Taylor's Theorem For function of two variables

Taylor series expansion of function $f(x)$ of one variable x is given by

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

where h is small.

Similarly, taylor series expansion of function $f(x,y)$ of two variables x & y is given by

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots \quad (*)$$

where h & k are very small.

- Replace x by x_0, y by y_0, h by $x-x_0$ & k by $y-y_0$ in $(*)$.

$$f(x,y) = f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} + (y-y_0)\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} + \frac{1}{2!}\left((x-x_0)^2\frac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} + 2(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial x \partial y}\Big|_{(x_0,y_0)} + (y-y_0)^2\frac{\partial^2 f}{\partial y^2}\Big|_{(x_0,y_0)}\right) + \cdots$$

Above is known as taylor series expansion of $f(x,y)$ around point (x_0, y_0).

- If we take point $(x_0, y_0) = (0,0)$ then above series is

$$f(x,y) = f(0,0) + x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\frac{\partial f}{\partial y}\Big|_{(0,0)} + \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} + 2xy\frac{\partial^2 f}{\partial x \partial y}\Big|_{(0,0)} + y^2\frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)}\right) + \cdots$$

and is known as Maclaurin series expansion of $f(x,y)$.

Q Expand $f(x,y) = e^{xy}$ about $(1,1)$ upto second degree terms.

→ Here, point $(x_0, y_0) = (1,1)$.

$$f(x_0, y_0) = f(1,1) = e^1 = e$$

$$\frac{\partial f}{\partial x} = y e^{xy} \; ; \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 1e$$

$$\frac{\partial f}{\partial y} = x e^{xy} \; ; \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = e$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy} \; ; \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = e$$

$$\frac{\partial^2 f}{\partial y \partial x} = xy e^{xy} \; ; \quad \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(1,1)} = e$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 e^{xy} \; ; \quad \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,1)} = e.$$

$$f(x,y) \approx f(x_0, y_0) + (x - x_0)\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} + (y - y_0)\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$$
$$+ \frac{1}{2!}\left[(x-x_0)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} + 2(x-x_0)(y-y_0)\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(x_0,y_0)} + (y-y_0)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)}\right]$$

$$\approx f(1,1) + (x-1)\left.\frac{\partial f}{\partial x}\right|_{(1,1)} + (y-1)\left.\frac{\partial f}{\partial y}\right|_{(1,1)} +$$
$$\frac{1}{2!}\left[(x-1)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} + 2(x-1)(y-1)\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(1,1)} + (y-1)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,1)}\right]$$

$$f(x,y) \approx e + (x-1)e + (y-1)e + \frac{1}{2}\left((x-1)^2 e + 2(x-1)(y-1)e + (y-1)^2 e\right)$$
$$= (1 + x - 1 + y - 1)e + \frac{e}{2}\left(x^2 - 2x + 1 + 2xy - 2x - 2y + 2 + y^2 - 2y + 1\right)$$
$$= (x + y - 1)e + \frac{e}{2}\left(x^2 + y^2 + 4 - 4x - 4y + 2xy\right) = (x+y-1)e + \frac{e}{2}(x+y-2)^2$$

Q Expand $f(x,y) = \tan^{-1}(xy)$ in powers of $(x-1)$ and $(y-1)$ and hence evaluate $f(0.9, 1.1)$.

$\longrightarrow$ Take point $(x_0, y_0) = (1,1)$.

$$f(x_0, y_0) = f(1,1) = \tan^{-1} 1 = \pi/4 .$$

$$\frac{\partial f}{\partial x} = \frac{y}{1+(xy)^2} \quad ; \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \frac{1}{1+1^2} = \frac{1}{2}.$$

$$\frac{\partial f}{\partial y} = \frac{x}{1+(xy)^2} \quad ; \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \frac{1}{1+1^2} = \frac{1}{2}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-y}{(1+(xy)^2)^2} \times 2xy^2 \quad ; \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = \frac{-2}{(1+1^2)^2} = \frac{-2}{2^2} = \frac{-1}{2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x}{(1+(xy)^2)^2} \times 2yx^2 \quad ; \quad \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,1)} = \frac{-2}{(1+1^2)^2} = \frac{-2}{2^2} = \frac{-1}{2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{+1}{1+(xy)^2} - \frac{y}{(1+(xy)^2)^2} \times 2xy \quad ; \quad \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(1,1)} = \frac{1}{1+1^2} - \frac{2}{(1+1^2)^2}$$

$$= \frac{1}{2} - \frac{2}{2^2} = \frac{1}{2} - \frac{1}{2} = 0$$

$$f(x,y) \approx f(1,1) + (x-1)\left.\frac{\partial f}{\partial x}\right|_{(1,1)} + (y-1)\left.\frac{\partial f}{\partial y}\right|_{(1,1)} +$$

$$\frac{1}{2!}\left[(x-1)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} + 2(x-1)(y-1)\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(1,1)} + (y-1)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,1)}\right] + \ldots$$

$$= \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2}\left[(x-1)^2\left(\frac{-1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(\frac{-1}{2}\right)\right] + \ldots$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1+y-1) + \frac{1}{2}\left(\frac{-1}{2}\right)\left[(x-1)^2 + (y-1)^2\right] + \ldots$$

$$f(x,y) = \frac{\pi}{4} + \frac{1}{2}(x+y-2) - \frac{1}{4}\left((x-1)^2 + (y-1)^2\right) + \ldots$$

$$f(0.9, 1.1) \pm \frac{\pi}{4} + \frac{1}{2}(0.9+1.1-2) + \frac{1}{4}\left((0.9-1)^2 + (1.1-1)^2\right) + \cdots$$

