Parallel Monte Carlo Synthetic Acceleration Methods for Discrete Transport Problems

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July 29, 2013

Outline

- Monte Carlo Synthetic Acceleration Methods
- Application to Neutron Transport
- Application to Fluid Flow
- Parallelization of MCSA

Acknowledgments

This work was performed under appointment to the Nuclear Regulatory Commission Fellowship program at the University of Wisconsin - Madison Engineering Physics Department

Statement of Work

The goal of this work is to improve the iterative performance and parallel scalability of solutions to discrete linear and nonlinear transport problems by researching and developing a new set of domain decomposed Monte Carlo Synthetic Acceleration methods.

Hardware-Based Motivation

- Modern hardware is moving in two directions (Kogge,2011):
  - Lightweight machines
  - Heterogeneous machines
  - Both characterized by low power and high concurrency
- Some issues:
  - Higher potential for both soft and hard failures (DOE,2012)
  - Memory restrictions are expected with a continued decrease in memory/FLOPS
- Potential resolution from Monte Carlo:
  - Soft failures buried within the tally variance
  - Hard failures are high variance events
  - Memory savings over conventional methods

Research Outline

- Development of a linear solver for discrete systems leveraging Monte Carlo Synthetic Acceleration
  - Application to neutron transport
  - Research is required to explore general solver development
  - Performance is of concern
- Development of a nonlinear solver for discrete systems leveraging Monte Carlo Synthetic Acceleration
  - Application to fluid flow
  - Potential memory benefits
  - Convergence of the linear model is of concern
- Parallelization of Monte Carlo Synthetic Acceleration
  - Parallel strategies taken from modern reactor physics methods
  - Research is required to explore varying parallel strategies
  - Scalability is of concern

Outline

- **Monte Carlo Synthetic Acceleration Methods**
- Application to Neutron Transport
- Application to Fluid Flow
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Monte Carlo Methods for Discrete Linear Systems

- First proposed by J. Von Neumann and S.M. Ulam in the 1940's
- Earliest published reference in 1950
- General lack of published work
- Modern work by Evans and others has yielded new applications

Thomas Evans and Scott Mosher, "A Monte Carlo Synthetic Acceleration method for the non-linear, time-dependent diffusion equation", American Nuclear Society - International Conference on Mathematics, Computational Methods and Reactor Physics, 2009.

Monte Carlo Linear Solver Preliminaries

- Split the linear operator

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \rightarrow \quad \mathbf{x} = \mathbf{H}\mathbf{x} + \mathbf{b}$$

$$\mathbf{H} = \mathbf{I} - \mathbf{A}$$

- Generate the *Neumann series*

$$\mathbf{A}^{-1} = (\mathbf{I} - \mathbf{H})^{-1} = \sum_{k=0}^{\infty} \mathbf{H}^k$$

- Require $\rho(\mathbf{H}) < 1$ for convergence

$$\mathbf{A}^{-1}\mathbf{b} = \sum_{k=0}^{\infty} \mathbf{H}^k\mathbf{b} = \mathbf{x}$$

Monte Carlo Linear Solver Preliminaries

- Expand the Neumann series

$$x_i = \sum_{k=0}^{\infty} \sum_{i_1}^{N} \sum_{i_2}^{N} \dots \sum_{i_k}^{N} h_{i,i_1} h_{i_1,i_2} \dots h_{i_{k-1},i_k} b_{i_k}$$

- Define a sequence of state transitions

$$\nu = i \rightarrow i_1 \rightarrow \cdots \rightarrow i_{k-1} \rightarrow i_k$$

- Use the adjoint Neumann-Ulam decomposition

$$\mathbf{H}^T = \mathbf{P} \circ \mathbf{W}$$

$$p_{ij} = \frac{|h_{ji}|}{\sum_j |h_{ji}|}, \; w_{ij} = \frac{h_{ji}}{p_{ij}}$$

The Hadamard product $\mathbf{A} = \mathbf{B} \circ \mathbf{C}$ is defined element-wise as $a_{ij} = b_{ij} c_{ij}$.

Evolution of a Solution

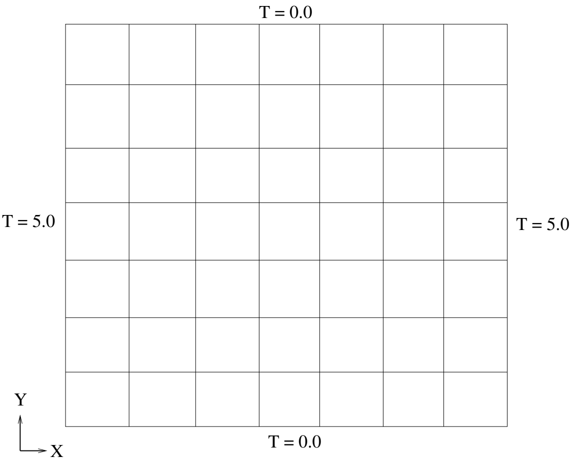

Figure: **Poisson Problem.** *Distributed source of 1.0 in the domain.*

Evolution of a Solution

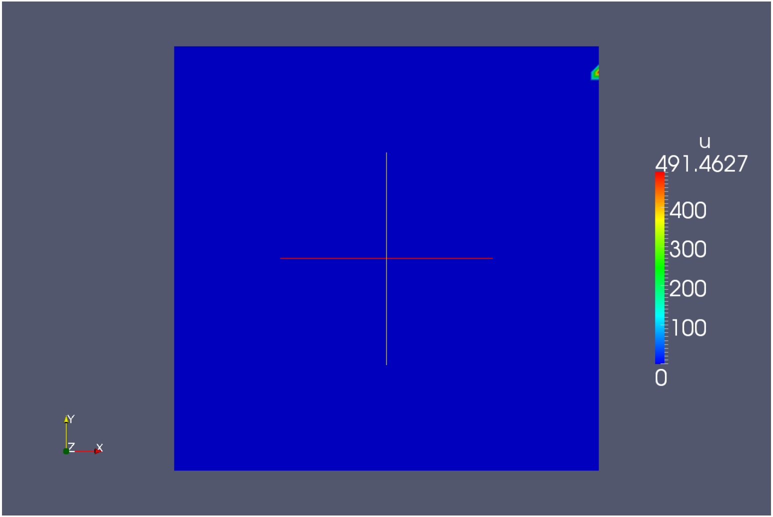

Figure: **Adjoint solution to Poisson Equation.** 1×10^0 *total histories, 0.286 seconds CPU time.*

