=\frac{3}{4}\times 3.291\times 10^{15}\times(28)^2$$

$$=1.94\times 10^{18}\text{ Hz}$$

The agreement is quite good.

(ii) Similarly, wavelength for L_α from platinum is 1.32 Å.

Thus the experimental value of ν is

$$\nu=\frac{c}{\lambda}=\frac{3\times 10^8}{1.32\times 10^{-10}}=2.27\times 10^{18}\text{ Hz}$$

Now for the L_α lines, $n_1=2$, $n_2=3$ and $b=7.4$

$$\nu_{k\alpha}=R(Z-7.4)^2\left(\frac{1}{2^2}-\frac{1}{3^2}\right)=\frac{5}{36}R(Z-7.4)^2$$

But for platinum, $Z=78$

$$\therefore\quad (\nu_{k\alpha})_{Pt}=\frac{5}{36}\times 3.291\times 10^{15}\times 70.6\times 70.6\text{ Hz}$$

$$=2.28\times 10^{18}\text{ Hz}$$

In this case, agreement is good within experimental limit.

of Moseley's Law

The researches carried out by Moseley attracted the attention of chemists as well as physicists towards atomic number. It was first experiment to show the supremacy of the atomic number over the atomic weight in the construction of atoms. The development of the entire periodic table is based on the "atomic number" which in turn finds its origin in Moseley's law. This led to the discovery of unknown elements. Further the difference between the atomic weight and atomic number was explained in the number of neutrons, a fundamental constituent of nucleus.

Moseley's law can be obtained from Bohr's theory. Thus in a way this law supports the Bohr theory.

The only negative point about the Moseley's law is that by this method we can not distinguish isotopes of an element as they give rise to same frequencies in X-rays region.

Example 10.3. *Calculate the wavelength of the K_α line of thallium ($Z=81$) if the K_α line of chlorine ($Z=17$) is 0.4718 nm. Given the screening factor $\sigma=1$.*

Solution. We know that from Molseley's law

$$\nu=\frac{c}{\lambda}=KR(Z-\sigma)^2$$

For chlorine, above equation becomes

$$\frac{c}{0.4718}=KR(17-1)^2$$

Let the wavelength of K_α line of thallium be λ, then for thallium, we have

$$\frac{c}{\lambda}=KR(81-1)^2$$

On dividing above two equations, we have

$$\lambda'=\left(\frac{17-1}{81-1}\right)^2\times 0.4718=\left(\frac{1}{5}\right)^2\times 0.4718$$

$$=0.0188\text{ nm}$$

Example 10.4. *The wavelength of the L_α X-rays line of platinum (atomic number 78) is 1.321 Å. An unknown substance emits L_α X-rays of wavelength 4.174 Å. Calculate the atomic number of the unknown substance. Given $b=7.4$ for L_α lines.*

Solution. According to Moseley's law

$$\sqrt{\nu}=a(Z-b)$$

Let ν_1 and ν_2 be the frequencies of the L_α line of platinum and unknown substance respectively. Suppose Z_1 and Z_2 are atomic numbers of Pt and unknown substance respectively.

$$\sqrt{\nu_1}=a(Z_1-b) \quad \text{...(i)}$$

and

$$\sqrt{\nu_2}=a(Z_2-b) \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii)

or

$$\sqrt{\frac{\nu_1}{\nu_2}}=\frac{(Z_1-b)}{(Z_2-b)}$$

or

$$(Z_2-b)=(Z_1-b)\sqrt{\frac{\nu_2}{\nu_1}}$$

We know that $\nu=\frac{c}{\lambda}$, we have $\frac{\nu_2}{\nu_1}=\frac{\lambda_1}{\lambda_2}$

$$(Z_2-b)=(Z_1-b)\sqrt{\left(\frac{\lambda_1}{\lambda_2}\right)}$$

or

$$Z_2=b+(Z_1-b)\sqrt{\left(\frac{\lambda_1}{\lambda_2}\right)}$$

Here $b=7.4$, $Z_1=48$, $\lambda_2=4.174$ Å and $\lambda_1=1.321$ Å

$$\therefore\quad Z_2=7.4+(78-7.4)\sqrt{1.321/4.174}=47.12$$

$$Z_2=47.12$$

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10.5 Properties of X-Rays

Some properties of X-rays are as follows :

1. X-rays are electromagnetic waves of wavelengths between 1 Å to 100 Å.
2. They travel with the velocity of light in free space or vacuum.
3. X-rays travel in straight lines like ordinary light.
4. X-rays are not deflected by electromagnetic fields.
5. X-rays can produce fluorescence in different substances *e.g.*, barium platin cyanide, uranium glass, rock salt, different compounds of calcium etc.
6. X-rays affect photographic plates.
7. They are highly penetrating and can pass through many thin sheets of metal, paper, wood, card boards etc.
8. They ionize gases through which they pass.
9. They exhibit the phenomenon of reflection, refraction, interference, diffraction etc polarization.
10. They destroy living tissues and produce biological effects.
11. They produce photoelectric effect and there by exhibit a corpuscular nature.

10.6 Amorphous and Crystalline Solids

All forms of matter condense in the solid state at a certain specific temperature. Solid State Physics (also called ***Condensed Matter Physics***) deals with the study of the physical properties of the matter in this state.

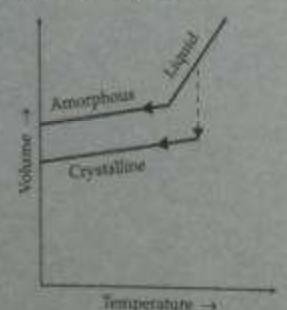


Fig. 10.10 The cooling curves for an amorphous and crystalline solid.

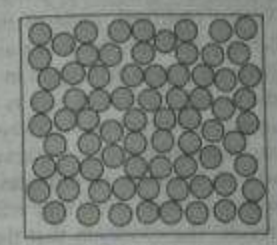
The solids can be classified into two categories :

- Amorphous solids
- Crystalline solids.

Both these names have their origin in Greek words, 'amorphous' from 'shapeless' and crystalline from 'clear ice'. The distinction between an amorphous and a crystalline solid can be visualised from their cooling curves (Fig. 10.10).

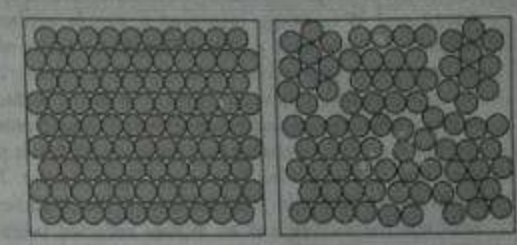
10.6.1. Amorphous Solids

In such solids, the atoms or molecules are strongly bonded yet hardly there is any geometrical regularity or periodicity in the way in which atoms are arranged in space. A two dimensional representation of such a solid is depicted in Fig. 10.11.



Amorphous solid

Fig. 10.11. Schematic sketches of the atomic arrangements in amorphous solid.



(a) Single crystal (b) Polycrystal

Fig. 10.12. Schematic sketches of the atomic arrangements in crystalline solid.

10.6.2 Crystalline solids

The solids are characterised by a perfect or nearly perfect periodicity of atomic structure. The geometrical regularity of structure provides a simple conceptual picture of a crystal and help a lot in gaining the knowledge of the physical properties of the solids. Such a solid in 2–*D* has been pictured in Fig. 10.12.

10.6.3 Distinction between Crystalline and Amorphous Solids

1. Crystalline solids are of a regular arrangement of particles, whereas the amorphous solids have a completely random particles arrangement.
2. Crystalline solids have different physical properties (thermal conductivity, electrical conductivity, refractive index etc.) in different directions *i.e.*, they are anisotropic while amorphous substances are isotropic *i.e.*, they have their physical properties same in all directions.
3. The cooling curve for amorphous substance is smooth, while the curve of crystalline substance has breaks, the middle of which correspond to the process of crystallisation. In the process of crystallization some energy is liberated which compensates the loss of heat and hence temperature remains constant.
4. In case of a crystalline substance, the melting point is very sharp, while the amorphous substance do not have sharp melting points. For examples, as the temperature in glass gradually raised, it soften and starts flowing without any sharp change from solid state to liquid state.

10.7 Bragg's Law

First of all Schuster pointed out that X-rays are electromagnetic waves with wavelengths thousand times smaller than the visible light. In order to measure the wavelengths, a grating of corresponding dimensions is required and hence simple grating can not be used. Moreover, it is rather impossible to prepare artificially a grating of such fine dimension. Laue and his collaborators showed that atoms in crystals are arranged in a regular manner and the spacing between them is comparable to the wavelength of X-rays

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and hence the crystal could act as suitable natural grating for diffracting the X-rays. W.H. Bragg and W.L. Bragg presented a simple explanation of the observed angle of the diffracted beams from a crystal. Consider a series of parallel row in which the atoms are arranged in a given plane of the crystal. Suppose parallel beam of X-rays is incident in a direction making a glancing angle θ_0 with the surfaces of the planes. Each atom acts as a centre of disturbance and sends spherical wave-fronts by Huygens construction. As X-rays are much more penetrating than ordinary light, it is necessary to consider the *rays reflected not from a single layer but from several layers together*. There will, be however, no diffracted beam unless the waves reflected from different planes are exactly in phase. Now they will reinforce themselves and an intense reflected beam will result. The condition that the reflected wavefronts be in same phase, is that path difference between reflected wave from one layer and that from the next must be an exact wavelength or the integral multiple of it. If there is a smallest disagreement in phase between the beams reflected from successive planes, it causes destructive interference.

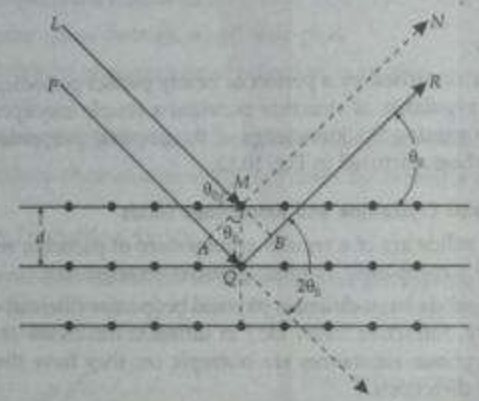

Fig. 10.13 Illustration for the derivation of the Bragg's law.

Let us consider two parallel rays *LMN* and *PQR*, which are reflected by two atoms *M* and *Q* as shown in Fig. 10.13. The atom *Q* is vertically below *M*. The length of the path *PQR* is greater than the length of the path *LMN*. The path difference is $(AQ+QB)$ and according to the condition of reflection, we have

$$(AQ+QB) = n\lambda \qquad ...(20)$$

But from Fig. 10.13,

$$AQ = BQ = d\sin\theta_0$$

$$\therefore \quad 2d\sin\theta_0 = n\lambda \qquad ...(21)$$

Eq. (21) is known as **Bragg's Law** and gives the condition for reflection of X-rays from series of atomic layers in a given plane. For a given set up with monochromatic

radiation, the wavelength λ is fixed and hence Eq. (21) has only a particular set of solutions namely

$$\left.\begin{aligned} &\text{For } n=1;\ \theta_0 = \sin^{-1}\left(\frac{\lambda}{2d}\right); \\ &n=2;\ \theta_0 = \sin^{-1} 2\left(\frac{\lambda}{2d}\right) \\ &n=3;\ \theta_0 = \sin^{-1} 3\left(\frac{\lambda}{2d}\right) ...\text{etc.} \end{aligned}\right\} \qquad ...(22)$$

NOTE

We call the condition for diffraction maxima the **Bragg condition**, after the British father-son team of W.H. Bragg and W.L. Bragg, who shared the 1915 Noble prize in Physics for their work on X-rays diffraction by crystals.

These are known as the first, second, third etc. reflection according to n is 1,2,3,..... etc. This shows that crystal cannot give rise to reflections at any angle but only at those discrete angles indicated by Eq. (22). Bragg's reflection can occur only for wavelength $\lambda \le 2d$. Due to this fact, the visible light wavelength cannot be used in diffraction.

Example 10.5. *An X-rays tube with Cu target is operated at 28 kV. The smallest glancing angle for NaCl crystal for the CuK_α line is 15.8°. Find the wavelength of this line. Also find the glancing angle for photons at the short wavelength limit. (d for NaCl = 0.282 nm).*

Solution. We know that from Bragg's law

$$2d\sin\theta = n\lambda, \text{ here } n=1$$

$$\Rightarrow \quad \lambda = 2d\sin\theta = 2\times(0.282\text{ nm})\times\sin 15.8° = 0.153\text{ nm}$$

$$\lambda_{min} = \frac{hc}{eV} = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.6\times10^{-19}\times28\times10^{3}}\text{ m} = 0.0496\text{ nm}$$

Let θ' be the glancing angle corresponding to λ_{min}, then

$$\theta' = \sin^{-1}\left(\frac{\lambda_{min}}{2d}\right) = \sin^{-1}\left(\frac{0.0496\text{ nm}}{2\times0.282\text{ nm}}\right)$$

$$\theta' = 5.1°$$

Example 10.6. *The spacing of one set of crystal planes in common salt (NaCl) is d = 0.282 nm. A monochromatic beam of X-rays produces a Bragg maximum when its glancing angle with these planes is θ = 7°. Assuming that this is the first order maximum (n = 1), find the wavelength of the X-rays. What is the minimum possible acceleration voltage V_0 that produced the X-rays ?*

Solution. We know that from Bragg's law

$$2d\sin\theta = n\lambda$$

or $$2d\sin\theta = \lambda \qquad \text{(for } n=1)$$

or $$\lambda = 2d\sin\theta = 2\times(0.282\text{ nm})\sin 7° = 0.069\text{ nm}$$

The Duane-Hunt law requires that the electron's kinetic energy eV_0 in the X-rays tube must be at least equal to the energy $h\nu$ of the X-rays photons.

Therefore $$eV_0 \ge h\nu = \frac{hc}{\lambda} = \frac{1240\text{ eVnm}}{0.069\text{ nm}} = 18{,}000\text{ eV} \quad \text{or} \quad V_0 \ge 18{,}000\text{ V}.$$

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10.8 Experimental Crystal Structure Determination

In practice, to satisfy Bragg's law for X-rays diffraction, it is necessary to vary either angle of inclination of the specimen to the beam or the wavelength of radiation.

The three standard methods of X-rays crystallography allows for this in the following ways :

(i) in the **Laue technique**, a stationary single crystal is irradiated by a range of X-rays wavelengths.

(ii) in the **rotating crystal method**, a single crystal specimen is rotated in a beam of monochromatic X-rays, and

(iii) in the **powder technique**, a polycrystalline powder is kept stationary in a beam of monochromatic radiation.

10.8.1 Laue Method

A single crystal is mounted on a goniometer, which enables the crystal to be rotated through known angles in two perpendicular planes, and maintained stationary in a beam of X-rays ranging in wavelength from about 0.2 to 2.0 Å. The crystal selects out and diffracts those values of λ for which planes exist, of spacing d and glancing angle satisfying the Bragg equation. A flat photographic film is placed to receive either the transmitted diffracted beam or the reflected diffracted beams as illustrated in Fig. 10.14.

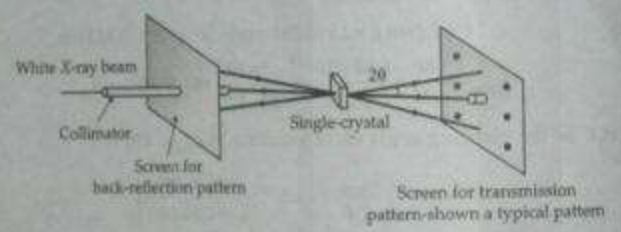

Fig. 10.14 Illustration for Laue technique.

As shown in the Fig. 10.14, the resulting Laue pattern consists of a series of spots. Sharp well-defined spots on the film are good evidence of a perfect crystal structure whereas diffuse, broken or extended spots indicate lattice distortion, defects or other departures from the perfect crystal lattice.

The Laue pattern reveals the symmetry of the crystal structure in the orientation used, for example, if a cubic crystal is oriented with a cube edge, *e.g.*, [100] axis, parallel to the incident beam, the Laue pattern will show the four fold symmetry appropriate to that axis.

10.8.2 Rotating Crystal Technique

A small single crystal (1 mm dimensions) is mounted on a goniometer which, in turn, is rigidity fixed to a spindle so that the crystal can be rotated about a fixed axis in a beam of monochromatic radiation. The specimen is usually oriented with one of the crystallographic axes parallel to the axis of the rotation. The resulting variation in θ brings different

lattice planes into position for reflection and diffracted images are recorded on a photographic film placed cylindrically, coaxial with the rotating spindle (Fig. 10.15).

To explain the general nature of the diffraction, consider a crystal mounted so that z-axis is parallel to the axis of rotation, then diffraction cannot occur from the planes of atoms parallel to the axis unless

$$z \cos \varphi_n = n\lambda \qquad ...(23)$$

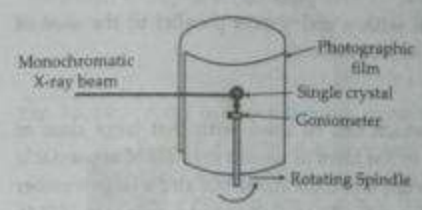

Fig. 10.15 Rotating crystal technique.

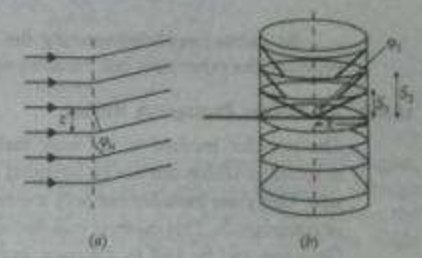

Fig. 10.16 Diffraction pattern in rotating crystal technique (*a*) Diffraction condition (*b*) Cones of diffraction.

where n is an integer [Fig. 10.16(*a*)]. The diffracted beams will, therefore, be along the surfaces of the family of cones whose vertices one at the crystal, and whose semi-vertical angles are given by the above Eq. (23) [Fig. 10.16(*b*)].

Of course, the diffracted beams will only occur along those specific directions lying on the cones for which the correct phase relationship also holds for planes parallel to the other two coordinate axes. When the film is flattened out after development, these diffraction images will lie on a series of lines called *layer lines*, as illustrated in Fig. 10.17. All the images on zero layer line come from planes parallel to the axis of rotation *i.e.*, planes with $l = 0$ in this example, and the other layer lines from planes with $l = \pm 1, \pm 2, \ldots$ etc. Diffraction images from planes with the same value of h and k but different values of l, all lie on one of a series of curves known as row lines which are transverse to the layer lines and in the particular case when the x and y axes are perpendicular to z, they intersect the zero layer line at right angles.

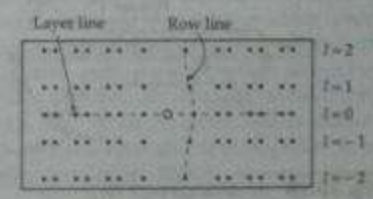

Fig. 10.17 Typical rotation photograph.

If S_n is the separation of these layer lines and R is the radius of the camera, then as seen from Fig. 10.16(*b*).

$$S_n = R \cot \varphi_n \qquad ...(24)$$

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For Eqs. (23) and (24)

$$S_n = \frac{R}{\sqrt{\left[1-\left(\frac{z}{n\lambda}\right)^2\right]}}$$

$$\therefore \quad z = \frac{n\lambda}{S_n}\sqrt{(R^2+S_n^2)} \qquad \text{...(26)}$$

By subsequently orientating the crystal with x and y axes parallel to the axes of rotation, the other unit cell parameters may be determined.

10.8.2 The Powder Photograph Method

The powder method is the only method which can be used with that large class of substances which can not be obtained easily in the form of perfect crystals of appreciable size. This class includes not only the most metals and their alloys but also a large number of compounds. This method was devised independently by Debye and Scherrer in 1916 in Germany and by Hill in America. In this method a monochromatic X-rays beam is used and instead of using a single crystal, fine powders of crystalline aggregates of all kinds having random of chaotic orientation are used. Such a powder requires no rotation because every atomic plane is present in every possible orientation and hence the diffraction depends upon the fact that in a fine powder of grains are arranged in an entirely chaotic manner. The entirely random orientations of the grains with respect to the beam means that some of them will be in a position to reflected the radiation from an important set of planes. Now diffracted rays go out from individual crystallites which happen to be oriented with planes making an angle θ with the beam satisfying the Bragg equation. Any fragment in which the normal to the plane in questions makes an angle (90 – θ) with the incident beam will be in position to reflect and since all orientations of the fragment are equal likely, the reflected rays will form a cone, concentric with the original beam and whose semi-vertical angle is 2θ. There is such a cone of diffracted rays for each set of planes. The cone intercept the film in a series of concentric circular halves, from the radii of which the angle θ and hence the spacing of the planes can be deduced.

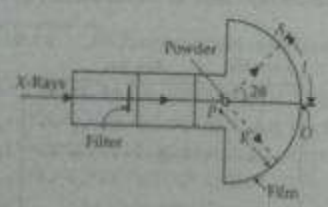

Fig. 10.18 An arrangement for taking powder photographs.

The formation of powder photograph is shown in Fig. 10.18.

The radiation is made approximately monochromatic with the help of filter as shown in Fig. 10.18. P is powder and O is the point where the direct beam would have struck the film. Point A on the film correspond at which a spectrum with glancing angle θ is formed. The diffracted maxima lie on cones coaxial with the directed beam and if a photographic plate is mounted normal to the direct beam, concentric circles are registered upon it as

shown in Fig. 10.19. Usually a plate or film in the form of cylindrical shape is employed whose axis is perpendicular to the beam. There appear arcs of the circles as shown in Fig. 10.20.

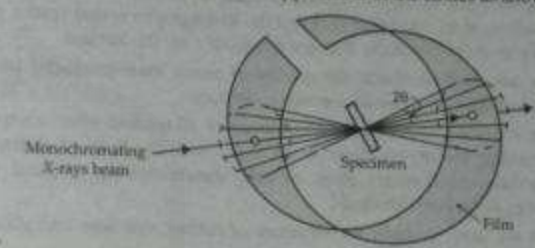

Fig. 10.19 X-ray powder diffraction camera.

From Fig. 10.20, it is observed that when rays are diffracted through small angles, they make arc around the central spot on the film. When the rays are diffracted through 90°, the cones become flat and the corresponding trace is a straight line. When the diffracted angle increases above 90°, the curvature is reversed and when the angle approaches to 180°, the traces are nearly circular. Thus the curvature of lines changes from the centre to the outside the film.

