Partial differential equation

A partial differential equation is an equation which contains two or more independent variable, a dependent variable and partial derivatives of dependent variable with respect to independent varaible.

eg $2\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} + 6\frac{\partial^2 z}{\partial x \partial y} = 0$.

- let z be a function of two independent variables x & y ie $z = f(x,y)$. Then z is dependent variable.

Denote $\frac{\partial z}{\partial x}$ by p and $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.

Formation of partial differential equation

by eliminating arbitrary constants

① Form a partial diff eq^n by eliminating arbitrary constants a and b from relation $z = (x-a)^2 + (y-b)^2$.

→ $z = (x-a)^2 + (y-b)^2$ —— ⊛

Compute partial derivate of z wrt x & y.

$\frac{\partial z}{\partial x} = 2(x-a) + 0 \Rightarrow \frac{1}{2}\frac{\partial z}{\partial x} = x - a$

$\frac{\partial z}{\partial y} = 0 + 2(y-b) \Rightarrow \frac{1}{2}\frac{\partial z}{\partial y} = y - b$.

Put $x - a = \frac{1}{2}\frac{\partial z}{\partial x}$ & $y - b = \frac{1}{2}\frac{\partial z}{\partial y}$ in ⊛

$z = \left(\frac{1}{2}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{2}\frac{\partial z}{\partial y}\right)^2$

$z = \frac{1}{4}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

$$4z = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

is required partial diff eq^n.

Q Form pde by eliminating arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

$\rightarrow$ $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Compute partial derivative of z wrt x & y.

$$\frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{-1}{x^2}\right)$$

$$\Rightarrow -\frac{x^2}{2}\frac{\partial z}{\partial x} = f'\left(\frac{1}{x} + \log y\right).$$

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(0 + \frac{1}{y}\right).$$

$$\frac{\partial z}{\partial y} = 2y + \frac{2}{y}\left[-\frac{x^2}{2}\frac{\partial z}{\partial x}\right].$$

$$\frac{\partial z}{\partial y} = 2y - \frac{x^2}{y}\frac{\partial z}{\partial x}$$

$$y\frac{\partial z}{\partial y} = 2y^2 - x^2\frac{\partial z}{\partial x}$$

$$\boxed{x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2y^2}$$

Q Form pde by eliminating arbitrary function f (3)
from $f(xy+z^2, x+y+z) = 0$

→ Let $u = xy+z^2$ & $v = x+y+z$.

Then $f(u,v) = 0$ where $u = xy+z^2$ & $v = x+y+z$.

$$\frac{\partial u}{\partial x} = y + 2z\frac{\partial z}{\partial x} \quad \Big| \quad \frac{\partial v}{\partial x} = 1+0+\frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial y} = x + 2z\frac{\partial z}{\partial y} \quad \Big| \quad \frac{\partial v}{\partial y} = 0+1+\frac{\partial z}{\partial y}.$$

Partially differentiate $f(u,v) = 0$ wrt x & y.

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial u}(y+2zp) + \frac{\partial f}{\partial v}(1+p) = 0 \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial u}(x+2zq) + \frac{\partial f}{\partial v}(1+q) = 0 \quad \text{—— (2)}$$

From (1) equation $\frac{\partial f}{\partial u}(y+2zp) = -\frac{\partial f}{\partial v}(1+p)$

$$\Rightarrow \frac{\partial f/\partial u}{\partial f/\partial v} = \frac{-(1+p)}{y+2zp} \quad \text{—— (3)}$$

From (2) equation $\frac{\partial f}{\partial u}(x+2zq) = -\frac{\partial f}{\partial v}(1+q)$

$$\Rightarrow \frac{\partial f/\partial u}{\partial f/\partial v} = \frac{-(1+q)}{x+2zq} \quad \text{—— (4)}$$

Equate (3) & (4)

$$\frac{\cancel{-}(1+p)}{y+2zp} = \frac{\cancel{-}(1+q)}{x+2zq}$$

(4)

$$(1+p)(x+2zq) = (1+q)(y+2zp)$$

$$x+2zq+px+\cancel{2zpq} = y+2zp+yq+\cancel{2zpq}$$

$$2zq-2zp+px-yq = y-x$$

$$(2z-y)q+p(x-2z) = y-x$$

$$\boxed{(2z-y)\frac{\partial z}{\partial y} + (x-2z)\frac{\partial z}{\partial x} = y-x}$$

Q Form partial diff eq^n by eliminating arbitrary function f and ϕ from eq^n $z = f(y/x) + \phi(xy)$.

$\longrightarrow$ $z = f(y/x) + \phi(xy)$.

Partially diff z wrt x & y.

$$p = \frac{\partial z}{\partial x} = f'(y/x)(-y/x^2) + \phi'(xy)(y).$$

$$p = -\frac{y}{x^2} f'(y/x) + y\phi'(xy). \Big] \times x \quad (1)$$

$$q = \frac{\partial z}{\partial y} = f'(y/x)\left(\frac{1}{x}\right) + \phi'(xy)\, x.$$

$$q = \frac{1}{x} f'(y/x) + x\phi'(xy). \Big] \times y \quad (2)$$

Add x times (1) & y times (2).

$$xp+yq = \cancel{-\frac{y}{x} f'(y/x)} + xy\phi'(xy) + \cancel{\frac{y}{x} f'(y/x)} + xy\phi'(xy)$$

$$xp+yq = 2xy\phi'(xy)$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xy\,\phi'(xy) \quad \text{—} \ (*)$$

Partially diff $(*)$ wrt x & y.