Evolution of a Solution

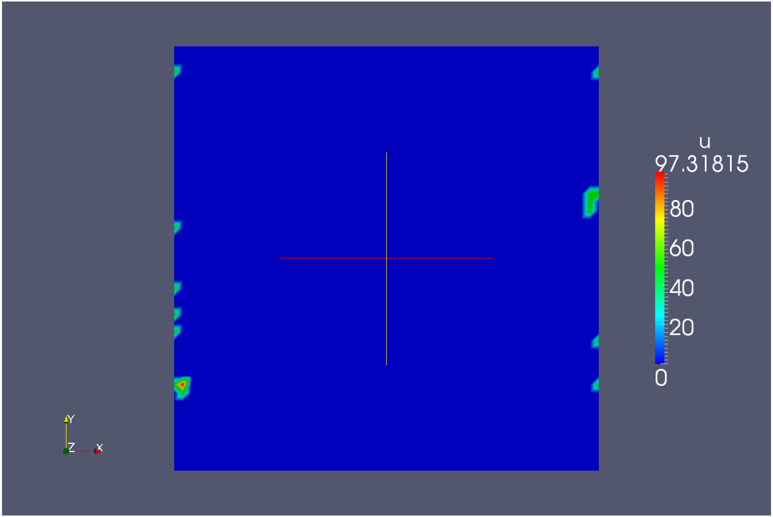

Figure: **Adjoint solution to Poisson Equation.** 1×10^1 *total histories, 0.278 seconds CPU time.*

Evolution of a Solution

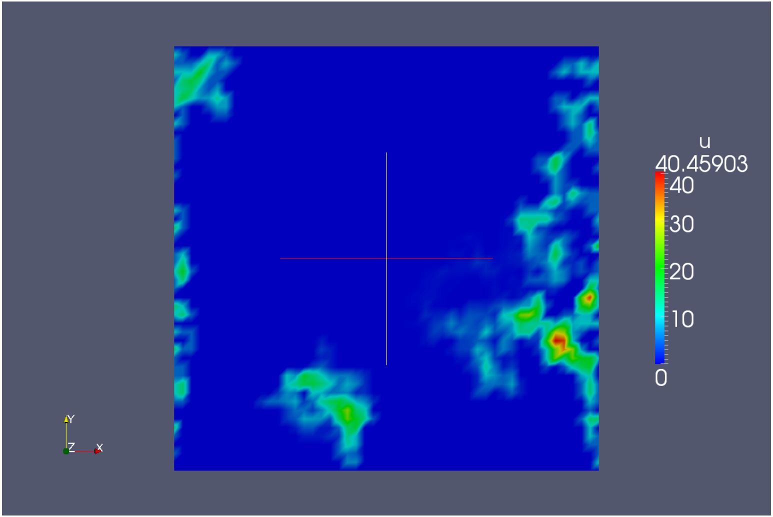

Figure: **Adjoint solution to Poisson Equation.** 1×10^2 *total histories, 0.275 seconds CPU time.*

Evolution of a Solution

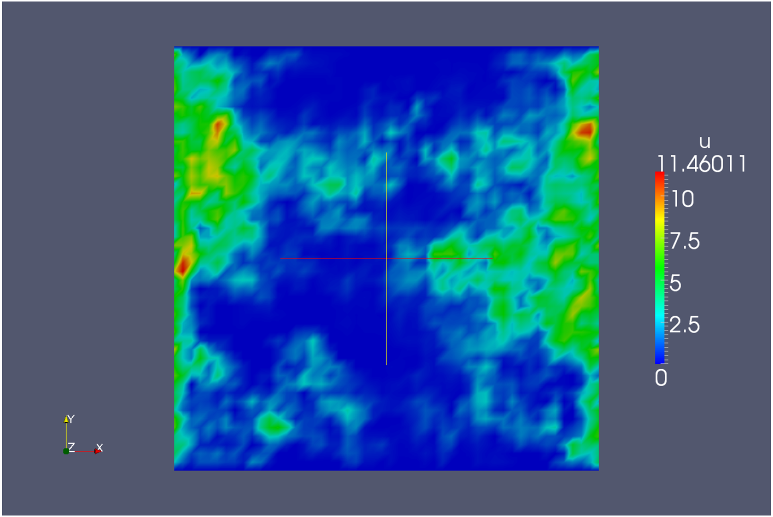

Figure: **Adjoint solution to Poisson Equation.** 1×10^3 *total histories, 0.291 seconds CPU time.*

Evolution of a Solution

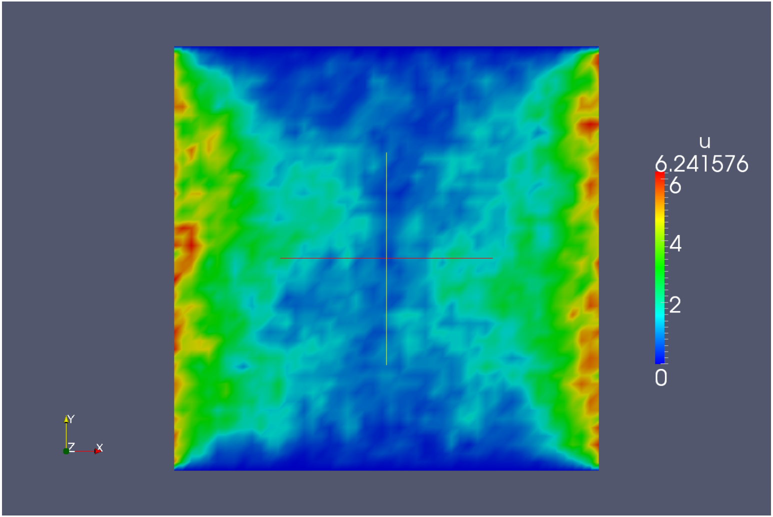

Figure: **Adjoint solution to Poisson Equation.** 1×10^4 *total histories, 0.428 seconds CPU time.*