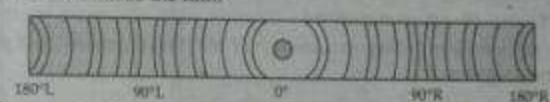

Fig. 10.20 Arrangement of lines in a powder photograph.

Now considering Fig. 10.18, if l is the distance from O to A measured on the film and R is the radius of camera, then $\theta = \frac{l}{2R}$. In this way by measuring l, the value of θ can be calculated.

This method is very useful in investigating the structures of simple crystals particularly belonging the cubic system of which spacings a, b, c of unit cell are all equal. In these crystals, there are certain definite relationship between the angles at which spectra can occur. The spacings of all planes parallel to faces of the same from (h, k, l) are equal and therefore produce spectra at the same angle. In most general case in which all h, k, l are different, there are 48 faces in the form and 24[1] sets of planes, all having the same spacing. These all co-operate to produce one line on the film. When the three axes of different lengths are at right angles to one another, the general form (h, k, l) corresponds to six different spacing and hence there are six different lines on the film.

10.9 Applications of X-rays

On account of their and distinctive properties, X-rays have been put to many uses in different fields of our daily life *i.e.*, in industry, medicine and research.

1. We know that any permutations of the three integers h, k, l will lead to the same value of θ. The same result would be obtained from the planes <100>, <010>, <001>, <$\bar{1}$00>, <0$\bar{1}$0>, <00$\bar{1}$> making six possible permutations. If all three indices are different, there will be 48 permutations.

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10.9.1 Industrial and Engineering Applications

1. X-rays are used to inspect products made of various kinds of materials, including aluminium, steel and other cast metals. Radiographs reveal cracks, defects and also cavities in these products that are not visible on the surface.
2. X-rays are used to check the quality of many mass-produced products, such as transistors and other small electronic devices.
3. X-rays are used to analyse the structure of alloys and other composite bodies by determining the crystal from in an ingot with the help of diffraction of X-rays. Alloys like cobalt-nickel steel, bronze, duraluminium, artificial pearls and old paintings have been analysed.
4. X-rays are used to study the structure of rubber, cellulose and plastics.

10.9.2 Applications for Pure Scientific Research

1. X-rays are used for the investigation of structure of atoms.
2. X-rays are used for studying the structure of crystalline solids and alloys.
3. Scientists use X-rays to help analyse the structure and make up of many complex chemical substances, such as enzymes and proteins.
4. Archaeologist have used X-rays to examine ancient objects that are covered by a heavy crust of dirt or corrosion.
5. They are used for testing genuineness of old documents and old paintings, pearls, diamonds gems etc.

10.9.3 Medical Applications

1. X-rays widely used to make radiographs (X-rays pictures) for detection of fracture of bones and for the diagnosis of ulcer, tumor, tuberculosis and stone in kidneys.
2. X-rays are used for treatment of abnormal growth of tissues.
3. X-rays are widely used to treat cancer.
4. X-rays have been used for the identification of different types of cells and tissues and for bringing about genetic mutations.

10.10 Difference between X-Rays Spectra and Optical Spectra

S.No.	X-rays spectra	Optical spectra
1.	X-rays spectra arise from electronic transitions among the inner closed shells of the atom.	Optical spectra arise from electronic transitions among the outer shells of the atom.
2.	X-rays spectra vary smoothly from element to element.	Optical spectra show periodic changes from element to element.
3.	X-rays spectra arise as a result of the complete removal of an electron from an inner shell of the atom.	The electron from the outer shell is merely raised to a higher level and not completely removed.
4.	X-rays emission spectra (consisting of discrete lines) are entirely different from the X-rays absorption spectra (consisting of continuous regions bounded by sharp edges).	Optical emission spectra are identical to the absorption spectra.

Formulae at a Glance

1. $E_{max} = eV = h\nu_{max} = \frac{hc}{\lambda_{min}}$

   or $\lambda_{min} = \frac{hc}{eV} = \frac{K}{V}$

   λ = wavelength, V = tube voltage,

   h = Planck's constant

   ν = Frequency, E = energy

2. $\nu_{max} = \frac{E_{max}}{h} = \frac{eV}{h}$

   $\lambda_{min} = \frac{12400\,\text{Å}}{V}$

   $K = \frac{ch}{e} = 12.400 \times 10^{-10}$ V - m.

3. $\nu = \frac{E - E'}{h}$

   where E and E' are the energies of the electron when it enters the target and comes out the charge cloud surrounding the nucleus.

4. $eV_C = W$

   V_C = critical voltage

   W = binding energy

   e = electronic charge

5. $W = \frac{13.6\, eZ^2}{n^2}$ (in eV)

   Z = atomic number

   n = principal quantum number.

6. Moseley's law

   (a) $\nu = a(Z - b)^2$

   (b) $\nu = R(Z - b)^2\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$

   $a = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$

7. Bragg's Law : $2d\sin\theta_n = n\lambda$

8. Rotating crystal technique

   $z\cos\varphi_n = n\lambda$

   $S_n = R\cot\varphi_n$

   $S_n = \frac{R}{\sqrt{\left[1 - \left(\frac{z}{n\lambda}\right)^2\right]}}$

   or $z = \frac{n\lambda}{S_n}(R^2 + S_n^2)^{1/2}$

   where R = Radius of the camera

   S_n = Separation of these layer line.

Miscellaneous Numerical Problems for Practice

Example 1. X-rays of wavelength 2×10^{-11} m suffer first order reflection from (111) crystal plane at an angle of 45°. What is interatomic spacing of the crystal ? **[GGSIPU May 2007 (2.5 Marks)]**

Solution. We know that $d_{hkl} = \frac{a}{\sqrt{(h^2 + k^2 + l^2)}}$

where a = interatomic distance, d = interplanar distance, h, k, l = Miller indices

According to problem $h = k = l = 1$ (being d_{111})

$$\theta = 45°, \quad \lambda = 10^{-11}\text{ m}, \quad n = 1$$

From Bragg's law $2d\sin\theta = n\lambda$,

We have $2\frac{a}{\sqrt{3}} \times \sin 45° = 2 \times 10^{-11}$

$$a = \sqrt{6} \times 10^{-11} = 2.45 \times 10^{-11}\text{ m}$$

$$a = 0.25\text{ Å}$$

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Example 2. The distance d between two adjacent atomic planes in a calcite crystal is 3.04 Å. Find the wavelength for which the first order Bragg's reflection occurs at θ = 30°, where θ is grazing angle.

Solution. We know that $2d\sin\theta = n\lambda$

Here $d = 3.04$ Å, $\theta = 30°$, $n = 1$

then $$\lambda = \frac{2d\sin\theta}{n}$$

$$= \frac{2\times 3.04\times 10^{-10}\times \sin 30°}{1} = 3.04 \text{ Å}$$

Example 3. Calculate the glancing angle of the (110) plane of simple cubic crystal (a = 2.814 Å) corresponding to second order diffraction maxima for the X-rays of wavelength 0.710 Å.

[GGSIPU May 2006 (4.5 Marks)]

Solution. For n^{th} order diffraction maximum for X-rays of wavelength λ from lattice planes of spacing d, the glancing angle θ is given by

$$2d\sin\theta = n\lambda \quad \text{...(i)}$$

The distance between consecutive lattice planes defined by Miller indices (hkl) in a cubic lattice

$$d_{hkl} = \frac{a}{\sqrt{h^2+k^2+l^2}}$$

Here $a = 2.814$ Å, $h = 1$, $k = 1$, $l = 0$

then $$d_{110} = \frac{2.814}{\sqrt{1^2+1^2+0}} = \frac{2.814}{\sqrt{2}} \text{ Å} = 1.989 \text{ Å}$$

Also we have $n = 2$ and $\lambda = 710$ Å

Substituting these values in Eq. (i)

$$2\times 1.989\sin\theta = 2\times 0.710$$

$$\sin\theta = \frac{0.710}{1.989} = 0.3569$$

$$\theta = \sin^{-1}(0.3569)$$

$$\theta = 20°55'.$$

Example 4. The K_α X-rays line from molybdenum has a wavelength of 0.7078 Å. Calculate the wavelength of the K_α line of copper.

Solution. From Moseley's law

$$\nu \propto (Z-1)^2$$

or $$\frac{c}{\lambda} = (Z-1)^2$$

or $$(Z-1)^2\lambda = \text{constant}$$

Thus $(42-1)^2\times 0.7078 = (29-1)^2\lambda'$

λ' = Wavelength of the K_α line of Cu

$$\Rightarrow \quad (41)^2 = 0.7078 \text{ Å} = (28)^2\lambda'$$

$$\Rightarrow \quad \lambda' = 1.517 \text{ Å}$$

Example 5. An X-rays tube operates on 50 kV. Calculate the shortest wavelength of the X-rays produced.

[GGSIPU May 2005 (2.5 Marks)]

Solution. We have $\lambda_{min} = \frac{hc}{eV}$

where $h = 6.623\times 10^{-34}$ Js, $c = 3\times 10^8$ ms^{-1}, $e = 1.6\times 10^{-19}$ C and $V = 50{,}000$ V

$$\lambda_{min} = \frac{6.623\times 10^{-34}\times 3\times 10^8}{1.6\times 10^{-19}\times 50000}$$

$$= 0.248\times 10^{-10} \text{ m} = 0.248 \text{ Å}.$$

Example 6. Calculate the wavelength of X-rays produced when the potential difference is 12400 volts.

Solution. We know that

$$\lambda = \frac{hc}{eV}$$

$$= \frac{12400}{V} \text{ Å} = \frac{12400}{12400} \text{ Å} = 1 \text{ Å}$$

Example 7. The spacing between principal planes of NaCl crystal is 2.82 Å. It is found that first order Bragg reflection occurs at an angle of 10°. What is the wavelength of X-rays ?

Solution. From Bragg's law $2d\sin\theta = n\lambda$

Here $d = 2.82\times 10^{-10}$ m, $n = 1$, $\theta = 10°$, $\lambda = ?$

$$\lambda = \frac{2d\sin\theta}{n} = \frac{2\times(2.82\times 10^{-10})\sin 10°}{1}$$

$$\Rightarrow \quad \lambda = 0.98\times 10^{-10} \text{ m} = 0.98 \text{ Å}$$

Example 8. The distance between adjacent atomic planes in calcite is 0.300 nm. Find the smallest angle of Bragg's scattering for a 0.030 nm X-rays.

[GGSIPU May 2008 (2.5 Marks)]

Solution. Since we know that

$$2d\sin\theta = n\lambda$$

Here $d = 0.300$ nm $= 3\times 10^{-10}$ m, $\lambda = 3.0\times 10^{-11}$ m, $n = 1$

$$2\times 3\times 10^{-10}\sin\theta = 3\times 10^{-11}$$

$$\sin\theta = \frac{1}{2}\times 10^{-1} = 0.05$$

$$\theta = \sin^{-1}(0.05) = 2.87°.$$

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Example 9. *While comparing the wavelength of two monochromatic X-rays line, it is found that line 'A' gives a first order Bragg's reflection maximum at a glancing angle 30° to the smooth face of a crystal. Line 'B' of wavelength $\lambda_B = 0.97$ Å gives a third order reflection maximum at a glancing angle of 60°. Find the wavelength of line 'A' [λ_A].* **[GGSIPU May 2005 (4.5 Marks)]**

Solution. For line A $\quad 2d\sin\theta = \lambda_A \quad (\because n = 1)$

$\Rightarrow \quad 2d\sin 30° = \lambda_A \quad ...(i)$

For line B $\quad 2d\sin\theta = 3\lambda_B \quad (\because n = 3)$

$\Rightarrow \quad 2d\sin 60° = 3\lambda_B$

$\Rightarrow \quad 2d\sin 60° = 3\times 0.97\times 10^{-10} \quad ...(ii)$

From Eq. (ii) $\quad d = \sqrt{3}\times 0.97\times 10^{-10}$ m

$= 1.68$ Å

From Eq. (i) $\quad \lambda_A = 2\times 1.68\times 10^{-10}\times \sin 30°$

$= 1.68$ Å.

Example 10. *Find the maximum frequency of the X-rays emitted by an X-ray tube operating at 30 kV.*

Solution. For maximum frequency, the total kinetic energy (eV) should be converted into X-rays photon.

Thus $\quad h\nu_{max} = eV$

or $$\nu_{max} = \frac{1.6\times 10^{-19}\times 30\times 10^{3}}{6.623\times 10^{-34}}$$

$$= 7.24\times 10^{18}\text{ Hz}.$$

Example 11. *Calculate the wavelength of X-rays from the following data :*

(i) *Glancing angle for NaCl for first order spectrum = 11.8°.*

(ii) *Atomic weight of Na = 23.*

(iii) *Atomic weight of Chlorine = 35.5.*

(iv) *Density of NaCl = 2.17 g/cm³.*

(v) *Mass of H_2 atom $= 1.64\times 10^{-24}$ g.*

Solution. On examining the structure of NaCl crystal, we find that each atom lies at the junction of eight cubes and that each cube has 8 atoms one at each corner, so that in the whole structure extending in all directions there is one atom per cube or half a molecule per cube.

$\therefore$ average mass of each cube

$$= \frac{1}{2}\times \text{molecular weight} \times \text{weight of } H_2 \text{ atom}$$

Now molecular weight of NaCl $= 23 + 35.5 = 58.5$

Mass of H_2 atom $= 1.64\times 10^{-24}$ g

$\therefore$ Average mass of each cube $= \frac{1}{2}\times 58.5\times 1.64\times 10^{-24}$ g

But volume of each cube

$$d^3 = \frac{\text{mass}}{\text{density}}$$

$$= \frac{\frac{1}{2}\times 58.5\times 1.64\times 10^{-24}\text{ g}}{2.17\text{ g cm}^{-3}}$$

where d is the side of the cube

whence $$d = \sqrt[3]{\frac{58.5\times 1.64\times 10^{-24}\text{ cm}^3}{2\times 2.17}}$$

$$= 2.81\times 10^{-8}\text{ cm}$$

Now the glancing angle θ_1 for the first order spectrum of NaCl = 11.8°

$$\lambda = 2d\sin\theta_1$$
$$= 2\times 2.8\times 10^{-8}\times \sin 11.8°$$
$$= 2\times 2.81\times 10^{-8}\times 0.204$$
$$= 1.15\times 10^{-8}\text{ cm}.$$

Example 12. *A neutron beam of K.E. 0.04 eV is diffracted at the (100) plane of the sylvine crystal ($d_{100} = 3.14$ Å). Calculate the glancing angle θ at which the first order Bragg spectrum will be observed. Mass of neutron $= 1.6748\times 10^{-24}$ g and Planck's constant, $h = 6.625\times 10^{-27}$ ergs.*

Solution. $\quad \text{K.E.}\left(\frac{1}{2}mv^2\right) = 0.04\times 0.04\times 1.6\times 10^{-12}$ ergs $\quad (\because 1\text{ eV} = 1.6\times 10^{-12}\text{ erg})$

$$\therefore \quad v = \sqrt{\frac{2\times 0.04\times 1.6\times 10^{-12}}{m}}$$

and $$mv = \sqrt{2\times 0.04\times 1.6\times 10^{-12}\,m}$$

$$= \sqrt{2\times 0.04\times 1.6\times 10^{-12}\times 1.6748\times 10^{-24}}$$

$$= 10^{-22}\sqrt{(1280\times 16748)}\text{ g cm s}^{-1}.$$

Now de Broglie wavelength

$$\lambda = \frac{h}{mv}$$

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substituting for h and m, we get

$$\lambda = \frac{6.625 \times 10^{-27} \text{ erg.s}}{10^{-22}\sqrt{(1280 \times 16748)} \text{ g cm s}^{-1}}$$

$$= \frac{6625 \times 10^{-8}}{4629} \text{ cm} = 1.431 \times 10^{-6} \text{ cm.}$$

From Bragg's Law

$$2d \sin\theta_1 = \lambda$$

or $$\sin\theta_1 = \frac{\lambda}{2d}$$

$$= \frac{1.431 \times 10^{-8}}{2 \times 3.14 \times 10^{-8}} = 0.2278$$

$$\theta_1 = 13°10'$$

Example 13. The radiation from a X-rays tube operated at 40 kV is analysed with a Bragg spectrometer using calcite crystal cut along the cleavage plane (100) :

(a) Calculate the shortest wavelength of X-rays coming from tube.

(b) What is the smallest glancing angle at which the wavelength will be reflected?

Given $h = 6.625 \times 10^{-27}$ erg-sec, $e = 4.803 \times 10^{-10}$ esu, $c = 3 \times 10^{10}$ cm/s and $d_{100} = 3.029$ Å.

Solution. (a) $$E_{max} = eV = 4.803 \times 10^{-10} \times \frac{40000}{300}$$

$$= 400 \times 1.601 \times 10^{-10} \text{ erg}$$

$$h\nu = \frac{hc}{\lambda_{min}} = E_{max}$$

$$\therefore \quad \lambda_{min} = \frac{hc}{E_{max}} = \frac{hc}{eV}$$

$$= \frac{6.625 \times 10^{-27} \times 3 \times 10^{10}}{400 \times 1.601 \times 10^{-10}} \text{ cm}$$

$$= \frac{1.9875}{6.404} \times 10^{-8} \text{ cm} = 0.3103 \times 10^{-8} \text{ cm} = 0.31 \text{ Å}$$

(b) $2d \sin\theta_1 = n\lambda$

or $$\sin\theta_1 = \frac{0.3103 \times 10^{-8}}{2 \times 3.029 \times 10^{-8}} = 0.0512$$

$$\therefore \quad \theta_1 = 2°57'.$$

Example 14. Detect the element whose K_α X-rays line has wavelength 0.18 nm ?

Solution. We know from Moseley's law

$$\nu = \frac{C}{\lambda} = \frac{3}{4} R(Z-1)^2$$

Given that $\lambda = 0.18$ nm, also $R = 3.291 \times 10^{15}$

$$\Rightarrow \quad \frac{3 \times 10^8}{0.18 \times 10^{-9}} = \frac{3}{4} \times 3.291 \times 10^{15} (Z-1)^2$$

or $$(Z-1)^2 = 676 \quad \Rightarrow \quad Z = 27$$

$\therefore$ The element must cobalt corresponding to given K_α X-rays line.

Example 15. Prove that the sum of the frequencies of L_α and M_α X-rays of a material is equal to the frequency of L_β X-rays of the same material.

Solution. The energy level diagram for L and M-lines is shown in Fig. 10.21.

From Fig. 10.21, it is clear that the

Energy of L_α-line is

$$E_{L_\alpha} = E_L - E_M \quad ...(i)$$

Energy of L_β-line is

$$E_{L_\beta} = E_L - E_N \quad ...(ii)$$

Energy of M_α-line is

$$E_{M_\alpha} = E_M - E_N \quad ...(iii)$$

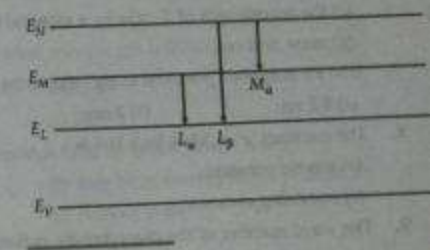

Fig. 10.21

From Eqs. (i), (ii) and (iii), we have

$$E_{L_\beta} = E_{L_\alpha} + E_{M_\alpha}$$

$$h\nu_{L\beta} = h\nu_{L_\alpha} + h\nu_{M_\alpha} \quad \text{or} \quad \nu_{L_\beta} = \nu_{L_\beta} + \nu_{M_\alpha}.$$

Question Bank

Multiple Choice Questions

1. Using X-rays of 2.29 Å, the first two reflections have Bragg angles of 30.06° and 65.51°. The Br angle for the third reflection is.
   (a) 68.1° (b) 75.9°
   (c) 89.9° (d) reflection absent.
   [GGSIPU May 2006 (2.5 M

2. X-rays beam can be deflected
   (a) by an electric field (b) by a magnetic field
   (c) by an electric field as well as a magnetic field
   (d) neither by an electric field nor by a magnetic field.

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3. If the shortest wavelength emitted by an X-ray tube is 0.25 nm the operating potential of the tube is approximately :
   (a) 50 kV (b) 5 kV
   (c) 500 kV (d) none of these.
4. If the angle between the direction of the incident X-rays and the diffracted one is 16°, the angle of incidence will be
   (a) 32° (b) 24°
   (c) 90° (d) 82°
5. X-rays are :
   (a) negatively charged particles (b) electromagnetic radiation
   (c) positively charged particles (d) a stream of neutrons.
6. Which one of the following statement is correct ?
   (a) X-rays are electromagnetic radiation of long wavelength
   (b) X-rays produced in a gas tube are different from these in a Coolidge tube.
   (c) the wavelength of X-rays by a material is longer than that of the incident beam
   (d) none of these.
7. If 50 kV is applied potential in an X-rays tube then the minimum wavelength of X-rays produced is
   (a) 0.2 nm (b) 2 nm (c) 0.02 nm (d) 2 Å
8. The constant 'a' in Moseley's law is :
   (a) atomic constant (b) Rydberg constant
   (c) screening constant (d) none of these.
9. The wave number of the characteristic radiation
   (a) decreases with increasing atomic number of the anode
   (b) increases with increasing atomic number of the anode
   (c) may be (a) and (b) both
   (d) none of these.
10. According to the Moseley's law, the frequency of the characteristic X-radiation is proportional to the square of
    (a) atomic weight of the element (b) atomic number of element
    (c) molecular weight of the element (d) none of these.
11. As the wavelength of X-rays is smaller than that of visible light, the speed of X-rays in vacuum is
    (a) same as that of visible light (b) larger than that of visible light.
    (c) smaller than that of visible light (d) none of these.
12. X-rays produced when an element of high atomic weight is bombarded by high energy :
    (a) protons (b) neutrons
    (c) photons (d) electrons.
13. Laue's experiment proves :
    (a) X-rays are electromagnetic radiation (b) X-rays are charged particles
    (c) the wavelength of X-rays is very high (d) X-rays will not penetrate through metal.