≈ 0.785 approx.

Errors and Approximations

Let u be function of two variables x & y; let δx & δy be small changes in value of x & y respectively. Then, corresponding change in value of u be δu.

$$\delta u = u(x+\delta x, y+\delta y) - u(x, y)$$

$$= [u(x+\delta x, y+\delta y) - u(x, y+\delta y)] + [u(x, y+\delta y) - u(x, y)]$$

$$= \frac{u(x+\delta x, y+\delta y) - u(x, y+\delta y)}{\delta x}\delta x + \frac{u(x, y+\delta y) - u(x, y)}{\delta y}\delta y.$$

If δx is very small change, then $\lim_{\delta x \to 0} \frac{u(x+\delta x, y+\delta y) - u(x, y+\delta y)}{\delta x} = \frac{\partial u}{\partial x}$

If δy is very small change, then $\lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x, y)}{\delta y} = \frac{\partial u}{\partial y}$

Then changes in δu becomes,

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y.$$

Above represents error in value of u corresponding to small error δx & δy in values of x & y respectively.

Q Compute approximate value of $(1.04)^{3.01}$.

$\rightarrow$ $(1.04)^{3.01} = (1+0.04)^{3+0.01}$

let $u = x^y$; $x=1$, $y=3$, $\delta x = 0.04$, $\delta y = 0.01$.

$\cancel{\hookrightarrow = f(x,y)}$

Then,

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y.$$

$$\delta u = y x^{y-1}\delta x + x^y \ln x \, \delta y.$$

$$\cancel{f}(x+\delta x, y+\delta y) = \cancel{f}(x,y) + \delta u.$$

$$u(x+\delta x, y+\delta y) = u(x,y) + y x^{y-1}\delta x + x^y \ln x \, \delta y.$$

$$u(1+0.04, 3+0.01) = u(1,3) + (3)(1)^{3-1}(0.04) + (1)^3 \ln 1 \,(0.01)$$

$$u(1.04, 3.01) = 1^3 + 3(1)^2(0.04) + 0$$

$$(1.04)^{3.01} = 1 + 0.12$$

$$= 1.12 \text{ approx.}$$

Q If $f(x,y,z) = x^2 y^3 z^{1/10}$, find approx value of f when $x_0 = 1.99$, $y_0 = 3.01$, $z_0 = 0.98$.

$\rightarrow$ $x_0 = 1.99 = \underbrace{2}_{x} - \underbrace{0.01}_{\delta x}$

$y_0 = 3.01 = \underbrace{3}_{y} + \underbrace{0.01}_{\delta y}$

$z_0 = 0.98 = \underbrace{1}_{z} \underbrace{-0.02}_{\delta z}$

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$\cancel{f(x_0+\delta x, y_0+\delta y, z_0+\delta z) - f(x_0,y_0,z_0)}$ $f(x,y,z) = x^2 y^3 z^{1/10}$.

$f(x+\delta x, y+\delta y, z+\delta z) = f(x,y,z) + \delta f$ where — (1)

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z.$$

$$\frac{\partial f}{\partial x} = 2xy^3 z^{1/10} \;;\; \left.\frac{\partial f}{\partial x}\right|_{(2,3,1)} = 2(2)(3)^3(1)^{1/10} = 108$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 z^{1/10} \;;\; \left.\frac{\partial f}{\partial y}\right|_{(2,3,1)} = 3(2)^2(3)^2(1)^{1/10} = 108.$$

$$\frac{\partial f}{\partial z} = \frac{1}{10}x^2y^3 z^{-9/10} \;;\; \left.\frac{\partial f}{\partial z}\right|_{(2,3,1)} = \frac{1}{10}(2)^2(3)^3(1)^{-9/10} = \frac{108}{10}.$$

Put $x=2, y=3, z=1, \delta x = -0.01, \delta y = 0.01, \delta z = -0.02$.

$$f(2-0.01, 3+0.01, 1-0.02) = f(2,3,1) + \cancel{108(-0.01) + 108(0.01)} + \frac{108}{10}(-0.02)$$

$$f(1.99, 3.01, 0.98) = (2)^2(3)^3(1)^{1/10} - 10.8(0.02)$$

$$= 108 - 0.216$$

$$= 107.784.$$

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