Evolution of a Solution

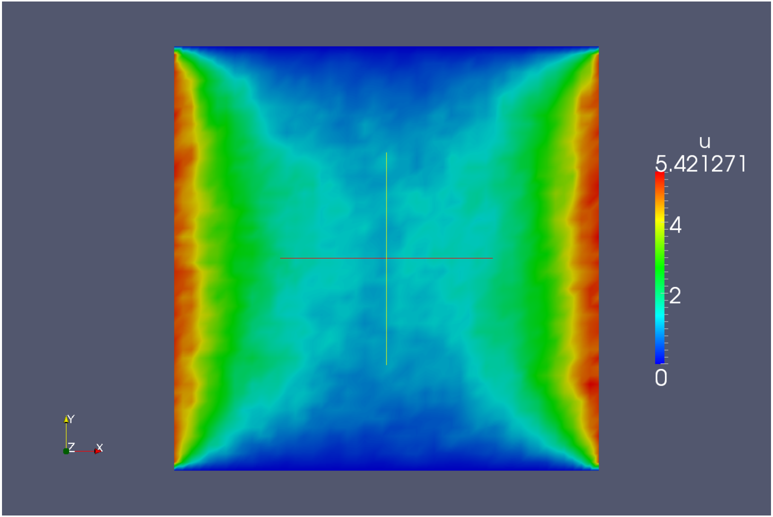

Figure: **Adjoint solution to Poisson Equation.** 1×10^5 *total histories, 1.76 seconds CPU time.*

Evolution of a Solution

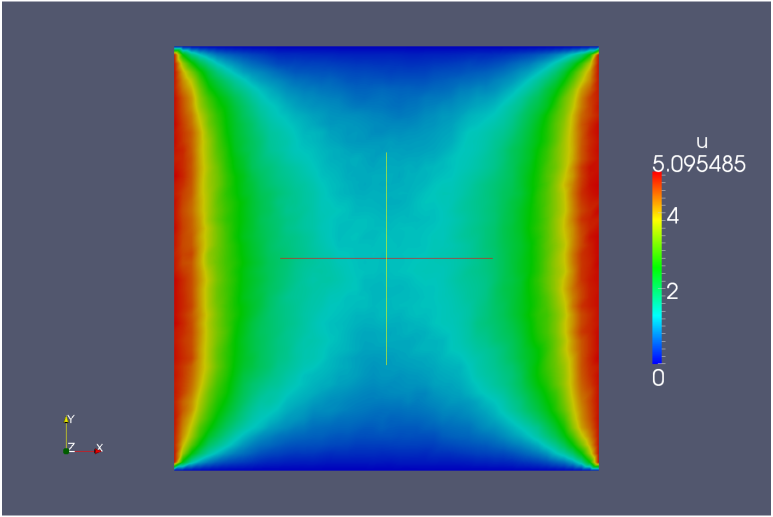

Figure: **Adjoint solution to Poisson Equation.** 1×10^6 *total histories, 15.1 seconds CPU time.*

Evolution of a Solution

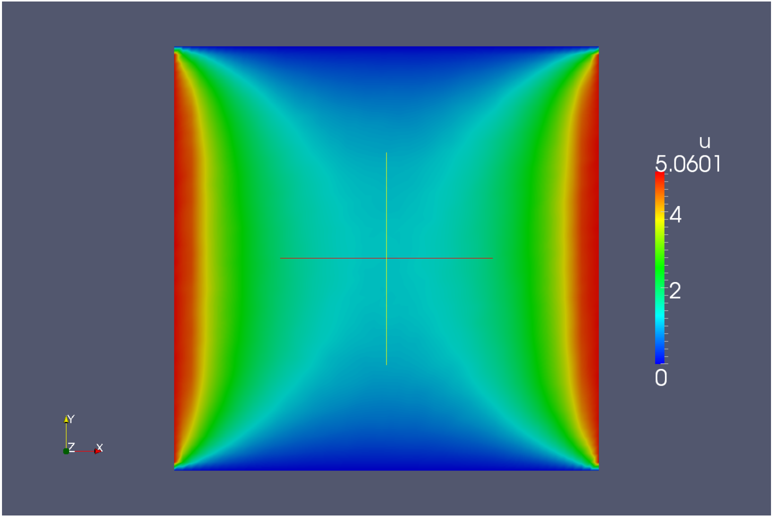

Figure: **Adjoint solution to Poisson Equation.** 1×10^7 *total histories, 149 seconds CPU time.*

Monte Carlo Synthetic-Acceleration

MCSA Iteration

$$\mathbf{r}^k = \mathbf{b} - \mathbf{A}\mathbf{x}^k$$

$$\mathbf{x}^{k+1/2} = \mathbf{x}^k + \mathbf{r}^k$$

$$\mathbf{r}^{k+1/2} = \mathbf{b} - \mathbf{A}\mathbf{x}^{k+1/2}$$

$$\hat{\mathbf{A}}\delta\mathbf{x}^{k+1/2} = \mathbf{r}^{k+1/2}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^{k+1/2} + \delta\mathbf{x}^{k+1/2}$$

- Neumann-Ulam methods bound by the Central Limit Theorem
- Build on Halton's 1962 Sequential Monte Carlo method
- Neumann-Ulam Monte Carlo solver computes the correction
- Decouples MC error from solution error, exponential convergence

Monte Carlo Linear Solvers Library (MCLS)

- Preliminary development during January of 2013
- General asynchronous MSOD MCSA implementation
  - Forward and adjoint Monte Carlo with method of expected values
  - Parallel row matrix/vector interface
  - General fixed point iteration strategy
  - Explicit algebraic preconditioner suite
- Implemented in C++
- Heavy use of the Trilinos scientific computing libraries
- Open-source BSD 3-clause license
- https://github.com/sslattery/MCLS