14. In a Coolidge tube, the electrons are focused on to target by using :
    (a) a curved cathode (b) magnetic lens
    (c) transformer (d) an electron gun.
15. X-rays and gamma rays are both electromagnetic waves. Which of the following statement is true ?
    (a) In general, X-rays have larger wavelength than that of gamma rays
    (b) X-rays have smaller wavelength than that of gamma rays
    (c) gamma rays have smaller frequency than that of X-rays
    (d) Wavelength and frequency of X-rays are both large than those of gamma rays.
16. Electromagnetic theory demands that the acceleration of any charged particle must be accompanied by the emission of electromagnetic waves. Why the target in a X-rays tube is bombarded with high speed electrons only ?
    (a) electrons have the smallest possible charges
    (b) electronic mass is very small
    (c) electrons are common constituent of all matters
    (d) electrons can be produced very conveniently.
17. An X-rays tube produces X-rays whose shortest wavelength is 0.2475 Å
    (a) 500 volts (b) 25 kV
    (c) 100 kV (d) 50 kV.
18. When an electron strikes the target in a Coolidge tube, its entire kinetic energy
    (a) is converted into a photon (b) may be converted into a photon
    (c) is converted into heat (d) none of these
19. For harder X-rays,
    (a) the wavelength is higher (b) the intensity is higher
    (c) the frequency is higher (d) the photon energy is higher.
20. Mark the correct options
    (a) An atom with a vacancy has smaller energy than a neutral atom
    (b) K X-rays is emitted when a hole makes a jump from the K shell to some other shell
    (c) The wavelength of K X-rays is smaller than the wavelength of L X-rays of same material
    (d) The wavelength of K_α X-rays is smaller than the wavelength of K_β X-rays of the same material.
21. Frequencies of K_α X-rays of different materials are measured. Which one of the graphs in Fig. 10.22 may represent the relation between the frequency v and the atomic :
    (a) 1
    (b) 2
    (c) 3
    (d) 4

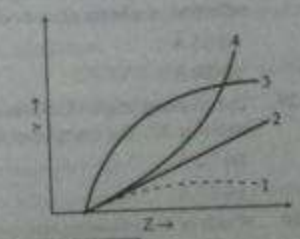

Fig. 10.22

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22. Fig. 10.23 shows intensity-wavelength relations of X-rays coming from two different Coolidge tubes. The solid curve represents the relation for the tube A in which the potential difference between the target and the filament is V_A and the atomic number of the target material is Z_A. These quantities are V_B and Z_B for other tube B. Then

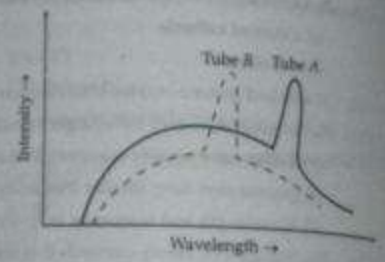

Fig. 10.23

(a) $V_A > V_B$; $Z_A > Z_B$
(b) $V_A > V_B$; $Z_A < Z_B$
(c) $V_A < V_B$; $Z_A > Z_B$
(d) $V_A < V_B$; $Z_A < Z_B$

23. Moseley's law for characteristic X-rays is $\sqrt{\nu^{1/2}} = a(Z - b)$. In this
(a) both a and b are independent of the material
(b) a is independent but b depends on the material
(c) b is independent but a depends on the material
(d) both a and b depend on the material.

24. Moseley's law is concerned with
(a) continuous X-rays (b) characteristic X-rays
(c) laser light (d) radioactivity.

25. Duane and Hunt limit for an X-rays tube operating at 25 kV is
(a) 0.481 Å (b) 6.453 Å
(c) 10.497 Å (d) 10.475 Å

26. Bragg's equation has no solution if
(a) $\lambda > 2d$ (b) $\lambda < d$
(c) $\lambda < 2d$ (d) $\lambda < \frac{d}{2}$

27. The shortest wavelength that can be generated by an X-rays tube depends mainly upon
(a) Current in beam (b) Nature of target material
(c) Accelerating voltage (d) All three above.

28. Spacing between the principal planes of a crystal is 2 Å. It is found that the first order Bragg reflection of a beam of monochromatic X-rays occurs at angle of 30°, wavelength of X-rays is
(a) 0.5 Å (b) 1.0 Å
(c) 2.0 Å (d) 4.0 Å

29. The glancing angle of incidence for first order reflection of X-rays of wavelength 0.6 Å from a crystal is 30°. The interplanar distance for the crystal is
(a) 1.2 Å (b) 0.346 Å
(c) 2.0 Å (d) 0.6 Å

30. Which of the following statement is not true for X-rays
(a) X-rays are electromagnetic radiation of very short wavelength

(b) X-rays are produces by energy changes in the inner electrons of the atom
(c) X-rays have electric and magnetic fields
(d) In X-rays tube entire energy of the incident electron is converted to X-rays.

31. The graph between the frequency of K_α line of X-rays and atomic number (Z) of target is
(a) straight line (b) parabola (c) hyperbola (d) ellipse.

32. The spacing between two principal planes of NaCl crystal is $\sqrt{2}$Å. It is found that the first order Bragg reflection of a beam of monochromatic X-rays occurs at an angle of 45°. The wavelength of X-rays is
(a) $\sqrt{2} \times 10^{-10}$ m (b) $\frac{1}{\sqrt{2}} \times 10^{-10}$ m (c) 2×10^{-10} m (d) $\frac{1}{2} \times 10^{-10}$ m

Answers

1. (c)	2. (d)	3. (a)	4. (b)	5. (b)	6. (c)
7. (c)	8. (c)	9. (b)	10. (b)	11. (a)	12. (d)
13. (a)	14. (b)	15. (a)	16. (b)	17. (d)	18. (b)
19. (c, d)	20. (b, c)	21. (d)	22. (b)	23. (a)	24. (b)
25. (a)	26. (a)	27. (c)	28. (c)	29. (d)	30. (c)
31. (b)	32. (c)				

Theoretical Questions

1. Distinguish between characteristic and continuous X-rays. Explain the powder diffraction method to study crystal structure. [GGSIPU, May 2007 (6 marks)]
2. What are characteristics of X-rays ? How is their production explained ? Describe with theory, Bragg's method for determining the wavelength X-rays. [GGSIPU, May 2006 (8 marks)]
3. Discuss the characteristics features of X-rays spectra. [GGSIPU, May 2008 (9 marks)]
4. What are Bragg's planes ? [GGSIPU, May 2008 (3.5 marks)]
5. State Bragg's law for X-rays diffraction. Indicate the quantities with a diagram. [GGSIPU, May 2005 (2.5 marks)]
6. Describe the Laue method or the powder method of X-rays diffraction. [GGSIPU, May 2005 (8 marks)]
7. Distinguish between continuous and characteristics X-rays and explain their origin.
8. Explain the origin of characteristic X-rays with the help of pictorial diagram.
9. Explain briefly continuous X-rays spectrum has a short wavelength limit.
10. What is meant by continuous and characteristic components in the spectrum of X-rays emitted by a target in an X-rays tube ?
11. Describe the salient features of continuous and characteristic X-rays. How are they explained ?

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12. Describe X-rays emission spectra and explain its origin.
13. Show that greater the diffraction angle, the greater is the accuracy in determining lattice parameter.
14. What is meant by X-rays diffraction. Show that $2d \sin\theta = n\lambda$, where the symbols have their usual meanings.
15. Write short notes on :
    (i) Production X-rays ; (ii) Properties of X-rays ; (iii) Applications of X-rays ;
    (iv) Moseley's law ; (v) Bragg's law.
16. What are continuous and characteristic X-rays ? How are they accounted for ?
17. State and explain Moseley law. Give example of its application.
18. Derive Bragg's law in X-rays diffraction. Give an account of powder method of crystal structure analysis.
19. Explain what continuous and characteristic X-rays and how they are generated ? What are main features of X-rays ?
20. How are the continuous and characteristic X-rays produced ? How can you control intensity and penetration power of X-rays ? Discuss some industrial and engineering applications of X-rays.

Numerical Problems

1. An X-rays machine uses an accelerating potential of 50,000 volts. Calculate the shortest wavelength present in X-rays produced. [Ans. $\lambda_{min} = 0.248$ Å]
2. Calculate the longest wavelength that can be analysed by a rock salt crystal of spacing $d = 2.82$ Å for the (i) first order and (ii) second order. [Ans. (i) 5.64 Å ; (ii) 2.82 Å]
3. An X-rays tube works at 18 kV. Find maximum speed of electron striking the anticathode. [Ans. $v = 8 \times 10^8$ m/s]
4. X-rays spectrum of cobalt (Z = 27) contains strong K_α line of wavelength 0.1785 nm and weak K_β lines having wavelengths of 0.2285 nm and 0.1507 nm due to impurities. Using Moseley law calculate the atomic numbers of the two impurities and identify their nature. the screening constant σ for K-series is unit. [Ans. Z = 24 (chromium), Z = 29 (copper)]
5. The K_α and L_α absorption edges of copper occur at wavelength 0.138 nm and 1.538 nm respectively. Calculate the atomic number of copper. Given that σ = 1 for K_α line and σ = 7.4 for L_α line. [Ans. Z = 27]
6. Calculate the energy of electrons that produced Bragg's diffraction of first order at an angle of 25° when incident on crystal with interplanar spacing of 1.8 Å. [Ans. 1.99×10^{-16} eV]
7. Electrons are accelerated in TV through a potential difference of about 10 kV. Find the highest frequency of electromagnetic waves emitted when these electrons strike the screen of the tube. What kind of waves are these ? [Ans. 2.4×10^{18} Hz, X-rays]
8. The wavelength of the L_α X-rays line of platinum (Z = 78) is 1.321 Å. An unknown substance emits L_α X-rays of wavelength 4.174 Å. Calculate the atomic number of unknown substance. Given k = 7.4 for L_α lines. [Ans. 47.12]

9. Calculate the wavelength of the K_α line from a copper target assuming Moseley's law. Rydberg constant for hydrogen = 10967800 m^{-1} and screening constant = 1. [Ans. 1.551 Å]
10. A TV tube operates with a 20 kV accelerating potential. What are the maximum energy X-rays from the TV set ? [Ans. 2×10^4 eV]
11. In a Laue photograph of an f.c.c. crystal whose unit cell has an edge of length 4.50 Å, what is the minimum distance from the centre of the pattern at which reflections can occur from the planes of maximum spacing, if the potential difference across the X-rays tube is 50 kV and the distance of the film from the crystal is 5.0 cm ? [Ans. 0.97 cm]
12. In a rotating crystal photograph, six layer lines were observed both above and below the zero layer line. If the heights of these layer lines above (or below) the zero layer were 0.29, 0.59, 0.91, 1.25, 1.65 and 2.12 cm, obtain the value for the repeat distance of the crystal along the axis of rotation. Radius of camera = 3.0 cm ; wavelength of X-ray = 1.54 Å. [Ans. 1.60 Å]
13. The Bragg angle corresponding to the first order reflection from (1, 1, 1) planes in a crystal is 30°, when X-rays of wavelength 1.75 Å are used. Calculate interatomic spacing. [Ans. a = 3.031 Å]
14. A certain crystal has a planar spacing of 0.25 nm. What energies are necessary to observe up to three interference maxima for (a) electrons and (b) neutrons ? [Ans. (a) 8.69×10^{-18} J = 54.3 eV ; (b) 4.71×10^{-21} J = 0.0294 eV]
15. Potassium chloride (KCl) has a set of crystal planes separated by a distance d = 0.31 nm. At what glancing angle θ to these planes would be first order Bragg maximum occur for X-rays of wavelength 0.05 nm ? [Ans. θ = 4.6°]
16. A student is told to analyse a crystal using Bragg diffraction. She finds that the ancient equipment has seized up and cannot turn to glancing angles below 30°. She bravely persists and using X-rays with λ = 0.0438 nm, find three weak maxima at θ = 36.7°, 52.8° and 84.5°. What are the order of these maxima, and what is the spacing of the crystal planes ? [Ans. n = 3, 4 and 5 ; d = 0.11 nm]
17. What is the voltage of an X-rays tube that produces X-rays with wavelengths down to 0.01 nm but no shorter ? [Ans. V_0 = 124 kV]
18. Bragg's spectrometer is set for the first order reflection to be received by the detector at glancing angle of 9°18'. Calculate the angle through which the detector is rotated to receive the second order reflection from the same face of the crystal. [Ans. $\theta_2 = 18°48'$, $\theta_2 - \theta_1 = 9°30'$, $2(\theta_2 - \theta_1) = 19°$]
19. X-rays of wavelength λ are reflected in first order at an angle of 8°35' by a NaCl crystal mounted on a Bragg spectrometer. Calculate λ, given density of NaCl, ρ = 2170 kg/m³, molecular weight of NaCl M = 58.46 and Avogadro's number $N = 6.02 \times 10^{26}$ kg molecules. [Ans. λ = 0.841 Å]
20. Electrons are accelerated to 728 volts and are reflected from a crystal. The first reflection maximum occurs when glancing angle is 8°. Determine the interplanar spacing of the crystal. [Ans. d = 1.635 Å]
21. The glacing angles of reflection for K_α X-rays from palladium are 5.4° from (100) plane ; 7.6° from (110) planes and 9.4° from (111) planes. From the above data determine the cubic lattice structure of the crystal. [Ans. $\frac{1}{d_1} : \frac{1}{d_2} : \frac{1}{d_3} = 0.0914 : 0.1323 : 0.1633 = 1 : \sqrt{2} : \sqrt{3}$: simple cubic]
22. The Bragg angle corresponding to a reflection for which $(h^2 + k^2 + l^2) = 8$ is found to be 14.35°. Determine the lattice parameter of the crystal. X-rays of wavelength 0.71 Å are used. If there are two other reflections with smaller Bragg's angles, what is the crystal structure. [Ans. 4.05 Å, f.c.c]

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23. Electrons bombarding the anode of a Coolidge tube produce X-rays of wavelength 1 Å. Find the energy of each electron at the movement of impact. [Ans. 1.24×10^4 eV]

24. What element has a K_α X-ray line of wavelength 0.71 Å ? [Ans. Z = 42 (molybdenum)]

25. The K_α line from molybdenum has a wavelength 0.71 Å. Calculate the wavelength of K_α line of copper. The atomic number of molybdenum is 42 and that of copper is 29. [Ans. 1.52 Å]

26. (a) What is the minimum wavelength in white radiation of X-rays if the applied voltage on the tube is 30 kV ?

(b) What is the wavelength associated with an electron of kinetic energy of 10 keV ?

(c) What is the wavelength associated with a neutron at 300 K (K.E. = $\frac{1}{2}k_B T$) ?

[Ans. (a) 0.40 Å ; (b) 3.85 Å ; (c) 2.52 Å]

27. Calculate the wavelength of the K_α line of thallium (Z = 81) if the K_α line of thallium (Z = 81) if the K_α line of chlorine (Z = 17) is 0.4718 nm, assuming the screening constant to be equal to one. [Ans. λ = 0.09436 nm]

28. An electron falls from *L* level to the *K* level in an atom of atomic number 100. Find the wavelength of the X-rays produced thereby. [Ans. 15.3 nm]

Chapter 11

Ultrasonics

11.1 Introduction

Acoustics. The science of vibrations and waves in elastic media. Elastic media comprises air, water and solid bodies such as metals, concrete, wood etc.

Ultrasonics. A branch of acoustics, which deals a study of everything in frequency range 20 kHz to 500 MHz.

Sound is a form of energy in the process of being transmitted through matter by means of mechanical vibration. It can therefore be propagated through gases, liquids and solids. However, as should be well known, but too often is forgotten, a vacuum can support sound.

Ultrasound is sound in which the frequency of vibration is too high for it to be audible to the human ear. The limit of audibility is about 20000 cps but varies from person to person, it tends to highest with young people and decreases as you get elder. It is general to define *ultrasound* as a sound of frequency above 20000 cps, since the S.I. unit for cps is Hz. Ultrasound, therefore has a frequency greater than 20000 Hz *i.e.*, 20 kHz.

Types of Ultrasound	
Longitudinal (alternately compressional)	**Transverse (after shear ultrasound)**
A wave composed of alternate surfaced of compression and rarefaction travelling perpendicular to these surfaces. Particle motion in the direction of travel.	A particle displacement at each point in a material perpendicular to the direction of wave propagation. Transverse waves not supported by liquids and gases.

- Sound frequencies that are higher than detectable with the human ear.
- Sound waves above frequency normally detectable by the human ear, *i.e.*, above 15 to 20 kHz are referred as ultrasonic waves. In most practical applications of ultrasonics, pulse or packets, containing a number of oscillation cycles are set through the solid or liquid under investigation.
- An appreciation of useful applications of ultrasound dates back to the development of RADAR and SONAR in the late 1930's and easily 1940s World War II era. One of very earliest uses was for detecting flaws in materials. Today's

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ultrasound is one of several effective methods used in the field of non-destructive testing (NDT) and inspection. Ultrasonic devices also used in industrial processing applications such as cleaning, drilling, emulsifying soldering and welding. Possibly the most nature worthy of ultrasonic devices are found in the medical fields for use in diagnostics where ultrasound views with X-rays and other diagnostic procedures. Ultrasound is also applied in some security systems to detect inducers.

- Sound waves with frequency above 20,000 Hz are called ultrasonic waves. Because of their very small wavelengths, narrow beams of ultrasonic waves can be sent out and reflected from small objects.
- Bats can emit and detect frequency up to about 120 kHz corresponding to a wavelength of 2.8 mm, which they use to locate small prey such as moths. Ships use a device called SONAR (Sound Navigation and Ranging) to detect the outlines of submarines and other submerged objects with ultrasonic waves.
- In medicine ultrasonic waves are used for diagnostic purposes.
- To create a sonogram, ultrasonic waves are passed through the body and information about the frequency and intensity of transmitted and reflected waves is processed to construct a three dimensional picture of the body interior.

The correct scenario of ultrasonic waves can be found in Fig. 11.1.

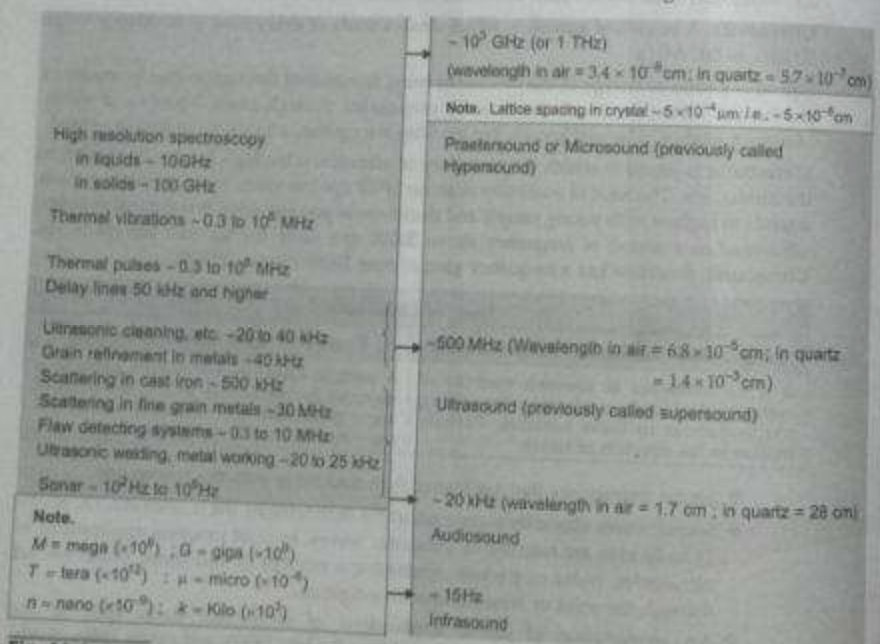

Fig. 11.1. Acoustic frequency scale.

11.2 Properties of Ultrasonic Waves

1. ***Frequency*** **(ν).** All the sound waves oscillate at a specific frequency, or number of vibrations or cycles per second, which we experience as pitch in the familiar range of audible sound. Human hearing extends to a maximum frequency about 20,000 cycles per second (20 kHz), while the majority of ultrasonic applications utilised frequency between 20 kHz to 500 MHz. At frequencies in the megahertz range, sound energy does not travel efficiently through air or other gases, but it travels freely through most liquids and common engineering materials. Hence ultrasonic waves are high frequency waves.

2. ***Velocity (v).*** The speed of a sound wave (a ultrasound wave varies depending on the medium through which it is travelling) affected by medium's density (ρ) and elastic properties [Elastic constant, (E)]. Different types of sound waves will travel at different velocities :

i.e.,
$$v = \sqrt{\frac{E}{\rho}} \quad ...(1)$$

3. ***Wavelength (λ).*** Any type of wave will have an associated wavelength, which is distance any two corresponding points in the wave cycle as it travels through a medium. Wavelength is related to frequency and velocity by simple equation

$$\lambda = v/\nu \quad ...(2)$$

Wavelength is a limiting factor that control the amount of information that can be derived from the behaviour of the wave. Hence ultrasonic waves are of low wavelength waves negligible diffraction is shown in ultrasonics.

4. ***Energy.*** We know from the relation $E = h\nu$; since frequency of ultrasonic waves is high. Hence ultrasonic waves are high energetic waves.

5. ***Mode of Propagation.*** Sound waves in solids can exist in various modes of propagation that are defined by the type of motion involved. Longitudinal waves and shear waves are the most common mode employed in ultrasonics testing. Surface waves and plate waves is also used on occasion.

- A *longitudinal* or *compressional wave* is characterised by particle motion in the same direction as wave propagation, as from a piston source. Audible sound exists as longitudinal wave.
- A *shear or transverse wave* is characterised by particle motion perpendicular to the direction of wave propagation.
- A *surface or Rayleigh wave* has an elliptical particle motion and it travels across the surface of a material, penetrating to a depth of approximately one wavelength.
- A *plate or Lamb wave* is a complex mode of vibration in thin plates where material thickness is less than one wavelength and wave fills the entire cross-section of the medium.

Sound waves may be converted from one form to another. Most commonly shear waves are generated in a test material by introducing longitudinal waves at a selected angle.

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6. ***Variable Limiting Transmission of Sound Waves.*** The distance that a wave of a given frequency and energy level will travel depends on the material through which it is travelling. As a general rule, material that are hard and homogeneous will transmit sound waves more efficiently than those that are soft and heterogeneous or granular. Three factors govern the distance of sound waves will travel in a given medium : beam spreading, attenuation and scattering. As the beam travels, the leading edge becomes wider, the energy associated with the wave is spread over a larger area, and eventually the energy dissipates. Attenuation is energy loss associated with sound transmission through a medium, essentially the degree to which energy is absorbed as the wavefront moves forward. Scattering is random reflection of sound energy from grain boundaries and similar micro-structure. As frequency goes up, beam spreading increases but the effect of attenuation and scattering are reduced.

7. ***Reflection at a Boundary.*** When sound energy travelling through a material encounters a boundary with another material, a portion of the energy will be reflected back and a portion will be transmitted through. The amount of energy reflected or reflection coefficient, is related to the relative acoustic impedance of the two materials. Acoustic impedance in turn is a material property defined as density multiplied by the speed of sound in a given material. For any two materials, the reflection coefficient as a percentage of incident energy pressure may be calculated through the formula

$$R = \frac{Z_1 - Z_2}{Z_2 + Z_1} \qquad \text{...(3)}$$

where R = reflection coefficient (percentage of energy reflected)

Z_1 = acoustic impedance of first material

Z_2 = acoustic impedance of second material.