$$\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(2xy\,\phi'(xy)\right).$$

$$1\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=2y\,\phi'(xy)+2xy^2\,\phi''(xy)$$

$$p+xr+ys=2y\,\phi'(xy)+2xy^2\,\phi''(xy). \quad \text{—} \ (3).$$

$$\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(2xy\,\phi'(xy)\right)$$

$$x\frac{\partial^2 z}{\partial y\partial x}+1\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=2x\,\phi'(xy)+2x^2y\,\phi''(xy) \quad \text{—} \ (4).$$

$$\hookrightarrow xs+q+yt=2x\phi'(xy)+2x^2y\,\phi''(x).$$

Subtract y times (4) from x times (3).

$$\begin{aligned} x(p+xr+ys) \\ -y(xs+q+yt) \end{aligned} = \begin{aligned} &2xy\,\cancel{\phi'(xy)}+2x^2y^2\cancel{\phi''(xy)} \\ &-2xy\,\phi'\cancel{(xy)}-2x^2y^2\,\cancel{\phi''(xy)} \end{aligned}$$

$$=0.$$

$$\begin{aligned} &xp+x^2r+\cancel{xy}s \\ &-\cancel{yx}s-qy-y^2t \end{aligned}$$

$$xp+x^2r=qy+y^2t.$$

$$\boxed{x\frac{\partial z}{\partial x}+x^2\frac{\partial^2 z}{\partial x^2}=y\frac{\partial z}{\partial y}+y^2\frac{\partial^2 z}{\partial y^2}}.$$

Solution of linear partial differential equation of order one.

A linear partial differential equation of first order with dependent variable z and independent variables x and y, is of form $Pp + Qq = R$ where $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.

$Pp + Qq = R$ — (*) is known as lagrangian form of partial differential equation.

To solve above equation, form a set of auxillary simultaneous equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

If $u = a$ and $v = b$ are two independent solution of these equations, then solution of partial diff eq^n (*) is $\phi(u,v) = 0$ where ϕ is an arbitrary function. [OR $u = f(v)$ OR $v = f(u)$ can be sol^n of (*) where f is arbitrary function].

Q Find general sol^n of pde:

$$y^2 z p - x^2 z q = x^2 y.$$

→ Compare above PDE with $Pp + Qq = R$ to get

$P = y^2 z$, $Q = -x^2 z$ and $R = x^2 y$.

Auxillary eq^n is

$$\frac{dx}{y^2 z} = \frac{dy}{-x^2 z} = \frac{dz}{x^2 y}.$$

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{-x^2 \cancel{z}}$$

$x^2 dx = -y^2 dy$

$x^2 dx + y^2 dy = 0$

Integrate both sides

$\int x^2 dx + \int y^2 dy = \int 0$

$\frac{x^3}{3} + \frac{y^3}{3} = \frac{a}{3}$

$x^3 + y^3 = a$ — (1) (a is constant).

$\frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$

$\cancel{x^2} y\, dy = -\cancel{x^2} z\, dz$

$y\,dy + z\,dz = 0$

Integrate both sides

$\int y\,dy + \int z\,dz = \int 0$

$\frac{y^2}{2} + \frac{z^2}{2} = \frac{b}{2}$

$\Rightarrow y^2 + z^2 = b$ — (2)

using (1) & (2), general solⁿ of given pde is $\phi(x^3+y^3, y^2+z^2) = 0$ where ϕ is an arbitrary function.

Q Find general solⁿ of linear pde
$(x^2-y^2-z^2)p + 2xyq = 2xz$.

→ Compare above pde with $Pp + Qq = R$ to get
$P = x^2-y^2-z^2$, $Q = 2xy$, $R = 2xz$.

Auxillary eqⁿ is

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \frac{dy}{y} - \frac{dz}{z} = 0$$

Integrate both sides

$$\int \frac{dy}{y} - \int \frac{dz}{z} = \int 0$$

$$\ln y - \ln z = \ln a$$

$$\ln\left(\frac{y}{z}\right) = \ln a$$

$$\frac{y}{z} = a \quad \text{— (1)}$$

Consider,

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}.$$

$$\therefore \quad \frac{dz}{2\cancel{x}z} = \frac{xdx + ydy + zdz}{\cancel{x}(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}.$$

$$0 = \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2} - \frac{dz}{z}$$

Integrate both sides

$$\int 0 = \int \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2} - \int \frac{dz}{z}$$

$$\ln b = \ln(x^2+y^2+z^2) - \ln z$$

$$\ln b = \ln\left(\frac{x^2+y^2+z^2}{z}\right)$$

$$b = \frac{x^2+y^2+z^2}{z} \longleftarrow (2)$$

General solⁿ of given pde is

$$\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$ where ϕ is an arbitrary function.

<u>Q</u> Solve $p - x^2 = q + y^2$.

$$\Rightarrow p - q = x^2 + y^2.$$

$\rightarrow$ Auxillary eqⁿ is

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{x^2+y^2} = \frac{-x^2dx + y^2dy + 1dz}{-x^2 - y^2 + x^2 + y^2}$$

$$= \frac{-x^2dx + y^2dy + dz}{0}$$

$$\frac{dx}{1} = \frac{dy}{-1}$$

$\Rightarrow dx - dy = 0$

Integrate both sides

$$\int dx - \int dy = \int 0$$

$$x - y = a \quad \text{—— } (1)$$

$$\frac{dx}{1} = \frac{-x^2dx + y^2dy + dz}{0}$$

$\Rightarrow 0 = -x^2dx + y^2dy + dz$

Integrate both sides

$$\frac{b}{3} = \frac{-x^3}{3} + \frac{y^3}{3} + z$$

$\Rightarrow b = -x^3 + y^3 + 3z$

$\therefore$ Solⁿ is $\phi(x-y, -x^3+y^3+3z) = 0$

where ϕ is arbitrary function.