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SP_N Equations

$$\hat{\Omega} \cdot \vec{\nabla}\psi(\vec{r},\hat{\Omega},E) + \sigma(\vec{r},E)\psi(\vec{r},\hat{\Omega},E) = \iint \sigma_s(\vec{r},E' \to E,\hat{\Omega}' \cdot \hat{\Omega})\psi(\vec{r},\hat{\Omega}',E')d\Omega' dE' + q(\vec{r},\hat{\Omega},E)\,, \quad (1)$$

$$-\nabla \cdot \left[\frac{n}{2n+1}\frac{1}{\Sigma_{n-1}}\nabla\left(\frac{n-1}{2n-1}\phi_{n-2} + \frac{n}{2n-1}\phi_n\right) + \frac{n+1}{2n+1}\frac{1}{\Sigma_{n+1}}\nabla\left(\frac{n+1}{2n+3}\phi_n + \frac{n+2}{2n+3}\phi_{n+2}\right)\right] + \Sigma_n\phi_n = q\delta_{n0} \qquad n = 0,2,4,\cdots,N\,, \quad (2)$$

$$-\nabla \cdot \mathbb{D}_n \nabla \mathbb{U}_n + \sum_{m=1}^{4} \mathbb{A}_{nm}\mathbb{U}_m = \frac{1}{k}\sum_{m=1}^{4} \mathbb{F}_{nm}\mathbb{U}_n \qquad n = 1,2,3,4\,. \quad (3)$$

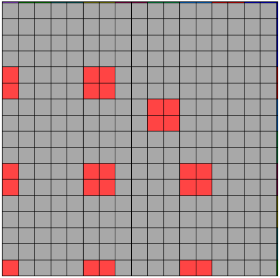
k-eigenvalue Solutions with MCSA

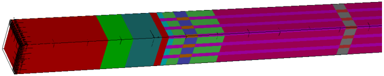
Algorithm 1 Power Iteration MCSA Scheme

$k_0 =$ initial guess
$\mathbf{\Phi}_0 =$ initial guess
$n = 0$
while $|\frac{k^n - k^{n-1}}{k^n}| < \epsilon$ **do**
  $\mathbf{M}\mathbf{\Phi}^{n+1} = \frac{1}{k^n}\mathbf{F}\mathbf{\Phi}^n$ {Solve for the new flux state with MCSA}
  $k^{n+1} = k^n \frac{\int \mathbf{F}\mathbf{\Phi}^{n+1} d\mathbf{r}}{\int \mathbf{F}\mathbf{\Phi}^n d\mathbf{r}}$
  $n = n + 1$
end while

- Swap in MCSA as the solver at each eigenvalue operation
- Use Thyra/Stratimikos interfaces for plug and play - Thanks Ross!
- Current Exnihilo implementation gives the full operator

Fuel Assembly Criticality Calculations





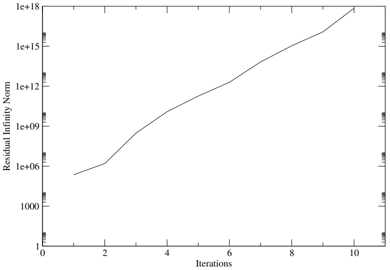
- CASL Problem 3: 17×17 quarter symmetry HZP LWR fuel assembly
- SP_N criticality calculation
- MCLS leveraged by the Exnihilo code base (ORNL)
- Comparison to Trilinos Aztec Krylov solvers with ILUT

Parameter	**Value**
Power Level	0 MW
Inlet Temperature	326.85C
Fuel Temperature	600C
Boron Concentration	1300 ppm
Moderator Density	0.743 g/cc
Helium Density	1.79×10^{-4} g/cc
Zirconium Density	6.56 g/cc
Stainless Steel Density	8.0 g/cc
Inconel Density	8.19 g/cc
UO2 Density	10.257 g/cc
Fuel Pin Radius (w/o clad)	0.4096 cm

Jacobi Preconditioned Calculations

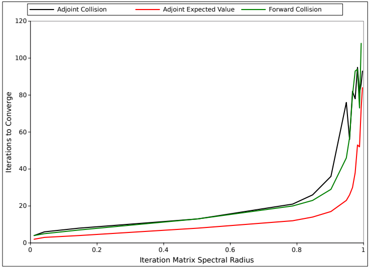
		SP_N Order			
		1	**3**	**5**	**7**
	0	0.0647	0.1275	0.1449	0.1514
	1	0.0686	0.1338	0.1484	0.1547
P_N Order	**3**	0.0687	0.1399	0.1582	0.1625
	5	0.0692	0.1399	0.1582	0.1657
	7	0.0678	0.1393	0.1624	0.166

Table: **Spectral radius results for the block Jacobi preconditioned iteration matrix with 10 energy groups and full downscatter for sample problem.**

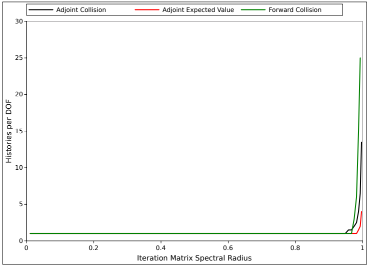


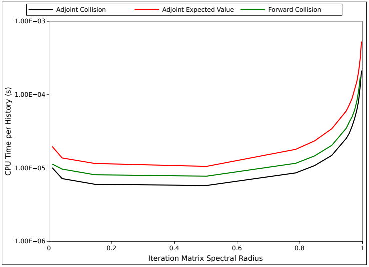
- Light water moderator creates a lot of scattering and $\rho(\mathbf{H}) \approx 1$
- Intractable number of histories required for MCSA convergence

MCSA Breakdown



- As $\rho(\mathbf{H}) \to 1$ terrible things happen...
- A more robust set of preconditioners is required for the SP_N equations