For the metal/air boundaries commonly seen in ultrasonic waves applications like flaw detection sodar etc., the reflection coefficient approaches 100%. Virtually all of the sound energy is reflected from a crack or other discontinuity in the path of the wave. This is fundamental principle that makes ultrasonic flaw detection, detection of any object in sea etc.

8. ***Angle of Reflection and Refraction.*** Sound energy at ultrasonic frequencies is ***highly directional*** and sound beams used for flaw detection in any metal, object detection in the sea are well defined. In situation where sound reflects off a boundary, the angle of reflection equals the angle of incidence. A sound beam that hits a surface at perpendicular incidence will reflect straight back. A sound beam that hits a surface at angle will reflect forward at the same angle.

Sound energy that is transmitted from one material to another bends in accordance with Snell's law of refraction. Again, a beam that is travelling straight will continue in a straight direction, but a beam that strikes a boundary at an angle will be bent according to the formula

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \qquad \text{...(4)}$$

where

θ_1 = incident angle in first material

θ_2 = refracted angle in second material

v_1 = sound velocity in first material

v_2 = sound velocity in second material

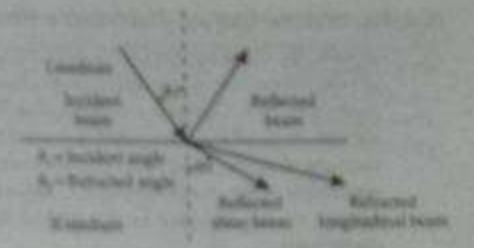

Fig. 11.2 Illustration of reflection and refraction.

This relationship is an important factor in angle beam testing.

11.3 Transducers

A transducer is a device that generates and receives (i.e., it senses) sound waves. The transducer essentially function as an energy converter. That is, it converts acoustical energy to other forms of energy (e.g., electrical, mechanical, or thermal). A transducer is said to be reversible if it can convert in either direction. Most high intensity ultrasonic generators in use are basically crystal oscillators or magnetostrictive devices.

Transducers fall into the following categories :

1. *Crystal Oscillators* that operate on the piezoelectric effect, which is reversible.
2. *Magnetostrictive Devices* based on magnetostriction phenomenon, which is also reversible.
3. *Mechanical generators and receivers* which include whistles and sirens acting as generators and also radiometers and Rayleigh discs serving as receivers.
4. *Electromagnetic Transducers* which operate on the same principle as the customary audio loudspeaker, but can function only in the lower range of ultrasonic frequencies.
5. *Miscellaneous Types*, such as chemical, thermal and optical transducers.

In addition there are ultrahigh frequency transducer that work in the megahertz (MHz) and gigahertz (GHz) ranges.

Here, we concentrate on crystal oscillators and magnetostriction devices that constitute a major portion of the transducer types of currently in principal use.

Ultrasonic receivers fall into two categories, namely

(i) *Receivers that terminate ultrasonic paths of propagation.* They have cross-sectional dimensions extending over several wavelength, with the result that the presence of such transducer will materially affect the acoustic field mainly through reflections.

(ii) *Receivers that serves as probe.* The purpose of an **ultrasonic probe** is to gauge the characteristic of an acoustic field, so its dimensions must be sufficiently small so as not to affect the field. A probe diameter is typically only about one-tenth of the wavelength.

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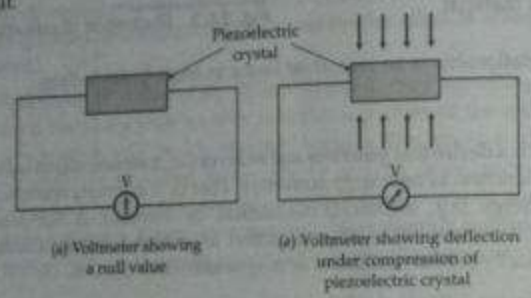
11.4 Piezoelectric Crystals (Piezoelectric Effect)

In 1880, the Curie brothers (J. Curie and P.Curie) discovered that *when a crystal having one or more polar axes or lacking axis symmetry is subjected to mechanical stress an potential difference occurs (Fig. 11.3).* Consider a segment of such a crystal, in the form of a slab or a disc, that is act with its parallel surface running normal to a polar axis. When this segment undergoes a mechanical stress, equal and opposite electric charges arise on the parallel surfaces. The magnitude of the charge density (*i.e.*, dielectric polarization is directly proportional to the applied stress), provided by the applied stresses does not strain the crystal beyond its elastic limit.

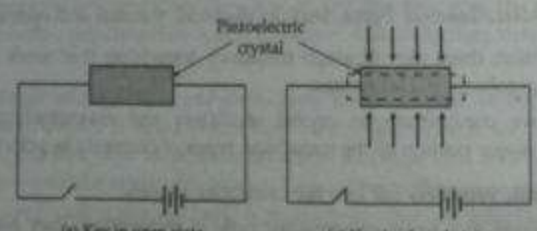


(a) Voltmeter showing a null value

(b) Voltmeter showing deflection under compression of piezoelectric crystal

Fig. 11.3 Direct piezoelectric effect.

The opposite effect, predicted by Lippmann in 1881 and verified experimentally by the Curie brothers in the same year when an electric field is applied in the direction of a polar axis, causing a mechanical strain in the crystal segment (Fig. 11.4). The amount of strain is directly proportional to the intensity of applied electric field. From the view point of the principle of conservation of energy, the piezoelectric effect and its converse may be deemed to be equal and opposite. Such effects occur in crystals such as quartz (a member of the trigonal system) as shown in Fig. 11.5 ; Rochelle salt and lithium sulphate.



(a) Key in open state

(a) Key in closed state

Fig. 11.4 Inverse piezoelectric effect.

Quartz is very commonly applied for ultrasonic generation. A quartz crystal is shown in Fig. 11.5 with a hexagonal cross-sectional normal to the non-polar optic axis, denoted by

the z-axis. The axes joining opposite edges are designated as x-axes and the associated axes, which are perpendicular to these and joining opposite faces are termed as y-axis. The x- and y-axes are *polar axes* and slab cut with their faces perpendicular to them manifest the piezoelectric effect. Crystals that are cut with their faces perpendicular to an x-axis and y-axis are termed ***x-cut*** and ***y-cut*** crystals respectively. The x-cut crystals are generally utilized to propagate *compression waves*, and the y-cut crystals are applied to generate *shear waves*.

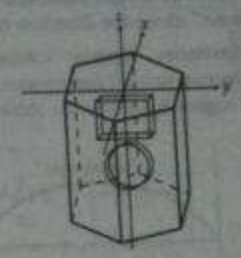
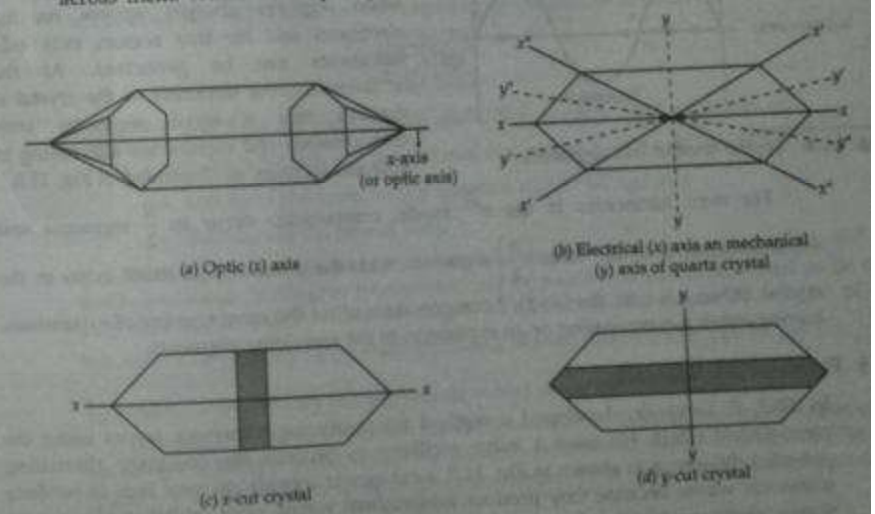


Fig. 11.5A A hexagonal quartz crystal with x-cut rectangular and circular plates.

Now consider an x-cut crystal in the form of a rectangular prism is shown in Fig. 11.5A. Applying an electric field along the x-axis yields a compression in that direction, while an expansion occurs simultaneously along the y-direction. If the direction of field is reversed, an expansion occurs along the x-axis with an associated compression along the y-axis. No strain, however, occurs along the z-axis. If a pair of surfaces normal to either of the polar axis (x- and y-axis) are coated with a conductive material to form electrodes, small amplitude oscillations will result when an alternating voltage of frequency v is applied across them. When the frequency equals one of the natural frequencies of mechanica



(a) Optic (x) axis

(b) Electrical (x) axis an mechanical (y) axis of quartz crystal

(c) z-cut crystal

(d) y-cut crystal

Fig. 11.5B Quartz Crystal.

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vibration for a particular axis, the response amplitude jumps to a considerably higher value. Crystals are generally operated at resonant frequencies for either "length" or "thickness" vibrations as denoted by resonance occurring in the direction parallel with or normal to the radiating surfaces, respectively. The natural frequencies for mechanical vibrations is proportional to the inverse of the dimension along which they occur, so it becomes obvious the lower frequencies are generated by "length" vibrations along the direction of the longer dimension while the higher frequencies are produced by "thickness vibrations" along the direction of the smaller dimensions.

Maximum acoustic intensities are obviously obtained by operating at the fundamental natural frequencies. But material constraints in crystals may necessitate the use of higher harmonics to obtain higher frequencies. For example an x-cut quartz plate can be only 0.15 mm thick in order to generate a fundamental "thickness" mode for 20 MHz. Such a quartz plate is extremely brittle and it can shatter under the impetus of exceedingly high applied voltage, or its dielectric properties may break down. To avoid this situation, it is customary to use thicker slabs of crystal with lower resonance frequencies and operate at one of the upper harmonics. An example is the vibration of a 1 cm thick quartz crystal at its 191st harmonic to generate 55 MHz ultrasound.

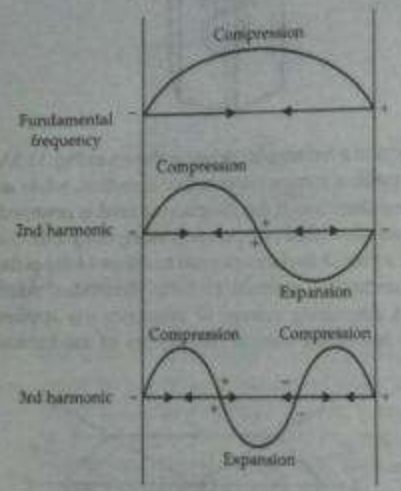

Fig. 11.6 Crystal divided into segments.

The piezoelectric effect occurs only when opposite charges appear on the electrodes and for that reason, only odd harmonics can be generated. At the nth harmonic, the thickness of the crystal is divided into n equal segments with compressions and expansions alternating in adjacent sections, as illustrated in Fig. 11.6.

For even harmonics in the nth mode, compression occur in $\frac{n}{2}$ segments and expansions occur in the other $\left(\frac{n}{2}\right)$ segments, with the result no net strain exists in the crystal. When n is odd, the $(n-1)/2$ compressions offset the same number of expansions, leaving either a compression or an expansion in the remaining segments.

11.5 Piezoelectric Generator

In 1917, P. Langevin developed a method for producing ultrasonic waves using the piezoelectric effect. He used a valve oscillator to produce the necessary alternating potential difference as shown in Fig. 11.7. x-cut quartz crystals are used here to produce ultrasonic waves because they produce longitudinal waves. y-cut crystal produce shear waves which can travel only in solids.

The x-cut quartz crystal Q is placed within two metallic foils which are connected to the anode A and grid G of a triode valve. An oscillator tuned circuit LC is connected to anode A. High tension battery (H.T.B.) is shunted by a bypass capacitor C_1 to prevent higher frequency currents from passing through the battery. A grid leak resistor R_g is used to regulate the action of grid.

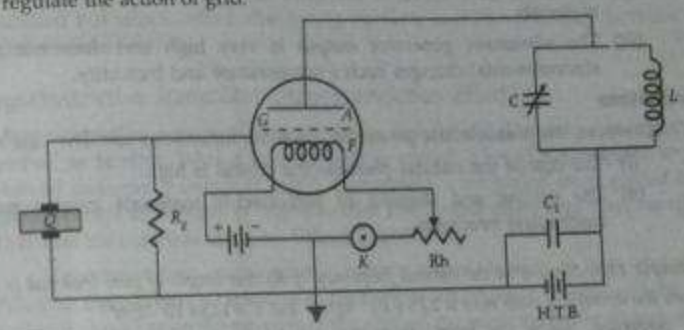

Fig. 11.7 Piezoelectric Generator.

When a high frequency current flows in anode circuit, an alternating potential difference of the same frequency is applied to the faces of the crystal. Consequently, the crystal undergoes linear expansions and contractions or is set into mechanical vibrations. The frequency of alternating potential difference can be varied by varying the capacitance of capacitor C. When this frequency becomes equal to the fundamental natural frequency of the crystal, the crystal is set into resonant mechanical vibrations, thus, producing ultrasonics. The expression for the frequency of ultrasonics is given by

$$\nu = \frac{P}{2t}\sqrt{\frac{Y}{\rho}}\ \text{Hz} \qquad ...(5)$$

where P = 1, 2, 3 etc. for fundamental mode, first, second, third overtones etc.

Y = Young's modulus of the material (for quartz $Y = 7.9 \times 10^{10}\ N/m^2$)

ρ = density of the material (for quartz $\rho = 2.65 \times 10^4\ kg/m^3$)

t = thickness of the crystal slice.

With a moderate slice of quartz crystal, ultrasonic waves of frequencies 5.4×10^5 Hz can be produced. Still higher frequencies may be possible in the fundamental mode with very thin plates but they are too fragile to withstand the strain. Tourmaline plates, however, can be used to produce ultrasonic waves upto 1.5×10^8 Hz. In Eq. (5) $\sqrt{\frac{Y}{\rho}}$ is equivalent to the velocity of longitudinal waves along the x-direction.

In a piezoelectric generator the crystal is made to oscillate at its natural frequency for maximum efficiency by adjusting the variable capacitor C in the tank circuit. Then the frequency of the oscillatory circuit gives the frequency of vibration of quartz crystal *i.e.*,

$$\nu = \frac{1}{2\pi\sqrt{LC}} \qquad ...(6)$$

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Advantages

The principal advantages of piezoelectric generator include :

(i) One can generate ultrasonic waves whose frequencies are of upto 500 MHz.

(ii) A wide range of frequencies can be generated at lower cost using synthetic materials.

(iii) The ultrasonic generator output is very high and does not depend on environmental changes such a temperature and humidity.

Limitations

However, the piezoelectric generator has some limitations too. They are :

(i) The cost of the natural piezoelectric crystal is high.

(ii) The cutting and shaping of piezoelectric materials involve monotonous mechanical processes.

Example 11.1. *Show that the natural frequency of 40 mm length of pure iron rod is 49.75 kHz. Given the density of pure iron is 2.75×10^3 kg/m^3 and $Y = 115\times10^9$ N/m^2.*

Solution. Since we know that the natural frequency with $Y = 115\times10^9$ N/m^2, $t = 40$ mm $= 40\times10^{-3}$ m, $\rho = 2.75\times10^3$ kg/m^3

$$v = \frac{1}{2t}\sqrt{\frac{Y}{\rho}}$$

$$= \frac{1}{2\times40\times10^{-3}}\left(\frac{115\times10^9}{7.25\times10^3}\right)^{\frac{1}{2}} \text{ Hz}$$

$$= 49750 \text{ Hz} \quad \Rightarrow \quad 49.75 \text{ kHz.}$$

Example 11.2. *Calculate the inductance to produce ultrasonics of 10^6 Hz with capacitance of 0.025 μF.*

Solution. Since we know that the natural frequency of LC circuit is given as

$$v = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

or

$$L = \frac{1}{4\pi^2 v^2 c}$$

$$L = \frac{1}{4\pi^2 v^2 C}$$

where $v = 10^6$ Hz, $C = 0.025$ μF

$$L = \frac{1}{4\times(3.14)^2\times(10^6)^2\times(0.025)\times10^{-6}} = 1 \text{ H}$$

11.6 The Electrostrictive Effect

The electrostrictive effect, which is the electrical analog of the magnetostricitve effect, occurs in all dielectrics, but it is not a very pronounced phenomenon in most materials except for a certain class of dielectrics. The effect is much more apparent in this class

namely the ferroelectrics. *An electric field applied along a given direction produces a mechanical strain.* The magnitude of the strain is proportional to the square of the strength of the applied electric field and is therefore independent of the sense of the field. The positive strain may thus result for positive and negative values of the excitation field. For a sinusoidally varying electric field, the waveform of the strain assumes that of a rectified but unsmoothed alternating current, and its frequency is twice that of the applied field.

11.7 Magnetostrictive Transducer (Magnetostriction Effect)

Magnetostriction occurs in ferromagnetic materials and certain non-metals which are termed as **ferrites**. *When a magnetic field is applied, a bar of ferromagnetic or ferrimagnetic material undergoes a change in length.* Conversely, *a mechanical stress applied to the bar will cause a change in intensity of magnetization.* The former effect was discovered by **Joule** in 1847 and the converse effect by **Villari** in 1868.

Magnetostriction occurs prominently in materials such as iron, nickel and cobalt. Whether there occurs an increase or decrease in length fully depends on the nature of the material as well as on the strength of the applied magnetic field. The change in length does not depend on the direction of the magnetic field. The magnitude of the strain varying as a function of the applied magnetic field is shown in Fig. 11.8 for four different materials, viz cost cobalt, permendur, nickel, and iron. Fig. 11.9 shows strain varying as a function of magnetic polarisation. The magnetostrictive effect generally decreases with a rise in temperature and disappears altogether at the Curie temperature.

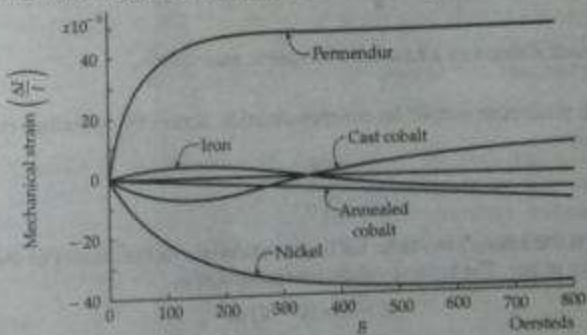

Fig. 11.8 Mechanical strain as a function of magnetic field.

When a sinusoidal varying magnetic field is applied in the direction of the axis of a bar of ferromagnetic material, the bar will oscillate at double the frequency of the applied field. In accordance with the associated curve in Fig. 11.9, a decrease in length occurs when a field is applied to nickel, regardless of the sense of the field. A negative strain occurs every half cycle. The waveform of the strain occurs as a rectified sine curve, with the result that unwanted harmonics may be generated. A purely sinusoidal wave corresponding to the frequency of the applied field, along with a markedly increased energy output, will be obtained if the bar is polarised. This is achieved by simultaneously applying the

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alternating field and a direct magnetic field of sufficiently high intensity for the value of the resultant field to remain above zero.

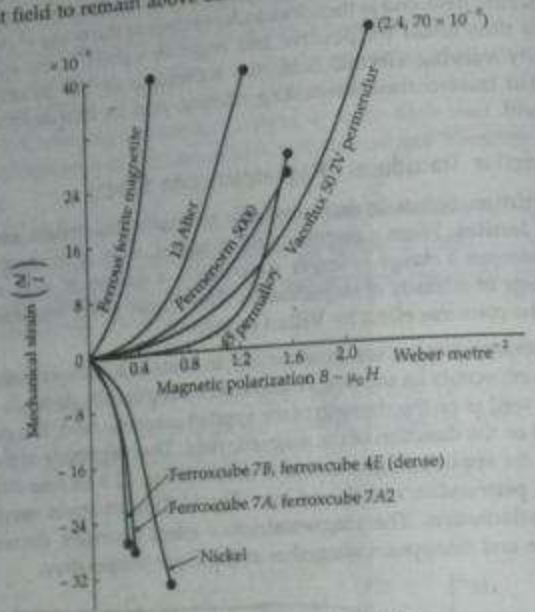

Fig. 11.9 Mechanical strain as a function of magnetic polarization.

The maximum output for magnetostriction occurs by operating at the fundamental frequency

$$\nu_r = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \qquad ...(7)$$

where Y is the Young's modulus for the bar material, ρ is the density of the material, and l is the length of bar. The typical values for quartz are as

$$Y = 7.9 \times 10^{10}\ N/m^2$$

$$\rho = 2.65 \times 10^4\ kg/m^3$$

The term $\sqrt{Y/\rho}$ in Eq. (7) be the propagation velocity of sound in the material.

11.8 Magnetostriction Oscillator

The oscillator generates ultrasonics based on magnetostriction phenomenon. This oscillator was first designed by **G.W.Pierce** and is shown in Fig. 11.10.

A nickel rod is clamped in the middle and two coils L_1 and L_2 wound round its two halves. Coil L_1 is included in the plate circuit and L_2 in the grid circuit of a triode oscillator and the two coils are indirectly coupled.

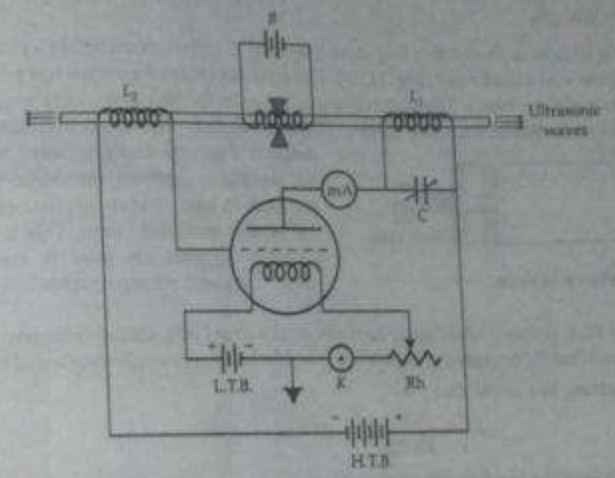

Fig. 11.10 Magnetostriction Oscillator.

When a circuit is closed the rod gets magnetized by the plate current passing through coil L_1. Any change in the plate current results in the change of magnetization of rod which causes a change in its length. This change in length gives rise to a change in flux linked with the grid coil L_2, thereby inducing an e.m.f. in it. This e.m.f. is amplified and passed on to the plate circuit so that oscillations are maintained. The frequency of oscillations $\nu = \frac{1}{2\pi\sqrt{L_1 C}}$ is controlled by the variable capacitor C. If this is frequency matches with the natural frequency of the longitudinal vibrations of the rod, the oscillations become vigorous because of the resonance effect. The variation in the deflection of the milliammeter (mA) will give a definite indication of the resonance. It should be noted here that the best effect is obtained when the nickel bar is previously magnetized by means of a second coil L_2 wound over it and carrying a direct current.

Advantages

Some of the merits of the magnetostriction oscillator are :

(*i*) The construction is very simple.

(*ii*) The cost of the generator is low.

(*iii*) At low ultrasonic frequencies, a large power output is possible without the risk of damage of the oscillatory circuits.

Disadvantages

(*i*) It cannot generate ultrasonics of frequency above 3×10^7 kHz.

(*ii*) The frequency of oscillations depends upon operating temperature. At higher temperatures the output from the oscillator will not be very stable.

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11.9 Galton's Whistle

Galton's whistle is similar to a boy scout's whistle. It differs in that it has a piston provided with screw and milled head (Fig. 11.11). The screw operates through a nut provided in the back plate of the whistle. The piston is moved so as to shorten its effective length. The pitch of the sound emitted increases as the effective length is shortened. Eventually a stage is reached when no sound is heard, when in fact the whistle is emitting ultrasonic sound. Dogs are able to hear sound frequencies far beyond the human audibility limit. This is reason why Galton's whistles are used to train dogs for patrol duty and reconnaissance operations.

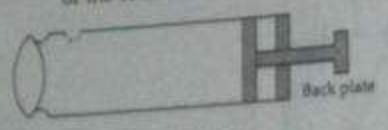

Fig. 11.11 Galton's Whistle.

Example 11.4. *Galton's whistle may be regarded as a closed tube. Calculate the least length of the whistle such that the frequency of the whistle is 30,000 Hz (Assume velocity of sound be 330 ms^{-1}).*

Solution. We know that $v = \nu\lambda$

or $$\lambda = \frac{v}{\nu} = \frac{330}{30000} = 11 \times 10^{-3}\ \text{m}$$

Therefore the least length of the tube is

$$l = \frac{\lambda}{4} = 2.75 \times 10^{-3}\ \text{m}.$$

11.10 Determination of the Velocity of Ultrasonic Waves in Liquid (Acoustical Grating)

When ultrasonic waves propagate in a liquid (suppose kerosine oil) medium, the alternating compressions and rarefactions changes the density of the medium. Due to this periodic variation of refractive index of liquid occurs. Such a liquid column subjected to ultrasonic waves constitutes an **acoustical grating**. When monochromatic light (suppose sodium light) passes through the liquid column at 90° to the waves, the liquid causes the light diffraction.

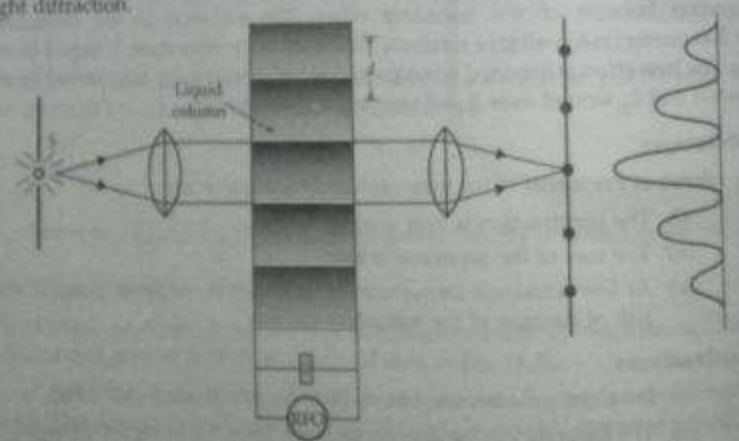

Fig. 11.12 A acoustic grating produced by a liquid column subjected to ultrasonics.

Fig. 11.12 shows the experimental arrangement. Stationary ultrasonics are produced in a liquid contained in a glass tube. The density and the refractive index of that particular liquid are minimum at antinodal and maximum at nodal points. So the nodal areas act as opaque regions, while antinodal areas act as transparent regions for light. Thus liquid column having a ruled grating.

When crystal is at rest, only single image of the slit is formed on the screen. When the crystal is in excited position, a diffraction pattern is constructed. This diffraction pattern consists of a central maxima flanked by first order, second order maxima and minima etc. The grating period 'd' equal to $\frac{\lambda_{ult}}{2}$ and is given by

$$d \sin\theta = m\lambda \qquad \text{...(8)}$$

where λ_{ult} = wavelength of ultrasonic waves

λ = wavelength of monochromatic light beam

m = order of maxima

$$\therefore \quad \lambda_{ult} = \frac{2m\lambda}{\sin\theta} \qquad \text{...(9)}$$

If we know the value of λ and m and measuring angle θ, λ_{ult} can be calculated using Eq. (9). The frequency ν of the waves is known from the frequency of the oscillator (Radio frequency oscillator). The velocity of waves in the liquid can be found from the relation

$$v = \nu\lambda_{ult} \qquad \text{...(10)}$$

The above method is known as acoustic diffraction method.

Example 11.4. *Find the velocity of the given liquid with the following data :*

(*i*) *Wavelength of sodium light* = 5896 Å

(*ii*) *angle* $\theta = 8°$.

(*iii*) *order of maximum* = 1

(*iv*) *frequency of the R.F. oscillator* = 4.2 MHz

Solution. Since we know that $\lambda_{ult} = \frac{2m\lambda}{\sin\theta}$

$$\lambda_{ult} = \frac{2 \times 5896 \times 10^{-10}}{\sin\left(\frac{8}{60}\right)^\circ}$$

Then $$v = f\lambda_{ult} = \frac{4.2 \times 10^6 \times 2 \times 5896 \times 10^{-10}}{\sin\left(\frac{8}{60}\right)^\circ} = \frac{4.2 \times 0.2 \times 5896}{2.327 \times 10^{-3}} = 2128\ \text{m/sec}$$

11.11 Ultrasonic Non-Destructive Techniques

Ultrasonic Testing Systems. There are *three* basic ultrasonic systems that are commonly used in industries :

1. Pulse-echo method 2. Through transmission method 3. Resonance system

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11.11.1 Pulse Echo System

In pulse-echo inspection, short burst of ultrasonic pulses or wave packets are introduced into a test piece at regular interval of time. If the pulses encounter a reflecting surface, some of the energy is reflected. The reflected energy is monitored ; both the amount of energy reflected in a specific direction and the time delay between transmission of the initial pulse and receipt of the echo are measured. In most ultrasonic test equipment the received signals are displayed on the screen of a cathode rays oscilloscope. As shown in Fig. 11.13 most pulse echo systems consists of the following :

(i) a master timer (an electronic clock),
(ii) an electronic pulse (signal) generator ;
(iii) signal transmitting transducer ;
(iv) reflected signal receiving transducer ;
(v) an echo-signal amplifier and
(vi) a display device such as CRT.

At regular intervals, the master timer (the electronic clock) triggers the signal generator. This imposes a short burst of high frequency alternating voltage on the transducer (probe). Simultaneously, the clock activates a time-measuring circuit (time base amplifier) connected to the display device. Using pulse repetition rate control operator can preselect constant interval between pulses.

The transducer converts the pulse of alternating voltage into pulse of mechanical vibration. The mechanical vibration (ultrasound) is introduced into the test piece through a couplant.

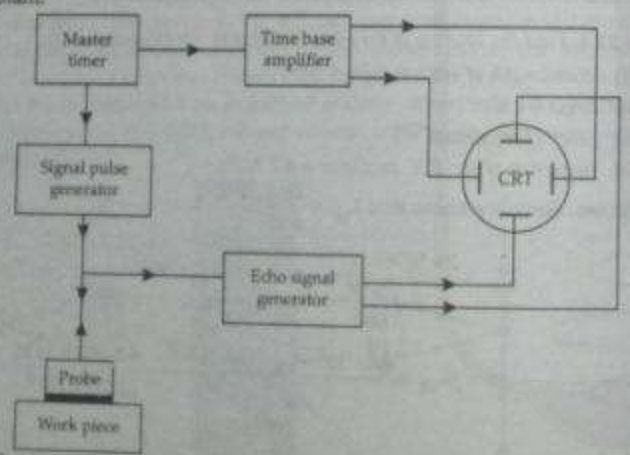

Fig. 11.13 Pulse Echo Technique.

When the pulse of ultrasound encounters a reflecting surface that is perpendicular to the direction of travel, ultrasonic energy is reflected and returns to the transducer. Upon reaching the transducer through the couplant, the returning pulse causes the transducer element to vibrate, which induces an alternating electrical voltage across the transducer.

The induced voltage is instantaneously amplified, then fed into the display device (CRT). Sometimes it is advantages to use separate sending and receiving transducers for pulse-echo inspection.

Depending mainly on geometrical consideration, these separate transducers may be housed in a single search unit or in two separate search units.

NOTE
For sound transmission inspection, separate transducers are always used.

Advantages

(i) Can reveal internal defects.
(ii) High sensitivity to most cracks and flaws.
(iii) High speed test with immediate results.
(iv) Can be automated and recorded
(v) Portable
(vi) High penetration in most important materials (upto 60' in steel).
(vii) Indicate flaw size and location.
(viii) Can also be used to measure thickness, Poisson's ratio, ultrasonic velocity, elastic moduli etc.
(ix) Presents no radiation or safety hazard.

Disadvantages

(i) Difficult to use with complex shapes.
(ii) External surfaces and defect orientation can affect the test (may need dual transducer or multiple inspection.).
(iii) A couplant is required.
(iv) The area of coverage is small (inspection of large areas requires scanning).
(v) Trained, experienced and motivated technician may be required.

Limitations

Material Limitations	Geometrical Limitations	Permanent Record
❑ Few can be used on metals, plastics, ceramics, glass, rubber, graphite and concrete as well as joints and interfaces between materials.	❑ Small, thin or complex-shape parts or parts with rough surfaces and non-homogeneous structure pose the greatest difficulty.	❑ Ultrasonic signals can be recorded for subsequent playback and analysis. Strip chart can also be used.

11.11.2 Ultrasonic Scanning Technique

We know that it is the scanning technique of RADAR, the storage and retrieval techniques of computers and TV display methods applied to ultrasound that have to the application of ultrasonics in medicine. Different methods of scanning are given below :

- The first method of scanning is to have a rotor which is mechanically turned, and is provided with a matched set of transducers. All the transducers possess the same frequency but only the one in contact with the skin gets activated.

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- Another method is to have an array of transducers activated in sequence. At the end of each row there is a fly back action ; the first element of the next row is activated and the transducers of that row are triggered in sequence.
- The third method is one in which only a few elements are present. They give an ultrasonic beam which can be steered, to scan different parts of the organs several times every second.

Figs. 11.14(*a*), (*b*) and (*c*) depict the active elements of the system.

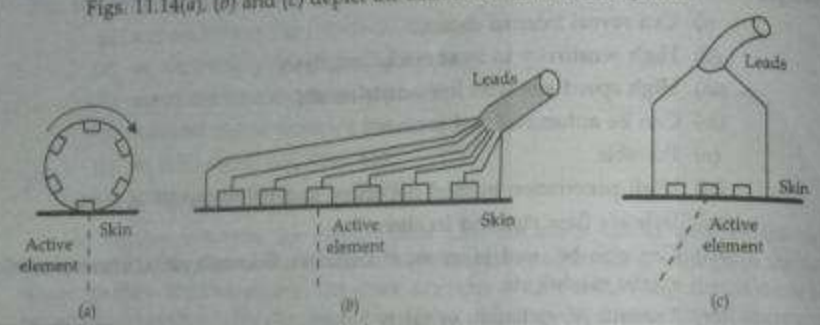

Fig. 11.14 Different methods of ultrasonic scanning.

The actual rate of scanning depends upon the part of the body being scanned as shown in Table 11.1.

Table 11.1 *Scanning rate of parts of the body.*

Part of the body	Scanning rate
Abdominal examination	4 frames per second (125 lines per frame)
Deep penetration (150 mm)	40 frames per second
Movements of heat values	60 frames per second
Gynaecological or cardiac examination	Ultrasonic frequency of 2-3 MHz
Blood flow studies	Ultrasonic frequency of 10 MHz

In an *A*-scan mode of operation, the amplitude data is represented and in the *B*-scan, the brightness data is represented. The image obtained during the scan is stored in the memory of a scan converter which displays it in rectangular or trapezoidal or sector form on a TV display screen. A digital or pixel display is also in use now. A possible flow diagram for an *A*-scan is shown in Fig. 11.15.

The flow diagram starts with a pulse generator. The time interval between pulses (*i.e.*, the repetition rate) can be altered to suit the purpose to which the scanner is put.

NOTE

In case of a blood flow of speed $100\ ms^{-1}$, the frequency $\nu = 10$ MHz.

This is connected to an attenuator through a transmitter. The attenuator is used to adjust the height and width of the pulse. The pulse generator also connected to *Swrep gain generator* which increases the gain with time. This is necessary because the farther the ultrasound travels into the body, the greater is the absorption and attenuation. The deeper the pulse travels, the longer it will take to return to the transreceiver. Hence the gain is increased. The probe is connected to *y*-plates and the pulse generator to the *x*-plates. The oscilloscope has been a long persistence screen. The probe scans the organ 40 to 60 frames per second and persistence of vision enables us to see the picture live.

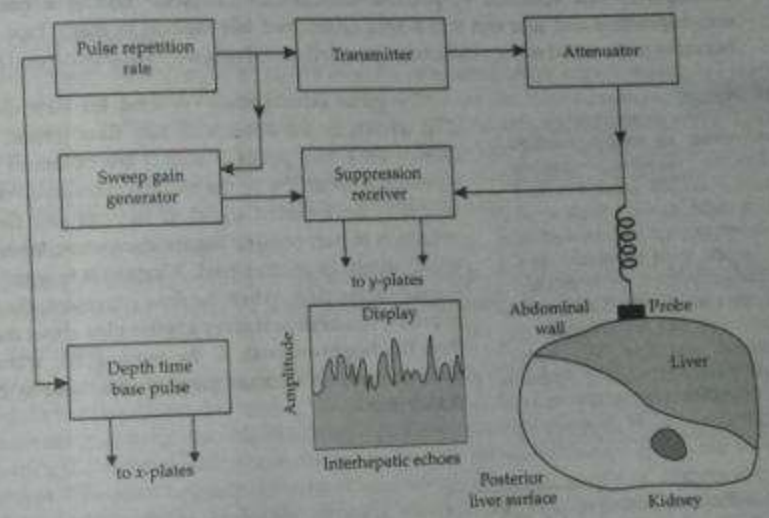

Fig. 11.15 Flow diagram of ultrascan (A-scan).

Applications

Ultrasound techniques have been replaced X-rays in almost all areas of medical diagnos Prenatal corrective surgery has been made possible by the ultrasound study in vivo bab

Ultrasound has many applications in the field of medical diagnostics. It is particula well known for its use in the following areas :

- Pregnancies and gynaecological problems
- Kidney examination (kidney stones can be pulverised by ultrasound and flushed out of the system)
- Examination of the scrotum, testes etc.
- Examination of thyroid, gullet, larynx and allied organs.

Craniotomy, or the study of cranium, layer by layer by a safe non-invasive meth

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11.12 Application of Ultrasonic Waves

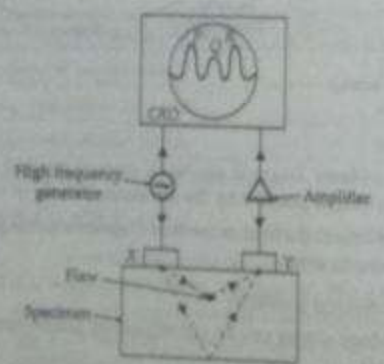
11.12.1 Ultrasonic Flaw Detection

Of all the applications of industrial ultrasonic testing, flaw detection is the oldest and the most common, since the 1940s, the laws of physics that govern the propagation of sound waves through the solid materials have been used to detect hidden cracks, voids, porosity, and other discontinuities in metals, composites, plastics and ceramics. High energy sound waves reflect from flaws in predictable way, producing distinctive echo patterns that can be displayed and recorded by portable instruments. Ultrasonic testing is completely non-destructive and safe and it is a well established test method in many basic manufacturing process and service industries, especially involving welds and structural metals.

PRINCIPLE

Sound waves are simply organised mechanical vibrations travelling through a medium which may be a solid, a liquid, or a gas. These waves will travel through a given medium at a specific speed or velocity, in a predictable direction, and when they encounter a boundary with a different medium they will be reflected or transmitted according to simple rules. This is the principle of Physics that underlies ultrasonic flaw detection.

The pulse echo method[1] is used for flaw detection (Fig. 11.16). In the absence of any flaw (crack, cavity, bubble etc.), two peaks P and Q are observed on the screen. The peak P is of the original transmission signal and Q is the reflected signal. If there is any flaw, the reflection of part or all of the incident wave takes place, which results in another peak R indicates to some extent the size of the flaw. When the flaw is present, the height of peak Q diminishes. It gives a better idea about the flaw, than the height of peak R. By moving the transducer probe defects in various parts of the sample can be detected.



P = incident pulse ;
Q = pulse from flaw ;
R = reflected pulse from the bottom of specimen ;
X = Transmitting transducer ;
Y = Receiving transducer.

Fig. 11.16 Flaw detection by Pulse Echo Technique

1. Pulse Echo Technique (PET) is discussed in Article 11.11.

11.12.2 Thickness Gauging

For ultrasonic thickness measurement either pulse or resonance technique is used. The principle is similar to flaw detection, the time taken for travelling the distance is shown on the digital display. In the beginning the gauge is calibrated with the standard sample. the important advantage is that access to only one surface is sufficient for the thickness measurement. The method is used for the assessment of internal corrosion of pipes, the measurement of the thickness of a ship's hull for external corrosion from inside the ship when it is at sea', the measurement of depths of fat on the bodies of live animals such as pigs and so on.

11.12.3 Depth Sounding

The depth of the sea can be measured from the ship, using ultrasound. The reflected waves are collected by receiver and the time interval between the transmission and reception of the corresponding echo is measured. Generally this time interval is fed to autocalculating electronic circuit, which displays the depth. The profile of the ocean bed can also be mapped by this method. The ultrasonic range finder also works on the same principle. In this method, either pulses or continuous waves are used.

11.12.4 Detection of submerged objects

Submerged objects, such as icebergs, rocks, a shoal of fish etc. can be detected using ultrasonic waves. It is called SONAR (Sound Navigation and Ranging) technique.

11.12.5 Cavitation

When a sound wave travels through a liquid, during the positive half cycle of wave, the liquid is compressed and during the negative half cycle, it expands. When the acoustic pressure is, greater than the hydrostatic pressure, voids are created. These voids expand during the negative half cycle and collapse abruptly. It releases energy, giving rise to shock waves. The local temperature rises to tens, hundreds or thousands of degrees for very short periods at the frequency of wave propagation. This is called **cavitation**. The cavitation process is used in emulsification of two immiscible liquids. In the food industry, this method is used in the preparation of dairy products, sauces and gravies, salad creams and synthetic creams. It is used in extraction of hops in brewing industry. The sterilisation of milk and tendering of meat by breaking down its fibers are some of other applications.

11.12.6 Ultrasonic Cleaning

In ultrasonic cleaning process both agitation and cavitation are involved. The cleaning of large components is carried out at low frequency range from 18 kHz to 40 kHz, here cavitation is most active, for delicate articles, higher frequency range 100 kHz to 1 MHz is used, where cavitation is almost absent. The process is used in removal of grease from engine components, oil, dust, buffering compounds etc. from printed circuit boards, jewelry and various instruments and devices.

11.12.7 Fatigue Testing, Drilling, Soldering and Welding

When high intensity ultrasonic waves pass through a solid, the strain developed finally results in fracture. The phenomenon is used in fatigue testing. The ultrasonic-drill is widely used for drilling materials, like glass, ceramics, germanium, tungsten carbide etc. The high intensity ultrasonic waves are used for treatment of melts, soldering and welding.

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11.12.8 Motion and Fire Sensing

One of the few ultrasonic applications in open air is that of the motion and fire sensing, which is restricted to the lower kHz range, where attenuation is not very much. A magnetostructive transducer placed at some point in a room emits pulses in all directions. The reflected signals from the walls and furniture are eventually picked up by a receiver, from which a constant indication is generated. Any variation in the sound field, caused by an intruder or a change in temperature, gives rise to a change in this indication, which triggers an alarm. One version of this device is a light switch that goes on automatically when someone enters an empty room and goes off automatically when no body is present in the room after a predetermined period of time.

11.12.9 Ultrasonic Imaging

The application of ultrasound to imaging process is extremely important in industry and in medicine. Because imaging entails low intensity ultrasound energy, it provides a valuable non-destructive testing technique. *Ultrasonic imaging which may be defined as any technique providing a visible display of the intensity and phase transitions in an acoustic field*, it falls into a number of categories :

(a) the electronic-acoustic imaging.

(b) B-scanning most commonly used in medical diagnosis.

(c) C-scanning widely used in non-destructive testing and inspection flat and cylindrical surface.

(d) liquid surface levitation presentations.

(e) liquid crystal display (LCD), photographic and similar display.

(f) light-refraction methods.

(g) acoustical holography.

11.12.10 Acoustic Lenses

Acoustic lenses are necessary in the use of electron-acoustic image converters. An acoustic lens can (a) increase the sensitivity of imaging system through energy concentration, and (b) provide coverage of a larger area by concentrating the image on the receiving piezoelectric element of image converter in almost the same fashion an optical lens reduces a larger picture to a smaller area. For optimal performance, the velocity of sound in the lens material must differ considerably from that of the surrounding media, and the reflection of energy at the boundary between the lens and the surrounding should be minimal. The latter condition is fulfilled automatically when the acoustic impedances of the lens materials and the surrounding medium match each other. Liquid lenses of carbon tetrachloride or chloroform have the same acoustic impedance as water, but their toxicity generally preclude their use in industry. Plastic lenses also have been developed, but the sound propagation velocity in these material exceeds that of water and they present an impedance mismatch between the lenses and water, but not to the degree of rendering them useless. The relatively high absorption of plastics limits their use to frequencies less than 15 MHz. Metallic lenses, which can be made of aluminium and other metals, possess low absorption characteristics, so they can be used at frequencies less than 15 MHz, but the impedance mismatch between metals and water essentially prevent their effective use.