Explicit Preconditioning

$$\mathbf{M}_L^{-1}\mathbf{A}\mathbf{M}_R^{-1}\mathbf{M}_R\mathbf{x} = \mathbf{M}_L^{-1}\mathbf{b} \quad \rightarrow \quad \mathbf{M}_L^{-1}\mathbf{A}\mathbf{M}_R^{-1}\mathbf{u} = \mathbf{M}_L^{-1}\mathbf{b}$$

$$\mathbf{x} = \mathbf{M}_R^{-1}\mathbf{u}$$

Left/Right Preconditioned MCSA Iteration

$$\mathbf{r}^k = \mathbf{M}_L^{-1}(\mathbf{b} - \mathbf{A}\mathbf{M}_R^{-1}\mathbf{u}^k)$$

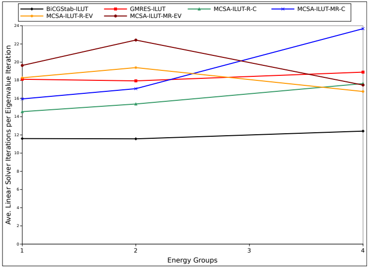
$$\mathbf{u}^{k+1/2} = \mathbf{u}^k + \mathbf{r}^k$$

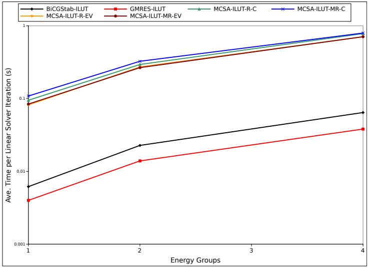
$$\mathbf{r}^{k+1/2} = \mathbf{M}_L^{-1}(\mathbf{b} - \mathbf{A}\mathbf{M}_R^{-1}\mathbf{u}^{k+1/2})$$

$$\mathbf{M}_L^{-1}\mathbf{A}\mathbf{M}_R^{-1}\delta\mathbf{u}^{k+1/2} = \mathbf{r}^{k+1/2}$$

$$\mathbf{u}^{k+1} = \mathbf{u}^{k+1/2} + \delta\mathbf{u}^{k+1/2}$$

Fuel Assembly Results





- MCLS generates the same k-eigenvalue and neutron flux in all groups and spatial locations
- MCSA converged in fewer iterations than GMRES, more iterations than BiCGStab
- Explicit preconditioning strategy destroys sparsity and elevates CPU times (Ifpack ILUT)
- Spectral radius and memory limitations combined to prevent solutions at finer discretizations

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- Monte Carlo Synthetic Acceleration Methods
- Application to Neutron Transport
- **Application to Fluid Flow**
- Parallelization of MCSA

Monte Carlo Synthetic Acceleration for Nonlinear Problems

- Many physics problems of interest are nonlinear
- Signficant research on Newton methods since the 1980's
- Newton methods often leverage Krylov solvers
  - Robust implementations
  - No operator required
- Monte Carlo methods need the full operator
- Automatic construction of the linear model is available
  - Operator overloading for nonlinear residual differentiation
  - Ideal for Monte Carlo
  - Potential scaling improvements
  - Resiliency benefits

Newton's Method

- Seek solutions of the general nonlinear problem

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}$$
$$\mathbf{u} \in \mathbb{R}^n,\ \mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$$

- Interpret the exact solution $\mathbf{u}$ to be the roots of $\mathbf{F}(\mathbf{u})$

$$\mathbf{F}(\mathbf{u}^{k+1}) = \mathbf{F}(\mathbf{u}^k) + \mathbf{F}'(\mathbf{u}^k)(\mathbf{u}^{k+1} - \mathbf{u}^k) + \frac{\mathbf{F}''(\mathbf{u}^k)}{2}(\mathbf{u}^{k+1} - \mathbf{u}^k)^2 + \cdots$$

- Form Newton's method

$$\mathbf{J}(\mathbf{u})\delta\mathbf{u}^k = -\mathbf{F}(\mathbf{u}^k)$$
$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\mathbf{u}^k$$

The FANM Method

Forward-Automated Newton-MCSA

Algorithm 2 FANM

1: $k := 0$
2: **while** $||\mathbf{F}(\mathbf{u}^k)|| > \epsilon ||\mathbf{F}(\mathbf{u}^0)||$ **do**
3: $\quad \mathbf{J}(\mathbf{u}^k) \leftarrow AD(\mathbf{F}(\mathbf{u}^k))$ {Automatic differentiation}
4: $\quad \mathbf{J}(\mathbf{u}^k)\delta\mathbf{u}^k = -\mathbf{F}(\mathbf{u}^k)$ {Solve for the Newton correction with MCSA}
5: $\quad \mathbf{u}^{k+1} \leftarrow \mathbf{u}^k + \delta\mathbf{u}^k$
6: $\quad k \leftarrow k + 1$
7: **end while**

- Robustness of Newton's method (inexact)
- Accuracy and convenience of FAD
- Potential parallelism, and resiliency benefits of MCSA
- Requires only nonlinear function evaluations
- Can utilize globalization and forcing term selection methods

FANM Numerical Experiments

- Sequence of Navier-Stokes benchmarks for the nonlinear methods
  - Thermal convection cavity problem (De Vahl Davis, 1983)
  - Lid driven cavity problem (Ghia et al., 1982)
  - Backward-Facing step problem (Gartling, 1990)
- Tuning benchmark parameters varies the strength of nonlinearities
- Newton-Krylov method leveraging Aztec GMRES used for comparisons
- All problems preconditioned with algebraic multigrid (ML) and leveraged some kind of globalization (e.g. backtracking)

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{T} - \rho \mathbf{g} = \mathbf{0}$$