The velocity of sound in solids is higher than in liquids, so solid concave lenses are convergent and solid convex lenses are divergent unlike the case of light travelling in a vacuum or a gas through optical lenses, (Fig. 11.17).

NOTE

It is interesting to understand the difference between optical and acoustic lenses : the velocity of light in solids is less than air. Hence a convex lens acts as a converging lens and a concave lens as a diverging lens. when the velocity of sound in solids is more than that in air, a concave lens acts as a converging acoustic lens.

Fig. 11.17 Focusing of the ultrasonic waves.

11.12.11 Liquid Crystal Imaging

Liquid crystals manifest properties of solid crystals that are not apparent in ordinary liquid. When a stress is placed on a liquid crystal, its optical properties changes. A certain class of liquid crystals known as *nematic crystals* are used to indicate the presence of an ultrasonic field with sensitivity equal to that of schlieren systems, high resolution, large area capability and handling ease.

11.12.12 Acoustical Holography

The wave used in the reconstruction does not have to be the original, and this allows the use of ultrasonic waves and the subsequent reconstruction of the image using light from a laser. But size of image changes in proportion to the ratio of the wavelength of the recons- tructing wave to the wavelength of the original illuminating waves. There are a number of methods of making acoustical holograms. One method in Fig. 11.18 uses liquid levitation, where the acoustic waves from below the surface of liquid forms an ultrasonic image at the surface of the liquid that can be rendered visible on a photographic plate. The reference beam onto the surface or by generating a separate wave through a second transducer. The height of the bulge of the liquid surface due to the impetus of an acoustic wave is critical, it should be small as compared to the wavelength of the light.

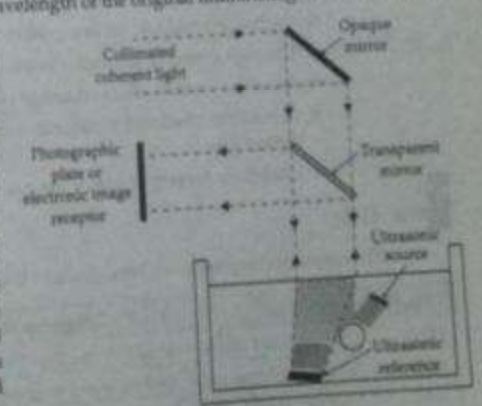

Fig. 11.18 Acoustical holography making use of liquid levitation.

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11.12.13 Medical Diagnosis

The technique used for flaw detection is used for medical diagnosis. The characteristic impedance and absorption coefficients of different parts of human body are different *e.g.*, muscles, bones, layers of fat etc. The difference in healthy and malignant tissues can be detected for early warning of cancer etc. The frequency range used for diagnosis is 1 MHz to 10 MHz. The scanning technique is used to obtain clear pictures of the foetus at different stages of pregnancy, malignant growths, damage etc. of the various organs such as liver, brain, breast etc. The technique is known as **sonography**. The tumors in the brain, breast, abdomen and structures like gallstones, ureteric calculus, foreign bodies etc. can be located.

11.12.14 Other Applications

- Decreasing metal parts by suspending them in a liquid containing fine abrasive powder and setting up an ultrasonic beam in the liquid.
- Agglomeration of minute particles ; an ultrasonic transducer set up in a chimney causes all the soot particles to clump together and drop to the bottom, significantly reducing air pollution.
- Immiscible liquids can be emulsified by ultrasonic agitation.
- High polymers are depolymerised by ultrasonics. (For example, polystyrene in toluene molecules are reduced from 5,000 to 10,000, when exposed to ultrasonic waves for about 30 minutes.)
- Low intensity ultrasonics accelerate the rate of chemical reaction due to increased formation of free ions.
- High intensity ultrasound destroys bacteria and low intensity ultrasonic waves stimulate growth.
- Non-destructive testing of coatings in metallurgy is an important application of ultrasonics.
- Cavitation breaks down living cells or even destroy them by means of ultrasonics.
- The high intensity ultrasonic waves are dangerous for human being.
- There may occur cellular damage or disruptive effects on chromosomes.
- In medical treatment ultrasonic waves are used to heat certain portion or kill certain tissues.
- Ultrasonic surgery is one of the emerging fields.

Table 11.2

The major developments in Acoustics and Ultrasonics :

Year	Major developments
1822	Colladen used underwater bell to calculate the speed of sound in waters of Lake Genava.
1830	Savart developed large, toothed wheel to generate very high frequencies.
1842	Magnetostrictive effect discovered by Joule.
1845	Stokes investigated effect of viscosity on attenuation.

1860	Tyndall developed the sensitive flame to detect high frequency waves.
1866	Kundt used dust figures in a tube to measure sound velocity.
1876	Galton invented the ultrasonic whistle.
1877	Rayleigh's "Theory of Sound" laid foundation for modern acoustics.
1880	Curie brothers discovered the piezoelectric effect.
1890	Koenig, studying audibility limits produced vibrations up to 90 kHz.
1903	Lebedev and coworkers developed complete ultrasonic system to study absorption of waves.
1912	Sinking to Titanic led to proposals on use of acoustic waves to detect icebergs.
1912	Richardson files first patent for an underwater echo ranging sonar.
1914	Fessenden built first working sonar system in the United states which could detect icebergs two miles away.
1915	Langevin originated modern science of ultrasonics through work on the "Hydrophone" for submarine detection.
1921	Cady discovered the quartz stabilized oscillator.
1922	Hartmann developed the air-jet ultrasonic generator.
1925	Pierce developed the ultrasonic interferometer.
1926	Boyle and Lehmann discovered the effect of bubbles and cavitation in liquids by ultrasound.
1927	Wood and Loomis described effects of intense ultrasound.
1928	Pierce developed the magnetostrictive transducer.
1928	Herzfeld and Rice developed molecular theory for dispersion and absorption of sound in gases.
1928	Sokolov proposed use of ultrasound for flaw detection.
1930	Debye and Sears and Lucas and Biquard discover diffraction of light by ultrasound.
1930	Harvey reported on the physical, chemical and biological effects of ultrasound in macromolecules, microorganisms and cells.
1931	Mulhauser obtained a patent for using two ultrasonic transducers to detect flaws in solids.
1937	Sokolov invented an ultrasonic image tube.
1938	Pierce and Griffin detect the ultrasonic cries of bats.
1939	Pohman investigated the therapeutic uses of ultrasonics.
1940	Firestone, in the United States and Sproule, in Britain, discovered ultrasonic pulse-echo metal-flaw detection.
1940	Sonar extensively developed and used to detect submarines.
1941	"Reflectoscopes" extensively developed for non-destructive metal testing.
1942	Dussik brothers made first attempt at medical imaging with ultrasound.
1944	Lynn and Putnam successfully used ultrasound waves to destroy brain tissue of animals.
1945	Newer piezoelectric ceramics such as barium titanate discovered.
1945	Start of the development of power ultrasonic processes.
1948	Start of extensive study of ultrasonic medical imaging in the United States and Japan.
1954	Jaffe discovered the new piezoelectric ceramics lead titanate-zirconate.

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Formulae at a Glance

1. Velocity $v = \sqrt{\frac{Y}{\rho}}$

2. Wavelength $\lambda = \frac{v}{\nu}$

3. Reflection coefficient $R = \frac{Z_2 - Z_1}{Z_2 + Z_1}$

4. $\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$

5. Piezoelectric Generator

(i) Frequency of ultrasonics

$$\nu = \frac{P}{2l}\sqrt{\frac{Y}{\rho}}$$

$P = 1, 2, 3$ ..... for fundamental, first, second, third overtones etc.

Y = Young's modulus, ρ = density.

(ii) For tank circuit

$$\text{Frequency } \nu = \frac{1}{2\pi\sqrt{LC}}$$

L = inductance ;

C = capacitance

6. Magnetostrictive Transducer

$$\text{Frequency of the rod } \nu = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$$

7. Velocity of ultrasonic waves

(i) Wavelength $\lambda_{ult} = \frac{2m\lambda}{\sin\theta}$

m = order of maxima

λ = wavelength of monochromatic light beam

(ii) Velocity $v = \nu\lambda_{ult}$

ν = frequency of ultrasonics.

Miscellaneous Numerical Problems for Practice

Example 1. Ultrasonic pulse echo method is employed to detect possible defect in steel bar of thickness 40 cm. If pulses arrival times are 30 μs and 80 μs, find the distance of the defect.

Solution. Let us consider that x is the distance of the defect from the end of the bar at which pulse enters the bar. The pulse arriving first is the one reflected from the defect and that arriving later on is the one reflected from the other end.

Obviously, the pulse takes 30 μs to cover a distance of $2x$ and 80 μs to cover a distance 80 cm.

Hence $$30 = \frac{2x}{v} \quad ...(i)$$

and $$80 = \frac{80}{v} \quad ...(ii)$$

From (ii) $v = 1$, then from (i) $x = \frac{30}{2} = 15$ cm.

Example 2. An ultrasonic source of 0.07 MHz sends down a pulse towards the sea bed, which returns after 0.65 s. The velocity of sound in sea water is 1700 m/s. Show that the depth of sea is 552.5 m and the wavelength of pulse is 0.0242 m.

Solution. We know that

$$2d = vt \quad \text{[where } d \text{ = depth of sea]}$$

$$d = \frac{vt}{2} = \frac{1700 \times 0.65}{2} = 552.5 \text{ m}$$

and $$v = \nu\lambda \quad \text{[where } \nu \text{ = frequency]}$$

$$\lambda = \frac{v}{\nu} = \frac{1700}{0.07 \times 10^6} = 2.43 \times 10^{-3} \text{ m}$$

$$= 0.0243 \text{ m}$$

Example 3. Determine the frequency of first and second modes of vibration for a quartz of piezoelectric oscillator. The velocity of longitudinal waves in quartz crystal is 5.5×10^3 m/s. Given the thickness of quartz crystal is 0.05 m.

Solution. The distance between the two faces of the crystal (t = thickness) in the lowest mode of vibration will be $\frac{\lambda}{2}$. Thus

$$\lambda = 2t = 2 \times 0.05 = 0.1 \text{ m}$$

∴ The lowest frequency

$$\nu_1 = \frac{v}{\lambda} = \frac{5.5 \times 10^3}{0.1} = 5.5 \times 10^4 \text{ Hz}$$

In the second mode, the frequency would be

$$\nu_2 = \nu_1 = 2 \times 5.5 \times 10^4$$

$$= 11 \times 10^4 = 1.1 \times 10^5 \text{ Hz}$$

Example 4. A quartz crystal of thickness 0.001 m is vibrating at resonance. Find the fundamental frequency. Given Y for quartz $= 7.9 \times 10^{10}$ N/m² and ρ for quartz = 2650 kg/m³.

Solution. For longitudinal vibration, we have

$$v = \sqrt{\frac{Y}{\rho}} \quad \text{with } Y = 7.9 \times 10^{10} \text{ N/m}^2 ;\ \rho = 2650 \text{ kg/m}^3$$

$$\therefore \quad v = \sqrt{\frac{7.9 \times 10^{10}}{2650}} = 5460 \text{ m/s}$$

For fundamental mode of vibration, we have

The thickness $(t) = \frac{\lambda}{2}$

$$\lambda = 2t = 2 \times 0.01 = 0.002 \text{ m}$$

$$\text{Frequency } \nu = \frac{v}{\lambda} = \frac{5460}{0.002}$$

$$= 2740000 \text{ Hz} = 2730 \text{ kHz}$$

Example 5. Calculate the natural frequency of ultrasonic waves using the following data :

(i) *Thickness of quartz plate* $= 5.5 \times 10^{-3}$ m

(ii) *Young's modulus of quartz* $= 8.0 \times 10^{10}$ N/m²

(iii) *Density* $= 2.65 \times 10^3$ kg/m³.

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Solution. Since we know that the frequency

$$v = \frac{P}{2t}\sqrt{\left[\frac{Y}{\rho}\right]} = \frac{1}{2\times 5.5\times 10^{-3}}\sqrt{\left[\frac{8.0\times 10^{10}}{2.65\times 10^{3}}\right]} = 90.9\times 5.49\times 10^{3}$$

$$v = 4.99 \text{ kHz}.$$

Example 6. *Find echo time of ultrasonic pulse, which travelling with velocity 5.9×10^{3} m/s in mild steel whose correct thickness measured by gauge meter is 18 mm.*

Solution. We now that

$$2d = vt \quad \text{or} \quad t = \frac{2d}{v}$$

or

$$t = \frac{2\times 18\times 10^{-3}}{5.9\times 10^{3}} = 6.1\times 10^{-6} \text{ second.}$$

Example 7. *A piezoelectric x-cut crystal plate has a thickness of 1.6 mm. If the velocity of sound wave along the x-direction is 5760 m/s, calculate the fundamental frequency of the crystal.*

Solution. In the lowest mode of vibration, the distance between the faces of the crystal of thickness t will be $\frac{\lambda}{2}$

$$\therefore \quad t = \frac{\lambda}{2}$$

or

$$\lambda = 2t = 2\times 1.6\times 10^{-3} = 3.2\times 10^{-3} \text{ m}$$

then

$$v = \frac{v}{\lambda} = \frac{5760}{3.2\times 10^{-3}} = 1800\times 10^{3} \text{ Hz} = 1.8 \text{ MHz}$$

Question Bank

Multiple Choice Questions

1. Which of the following waves does not belong to the electromagnetic spectrum :
   (a) X-rays (b) Microwave (c) Ultrasonics (d) Infrared.
2. The technique which is not concerned with the application of ultrasonic waves :
   (a) ultrasonic sonography (b) Sound ranging
   (c) depth sounding (d) MRI.
3. The wave which does not belong to electromagnetic spectrum is
   (a) heat (b) ultraviolet (c) ultrasonics (d) X-rays.
4. Decibel is
   (a) a musical instrument (b) the wavelength of noise
   (c) a measure of sound level (d) a musical note.
5. The frequency range of ultrasonic waves is
   (a) 0 – 15 Hz (b) 20 kHz – 500 MHz
   (c) 15 Hz – 20 kHz (d) 500 MHz – 1000 GHz.

6. In the piezoelectric effect, what is the meaning of word "piezo" ?
   (a) Mass (b) pressure (c) volume (d) None of these.
7. Who invented the direct piezoelectric effect ?
   (a) Curie brothers (b) Wright brothers (c) Joule and Villari (d) None of these.
8. The opposite piezoelectric effect is predicted by :
   (a) J.Curie and P.Curie (b) Lippmann (c) Joule (d) None of these.
9. In 1847, who discovered direct magnetostriction effect ?
   (a) Joule (b) Villari (c) Galton (d) Curie brother.
10. In 1868, Villari discovered :
    (a) Direct magnetostrictive effect (b) Converse magnetostrictive effect
    (c) Direct piezoelectric effect (d) Converse piezoelectric effect
11. The wavelength of ultrasonic waves by acoustic grating is
    (a) $\lambda_{ult} = \frac{2m\lambda}{\sin\theta}$ (b) $\lambda_{ult} = \frac{m\lambda}{\sin\theta}$ (c) $\lambda_{ult} = \frac{m\lambda}{\cos\theta}$ (d) None of these.
    where symbol are having their usual meanings.
12. The relation between velocity (v), frequency (v) and wavelength (λ_u) is
    (a) $v = v\lambda_{ult}$ (b) $v = v\lambda_{ult}$ (c) $\lambda_{ult} = vv$ (d) None of these.
13. Which of the following shows the piezoelectric effect
    (a) aluminium (b) quartz (c) nickel (d) manganese.
14. Ultrasonic waves can not travel through
    (a) metal (b) air (c) vacuum (d) water.
15. Sound waves having following frequency range are audible to human being
    (a) less than 20 Hz (b) 15 Hz – 20 kHz
    (c) more than 20 kHz (d) more than 20 MHz.
16. SONAR is an application
    (a) infrasonics (b) X-rays (c) Laser (d) Ultrasonics.
17. Ultrasonic waves can be detected by
    (a) Piezoelectric detector (b) Kundt's tube
    (c) Thermal detector (d) All the above.
18. Ultrasonic waves are of
    (a) transverse waves (b) longitudinal waves
    (c) radio waves (d) light waves.
19. Material used for the production of ultrasonic waves in magnetostriction effect is
    (a) paramagnetic (b) diamagnetic (c) ferromagnetic (d) ferrimagnetic.
20. Ultrasonic waves can be detected by :
    (a) Quincke's tube (b) Hebb's tube (c) Kundt's tube (d) Nosal tube.
21. Ultrasonics
    (a) show directive property (b) produce luminescence
    (c) have short wave length (d) all are true.

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22. Ultrasonic waves are those which
(a) man can hear (b) man can not hear
(c) which are of high amplitude (d) which are of high wavelength.

23. Ultrasonics are reflected when medium changes. This phenomenon is used in
(a) flaw detection (b) depth measurement
(c) detection of submarine, ice berg etc. (d) all the above.

Answers

1. (c)	2. (d)	3. (c)	4. (c)	5. (b)	6. (b)
7. (a)	8. (b)	9. (a)	10. (b)	11. (a)	12. (a)
13. (b)	14. (c)	15. (b)	16. (d)	17. (d)	18. (b)
19. (c)	20. (c)	21. (d)	22. (b)	23. (d)	

Theoretical Questions

1. What frequency ranges constitute "Ultrasonics" and "Infrasonics" ? [GGSIPU, May 2005 (2.5 marks)]
2. Discuss the advantages and disadvantages of piezoelectric and magnetostriction methods in production of ultrasonic waves ? [GGSIPU, May 2005 (7.5 marks)]
3. Draw the circuit diagram of a magnetostriction oscillator and explain its working. [GGSIPU, May 2005 (5 marks)]
4. Discuss the properties of ultrasonics. Mention two of its important applications. [GGSIPU, May 2008 (12.5 marks)]
5. Write notes on :
(a) Applications of ultrasonic waves [GGSIPU, May 2007 (2.5 marks)]
(b) Piezoelectric effect (c) Magnetostriction effect. (d) Transducer.
6. What are ultrasonic waves ? Describe the piezoelectric method for producing ultrasonic waves. Give three names of piezoelectric transducers. [GGSIPU, May 2006 (8.5 marks)]
7. Explain in detail how the ultrasonic pulse technique is used for non-destructive testing of material and for depth exploration. [GGSIPU, May 2006 (4 marks)]
8. What are ultrasonic waves ? How are they produced ?
9. Describe laboratory method with diagram for the determination of speed of the ultrasonic wave.
10. What is magnetostriction ? Describe magnetostriction oscillator for generating ultrasonics.
11. Describe the method of finding fault inside a metallic slab with the help of ultrasonic transducer. What are x-cut and y-cut piezoelectric crystals ?
12. Discuss one method for generation of ultrasonics.
13. Explain how one can determine the velocity of ultrasonic waves in a liquid using acoustic grating.
14. Describe the experiment where light is diffracted by ultrasonics in a liquid and this is used to determined the velocity of ultrasonic waves in the liquids.

15. What are ultrasonic waves ? How will you measure the speed of these waves in liquids ?
16. What are ultrasonic waves ? Discuss Piezoelectric effect.
17. What is the principle involved in ultrasonic testing of materials ?
18. Describe ultrasonic testing systems and methods involved in non-destructive testing of materials.
19. Explain the principle of the following : (i) Magnetostriction effect and (ii) Piezoelectric effect.
20. Describe the brief scientific applications of ultrasonic waves.
21. What are the direct piezoelectric and inverse piezoelectric effect ? Explain their importance and use.
22. State the applications of ultrasonic waves using principle of (i) echo sounding and (ii) cavitation.
23. What are the applications of ultrasonics ? Describe the non-destructive testing application in detail.
24. Distinguish between low power application and high power applications of ultrasonics and give one example each.
25. What are ultrasonics ? Explain the magnetostriction method of producing ultrasonic energy and hence describe its advantages over the piezoelectric method.
26. Discuss the advantages and disadvantages of piezoelectric and magnetostriction methods in production of ultrasonic waves.
27. What is magnetostriction effect ? Draw circuit diagram of magnetostriction oscillator ? Explain its working.
28. In light, wavelengths of the visible region are mentioned for limits of visible radiation, whereas in acoustics, frequencies are mentioned as limits of audibility. Can you think of a reason for this usage ? What are the frequency limits of visible radiation ? What are the wavelengths of audible sound ?
29. Is the fundamental frequency of an x-cut quartz crystal plate related to its thickness ? If so, explain.
30. In magnetostriction, does the length of the ferrite rod increase on magnetisation ? Give a plausible reason for what actually happens ?
31. How does the area of a transducer plate affect its ultrasonic output.
32. In acoustic convex lenses are divergent and concave lenses are convergent. What is the reason for this.

Numerical Problems

1. Calculate (a) the velocity (b) the wavelength of the longitudinal waves (c) the fundamental frequency in a rod of material density $\rho = 2650\ kg/m^3$. Young's modulus $7.9 \times 10^{10}\ N/m^2$, the thickness of the rod is 0.001 m. [Ans. (a) 5461 m/s ; (b) $\lambda = 0.002$ m ; (c) $\nu = 2730$ kHz]
2. A quartz crystal of thickness 0.5 cm and length 5 cm is vibrating at resonance for its fundamental mode. Calculate the frequency of vibration of the crystal. Given $Y = 8 \times 10^{11}$ dyne/cm, $\rho = 2.654\ g/cc$. [Ans. $\nu_t = 5.5 \times 10^5$ Hz, $\nu_l = 5.5 \times 10^4$ Hz]
3. Two quartz transmitters 50 cm apart, which are connected in parallel to the same oscillator generates ultrasonic waves of the same frequency 10^5 Hz in a liquid placed between them. Cork dust sprinkled in the liquid settled in lines with separation of 0.792 cm. Determine the velocity of ultrasonic waves in the liquid.

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4. A piezoelectric x-cut quartz crystal plate has a thickness of 1.6 mm. If the velocity of propagation of sound waves along the x-direction is 5760 m/s, calculate the fundamental frequency of the crystal. [Ans. 18 MHz]

5. Longitudinal standing waves are set up in a quartz plate with antinodes at opposite faces. The fundamental frequency of vibration is given by $\nu = \frac{2.87 \times 10^5}{t}$; where ν is in Hz and t is the thickness of the plate in cm. Evaluate (a) Young's modulus of the quartz plate (b) the thickness of the plate required for a frequency of the 1200 kHz. The density of quartz is $2.66 \times 10^3 \text{ kg/m}^3$. [Ans. $Y = 8.76 \times 10^{10} \text{ N/m}^2$, $t = 0.24$ cm]

6. An ultrasonic interferometer works at a frequency of 2 MHz in water. When the reflector is moved through the 100th maxima, the distance travelled by it is 3.5×10^{-2} m. Show that the velocity of sound in water is 140 ms^{-1}.