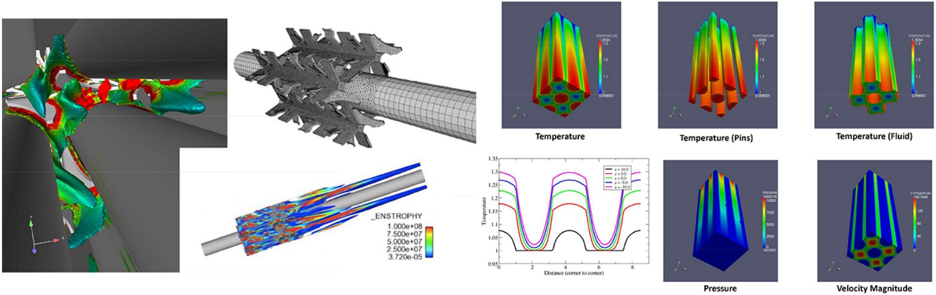
$$\nabla \cdot \mathbf{u} = 0$$

$$\rho C_p \mathbf{u} \cdot \nabla T + \nabla \cdot \mathbf{q} = 0$$

$$\mathbf{T} = -P\mathbf{I} + \mu[\nabla \mathbf{u} + \nabla \mathbf{u}^T]$$

$$\mathbf{q} = -k \nabla T$$

FANM Numerical Experiments



- Drekar is a production physics suite that leverages Newton methods with FAD
- MCLS incorporated into the Drekar nonlinear solver scheme using Thyra/Stratimikos to implement FANM
- Thanks Roger!

Images source: www.casl.gov

FANM Nonlinear Iteration Results

Benchmark	Newton-Krylov	FANM
Convection, Ra=1×10^3	5	5
Convection, Ra=1×10^4	7	7
Convection, Ra=1×10^5	9	10
Convection, Ra=1×10^6	11	11
Lid Driven, Re=100	6	6
Lid Driven, Re=300	9	9
Lid Driven, Re=500	11	11
Lid Driven, Re=700	14	10
Backward Step, Re=200	10	9
Backward Step, Re=300	15	14
Backward Step, Re=400	10	10
Backward Step, Re=500	19	20

Table: **Navier-Stokes benchmark comparison for nonlinear iterations.** *Over all benchmarks, FANM performed better in terms of nonlinear iterations for 1 more case than the Newton-Krylov method.*

FANM Linear Solver Iteration Results

Benchmark	Newton-Krylov	FANM
Convection, Ra=1×10^3	32	18
Convection, Ra=1×10^4	23	17
Convection, Ra=1×10^5	25	20
Convection, Ra=1×10^6	39	25
Lid Driven, Re=100	27	42
Lid Driven, Re=300	35	52
Lid Driven, Re=500	41	56
Lid Driven, Re=700	21	14
Backward Step, Re=200	24	13
Backward Step, Re=300	23	17
Backward Step, Re=400	18	12
Backward Step, Re=500	30	52

Table: **Navier-Stokes benchmark comparison for total linear solver iterations.** *Over all benchmarks, FANM performed better in terms of linear solver iterations for twice as many cases as the Newton-Krylov method.*

FANM CPU Time Results

Benchmark	Newton-Krylov Speedup
Convection, Ra=1×10^3	338
Convection, Ra=1×10^4	336
Convection, Ra=1×10^5	346
Convection, Ra=1×10^6	465
Lid Driven, Re=100	299
Lid Driven, Re=300	322
Lid Driven, Re=500	288
Lid Driven, Re=700	488
Backward Step, Re=200	400
Backward Step, Re=300	593
Backward Step, Re=400	825
Backward Step, Re=500	1057

Table: **Newton-Krylov speedup over FANM.** *For all benchmarks the explicit MCSA preconditioning strategy caused significantly larger CPU times for FANM when compared to the Newton-Krylov solutions.*

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Parallelization of Monte Carlo Methods

- No literature observed for parallel Neumann-Ulam solvers beyond history-level parallelism
- Numerous references for modern parallel Monte Carlo methods in reactor physics
- Build a strategy for applying modern methods to the Neumann-Ulam method
- MCSA iteration-level parallelism comes from parallel matrix/vector operations

Domain Decomposed Monte Carlo

- Each parallel process owns a piece of the domain (linear system)
- Random walks must be transported between adjacent domains through parallel communication
- Domain decomposition determined by the input system

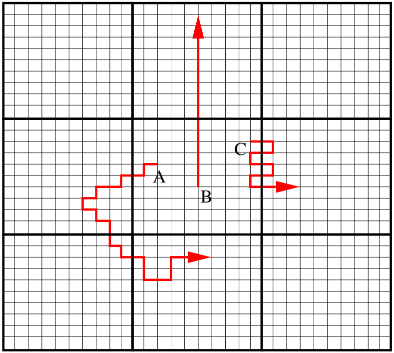
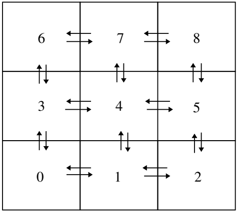
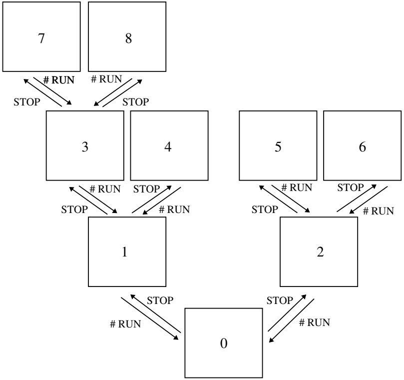


Figure: *Domain decomposition example illustrating how domain-to-domain transport creates communication costs.*

Asynchronous Monte Carlo Transport Kernel

- Developed by Brunner and Brantley in 2009
- Asynchronous nearest neighbor communication of histories
- Binary asynchronous communication tree for completing transport
- Extensible to problems where histories may be created (i.e. variance reduction)





Thomas A. Brunner and Patrick S. Brantley, "An efficient, robust, domain-decomposition algorithm for particle Monte Carlo", Journal of Computational Physics, vol. 228, pp.3882-3890, 2009.

Multiple-Set Overlapping-Domain Decomposition

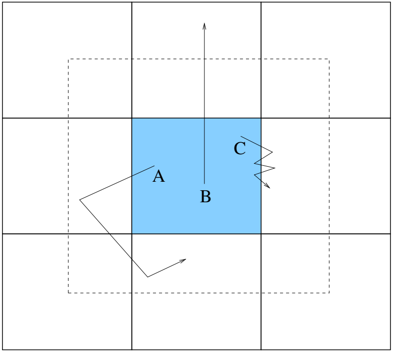


Figure: *Overlapping domain example illustrating how domain overlap can reduce communication costs.*

- Developed by Wagner and colleagues in 2010
- Each set contains the full domain
- Multiple sets replicate the domain
- Domains overlap within a set
- Redundancy for resiliency (and useful work)

Wagner et. al., "Hybrid and parallel domain-decomposition methods development to enable Monte Carlo for reactor analysis", Joint International Conference on Supercomputing in Nuclear Applications and Monte Carl (SNA+MC 2010), 2010.