7. An ultrasonic transducer is immersed in water. What fraction of energy is transferred to the water? Density of the ceramic transducer material = $2.8 \times 10^3 \text{ kg/m}^3$ and velocity of sound in ceramic is 6.2×10^3 m/s.

8. An ultrasonic flaw detector operates at a frequency of 1 MHz and emits 50 pulses per second. What should be the minimum distance of the flaw from the surface so that it can be detected?

9. A magnetostriction oscillator has a frequency of 20 kHz. Find the length of the ferrite rod.

10. TV sets have a scanning rate of 650 lines per frame and 20 frames per second. How does this compare with the scanning rate in ultrasonic techniques employed in medical diagnostic?

11. Calculate the capacitance to produce ultrasonic wave of 10^6 Hz with inductance of 1 Henry. [Ans. $C = 0.025$ pF]

12. An ultrasonic beam is used to determine the thickness of a steel plate. It was found that difference in two adjacent harmonic frequencies is 50 kHz. The velocity of sound in steel = 5000 m/s. Show that the thickness of steel plate is 0.05 m. [Ans. $d = 0.05$m]

13. Bats emit ultrasonic waves. The shortest wavelength emitted in air by a bat is about 0.33 cm. What is the highest frequency that a bat can emit? [Ans. 1.05×10^5 Hz]

14. A piezoelectric x-cut crystal has a thickness of 0.002 m. If the velocity of sound waves in a crystal is 5750 m/s, calculate the fundamental frequency of the crystal. [Ans. 1.44 MHz]

15. Calculate the natural frequency of iron rod of 0.03 m length. The density of iron is $7.23 \times 10^3 \text{ kg/m}^3$ and Young's modulus $11.6 \times 10^{10} \text{ N/m}^2$.

16. A quartz crystal of thickness 0.05 m is vibrating at resonance. Calculate the fundamental frequency. Given the Young's modulus for quartz as $7.9 \times 10^{10} \text{ N/m}^2$ and density of quartz 2650 kg/m^3. [Ans. 5.46×10^5 Hz]

17. Calculate the natural frequency of 40 mm length of a pure iron rod. Given the density of pure iron is $7.25 \times 10^3 \text{ kg/m}^3$ and its Young's modulus is $115 \times 10^9 \text{ N/m}^2$. Can you use it in magnetostriction oscillator to produce ultrasonic waves? [Ans. 49.75 kHz]

Conceptual Questions

Chapter 1 Motion of the Charged Particles in Electromagnetic Fields

1. *What is the construction of the cathode ray tube?*

Ans. It consists of an evacuated glass tube, a filament, an electron gun, a pair of horizontal and vertical deflecting plates and a fluorescent screen.

2. *What are functions of CRO's parts?*

Ans. Electrons emitted by filament are concentrated into a narrow pencil by electron gun. These electrons are then subjected to deflection as a result of voltage applied at vertical and horizontal deflection as a result of voltage applied at vertical and horizontal deflection plates. The electron beam then strikes the fluorescent screen and produced visual indication.

3. *Why the electron beam is deflected between deflected plates in e/m experiment?*

Ans. This is because of the force exerted by electric and magnetic field.

4. *How is the electron deflected under the action of this force in e/m experiment?*

Ans. Path of electron is circular because this force is always perpendicular to the direction of motion.

5. *What is use of determining e/m in Thomson experiment?*

Ans. It proves that all cathode ray particles are electrons.

6. *What are the differences of an optical lens and a magnetic lens?*

Ans. An optical lens focusses light, whereas the magnetic lens focusses an electron beam. The focal length of an optical lens is fixed, while the focal length of a magnetic lens can be changed.

7. *In e/m experiment, what type of magnetic field acts on electrons? What is the effect of this field on electrons?*

Ans. A uniform magnetic field perpendicular to the axis of the cathode ray tube acts on the electrons. The electrons experience a force $\mathbf{F} = -e(\mathbf{v} \times \mathbf{B})$ perpendicular to the axis of the cathode ray tube and the direction of the magnetic field and follows a parabolic path.

8. *Why is the Hall potential difference developed when a transverse magnetic field is applied to a current carrying conductor?*

Ans. When a current carrying conductor is placed in a transverse magnetic field, the magnetic field exerts a deflecting force **F** on the charge carriers $[\mathbf{F} = e(\mathbf{v}_d \times \mathbf{B})]$ in a direction perpendicular to both drift velocity $\mathbf{v}_d$ and applied magnetic field **B**. This force cause charges to drift from one surface to other and as a result excess negative charge at one surface and excess positive charge on opposite surface creates an electric field.

9. *What is Fleming's left hand rule?*

Ans. If we stretch thumbs, fore finger and middle finger of the left hand mutually at right angles to one another such that fore finger points in the direction of magnetic field **B** and middle finger in the direction of current I, then the thumb will point in the direction of force **F** acting on the conductor.

10. *How does the mobility depend on the electrical conductivity of the specimen?*

Ans. Mobility of a charge carrier increases with the increase of the electrical conductivity of the specimen.

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11. *What do you mean by cyclotron frequency ?*

Ans. The number of revolutions made by the particle per second is known as cyclotron frequency and is given by

$$f = \frac{Bq}{2\pi m}$$

12. *What is trajectory of charged particle in a plane perpendicular to a magnetic field* **B** *?*

Ans. A charged particle, velocity of which is in a plane perpendicular to a magnetic field **B**, describes a circular trajectory.

The radius r of this circular path is given by

$$r = \frac{mv}{Bq}$$

where m is the mass of the charged particle.

13. *When the direction of the charged particle is neither parallel nor perpendicular to the direction of magnetic field. Which is trajectory of this particle ?*

Ans. A helical trajectory.

14. *What is the Lorentz force ?*

Ans. The motion of a charged particle, moving through a combination of electric and magnetic fields, is described by Lorentz force

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$$

Chapter 2 Gauss's Law and Its Applications

1. *What is a solid angle ?*

Ans. Solid angle is a measure of a cone. Consider the intersection of a given cone with a sphere of radius R. The solid angle $\Delta\omega$ of the cone is defined to be equal to $\Delta S / R^2$, where ΔS is the area on the sphere cut out of the cone.

2. *Give statement of Gauss theorem.*

Ans. The electric flux (ϕ) through a closed surface is equal to ($1/\varepsilon_0$) times the net charge (q) enclosed by the surface.

3. *What do you mean by electric flux ?*

Ans. The electric flux is defined as the number of lines of force that pass through a surface placed in the electric field. The electric flux ($d\phi$) through an elementary area dS is defined as the product of the area and the component of electric field strength normal to the area.

The electric flux normal to the area $(dS) = d\phi = \mathbf{E} \cdot d\mathbf{S}$

4. *What are the electric lines of force ?*

Ans. An electric field may described in terms of lines of force in much the same way as a magnetic field. The electric field around a charged body is represented by imaginary lines called electric lines of force. The direction of lines of force at any point is the direction along which a unit positive charge (+ 1C) placed at that point would move or tend to move.

5. *What are the properties of electric lines of force ?*

Ans. (i) Every line originates from a positive charge and terminates on a negative charge.

(ii) Lines of force never intersect.

(iii) The tangent to line of force at any point gives the direction of electric field **E** at that point.

(iv) The number of lines of force per unit area at right angles to the lines is proportional to the magnitude of **E**.

(v) Each unit positive charge gives rise to $1/\varepsilon_0$ times of force in free space.

6. *What are the applications of Gauss's theorem ?*

Ans. This theorem is especially useful in determining electric field **E**, when the source distribution has simple symmetry.

(i) *Thin infinitely long straight wire of uniform linear charge density*

$$E = \frac{\beta}{2\pi\varepsilon_0 r}\hat{e}$$

where r is the perpendicular distance of the point from the wire and $\hat{e}$ is the radial unit vector in the plane normal to the wire passing through the point.

(ii) *Infinite thin plane sheet of uniform surface charge density* σ

$$E = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

where $\hat{n}$ is a unit vector normal to the plane, outward on either side.

(iii) *Thin spherical shell of uniform surface charge density* σ

$$E = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad (r \geq R)$$

$$= 0 \qquad (r < R)$$

where r is the distance of the point from the centre of the shell and R the radius of the shell. q is the total charge of the shell : $q = 4\pi R^2\sigma$.

7. *Is Gauss's law useful in calculating the field due to three equal charges located at the corners of an equilateral triangle ? Explain.*

Ans. Gauss's law is not useful for calculating the field due to three equal charges located at the corners of an equilateral triangle because it is difficult to find a surface of appropriate symmetry over which the electric field can be taken constant and thus to evaluate the integral.

8. *Suppose that a Gaussian surface, encloses no net charge*

(a) *Does Gauss's law require* **E** *is zero for all points on the surface ?*

(b) *If the converse of this statement is true, i.e., if* **E** *equals zero everywhere on the surface, does Gauss's law require that there be no net charge inside ?*

Ans. (a) When the Gaussian surface contains no net charge, Gauss's law becomes $\mathbf{E} \cdot d\mathbf{S} = 0$, which does not $\mathbf{E} = 0$. Here **E** and $d\mathbf{S}$ may be at right angles.

(b) When $\mathbf{E} = 0$ everywhere on the surface, Gauss's law requires that there be no net charge inside.

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9. *A point charge is placed at the centre of a spherical Gaussian surface. Is flux changed ?*

(a) If the surface is replaced by a cube of the same volume.

(b) If the sphere is replaced by a cube of one tenth the volume.

(c) If the charge is moved off centre in the original sphere but still remaining inside.

(d) If the charge is moved just outside the original sphere.

(e) If a second charge is placed near and outside the original sphere, and

(f) If a second charge is placed inside the Gaussian surface ?

Ans. (a) No, (b) No (c) No (d) Yes (e) No and (f) Yes.

Flux $\phi = \frac{q}{\varepsilon_0}$. Flux depends upon only on charge enclosed. Remember for charge external to the surface, as much flux enters the surface as that leaves the surface as shown in Fig. CQ.1

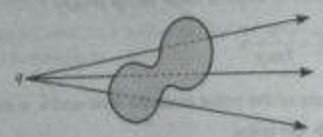

Fig. CQ.1

The net number of lines emerging from a volume due to a charge outside the volume is zero. (The number of lines entering = Number of lines emerging)

Chapter 3 Scalar and Vector Fields

1. *What do you mean scalar and vector quantities ?*

Ans. *Scalar Quantities.* The quantities, which have magnitude but no direction are termed as scalar quantities or simply scalars.

Examples : mass, time, temperature, work, pressure.

Vector Quantities. The quantities, which have magnitude as well as direction are termed as vector quantities or simply vectors.

Examples : velocity, acceleration, torque, impulse.

2. *What is scalar field ? Give examples to understand the concept of scalar field.*

Ans. *Scalar field.* If a scalar physical quantity is assigned to each point in space then we have a scalar field in that region of space. The scalar field in three dimensions can be represented by a scalar point function $\phi(x, y, z)$.

The concept of a scalar field can easily be understood with the help of the following examples :

(i) Consider a solid block of material, whose faces are maintained at different temperatures. Now the temperature of the body will vary from point to point, *i.e.*, temperature will be a function of position coordinates (x, y, z) in rectangular system. Hence, temperature is scalar field.

(ii) The density at any point inside a body occupying given region is a scalar field.

3. *What is a vector field ? Give examples to understand the concept of vector field.*

Ans. When a vector physical quantity is expressed from point to point in the region of space by a continuous vector function $\mathbf{A}(x, y, z)$ then the region is a vector field.

The concept of vector field can easily be understood with the help of the following examples :

(i) Consider a body which is rotating about an axis. The velocity of the body is different at different points *i.e.*, the velocity is a function of position of the point. Hence velocity is a vector field.

(ii) Consider the heat flow through a material whose faces are maintained at different temperatures. It will be flow of heat from hotter part of the block to the colder part. The heat will be flowing in different directions in different parts of the block. Thus, heat flow is a vector field.

4. *What do you understand by gradient of a scalar field ? what is its significance ?*

Ans. *Gradient.* The gradient is a differential operator by means of which we can associate a vector field with a scalar field.

Significance of gradient :

The gradient of a scalar field S is a vector whose magnitude at any point is equal to the maximum rate of increase of S at least point and whose direction is along the normal to the level surface at that point. This gives the physical significance of the gradient of scalar field.

5. *What is divergence of a vector field ? Write its physical significance.*

Ans. *Divergence.* The divergence of a vector field at any point is defined as the amount of flux per unit volume diverging from that point.

Physical significance of divergence :

The physical significance of divergence of a vector field is that at a point it gives the amount of flux per unit volume diverging from that point.

6. *Define curl of a vector field.*

Ans. (i) The rate of change of a vector field is also called curl, which means circular rotation.

(ii) The curl of a vector field is defined as the maximum line integral of the vector per unit area.

7. *What do you mean by (i) solenoidal and (ii) irrotational ?*

Ans. (i) *Solenoidal.* A vector whose divergence is zero is called solenoidal.

(ii) *Irrotational.* A vector whose curl is zero is called irrotational.

Chapter 4 Maxwell's Equations

1. *Can a point in space have a non-zero potential even when the electric field in the space is zero ?*

Ans. Yes, electric intensity at a given point is given by $E = -\frac{dV}{dt}$. If potential V is constant in space, t potential gradient in any direction is zero, hence the intensity is zero. Obviously, potential can exist whe there is no electric field.

2. *What do you understand by displacement current ?*

Ans. In one of the Maxwell's equations, the changing electric field term, is known as the *displacem current* D. This was an analogy with a dielectric material. If a dielectric material is placed in an electric field, molecules are distorted, their positive charges moving slightly to the right, say, the negative charges sligh

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to the left. Now consider, what happens to dielectric in an increasing electric field. The positive charges will be displaced to the right by a continuous increasing distance, so, as long as the electric field is increasing in strength, these charges are moving ; there is actually a displacement current. Meanwhile, the negative charges are moving the other way, so adds to the effect of the positive charges motion.

3. *What is absolute index of refraction ?*

Ans. $\mu = \frac{c}{v} = \frac{1}{\sqrt{\epsilon_0\mu_0}}\sqrt{\epsilon\mu} = \sqrt{\left(\frac{\epsilon\mu}{\epsilon_0\mu_0}\right)} = \sqrt{(\epsilon_r\mu_r)}$

4. *Is gamma rays are electromagnetic radiations ?*

Ans. Yes, these waves are electromagnetic radiations.

5. *What is Faraday's laws of electromagnetism ?*

Ans. Faraday's laws of electromagnetic induction are as below :

First law : Whenever the magnetic flux linked with a closed circuit changes, an induced e.m.f. is setup in the circuit whose magnitude, at any instant, is proportional to the rate of change of magnetic flux ϕ_B linked with the circuit

$$e \propto -\frac{d\phi_B}{dt}$$

Second law (Lenz's law). The direction of induced e.m.f. or current in the circuit is such that it opposes the change in flux that produced it.

6. *What is Ampere's circuital law ?*

Ans. Ampere's law in magnetostatics is analogous to Gauss's law in electrostatics.

The line integral $\int \mathbf{B}\cdot d\mathbf{l}$ of magnetic induction **B** for a closed path is numerically equal to μ_0 times the current through the area bounded by the path.

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i$$

where μ_0 = permeability constant.

7. *What do you understand by electromagnetic waves ?*

Ans. Electromagnetic waves are coupled electric field and magnetic field vectors that move with speed of light and exhibit typical wave behaviour *i.e.*,

- they travel with speed of light.
- they are transverse in nature.
- the ratio of electric to magnetic field vectors in an electromagnetic wave equals to the speed of light.
- they carry both energy and momentum.

8. *Write down Maxwell's equations.*

Ans. There are four fundamental equations of electromagnetism known as Maxwell's equations, which may be written in differential form as :

(i) $\nabla\cdot\mathbf{D} = \rho$ (Differential form of Gauss's law in electrostatics)

(ii) $\nabla\cdot\mathbf{B} = 0$ (Differential form of Gauss's law in magnetostatics)

(iii) $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ (Differential form of Faraday's law in electromagnetic induction)

(iv) $\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$ (Maxwell's modification in Ampere's law)

In above equations the notations have the following meanings :

D = electric displacement vector in C/m^2

ρ = charge density in C/m^3

B = magnetic induction in Wb/m^2 or Tesla

E = electric field intensity in V/m or N/C

H = magnetic field intensity or strength in A/turn-m

9. *What do you understand by Poynting vector ?*

Ans. Poynting vector is defined as the vector product of electric field and magnetic field vectors, which gives the time rate of flow of wave energy per unit area of the medium.

10. *Why does not one see the portion other than visible one of electromagnetic spectrum ?*

Ans. The retina of the eye is sensitive only to colours in the visible region *i.e.*, wavelength lying between 3900 Å and 7800 Å. This region corresponds to visible part of the spectrum.

11. *An electromagnetic wave carries momentum. What it signifies ?*

Ans. An electromagnetic interaction between two electric charges means an exchange of energy and momentum between the charges.

12. *Distinguish between conduction current and displacement current.* **[GGSIPU, May 2009 (2.5 marks)]**

Ans.

S.No.	Conduction current	Displacement current
1.	Actually flows through in conductivity medium and obeys Ohm's law.	Set up in a dielectric medium due to changing electric field across the dielectric, which leads to variation of induced displacement of charge.
2.	$J_c = \sigma E$, $I = \frac{V}{R}$	$J_d = \frac{\partial D}{\partial t} = \epsilon\frac{\partial E}{\partial t}$
3.	For constant E, $J_c \neq 0$	For constant E, $J_d = 0$

13. *A plane monochromatic wave, travelling in a homogeneous medium, meets a denser medium. What are the changes in*

(i) *amplitude* (ii) *frequency* (iii) *speed of propagation and*

(iv) *phase of reflected and transmitted waves in comparison with the corresponding properties of the incident wave ?*

Ans. (i) Amplitude decreases for both cases. (ii) Frequency remains same for both cases.

(iii) Speed same for reflected wave and less for transmitted waves in comparison to reflected waves.

(iv) Phase change by π for reflected wave. No phase change for transmitted wave.

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Chapter 5 Wave-Particle Dualism

1. *Explain how the wave nature of particles gives rise to uncertainty principle.*

Ans. Wave nature of a particle requires that the particle is to be represented by means of a wave group or a wave packet. A wave group is obtain by a superposition of a large number of monochromatic waves with slightly varying frequencies or wave vectors. The wave group has a finite spatial extent which implies that the position of a particle cannot be pin-pointed, it must have some uncertainty. Again the momentum p of a particle is related to the wave vector k by $p = \frac{hk}{2\pi}$. Since a wave group is made up of several waves of slightly varying wave vector, it is clear that a wave group has a certain spread Δk in wave vector. This means that momentum of the particle is not definite but has an uncertainty $\Delta p = \hbar\Delta k$. We know that a narrow wave group has a broader range of wave-vector. So smaller positional uncertainty is associated with large momentum uncertainty. Thus the wave nature of particle gives rise to the uncertainty principle.

2. *What do you mean by matter waves ?*

Ans. Because of such a dual nature of radiation, Louis de Broglie of France in 1924, put forward bold hypothesis that since nature loves symmetry, if the radiation behaves as particle under certain circumstances and as a waves under certain other circumstances, then one can even expect that entities which ordinary behaves as particles to exhibit properties attributable to only waves under appropriate circumstances and he termed them as matter waves.

3. *What is de Broglie wavelength ?*

Ans. The wavelength of matter waves is called de Broglie wavelength and is given by

$$\lambda = \frac{h}{p}$$

where h = Planck's constant.

4. *What is meant by travelling waves ?*

Ans. The wavemotion associated with the oscillating mass under such a condition, has a close resemblance to the progressive motion of waves, caused in water by some disturbance, or to those set up in a rope by trying it at one end, and oscillating it at the free end. Such progressive waves are called **travelling waves**.

5. *What do you understand by (i) phase velocity and (ii) group velocity.*

Ans. (i) *Phase velocity.* If a point is imagined to be marked on a travelling wave, then it becomes a representative point for a particular phase of the wave, the velocity with which it is transported owing to the motion of the wave, is called the phase velocity.

(ii) *Group velocity.* Group velocity is the velocity with which the envelope enclosing a wave group called wave packet, formed due to superposition of two or more travelling waves of slightly different wavelengths, is transported. It is the velocity with which the energy transmission occurs in a wave.

6. *Write statement of Heisenberg's uncertainty principle.*

Ans. In any simultaneous determination of the position and momentum of the particle, the product of the corresponding uncertainty inherently present in the measurement is equal to, or greater than $(h/2\pi)$.

7. *What was the aim of Davisson-Germer experiment ?*

Ans. Davisson and Germer in 1927 designed an apparatus to determine the wavelength associated with electrons.

Chapter 6 The Equation of Motion of Matter Waves : Schrödinger's Equation

1. *What is a wave function ?*

Ans. In quantum mechanics every physical system is represented by means of function which implicitly contains all the information about the system. This function is known as the wave function of the system. For a single particle the wave function is a function of its position r and time t.

2. *What is physical significance of the wave function ?*

Ans. In classical mechanics, the square of a wave amplitude associated with electromagnetic radiation is interpreted as a measure of radiation intensity. This suggests that we should make a similar interpretation for de Broglie waves associated with electron or any particle. Hence, if we consider a system of electrons, and if ψ is the wave function associated with the system, then $|\psi|^2$ may be regarded as a measure of density of electrons. Also, if τ is a volume inside which, an electron is known to be present as shown in Fig. CQ.2, but where exactly the electron is situated inside volume τ is not known, and if ψ is the wave function associated with the electron, then the probability of finding the electron in a certain element of volume $d\tau$ of τ is given equal to the $|\psi|^2 d\tau$. For this reason $|\psi|^2$ is called the probability function. This interpretation was first given by Max Born in 1926.

Fig. CQ.2 Presence of electron in a certain region.

Since the electron must be somewhere inside the volume τ, the integration of $|\psi|^2$ over the whole volume τ must be unity, so that

$$\int |\psi|^2 d\tau = 1$$

In a quantum mechanics it is postulated that the state of a system is completely characterised by a wave function.

3. *What are the properties of the wave function ?*

Ans.
- $\psi(x)$ must be single valued, finite and continuous for all values of x.
- $\frac{d\psi(x)}{dt}$ must be finite and continuous for all values of x, except at those point where $V \to \infty$. At these points $\frac{d\psi(x)}{dt}$ has a finite discontinuity but ψ remains continuous.
- For bound state and probability of finding the particle between x and $(x + dx)$, *i.e.*, $|\psi|^2 dx$ must vanish as $|x| \to \infty$.

  Hence $|\psi(x)| \to 0$ as $|x| \to \infty$ *i.e.*, $\psi(x)$ is a square integrable wave function.

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4. *Why the wave function ψ itself has no direct physical significance ?*

Ans. ψ cannot be interpreted in terms of an experiment. The probability that something be in a certain place at a given time must be lie between 0 (the object is definitely there) and 1 (the object is definitely there). An intermediate probability, say 0.2 means that there is a 20% chance of finding the object. But the amplitude of a wave can be negative as well as positive and negative probability is meaningless. Hence ψ by itself cannot be an observable quantity.

5. *What is zero point energy ?* **IGGSIPU, May 2009 (2.5 marks)**

Ans. The possible energies of a particle in a box of length '*a*' is given by

$$E_n = \frac{n^2h^2}{8ma^2} \qquad \text{(where } n = 1, 2, 3 \ldots)$$

If $n = 1$, then $$E_1 = \frac{h^2}{8ma^2}$$

This is the energy of the ground state of particle. Since, the particle in a box cannot be at rest, its minimum energy is positive and is often called the zero point energy.

6. *What are eigen values and eigen functions ?*

Ans. From equation, $E_n = \frac{n^2h^2}{8ma^2}$ (where $n = 1, 2, 3, 4 \ldots$)

It is clear that inside an infinitely deep potential well (or in an infinite square well), the particle cannot have an arbitrary energy, but can have only certain discrete energy corresponding to $n = 1, 2, 3, 4 \ldots$ Each permitted energy level energy is called eigen value of the particle and constitutes the energy level of the system. The wave function ψ corresponding to each eigen value are called eigen functions.

7. *Is Schrödinger equation is linear in the wavefunction ψ ? What is meaning of it ?*

Ans. Yes, Schrödinger equation is linear in wave function ψ. By this we mean that the equation has terms that contain ψ and its derivatives but no terms independent of ψ or that involves higher power of ψ or its derivative.

8. *What is expectation value ?*

Ans. In quantum mechanics a physical system is represented by means of a wave function. So the physical variables like the position, momentum and energy of the system do not have definite values, instead they have a probable value. Thus results of two successive measurements of the position of a particle may not yield the same value. The average value of a physical variable is called its expectation value. This may be obtained by a large number of independent measurements on the same system or by a single measurement on each of the large number of identical systems.

Chapter 7 Statistical Mechanics

1. *Which type of statistics shall be applicable for a gas of photons ? Justify your answer.*

IGGSIPU, May 2008 (2.5 marks)

Ans. Bose-Einstein statistics shall be applicable for a gas of photons. Bose-Einstein statistics is obeyed by indistinguishable particles of integral spin that have symmetric wave functions and does not obey Pauli exclusion principle.

2. *Distinguish between a Bosin and a Fermion.* **IGGSIPU, April 2008 (2 marks)**

Ans. *Bosons.* The identical particles having an integral quantum number of their intrinsic spin by symmetric wave functions, such as

$$\psi_s(1,2) = +\psi(2,1)$$

This type of particles described by symmetric wave functions are known as bosons as they obey Bose-Einstein statistics.

Fermions. The particle which have half integral quantum number for their intrinsic spin are described by anti-symmetric wave functions, such as

$$\psi_A(1,2) = -\psi(2,1)$$

This type of particles described by anti-symmetric wave functions are called fermions as they obey Fermi-Dirac statistics.

3. *Example the terms : position space, momentum space and phase space.* **IGGSIPU, April 2008 (2.5 marks)**

Ans. Refer to article 7.2.2 at page 246.

4. *What do you mean by Fermi Dirac probability function ?*

Ans. The statistical distribution of electrons among available energy state is described by Fermi Dirac probability function $f(E)$. This probability function gives the probability of occupying an allowed energy state by electrons.

5. *Define Fermi energy.*

Ans. Fermi energy is defined as the topmost occupied energy level by electrons at 0 K. At temperature exceeding absolute zero, the Fermi energy may be defined as that energy level for which occupation probability of an electron is half.

6. *What is density of states function ?*

Ans. It is defined as the number of energy states per unit energy per unit volume.

7. *What are the differences among particles belongs to (i) Maxwell-Boltzmann statistics ; (ii) Bose-Einstein statistics ; and (iii) Fermi Dirac statistics ?*

Ans. (*i*) Maxwell-Boltzman statistics apply to systems of distinguishable particles, such as gases.

(*ii*) Bose-Einstein statistics apply to systems of indistinguishable particles not obeying Pauli's exclusion principle, such as photons, phonons and liquid helium at low temperature.

(*iii*) Fermi-Dirac statistics apply to systems of indistinguishable particles obeying Pauli exclusion principle such as free electrons in metal electrons in a star (White dwarf star).

8. *Write a short notes on electrons in White dwarf stars.*

Ans. Electrons in White Dwarf Stars :

In a normal star hydrogen nuclei fuse together to form helium, the reaction giving out huge amounts of nuclear energy. It is the release of all this energy that keeps the star hot and inflated. When the star has used up all its hydrogen, it loses this source of energy, cools down, and collapses. When this happens the atoms of helium are strongly compressed together under gravitational forces, and become very compressed. An electron orbiting a helium atom would become more tightly localized, which leads, via the uncertainty principle, to an increase in its kinetic energy.

If the density is increased sufficiently, the kinetic energy of the electrons overcomes the potential energy which binds them to the atomic nuclei and the electrons become free particles, the helium atom turns into alpha particle plus two free electrons. There is a huge density of electrons which are relatively cold since the hydrogen fuel has been used. The large star has shrunk to what is called a white dwarf.

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Chapter 8 Semiconductor Physics

1. *In doped semiconducter how does the free carrier concentration change with increase in temperature ?*

Ans. In doped semiconductors there are three regions as far as the carrier concentration versus temperature variation is concerned.

- At low temperature the electron (hole) from the donor (acceptor) is bound to the dopant. This is the freezeout region.
- With increase of temperature the fraction of ionized donors or acceptors increases and ultimately all the donors and acceptors are ionized and the free carrier concentration becomes equal to the dopant concentration. This is called saturation region. Semiconductor devices are normally preferred to operate in the temperature range at which saturation occurs.
- As the temperature is further increased beyond saturation region, the carrier concentration again increases due to increases the intrinsic generation across the band gap and becomes larger then the doping concentration. This is called intrinsic region.

2. *What are compensated semiconductor ?*

Ans. A compensated semiconductor is one that contains both donor and acceptor impurity atoms. If acceptor impurities are diffused in N-type semiconductor or donor impurities are diffused in P-type semiconductor then compensated semiconductor are created. A completely compensated semiconductor is one for which donor impurity concentration equals acceptor impurity concentration *i.e.*, $N_D = N_A$.

3. *What are degenerate semiconductors ?*

Ans. With increasing doping concentration of donors (acceptors) a stage will be reached when the electron (hole) concentration will exceed the effective density of states of electrons (holes) and Fermi level goes into the conduction band (N-type) or in valence band (P-type) then the semiconductor is called degenerate semiconductor. This can be explained as follows :

As the doping concentration increases the distance between impurity atoms decreases and concentration increases the distance between impurity atoms decreases and a stage will be reached when the donor electrons will start to interact among themselves. This will lead to splitting of single discrete donor energy level into an energy band which will widen with the increase of impurity concentration and ultimately overlap the edge of the conduction band.

4. *What do you mean by thermal equilibrium in a semiconductor ?*

Ans. Equilibrium or thermal equilibrium in a semiconductor means that no external forces such as electric field, magnetic field, radiation or temperature gradient act on the semiconductor. All properties of the semiconductor will be independent of time in thermal equilibrium.

5. *Where is Fermi energy level located in an intrinsic semiconductor ?*

Ans. The Fermi level is located very close to the middle of the band gap slightly upwards towards the edge of the conduction band due to the difference of the effective mass of the electrons and holes.

6. *Why do metals have a positive temperature coefficient of resistance and semiconductor have negative temperature coefficient of resistance ?*

Ans. A large number of free electrons participate in conduction in metals. With increase of temperature these electrons are scattered more frequently by increasing number of phonons and the relaxation time for collision decreases. As a result mobility decreases and resistivity increases with the increase mobility decreases and resistivity increases with the increase of temperature in metals. In case of semiconductors, as temperature increases large number of electron-hole pairs are generated thermally. Therefore, conductivity increases and resistivity decreases.

7. *How does a doped semiconductor behave with the increase of temperature ?*

Ans. A doped or extrinsic semiconductor tends to become an intrinsic one as temperature increases.

8. *How does the Fermi level of doped semiconductor shift its position with increase of temperature ?*

Ans. As temperature increases the Fermi level of a doped semiconductor shifts towards the intrinsic level *i.e.*, towards the midgap.

9. *What is an ideal P-N junction diode ?*

Ans. An ideal diode has zero resistance when forward biased and infinite resistance when reverse biased. It acts like an open switch in reverse bias and closed switch in forward bias.

10. *Define Knee voltage or cut-in-voltage from current voltage characteristics of a P-N junction diode.*

Ans. The forward bias characteristics of a P-N junction diode indicate that practically no diode current flows till the point *A* in the characteristics is reached. As the forward voltage increases beyond the point *A* the current increases rapidly. The voltage at which the diode starts conducting is known as knee voltage or cut-in voltage, which can be obtained from the forward characteristics by extending the part *BA* of the curve backwards until it meets the voltage axis. The value of the voltage where the curve meets the voltage axis is equal to knee voltage (V_k). Its value is 0.6 V for silicon diodes and 0.2 V for germanium diodes.

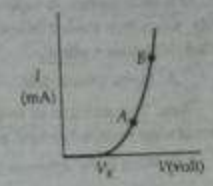

Fig. CQ.3

11. *Mention few important applications of a P-N junction diode.*

Ans. A P-N junction diode is a basic semiconductor device which finds various applications in electronic circuits. A few such applications are given below :

- As a switch in digital circuits.
- As detector in communication circuits.
- As Zener diode in regulated power supply.
- As Varactor diode in radio and TV tuning circuits.
- As rectifier in dc power supply.

12. *Distinguish between Zener breakdown and avalanche breakdown.*

Ans.

S.No.	Zener breakdown	Avalanche breakdown
1.	This type of breakdown occurs in heavily doped P-N junction.	This break down occurs in lightly doped junction.
2.	The phenomenon is caused by tunneling or field emission.	The phenomenon is caused by impact ionization and avalanche multiplication.
3.	Breakdown voltage is low.	Breakdown voltage is high.
4.	Temperature coefficient of breakdown voltage is negative.	Temperature coefficient of breakdown voltage is positive.

13. *Mention few applications of Zener diode.*

Ans. Zener diode finds some important applications in practical electronic circuits. Some of these are mentioned below :

- As a voltage regulator
- As a fixed reference voltage in transfer biasing circuit.
- As a peak clippers or limiters in waveshaping circuits.
- For protection of galvanometer, ammeter and voltmeter against damage.

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Chapter 9 Superconductivity

1. *What do you mean by superconductivity and critical temperature ?*

Ans. The resistance offered by certain metals, alloys and chemical compounds to the flow of electric current abruptly drops to zero below a threshold temperature. This phenomenon is called superconductivity and threshold temperature is called critical temperature.

2. *What are the Isotope effect and the Meissner's effect ?*

Ans. Isotope Effect. The variation of critical temperature of an element with its isotopic mass is known as isotopic effect *i.e.*, $T_c \propto M^{-\beta}$ $[0 < \beta \le 0.5]$

Meissner's Effect. A superconducting material kept in a magnetic field expects the magnetic flux out of its body when it is cooled below the critical temperature and thus becomes perfect diamagnet. This effect is called 'Meissner's effect'.

3. *Define critical field.*

Ans. The strength of magnetic field required to just switch a material from superconducting state to normal state is called critical magnetic field.

4. *What is the difference between perfect diamagnetic and superconductor ?*

Ans. If a conductor already had a steady magnetic field through it and was then cooled through the transition to a zero resistance state, becoming a perfect diamagnet, the magnetic field would be expected to stay the same. The magnetic behaviour of a superconductor is distinct from perfect dimagnetism. It will actively exclude any magnetic field present when it makes the phase change to the superconducting state.

5. *Define Cooper pair. Give their physical significance.*

Ans. Cooper pair is a bound pair of electrons formed by the interaction between the electrons with opposite spin and momenta in a phonon field.

When the electrons flows in the form of Cooper pairs in materials. They do not encounter any scattering and the resistance factor vanishes or, in other words, conductivity becomes infinity, which is named as superconductivity.

6. *Why type II superconductors are more important ?*

Ans. Type II superconductors or hard superconductors can carry high current densities, as a result these have great technological values. To obtain high critical current densities, a superconductor should be fine grain material or heavily cooled-worked.

7. *What should be two distinct properties of material to be superconductor ?*

Ans. Zero resistivity *i.e.*, infinite conductivity and $B = 0$ inside the superconductor.

8. *What is BCS theory.*

Ans. In 1956 Cooper introduced first pairing of electrons called Cooper pairs. Then the next year in 1957, the microscopic theory called reliable theory was first constructed, only 46 years after the discovery of superconductivity. It is known as Bardeen, Cooper and Schrieffer (BCS) theory. This theory concludes the following important results :

The critical temperature of a superconductor is Debye temperature (θ_D) dependent *i.e.*,

$$T_c = 1.14\theta_D e^{-1/\lambda} \quad [\lambda \text{ is coupling constant}]$$

The energy gap in superconductor at transition temperature is

$$E_g = 3.52\, k_B T_c$$

mag lev

The specific heat capacity at transition temperature is

$$C_v = 1.43\, \gamma T_c$$

where γ is the coefficient of electronic specific heat.

T_c depends on mass and observed as $T_c \propto M^{-1/2}$ called isotope effect.

9. *Define the penetration depth.*

Ans. The penetration depth can be defined as the perpendicular distance from the surface of a superconductor to a point inside, where the superconducting density is $1/e$ times of its value on the surface.

10. *What is coherence length ?*

Ans. Coherence length is a distance within which the order parameter (ratio of superconducting electrons and normal electrons) changes from its maximum value in the bulk superconductor region to zero in the normal region.

11. *What is magnetic levitation ?*

Ans. Magnetic levitation or maglev is the process by which an object is suspended above another object with no other support but magnetic field.

Chapter 10 X-rays

1. *What are X-rays ? Classify them.*

Ans. X-rays are electromagnetic waves having wavelengths from 0.5 Å to 1.5 Å and are not deflected by electric and magnetic field. X-rays are primarily classified into two.

- X-rays of long wavelength are called soft X-rays.
- X-ray of short wavelength are called hard X-rays.

2. *What do you mean by continuous and characteristic X-rays ? Give their important applications.*

Ans. It is evident from the Fig. CQ. 4, that at lower value of V (potential difference between cathode and anode), smooth and continuous curves starting from a minimum wavelength (λ_{min}) and extending upto very high wavelength are obtained. As V is increased, λ_{min} shifts towards lower wavelength side. The continuous spectrum of emitted radiation is termed as **continuous X-rays** when V is further increased, bumps start appearing at fixed wavelengths in I versus λ curve. These bumps at fixed wavelength are called the **characteristic X-rays**.

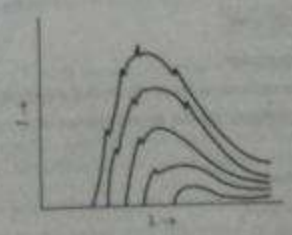

Fig. CQ.4 I vs. λ curve

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Important Applications :

The study of the continuous X-rays spectrum is more important in the field of medicine, surgery and industry because greater part of the energy is to be found in that part, while study of characteristic X-rays is important for scientific and research point of view.

3. *Why are X-rays used to study the crystal structure ?*

Ans. First of all Schuster pointed out X-rays are e.m.waves with wavelengths thousand times smaller than those of the visible light. In order to measure the wavelengths, a grating of corresponding dimensions is required and simple grating can not be used. Moreover, it is rather impossible to prepare artificially a grating of such fine dimension. Laue and his collaborators showed the atoms in crystals are arranged in a regular manner and the spacing between them is comparable to the wavelength of X-rays and hence the crystal could act as suitable natural grating for diffraction grating. In this way X-rays are used to study the crystal structure.

4. *What is the difference between crystalline solid and amorphous solids ?*

Ans.

S.No.	Crystalline solids	Amorphous solids or Non-crystalline
1.	A substance is said to be crystalline when the arrangement of units of matter is regular and periodic.	In amorphous solids, the constituent particles are not arranged in an orderly manner. They are randomly distributed.
2.	A crystalline material has directional properties and therefore called as *anisotropic* substance.	They do not have directional properties, so they are called as *'isotropic'* substance.
3.	A crystal has a sharp melting point.	Amorphous has wide range of melting point.
4.	They possess regular shape.	They do not possess regular shape.
5.	*Example* : Copper, silver, carbon etc.	*Example* : Glass, plastics, rubber etc.

5. *What are Bragg's planes ?*

Ans. The atoms in a crystal may be visualised as families of parallel planes where each family has a certain characteristic separation between its component planes. These planes are known as **Bragg's planes**.

6. *Define Bragg's law.*

Ans. The law which relates the wavelength of X-rays (λ), interplanar distance (d) and glancing angle (θ_B) is known as Bragg's law. This is

$$2d \sin \theta_B = n\lambda$$

where $n = 1, 2, 3, \ldots$ is the order of spectrum.

7. *Name some X-rays diffraction methods to study the crystal structures.*

Ans. The widely used X-rays methods for the determination of crystal structures are :

- Laue method
- Bragg's X-ray spectrometer
- Powder photograph method
- Rotating crystal method, and
- Back reflection method.

8. *What are the applications of Laue method ?*

Ans.
- This method is mainly application to single crystal.
- This method is used to study the crystal orientations and to verify the crystal symmetry.
- It is used to identify the lattice strains and lattice imperfections.
- An interpretation of the reflection spots leads to know about the crystal structure of the specimen.

9. *Differentiate between characteristic and continuous X-rays.*

Ans.

S.No.	Continuous X-rays spectra	Characteristic X-rays spectra
1.	This type of spectrum consists of radiation of all possible wavelengths within a range starting with a minimum wavelength λ_{min} called short wavelength limit.	The characteristic curve or spectrum has a few sharp peaks at certain wavelengths. These peaks appearing in characteristic spectrum form line spectrum.
2.	The fast moving electrons during deacceleration radiates energy in the form of X-rays.	The fast moving electrons remove an inner electron and radiated electron makes a transition from higher state of lower state.
3.	This type of spectrum consists of radiation with λ_{min} depending upon the potential applied.	The characteristic spectrum depends on the atom through X-rays are emitted.

Chapter 11 Ultrasonics

1. *What are ultrasonic waves ?*

Ans. The longitudinal mechanical waves having frequency greater than the highest audible frequency 20 kHz are called ultrasonic waves.

2. *When the ultrasonic generator delivers maximum power in piezoelectric method of production of ultrasonics ?*

Ans. The ultrasonic generator delivers maximum power when it is operated at fundamental frequency of crystal. In order to generate higher ultrasonic waves, the *L-C* circuit is made to oscillate at a frequency equal to one of the harmonics of the crystals.

3. *What happens when ultrasonic waves pass through a substance ?*

Ans. Intense heat is produced.

4. *What is piezoelectric effect ?* **[GGSIPU, May 2009 (4 marks)]**

Ans. 'Piezo' is a Greek word having meaning pressure. It has been observed in a class of crystal like quartz, that if mechanical pressure is applied along a characteristic axis called mechanical axis then e.m.f. is automatically generated along a perpendicular axis called electric axis. The phenomenon is called piezoelectric effect. Similarly, if e.m.f. is applied along electric axis the mechanical pressure (compression or extension) is generated along mechanical axis. It is called inverse piezoelectric effect.

5. *What is the shortest wavelength of the ultrasonic waves in air at room temperature ?*

Ans. The ultrasonic waves in air travel with the speed of sound which is about 331 m/s at room temperature. Thus the shortest wavelength is 331/2000 m. [smallest ultrasonic wavelength = 2000 Hz]

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6. *What is acoustic grating ?*

Ans. When the ultrasonic waves are created in a container. Stationary waves are created as a result of superposition of the direct and the waves that are reflected from the walls of the container, due to which at the points of rarefaction the density is more. Thus the liquid acts as diffraction grating. This type of grating is called acoustic grating.

7. *Is it possible to find the compressibility of the liquid from the acoustic grating experiment ?*

Ans. Yes, once velocity of ultrasonic waves 'v' is determined. One can use the formula :

Compressibility $C = \frac{1}{E} = \frac{1}{\rho v^2}$, where ρ is the density of liquid.

8. *Why is it necessary to use R.F. oscillator in the experiment of acoustic grating ?*

Ans. As the natural frequency of the quartz crystal is of the order of 10^6 Hz. R.F. oscillator is necessary as it gives a.c. voltages of this range of frequency which is requisite for resonance.

9. *How does a ferroelectric and piezoelectric material respond to an applied alternating electric field ?*

Ans. The material starts vibrating with the frequency of applied electric field.

10. *What is magnetostriction effect ?*

Ans. When a magnetic field is applied, a bar of ferromagnetic or ferrimagnetic material undergoes a change in length. Conversely, a mechanical stress is applied to the bar will cause a change in intensity of magnetisation.

11. *Whether you are using piezoelectric or inverse piezoelectric effect in the experiment of acoustic grating ?*

Ans. Inverse piezo-electric effect.

12. *What are the advantages of ultrasonic inspection method ?*

Ans. *Advantages :*

- Can reveal internal defects.
- high sensitivity to most cracks and flaws.
- high speed test with immediate results
- can be automated and recorded.
- portable
- high penetration in most important materials (upto 60 feet in steel)
- indicate flaw size and location
- access to only one side is required
- can also to used to measure thickness, Poisson's ratio or elastic modulus.

13. *What are the properties of ultrasonic waves ?*

Ans.
- The energy of ultrasonic waves is very high.
- By increasing the frequency of the wave, the speed of propagation can be increased.
- They can be transmitted over long distance without any loss.
- Due to small in wavelength, the ultrasonic waves diffract less.
- They produce intense heating effect when it is passed through a substance.
- They are highly absorbed by gas and considerably lesser in liquid.
- Ultrasonics have a crushing action and hence they can be used to crush solid bodies, living organism, large molecules.