Multiple-Set Overlapping-Domain Decomposition

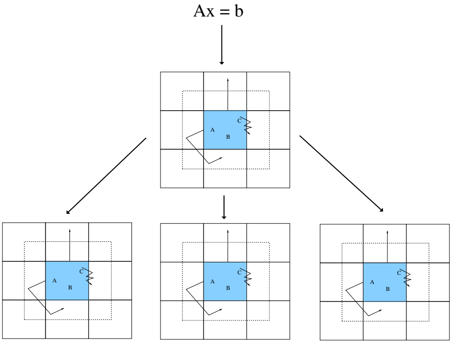


Figure: *MSOD construction.*

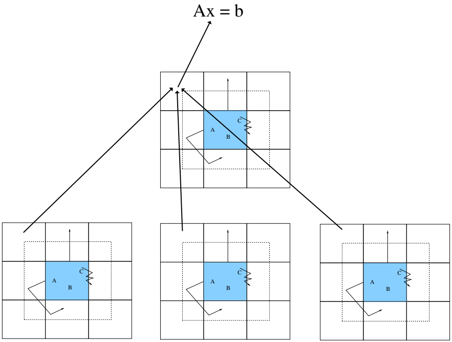


Figure: *MSOD tally reduction.*

Parallel Scaling

- Simple 2D neutron diffusion problem for control - spectral radius is maintained as global problem size grows
- Comparison to Trilinos Belos Krylov solvers with Jacobi preconditioning - conjugate gradient and GMRES
- Strong scaling - Global size fixed at 1.6E7 DOFs
- Weak scaling - Local size fixed at 4.0E4 DOFs
- Calculations performed on the Titan Cray XK7 machine at ORNL (MPI only)
- Limited MCLS arithmetic optimization artificially inflates efficiencies

Strong Scaling Results

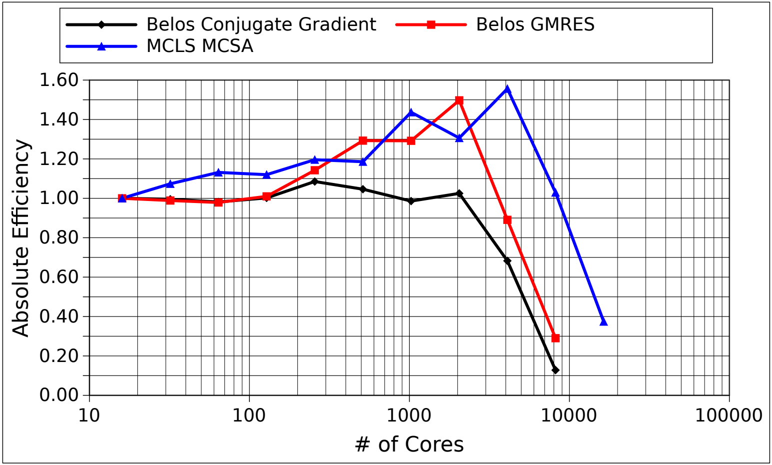
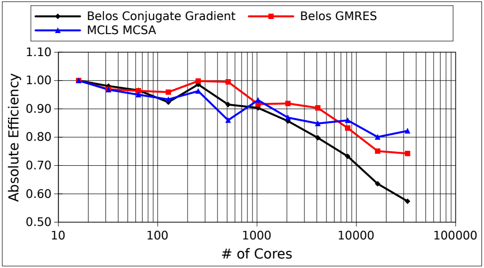
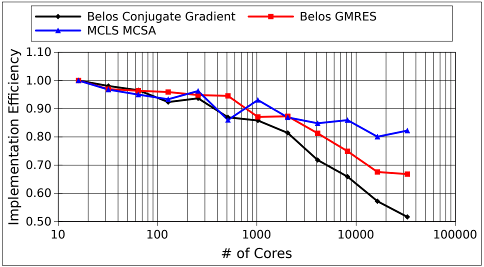


Figure: **Pure domain decomposition.** *Super-linear speed-up from memory thrashing in base case. MCLS is an order of magnitude slower arithmetically.*

Weak Scaling Results





- CG demonstrates poor scaling due to the cheaper iteration sequence
- Implementation efficiency is effectively the parallel efficiency of a single iteration
- Improved implementation efficiency means improved iterative performance will potentially give a better time to solution

Strong Scaling Results with Multiple Sets

Splitting: same global number of histories as the single set

Replicating: set-multiple of single set problem global histories

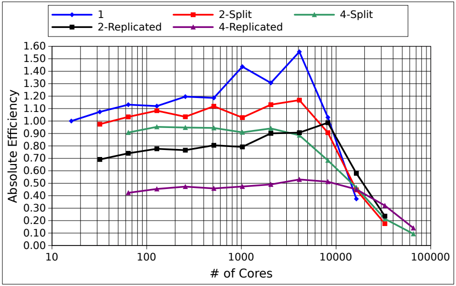


Figure: Absolute parallel efficiency relative to 16-core 1-set base case.

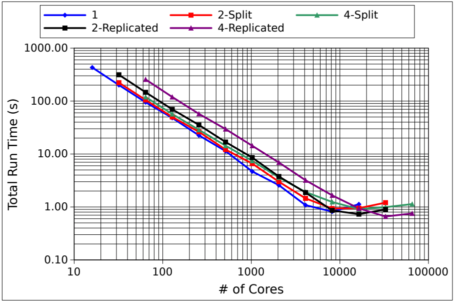


Figure: Wall time in seconds to solution for each case.

Weak Scaling Results with Multiple Sets

- Need to consider adding sets is a strong scaling exercise
- Modify the weak scaling efficiency computation to account for these extra resources
- Superposition of Monte Carlo results enhances time to solution!

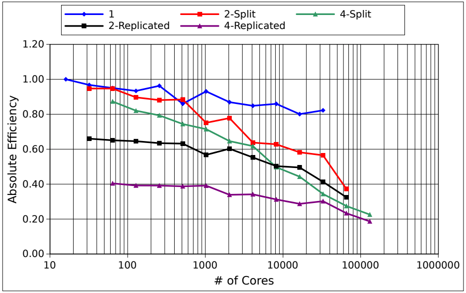


Figure: Absolute parallel efficiency relative to 16-core 1-set base case.

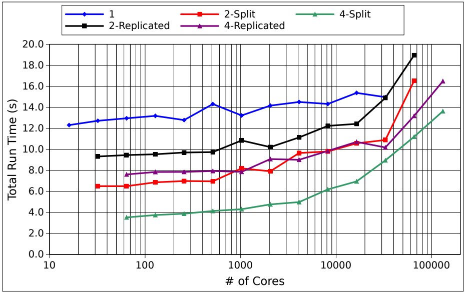


Figure: Wall time in seconds to solution for each case.

Strong Scaling Results with Overlap

- Overlap values selected based on average 'diffusion length' of a history in the system of 2.6 discrete states
- Overlap eliminates communication in the Monte Carlo sequence but simply differs it to an overlapping tally vector reduction

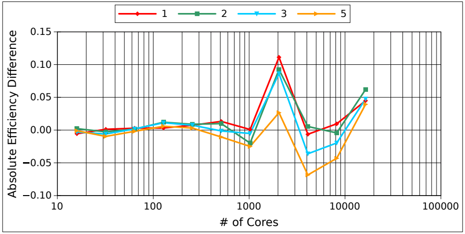


Figure: Strong scaling efficiency difference compared to the 0 overlap case.

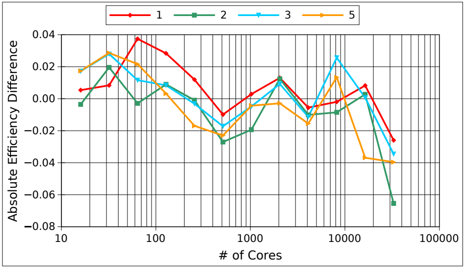


Figure: Weak scaling efficiency difference compared to the 0 overlap case.

MCSA as a Stochastic Additive Schwarz Method

- No domain-to-domain communication in Monte Carlo sequence
- Fixed point iteration acts as a smoother
- Observed to converge in the same number of iterations
- Can add overlap to preserve iterative performance for more ill-conditioned problems

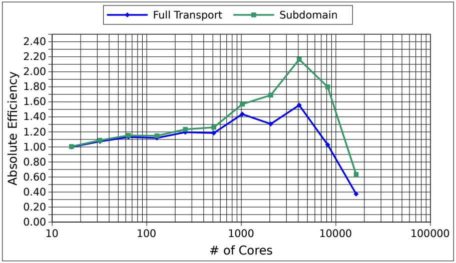


Figure: Strong scaling absolute efficiency for pure domain decomposition.

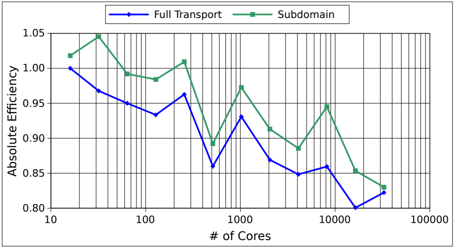


Figure: Weak scaling absolute efficiency for pure domain decomposition.

MCSA Application to Neutronics Summary

- MCSA can solve the asymmetric system generated by the SP_N equations
- Light water reactor problems are difficult to solve with MCSA as they have large spectral radii due to the neutron scattering in the moderator
- Advanced algebraic preconditioning strategies were applied to the SP_N equations to obtain convergence with ILUT chosen for subsequent investigations
- MCSA was verified to produce the same flux distribution and k-eigenvalue for the fuel assembly as production Krylov methods
- MCSA was observed to converge in fewer iterations per eigenvalue iteration than GMRES for the fuel assembly criticality problem and more than Bi-CGStab using the same preconditioning

MCSA Application to Fluid Flow Summary

- Forward-Automated Newton-MCSA (FANM) has been developed
- The FANM method has been verified to produce the same solutions as a production Newton-Krylov method for three difficult benchmark problems
- The FANM method has better iterative performance than the Newton-Krylov method for convection dominated problems, converging in fewer linear solver iterations with the same preconditioning for high and low Rayleigh numbers
- The spectral radius convergence restriction on MCSA was observed to be a significant hindrance by preventing solutions to forced flow problems at high Reynolds numbers
- More Monte Carlo histories at every FANM iteration can reduce the number of linear and nonlinear iterations required to converge the problem

Parallelization of MCSA Summary

- The multiple-set overlapping-domain (MSOD) parallel algorithm for particle transport has been adapted to parallelize MCSA
- MCSA scales favorably compared to production Krylov methods for both strong and weak scaling cases
- Overlap in small quantities can provide parallel efficiency boosts of a few percent in strong scaling cases but is ineffective in weak scaling cases
- Multiple sets offers a means to reduce time to solution by solving multiple copies of the original problem and combining the solutions using superposition
- MCSA is most efficiently used in parallel as a stochastic realization of an additive Schwarz method

Future Work

- Shortcomings observed on real problems
  - Significant optimization required to determine production feasibility and true scalability
  - Explicit algebraic preconditioning methods not sufficient
  - Spectral radius limitation is severe
- Performance improvements
  - Random walk optimizations
  - Alias sampling implementations
  - Multiple set reduction analysis
  - FANM forcing term and MCSA history relationships
- Preconditioning improvements
  - Alternative left/right sampling schemes
  - Variance reduction based strategy
  - Reduced order physics/PDE models for acceleration
- Breaking away from $\rho(\mathbf{H}) < 1$
  - Monte Carlo methods of the second degree
  - Stochastic projection methods

Discussion